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CONSISTENT DISCRETIZATION AND CANONICAL CLASSICAL AND QUANTUM REGGE CALCULUS

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We apply the “consistent discretization” technique to the Regge action for (Euclidean and Lorentzian) general relativity in arbitrary number of dimensions. The result is a well defined canonical theory that is free of constraints and where the dynamics is implemented as a canonical transformation. In the Lorentzian case, the framework appears to be naturally free of the “spikes” that plague traditional formulations. It also provides a well defined recipe for determining the integration measure for quantum Regge calculus.

Keywords: Consistent discretization; Regge calculus

Dedicated to Rafael Sorkin on his 60th birthday.

1. Introduction

Regge calculus¹ has been proposed as an approach to classical and quantum general relativity. It consists in approximating space-time by a simplicial decomposition. The fundamental variables of the theory are the lengths of the edges of the simplices. This approach has been demonstrated in numerical simulations of classical general relativity and also has inspired attractive ideas for the quantization of gravity. For instance an extension of this framework led to the successful quantization of 2 + 1 dimensional Euclidean gravity through the Ponzano–Regge model², which can also be seen as one of the key motivations for the “spin-foam” approaches to 3 + 1 dimensional quantum gravity. There has been quite a bit of work devoted over the years to Regge calculus, for a recent review including related formulations see Loll³, and for a earlier pedagogical presentations see Misner, Thorne and Wheeler⁴.

A canonical formulation for Regge calculus has nevertheless, remained elusive (for a review see Williams and Tuckey⁵). We have recently introduced a methodology

to treat discrete constrained theories in a canonical fashion, which has been usually called “consistent discretizations”. The purpose of this paper is to show that this methodology can be successfully applied to Regge calculus without any need for modifications of the Regge action. The resulting theory is a proper canonical theory that is consistent, in the sense that all its equations can be solved simultaneously. As is usually the case in “consistent discretizations” the theory is constraint-free (although as is usual in Regge calculus there are triangle inequalities to be satisfied among the variables). We will see that the treatment can be applied in both the Euclidean and Lorentzian case. In the latter case there is an added bonus: in order to have a well defined canonical structure one naturally eliminates “spikes” that have been a problem in Regge formulations in the past⁶ at the time of considering the continuum limit. This is due to the fact that our simplices only have one time-like hinge. It is therefore not possible to construct simplices with infinitesimal volume and arbitrary length. If one lengthens the time-like hinge one necessarily has to lengthen the space-like hinges and therefore increase the volume. Therefore one will not see the quantum amplitude dominated by long simplices of vanishing volume.

2. Consistent discretization

To make the calculations and illustrations simpler, we will concentrate on three dimensional gravity, but the reader will readily notice that there is no obstruction to applying the same reasonings in 3 + 1 dimensions. Given a simplicial approximation to a three dimensional manifold, one can approximate the Einstein action (with a cosmological term), as a sum over the edges (“hinges”) of the decomposition plus a sum over the simplices¹,

$$S = k \sum_h \ell_h \delta_h + \lambda \sum_\sigma V_\sigma \quad (1)$$

where the first sum is over all hinges and the second over all simplices, ℓ_h is the length of the hinge h and δ_h is the deficit angle around the hinge, i.e. $\delta_h = 2\pi - \sum_{\sigma_h} \Theta(\sigma_h)$ where $\Theta(\sigma_h)$ is the angle formed by the two faces of the simplices σ_h that end in the hinge h . V_σ is the volume of the simplex σ (in our three dimensional case, a tetrahedron). The constants k and λ are related to Newton’s constant and the cosmological constant. A more explicit expression (see for instance David⁷) can be given involving the values of the volumes of the two (in three dimensions) faces which share the hinge h , $\sin \Theta(\sigma_h) = 3/2V_{\sigma_h} \ell_h / (A_{\sigma_h} A'_{\sigma_h})$, where A and A' are the areas of the two triangles adjacent to h in the simplex σ_h . This in turn can be used to give an expression that is purely a function of the lengths of the hinges, using the Cayley–Menger determinants. We do not quote its explicit expression for brevity.

In order to have a formulation that is amenable to a canonical treatment that is uniform, in the sense that one has the same treatment at all points on the lattice, one needs to make certain assumptions about the regularity of the simplicial decomposition chosen. This requirement can be somewhat relaxed and our method still applies, but in a first approach we will consider a regular decomposition as shown in

figure 1. We have divided space-time in prisms (1 and 2 in the figure, for example), and each prism in turn can be decomposed into three tetrahedra (in the case of prism 2 the tetrahedra would be given by vertices $ABB'D$, $AB'D'D$, $A'E'DD'$).

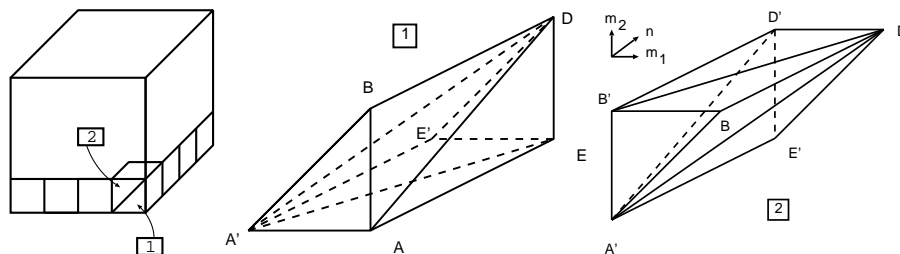


Fig. 1. The simplicial decomposition considered. The figures on the right show prisms number 1 and 2 respectively, the other prisms are obtained by reflection and periodicity. The hinge length variables ℓ_i are assigned to the hinges in the following way: $A'A \mapsto \ell_1$, $A'B \mapsto \ell_2$, $A'B' \mapsto \ell_3$, $A'E \mapsto \ell_4$, $A'D \mapsto \ell_5$, $A'D' \mapsto \ell_6$, $A'E' \mapsto \ell_7$.

To construct a Lagrangian picture for the previous action we consider two generic “instants of time” n and $n+1$, as indicated by the direction labeled n in figure 1. We wish to construct an action of the form $S = \sum_n L(n, n+1)$ where the Lagrangian $L(n, n+1)$ depends on variables only at instants n and $n+1$. We choose one of the fundamental cubes (union of prisms 1 and 2 in the figure), choose a conventional vertex in the cube labeled by n, m_1, m_2 in the lattice. Notice that the use of the cubes is just for convenience, the framework is based on prisms that have a triangular spatial basis and therefore can tile any bidimensional spatial manifold. The variables we will consider are the lengths ℓ_1, \dots, ℓ_7 emanating from the vertex, as designated in the figure. A similar construction is repeated for each fundamental cube. The Lagrangian that reproduces the Regge action is given by a function

$$L(n, n+1) = \sum_{m_1, m_2} L \left(\ell_1(n, m_1, m_2), \dots, \ell_7(n, m_1, m_2), \right. \\ \left. \ell_1(n+1, m_1, m_2), \ell_2(n+1, m_1, m_2), \ell_3(n+1, m_1, m_2) \right), \quad (2)$$

that includes step functions that enforce the triangle inequalities between the hinge length variables.

Up to now we have kept the discussion generic, but we should now make things more precise, in dealing with either the Euclidean or the Lorentzian case. In the former, all angles and quantities involved are real. In the Lorentzian case, angles can become complex. Moreover lengths can be time-like or space-like. Null intervals can also be considered, but make the formulas more complicated, so for simplicity we do not consider them here. We will take all lengths as positive numbers, irrespective of the space-like or time-like character of the underlying hinge. In the above construction we have chosen the decomposition in such a way that the hinge ℓ_7 is time-like and all other hinges are space-like. The formulas presented above (for the angles,

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for instance) are valid in both the Euclidean and Lorentzian case, but in the latter volumes, areas and length may have to be considered as imaginary numbers. All volumes involving a time-like direction are real, and in the construction these are the only ones involved. Areas are imaginary if they involve one time-like direction and real if they do not. Lengths are real if they are time-like and purely imaginary if they are space-like. With these conventions, dihedral angles around time-like directions are real (for instance around ℓ_7), and dihedral angles around space-like directions are complex. Some can be purely imaginary (for instance rotation around $A'A$ in tetrahedron $AA'BD$) which correspond to Lorentz boosts, or complex (for instance rotation around AB in the aforementioned tetrahedron) which does not correspond to a Lorentz transformation (it traverses the light cone). There is one further point to consider. In the expression for the deficit angle the term 2π is present for hinges that span from the base of the elementary cube to the top cover. For hinges lying entirely within the base or the top cover the term is π . With these conventions the Lagrangian $L(n, n+1)$ turns out to be real and the sum yields the correct action avoiding over counting. For a more detailed discussion of angles in the Lorentzian case see Sorkin⁸.

We now proceed to treat this action with the “consistent discretization” approach. We consider as configuration variables ℓ_1, \dots, ℓ_7 and define their canonical momenta,

$$P_{\ell_i}(n+1) = \frac{\partial L(n, n+1)}{\partial \ell_i(n+1)}, \quad (3)$$

$$P_{\ell_i}(n) = -\frac{\partial L(n, n+1)}{\partial \ell_i(n)}. \quad (4)$$

Here one is faced with several constraints. Notice that variables ℓ_4, \dots, ℓ_7 are “Lagrange multipliers” since the Lagrangian does not depend on their value at instant $n+1$ and therefore their canonical momenta vanish. The $P_{\ell_1}, \dots, P_{\ell_3}$ only depend on links at level n and therefore are constraints among the variables. The system of equations determines variables ℓ_4, \dots, ℓ_7 and their momenta in terms of the other variables so they can be eliminated. The resulting canonical pairs are $\ell_1, \dots, \ell_3, P_{\ell_1}, \dots, P_{\ell_3}$. The remaining equations are evolution equations for these variables and there are no constraints left (in the sense of dynamical constraints, the variables are still constrained by the usual triangle inequalities). The evolution equations are a true canonical transformation from the variables at instant n to the variables at instant $n+1$. This canonical transformation has as generating function $-L(n, n+1)$, viewed as a type 1 canonical transformation, where in the Lagrangian the variables ℓ_4, \dots, ℓ_7 have been replaced via the equations that determine them. The reader unfamiliar with the “consistent discretization” approach may question the legitimacy of this procedure in the sense of yielding a true canonical structure, however it was discussed⁹ how the canonical structure arises in detail through a generalization of the Dirac procedure for discrete systems.

This concludes the classical discussion. We have reduced the Regge formulation to a well defined, unconstrained canonical system where the discrete time evolution is implemented as a canonical transformation. Some of the original dynamical variables are eliminated from the formulation using the equations of motion. In the usual “consistent discretizations” the variables that are eliminated are the Lagrange multipliers. Here the links that get determined can be viewed as playing a similar role. The equations that determine these variables are a complicated non-linear system. As in the usual “consistent discretization” approach, one may have concerns that the solutions of the non-linear system could fail to be real, or could become unstable. We now have experience with consistent discretizations of mechanical systems and of Gowdy cosmologies, which have field theoretic degrees of freedom and the evidence suggests that one can approximate the continuum theory well in spite of potential complex solutions and multi-valued branches¹⁰. We expect a similar picture to occur in Regge calculus.

3. Quantization

Turning our attention to quantization, as usual in the “consistent discretization” approach, the hard conceptual issues are sidestepped since the theory is constraint-free. The task at hand is to implement the canonical transformation that yields the discrete time evolution as a unitary quantum operator that implements the discrete classical equations of motion as quantum operatorial equations. This will in a generic situation be computationally intensive, but conceptually clear. It should be noted that the resulting unitary operator differs significantly from the ones that have been historically proposed in path integral approaches based on Regge calculus. The usual approach to a path integral would be to compute,

$$\int \prod_{i=1,7,n,\vec{m}} d\ell_i(n, \vec{m}) \mu(\ell_1(n, \vec{m}), \dots, \ell_7(n, \vec{m})) \times \quad (5)$$

$$\times \exp \left(i \sum_{n', \vec{m}'} L(\ell_1(n, \vec{m}), \dots, \ell_7(n, \vec{m}), \ell_1(n+1, \vec{m}'), \dots, \ell_3(n+1, \vec{m}')) \right),$$

with μ a measure that presumably should enforce the constraints of the theory. On the other hand, in our approach one would have something like

$$\int \prod_{i=1,3,n,\vec{m}} d\ell_i(n, \vec{m}) \mu(\ell_1(n, \vec{m}), \dots, \ell_3(n, \vec{m})) \times \quad (6)$$

$$\times \exp \left(i \sum_{n', \vec{m}'} L'(\ell_1(n, \vec{m}), \dots, \ell_3(n, \vec{m}), \ell_1(n+1, \vec{m}'), \dots, \ell_3(n+1, \vec{m}')) \right),$$

where L' is obtained by substituting in L the values of the “Lagrange multipliers” ℓ_4, \dots, ℓ_7 obtained from their equations of motion. μ is uniquely determined when one determines the unitary transformation that implements the dynamics (examples of this in cosmological situations can be seen in our paper¹¹). So we see that we

have eliminated some of the variables and the constraints of the theory and the path integral is uniquely defined by the consistent discretization approach.

4. Discussion

Some concerns might be raised about the limitations imposed on our framework by the choice of initial lattice. We have chosen to use a lattice that is topologically cubical. This sets a well defined framework in which to construct a Lagrangian evolution between two spatial hypersurfaces. The cubic lattice is not a strict requirement. It would be enough to have two “close in time” space-like hypersurfaces with the same simplicial decomposition in both for us to be able to set up our framework and start evolving. This can encompass quite a range of geometrical situations. It is however, inevitable that one should give up some arbitrariness in the space-time simplicial decomposition if one wishes to have a canonical structure. It is interesting that the structure imposed is such that it automatically eliminates the “spikes” (thin simplices arbitrarily large in the time-like direction).

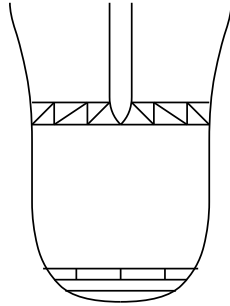


Fig. 2. The framework can handle topology change.

It is also worthwhile emphasizing that the framework can, with relatively simple additions, incorporate topology change. The idea is depicted in figure 2. There we see a point where there is topology change where the legs of the pair of pants separate. For that to happen one would need to modify at the hypersurface the explicit form of the Lagrangian $L(n, n + 1)$. It is interesting that from there on one can continue without further altering the framework and that at all times the number of variables involved has not changed. The picture also shows how one would handle an initial “no boundary” type singularity. Here one would have to “by hand” add links as the time evolution progresses forward. These variables are free as long as one does not wish to match some final end state for the evolution. If one has, however, specified initial and final data for the evolution, one finds that constraints appear that determine the values of the lengths of the extra links added.

As an example of the framework, one can work out explicitly the evolution of a $2 + 1$ dimensional space-time consistent of a four adjacent “unit cubes” of the

type we have considered with fixed outer boundary conditions. In this case the initial data consists of eight lengths, the other initial lengths are determined by the boundary conditions. As we discussed, having the data at level n and $n + 1$ one can determine the “Lagrange multiplier” links that have to be substituted in the Lagrangian to generate the canonical transformation between the initial and final data. This transformation is later to be implemented unitarily upon quantization. In our formalism it can happen that the “Lagrange multipliers” are not entirely determined by this procedure (for other examples where this happens, including BF theory, see our paper¹²). In such case the resulting theory has true constraints and true Lagrange multipliers. In this example this happens. One is finally left with a canonical transformation dependent on 3 parameters (this result is also true if one considers an $N \times N$ adjacent unit cube system). The presence of free parameters also requires modifications in the path-integral formulas listed above as well. It should be emphasized that the equations determining the Lagrange multipliers, even in this simplified case, are complicated coupled non-linear equations that have a complexity not unlike those in $3 + 1$ dimensions. What makes them easy to solve is the knowledge that the Regge equations of motion correspond in this case to flat space-time. The canonical transformation can be implemented unitarily and the quantization completed. We will discuss the example in detail in a separate publication.

5. Conclusions

We have applied the “consistent discretization” approach to Regge calculus. We see that it leads to a well defined constraint-free canonical formulation, that is well suited for quantization. The approach can incorporate topology change. Although we have limited the equations to the three dimensional case for simplicity, we have never used any of the special properties of three dimensional gravity and it is clear that the construction can be carried out in an arbitrary number of dimensions. It is interesting to notice that one of the original motivations for the construction of the “consistent discretization” approach was the observation by Friedman and Jack¹³ that in canonical Regge calculus the Lagrange multipliers failed to be free. It can be viewed as if this point of view has now been exploited to its fullest potential, offering a well defined computational avenue to handle classical and quantum gravity.

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