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Consistent discretizations as a road to quantum gravity

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Abstract

We present a brief description of the “consistent discretization” approach to classical and quantum general relativity. We exhibit a classical simple example to illustrate the approach and summarize current classical and quantum applications. We also discuss the implications for the construction of a well defined quantum theory and in particular how to construct a quantum continuum limit.

1.1 Consistent discretizations: the basic idea

There has long been the hope that lattice methods could be used as a non-perturbative approach to quantum gravity. This is in part based on the fact that lattice methods have been quite successful in the treatment of quantum chromodynamics. However, one needs to recall that one of the appeals of lattice methods in QCD is that they are gauge invariant regularization methods. In the gravitational context this is not the case. As soon as one discretizes space-time one breaks the invariance under diffeomorphisms, the symmetry of most gravitational theories of interest. As such, lattice methods in the gravitational context face unique challenges. For instance, in the path integral context, since the lattices break some of the symmetries of the theory, this may complicate the use of the Fadeev–Popov technique. In the canonical approach if one discretizes the constraints and equations of motion, the resulting discrete equations are inconsistent: they cannot be solved simultaneously. A related problem is that the discretized constraints fail to close a constraint algebra.

To address these problems we have proposed (Gambini & Pullin 2003b,

Di Bartolo *et. al.* 2002) a different methodology for discretizing gravitational theories (or to use a different terminology “to put gravity on the lattice”). The methodology is related to a discretization technique that has existed for a while in the context of unconstrained theories called “variational integrators” (Lew *et. al.* 2004). In a nutshell, the technique consists in discretizing the action of the theory and working from it the discrete equations of motion. Automatically, the latter are generically guaranteed to be consistent. The resulting discrete theories have unique features that distinguish them from the continuum theories, although a satisfactory canonical formulation can be found for them (Di Bartolo *et. al.* 2005). The discrete theories do not have constraints associated with the space-time diffeomorphisms and as a consequence the quantities that in the continuum are the associated Lagrange multipliers (the lapse and the shift) become regular variables of the discrete theories whose values are determined by the equations of motion. We call this approach in the context of constrained theories “consistent discretizations”.

The consistently discretized theories are both puzzling and attractive. On the one hand, it is puzzling that the Lagrange multipliers get fixed by the theory. Don’t the Lagrange multipliers represent the gauge freedom of general relativity? The answer is what is expected: the discretization breaks the freedom and solutions to the discrete theory that are different correspond, in the continuum limit, to the same solution of the continuum theory. Hence the discrete theory has more degrees of freedom. On the other hand, the lack of constraints make the consistently discretized theories extremely promising at the time of quantization. Most of the hard conceptual questions of quantum gravity are related to the presence of constraints in the theory. In comparison, the consistently discretized theories are free of these conceptual problems and can be straightforwardly quantized (to make matters even simpler, as all discrete theories, they have a finite number of degrees of freedom). In addition, they provide a framework to connect the path integral and canonical approaches to quantum gravity since the central element is a unitary evolution operator. In particular they may help reconcile the spin foam and canonical loop representation approaches. They also provide a natural canonical formulation for Regge calculus (Gambini & Pullin 2005c).

In this article we would like to briefly review the status of the consistent discretization approach, both in its application as a classical approximation to gravitational theories and as a tool for their quantization. Other brief reviews with different emphasis can be seen in Gambini & Pullin (2005a, 2005b). The organization of the article is as follows. In

section 1.2 we consider the application of the technique to a simple, yet conceptually challenging mechanical model and discuss how features that one observes in the model are actually present in more realistic situations involving general relativity. In section 1.3 we outline various applications of the framework. In section 1.4 we discuss in detail the quantization of the discrete theories and in section 1.5 we outline how one can define the quantum continuum limit. We end with a summary and outlook.

1.2 Consistent discretizations

To introduce and illustrate the method in a simple —yet challenging— model we consider the model analyzed in detail by Rovelli (1990) in the context of the problem of time in canonical quantum gravity: two harmonic oscillators with constant energy sum. We have already discussed this model in some detail in Gambini & Pullin (2005b) but we would like to revisit it here to frame the discussion with a different emphasis.

The model has canonical coordinates q^1, q^2, p^1, p^2 with the standard Poisson brackets and a constraint given by,

$$C = \frac{1}{2} ((p^1)^2 + (p^2)^2 + (q^1)^2 + (q^2)^2) - M = 0, \quad (1.1)$$

with M a constant. The model is challenging since no standard unconstrained Hamiltonian formulation can correspond to this dynamical system since the presymplectic space is compact and therefore cannot contain any $S \times R$ structure. Nevertheless, we will see that the consistent discretization approach does yield sensible results. This helps dispel certain myths about the consistent discretization scheme. Since it determines Lagrange multipliers, a lot of people tend to associate the scheme with some sort of “gauge fixing”. For this model however, a gauge fixing solution would be unsatisfactory, since it would only cover a portion of phase space. We will see that this is not the case in the consistent discretization scheme. We will also see that the evolution scheme is useful numerically in practice.

We start by writing a discrete Lagrangian for the model,

$$\begin{aligned} L(n, n+1) = & p_n^1 (q_{n+1}^1 - q_n^1) + p_n^2 (q_{n+1}^2 - q_n^2) \\ & - \frac{N_n}{2} ((p_n^1)^2 + (p_n^2)^2 + (q_n^1)^2 + (q_n^2)^2 - 2M), \end{aligned} \quad (1.2)$$

and working out the canonical momenta for all the variables, i.e., P_q^1 ,

P_q^2, P_p^1, P_p^2 . The momenta of a variable at level n are obtained by differentiating $L(n, n+1)$ with respect to the variable at level $n+1$. One then eliminates the $p^{1,2}$ and the $P_p^{1,2}$ and is left with evolution equations for the canonical pairs,

$$q_{n+1}^1 = q_n^1 + N_n (P_{q,n}^1 - 2q_n^1) \quad (1.3)$$

$$q_{n+1}^2 = q_n^2 + N_n (P_{q,n}^2 - 2q_n^2) \quad (1.4)$$

$$P_{q,n+1}^1 = P_{q,n}^1 - N_n q_n^1 \quad (1.5)$$

$$P_{q,n+1}^2 = P_{q,n}^2 - N_n q_n^2. \quad (1.6)$$

The Lagrange multiplier gets determined by the solution(s) of a quadratic equation that is obtained working out the momenta of the Lagrange multipliers,

$$\begin{aligned} & ((q_n^1)^2 + (q_n^2)^2) (N_n)^2 - 2 (P_{q,n}^1 q_n^1 + P_{q,n}^2 q_n^2) N_n + \\ & + (P_{q,n}^1)^2 + (P_{q,n}^2)^2 + (q_n^1)^2 + (q_n^2)^2 - 2M = 0. \end{aligned} \quad (1.7)$$

The resulting evolution scheme when one eliminates the Lagrange multipliers using equation (1.7) constitutes a canonical transformation between instants n and $n+1$. This result may appear puzzling at first, a general discussion of how this can be framed in a Dirac-like approach for discrete theories can be seen in Di Bartolo *et. al.* (2005).

We would like to use this evolution scheme to follow numerically the trajectory of the system. For this, we need to give initial data. Notice that if one gives initial data that satisfy the constraint identically at level n , the quadratic equation for the lapse has a vanishing independent term and therefore the solution is that the lapse N vanishes (the non-vanishing root will be large and would imply a large time evolution step that puts us away from the continuum generically). To construct initial data one therefore considers a set for which the constraint vanishes and introduces a small perturbation on one (or more) of the variables. Then one will have evolution. Notice that one can make the perturbation as small as desired. The smaller the perturbation, the smaller the lapse and the closer the solution will be to the continuum.

For concreteness, we choose the following initial values for the variables, $M = 2, q_0^1 = 0, q_0^2 = (\sqrt{3} - \Delta) \sin(\frac{\pi}{4}), P_{q,0}^1 = 1, P_{q,0}^2 = (\sqrt{3} - \Delta) \cos(\frac{\pi}{4})$.

We choose the parameter Δ to be the perturbation, i.e., $\Delta = 0$ corresponds to an exact solution of the constraint, for which the observable $A = 1/2$ (see below for its definition). The evolution scheme can eas-

ily be implemented using a computer algebra program like Maple or Mathematica.

Before we show results of the evolution, we need to discuss in some detail how the method determines the lapse. As we mentioned it is obtained by solving the quadratic equation (1.7). This implies that for this model there will be two possible solutions and in some situations they could be negative or complex. One can choose any of the two solutions at each point during the evolution. This ambiguity can be seen as a remnant of the re-parameterization invariance of the continuum. It is natural numerically to choose one “branch” of the solution and keep with it. However, if one encounters that the roots become complex, we have observed that it is possible to backtrack to the previous point in the iteration, choose the alternate root to the one that had been used up to that point and continue with the evolution. A similar procedure could be followed when the lapse becomes negative. It should be noted that negative lapses are not a problem per se, it is just that the evolution will be retraced backwards. We have not attempted to correct such retracings, i.e. in the evolutions shown we have only “switched branches” whenever the lapse becomes complex. This occurs when the discriminant in the quadratic equation (1.7) changes sign.

We would like to argue that in some sense the discrete model “approximates” the continuum model well. This, however, turns out to be a challenging proposition in re-parameterization invariant theories. The first thing to try, to study the evolution of the quantities as a function of n is of course meaningless as a grounds to compare with the continuum. In the discrete theory we do not control the lapse, therefore plots of quantities as a function of n are meaningless. To try to get more meaningful information one would like to concentrate on “observables”. In the continuum theory, these are quantities that have vanishing Poisson brackets with the constraints (also sometimes known as “perennials”). Knowing these quantities as functions of phase space allows to know any type of dynamical physical behavior of the system. One can use them, for instance, to construct “evolving constants” (Rovelli 1990). The existence of perennials in the continuum theory is associated with symmetries of the theory. If such symmetries are not broken by the discretization process, then in the discrete theory one will have exact conserved quantities that correspond to the perennials of the continuum theory. The conserved quantities will be given by discretizations of the perennials of the continuum. It should be noted that in the continuum theory perennials as functions of phase space are defined up to the addition of multiples

of the constraints. There are therefore infinitely many versions of a given perennial. When discretized these versions are inequivalent (since in the discrete theory the constraints of the continuum theory do not hold exactly) and only one of these versions will correspond to an exact conserved quantity of the discrete theory.

In this model there are two independent perennials in the continuum. One of them becomes straightforwardly upon discretization an exact conserved quantity of the discrete theory,

$$O_1 = p^1 q^2 - p^2 q^1. \quad (1.8)$$

Another perennial is given by

$$O_2 = (p^1)^2 - (p^2)^2 + (q^1)^2 - (q^2)^2. \quad (1.9)$$

This quantity is not an exact conserved quantity of the discrete model, it is conserved approximately, as we can see in figure (1.1). We at present do not know how to find an exact conserved quantity in the discrete theory that corresponds to a discretization of this perennial (plus terms proportional to the constraint). In the end, this will be the generic situation, since in more complicated models one will not know exact expressions either for the perennials of the continuum theory or the constants of motion of the discrete theory. Notice also that in the continuum, in order to recover physical information about the system, one generically needs the two perennials plus combinations involving the constraints. In the discrete theory these combinations will not be exactly preserved. Therefore even if we found exact conserved quantities for both perennials in the discrete theory, the extracted physics would still only be approximate, and the measure of the error will given by how well the constraint of the continuum theory is satisfied in the discrete theory. It is in this sense that one can best say that the discrete theory “approximates the continuum theory well”.

Figure (1.1) depicts the relative errors throughout evolution in the value of the second perennial we discussed. Interestingly, although in intermediate steps of the evolution the error grows, it decreases later.

As we argued above, in the discrete theory quantities approximate the ones of the continuum with an error that is proportional to the value of the constraint. Therefore the value of the constraint is the real indicator of how accurately one is mirroring the continuum theory. It is a nice feature to have such an error indicator that is independent of the knowledge of the exact solution. Using this indicator one can, for instance, carry out convergence studies and show that the method does

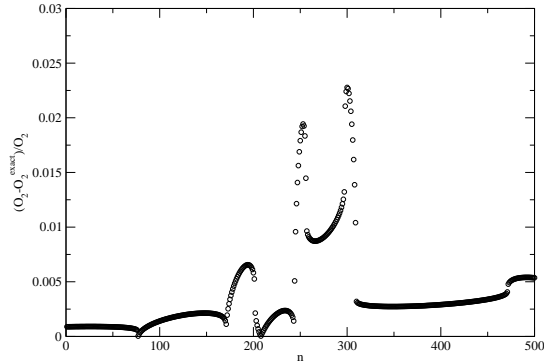


Fig. 1.1. The model has two “perennials”. One of them is an exact conserved quantity of the discrete theory, so we do not present a plot for it. The second perennial (O_2) is approximately conserved. The figure shows the relative error in its computation in the discrete theory. It is worthwhile noticing that, unlike what is usual in free evolution schemes, errors do not accumulate, they may grow for a while but later they might diminish.

indeed converge for this model in a detailed way (Gambini & Pullin 2005b).

Figure (1.2) shows the trajectory in configuration space. As we see, the complete trajectory is covered by the discretized approach. This is important since many people tend to perceive the consistent discretization approach as “some sort of gauge fixing”. This belief stems from the fact that when one gauge fixes a theory, the multipliers get determined. In spite of this superficial analogy, there are many things that are different from a gauge fixing. For instance, as we discussed before, the number of degrees of freedom changes (for more details see Gambini & Pullin 2003c). In addition to this, this example demonstrates another difference. If one indeed had gauge fixed this model, one would fail to cover the entire available configuration space, given its compact nature.

To conclude this section, let us point out to some hints that this model provides. To begin with, we see that the consistent discretization scheme successfully follows the classical continuum trajectory. One has control of how accurate things are by choosing the initial data. One can show that the approach converges using estimators of error that are independent of knowledge of exact solutions or other features generically

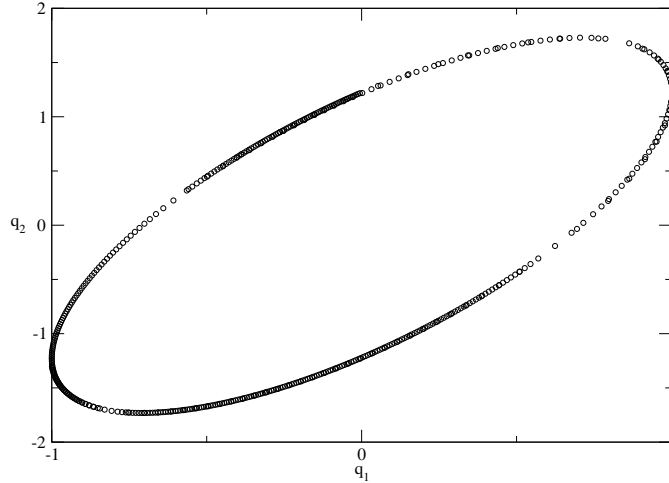


Fig. 1.2. The orbit in configuration space. As it is readily seen, the consistent discrete approach covers the entire available configuration space. This clearly exhibits that the approach is not a “gauge fixing”. Gauge fixed approaches cannot cover the entire configuration space due to its compact nature. The dynamical changes in the value of the lapse can be seen implicitly through the density of points in the various regions of the trajectory. Also apparent is that the trajectory is traced on more than one occasion in various regions. Deviation from the continuum trajectory is not noticeable in the scales of the plot.

not available. The solution of the equations for the Lagrange multipliers may develop branches, and one can use this to one’s advantage in tackling problems where the topology of phase space is not simple.

What is the state of the art in terms of applying this approach as a classical numerical relativity tool? We have applied the method in homogeneous cosmologies and also in Gowdy cosmologies (Gambini, Ponce & Pullin 2005) where one has spatial dependence of the variables. All of the features we have seen in the model described in this section are present in the more complicated models, the only difference is computational complexity. How well does it compete with more traditional numerical relativity approaches? At the moment the method is too costly to compete well, since the evolution equations are implicit. But as traditional

“free evolution” methods in numerical relativity keep on encountering problems of instabilities and constraint violations, and as computational power increases, the costliness of the consistent discretization approach may become less of a problem. A challenge to be overcome is that in situations of interest the problems have boundaries, and the approach has not yet been worked out in the presence of boundaries, although we are actively considering this point.

1.3 Applications

1.3.1 *Classical relativity*

As we argued before, our approach can be used to construct discrete theories that approximate general relativity. It is therefore suitable for doing numerical relativity. The main problem is that the resulting numerical schemes are implicit, and therefore very costly in situations of physical interest where there are no symmetries. Most of present numerical relativity is being pursued with explicit algorithms for that reason. In spite of this, our experience with the model analyzed by Rovelli and the Gowdy cosmologies indicates that our discretizations may have attractive features that are not present in more traditional discretization schemes. In particular the fact that errors do not seem to accumulate but rather grow and decrease in cycles as one evolves, could offer unique promises for long term evolutions like the ones desired in binary systems that emit gravitational waves. In addition to this, it has been shown (Di Bartolo *et. al.* 2005a) that our approach applied to linearized gravity yields a discretization that is “mimetic”, that is, the constraints are automatically preserved without determining the Lagrange multipliers. This may suggest that at least at linearized level, our discretizations may perform better than others.

In spite of these hints of a promise, there is a lot of terrain yet to cover before one could consider seriously using one of these schemes in problems of current interest. In particular, it has growingly been recognized in numerical relativity the importance of having symmetric hyperbolic formulations (see Reula (1998) for a review) and in particular of incorporating constraint preserving boundary conditions. Most symmetric hyperbolic formulations are constructed at the level of equations of motion and do not derive from an action principle. Therefore our discretization technique is not directly applicable. More work is clearly needed in this area.

Another area of recent progress (Gambini & Pullin 2005c) has been the application of these ideas to Regge calculus. In Regge calculus it had been observed that the canonical formulation was problematic. In particular it seemed to require that the Lagrange multipliers be fixed (Friedman & Jack 1986). This is exactly the statement that we use as a starting point for our discrete construction. We have recently shown how one can construct an unconstrained version of canonical Regge calculus in which some of the lengths of the links are determined precisely mirroring what happens with the Lagrange multipliers in other theories. Although this is only a beginning, it suggests a novel technique to have a canonical formulation of Regge calculus that may have attractive implications quantum mechanically (for instance it contains a new prescription to define the path integral).

1.3.2 The problem of time

Since the discrete theory that one constructs through our procedure is constraint-free, it immediately circumvents most of the hard conceptual problems of canonical quantum gravity including the “problem of time”. The issue is a bit more subtle than it initially appears. One indeed has a theory without constraints and a “genuine evolution”, except that the latter is cast in terms of the discrete parameter n . This parameter cannot be accessed physically, it is not one of the variables one physically observes for the systems under study. This forces us to consider a “relational” formulation, in the same spirit as Page and Wootters (1983) considered. The idea is to pick one of the physical variables and use it as a clock. One then asks relational questions, for instance “what is the conditional probability than one of the other variables takes a given value when the clock variable indicates a certain time”. These questions can of course also be asked in continuum general relativity, but the detailed construction of the conditional probabilities is problematic, due to the difficulties of having a probabilistic interpretation of quantum states in canonical quantum gravity (see the discussion in Kuchař 1992). In our approach, on the other hand, the conditional probabilities are well defined, since there are no constraints to generate problems with the probabilistic interpretation of states. For more details see (Gambini, Porto & Pullin 2003).

1.3.3 *Cosmological applications*

We have applied the technique to cosmological models. The use of these discrete theories in cosmology has an attractive consequence. Since the lapse, and therefore the “lattice spacing in time” is determined by the equations of motion, generically one will avoid the singularity classically. Or to put it in a different way, one would have to “fine tune” the initial data to reach the singularity (unless one uses variables in which the singularity is on a boundary of phase space). Quantum mechanically, this implies that the singularity will be probabilistically suppressed. As the discrete theory tunnels through the singularity, there is a precise sense in which one can claim that the lattice spacing changes qualitatively. This could be used to argue that physical constants change when tunneling through a singularity since in lattice theories the “dressed” value of the coupling constants is related to the lattice spacing. Therefore this provides a concrete mechanism for Smolin’s “The life of the cosmos” proposal (1992). For more details see Gambini & Pullin (2003a).

1.3.4 *Fundamental decoherence, black hole information puzzle, limitations to quantum computing*

Once one has solved the problem of time in the relational fashion discussed above, one notices that the resulting quantum theory fails to be unitary. This is reasonable. In our approach, when one quantizes, one would have a unitary evolution of the states as a function of the discrete parameter n . In the relational approach one picked some dynamical variable and called it time T . Suppose one chose a state in which this variable is highly peaked as a function of n . If one lets the system evolve, the variable will spread and at a later instant one would have a distribution of values of n that correspond to a given T (or vice-versa). That means that if one started with a “pure” state, one ends with a mixed state. The underlying reason is that the physical clock T cannot remain in perfect lock-step with the evolution parameter n .

A detailed discussion of the implications of this lack of unitarity is in Gambini, Porto & Pullin (2004a, 2004b, 2005a). Of course, this is not the first time that quantum gravity effects have been associated with loss of unitarity. However, unlike previous proposals (see Banks *et. al* 1984), the detailed evolution implied by the relational description we find conserves energy, which is a very desirable feature. One can give a bound on the smallness of the effect by taking into account what is the

“best” clock one can construct from fundamental physical principles (Ng & van Dam 1995). The lack of unitarity makes the off diagonal elements of the density matrix go to zero exponentially. The exponent (for a system with two energy levels, for simplicity) is proportional to minus the Bohr frequency between the levels squared, to the Planck time to the $(4/3)$ power and to the time one waits for the state to lose coherence to the $(2/3)$ power (these results appear not even to be Galilean invariant, but this is not the case as discussed in detail in Gambini, Porto & Pullin 2004c). It is clear that the effect is negligible for most quantum systems. Chances of observing the effect in the lab (see for instance Simon & Jaksch 2004) are at the moment remote, one would require a quantum system of macroscopic size. If one assumes energy differences of eV size, one would roughly need 10^{13} atoms. Bose-Einstein condensates at present can achieve states of this sort with perhaps hundreds of millions of atoms, but they do not involve energy differences of eV 's per atom. Another important caveat of these types of discussions is that they have been carried out at a very naive level of Newtonian quantum mechanics. If one were to consider relativistic quantum field theory, one would have to have a “clock” variable per spatial point. This would imply that quantum states would lose coherence not only as time evolves, but also between points in space. Such effects could potentially have consequences that are much more amenable to experimental testing (Simon & Jaksch 2004).

Once one accepts that quantum mechanics at a fundamental level contains loss of unitarity one may wish to reconsider the black hole information paradox. After all, the reason one has a paradox is that when a black hole evaporates, the final result is a mixed state, even if one built the black hole by collapsing a pure state. The question is: does this loss of unitarity occur faster or slower than the one we have found? If it is slower, then it will be unobservable. A priori one could expect that the effect we discussed should not be too important. We just argued in the previous paragraph that it is very small. However, black holes take a long time to evaporate. And as they evaporate their energy levels become more separated as the temperature increases. A detailed calculation shows that the order of magnitude of the off-diagonal elements of the density matrix at the time of complete evaporation would be approximately $M_{BH}^{-2/3}$ with M_{BH} the black hole mass in Planck mass units (Gambini, Porto & Pullin 2005a). For an astrophysical size black hole therefore the loss of unitarity is virtually complete and the paradox cannot be realized physically. What happens if one takes, say, a very

small black hole? Can one reformulate the paradox in that case? The formulation we have is not precise enough to answer this question. We have only roughly estimated the magnitude of the decoherence just to give an order of magnitude estimate. Many aspects of the calculation are also questionable for small black holes, where true quantum gravity effects are also important.

An interesting additional observation (Gambini, Porto & Pullin 2005b) is that the loss of quantum coherence we found can provide a fundamental limitation to how fast quantum computers can operate that is more stringent than other fundamental limits considered.

1.4 Constructing the quantum theory

As we argued above, the construction of the quantum theory starts by implementing the canonical transformation that gives the evolution in terms of the discrete parameter n as a unitary transformation. Before doing this one constructs the canonical theory that results from the elimination of the Lagrange multipliers. The resulting canonical theory generically has no constraints, and has evolution equations for its canonical variables. One picks a polarization, for instance $\Psi(q)$ where q is a set of configuration variables, and considers the unitary transformation as operating on the space of wavefunctions chosen. Since generically there are no constraints, one can pick as physical inner product the kinematical one and construct a Hilbert space of wavefunctions that are square integrable. If one is in the Schrödinger representation states evolve, so we label them as $\Psi_n(q)$ and the evolution is given by,

$$\Psi_{n+1}(q) = \int dq' U(q|q') \Psi_n(q'). \quad (1.10)$$

The transformation has to be such that it implements the evolution equations as operatorial relations acting on the space of wavefunctions in the Heisenberg representation, where

$$U(q|q') = \langle n+1, q' | n, q \rangle, \quad (1.11)$$

and where $|n+1, q' \rangle$ and $|n, q \rangle$ are the eigenvectors of the configuration operators \hat{q} in the Heisenberg representation at levels $n+1$ and n respectively. The evolution equations take the form,

$$\langle n+1, q | \hat{q}_{n+1} - f(\hat{q}_n, \hat{p}_n) | n, q' \rangle = 0, \quad (1.12)$$

$$\langle n+1, q | \hat{p}_{n+1} - g(\hat{q}_n, \hat{p}_n) | n, q' \rangle = 0, \quad (1.13)$$

with f, g the quantum evolution equations, which are chosen to be self-adjoint in order for the transformation to be unitary. Explicit examples of this construction for cosmological models can be seen in (Gambini & Pullin 2003c).

If at the end of this process one has constructed a transformation that is truly unitary the quantization is complete in the discrete space and one has a well defined framework to rigorously compute the conditional probabilities that arise when one uses a relational time to describe the physical system. This is a major advantage over attempts to construct the relational picture with systems where one has constraints.

There are some caveats to this construction that are worth pointing out. As we mentioned, our construction generically yields discrete theories that are constraint-free. To be more precise, the theories do not have the constraints associated with space-time diffeomorphisms. If the theory under consideration has other symmetries (for instance the Gauss law of Yang–Mills theory or gravity written in the new variable formulation), such symmetries may be preserved upon discretization (we worked this out explicitly for Yang–Mills and BF theory in Di Bartolo *et. al* 2002). The resulting discrete theory therefore will have some constraints. If this is the case, the above construction starts by considering as wavefunctions states that are gauge invariant and endowed with a Hilbert space structure given by a gauge invariant inner product. The resulting theory has true (free) Lagrange multipliers associated with the remaining constraints. The unitary transformation will depend on such parameters. An alternative is to work in a representation where the constraints are solved automatically (like the loop representation for the Gauss law). There one has no constraints left and the inner product is the kinematical one in the loop representation and the unitary transformation does not depend on free parameters.

Other issues that may arise have to do with the fact that in many situations canonical transformation do not correspond quantum mechanically to unitary transformations. This problem has been discussed, for instance, by Anderson (1994). He noted that the only canonical transformations that can be implemented as unitary transformations are those that correspond to an isomorphism of a phase space into itself. This is important for the discrete theories in the following way. If one has a continuum constrained theory, its physical phase space is on the constraint surface. The discrete theories have a phase space that includes the constraint surface of the continuum theory. However, the discrete phase space variables cover only a subspace of the kinematical phase

space of the continuum theory. There are inaccessible sectors that correspond to complex values of the Lagrange multipliers in the discrete theory. Therefore, in order to have the canonical transformation of the discrete theory be an isomorphism, one may have to choose a physical Hilbert space for the discrete theory that is a subspace of the kinematical space instead of just taking it to be coincident. This has to be done carefully, since restricting the Hilbert space may imply that some physical quantities fail to be well defined in the physical Hilbert space. We have explored some of these issues in some quantum mechanical models that have a relational description. We have shown that one can successfully recover the traditional quantum mechanical results in a suitable continuum limit by carefully imposing a restriction on the kinematical Hilbert space, and that one can define variables that approximate any dynamical variable of the continuum theory in the continuum limit in the restricted Hilbert space (see Di Bartolo *et. al* 2005b).

1.5 The quantum continuum limit

As we argued in the discussion of the model analyzed by Rovelli, a good measure of how close one is to the continuum theory in a given solution of the discrete theory is to evaluate the constraint of the continuum theory. Such constraint is only exactly satisfied in the continuum limit. An alternative way of presenting this is to consider the construction of a “Hamiltonian” such that exponentiated would yield the unitary evolution between n and $n + 1$, $\hat{U} = \exp(i\hat{H})$ where $\hbar = 1$ and \hat{H} has units of action. Such Hamiltonian can only be constructed locally since in some points of the evolution the logarithm of the unitary transformation is not well defined. Such Hamiltonian can be written as a formal expansion in terms of the constraint of the continuum theory (a way of seeing this is to notice that in the continuum limit this Hamiltonian has to vanish since it incorporates the timestep). If one chooses an initial state such that $\langle \hat{H} \rangle \ll 1$ the evolution will preserve this (\hat{H} is an exact constant of the motion). This will continue until one reaches a point where \hat{H} is not well defined. The evolution will continue, but it will not necessarily remain close to the continuum limit. In certain cosmological examples this point coincides with the point where the continuum theory has the singularity, for example (Gambini & Pullin 2003c). Therefore a first condition on the quantum states in the continuum limit $\langle \hat{H} \rangle \ll 1$. A second condition is that the expectation values of the physical variables should not take values in the points where \hat{H} is not well defined. A third

condition is not to make measurements with “too much accuracy” on variables that do not commute with \hat{H} . This requirement stems from the fact that such measurements would introduce too much dispersion in \hat{H} and one would violate the first requirement. In examples we have seen that this condition translates in not measuring q, p with sharper accuracy than that of the step of the evolution in the respective variable. This appears reasonable, a discrete theory should not allow the measurement of quantities with accuracies smaller than the discretization step. The variables that do not commute with \hat{H} play a crucial role in the relational description since they are the variables that can be used as “clocks” as they are not preserved under evolution as constants of the motion.

1.6 Summary and outlook

One can construct discrete canonical theories that are constraint free and nevertheless approximate continuum constrained theories in a well defined sense. The framework has been tested at a classical level in a variety of models, including gravitational ones with infinitely many degrees of freedom. Further work is needed to make the framework computationally competitive in numerical relativity. In particular the use of better discretizations in time, including higher order ones, appears as promising. Initial explorations we are carrying out in simple models indicate that one can achieve long-term stable and accurate evolutions using moderately large timesteps. This could be very attractive for numerical relativity if it turns out to be a generic property.

Since the discrete theories are constraint free, they can be quantized without serious conceptual obstacles. In particular a relational time can be introduced in a well defined way and quantum states exhibit a non-unitary evolution that may have implications experimentally and conceptually (as in the black hole information puzzle). There is a reasonable proposal to construct the quantum continuum limit that has been tested in simple constrained models. The main challenge is to apply the framework at a quantum level in systems with field theoretic degrees of freedom. The fact that one has a well defined framework that is computationally intensive suggests that this is an avenue for conducting numerical quantum gravity.

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