Abstract Wiener Space Approach to Hida Calculus (Brownian Motion).

Youngsook Lee Shim
Louisiana State University and Agricultural & Mechanical College

Follow this and additional works at: https://repository.lsu.edu/gradschool_disstheses

Recommended Citation
https://repository.lsu.edu/gradschool_disstheses/4379

This Dissertation is brought to you for free and open access by the Graduate School at LSU Scholarly Repository. It has been accepted for inclusion in LSU Historical Dissertations and Theses by an authorized administrator of LSU Scholarly Repository. For more information, please contact gradetd@lsu.edu.
INFORMATION TO USERS

While the most advanced technology has been used to photograph and reproduce this manuscript, the quality of the reproduction is heavily dependent upon the quality of the material submitted. For example:

- Manuscript pages may have indistinct print. In such cases, the best available copy has been filmed.

- Manuscripts may not always be complete. In such cases, a note will indicate that it is not possible to obtain missing pages.

- Copyrighted material may have been removed from the manuscript. In such cases, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, and charts) are photographed by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each oversize page is also filmed as one exposure and is available, for an additional charge, as a standard 35mm slide or as a 17"x 23" black and white photographic print.

Most photographs reproduce acceptably on positive microfilm or microfiche but lack the clarity on xerographic copies made from the microfilm. For an additional charge, 35mm slides of 6"x 9" black and white photographic prints are available for any photographs or illustrations that cannot be reproduced satisfactorily by xerography.
Abstract Wiener space approach to Hida calculus

Shim, Youngsook Lee, Ph.D.

The Louisiana State University and Agricultural and Mechanical Col., 1987
PLEASE NOTE:

In all cases this material has been filmed in the best possible way from the available copy. Problems encountered with this document have been identified here with a check mark √.

1. Glossy photographs or pages ______
2. Colored illustrations, paper or print ______
3. Photographs with dark background ______
4. Illustrations are poor copy ______
5. Pages with black marks, not original copy ______
6. Print shows through as there is text on both sides of page ______
7. Indistinct, broken or small print on several pages ______
8. Print exceeds margin requirements ______
9. Tightly bound copy with print lost in spine ______
10. Computer printout pages with indistinct print ______
11. Page(s) ______ lacking when material received, and not available from school or author.
12. Page(s) ______ seem to be missing in numbering only as text follows.
13. Two pages numbered ______. Text follows.
14. Curling and wrinkled pages ______
15. Dissertation contains pages with print at a slant, filmed as received ______
16. Other _______________________________________________________________________

University
Microfilms
International
ABSTRACT WIENER SPACE APPROACH TO HIDA CALCULUS

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The Department of Mathematics

by

Youngsook Lee Shim
B.S., The Seoul National University of Korea, 1974
May 1987
ACKNOWLEDGEMENTS

I would like to express my deepest appreciation to Professor Hui-Hsiung Kuo for his assistance and encouragement in the preparation of this dissertation.

Also I would like to thank my husband for his support which made it possible for me to finish my higher education.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGEMENTS</td>
<td>ii</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>iv</td>
</tr>
<tr>
<td>INTRODUCTION</td>
<td>v</td>
</tr>
<tr>
<td>CHAPTER 1: Background</td>
<td></td>
</tr>
<tr>
<td>§1. Hida's theory</td>
<td>1</td>
</tr>
<tr>
<td>§2. Abstract Wiener space</td>
<td>5</td>
</tr>
<tr>
<td>CHAPTER 2: The space $(L^2)^{-}$ of generalized random variables</td>
<td></td>
</tr>
<tr>
<td>§1. Ordinary multiple Wiener integrals</td>
<td>10</td>
</tr>
<tr>
<td>§2. S-transform on $L^2(B,\mu)$</td>
<td>20</td>
</tr>
<tr>
<td>§3. The space $(L^2)^{-}$ of generalized random variables</td>
<td>25</td>
</tr>
<tr>
<td>CHAPTER 3: Operators acting on $(L^2)^{-}$</td>
<td></td>
</tr>
<tr>
<td>§1. $\Theta(t)$-differentiation and its adjoint operator</td>
<td>36</td>
</tr>
<tr>
<td>§2. Laplacian operators</td>
<td>45</td>
</tr>
<tr>
<td>CHAPTER 4: Fourier transform on $(L^2)^{-}$</td>
<td></td>
</tr>
<tr>
<td>§1. Fourier transform</td>
<td>52</td>
</tr>
<tr>
<td>§2. Relation between Fourier transform and the Levy Laplacian</td>
<td>64</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>66</td>
</tr>
<tr>
<td>VITA</td>
<td>68</td>
</tr>
</tbody>
</table>
ABSTRACT

Let $\mathcal{S}$ be the Schwartz space of rapidly decreasing real functions on $\mathbb{R}$. The dual space $\mathcal{S}^*$ of $\mathcal{S}$ consists of tempered distributions. The inclusion maps

$$\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^*$$

are continuous.

Hida's theory of Brownian and generalized Brownian functionals is the study of functionals defined on $\mathcal{S}^*$. In this dissertation, the triple $\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^*$ is replaced by an abstract Wiener space $B^* \subset H \subset B$ and an abstract version of Hida's theory is developed. The Gaussian measure on $\mathcal{S}^*$ in Hida's calculus is replaced by the standard Gaussian measure $\mu$ on the space $B$. The $\mathcal{S}^*$ valued curve $\{\delta_t; t \in \mathbb{R}\}$ in Hida calculus is replaced by a $B$-valued curve $\{\mathcal{O}(t); t \in \mathbb{R}\}$. The coordinate system, differential operator, and Laplacina operators with respect to $\{\delta_t\}$ in Hida calculus can be defined with respect to $\{\mathcal{O}(t)\}$ in the Abstract Wiener space setup. Similar properties and theorems as in Hida calculus are obtained.
INTRODUCTION

In 1975, Hida [3,5] initiated the study of Brownian functionals from the white noise point of view. This study leads to the theory of generalized Brownian functionals, which is referred nowadays as the Hida calculus. It is related to the curve $\delta_t$, $t \in \mathbb{R}$, in the space $\mathcal{S}^*$ of tempered distributions. The purpose of this dissertation is to develop an abstract version of Hida calculus. Therefore, the space $\mathcal{S}^*$ is replaced by an abstract Wiener space $B$ and $\delta_t$ by a curve $\delta_t$ in $B$.

The space $\mathcal{S}^*$ of tempered distributions is the dual space of the space $\mathcal{S}$ of rapidly decreasing smooth functions on $\mathbb{R}$. Then we have the triple $\mathcal{S} \subset L^2(\mathbb{R}) \subset \mathcal{S}^*$. There exists a unique probability measure $\mu$ on $\mathcal{S}^*$ such that

$$\int_{\mathcal{S}^*} e^{i\langle x, \xi \rangle} d\mu(x) = e^{-\frac{1}{2} \| \xi \|^2}, \text{ for } \xi \in \mathcal{S},$$

where $\| \cdot \|$ is the $L^2(\mathbb{R})$ norm. We call $\mu$ the standard white noise measure on $\mathcal{S}^*$. For each $\xi$ in $\mathcal{S}$, the random variable $\langle x, \xi \rangle$ on $(\mathcal{S}^*, \mu)$ is normally distributed with mean 0 and variance $\| \xi \|^2$.

Moreover, for $f$ in $L^2(\mathbb{R})$, we can define $\langle x, f \rangle$ to be the $L^2(\mathcal{S}^*)$ limit of $\langle x, f_n \rangle$, where $\{f_n\}$ is any sequence of $\mathcal{S}$ such that $\| f_n - f \| \to 0$. We can easily see that $\langle \cdot, f \rangle$ is a Gaussian random variable with mean 0 and variance $\| f \|^2$.

On the other hand, let $(\mathcal{H}, B)$ be an abstract Wiener space [11,12]. Then we have the triple $B^* \subset \mathcal{H} \subset B$, where $B^*$ is
the dual space of $B$. There exists a unique probability measure $\mu$ on $B$ such that

$$
\int_B e^{i \langle x, z \rangle} \mu(x) = e^{-\frac{1}{2} \|z\|^2}, \text{ for } z \in B^*
$$

where $\|\cdot\|$ denotes the $H$-norm, $\mu$ is the standard Gaussian measure on $B$. For each $z \in B^*$, the random variable $\langle x, z \rangle$ on $(B, \mu)$ is defined everywhere and is normally distributed with mean 0 and variance $\|z\|^2$. If $h \in H$, then $\langle x, h \rangle$ is defined as the $L^2(\mu)$ limit of random variable $\langle x, z_n \rangle$, where $\{z_n\}$ is any sequence in $B^*$ such that $\|z_n - h\| \to 0$. The random variable $\langle \cdot, h \rangle$ is normally distributed with mean 0 and variance $\|h\|^2$.

In Hida calculus we have the space $(L^2)^+$ of test functionals and the space $(L^2)^-$ of generalized Brownian functionals over the space $\mathcal{M}^*$, and $(L^2)^+ \subset L^2(\mathcal{M}^*) \subset (L^2)^-$. [3,5]. In Chapter 2 we will develop ordinary multiple integrals with respect to $\Theta(t)$ for $L^2(B)$ and define the space $(L^2)^+$ of test random variables and the space $(L^2)^-$ of generalized random variables by using the fact that $L^2(B)$ has the orthogonal decomposition $\bigoplus_{n=0}^{\infty} \Theta K_n$, where $K_n$ is the space of multiple Wiener integrals of order $n$. $K_n$ can be identified with the space of $n$-fold symmetric tensor product of $H$ [12]. In Hida calculus $\{B(t); t \in \mathbb{R}\}$ is taken as a coordinate system. It is a family of independent, identically distributed generalized random variables and takes the time propagation into account. By the same way we will take $\langle \cdot, \Theta(t) \rangle; t \in \mathbb{R}$ as a coordinate system for our calculus. This system enjoys the same properties as $\{B(t); t \in \mathbb{R}\}$ in the Hida calculus.
One of the conditions assumed on $\theta(t)$ allows us to extend the $H$-inner product to $B$ formally as

$$\langle x, y \rangle \equiv \int_{\mathbb{R}} \langle x, \theta(t) \rangle \langle y, \theta(t) \rangle \, dt, \quad x, y \in B.$$ 

Although it is only a formal expression, it plays a very important role in developing our calculus. In Chapter 3 we will define several operators such as $\theta(t)$- differentiation operator $\partial \theta(t)$, the adjoint operator $\partial^* \theta(t)$, $\langle \cdot, \theta(t) \rangle$ multiplication operator, and the various Laplacian operators. We will obtain similar results concerning these operators as in the Hida calculus. We will also define the Fourier transform in our setup. The connection between the Fourier transform and the Levy Laplacian will be studied in Chapter 4.

The following chart gives a comparison among the finite dimensional case, Hida calculus, and the abstract Wiener space setup of Hida calculus.

<table>
<thead>
<tr>
<th>Finite dimensional case</th>
<th>Hida calculus</th>
<th>Abstract Wiener space</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}^n$</td>
<td>$\mathbb{R}^n$</td>
<td>$B$</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
<td>$\mathbb{R}^n$</td>
<td>$B$</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
<td>$\mathbb{R}^n$</td>
<td>$B$</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
<td>$\mathbb{R}^n$</td>
<td>$B$</td>
</tr>
<tr>
<td>$(x_1, \ldots, x_n)$</td>
<td>$B(t)$</td>
<td>$B$</td>
</tr>
<tr>
<td>$\partial_j = \partial/\partial x_j$</td>
<td>$\partial_{B(t)}$</td>
<td>$\delta_{\theta(t)} \equiv \partial_{\cdot, \theta(t)}$</td>
</tr>
<tr>
<td>$x_j$</td>
<td>$B(t)$</td>
<td>$\delta_{\theta(t)} \equiv \partial_{\cdot, \theta(t)}$</td>
</tr>
<tr>
<td>$\Delta$ (Laplacian)</td>
<td>$\Delta_G$ (Gross)</td>
<td>$\Delta_G$</td>
</tr>
<tr>
<td>$\Delta_B$ (Beltrami)</td>
<td>$\Delta_B$</td>
<td>$\Delta_B$</td>
</tr>
<tr>
<td>$\Delta_L$ (Levy)</td>
<td>$\Delta_L$</td>
<td>$\Delta_L$</td>
</tr>
<tr>
<td>$\Delta_V$ (Volterra)</td>
<td>$\Delta_V$</td>
<td>$\Delta_V$</td>
</tr>
</tbody>
</table>
CHAPTER 1. Background

In this chapter we will give a brief review of the background for our work. Definitions and theorems are stated without proof and can be found in [3,5] and [11,12].

§1. Hida's theory.

Let $\mathcal{A}$ be the space of rapidly decreasing smooth real valued functions on $\mathbb{R}$. The dual space $\mathcal{A}^*$ of $\mathcal{A}$ consists of the tempered distributions. Thus we have the continuous inclusions $\mathcal{A} \subset L^2(\mathbb{R}) \subset \mathcal{A}^*$. The canonical bilinear form connecting $\mathcal{A}$ and $\mathcal{A}^*$ will be denoted by $\langle x, \xi \rangle$, $x \in \mathcal{A}$, $\xi \in \mathcal{A}^*$.

Theorem 1.1 (Bochner-Minlos theorem)

Let $C(\xi)$ be a functional on $\mathcal{A}$ satisfying (i) continuous; (ii) positive-definite, and; (iii) $C(0) = 1$. Then there exists a unique probability measure $\mu$ on $(\mathcal{A}^*, \mathcal{B})$ such that

$$C(\xi) = \int_{\mathcal{A}^*} \exp[i\langle x, \xi \rangle] d\mu(x)$$

where $\mathcal{B}$ is the $\sigma$-field generated by the cylinder subsets of $\mathcal{A}^*$ of the form $\{x; \langle x, \xi_1 \rangle, \ldots, \langle x, \xi_n \rangle \} \in \mathcal{B}$, $\mathcal{B}$ is a Borel subset of $\mathbb{R}^n$, $\xi_1, \ldots, \xi_n \in \mathcal{A}$, $n = 1, 2, \ldots$. 
Definition 1.1 [3]

The measure space \((\mathcal{X}, \mathcal{B}, \mu)\) determined by the characteristic functional \(C(\xi) = \exp[-i \xi \xi^2 / 2]\) is called a white noise space. Here \(\| \cdot \|\) is the \(L^2(\mathbb{R})\) norm.

Remark: By this definition, sometimes we call Hida calculus White Noise calculus. Moreover,

\[
B(t,x) = \begin{cases} 
\langle x, 1_{(0,t)} \rangle, & t > 0 \\
-\langle x, 1_{(t,0)} \rangle, & t < 0 
\end{cases}
\quad x \in \mathcal{X}^*
\]

is a Brownian motion, and formally we have \(B(t) = \int_0^t x(u)du\) and \(\dot{B}(t) = x(t)\) where \(x \in \mathcal{X}^*\). Therefore we will use \(\dot{B}\) for the element in \(\mathcal{X}^*\). Then White Noise is the probability distribution of \(\{B(t); t \in \mathbb{R}\}\).

In distribution theory of finite dimensional space \(\mathbb{R}^n\), we are concerned with a triple \(X \subset L^2(\mathbb{R}^n) \subset X^*\), where \(X\) is the space of test functions and \(X^*\) is the space of generalized functions. Now we have the space \(L^2(\mathcal{X}^*)\) of Brownian functionals and Gaussian measure \(\mu\) as Theorem 1.1. So we will introduce the space \(X\) and \(X^*\) such that \(X \subset L^2(\mathcal{X}^*) \subset X^*\).

Theorem 1.2. (Wiener-Ito decomposition theorem) [5]

\(L^2(\mathcal{X}^*)\) has the direct orthogonal decomposition

\[
L^2(\mathcal{X}^*) = \bigoplus_{n=0}^{\infty} K_n
\]

where \(K_n\) is the space of n-tuple Wiener integrals, i.e. each \(\phi\) in \(K_n\) has the following form

\[
\phi(x) = \int_{\mathbb{R}^n} \ldots \int f(u_1, \ldots, u_n) dB(u_1, x) \ldots dB(u_n, x)
\]

where \(f \in L^2(\mathbb{R}^n):\) the symmetric \(L^2(\mathbb{R}^n)\) function. Moreover;

\[|\phi| = \sqrt{n!} |f|\].
**Definition 1.2**

The $S$-transform on $\mathcal{L}^2(\mathcal{A}^*)$ is defined by

$$(S\phi)(\xi) = \int_{\mathcal{A}^*} \phi(x + \xi) d\mu(x).$$

**Theorem 1.3**

Suppose $\phi \in K_n$ is of the form

$$\phi(x) = \int_{\mathbb{R}^n} \cdots \int f(u_1, \ldots, u_n) dB(u_1, x) \cdots dB(u_n, x), \quad f \in \mathcal{L}^2(\mathbb{R}^n).$$

Then

$$(S\phi(x))(\xi) = \int_{\mathbb{R}^n} \cdots \int f(u_1, \ldots, u_n) \xi(u_1) \cdots \xi(u_n) du_1 \cdots du_n.$$

**Remark:** From $(S\phi)(\xi) = \int_{\mathbb{R}^n} \cdots \int f(u_1, \ldots, u_n) \xi(u_1) \cdots \xi(u_n) du_1 \cdots du_n$ we can see $S\phi$ makes sense not only for $f \in \mathcal{L}^2(\mathbb{R}^n)$, but also for any $f \in \mathcal{K}(\mathbb{R}^n)$.

Noting the above remark we can consider following diagram

$$
\begin{array}{ccc}
K_n, a_n & \downarrow & K_n, \frac{a_n}{c_n} \\
\sqrt{n!} \mathbb{H}^{-a_n}(\mathbb{R}^n) & \subset & \sqrt{n!} \mathbb{L}^{-a_n}(\mathbb{R}^n) \\
\uparrow & & \uparrow \\
\mathbb{H}^{-a_n}(\mathbb{R}^n) & \subset & \sqrt{n!} \mathbb{H}^{-a_n}(\mathbb{R}^n)
\end{array}
$$

where $\mathbb{H}^{-a}(\mathbb{R}^n)$ is the symmetric Sobolev space of order $a > 0$, i.e.

$$\mathbb{H}^{-a}(\mathbb{R}^n) = \{ f \in \mathcal{S}^* \mid \| f \|_{-a}^2 = \int_{\mathbb{R}^n} (1 + |\lambda|^2)^a |\hat{f}(\lambda)|^2 d\lambda < \infty \}$$

where $\hat{f}(\lambda)$ is the Fourier transform of $f$.

Let $c_n$ be a bounded sequence of positive numbers. Then define

$$(\mathcal{L}^2)^{+}_{a_n, c_n} = \sum c_n K_{n, a_n},$$

$$(\mathcal{L}^2)^{-}_{a_n, c_n} = \sum c_n K_{n, -a_n}.$$
\((L^2)^+_{a_n, c_n} \subset (L^2(\mathbb{A}^*)) \subset (L^2)^-_{a_n, c_n}\)

Now \((L^2)^+_{a_n, c_n}\) and \((L^2)^-_{a_n, c_n}\) are called the space of test functionals and generalized functionals, respectively. Especially, in Hida calculus, we will assume that \(c_n = 1\) for every \(n\), \(a_n = \frac{n+1}{2}\) and \(K_0 = a_0\), the nonstandard real number system.

**Definition 1.3**

We define U-functional on \((L^2)^-\) by

\[
U(\phi)(\xi) = (S\phi)(\xi) \quad \text{for} \quad \phi \in (L^2(\mathbb{A}^*)).
\]

In general, we define \(U(\phi)(\xi) = \sum_{n=0}^{\infty} U_n(\xi)\) for \(\phi \in \sum_{n=0}^{\infty} \phi_n\) with

\[
\phi_n = \int \ldots \int f(u_1 \ldots u_n) dB(u_1) \ldots dB(u_n) \quad \text{for} \quad f \in \mathcal{A} \cdot \frac{2}{n+1} (\mathbb{H}^n), \text{where}
\]

where \(U_n(\xi) = \int_{\mathbb{H}^n} f(u_1, \ldots, u_n) \xi(u_1) \ldots \xi(u_n) du_1 \ldots du_n\).

For \(\phi \in (L^2)^-\) its U-functional is usually computed through renormalization. The procedure of renormalization is to find a sequence \(\phi_n\) in \(L^2(\mathbb{A}^*)\) such that \(\phi_n\) is Cauchy w.r.t. \((L^2)^-\) norm. So U-functional of \(\phi\) is the limit of the U-functionals of \(\phi_n\).

In Hida calculus, \(\hat{B}(t); t \in \mathbb{H}\) is taken to be a system of coordinates since \((\hat{B}(t); t \in \mathbb{H})\) is a family of independent random variables and often we have to consider time propagation. Therefore we need to define the coordinate differentiation \(\partial_t = \partial_{\hat{B}(t)}\).

**Definition 1.4**

Let \(U\) be the U-functional of \(\phi \in (L^2)^-\). Then the first variation of \(U\) is defined by

\[
(\partial_t U)(\xi) = \sum_{n=0}^{\infty} (\partial_t U_n)(\xi)\]

\(\text{where} \quad (\partial_t U_n)(\xi) = \int_{\mathbb{H}^n} f(u_1, \ldots, u_n) \partial_t \xi(u_1) \ldots \xi(u_n) du_1 \ldots du_n\).
\[ U(\xi + \delta \xi) = U(\xi) + \delta U + o(\delta \xi) \]

where

\[ \delta U = \int_{\mathbb{R}} U'_\xi(u) \delta \xi(u) \, du. \]

If \( U'_\xi(t) \) is a \( U \)-functional of some function in \((L^2)^-\), then we define \( \partial_t \phi \) to be the generalized Brownian functional with \( U \) functional \( U'_\xi(t) \), i.e., \( U(\partial_t \phi)(\xi) = U'_\xi(t) \).

**Remark:** We have a formal expression \( \partial_t B(u) = \delta_t (u) \) which is the analog of \( \partial_j x_k = \delta_j^k \) in the finite dimensional case \( \mathbb{R}^n \).

**Definition 1.5**

1. The adjoint operator \( \partial_t^* \) is defined by
   \[ \langle \partial_t^* \phi, \psi \rangle = \langle \phi, \partial_t \psi \rangle \text{ for } \phi \in (L^2)^-, \psi \in (L^2)^+ \]

2. \( \mathcal{B}(t) \) multiplication is defined by
   \[ \mathcal{B}(t)\phi = \partial_t \phi + \partial_t^* \phi \]

The Laplacian operator and the Fourier transform will be studied in Chapter 3 and Chapter 4.

**§2. Abstract Wiener space**

Let \( H \) be a real separable Hilbert space with norm \( \| \cdot \| \) and inner product \( ( \cdot, \cdot ) \). Let \( \mathcal{O} \) denote the partially ordered set of finite dimensional orthogonal projections of \( H \). A cylinder set \( E \) in \( H \) is a subset of \( H \) of the form \( E = \{ x \in H ; Px \in D, D \text{ is the Borel subset of } PH \} \). Let us denote \( \mathcal{W} \) as the collection of the cylinder sets. Then \( \mathcal{W} \) is a field, but it is not a \( \sigma \)-field if \( \dim H = \infty \) [11].
Definition 1.6.

The Gauss measure $\mu$ on $H$ is the set function $\mu: W \times [0,1]$ defined by: for $E = \{x \in H; Px \in D\}$,

$$\mu(E) = \left(\frac{1}{\sqrt{2\pi}}\right)^{\dim PH} \int_D e^{-\frac{\|x\|^2}{2}} \, dx,$$

where $dx$ is the Lebesgue measure on $PH$.

Lemma [11]

$\mu$ is not $\sigma$-additive if $\dim H = \infty$.

Definition 1.7

A norm $\|\cdot\|$ on $H$ is called measurable if for any $\epsilon > 0$, there exists $P_\epsilon \in \mathcal{P}$ such that

$$\mu(x \in H; |Px| > \epsilon) < \epsilon \quad P \in \mathcal{P} \quad \text{and} \quad P \perp P_\epsilon.$$

Remark: 1. $\|\cdot\|$ is not a measurable norm if $\dim H = \infty$.

2. There exists a constant $c$ such that $\|\cdot\| < c|x|$ for $x \in H$.

Let $B$ be the completion of $H$ w.r.t. $\|\cdot\|$. $B$ is a separable real Banach space and $H$ is dense in $B$.

Definition 1.8

The pair $(H,B)$ is called an abstract Wiener space.

$H \subset B$ implies $B^* \subset H^*$. Therefore if we identify $H^*$ with $H$ by the Riesz's representation theorem, we have $B^* \subset H^* = H \subset B$. Therefore, $\langle z,h \rangle = (z,h)$ for all $z \in B^*, h \in H$. 
Note: $<\cdot,\cdot>$ denotes the dual pairing, and $(\cdot,\cdot)$ denotes the inner product in $H$.

Now let $\mathcal{W}_B$ denote the cylinder subsets of $B$, i.e., $\mathcal{W}_B$ is the collection of the set of the form

$$\{x \in B; (<x,z_1>,\ldots,<x,z_n>) \in D\}$$

where $D$ is the Borel subset of $\mathbb{R}^n$, and $z_n \in B^*$, $n = 1,2,\ldots$

Define $\hat{\mu} : \mathcal{W}_B \rightarrow [0,1]$ by

$$\hat{\mu}[x \in B : (<x,z_1>,\ldots,<x,z_n>) \in D] = \mu[x \in H : ((x,z_1),\ldots,(x,z_n)) \in D].$$

**Theorem 1.4** [11]

$\hat{\mu}$ has a unique $\sigma$-extension to the $\sigma$-field $\sigma[\mathcal{W}_B]$ generated by $\mathcal{W}_B$.

Note: From now on, we will use $\hat{\mu}$ for the unique $\sigma$-extension of $\hat{\mu}$ throughout this paper.

It is a well known fact that $<x,z>$, for $x \in B$, $z \in B^*$, is a random variable which is normally distributed with mean 0 and variance $|z|^2$ w.r.t. $\mu$. Define a map $n : H \rightarrow L^2(B,\mu)$ as follows: For $z \in B^*$, $n(z) = <\cdot,z>$. Since $|n(z)|^2 = |z|^2$ we can extend $n$ to $H$ by continuity. So for $h \in H$, $<\cdot,h>$ is defined a.e. $(\mu)$ in $B$. Obviously

$$\int_B e^{i<\cdot,h>}d\mu(\cdot) = e^{-\frac{1}{2}||h||^2}.$$

Now for each $x \in B$, define $\omega(x,D) = \mu(D - x)$
Theorem 1.5 [12]

If $h \in H$, then $\mu(h,*)$ is absolutely continuous w.r.t. $\mu(*)$ and

$$\frac{d\mu(h,*)}{d\mu(*)}(x) = \exp\{<h,x> - \frac{1}{2} Hx^2\}, x \in B.$$

**Orthogonal decomposition of $L^2(B,\mu)$**

First recall that the Hermite polynomials $H_n$, $n = 0, 1, 2, \ldots$ are defined by $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$. They satisfy the recursion formula:

$$H_0(x) = 1$$

$$H_{n+1}(x) = 2x \cdot H_n(x) - 2n H_{n-1}(x).$$

Let $\{e_k\}$ be a fixed orthonormal basis for $H$. For $n_1 + \ldots + n_r = n$ and $i_1 < \ldots < i_r$, define a normalized Hermite polynomial

$$H_{i_1,\ldots,i_r}^{n_1,\ldots,n_r}$$

by

$$H_{i_1,\ldots,i_r}^{n_1,\ldots,n_r}(x) = \frac{1}{\sqrt{2^n n_1! \ldots n_r!}} H_{i_1}^{n_1} \left(\frac{\sqrt{2}}{2}\right) H_{i_2}^{n_2} \left(\frac{\sqrt{2}}{2}\right) \ldots H_{i_r}^{n_r} \left(\frac{\sqrt{2}}{2}\right), x \in B.$$

Theorem 1.6 [12]

$$L^2(B,\mu) = \bigoplus_{n=0}^\infty K_n,$$ where $K_n$ has the orthonormal basis

$$\{H_{i_1,\ldots,i_r}^{n_1,\ldots,n_r}; n_1 + \ldots + n_r = n, i_1 < \ldots < i_r\}.$$
Theorem 1.7 [12]

$K_n$ can be identified with the symmetric $n$-tensor product $\otimes H$ of $H$ by the following map:

$$
\begin{align*}
\binom{n_1, \ldots, n_r}{H_{1_1, \ldots, 1_r}} &\rightarrow \frac{\sqrt{n!}}{\sqrt{n_1! \ldots n_r!}} e_{1_1} \otimes \cdots \otimes e_{1_r}.
\end{align*}
$$
CHAPTER 2. The space \( (L^2)^* \) of generalized random variables

§1. Ordinary multiple Wiener integrals.

**Definition 2.1**

\( \{\theta(t); t \in \mathbb{R}\} \) is called a \( B \)-valued curve in \( B \) if it satisfies the following conditions:

(i) \( \theta : t \mapsto \theta(t) \in B \) is a continuous \( B \) valued function

(ii) \( \int_E \theta(u)du \in \mathbb{H} \) for \( m(E) < \infty \) and \( \int_E \theta(u)du^2 = m(E) \)

(iii) \( \int_{\mathbb{R}} <h, \theta(t)> <k, \theta(t)> dt = (h,k) \) for \( h,k \in \mathbb{H} \),

where \( m \) is Lebesgue measure on \( \mathbb{R} \).

**Remark:** From (iii) we can formally write

\[ <x,\xi> = \int_{\mathbb{R}} <x,\theta(t)> <\xi,\theta(t)> dt \quad \text{for} \quad x \in B, \xi \in B^*. \]

Even more generally we will use formal representation

\[ <x,y> = \int_{\mathbb{R}} <x,\theta(t)> <y,\theta(t)> dt \quad \text{for} \quad x,y \in B \]

**Corollary 2.1**

If \( E \cap E' = \phi \), with \( m(E) < \infty \), \( m(E') < \infty \), then

\[ (\int_E \theta(u)du, \int_{E'} \theta(u)du) = m(E \cap E') = 0. \]
Proof: By the definition 2.1, (ii)

\[
\left( \int_{E \cup E'} \Theta(u) \, du, \int_{E \cup E'} \Theta'(u) \, du \right) = m(E \cup E') = m(E) + m(E')
\]

\[ \text{since } E \cap E' = \emptyset. \]

On the other hand

\[
\left( \int_{E \cup E'} \Theta(u) \, du, \int_{E \cup E'} \Theta'(u) \, du \right)
= \left( \int_{E} \Theta(u) \, du + \int_{E'} \Theta(u) \, du, \int_{E} \Theta'(u) \, du + \int_{E'} \Theta'(u) \, du \right)
= \int_{E} \Theta(u) \, du + \int_{E'} \Theta(u) \, du + 2 \int_{E} \Theta'(u) \, du + \int_{E'} \Theta'(u) \, du
\]

Therefore we conclude that

\[
\left( \int_{E} \Theta(u) \, du, \int_{E} \Theta'(u) \, du \right) = 0 \text{ when } E \cap E' = \emptyset. \quad (Q.E.D.)
\]

**Corollary 2.2**

\[
\left( \int_{E} \Theta(u) \, du, \int_{E} \Theta'(u) \, du \right) = m(E \cap E') \text{ if } m(E) < \infty, m(E') < \infty
\]

Proof:

**Observation:** \( E = (E \setminus E') \cup (E \cap E') \) and \( E' = (E' \setminus E) \cup (E \cap E') \)

\[
\left( \int_{E} \Theta(u) \, du, \int_{E} \Theta'(u) \, du \right)
= \left( \int_{(E \setminus E') \cup (E \cap E')} \Theta(u) \, du, \int_{(E' \setminus E) \cup (E \cap E')} \Theta'(u) \, du \right)
= \left( \int_{E \setminus E'} \Theta(u) \, du + \int_{E \cap E'} \Theta(u) \, du, \int_{E' \setminus E} \Theta'(u) \, du + \int_{E \cap E'} \Theta'(u) \, du \right)
= \left( \int_{E \setminus E'} \Theta(u) \, du, \int_{E' \setminus E} \Theta'(u) \, du \right) + \left( \int_{E \cap E'} \Theta(u) \, du, \int_{E \cap E'} \Theta'(u) \, du \right)
+ \left( \int_{E \setminus E'} \Theta(u) \, du, \int_{E \cap E'} \Theta'(u) \, du \right) + \left( \int_{E \cap E'} \Theta'(u) \, du, \int_{E \cap E'} \Theta'(u) \, du \right)
= \left( \int_{E \cap E'} \Theta(u) \, du, \int_{E \cap E'} \Theta'(u) \, du \right)
= m(E \cap E').
\]
since

\[(E \setminus E') \cap (E' \setminus E) = \phi \]

\[(E \setminus E') \cap (E \cap E') = \phi \]

\[(E' \setminus E) \cap (E \cap E') = \phi. \]

(Q.E.D.)

From now on, we will assume that \((H, B)\) satisfies the following conditions (i) and (ii):

(i) There exists a Hilbert space \(F \subset B\) such that \(H \subset F \subset B\) and \((H, F)\) is also an abstract Wiener space.

(ii) \((\Theta(t); t \in \mathbb{R})\) is an \(F\) valued curve.

Since we assume that there exists \(F\) such that \(H \subset F \subset B\) and \((H, F)\) is an abstract Wiener space, there exists an injective Hilbert Schmidt operator \(T\) on \(H\) such that \((x, y)_F = (Tx, Ty)_H\) for \(x, y \in H\).

Then there exists an orthonormal basis \(\{e_n\} \subset H\) such that, \(T e_n = \lambda_n e_n\) with \(\sum \lambda_n^2 < \infty\) and \(e_n \in B^\star\), i.e. \(\lambda_n^\prime s\) are eigenvalues of \(T\), \(e_n\)'s are eigenvectors of \(T\). Obviously \(\|e_n\|_F = \lambda_n\).

We can easily see that \(\{\xi_n = \frac{1}{\lambda_n} e_n, n = 1, 2, \ldots\}\) is an orthonormal basis for \(F\). Now for \(x \in F\),

\[x = \sum_{n=1}^{\infty} (x, \xi_n)_F \xi_n\]

\[= \sum_{n=1}^{\infty} \left( x, \frac{1}{\lambda_n} e_n \right)_F \frac{1}{\lambda_n} e_n\]

\[= \sum_{n=1}^{\infty} \left( \frac{1}{\lambda_n} \right)^2 (x, e_n)_F e_n\]
Therefore for fixed \( t \in \mathbb{R} \), \( \Theta(t) \) can be represented by
\[
\Theta(t) = \sum_{n=1}^{\infty} (\Theta(t), e_n) H e_n \quad \text{even though} \quad \sum_{n=1}^{\infty} |(\Theta(t), e_n)|^2 = \infty. \quad \text{Here}
\]
\((\Theta(t), e_n)\) does not make sense, since \( \Theta(t) \in F \setminus H \). But we will understand this \( <\Theta(t), e_n> \) where \(<,>\) means dual pairing between \( \Theta(t) \in B \) and \( e_n \in B^* \).

**Note:**
\[
\|\Theta(t)\|_F^2 = \sum_{n=1}^{\infty} |(\Theta(t), \xi_n)|^2 = \sum_{n=1}^{\infty} |(\Theta(t), e_n)|^2 = \sum_{n=1}^{\infty} \left\{ \frac{1}{\lambda_n} \right\}^2 \lambda_n^2 \|\Theta(t), e_n\|_H^2 < \infty
\]
i.e., even though \( \sum_{n=1}^{\infty} \|\Theta(t), e_n\|_H^2 = \infty \), \( \sum_{n=1}^{\infty} \lambda_n^2 \|\Theta(t), e_n\|_H^2 < \infty \) for each fixed \( t \in \mathbb{R} \).

Now we can define ordinary multiple Wiener integrals, w.r.t., \( \Theta(t) \) in Ito's sense [8]. Let's define \( \beta(E,x) \) as follows
\[
\beta(E,x) = <x, \int_E \Theta(u)du> \quad \text{for} \quad E \in \mathcal{B}(\mathbb{R}), \; x \in B.
\]

Then \( \beta(E,x) \) is a normal random measure by Corollary 2.2., i.e.,
\[
\int_B \beta(E,x)\beta(E',x)du(x) = m(E \cap E')
\]
for any \( E, E' \) with \( m(E) < \infty, m(E') < \infty \) where \( m \) is Lebesgue measure on \( \mathbb{R} \).
Since Lebesgue measure satisfies the continuity condition we can define multiple Wiener integral in Itô's sense.

**Note:** Continuity condition. For any $E \in \mathscr{B}(\mathbb{R})$ with $m(E) < \infty$ and any $\varepsilon > 0$ there exists a decomposition of $E$:

$$E = \sum_{i=1}^{n} E_i \quad \text{such that} \quad m(E_i) < \varepsilon, \quad i = 1,2,\ldots,n.$$ 

Now suppose $f \in L^2(\mathbb{R}^p)$ is a special elementary function, i.e.,

$$f(t_1,\ldots,t_p)\begin{cases} a_{i_1,\ldots,i_p} \quad &\text{for } (t_1,\ldots,t_p) \in T_{i_1} \times \cdots \times T_{i_p} \\ 0 &\text{elsewhere} \end{cases}$$

where $T_1,T_2,\ldots,T_n$ are disjoint and $m(T_i) < \infty, \quad i = 1,2,\ldots,n$ and $a_{i_1,\ldots,i_p} = 0$ if any two of $i_1,\ldots,i_p$ are equal. Then define

$$I_p(f) = \int \cdots \int f(t_1,\ldots,t_p) \, d\beta(t_1) \cdots d\beta(t_p)$$

$$= \sum a_{i_1,\ldots,i_p} \beta(T_{i_1}) \cdots \beta(T_{i_p}).$$

We have the following properties

1. $I_p(af + bg) = aI_p(f) + bI_p(g)$

2. $I_p(f) = I_p(\hat{f})$

where $\hat{f}$ is the symmetrization of $f$, i.e.,

$$\hat{f}(t_1,\ldots,t_p) = \frac{1}{p!} \sum_{\pi} f(t_{\pi(1)},\ldots,t_{\pi(p)}),$$

where the summation runs over all permutations $\pi$ of $(1,2,\ldots,p)$.

3. $(I_p(f), I_p(g)) = p!(\hat{f},\hat{g})$
(4) \( (I_p(f), I_q(g)) = 0 \) if \( p \neq q \).

Note: For the proof of the above properties, see the paper by Ito [8].

Ito showed also that this integral can be extended to the functions \( f \) in \( L^2(\mathbb{R}^n) \).

**Theorem 2.1 [8]**

Let \( \xi_1(t), \xi_2(t), \ldots \) be an orthogonal system of real valued functions in \( L^2(\mathbb{R}) \). Then

\[
\int \cdots \int \xi_1(t_1) \cdots \xi_1(t_{p_1}) \xi_2(t_{p_1+1}) \cdots \xi_2(t_{p_1+p_2}) \xi(t_{p_1+p_2+1}) \cdots \\
\ldots \xi_n(t_{p_1+\cdots+p_n}) d\theta(t_1) \ldots d\theta(t_n) \\
= \frac{1}{\sqrt{2}} \prod_{\nu=1}^{n} \frac{H_{p_{\nu}} \left( \frac{1}{\sqrt{2}} \int \xi_{\nu}(t) d\theta(t) \right)}{(\sqrt{2})^{p_{\nu}}}.
\]

**Theorem 2.2 [8]**

For any \( f(\beta(E,x)), E \in \mathcal{B}(\mathbb{R}) \) with \( m(E) < \infty \) satisfying

\[
\int_{\mathbb{R}} |f(\beta(E,x))|^2 d\mu(x) < \infty,
\]

it can be represented by

\[
f(\beta(E,x)) = \sum_{p} \sum_{p_1+\cdots+p_n=p} \frac{a_1 \cdots a_n}{\prod_{\nu=1}^{n} (\sqrt{2})^{p_{\nu}}} \frac{1}{\sqrt{2}} \int \xi_{\nu}(t) d\theta(t) \]

**Theorem 2.3**

If \( \phi \in K_n \), then

\[
\phi(x) = \int_{\mathbb{R}^n} f(u_1, \ldots, u_n) d\beta(u_1, x) \cdots d\beta(u_n, x), f \in L^2(\mathbb{R}^n),
\]

\[
f(u_1, \ldots, u_n) \in L^2(\mathbb{R}^n)
\]

where \( L^2(\mathbb{R}^n) \) is the space of symmetric functions in \( L^2(\mathbb{R}^n) \).
Proof: We know \( k_n \) has the orthonormal basis of the form

\[
\{ \Pi_{i=1}^{k} \frac{1}{\sqrt{2^n n_i! \cdots n_k!}} H_{n_i} \left( \frac{<x, e_{\ell_1}>}{\sqrt{2}} \right); \Sigma n_i = n, \ell_1 < \ell_2 < \ldots \}
\]

where \( \{e_{\ell_1}\} : \text{ONB for } H \). Therefore it is enough to show that

\[
<x, e_{\ell_1}> = \int_{\mathbb{R}} \xi_{\ell_1}(t) d\beta(t, x)
\]

by the Theorem 2.1. Also it is enough to show that

\[
\int_{\mathbb{R}} \xi_{\ell_1}(u) \Theta(u) du \in H \text{ and } \{ \int_{\mathbb{R}} \xi_{\ell_1}(u) \Theta(u) du \}_{i=1}^{\infty} \text{ is ONB for } H
\]

since

\[
\int_{\mathbb{R}} 1_E(u) d\beta(u, x) = \beta(E, x) = <x, \int_{\mathbb{R}} \Theta(u) du> = <x, \int_{\mathbb{R}} 1_E(u) \Theta(u) du>
\]

Now define \( i: L^2(\mathbb{R}) \rightarrow H \) such that \( i(1_E) = \int_{\mathbb{R}} \Theta(u) du \). Since

\[
\int_{\mathbb{R}} 1_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \Theta(u) du \in H, \quad i \text{ is a densely defined isometry. Therefore}
\]

we can naturally extend \( i \) to \( L^2(\mathbb{R}) \). Then

\[
i(\xi_{\ell_1}) = \int_{\mathbb{R}} \xi_{\ell_1}(u) \Theta(u) du \in H \text{ for } \{\xi_{\ell_1}\} \text{ ONB for } L^2(\mathbb{R}) \text{ and}
\]

\[
[\int_{\mathbb{R}} \xi_{\ell_1}(u) \Theta(u) du, \int_{\mathbb{R}} \xi_{\ell_j}(u) \Theta(u) du]_H = [\xi_{\ell_1}(u), \xi_{\ell_j}(u)]_{L^2(\mathbb{R})} = \delta_{i,j}
\]

Therefore \( \{\int_{\mathbb{R}} \xi_{\ell_1}(u) \Theta(u) du\}_{i=1}^{\infty} \) is ONB for \( H \). So let

\[
i(\xi_{\ell_1}) = \int_{\mathbb{R}} \xi_{\ell_1}(u) \Theta(u) du \equiv e_{\ell_1}
\]

then we can fix \( \{e_{\ell_1}\}_{i=1}^{\infty} \), ONB for \( H \), for each \( \{\xi_{\ell_1}\} \) ONB for \( L^2(\mathbb{R}) \). (Q.E.D.)

Note: Symbolically we can write

\[
\phi(x) = \int_{\mathbb{R}} f(u_1, \ldots, u_n) d\beta(u_1, x) \ldots d\beta(u_n, x)
\]
\begin{align*}
\int_{\mathbb{R}^n} f(u_1, \ldots, u_n) : & \langle x, \Theta(u_1) \rangle \cdots \langle x, \Theta(u_n) \rangle : du_1 \cdots du_n \\
\text{for } \phi \in K_n, \text{ where } : : \text{ denotes the renormalization which we will define later.}
\end{align*}

**Example 1**

We have $\mathcal{H} \subset \mathcal{H}_p \subset L^2(\mathbb{R}) \subset \mathcal{H}_p^* \subset \mathcal{H}_p^*$ as before. In the Hida calculus, $\Theta(t) = \delta_t \in \mathcal{H}_p^*$ when $p > 1$, i.e., $\langle \Theta(t), \xi \rangle = \langle \delta_t, \xi \rangle = \xi(t)$ where $\xi \in \mathcal{H}$. We show that $\delta_t$ satisfies the conditions for $\Theta(t)$ in Definition 2.1. First we need to verify $\delta_t : t \mapsto \delta_t \in \mathcal{H}_p$ is continuous. It is a well-known fact that $\delta_t \in \mathcal{H}_{-1} = \mathcal{H}_1^*$, so we need to show that $\| \delta_{t+\Delta}(x) - \delta_t(x) \|_{H_{-1}} \to 0$ as $\Delta \to 0$ where $\| \cdot \|_{H_{-1}}$ is $H_{-1}$ norm

\[
\delta_{t+\Delta}(x) - \delta_t(x) = \int_{\mathbb{R}} e^{-ixy}(\delta_{t+\Delta}(y) - \delta_t(y))dy
\]

\[
= e^{-i(t+\Delta)x} - e^{-itx}
\]

\[
= e^{-itx} [e^{-i\Delta x} - 1].
\]

Since $|e^{i\Delta x} - 1|^2 < 4$, we can apply the Lebesgue Dominated Convergence theorem. Therefore

\[
\| \delta_t(x) - \delta_{t+\Delta}(x) \|^2_{H_{-1}} = \int_{\mathbb{R}} \frac{1}{1+x^2} \cdot |\delta_{t+\Delta}(x) - \delta_t(x)|^2 dx
\]

\[
= \int_{\mathbb{R}} \frac{1}{1+x^2} |e^{-i(t+\Delta)x} - e^{-itx}|^2 dx
\]

\[
= \int_{\mathbb{R}} \frac{1}{1+x^2} |(1 - e^{-it\Delta})|^2 dx
\]

\[
= \int_{\mathbb{R}} \frac{1}{1+x^2} |e^{-i\Delta x} - 1|^2 dx
\]
\[
\lim_{\Delta \to 0} \left| \delta_{t+\Delta}(x) - \delta_t(x) \right|_2^2 = \lim_{\Delta \to 0} \int_{\mathbb{R}} \frac{1}{1+x^2} |e^{-i\Delta x} - 1|^2 dx \\
= \int_{\mathbb{R}} \lim_{\Delta \to 0} \left[ \frac{1}{1+x^2} |(e^{-i\Delta x} - 1)|^2 \right] dx = 0.
\]

Next we need to check that \( \int_E \delta_t(u) du \in H = L^2(\mathbb{R}) \) and
\[
\int_E \delta_t(u) du = m(E) \quad \text{when} \quad m(E) < \infty.
\]
By the distribution theory \( \int_E \delta_t(u) du = \int_{\mathbb{R}} l_E(u) \cdot \delta_t(du) = l_E(t) \) and \( l_E(t) \in L^2(\mathbb{R}) \) when \( m(E) < \infty \). Obviously \( \|l_E(t)\|_{L^2(\mathbb{R})}^2 = m(E) \). Finally we need to check that
\[
\int_{\mathbb{R}} \langle \delta_t, h \rangle \langle \delta_t, k \rangle dt = (h, k)
\]
where \( h, k \in L^2(\mathbb{R}) \) and \( \langle , \rangle \) is inner product in \( L^2(\mathbb{R}) \)
\[
\int_{\mathbb{R}} \langle \delta_t, h \rangle \langle \delta_t, k \rangle dt = \int_{\mathbb{R}} (\int_{\mathbb{R}} \delta_t(u) h(u) du) (\int_{\mathbb{R}} \delta_t(v) k(v) dv) dt \\
= \int_{\mathbb{R}} h(t) k(t) dt \\
= (h, k)_{L^2(\mathbb{R})}
\]

**Example 2**

Let \( \mathcal{A}_p \) denote the Sobolev space \( H_p(\mathbb{R}) \), \( p > 0 \). Then
\( \mathcal{A}^* \subset \mathcal{A}_p \subset L^2(\mathbb{R}) \subset \mathcal{A}_p^* \subset \mathcal{A}^* \) where \( \mathcal{A} \) and \( \mathcal{A}^* \) denote the spaces of all rapidly decreasing functions with the Schwartz topology and the dual of \( \mathcal{A} \), respectively. When \( p > 1 \), \( L^2(\mathbb{R}) \subset \mathcal{A}_p^* \) is an abstract Wiener space.

Define \( \langle \Theta(u), \xi \rangle = \xi(\psi(u))|\psi'(u)|^{1/2} \) where \( \psi(u) \) is a strictly monotonic real valued function with \( \psi(\mathbb{R}) = \mathbb{R} \) such that \( u \to \Theta(u) \) is continuous and \( \Theta(u) \in \mathcal{A}_p^* \). This example depends on \( \psi(u) \) and is related to flows (rotation group) in Hida's work [5].
Corollary 2.3

If \( \{e_n(t)\} \) is an ONB for \( L^2(\mathbb{R}) \), then \( \{e_n(\psi(u))|\psi'(u)|^{1/2}\}_{n=1}^\infty \) is also an ONB for \( L^2(\mathbb{R}) \) where \( \psi(u) \) is given as above definition.

Proof:

\[
\langle e_n(\psi(u))|\psi'(u)|^{1/2}, e_m(\psi(u))|\psi'(u)|^{1/2}\rangle_{L^2(\mathbb{R})}
= \int_{\mathbb{R}} e_n(\psi(u))|\psi'(u)|^{1/2} \cdot e_m(\psi(u))|\psi'(u)|^{1/2} du
= \int_{\mathbb{R}} e_n(\psi(u))e_m(\psi(u))\psi'(u) du
= \int_{\mathbb{R}} e_n(t)e_m(t) dt
= \delta_{m,n}
\]

since \( \psi \) is a strictly monotone function. \(\text{(Q.E.D.)}\)

Now we check that the condition (ii) is satisfied by using the above Corollary 2.3

\[
\langle \int_E \Theta(u) du, \int_E \Theta(u) du \rangle_H
= \sum_{n=1}^{\infty} \langle \int_E \Theta(u) du, e_n \rangle \langle \int_E \Theta(u) du, e_n \rangle
= \sum_{n=1}^{\infty} \| \int_E e_n(\psi(u))|\psi'(u)|^{1/2} du \|^2
= \| \int_E (u) \|^2_{L^2(\mathbb{R})}
= m(E).
\]
For the condition (iii)

\[ \int_{\mathbb{R}} (\theta(u), h)(\theta(u), k) du \text{ for } h, k \in L^2(\mathbb{R}) \]

\[ = \int_{\mathbb{R}} (\theta(u), \sum_{n=1}^{\infty} (h, e_n)e_n)(\theta(u), \sum_{\ell=1}^{\infty} (k, e_\ell)e_\ell) du \]

where \( |e_n|^{\infty}_{1=1} \) is an ONB for \( L^2(\mathbb{R}) \)

\[ = \int_{\mathbb{R}} \sum_{n=1}^{\infty} (h, e_n)(\theta(u), e_n) \sum_{\ell=1}^{\infty} (k, e_\ell)(\theta(u), e_\ell) du \]

\[ = \int_{\mathbb{R}} \sum_{n=1}^{\infty} (h, e_n)e_n(\psi(u))|\psi'(u)|^{1/2} \sum_{\ell=1}^{\infty} (k, e_\ell)e_\ell(\psi(u))|\psi'(u)|^{1/2} du \]

where

\[ [\xi_1(u) = e_1(\psi)(u))|\psi'(u)|^{1/2}]^{\infty}_{1=1} \]

is another ONB for \( L^2(\mathbb{R}) \) by Corollary 2.3

\[ = \sum_{n=1}^{\infty} (h, e_n)(k, e_n) \]

\[ = (h, k) \in L^2(\mathbb{R}) \]

\section{S-transform on \( L^2(B, \mu) \)}

Define \((S\phi)(\xi) = \int_B \phi(x + \xi) du(x)\) for \( \phi \in L^2(B, \mu), \xi \in B^* \). Since \( y \in B^* \subset H \)

\[ \int_B \phi(x + \xi) du(x) = \int_B \phi(x) du(x - \xi) \]

\[ = \int_B \phi(x)e^{-1/2||\xi||^2 + \langle \xi, x \rangle} du(x) \]

\[ = e^{-1/2||\xi||^2} \int_B \phi(x)e^{\langle x, \xi \rangle} du(x), \]

where \( 1:1 \) - \( H \) norm and \( \langle x, \xi \rangle \) is dual pairing. (This can be verified by using the translation formula [11]).
We can easily see that \( \phi(x) e^{\langle x, \xi \rangle} \) is also in \( L^2(B) \). Therefore the S-transform is well-defined. Furthermore we will see that this S-transform carries the random variables on \( B \) to functions defined on \( B^* \).

We already know that \( K_n \) is identified with the symmetric n-tensor product \( \otimes^n H \), of \( H \) [12].

Define the functional on \( B^* \) in the following way. For \( h_1 \otimes \ldots \otimes h_n \in \otimes^n H \), define

\[
(h_1 \otimes \ldots \otimes h_n, \xi) = (h_1, \xi)(h_2, \xi) \ldots (h_n, \xi).
\]

**Theorem 2.4**

If \( \phi \in K_n \) is represented by

\[
h_1 \otimes \ldots \otimes h_n \in \otimes^n H,
\]

then \( S\phi \) is given by

\[
(S\phi)(\xi) = (h_1 \otimes \ldots \otimes h_n, \xi).
\]

Before proving the theorem, we need the following lemma.

**Lemma 2.1**

Suppose \( \phi(x) = \frac{1}{(\sqrt{2})^n} H_n(\frac{x}{\sqrt{2}}) \) then

\[
(S\phi)(\xi) = \langle \xi, e_\xi \rangle^n = \langle e_\xi \otimes \ldots \otimes e_\xi, \xi \rangle, \quad x \in B, \ \xi \in B^*
\]

where \( \{e_\xi\} \) is an ONB for \( H \).

**Proof:** The generating function for \( H_n(\frac{x}{\sqrt{2}}) \) is

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(\frac{x}{\sqrt{2}}) = e^{-t^2+\sqrt{2}tx}.
\]
Let \( \phi(x) = e^{-t^2 + \sqrt{2t} \langle x, e_x \rangle} \). Then,

\[
\phi(x) = e^{-t^2 + \sqrt{2t} \langle x, e_x \rangle} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n \left( \frac{\langle x, e_x \rangle}{\sqrt{2}} \right)
\]

\((S\phi(x))(\xi) = \int_B \phi(x + \xi) \, d\mu(x)\)

\[
\int_B e^{-t^2 + \sqrt{2t} \langle x+\xi, e_x \rangle} \, d\mu(x)
\]

\[
= e^{-t^2} e^{\sqrt{2t} \langle \xi, e_x \rangle} \int_B e^{\sqrt{2t} \langle x, e_x \rangle} \, d\mu(x)
\]

\[
= e^{-t^2} e^{\sqrt{2t} \langle \xi, e_x \rangle} \int_B e^{-1/2} e^{-i\sqrt{2t} e_x L_H^1} \, d\mu(x)
\]

\[
= e^{-t^2} e^{\sqrt{2t} \langle \xi, e_x \rangle} \int_B e^{-1/2} e^{-(-1)^2 t^2} \, d\mu(x)
\]

\[
= e^{-t^2} e^{\sqrt{2t} \langle \xi, e_x \rangle} \sum_{n=0}^{\infty} \frac{(\sqrt{2})^n t^n}{n!} \langle \xi, e_x \rangle^n.
\]

On the other hand

\[
S(\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n \left( \frac{\langle x, e_x \rangle}{\sqrt{2}} \right))(\xi)
\]

\[
= \int_B \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n \left( \frac{\langle x+\xi, e_x \rangle}{\sqrt{2}} \right) d\mu(x)
\]

(by Lebesgue Dominated Convergence Theorem)

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_B H_n \left( \frac{\langle x+\xi, e_x \rangle}{\sqrt{2}} \right) d\mu(x)
\]

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} S(H_n \left( \frac{\langle x, e_x \rangle}{\sqrt{2}} \right))(\xi).
\]

Therefore
\[ \sum_{n=0}^{\infty} \frac{(\sqrt{2})^n}{n!} \langle \xi, e_k \rangle^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} S(H_n\left(\frac{\langle x, e_k \rangle}{\sqrt{2}}\right))(\xi) \]

implies

\[ S(H_n\left(\frac{\langle x, e_k \rangle}{\sqrt{2}}\right))(\xi) = (\sqrt{2})^n \langle \xi, e_k \rangle^n \]

i.e.

\[ S\left(\frac{1}{(\sqrt{2})^n} H_n\left(\frac{\langle x, e_k \rangle}{\sqrt{2}}\right)\right)(\xi) = \langle \xi, e_k \rangle^n = \langle e_k \otimes \ldots \otimes e_k, \xi \rangle. \]

(Q.E.D.)

**Proof of Theorem 2.4**

It has been shown that

\[ [H_{i_1}, \ldots, H_{i_r}(x) = \frac{1}{\sqrt{2}^{n_1!}n_2! \ldots n_r!} H_{i_1}(\frac{1}{\sqrt{2}}) \ldots H_{i_r}(\frac{1}{\sqrt{2}}); \]

\[ n_1 + \ldots + n_r = n \text{ and } i_1 < \ldots < i_r, x \in B, \{e_i\} : \text{ONB for } H] \]

is an ONB for \( K_n \), and

\[ [h_{i_1}, \ldots, h_{i_r} = \frac{\sqrt{n_1! \ldots n_r!}}{\sqrt{n_1! \ldots n_r!}} e_{i_1} \otimes \ldots \otimes e_{i_1} \otimes e_{i_2} \otimes \ldots \otimes e_{i_2} \otimes \ldots \otimes e_{i_r} \otimes \ldots \otimes e_{i_r}; \]

\[ n_1 + \ldots + n_r = n, i_1 < \ldots < i_r \] is an ONB for \( K_n \).

Therefore it is enough to show that

\[ S(H_{i_1}, \ldots, H_{i_r}(x))(\xi) = \frac{1}{\sqrt{n!}} H_{i_1}, \ldots, H_{i_r}(\xi) \]

also, it suffices to show that for

\[ \phi(x) = \frac{1}{(\sqrt{2})^{n+m}} H_n(\frac{\langle x, e_k \rangle}{\sqrt{2}}) H_m(\frac{\langle x, e_k \rangle}{\sqrt{2}}) \]
Consider the generating function

\[ e^{-t^2+\sqrt{2}tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n\left(\frac{x}{\sqrt{2}}\right) \]

and

\[ e^{-t^2+\sqrt{2}tx_1} = \sum_{m=0}^{\infty} \frac{t^m}{m!} H_m\left(\frac{x_1}{\sqrt{2}}\right). \]

Therefore

\[ e^{-2t^2+\sqrt{2}t(x+x_1)} = \sum_{n,m=0}^{\infty} \frac{t^n t^m}{n! m!} H_n\left(\frac{x}{\sqrt{2}}\right) H_m\left(\frac{x_1}{\sqrt{2}}\right). \]

Now let \( \phi(x) = e^{-t^2+\sqrt{2}tx_1} \) then

\[ (S\phi(x))(\xi) = \int_B \phi(x + \xi) d\mu(x) \]

\[ = e^{-2t^2+\sqrt{2}t(x,e_k)+e_k} \int_B e^{-2t^2+\sqrt{2}t(x,e_k)+e_k} d\mu(x) \]

\[ = e^{-2t^2+\sqrt{2}t(x,e_k)+e_k} \frac{1}{e^{-2t^2+\sqrt{2}t(x,e_k)+e_k} - 1/2(-2t^2 e_k e_k^2)} \]

\[ = e^{-2t^2+\sqrt{2}t(x,e_k)+e_k} \frac{1}{e^{-2t^2+\sqrt{2}t(x,e_k)+e_k} - 1/2(-2t^2 e_k e_k^2)} \]

\[ = e^{-2t^2+\sqrt{2}t(x,e_k)+e_k} \frac{1}{e^{-2t^2+\sqrt{2}t(x,e_k)+e_k} - 1/2(-2t^2 e_k e_k^2)} \]

\[ = \sum_{j=0}^{\infty} \frac{(-1)^j (\sqrt{2})^j}{j!} \langle \xi, e_k + e_k \rangle^j \]

\[ = \sum_{j=0}^{\infty} \frac{t^n}{n!} (\sqrt{2})^n [\langle \xi, e_k \rangle + \langle \xi, e_k \rangle]^j \]

\[ = \sum_{n,m=0}^{\infty} \frac{t^n t^m}{n! m!} (\sqrt{2})^{n+m} \langle \xi, e_k \rangle^n \langle \xi, e_k \rangle^m. \]
Therefore

\[
S\left(\frac{1}{(\sqrt{2})^{n+m}}\right) H_n\left(\frac{\langle x, e_k \rangle}{\sqrt{2}}\right) H_m\left(\frac{\langle x, e_k \rangle}{\sqrt{2}}\right)(\xi) = (e_k, \xi)^n (e_k, \xi)^m
\]

\[
= (\xi, e_k \otimes \ldots \otimes e_k) \otimes (\xi, e_k \otimes \ldots \otimes e_k)
\]

(Q.E.D.)

Moreover, suppose \((S\phi)(\xi)\) is represented by \(h_1 \otimes \ldots \otimes h_n \in \mathbb{H}\), then

\[
\int_{L^2(B)} h_1^{\otimes 2} = n! h_1^{\otimes 2} \ldots h_n^{\otimes 2}
\]

This is clear since \(S(H_{11}, \ldots, 1_r) (x)\) is represented by

\[
\frac{1}{\sqrt{n!}} h_1^{n_1, \ldots, n_r}, \quad 1_{H_{11}, \ldots, 1_r} (x) \int_{L^2(B)} h_1^{n_1, \ldots, n_r} \otimes H^{n_1, \ldots, n_r} = 1.
\]

Also since the \(S\) transform carries the ONB for \(K_n\) to the ONB for \(H\) up to the constant, the \(S\) transform is an isomorphism between \(K_n\) and \(H\). Thus the following theorem has already been shown.

**Theorem 2.5**

\[L^2(B, \mu) = e \otimes H,\]  where \(e \otimes H\) is the direct sum of \(H\) with weight \(\sqrt{n!}\).

§3. The space \((L^2)^n\) of generalized random variables.

In the previous section, it was shown that \(\phi \in K_n\) has the representation \(h_1 \otimes \ldots \otimes h_n \in \mathbb{H}\) under the \(S\) transform, i.e.

\[(S\phi)(\xi) = (h_1 \otimes \ldots \otimes h_n, \xi) = (h_1, \xi) \ldots (h_n, \xi).\]
Note: Observe that \((h_1, \xi), \ldots, (h_n, \xi)\) is an ordinary random variable.

Furthermore, \(\langle f_1, \xi \rangle, \ldots, \langle f_n, \xi \rangle\) is meaningful when \(f_i \in F\) for \(i = 1, \ldots, n\) (\(H \subset F\) is also an abstract Wiener space, and \(H\) is dense in \(F\)). Therefore we can define natural extension \(U^{-1}\) of \(S^{-1}\) since \(H \subset F\) is dense.

Define

\[(U\phi)(\xi) = (S\phi)(\xi) \text{ for } \phi \in L^2(B),\]

and

\[K_n^{-} = U^{-1}(\langle \delta F, \xi \rangle_n),\]

\[K_n^{+} = U^{-1}(\langle \delta F^*, \xi \rangle_n).\]

Then we have the following diagram

\[
\begin{array}{c}
K_n^{+} \subset K_n \subset K_n^{-} \\
\uparrow \quad \uparrow \quad \uparrow \\
\sqrt{n!} \delta F^* \subset \sqrt{n!} \delta H \subset \sqrt{n!} \delta F
\end{array}
\]

Define

\[(L^2)^{-} = \bigoplus_{n=0}^{\infty} K_n^{-}, \quad (L^2)^{+} = \bigoplus_{n=0}^{\infty} K_n^{+}.\]

Then

\[(L^2)^{+} \subset L^2(B, \mu) \subset (L^2)^{-}.\]

We will call \((L^2)^{+}\) the space of test random variables \((L^2)^{-}\) and the space of generalized random variable.
Examples of generalized random variables.

1. The random variable \( <x, \Theta(t)> \).

**Note:** 1) \( \Theta(t) \in F \setminus H \), therefore \( <\cdot, \Theta(t)> \) is not an ordinary random variable.

2) \( \Theta(t) = \sum_{n=1}^{\infty} (\Theta(t), e_n) e_n \) with \( \{e_n\} \) : fixed ONB for \( H \).

Let's define \( \phi_N(x) = \sum_{i=1}^{N} (\Theta(t), e_i) e_i, x \) where \( \{e_i\} \) is a fixed ONB for \( H \). Obviously \( \phi_N(x) \) is an ordinary random variable for \( N = 1, 2, \ldots \), i.e., \( \phi_N(x) \in L^2(B) \). Therefore

\[
(S\phi_N(x))(\xi) = \int_B \phi_N(x + \xi) d\nu(x)
\]

\[
= \int_B \left( \sum_{i=1}^{N} (\Theta(t), e_i) e_i, x + \xi \right) d\nu(x)
\]

\[
= \sum_{i=1}^{N} (\Theta(t), e_i) \int_B \langle e_i, x \rangle d\nu(x)
\]

\[
= \sum_{i=1}^{N} (\Theta(t), e_i) \langle e_i, \xi \rangle
\]

\[
= \sum_{i=1}^{N} (\Theta(t), e_i)(\xi, e_i).
\]

Therefore, \( (S\phi_N(x))(\xi) = \sum_{i=1}^{N} (\Theta(t), e_i)e_i, \xi \).

**Note:** \( \sum_{i=1}^{N} (\Theta(t), e_i)e_i \) is not a Cauchy sequence in \( H \). But we know that

\[
\sum_{i=1}^{N} (\Theta(t), e_i)e_i \] is a Cauchy sequence in \( F \); i.e., \( (S\phi_N(x))(\xi) \) converges to \( \sum_{i=1}^{\infty} (\Theta(t), e_i)e_i, \xi \) = \( \langle \Theta(t), \xi \rangle \) in \( F \). Therefore we can find the element \( <x, \Theta(t)> \) in \( K(-) \) such that \( U(\langle x, \Theta(t)>)(\xi) = \langle \Theta(t), \xi \rangle \).

Actually \( \mathbb{E}\phi_N(x) = \int_B \phi_N(x) d\nu(x) = 0 \) yields that \( <x, \Theta(t)> = <x, \Theta(t)> \) where \( \mathbb{E} \) is the expectation of \( \langle \cdot \rangle \) w.r.t. \( \nu \).
Hida calculus approach for $<x, 0(t)>$

Since $0 : u \rightarrow 0(u)$ is continuous,

$$<x, 0(t)> = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \left[ \int_0^{t+\Delta} 0(u), x > du - \int_0^t 0(u), x > du \right]$$

$$= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \int_t^{t+\Delta} 0(u), x > du$$

$$= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \langle x, \int_t^{t+\Delta} 0(u) du \rangle.$$

Let $\phi_\Delta(x) = <x, \frac{1}{\Delta} \int_t^{t+\Delta} 0(u) du >$. Then $\phi_\Delta(x) \in L^2(B)$, since

$$\frac{1}{\Delta} \int_t^{t+\Delta} 0(u) du \in H \text{ and } \mathbb{E}[\phi_\Delta(x)] = \int_B <x, \frac{1}{\Delta} \int_t^{t+\Delta} 0(u) du > du(x) = 0. \text{ Now}$$

$$(S\phi_\Delta(x))(\xi) = \int_B <x + \xi, \frac{1}{\Delta} \int_t^{t+\Delta} 0(u) du > du(x) = <\xi, \frac{1}{\Delta} \int_t^{t+\Delta} 0(u) du >$$

so, $(S\phi_\Delta(x))(\xi)$ is represented by $\frac{1}{\Delta} \int_t^{t+\Delta} 0(u) du \in H$. Since

$$\frac{1}{\Delta} \int_t^{t+\Delta} 0(u) du \in H$$

does not converge in $H$ when $\Delta \rightarrow 0$. But $\frac{1}{\Delta} \int_t^{t+\Delta} 0(t) du + O(t)$ as $\delta \rightarrow 0$ in $F$. Therefore let us denote $U^{-1}(0(t), \xi)) = :<x, 0(t)>$. Similarly, we can say that

$$<x, 0(t)> = :<x, 0(t)>.$$  

2. The Random variable $<x, 0(t)>^2$

Since $0(t) \in F \setminus H$, $<x, 0(t)>^2$ is not an ordinary random variable.

Let $\phi_N(x)$ be defined by $\phi_N(x) = \sum_{i=1}^n (0(t), e_i) e_i, x >$. Then $\phi_N(x) \in L^2(B)$ and

$$\mathbb{E}[\phi_N(x)] = \int_B \sum_{i=1}^n (0(t), e_i) e_i, x >^2 du(x)$$

$$= \sum_{i=1}^n (0(t), e_i) e_i^2$$

$$= \sum_{i=1}^n |(0(t), e_i)|^2.$$
Notation: We know that \( \sum_{n=1}^{\infty} |(0(t), e_n)|^2 = \) since \( 0(t) \in F \setminus H \) and \( \{e_i\}_{i=1}^{\infty} \) is an O.N.B. for \( H \). But we can consider \( \sum_{n=1}^{\infty} |(0(t), e_n)|^2 \) as a number in the nonstandard real number system. So let's denote \( \sum_{n=1}^{\infty} |(0(t), e_n)|^2 = \frac{1}{D(t)} \), where \( \frac{1}{D(t)} \) is a nonstandard real number.

Now

\[
S(\phi_n(x))(\xi) = \int_B <x + \xi, \sum_{n=1}^{N} (0(t), e_n) e_n >^2 d\mu(x)
\]

\[
= \int_B <x, \sum_{n=1}^{N} (0(t), e_n) e_n >^2 d\mu(x) + \int_B <\xi, \sum_{n=1}^{N} (0(t), e_n) e_n >^2 d\mu(x)
\]

\[
+ 2\int_B <\xi, \sum_{n=1}^{N} (0(t), e_n) e_n > <x, \sum_{n=1}^{N} (0(t), e_n) e_n > d\mu(x)
\]

\[
= \sum_{n=1}^{N} |(0(t), e_n)|^2 + <\xi, \sum_{n=1}^{N} (0(t), e_n) e_n >^2.
\]

Therefore \( S(\phi_n(x) - \sum_{n=1}^{N} |(0(t), e_n)|^2)(\xi) = <\xi, \sum_{n=1}^{N} (0(t), e_n) e_n >^2 \). Hence

\[
S(\phi_n(x) - \sum_{n=1}^{N} |(0(t), e_n)|^2)(\xi) \text{ has the representation }
\]

\[
\sum_{n=1}^{N} (0(t), e_n) \otimes \sum_{n=1}^{N} (0(t), e_n) e_n \text{ in } H \otimes H.
\]

Now we know that

\[
\sum_{n=1}^{N} (0(t), e_n) e_n \otimes \sum_{n=1}^{N} (0(t), e_n) e_n - (0(t) \otimes 0(t)) \rightarrow 0 \text{ as } N \rightarrow \infty.
\]

Therefore we can find an element \( \phi(x) \) in \( K_2^{(-)} \) such that \( U(\phi(x))(\xi) \) has the representation \( 0(t) \otimes 0(t) \) in \( F \otimes F \). Let's denote \( \phi(x) \) as \( <x, 0(t) >^2 \) and we call this the renormalization of \( <x, 0(t) >^2 \). We can easily see that \( \phi(x) \equiv <x, 0(t) >^2 \) is the \( (L^2) \)-limit of
\( \phi_N(x) = \sum_{n=1}^{N} |(\Theta(t), e_n)|^2 \) i.e.,

\[
\phi(x) \equiv :<x, \Theta(t) >^2 = (L^2)^{-} \lim_{N \to \infty} \phi_N(x) = \sum_{n=1}^{N} |(\Theta(t), e_n)|^2
\]

\[
= :<x, \Theta(t) >^2 - \sum_{n=1}^{\infty} |(\Theta(t), e_n)|^2
\]

\[
\equiv :<x, \Theta(t) >^2 - \frac{1}{D(t)}.
\]

Therefore, \( U :<x, \Theta(t) >^2 \) \& \( \xi \) \& \( :<x, \Theta(t) >^2 \in K_2^{(-)} \) since \( \Theta(t) \in \Theta(t) \in \Phi \in \Phi \). From this example motivates why we choose \( K_2^{(-)} = \mathbb{R}^e \), is the nonstandard real number system.

3. The Random variable \( :<x, \Theta(t) >^n \):

Let \( H_n(x; \sigma^2) \) be Hermite polynomials with parameter i.e.,

\[
H_n(x; \sigma^2) = \left( \frac{-\sigma^2}{n!} \right)^n e^{x^2/2\sigma^2} \frac{d^n}{dx^n} e^{-x^2/2\sigma^2}, \sigma > 0
\]

Then the generating function is

\[
= \sum_{n=0}^{\infty} \frac{y^n}{n!} H_n(x; \sigma^2) = e^{-\sigma^2 y^2/2 + \gamma x}, \gamma \in \mathbb{C}.
\]

Now let \( \phi_N(x) = <x, \sum_{j=1}^{N} <\Theta(t), e_j | e_j > >\).

Let \( \sum_{n=1}^{N} |<\Theta(t), e_j >|^2 \) be denoted by \( \rho_N(t) \). Then \( \rho_N(t) \to \frac{1}{D(t)} \) as \( N \to \infty \), where \( \frac{1}{D(t)} \equiv :<\Theta(t), e_j >^2 \). Then

\[
= \sum_{n=0}^{\infty} \frac{y^n}{n!} H_n(\phi_N(x); \rho_N(t)) = \exp \left[ -\frac{\rho_N(t) y^2}{2} + \gamma \phi_N(x) \right].
\]

Now let us compute the S-transform
\[
\left[- \frac{\rho_N(t)\gamma^2}{2} + \gamma \phi_N(x) \right]((\xi)) = \int_B e \left[- \frac{\rho_N(t)\gamma^2}{2} + \gamma \phi_N(x+\xi) \right] d\mu(x)
\]

\[
= -\frac{\rho_N(t)\gamma^2}{2} e^{-\frac{\gamma}{\xi}} \sum_{j=1}^N \langle \theta(t), e_j \rangle e_j \gamma x, \sum_{j=1}^N \langle \theta(t), e_j \rangle e_j \gamma^2 \rho_N(t)
\]

\[
= e^{-\frac{\gamma}{\xi}} e^{-\frac{\gamma}{\xi}} \sum_{j=1}^N \langle \theta(t), e_j \rangle e_j \gamma^2 \rho_N(t)
\]

Therefore
\[
\left[- \frac{\rho_N(t)\gamma^2}{2} + \gamma \phi_N(x) \right]((\xi)) = e^{\gamma \phi_N(\xi)}. \tag{a}
\]

Now
\[
S\left( \sum_{n=0}^\infty \frac{\gamma^n}{n!} H_n(\phi_N(x); \rho_N(t)) \right)(\xi)
\]

(by Lebesgue Dominated Convergence Th.)

\[
= \sum_{n=0}^\infty \frac{\gamma^n}{n!} S(H_n(\phi_N(x); \rho_N(t)))(\xi)
\]

\[= e^{\gamma \phi_N(\xi)} \tag{by (a)}
\]

\[= \sum_{n=0}^\infty \frac{\gamma^n}{n!} (\phi_N(\xi))^n.
\]

Therefore by the uniqueness of the Taylor series,
\[
S(H_n(\phi_N(x); \rho_N(t)))(\xi) = \frac{1}{n!} (\phi_N(\xi))^n = \frac{1}{n!} \langle \theta(t), \theta(t) \rangle \sum_{j=1}^N \langle \theta(t), e_j \rangle e_j \]

Now note that
\[
\sum_{j=1}^N \langle \theta(t), e_j \rangle e_j \xrightarrow{n \to \infty} \theta(t) \oplus \cdots \oplus \theta(t)
\]

in
\[
\mathcal{F} \oplus \cdots \oplus \mathcal{F} \text{ as } N \to \infty.
\]
It is reasonable to denote the \((L^2)^-\) limit of \(H_n(\phi_n(x); \rho_n(t))\) by \(H_n(\langle x, \Theta(t)\rangle; \frac{1}{D(t)})\). Let us denote \(n! H_n(\langle x, \Theta(t)\rangle; \frac{1}{D(t)}) \equiv \langle x, \Theta(t) \rangle^n\). Then we can see

\[
U(\langle x, \Theta(t) \rangle^n): (\xi) = \langle \xi, \Theta(t) \rangle^n
\]

and

\[
\langle x, \Theta(t) \rangle^n = n! H_n(\langle x, \Theta(t)\rangle; \frac{1}{D(t)}) \in K_n^(-).
\]

4. **The random variable \(\exp(\langle x, \Theta(t) \rangle)\):**

Let \(\phi_N(x)\) be defined by

\[
\phi_N(x) = e^{\langle x, \frac{1}{N} \langle \Theta(t), e_j^j-e_j \rangle \rangle} \in (L^2).
\]

Then

\[
S(\phi_N(x))(\xi) = \int_{B^e} e^{\langle x+\xi, \frac{1}{N} \langle \Theta(t), e_j^j-e_j \rangle \rangle} d\mu(x)
\]

\[
= e^{\langle \xi, \frac{1}{N} \langle \Theta(t), e_j^j-e_j \rangle \rangle} \int_{B^e} e^{\langle x, \frac{1}{N} \langle \Theta(t), e_j^j-e_j \rangle \rangle} d\mu(x)
\]

\[
= e^{\langle \xi, \frac{1}{N} \langle \Theta(t), e_j^j-e_j \rangle \rangle} \frac{1}{2^{N-1}} \sum_{j=1}^{N} \frac{1}{2} \langle \Theta(t), e_j^j-e_j \rangle^2
\]

\[
= e^{\langle \xi, \frac{1}{N} \langle \Theta(t), e_j^j-e_j \rangle \rangle} \frac{1}{2} \sum_{j=1}^{N} |\langle \Theta(t), e_j^j-e_j \rangle|^2
\]

Now we see easily that

\[
\sum_{j=1}^{N} \langle \Theta(t), e_j^j-e_j \rangle \rightarrow \Theta(t) \quad \text{as} \quad N \rightarrow \infty
\]

in \(\mathbb{F}\). Therefore

\[
\sum_{n=0}^{N} \langle \xi, \frac{1}{N} \langle \Theta(t), e_j^j-e_j \rangle \rangle \rightarrow \langle \xi, \Theta(t) \rangle
\]

in \((L^2)^-\). Let us denote the \((L^2)^-\)-limit of
by $e^{\langle x, \theta(t) \rangle}$. Then $U(e^{\langle x, \theta(t) \rangle}) = e^{\xi, \theta(t)}$. It follows that 

$$e^{\langle x, \theta(t) \rangle} = e^{-\frac{1}{2} D(t)} + \langle x, \theta(t) \rangle.$$

5. **The random variable** $e^{\sum_{n=1}^{\infty} \langle x, e_n \rangle^2}$: \(c > \frac{1}{2}\)

Remarks: 1) This is related to the Gaussian Brownian functional $e^{\sum_{n=1}^{\infty} \langle x, e_n \rangle^2}$: in Hida Calculus.

Proof of Remark 2):

$$\exp[c \int_{\mathbb{R}} \frac{1}{2} \sum_{j=1}^{N} <x, e_j>^2 <0(u), e_j>^2 du] = \exp[c \int_{\mathbb{R}} \frac{1}{2} \sum_{j=1}^{N} <x, e_j>^2 <0(u), e_j>^2 du] + \int_{\mathbb{R}} \frac{1}{2} \sum_{j \neq k} <x, e_j> <0(u), e_j> <x, e_k> <0(u), e_k> du]$$

$$= \exp[c \int_{\mathbb{R}} \frac{1}{2} \sum_{j=1}^{N} <x, e_j>^2 <0(u), e_j>^2 du] + \exp[c \int_{\mathbb{R}} \frac{1}{2} \sum_{j \neq k} <x, e_j> <0(u), e_j> <x, e_k> <0(u), e_k> du]$$

$$= \exp[c \int_{\mathbb{R}} \frac{1}{2} \sum_{j=1}^{N} <x, e_j>^2].$$

Note: By Remark 2), we can formally write

$$\exp[c \sum_{j=1}^{\infty} <x, e_j>^2]: as: \exp[c \int_{\mathbb{R}} <x, \theta(u)>^2 du].$$

So we have a formal relation between this random variable and the coordinate system $\langle x, \theta(t) \rangle_{t \in \mathbb{R}}$. 
**Terminology:** We will call this random variable \( \exp \left[ c \sum_{j=1}^{\infty} \langle x, e_j \rangle^2 / j! \right] \) a generalized Gaussian random variable.

Now let us find the U transform of \( \exp \left[ c \sum_{j=1}^{\infty} \langle x, e_j \rangle^2 / j! \right] \).

Let \( \phi_N(x) = \exp \left[ c \sum_{j=1}^{N} \langle x, e_j \rangle^2 \right] \). Obviously \( \phi_N(x) \in L^2 \).

\[
(S\phi_N(x))(\xi) = \int_{B} e^{j=1} \langle x, e_j \rangle^2 d\mu(x)
\]

\[
= \int_{B} e^{j=1} \left[ \langle x, e_j \rangle^2 + 2\langle x, e_j \rangle \langle \xi, e_j \rangle + \langle \xi, e_j \rangle^2 \right] d\mu(x)
\]

\[
= e^{j=1} \int_{B} \left[ \langle x, e_j \rangle^2 + 2\langle x, e_j \rangle \langle \xi, e_j \rangle \right] d\mu(x)
\]

Now let \( \langle x, e_j \rangle = u \). Then \( \{\langle x, e_j \rangle\}_{j=1}^{\infty} \) is a Gaussian random variable,

\[
= e^{j=1} \int_{B} \frac{1}{\sqrt{2\pi}} e^{\frac{u^2}{2}} \left[ \frac{1}{\sqrt{2\pi}} \int_{B} e^{j=1} \frac{1}{\sqrt{2\pi}} e^{- \frac{1}{2} [(1-2c)u^2 - 4c \langle x, e_j \rangle u]} \right. \\
\overline{\left. - \frac{1}{2} [(1-2c)u^2 - 4c \langle x, e_j \rangle u] \right]} du \]

\[
= e^{j=1} \int_{B} \frac{1}{\sqrt{1-2c}} e^{\frac{2c^2 \langle x, e_j \rangle^2}{1-2c}} \left( c + \frac{1}{2} \right)
\]

\[
= \frac{1}{\sqrt{1-2c}} e^{j=1} \left( c \langle x, e_j \rangle^2 + \frac{2c^2}{1-2c} \langle x, e_j \rangle^2 \right)
\]

\[
= \frac{1}{\sqrt{1-2c}} e^{j=1} \left( c \langle x, e_j \rangle^2 + \frac{2c^2}{1-2c} \langle x, e_j \rangle^2 \right)
\]
Therefore

\[
\left( \frac{c}{1-2c} \right)^N \sum_{j=1}^{N} \left( \frac{c}{1-2c} \right)^2 \quad \text{for } c \neq \frac{1}{2}.
\]

Therefore

\[
S(\sqrt[1-2c]{c})^N \sum_{j=1}^{N} \left( \frac{c}{1-2c} \right)^2 \quad \text{for } c \neq \frac{1}{2}.
\]

Now we have

\[
\frac{c}{1-2c} \sum_{j=1}^{N} \left( \frac{c}{1-2c} \right)^2 \rightarrow e^N \quad \text{as } N \rightarrow \infty \quad \text{in } (L^2).\]

In \((L^2)\), where \(c \neq \frac{1}{2}\). Therefore let us denote \((L^2)\)-limit of

\[
\sum_{j=1}^{\infty} \left( \frac{c}{1-2c} \right)^2 \quad \text{as } e^j \quad \text{for } c \neq \frac{1}{2}.
\]

i.e.

\[
\sum_{j=1}^{\infty} \left( \frac{c}{1-2c} \right)^2 \quad \text{as } e^j \quad \text{for } c \neq \frac{1}{2}.
\]

Here we have another nonstandard real number \((\sqrt[1-2c]{c})\). Let us denote

\[
\sum_{j=1}^{\infty} \left( \frac{c}{1-2c} \right)^2 \quad \text{as } v(c) \quad \text{in the nonstandard real number system.}
\]

Then we have the relation

\[
\sum_{j=1}^{\infty} \left( \frac{c}{1-2c} \right)^2 \quad \text{as } v(c) e^j \quad \text{for } c \neq \frac{1}{2}.
\]
CHAPTER 3. Operators acting on \((L^2)^-\)

§1. \(\Theta(t)\)-differentiation and its adjoint operator.

It is time to discuss differentiation not only for ordinary random variables but also for generalized random variables. It will be defined through U-transform.

Definition 3.1

Suppose \((U\phi)'(\xi)\) has first functional derivative \((U\phi)'(\xi)\) at \(\xi\), with \((U\phi)'(\xi) \in F^*,\) i.e.,

\[
|U(\xi + n) - U(\xi) - ((U\phi)'(\xi), n)| = o(n1_F) \quad \text{and} \quad (U\phi)'(\xi) \quad \text{is a U-transform of a generalized random variable. We define} \quad \Theta(\xi)\phi \quad \text{to be the generalized random variable with U-transform}
\]

\[
U(\Theta(\xi)\phi)(\xi) = \langle(U\phi)'(\xi), \Theta(t)\rangle.
\]

Observation: Since \((U\phi)'(\xi)\) is a continuous linear functional, \((U\phi)'(\xi) \in F^*\) and \(\langle(U\phi)'(\xi), \Theta(t)\rangle\) makes sense.

Theorem 3.1

Suppose \(\phi(x) \in (L^2)\) has F-Fréchet derivative, i.e.,

\[
|\phi(x + y) - \phi(x) - (\phi'(x), y)| = o(||y||_F), \quad y \in F.
\]

Then \(\Theta(\xi)\phi(x) = \langle\phi'(x), \Theta(t)\rangle\)
Proof: We need to show that

\[ U(<\phi'(x),\Theta(t))>(\xi) = <(U\phi')(\xi),\Theta(t)> \]

thus

\[ U(<\phi'(x),\Theta(t))>(\xi) = \int_B <\phi'(x + \xi),\Theta(t)> du(x) \]

(by definition of \( U \) transform)

\[ = \int_B \phi'(x + \xi)du(x),\Theta(t)>. \]

\[ = \frac{\delta}{\delta\xi} [\int_B \phi(x + \xi)du(x)],\Theta(t)> \]

\[ = <(U\phi')(\xi),\Theta(t)> . \quad (\text{Q.E.D.}) \]

Remarks: 1) If \( U\phi'(\xi) \) exists, then

\[ <U\phi'(\xi),n> = \lim_{\lambda \to 0} \frac{1}{\lambda} \{ U(\xi + \lambda n) - U(\xi) \} \quad \text{where } \xi,n \in B^* \]

(variation in the Gateaux sense).

2) In Hida calculus, \( U(\partial_{\Theta(t)}\phi(x))(\xi) = U'_\xi(\xi,t) \), the functional derivative of \( U \).

Example 1. \( \phi(x) = <x,\xi_0> \) with \( \xi_0 \in B^* \). Then \( (U\phi)(\xi) = <\xi,\xi_0> \), so

\[ \lim_{\lambda \to 0} \frac{1}{\lambda} \{ U\phi(\xi + \lambda n) - U\phi(\xi) \} = <n,\xi_0>. \]

Therefore, \( <(U\phi)'(\xi),n> = <\xi_0,n> \)

and thus \( U<(\partial_{\Theta(t)}\phi(x))>(\xi) = <\xi_0,\Theta(t)> \).

Example 2. Suppose \( \phi(x) = :<x,\xi_1><x,\xi_2>: \) where \( \xi_1,\xi_2 \in B^* \).

Then \( (U\phi)(\xi) = <\xi,\xi_1><\xi,\xi_2>, \)

and

\[ (U\phi)'(\xi;\cdot) = <\xi_1,\cdot><\xi_2,\xi> + <\xi_2,\cdot><\xi_1,\xi> \]

i.e.,
\[
\langle (U\phi(x))'(\xi), \Theta(t) \rangle = U(\partial_{\Theta(t)}\phi)(\xi)
\]
\[
= \langle \xi_1, \Theta(t) \rangle \langle \xi_2, \xi \rangle + \langle \xi_2, \Theta(t) \rangle \langle \xi_1, \xi \rangle.
\]

**Remark:** Suppose \( \phi(x) = \langle x, \Theta(u) \rangle \). Then \( (U\phi)(\xi) = \langle \xi, \Theta(u) \rangle \). Therefore, \( \langle (U\phi)'(\xi), \eta \rangle = \langle \eta, \Theta(u) \rangle \), and \( U(\partial_{\Theta(t)}\phi)(\xi) = \langle \Theta(t), \Theta(u) \rangle \).

Similarly if \( \phi(x) = \langle x, \Theta(t) \rangle \), then
\[
U(\partial_{\Theta(t)}\phi)(\xi) = \langle \Theta(t), \Theta(t) \rangle \equiv \frac{1}{D(t)}
\]
is a nonstandard real number, i.e.,
\[
\partial_{\Theta(t)}(\langle x, \Theta(u) \rangle) = \langle \Theta(t), \Theta(u) \rangle \quad \text{when} \ t \neq u
\]
\[
= \frac{1}{D(t)} \quad \text{when} \ t = u.
\]

This is the infinite dimensional analog of \( \partial_{ij} \delta_{ij} = \delta_{ij} \) in the finite dimensional case. In the Hida calculus, i.e. \( \Theta(u) = \delta^{(*)}_{u} \),
\[
\partial_{\Theta(t)}(\delta^{(*)}_{u}) = \delta^{(*)}_{t(u)} \quad \text{if} \ t \neq u
\]
\[
= \delta^{(*)}_{t} \frac{1}{dt} \quad \text{if} \ t = u.
\]

**Example 3.** Let \( \phi(x) = H_n\left(\frac{\langle x, e_1 \rangle}{\sqrt{2}}\right) \) with \( e_1 \in \mathbb{B}^* \), \( H_n \) be the Hermite polynomial as before. Then \( (U\phi)(\xi) = (\sqrt{2})^n(\xi, e_1)^n \). Then
\[
U(\partial_{\Theta(t)}\phi)(\xi) = (\sqrt{2})^n \langle \xi, e_1 \rangle^{n-1} \langle e_1, \Theta(t) \rangle,
\]
and we have
\[
U(\partial_{\Theta(t)}\phi)(x) = n \langle e_1, \Theta(t) \rangle H_{n-1}\left(\frac{\langle x, e_1 \rangle}{\sqrt{2}}\right).
\]

**Theorem 3.2.** If \( \phi(x) \in K_n \), then \( \partial_{\Theta(t)}\phi \in K_{n-1} \).

**Proof:** Suppose \( \psi \) is a random variable of degree less than or equal to \( n - 2 \), then
\[ \int_B \psi(x) \partial_{\Theta(t)} \phi(x) d\mu(x) = \int_B \psi(x) \langle \phi'(x), \Theta(t) \rangle d\mu(x) \]

(by integration by parts formula [10])

\[ = \int_B \phi(x) \{ \psi(x) \langle \Theta(t), x \rangle - \langle \Theta(t), \psi(x) \rangle \} d\mu(x) \]

\[ = 0 \]

since \( \phi \in \mathbb{K}_n \) and the quantity inside \{ \} is a random variable whose degree is less than or equal to \( n - 1 \). Therefore \( \partial_{\Theta(t)} \phi \) is orthogonal to \( \mathbb{K}_0, \ldots, \mathbb{K}_{n-2} \). Similar computation as Example 3 leads us to conclude \( \partial_{\Theta(t)} \phi \in \mathbb{K}_{n-1} \).

(Q.E.D.)

**Definition 3.2.** Define the adjoint \( \partial_{\Theta(t)}^{*} \) of \( \partial_{\Theta(t)} \) by

\[ \langle \partial_{\Theta(t)}^{*} \phi, \psi \rangle = \langle \phi, \partial_{\Theta(t)} \psi \rangle \quad \text{where} \quad \phi \in (L^2)^-, \psi \in (L^2)^+. \]

**Corollary 3.1**

\[ U(\partial_{\Theta(t)}^{*} \phi)(\xi) = \langle \xi, \Theta(t) \rangle (U\phi)(\xi). \]

**Proof:**

\[ U(\partial_{\Theta(t)}^{*} \phi)(\xi) = \int_B \partial_{\Theta(t)}^{*} \phi(x + \xi) d\mu(x) \]

\[ = \int_B \partial_{\Theta(t)} \phi(y) d\mu(y - \xi) \]

(by the translation formula [11])

\[ = \int_B \partial_{\Theta(t)} \phi(y) \left( -\frac{1}{2} \| \xi \|^2 + \langle y, \xi \rangle \right) d\mu(y) \]

\[ = \int_B \phi(y) \partial_{\Theta(t)} \left[ e^{-\frac{1}{2} \| \xi \|^2 + \langle y, \xi \rangle} \right] d\mu(y) \]

\[ = \int_B \phi(y) \cdot e^{-\frac{1}{2} \| \xi \|^2 + \langle y, \xi \rangle} \langle \xi, \Theta(t) \rangle d\mu(y) \]
\[ <\xi, 0(t)> \Phi(y) = \frac{1}{2} \| \xi \|^2 + <y, \xi> \]
\[ = <\xi, 0(t)> \int_{B} \Phi(y) e^{-\frac{1}{2} \| y - \xi \|^2} \, d\mu(y) \]
\[ = <\xi, 0(t)> \int_{B} \Phi(y) d\mu(y - \xi) \]
\[ = <\xi, 0(t)> \int_{B} \Phi(x + \xi) d\mu(x) \]
\[ = <\xi, 0(t)>(\Phi)(\xi). \quad (Q.E.D.) \]

**Definition 3.3 (Multiplication by \( \cdot, 0(t) \))**

Suppose \( \phi \in (L^2)^- \). Define,

\[ <\cdot, 0(t)> \cdot \phi = a_0(t) \phi + a^*_0(t) \phi. \]

**Observation:** Let \( \phi \in (L^2)^- \), \( \psi \in (L^2)^+ \).

\[ <a^*_0(t) \phi, \psi> = <\phi, a_0(t) \psi>, \]

i.e.,

\[ \int_{B} (a_0^* \phi)(x) \psi(x) d\mu(x) = \int_{B} \phi(x) (a_0 \psi)(x) d\mu(x) \]
\[ = \int_{B} \phi(x) \psi'(x), 0(t) d\mu(x) \quad \text{(by Theorem 3.1)} \]
\[ = \int_{B} [\phi(x) \psi(x) <x, 0(t)> - \psi(x) \psi'(x), 0(t)] d\mu(x) \]
\[ = \int_{B} \psi(x) [- (a_0 \phi)(x) + <x, 0(t) \psi(x)] d\mu(x). \]

Therefore \( (a_0^* \phi)(x) = -(a_0 \phi)(x) + <x, 0(t) \psi(x) \) i.e.,

\[ a_0^* \phi + a_0 \phi = \cdot, 0(t) \phi. \]

**Theorem 3.3.**

(1) If \( \phi(x) \in K_n^{(+)} \) is represented by

\[ k_1^* \otimes \ldots \otimes k_n^* \in \otimes \mathbb{F}_n \]

then \( a_0 \phi \) is represented by
\[ \sum_{j=1}^{n} (k_j, \theta(t)) k_j^* \otimes \cdots \otimes k_n^* \]

where

\[ k_1^* \otimes \cdots \otimes k_n^* = k_1^* \otimes k_2^* \otimes \cdots \otimes k_{j-1}^* \otimes k_{j+1}^* \otimes \cdots \otimes k_n^* \]

(2) If \( \phi(x) \in K_n^* \) is represented by \( k_1 \otimes \cdots \otimes k_n \in K_n \) then \( \partial_{\theta(t)}^* \phi \) is represented by \( k_1 \otimes \cdots \otimes k_n \otimes \theta(t) \).

Proof: (1) Since \( \phi \in K_n^* \) is represented by \( k_1^* \otimes \cdots \otimes k_n^* \),

\[ (U\phi)(\xi) = (k_1^*, \xi)(k_2^*, \xi) \cdots (k_n^*, \xi) \]

Then

\[ \lim_{\lambda \to 0} \frac{1}{\lambda} \{ (U\phi)(\xi + \lambda n) - (U\phi)(\xi) \} \]

\[ = (k_1^*, n)(k_2^*, \xi) \cdots (k_n^*, \xi) + \cdots + (k_1^*, \xi) \cdots (k_{n-1}^*, \xi)(k_n^*, n) \]

Therefore

\[ \langle (U\phi)'(\xi), \theta(t) \rangle = \sum_{j=1}^{n} (k_j^*, \theta(t)) \langle k_1^*, \xi \rangle \cdots \langle k_{j-1}^*, \xi \rangle \langle k_{j+1}^*, \xi \rangle \cdots \langle k_n^*, \xi \rangle. \]

Therefore \( \partial_{\theta(t)}^* \phi \) is represented by \( \sum_{j=1}^{n} (k_j^*, \theta(t)) k_j^* \otimes \cdots \otimes k_n^* \).

(2) Since \( \phi(x) \in K_n^* \) is represented by \( k_1 \otimes \cdots \otimes k_n \),

\[ (U\phi)(\xi) = (k_1, \xi)(k_2, \xi) \cdots (k_n, \xi) \]

where \( k_i \in F \) for \( i = 1, \ldots, n \). Now let \( \psi \in K_{n+1}^* \) be represented by \( k_1^* \otimes \cdots \otimes k_{n+1}^* \) where \( k_i^* \in F^* \) for \( i = 1, \ldots, n + 1 \). Then by (1)

\[ \partial_{\theta(t)} \psi \) is represented by \[ \sum_{j=1}^{n+1} \langle k_j^*, \theta(t) \rangle \langle k_1^* \otimes \cdots \otimes k_{n+1}^* \rangle. \]

Now
\[<\partial_\theta(t)\phi,\psi> = <\phi,\partial_\theta(t)\psi>\]
\[= n! <k_1 \otimes \ldots \otimes k_n, \bigoplus_{j=1}^{n+1} <k_j^*, \theta(t)>(k_1^* \otimes \ldots \otimes k_{n+1}^*)>\]
\[= n! \sum_{j=1}^{n+1} <k_j^*, \theta(t)> <k_1 \otimes \ldots \otimes k_n, k_1^* \otimes \ldots \otimes k_{n+1}^*>\]
\[= n! \sum_{j=1}^{n} <k_j^*, \theta(t)> \frac{1}{n!} \sum_{\sigma} k_{\sigma(1)} \otimes \ldots \otimes k_{\sigma(n)}, k_1^* \otimes \ldots \otimes k_{n+1}^*>\]

where the second summation \(\sigma\) is over all permutations of \((1,2,\ldots,n)\)

\[= \sum_{j=1}^{n} <k_j^*, \theta(t)> \left[\sum_{\sigma} k_{\sigma(1)} \otimes \ldots \otimes k_{\sigma(n)}, k_1^* \otimes \ldots \otimes k_{n+1}^*>\right]\]
\[= \sum_{\sigma} k_{\sigma(1)} \otimes \ldots \otimes k_{\sigma(j-1)} \otimes \theta(t) \otimes k_{\sigma(j+1)} \otimes \ldots \otimes k_{\sigma(n)}, k_1^* \otimes \ldots \otimes k_{n+1}^*>\]
\[= (n+1)! <k_1 \otimes \ldots \otimes k_n \otimes \theta(t), k_1^* \otimes \ldots \otimes k_{n+1}^*>.\]

Therefore, \(\partial_\theta(t)\phi\) is represented by \(k_1 \otimes \ldots \otimes k_n \otimes \theta(t)\).

**Theorem 3.4**

1) \([\partial_\theta(t), \partial_\theta(s)] = [\partial_\theta(s), \partial_\theta(t)] = 0\]

2) \([\partial_\theta(t), \partial_\theta(s)] = <\theta(s), \theta(t)>I\]

where \([a,b] = ab - ba\) and \(I\) is the identity.

Proof: 1) Let \(\phi(x) \in K_n^{(+)}\) and \(k_1^* \otimes \ldots \otimes k_n^*\) be the representation for \(\phi(x)\) where \(k_i^* \in F^*\) for \(i = 1,\ldots,n\). Then

\((U\phi)(\xi) = (k_1^*, \xi) \ldots (k_n^*, \xi)\). By Theorem 5.3

\[U(\partial_\theta(t)\phi)(\xi) = \bigoplus_{j=1}^{n} (k_j^*, \theta(t))(k_1^*, \xi) \ldots (k_{j-1}^*, \xi)(k_{j+1}^*, \xi) \ldots (k_n^*, \xi).\]

Now
\[ \lim_{\lambda \to 0} \frac{1}{\lambda} [u(\partial_O(t) \phi)(\xi + \lambda \eta) - u(\partial_O(t) \phi)(\xi)] \]

\[ = (k_1, O(t))[\langle k_2, n \rangle \langle k_3, \xi \rangle \ldots \langle k_n, \xi \rangle + \ldots + \langle k_2, \xi \rangle \ldots \langle k_{n-1}, \xi \rangle \langle k_n, n \rangle] \]

\[ + \ldots + (k_n, O(t))[\langle k_1, n \rangle \langle k_2, \xi \rangle \ldots \langle k_{n-1}, \xi \rangle] \]

\[ + \ldots + (k_1, \xi) \ldots \langle k_{n-2}, \xi \rangle \langle k_{n-1}, n \rangle]. \]

Therefore,

\[ u(\partial_O(s) \partial_O(t) \phi)(\xi) = \sum_{i=1}^{n} \langle k_i, O(t) \rangle \sum_{j=1}^{n} \langle k_j, O(s) \rangle \langle k_1, \xi \rangle \ldots \langle k_n, \xi \rangle \]

where

\[ (k_1, \xi) \ldots \langle k_n, \xi \rangle \equiv (k_1, \xi) \ldots \langle k_{i-1}, \xi \rangle \]

\[ (i, j) \]

\[ \ldots \langle k_{i+1}, \xi \rangle \ldots \langle k_{j-1}, \xi \rangle \langle k_{j+1}, \xi \rangle \ldots \langle k_n, \xi \rangle. \]

Similarly,

\[ u(\partial_O(t) \partial_O(s) \phi)(\xi) = \sum_{i=1}^{n} \langle k_i, O(s) \rangle \sum_{j=1}^{n} \langle k_j, O(t) \rangle \langle k_1, \xi \rangle \ldots \langle k_n, \xi \rangle \]

Therefore

\[ [\partial_O(t), \partial_O(s)] = (\partial_O(t) \partial_O(s) - \partial_O(s) \partial_O(t))(\phi) = 0, \]

since

\[ u(\partial_O(t) \partial_O(s) \phi)(\xi) = u(\partial_O(s) \partial_O(t) \phi)(\xi). \]

Now, let's show that \[ [\partial_O(s)^*, \partial_O(t)^*] = 0. \] It is enough to show this for \( \phi \in K_n^{(-)}. \) So let \( \phi \) be represented by \( k_1 \otimes \ldots \otimes k_n \otimes O(t) \) and \( \partial_O(s)^* \partial_O(t)^* \phi \) be represented by \( k_1 \otimes \ldots \otimes k_n \otimes O(t) \otimes O(s). \) Similarly \( \partial_O(t)^* \partial_O(s)^* \phi \) is represented by \( k_1 \otimes \ldots \otimes k_n \otimes O(s) \otimes O(t). \) Therefore

\[ [\partial_O(s)^*, \partial_O(t)^*] = \partial_O(s)^* \partial_O(t)^* - \partial_O(t)^* \partial_O(s)^* = 0. \]
(2) Let $\phi \in K_n^{(+)}$ be represented by $k_1^* \otimes \ldots \otimes k_n^* \otimes \Theta(s)$, i.e.,

$$U(\partial_{\Theta(s)} \phi)(\xi) = (k_1^*, \xi) \ldots (k_n^*, \xi)(\Theta(s), \xi)$$

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \left\{ U(\partial_{\Theta(s)} \phi)(\xi + \lambda \eta) - U(\partial_{\Theta(s)} \phi)(\xi) \right\}$$

$$= (k_1^*, \eta)(k_2^*, \xi) \ldots (\Theta(s), \xi) + \ldots + (k_n^*, \eta)(k_1^*, \xi) \ldots (k_{n-1}^*, \xi)(\Theta(s), \xi)$$

$$+ (k_1^*, \xi) \ldots (k_n^*, \xi)(\Theta(s), \Theta(t)).$$

Therefore,

$$U(\partial_{\Theta(t)} \partial_{\Theta(s)} \phi)(\xi) = (k_1^*, \Theta(t))(k_2^*, \xi) \ldots (k_n^*, \xi)(\Theta(s), \xi)$$

$$+ \ldots + (k_n^*, \Theta(t))(k_1^*, \xi) \ldots (k_{n-1}^*, \xi)(\Theta(s), \xi)$$

$$+ (k_1^*, \xi) \ldots (k_n^*, \xi)(\Theta(s), \Theta(t)).$$

Now $\partial_{\Theta(t)} \phi$ is represented by

$$\sum_{j=1}^{n} <k_j^*, \Theta(t)> k_1^* \otimes \ldots \otimes k_n^*.$$ 

Therefore $\partial_{\Theta(s)} \partial_{\Theta(t)} \phi$ is represented by

$$\sum_{j=1}^{n} <k_j^*, \Theta(t)> k_1^* \otimes \ldots \otimes \Theta(s),$$

i.e.,

$$U(\partial_{\Theta(s)} \partial_{\Theta(t)} \phi)(\xi) = (k_1^*, \Theta(t))(k_2^*, \xi) \ldots (\Theta(s), \xi)$$

$$+ \ldots + (k_n^*, \Theta(t))(k_1^*, \xi) \ldots (\Theta(s), \xi).$$

Therefore

$$U(\partial_{\Theta(t)} \partial_{\Theta(s)} \phi)(\xi) - U(\partial_{\Theta(s)} \partial_{\Theta(t)} \phi)(\xi) = (k_1^*, \xi) \ldots (k_n^*, \xi)(\Theta(s), \Theta(t)).$$
Thus
\[ \partial \theta(t) \partial \theta(s) = \partial \theta(t) \partial \theta(s) = (\theta(s), \theta(t))I. \] (Q.E.D.)

§2. Laplacian operators.

In the theory of abstract Wiener space, the Gross Laplacian \( \Delta_G \) [11] and the Beltrami Laplacian \( \Delta_B \) [16] are defined as follows:

\[ (\Delta_G f)(x) = \text{trace}_H f''(x) \]
\[ (\Delta_B g)(x) = \text{trace}_H g''(x) - \langle x, g'(x) \rangle \]

where the primes denote the H-Fréchet derivatives. \( f''(x) \) is assumed to be a trace class operator of \( H \). \( g''(x) \) is assumed for \( g \).

We denote \( \phi'(x), \phi''(x) \) as F-Fréchet derivative of \( \phi(x) \) in the previous section (Theorem 3.1.1) and the \( \Theta(t) \)-differentiation is related with F-directional derivatives. On the other hand, if \( \phi'(x) \) is the F-directional derivative of \( \phi(x) \), then \( \phi'(x) \in F^* \) and the restriction of \( \phi'(x) \) to \( H \) is the derivative of \( \phi(x) \) in H-direction \( \phi(x) \).

Since \( f''(x) \) is assumed to be a trace class operator of \( H \), \( \Delta_G f \) is defined pointwise. On the other hand, it has been shown in [16] that \( \text{trace}_H g''(x) - \langle x, g'(x) \rangle \) is the \( L^2(B) \)-limit of \( \text{trace}_H P g''(x) - (Px, g'(x)) \) as \( P \) converges strongly to the identity in the set of finite dimensional orthogonal projections of \( H \).

**Theorem 3.5**

Suppose \( \phi \) is a twice differentiable function in F-direction on \( B \) with \( \phi'(x) \in F^* \) and \( \phi''(x) \) a trace class operator of \( H \) for all \( x \in B \), then \( \phi \) is twice \( \Theta(t) \)-differentiable and
\[ \Delta_G \phi = \int_{\mathbb{R}} \partial^2_{\Theta(t)} \phi dt, \quad \Delta_B \phi = -\int_{\mathbb{R}} \partial^*_\Theta(t) \cdot \partial_{\Theta(t)} \phi dt, \]

whenever the integral exists.

**Proof:** We have \( \partial_{\Theta(t)} \phi(x) = \langle \phi'(x), \Theta(t) \rangle \) where \( \phi'(x) \in F^* \). Then

\[ \partial^2_{\Theta(t)} \phi(x) = \sum_{n=1}^{\infty} \langle \phi''(x), \xi_n \rangle \langle \Theta(t), \xi_n \rangle_F, \]

where \( \{\xi_n\}_{n=1}^{\infty} \) is an ONB for \( F \). Therefore

\[ \partial^2_{\Theta(t)} \phi(x) = \sum_{k,n=1}^{\infty} \langle \phi''(x) \xi_n, \xi_k \rangle \langle \Theta(t), \xi_n \rangle_F \langle \Theta(t), \xi_k \rangle_F \]

\[ = \sum_{k,n=1}^{\infty} \langle \phi''(x) \frac{1}{\lambda_n} e_n, \frac{1}{\lambda_k} e_k \rangle \langle \Theta(t), \frac{1}{\lambda_n} e_n \rangle_F \langle \Theta(t), \frac{1}{\lambda_k} e_k \rangle_F \]

\[ = \sum_{k,n=1}^{\infty} \langle \phi''(x) e_n, e_k \rangle \langle \Theta(t), e_n \rangle_F \langle \Theta(t), e_k \rangle_F. \]

Then it follows that

\[ \int_{\mathbb{R}} \partial^2_{\Theta(t)} \phi dt = \sum_{n=1}^{\infty} \langle \phi''(x) e_n, e_n \rangle (\text{since } \int_{\mathbb{R}} \partial^2_{\Theta(t)} \phi(x) dt < \infty) = (\Delta_g \phi)(x). \]

We give a formal proof for the second part.

\( (\Delta_B \phi)(x) = (\Delta_G \phi(x) - \langle x, \phi'(x) \rangle \)

(Let us write \( \langle x, \phi'(x) \rangle \) as \( \int_{\mathbb{R}} \langle x, \Theta(t) \rangle \langle \phi'(x), \Theta(t) \rangle dt \) formally)

\[ = \Delta_G \phi(x) - \int_{\mathbb{R}} \langle x, \Theta(t) \rangle \langle \phi'(x), \Theta(t) \rangle dt \]

\[ = \int_{\mathbb{R}} \langle \partial^2_{\Theta(t)} \phi \rangle dt - \int_{\mathbb{R}} \langle \partial^*_\Theta(t) + \partial_{\Theta(t)} \phi \rangle \langle \phi'(x), \Theta(t) \rangle dt \]

\[ = \int_{\mathbb{R}} \langle \partial^2_{\Theta(t)} \phi \rangle dt - \int_{\mathbb{R}} \langle \partial^*_\Theta(t) + \partial_{\Theta(t)} \phi \rangle \partial_{\Theta(t)} \phi dt \]

\[ = -\int_{\mathbb{R}} \langle \partial^*_\Theta(t) \partial_{\Theta(t)} \phi \rangle dt. \]

(Q.E.D.)
Corollary 3.2

Suppose \( \phi(x) \in K_n \) is represented by \( h_1 \otimes \ldots \otimes h_n \), \( h_i \in H, i = 1, 2, \ldots, n \) and \( \int_{\mathbb{R}} \partial O(t)^* \partial O(t) \phi(x) dt \) exists, then \( \Delta_B \phi(x) = -n \phi(x) \).

Proof: By Theorem 3.5 we know that

\[ \Delta_B \phi(x) = -\int_{\mathbb{R}} \partial O(t)^* \partial O(t) \phi(x) dt. \]

So it is enough to show that

\[ U(\Delta_B \phi(x))(\xi) = U(-\int_{\mathbb{R}} \partial O(t)^* \partial O(t) \phi(x) dt)(\xi) = -n U(\phi(x))(\xi). \]

Now since \( \phi(x) \) is represented by \( h_1 \otimes \ldots \otimes h_n \),

\[ U(\phi(x))(\xi) = (h_1, \xi) \ldots (h_n, \xi). \]

Then by Theorem 3.3

\[ U(\partial O(t)^* \partial O(t) \phi(x))(\xi) = \sum_{j=1}^{n} \langle \partial O(t), \xi \rangle < h_j, \xi > \ldots < h_n, \xi > < h_j, \partial O(t) \rangle, \]  

Therefore

\[ U(\int_{\mathbb{R}} \partial O(t)^* \partial O(t) \phi(x) dt)(\xi) = \int_{\mathbb{R}} U(\partial O(t)^* \partial O(t) \phi(x))(\xi) dt \]

\[ = \sum_{j=1}^{n} \langle h_j, \xi \rangle \ldots < h_n, \xi > < h_j, \xi > \]

\[ = nU(\phi(x))(\xi). \] (Q.E.D.)

Remark: By Theorem 3.5, we can generalize the definition of Beltrami Laplacian \( \Delta_B \phi \equiv -\int_{\mathbb{R}} \partial O(t)^* \partial O(t) \phi dt \) for \( \phi \in (L^2)^- \) as long as \( \int_{\mathbb{R}} \partial O(t)^* \partial O(t) \phi dt \) exists. Then for \( \phi \in K_n^{(-)} \)

\[ \Delta_B \phi = -n \phi. \]
For the space \((L^2)^-\) of generalized random variables, we can define another two Laplacians as in Hida calculus [15].

**Definition 3.4**

Suppose \(U(\phi(t)^2)(\xi)\) is given by

\[
U(\phi(t)^2)(\xi) = U_1''(\xi,\phi(t))\phi(t)\phi(t) + U_2''(\xi,\phi(t))\phi(t)\phi(t),\xi,\phi(t)\phi(t),\xi
\]

and \(U_1''(\xi,\phi(t))\) and \(U_2''(\xi,\phi(u))\) are \(U\)-transforms of generalized random variables, then we define the Levy Laplacian \(\Delta_L\) and the Volterra Laplacian \(\Delta_V\) as follows:

\[
U(\Delta_L \phi)(\xi) = \int_{\mathbb{R}} U_1''(\xi,\phi(t)) dt
\]

\[
U(\Delta_V \phi)(\xi) = \int_{\mathbb{R}} U_2''(\xi,\phi(t))\phi(t)\phi(t),\xi,\phi(t)\phi(t),\xi dt.
\]

**Remark:** From the non-standard analysis point of view the above definition can be interpreted as follows:

\[
U(\Delta_L \phi)(\xi) = \int_{\mathbb{R}} U(\phi(t)^2)(\xi)(\phi(t) dt).
\]

(This is the analog of \(\Delta_L \phi = \int_{\mathbb{R}} \phi^2(dt)^2\) in Hida Calculus [15]).

**Definition 3.5**

Suppose \(\phi\) is a generalized random variable in \((L^2)^-\) and satisfies

\[
U(\phi)(\xi) = \int_{\mathbb{R}^k} f(u_1,\ldots,u_k)\phi(u_1)_{n_1} \ldots \phi(u_k)_{n_k} du_1 \ldots du_k
\]

where \(\sum_{j=1}^{n_j} n_j = n\), and \(f(u_1,\ldots,u_k)\) is such that

\[
\int_{\mathbb{R}^k} f(u_1,\ldots,u_k)\phi(u_1)_{n_1} \ldots \phi(u_2)_{n_1} \ldots \phi(u_k)_{n_1} du_1 \ldots du_k \in \mathbb{F}.
\]
Then we call $\phi$ a normal generalized random variable.

**Remark:** From the definition $\phi$ can be written as

$$\phi = \int_{\mathbb{R}^k} f(u_1, \ldots, u_k) : <\xi, \omega(u)>^1 \ldots <\xi, \omega(u)>^k : du_1 \ldots du_k.$$ 

**Example 1.**

Suppose $\phi = <\xi, h_1> \ldots <\xi, h_n>$ is an ordinary random variable in $\mathbb{K}_n$, i.e. $U(\phi)(\xi) = (\xi, h_1) \ldots (\xi, h_n)$. Then

$$U(\partial_0(t)\phi) = <\xi, \omega(t)> \sum_{j=1}^{n} <\xi, h_j> \ldots <\xi, h_n> <\xi, \omega(t), h_j>$$

$$U(\partial_0(t)\partial_0(t)\phi) = <\xi, \omega(t)> \sum_{j=1}^{n} <\xi, h_j> \ldots <\xi, h_n> <\xi, \omega(t), h_j>$$

Therefore

$$\Delta^g_\phi = \int_{\mathbb{R}} \partial^2_0(t)\phi dt = \sum_{i,j}^{n} <\xi, h_i> \ldots <\xi, h_n> <\xi, h_j>$$

$$\Delta^B_\phi = -n\phi$$

$$\Delta^L_\phi = 0$$

$$\Delta^V_\phi = \Delta^g_\phi.$$

**Example 2.**

Suppose $\phi(x) = \int_{\mathbb{K}_n} f(u) : <\xi, \omega(u)>^n : du$ is a normal generalized random variable in $\mathbb{K}_n$, i.e., $\int_{\mathbb{R}} f(u)\omega(u) \otimes \ldots \otimes \omega(u) du \in \mathbb{F}$. Then

$$U(\partial_0(t)\phi)(\xi) = n\int_{\mathbb{R}} f(u) <\xi, \omega(u)>^{n-1} <\xi, \omega(t), \omega(u)> du$$

$$U(\partial_0(t)\partial_0(t)\phi)(\xi) = n\int_{\mathbb{R}} f(u) <\xi, \omega(u)>^{n-1} <\xi, \omega(t), \omega(u)> \omega(t) du$$
\[ U(\partial_{\partial(t)}^2 \phi)(\xi) = n(n-1)\int f(u)\langle \xi, \phi(u) \rangle^{n-2} \langle \theta(t), \phi(u) \rangle \langle \theta(t), \phi(u) \rangle \, du. \]

Therefore,

\[ \Delta_G : \text{does not exist} \]

\[ \Delta_B = -n\phi \text{ since} \int_{B} \langle \xi, \theta(t) \rangle \langle \theta(t), \phi(u) \rangle \, dt = \langle \xi, \phi(u) \rangle \]

\[ \Delta_L = n(n-1)\int f(u)\langle *, \phi(u) \rangle^{n-2} \, du \]

\[ \Delta_V = 0. \]

**Example 3**

\[ \phi = \exp\left[ c \sum_{j=1}^{\infty} <*, e_j>^2 \right]: c \neq \frac{1}{2} \text{ is a generalized Gaussian random variable:} \]

\[ (U\phi)(\xi) = \exp\left[ -\frac{c}{1-2c} \| \xi \|^2 \right]. \]

Therefore

\[ (U\phi)'(\xi; \theta(t)) = \frac{2c}{1-2c} \langle \xi, \theta(t) \rangle \exp\left[ -\frac{c}{1-2c} (\xi, \xi) \right] = \frac{2c}{1-2c} U(\partial_{\partial(t)}^* \phi). \]

Thus

\[ U(\partial_{\partial(t)}^2 \phi) = (U''\phi)(\xi; \theta(t), \phi(t)) \]

\[ = \left( \frac{2c}{1-2c} \right)^2 \langle \xi, \theta(t) \rangle \langle \xi, \theta(t) \rangle \exp\left[ -\frac{c}{1-2c} (\xi, \xi) \right] \]

\[ + \left( \frac{2c}{1-2c} \right) \langle \theta(t), \phi(u) \rangle \exp\left[ -\frac{c}{1-2c} (\xi, \xi) \right] \]

\[ U(\partial_{\partial(t)}^* \partial_{\partial(t)}^* \phi)(\xi) = \frac{2c}{1-2c} \langle \xi, \theta(t) \rangle \langle \xi, \theta(t) \rangle \exp\left[ -\frac{c}{1-2c} (\xi, \xi) \right]. \]

Therefore,

\[ \Delta_G \text{ does not exist} \]

\[ \Delta_B \phi = \frac{2c}{2c-1} \sum_{j=1}^{\infty} <*, e_j>^2 \phi(x) \text{ where} \ U\left( \sum_{j=1}^{\infty} <*, e_j>^2 \right)(\xi) = (\xi, \xi)_{\text{H}} \]
\[ \Delta_L \phi = \frac{2c}{1-2c} \phi(x) \int_{\mathbb{R}^1} dt \]

\[ \Delta_V \phi = \left( \frac{2c}{1-2c} \right)^2 \sum_{j=1}^{\infty} <\cdot, \xi_j^2: \phi(x) = \left( \frac{2c}{2c-1} \right) \Delta_B \phi \]

where \( \int_{\mathbb{R}^1} dt \) is a non-standard real number. (Actually we avoid \( \int_{\mathbb{R}^1} dt \) by considering \( \phi = \exp c \int_T <\cdot, \Theta(t) \Theta(t)^* \geq dt \))

**Remark:** The generalized Gaussian random variable \( \phi \) in Example 3 is an eigenfunction of \( \Delta_L \).
CHAPTER 4. Fourier transform on \((L^2)^{-}\)

§1. Fourier transform.

A Fourier transform introduced by H.-H. Kuo is very useful for casual calculus. This Fourier transform carries \(\dot{b}(t)\)-differentiation into multiplication by \(ib(t)\). Such a Fourier transform can be defined in the space \((L^2)^{-}\) of generalized random variables. We will use \(x\) and \(y\) to denote variables in the domain and range spaces respectively.

**Renormalization of \(\exp[-i\langle x,y\rangle]\)**

If \(x \in B\) and \(y \in H\), then \(\langle x,y \rangle\) is an ordinary random variable and \(\exp[-i\langle x,y \rangle]\) is also an ordinary random variable. But if \(x \in B\), \(y \in B \setminus H\), then \(\langle x,y \rangle\) is not a random variable in the ordinary sense. So renormalization is required to define \(\exp[-i\langle x,y \rangle]:_{y}\). Let

\[
\phi_{N}(y) = \exp[-i \sum_{j=1}^{N} \langle x,e_{j}\rangle \langle y,e_{j}\rangle].
\]

Then obviously \(\phi_{N}(y) \in (L^2)^{+}\). Now

\[
S(\phi_{N}(y))(\xi) = \int_{B} \exp[-i \sum_{j=1}^{N} \langle y + \xi, e_{j}\rangle \langle x,e_{j}\rangle]d\mu(y)
\]

\[
= \int_{B} \exp[-i \sum_{j=1}^{N} (\langle x,e_{j}\rangle \langle y,e_{j}\rangle + \langle \xi,e_{j}\rangle \langle x,e_{j}\rangle)]d\mu(y)
\]

52
\[
\begin{align*}
\exp[-i \sum_{j=1}^{N} \langle \xi, e_j \rangle \langle x, e_j \rangle] & \to \exp[-i \langle x, \xi \rangle] \quad \text{as } N \to \infty.
\end{align*}
\]

**Note:** \(\exp[-i \langle x, \xi \rangle] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \langle x, \xi \rangle^k \) and

\[
\frac{(-1)^k}{k!} \mathcal{A} \cdots \mathcal{A} x \in \mathcal{A} F \text{ a.e. } u(x) \quad (\text{Here } x \in B). \text{ But we can assume } x \in F \text{ a.e. } (u) \quad (\text{since } \mathcal{H} \subset F \text{ is an abstract Wiener space}). \text{ Therefore,}
\]

\[
\frac{1}{2} \sum_{j=1}^{N} \langle x, e_j \rangle^2
\]

we can have \((L^2)^{-}\) limit of \(\exp[-i \langle x, y \rangle]_{y} \cdot \phi_N(y)\), denoted by

\[
U(\exp[-i \langle x, y \rangle]_{y})(\xi) = \exp[-i \langle x, \xi \rangle].
\]

We also see that

\[
\exp[-i \sum_{j=1}^{N} \langle x, e_j \rangle \langle y, e_j \rangle] = \exp[-i \sum_{j=1}^{\infty} \langle x, e_j \rangle \langle y, e_j \rangle] = \exp[-i \langle x, y \rangle].
\]

Formally, this can be written as
\[ \frac{1}{2i} \sum_{j=1}^{2} \frac{x_j^2}{H} \exp[-i<x,y>]. \]

Here it is obvious \( \frac{1}{2i} \sum_{j=1}^{2} \frac{x_j^2}{H} = e^{-i<x,y>} \) in the ordinary sense. But we can regard \( e \) as \( \sum_{j=1}^{2} \frac{x_j^2}{H} \) in the non-standard real number. Therefore,

\[ :\exp[-i<x,y>]; y \equiv e^{\frac{1}{2i} \sum_{j=1}^{2} \frac{x_j^2}{H} \exp[-i<x,y>]}. \]

**Definition 4.1**

The renormalization of \( \exp[-i<x,y>] \) with respect to the \( y \)-variable, denoted by \( :\exp[-i<x,y>]; y \), is the generalized random variable such that \( U(:\exp[-i<x,y>]; y)(\xi) = \exp[-i<\xi,x>] \).

**Note:** It is easy to see that \( y \rightarrow :\exp[-i<x,y>]; y \) is a measurable map from \( B \) into \( (L^2)_y \).

**Lemma 4.1.**

If \( \phi(y) \in K_{n,y}^{(+)} \) is represented by \( k_1 \otimes \cdots \otimes k_n \) with \( k_i \in F^* \) for \( i = 1,2,\ldots,n \), then

\[ :\exp[-i<x,y>]; y, \phi = (-1)^n <x,k_1^*> \cdots <x,k_n^*> \]

and

\[ :\exp[-i<x,y>]; y, \phi(y) > = K_{n,x}^{(+)} \cdot \]

**Proof:** Observe that

\[ U(:\exp[-i<x,y>]; y)(\xi) = \exp[-i<\xi,x>] = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} <\xi,x>_k \]

and
Therefore

\[ \langle :\exp[-i<x,y>] : y, \phi \rangle \]

\[ = n! \langle \frac{(-1)^n}{n!} x \otimes \ldots \otimes x, k^*_1 \otimes \ldots \otimes k^*_n \rangle \]

\[ = (-1)^n \langle x \otimes \ldots \otimes x, k^*_1 \otimes \ldots \otimes k^*_n \rangle \]

\[ = (-1)^n \langle x, k^*_1 \rangle \ldots \langle x, k^*_n \rangle. \]

Also \( k^*_1 \otimes \ldots \otimes k^*_n \in \mathcal{O}^* \) shows that \( \langle :\exp[-i<x,y>] : y, \phi \rangle \in K^{(+)\ r}_{n,x}. \)

(Q.E.D.)

**Lemma 4.2**

For any \( \phi(y) \in (L^2)^* \),

\[ \partial^{\alpha} \langle x, \Omega(t) \rangle \langle :\exp[-i<x,y>] : y, \phi \rangle = i\langle :\exp[-i<x,y>] : y, \langle y, \Omega(t) \rangle \phi \rangle. \]

Proof: It is sufficient to prove this for \( \phi(y) \in K^{(+)\ r}_{n,x}. \) Let us assume \( \phi \) to be represented by \( k^*_1 \otimes \ldots \otimes k^*_n \in \mathcal{O}^* \). Then by Lemma 4.1,

\[ \langle :\exp[-i<x,y>] : y, \phi(y) \rangle \in K^{(+)\ r}_{n,x} \]

and

\[ \langle :\exp[-i<x,y>] : y, \phi(y) \rangle = (-1)^n (x, k^*_1) \ldots (x, k^*_n) \]

i.e., \( \langle :\exp[-i<x,y>] : y, \phi(y) \rangle \) is represented by

\[ (-1)^n k^*_1 \otimes \ldots \otimes k^*_n. \]

Now by Theorem 3.3 (1), \( \partial^{\alpha} \langle x, \Omega(t) \rangle \langle :\exp[-i<x,y>] : y, \phi \rangle \) is represented by

\[ (-1)^n \sum_{j=1}^{n} (k^*_j, \Omega(t)) \circ (j) k^*_1 \otimes \ldots \otimes k^*_n \]
On the other hand,

\[ \langle y, O(t) \rangle \phi = \partial \langle y, O(t) \rangle \phi + \partial^* \langle y, O(t) \rangle \phi \]

\[ = \sum_{j=1}^{n} \langle k_j^*, O(t) \rangle \langle k_1^*, y \rangle \ldots \langle k_n^*, y \rangle + \langle k_1^*, y \rangle \ldots \langle k_n^*, y \rangle \langle O(t), y \rangle. \]

Therefore,

\[ \langle \exp[-i<x,y>] : y, \langle y, O(t) \rangle \phi \]

\[ \langle \exp[-i<x,y>] : y, \langle y, O(t) \rangle \phi + \partial \langle y, O(t) \rangle \phi \]

\[ = \langle \exp[-i<x,y>] : y, \partial \langle y, O(t) \rangle \phi \rangle + \langle \exp[-i<x,y>] : y, \partial^* \langle y, O(t) \rangle \phi \rangle \]

\[ = (n-1)! (-1)^{n-1} \cdot \frac{1}{(n-1)!} x \otimes \ldots \otimes x, \sum_{j=1}^{n} \langle k_j^*, O(t) \rangle k_1^* \otimes \ldots \otimes k_n^* \]

\[ + (n+1)! (-1)^{n+1} \cdot \frac{1}{(n+1)!} x \otimes \ldots \otimes x, k_1^* \otimes \ldots \otimes k_n^* \otimes O(t) \]

\[ = (-1)^{n-1} \sum_{j=1}^{n} \langle k_j^*, O(t) \rangle \langle k_1^*, x \rangle \ldots \langle k_{j-1}^*, x \rangle \langle k_{j+1}^*, x \rangle \ldots \langle k_n^*, x \rangle \]

\[ + (-1)^{n+1} \langle x, O(t) \rangle \langle x, k_1^* \rangle \ldots \langle x, k_n^* \rangle, \]

and so

\[ i \langle \exp[-i<x,y>] : y, \langle y, O(t) \rangle \phi \]

\[ = \partial \langle x, O(t) \rangle \langle \exp[-i<x,y>] : y, \phi \rangle + \langle x, O(t) \rangle \langle \exp[-i<x,y>] : y, \phi \rangle \]

\[ = \partial^* \langle x, O(t) \rangle \langle \exp[-i<x,y>] : y, \phi \rangle. \]

(Q.E.D.)

**Definition 4.2**

For \( \phi(x) \in (L^2_x)^{-} \), the Fourier transform \( \hat{\phi}(y) \) of \( \phi \) is defined by

\[ \hat{\phi}(y) = \int_B \exp[-i<x,y>] : y \phi(x) du(x). \]
Remark: The above definition can be interpreted as follows:

\[ \hat{\phi}(y) \in (L^2)^*_y \]
satisfies

\[ \langle \hat{\phi}(y), \psi(y) \rangle = \int_B \langle \exp[-i<x,y>] : y, \phi(x) \rangle \psi(y) \, d\mu(y) \]
\[ = \int_B \langle \exp[-i<x,y>] : y, \psi(y) \rangle \phi(x) \, d\mu(x) \]
\[ = \langle \phi, \langle \exp[-i<x,y>] : y, \psi \rangle \]

for \( \psi(y) \in (L^2)^+_y \).

(Since \( \langle \exp[-i<x,y>] : y, \psi \rangle \in (L^2)^+_x \) by Lemma 4.1, this is well-defined).

Theorem 4.1.

\[ (\partial_{x,0(t)} \phi(x))^\wedge(y) = i<y,0(t)> \hat{\phi}(y). \]

Remark: This theorem states that the Fourier transform carries \( <x,0(t)> \)-differentiation into multiplication by \( i<y,0(t)> \).

Proof: Let \( \psi(y) \in (L^2)^+_y \). Then by Lemma 4.2

\[ \partial_{x,0(t)}^* \langle \exp[-i<x,y>] : y, \psi \rangle = i(\exp[-i<x,y]> : y, <y,0(t)> \psi). \]

Therefore, for all \( \psi \in (L^2)^+_y \)

\[ \langle (\partial_{x,0(t)} \phi(x))^\wedge, \psi \rangle = \langle \partial_{x,0(t)} \phi(x), \langle \exp[-i<x,y>] : y, \psi \rangle \]
\[ = \langle \phi, \partial_{x,0(t)}^* \langle \exp[-i<x,y>] : y, \psi \rangle \]
\[ = \langle \phi, i(\exp[-i<x,y]> : y, <y,0(t)> \psi) \rangle \]
\[
- \langle \hat{\phi}, i \langle y, \hat{0}(t) \rangle \psi \rangle \\
- \langle i \langle y, \hat{0}(t) \rangle \hat{\phi}, \psi \rangle
\]

Therefore, \((\partial_{\langle x, \hat{0}(t) \rangle} \phi(x))^\wedge(y) = \langle y, \hat{0}(t) \rangle \hat{\phi}(y)\).

(Q.E.D.)

**Theorem 4.2**

For \(\phi(x) \in (L^2)^-\)

\[
U(\hat{\phi}(y))(n) = \exp(-\frac{1}{2} \|n\|^2)U(\phi(x))(-in).
\]

Proof: By definition

\[
U(\hat{\phi}(y))(n) = \int_B \hat{\phi}(y + n) d\mu(y)
\]

\[
= \int_B \int_B \exp[-i \langle x, y + n \rangle]: \phi(x) d\mu(x) d\mu(y)
\]

\[
= \int_B \int_B \exp[-i \langle x, y + n \rangle]: d\mu(y) \phi(x) d\mu(x)
\]

\[
= \int_B U(\exp[-i \langle x, y \rangle]): y(n) \phi(x) d\mu(x)
\]

\[
= \int_B \exp[-i \langle n, x \rangle] \phi(x) d\mu(x).
\]

Now

\[
U(\hat{\phi}(y))(in) = \int_B \exp[i \langle n, x \rangle] \phi(x) d\mu(x).
\]

Let \(x = y + n\), and use \(d\mu(y + n) = e^{-\frac{1}{2} \|n\|^2} - \langle n, y \rangle d\mu(y)\), then the above is equal to

\[
\int_B \exp[i \langle n, y + n \rangle] \phi(y + n) e^{-\frac{1}{2} \|n\|^2} - \langle n, y \rangle d\mu(y)
\]

\[
= \exp[\frac{1}{2} \|n\|^2] \int_B \phi(y + n) d\mu(y)
\]

\[
= \exp[\frac{1}{2} \|n\|^2] U\phi(n).
\]

(Q.E.D.)
Corollary 4.2

If \( \phi(x) \in (L^2)_x \), then

\[
U(\hat{\phi}(y))(\eta) = \int_B e^{-i\langle x, \eta \rangle} \phi(x) \, d\mu(x) \quad \text{for} \quad \eta \in B^*.
\]

Theorem 4.3

Suppose \( \phi(x) \in K_{n,x}^\ominus \) is represented by \( k_1 \otimes \ldots \otimes k_n \), where

\( k_i \in F \) for \( i = 1, \ldots, n \). Then

\[
U(\hat{\phi}(y))(\eta) = e^{\frac{1}{2} \| \eta \|^2} (-1)^n \langle \eta, k_1 \rangle \ldots \langle \eta, k_n \rangle.
\]

Proof: From Corollary 4.2 we see that

\[
U(\hat{\phi}(y))(\eta) = \langle e^{-i\langle x, \eta \rangle}, \phi(x) \rangle \quad \text{for} \quad \phi \in (L^2)_x.
\]

Also

\[
U(e^{-i\langle x, \eta \rangle})(\xi) = \int_B e^{-i\langle x+\xi, \eta \rangle} \, d\mu(x)
\]

\[
= e^{-i\langle \xi, \eta \rangle} \int_B e^{-i\langle x, \eta \rangle} \, d\mu(x)
\]

\[
= e^{-i\langle \xi, \eta \rangle} \cdot e^{-\frac{1}{2} \| \eta \|^2}
\]

\[
= e^{-\frac{1}{2} \| \eta \|^2} \sum_{n=0}^{\infty} \frac{(-1)^n \langle \xi, \eta \rangle^n}{n!}
\]

Therefore,

\[
\langle e^{-i\langle x, \eta \rangle}, \phi(x) \rangle = n! e^{-\frac{1}{2} \| \eta \|^2} \frac{(-1)^n}{n!} \langle \eta, \otimes \ldots \otimes n, k_1 \otimes \ldots \otimes k_n \rangle
\]

\[
= (-1)^n e^{-\frac{1}{2} \| \eta \|^2} \langle n, k_1 \rangle \ldots \langle n, k_n \rangle. \quad \text{(Q.E.D.)}
\]
**Theorem 4.4**

Let \( \phi \in (L^2)^{-} \) and

\[
F(u_1, \ldots, u_n) = \int_B \langle x, \Theta(u_1) \rangle \langle x, \Theta(u_2) \rangle \cdots \langle x, \Theta(u_n) \rangle \phi(x) \, d\mu(x) \in L^2(\mathbb{R}^n).
\]

Then for any \( \psi \in K_n \) represented by \( G(u_1, \ldots, u_n) \) with \( G(u_1, \ldots, u_n) \in L^2(\mathbb{R}^n) \), we have

\[
\langle \phi, \psi \rangle = (-1)^n \int_{\mathbb{R}^n} F(u_1, \ldots, u_n) G(u_1, \ldots, u_n) \, du_1 \cdots du_n.
\]

**Remark:** Suppose \( \psi \in K_n \) has the representation \( G(u_1, \ldots, u_n) \) then

\[
\psi(\cdot) = \int_{\mathbb{R}^n} G(u_1, \ldots, u_n) \, d\beta(u_1) \cdots d\beta(u_n)
\]

\[
= \int_{\mathbb{R}^n} G(u_1, \ldots, u_n) \langle \cdot, \Theta(u_1) \rangle \langle \cdot, \Theta(u_2) \rangle \cdots \langle \cdot, \Theta(u_n) \rangle \, du_1 \cdots du_n
\]

and

\[
\| \psi \|_{(L^2)^{-}} = \left( \frac{1}{n!} \right)^{\frac{1}{2}} \| G \|_{L^2(\mathbb{R}^n)}.
\]

**(Proof)**

\[
U(\hat{\phi}(y))(\eta) = \int_B e^{-i \langle x, \eta \rangle} \phi(x) \, d\mu(x)
\]

\[
= \int_B \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \langle x, \eta \rangle^n \phi(x) \, d\mu(x)
\]

\[
= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_B \left( \int_{\mathbb{R}^n} \langle x, \Theta(u_1) \rangle \cdots \langle x, \Theta(u_n) \rangle \, du_1 \cdots du_n \right) \phi(x) \, d\mu(x)
\]

\[
= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{\mathbb{R}^n} \left[ \int_B \langle x, \Theta(u_1) \rangle \cdots \langle x, \Theta(u_n) \rangle \phi(x) \, d\mu(x) \right] \, du_1 \cdots du_n.
\]

Therefore,
\[ \langle \hat{\varphi}, \psi \rangle = n! \frac{(-1)^n}{n!} \int_B <x, \Theta(u_1)> \cdots <x, \Theta(u_n)> \varphi(x) du(x) \in L^2(\mathbb{R}^n) \]

= \frac{(-i)^n}{n!} \int_{\mathbb{R}^n} G(u_1, \ldots, u_n) F(u_1, \ldots, u_n) du_1 \cdots du_n

where

\[ F(u_1, \ldots, u_n) \equiv \int_B <x, \Theta(u_1)> \cdots <x, \Theta(u_n)> \varphi(x) du(x). \quad (Q.E.D.) \]

**Theorem 4.5**

Suppose \( \varphi \in K_{m,x} \) and \( \psi \in K_{n,y} \) are represented by \( F(u_1, \ldots, u_m) \) and \( G(u_1, \ldots, u_n) \), respectively. Then

\[ \langle \hat{\varphi}, \psi \rangle = \begin{cases} 0 & \text{if } m > n \\ (-i)^n! \langle F, G \rangle & \text{if } m = n \end{cases} \]

Proof: By Theorem 4.4

\[ \langle \hat{\varphi}, \psi \rangle = (-i)^n! \int_{\mathbb{R}^n} \left[ \int_B <x, \Theta(u_1)> \cdots <x, \Theta(u_n)> \varphi(x) du(x) \right] \]

\[ \int_B G(u_1, \ldots, u_n) du_1 \cdots du_n \]

= \frac{(-i)^n}{n!} \int_{\mathbb{R}^n} G(u_1, \ldots, u_n) <x, \Theta(u_1)> \cdots <x, \Theta(u_n)> du_1 \cdots du_n \]

* \( \varphi(x) du(x) \).

Now let \( P_n \) be the orthogonal projection of \( L^2(\mathbb{R}) \) onto \( K_n \). Then

\[ P_m \left( \int_{\mathbb{R}^n} G(u_1, \ldots, u_n) <x, \Theta(u_1)> \cdots <x, \Theta(u_n)> du_1 \cdots du_n \right) = 0 \text{ if } m > n \]

and

\[ P_m \left( \int_{\mathbb{R}^n} G(u_1, \ldots, u_n) <x, \Theta(u_1)> \cdots <x, \Theta(u_n)> du_1 \cdots du_n \right) \]

= \int_{\mathbb{R}^n} G(u_1, \ldots, u_n) <x, \Theta(u_1)> \cdots <x, \Theta(u_n)> du_1 \cdots du_n
Therefore \[ \int_{\mathbb{R}^n} \mathbf{G}(u_1, \ldots, u_n) d\mathbf{\beta}(u_1) \ldots d\mathbf{\beta}(u_n) \quad \text{if } m = n. \]

Therefore \[ \langle \hat{\phi}, \psi \rangle = \begin{cases} 0 & \text{if } m > n \\ (-1)^n n! \langle F, G \rangle & \text{if } m = n. \end{cases} \quad \text{(Q.E.D.)} \]

**Example 1.** Let \( \phi(x) \equiv 1 \). Then

\[ U(\hat{\phi}(y))(n) = \int_B e^{-i \langle x, \eta \rangle} d\mu(x) = \int_B e^{-i \langle x, \eta \rangle} d\mu(x) = e^{\frac{1}{2} \eta \eta^T} = e^{\frac{1}{2} \eta \eta^T} = \sum_{n=0}^{\infty} \frac{1}{n!} \eta_n n^n = \sum_{n=0}^{\infty} \frac{1}{n!} \eta_n n^n \]

\[ \mathcal{L} \sum_{n=0}^{\infty} \frac{1}{n!} \eta_n n^n \int_{\mathbb{R}^n} <\eta, \mathcal{O}(u_1)>^2 <\eta, \mathcal{O}(u_2)>^2 \ldots <\eta, \mathcal{O}(u_n)>^2 du_1 \ldots du_n. \]

Therefore

\[ \hat{\phi}(y) = \sum_{n=0}^{\infty} \frac{1}{n!} \eta_n n^n \int_{\mathbb{R}^n} :<\eta, \mathcal{O}(u_1)>^2 \ldots <\eta, \mathcal{O}(u_n)>^2 :du_1 \ldots du_n. \]

**Example 2.** Suppose \( \phi(x) = <x, \xi_0> \) with \( \xi_0 \in \mathcal{B}^* \). Then, by Theorem 4.3,

\[ U(\hat{\phi}(y))(n) = e^{\frac{1}{2} \eta \eta^T} \cdot (-1)^n <\eta, \xi_0> \]

Therefore,

\[ \hat{\phi}(y) = \sum_{n=0}^{\infty} \frac{-1}{n!} \eta_n n^n \int_{\mathbb{R}^n} :<\eta, \mathcal{O}(u_1)>^2 \ldots <\eta, \mathcal{O}(u_n)>^2 :<\eta, \xi_0> du_1 \ldots du_n. \]

**Example 3.** Suppose

\[ \phi(x) = \int_{\mathbb{R}^k} F(u_1, \ldots, u_k) :<x, \mathcal{O}(u_1)>^n_1 \ldots <x, \mathcal{O}(u_k)>^n_k du_1 \ldots du_k \]

is normal generalized random variable with \( \sum_{j=1}^{k} \eta_j = n \) then
This can be verified as follows. We know

\[ U(\phi(x))(\xi) = \int_{\mathbb{R}^k} F(u_1,\ldots,u_k) \langle \xi, \partial(u_1) \rangle \cdots \langle \xi, \partial(u_k) \rangle \, du_1 \cdots du_k. \]

Therefore,

\[ U(\phi(x))(-n) = (-1)^n \int_{\mathbb{R}^k} F(u_1,\ldots,u_k) \langle n, \partial(u_1) \rangle \cdots \langle n, \partial(u_k) \rangle \, du_1 \cdots du_k. \]

Then

\[ U(\hat{\phi}(y))(n) = e^\frac{1}{2} \|n\|^2 U(\phi(x))(-n) \quad \text{by Theorem 4.2} \]

\[ = \frac{1}{2} \|n\|^2 \cdot (-1)^n \int_{\mathbb{R}^k} F(u_1,\ldots,u_k) \langle n, \partial(u_1) \rangle \cdots \langle n, \partial(u_k) \rangle \, du_1 \cdots du_k \]

\[ = \sum_{j=0}^{\infty} \frac{1}{j!} \langle n, n \rangle^j \cdot (-1)^n \int_{\mathbb{R}^k} F(u_1,\ldots,u_k) \langle n, \partial(u_1) \rangle \cdots \langle n, \partial(u_k) \rangle \, du_1 \cdots du_k \]

\[ = \sum_{j=0}^{\infty} \frac{(-1)^n}{j!} \int_{\mathbb{R}^k} F(u_1,\ldots,u_k) \langle n, \partial(u_1) \rangle \cdots \langle n, \partial(u_k) \rangle \, du_1 \cdots du_k \]

\[ \cdot \langle n, \partial(u_{k+1}) \rangle^2 \cdots \langle n, \partial(u_{k+j}) \rangle^2 \, du_1 \cdots du_{k+j}. \]

Therefore,

\[ \hat{\phi}(y) = \sum_{j=0}^{\infty} \frac{(-1)^n}{j!} \int_{\mathbb{R}^k} F(u_1,\ldots,u_k) :\langle y, \partial(u_1) \rangle \cdots \langle y, \partial(u_k) \rangle \, du_1 \cdots du_k \]

\[ \cdot \langle y, \partial(u_{k+1}) \rangle^2 \cdots \langle y, \partial(u_{k+j}) \rangle^2 \, du_1 \cdots du_{k+j}. \]
§2. Relation between the Fourier transform and the Levy Laplacian.

There are several operators that may be viewed as infinite dimensional analogs of the ordinary Laplacian in $\mathbb{R}^n$. T. Hida and K. Saito [7] have obtained a relation between Kuo's Fourier transform and the Levy Laplacian in white noise calculus. Similarly we can obtain a similar relation between the Fourier transform given in Definition 4.2 and the Levy Laplacian $\Delta_L$.

Now let us specify the domain of $\Delta_L$. Let $W$ be the vector space spanned by the normal random variables given in Definition 3.5. Define a norm $\|\cdot\|_e$ in $W$ as follows:

$$\|\phi\|_e^2 = \|\phi\|_e^2 + \int \|\partial_\theta(t)\phi\|_e^2 \, dt + \int \|\partial_\theta(t)\phi D(t)\|_e^2 \, dt.$$ 

We see easily $\|\phi\|_e < \infty$ for $\phi \in W$. The completion of $W$ with respect to the norm $\|\cdot\|_e$ will be denoted by $\overline{W}$ and will be taken to be the domain of the Levy Laplacian $\Delta_L$.

**Theorem 4.6**

For $\phi \in \overline{W}$

1) $\|\Delta_L \phi\|_e \leq \|\phi\|_e$

2) $\Delta_L \phi = \int \partial_\theta(t)\phi D(t) dt$.

3) $\Delta_L \phi = 0$ for $\phi \in L^2$.

Proof: (i) It is enough to show this for $\phi \in W$. Then it is obvious since

$$\|\Delta_L \phi\|_e^2 = \int \|\partial_\theta(t)\phi D(t) dt\|_e^2 \leq \int \|\partial_\theta(t)\phi D(t)\|_e^2 \, dt.$$

(ii) is mentioned in Remark 1 after Definition 3.4;

(iii) is proved in Example 1 in §2, Chapter 3.
Theorem 4.7

\((\Delta_L - 1)^\hat{\phi} = - (\Delta_L \phi)^\wedge\) for \(\phi \in \hat{W}\).

Proof: It is sufficient to show that the theorem is true for \(\phi\) of the form in Definition 3.5. Then from Example 3 in §1

\[
\begin{align*}
U(\hat{\phi}(y))(n) &= e^{\frac{1}{2} \eta_1^2} U(\phi(x))(-\in)
\end{align*}
\]

\[
\begin{align*}
= e^{\frac{1}{2} \eta_1^2} \cdot (-1)^n \int_{\mathbb{R}^k} F(u_1, \ldots, u_k) <n, \bar{\theta}(u_1)>^{n_1} \ldots <n, \bar{\theta}(u_k)>^{n_k} du_1 \ldots du_k
\end{align*}
\]

On the other hand,

\[
\begin{align*}
U(\Delta_L \hat{\phi}(x))(n) &= e^{\frac{1}{2} \eta_1^2} \cdot (-1)^n \sum_{k=1}^{k}(n_k)(n_k - 1) \int_{\mathbb{R}^k} F(u_1, \ldots, u_k)
\end{align*}
\]

\[
\begin{align*}
\cdot <n, \bar{\theta}(u_1)>^{n_1} \ldots <n, \bar{\theta}(u_k)>^{n_k - 2} <n, \bar{\theta}(u_k)>^{n_k} du_1 \ldots du_k
\end{align*}
\]

Therefore,

\[
\begin{align*}
U(\Delta_L \phi(x))^\wedge(n) &= e^{\frac{1}{2} \eta_1^2} \cdot (-1)^{n-2} \sum_{k=1}^{k}(n_k)(n_k - 1) \int_{\mathbb{R}^k} F(u_1, \ldots, u_k)
\end{align*}
\]

\[
\begin{align*}
\cdot <n, \bar{\theta}(u_1)>^{n_1} <n, \bar{\theta}(u_k)>^{n_k - 2} <n, \bar{\theta}(u_k)>^{n_k} du_1 \ldots du_k
\end{align*}
\]

and so,

\[(\Delta_L - 1)^\hat{\phi} = -\Delta_L \phi^\wedge.\]
REFERENCES


VITA

Youngsook Lee Shim was born on December 21, 1951, in Seoul, Korea. She graduated from Kyeong-Ki Girl's High School in 1970. In February, 1974 she received a Bachelor of Arts degree in Mathematics Education from Seoul National University. She served as a teacher in Sungsoo Junior High School for 4 years and in Chinhae Girl's Senior High School for 3 years. She entered Louisiana State University in June, 1982.
Candidate: Youngsook Lee SHIM

Major Field: Mathematics

Title of Dissertation: Abstract Wiener Space Approach to Hida Calculus

Approved:

[Signatures]

Date of Examination: April 6, 1977