Extreme Elements in Semigroups (Quasiorders, Green's).

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EXTREME ELEMENTS IN SEMIGROUPS

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by

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ABSTRACT

We study the elements of a semigroup which are minimal or maximal with respect to Green's quasiorders.

Part 1 begins with a preliminary review. The sets of minimal elements are characterized in terms of minimal ideals. We discuss the relationship between the min set of a semigroup and the min set of a subsemigroup. The sets of maximal elements are characterized, and it is shown that these sets do not necessarily satisfy any inclusion relationships to each other. We discuss the max sets of subsemigroups and product semigroups. Conditions are given under which the max sets and min sets can intersect. We define the concept of a paved semigroup and present conditions under which homomorphisms preserve sets of maximal elements.

The translational hull is discussed in Part 2. We compare the condition that a semigroup S is $H$ paved with the condition that $S = ESE$, where E is the set of idempotents of S. We prove that if S is a subsemilattice of a finite semilattice T and if their max sets are equal, then the degree of S is at most the degree of T.

Topological results appear in Part 3. Extreme sets of compact semigroups are discussed. An example is given in which the set of nonmaximal elements can be extended in more than one way. We compare the max set with various topological notions of boundary.

Part 4 contains results on divisibility and on the Nambooripad partial order. Conditions are given under which the minimal sets inherit divisibility properties of the semigroup. We prove that divisibility of a semigroup very strongly implies divisibility of its max sets. Finally, we show that any element of a regular semigroup which is maximal with respect to the $H$, $R$, or $L$ quasiorder is maximal with respect to the Nambooripad partial order.
INTRODUCTION

This study began as an investigation of the extent to which the set of $H$ maximal elements of an non-monoid mimics the properties of the groups of units of a monoid. In our study, we also include maximal sets with respect to the $R$, $L$, and $J$ quasiorders.

Correspondingly, we determine the relationships between the $R$, $L$, $H$, and $J$ minimal sets and the various minimal ideals.
PART 1: GREEN'S QUASIORDERS AND EXTREME SETS

The relations discussed by Green in his 1951 paper give rise to four quasiorders. We characterize the sets of minimal elements associated with these quasiorders in terms of minimal ideals. In the case that the semigroup is a monoid, the sets of maximal elements are the corresponding classes of the identity. Since the $H$ class of the identity is the group of units, the set of $H$ maximal elements is a generalization of the group of units for a non-monoid. We introduce the notion of a paved semigroup since the sets of maximal elements of a paved semigroup more closely resemble the classes of the identity of a monoid.

CHAPTER 1: GREEN'S QUASIORDERS

Four quasiorders arise from Green's relations on a semigroup. The $R$, $L$, and $J$ quasiorders can be described in terms of ideals. We present an analogous description of the $H$ quasiorder in terms of bi-ideals.

§1.1 REVIEW

A semigroup is a set together with an associative multiplication. For any semigroup $S$, for $a, b \in S$, we define

- $a \leq_R b$ if and only if there exists an element $x \in S^1$ such that $a = bx$;
- $a \leq_L b$ if and only if there exists an element $x \in S^1$ such that $a = xb$;
- $a \leq_H b$ if and only if there exist elements $x, y \in S^1$ such that $a = xb = by$; and
- $a \leq_J b$ if and only if there exist elements $x, y \in S^1$ such that $a = xby$.

These relations are called Green's quasiorders.
Also, \( aRb \) if and only if \( a \leq_R b \) and \( b \leq_R a \); and \( R_a = \{ b : aRb \} \). Similar statements hold for \( L, H, \) and \( J \). Notice that each of \( a \leq_R b, a \leq_L b, \) and \( a \leq_H b \) implies \( a \leq_J b \). Also, \( a \leq_H b \) if and only if \( a \leq_R b \) and \( a \leq_L b \).

**Fact:** If \( S \) is normal (i.e., \( xS = Sx \) for all \( x \in S \)), then \( \leq_R = \leq_L = \leq_H = \leq_J \), and so \( R = L = H = J \).

**Proof:** It suffices to show that \( a \leq_R b \Rightarrow a \leq_L b \) and that \( a \leq_J b \Rightarrow a \leq_L b \).

Suppose \( a \leq_R b \). Then there exists an element \( x \in S^1 \) such that \( a = bx \). Since \( bS^1 = S^1b \), there exists \( y \in S^1 \) such that \( a = bx = yb \). Thus \( a \leq_L b \).

Now suppose \( a \leq_J b \). Then there exists \( x, y \in S^1 \) such that \( a = xyb \). Since \( bS^1 = S^1b \), there exists \( z \in S^1 \) such that \( by = zb \). Then \( a = xyb = xzb \), and so \( a \leq_L b \).

These orders are quasiorders (reflexive and transitive). They are not necessarily symmetric or antisymmetric. For example, let \( S \) be (the unit circle \( \times \mathbb{I}_u \)) \( \subseteq \mathbb{C} \times \mathbb{R} \). This semigroup is commutative and so is normal. Let \( \leq \) be one of the four equivalent orders. Since \( (1,0) \leq (1,1) \) and \( (1,1) \not\leq (1,0) \), we see \( \leq \) is not symmetric. Further, \( (1,0) \leq (i,0) \) and \( (i,0) \leq (l,0) \) but \( (1,0) \neq (1,0) \), so \( \leq \) is not antisymmetric either.

§1.2 **BI-IDEALS AND THE H QUASIORDER**

We know that in a semigroup \( S \), \( a \leq_J b \) if and only if \( a \) is in the principal ideal generated by \( b \); \( a \leq_R b \) if and only if \( a \) is in the principal right ideal generated by \( b \); and \( a \leq_L b \) if and only if \( a \) is in the principal left ideal generated by \( b \). In this section, we show that for a regular semigroup \( S \), (i.e., a semigroup \( S \) satisfying \( a \in aSa \) for each element \( a \)), \( a \leq_H b \) if and only if \( a \) is in the principal bi-ideal generated by \( b \).

Let \( S \) be a semigroup. A subset \( B \) of \( S \) is called a *bi-ideal* of \( S \) if \( BS^1B \subseteq B \).
Notice that for all $a, b \in S$, $aSb$ is a bi-ideal. If $C$ is a nonempty subset of $S$, the \textit{bi-ideal generated by} $C$ is the smallest bi-ideal of $S$ containing $C$. We denote it by $B(C)$. If $C = \{x\}$, we write $B(x)$ instead.

For a reference on bi-ideals, see Clifford and Preston, pages 84 and 85. This section is the consequence of a suggestion by R. J. Koch.

**Lemma:** Let $S$ be a semigroup and let $C$ be a nonempty subset of $S$. Then:

1) $B(C)$ is the intersection of all bi-ideals of $S$ containing $C$.
2) $B(C) = C \cup CS^1C$.

**Proof:** 1) Let $\{B_j : j \in J\}$ be the set of all bi-ideals of $S$ containing $C$. Let $B = \bigcap\{B_j : j \in J\}$. If $a, b \in B$, then $aS^1b \subseteq B_jS^1B_j \subseteq B_j$ for each $j \in J$. So $aS^1b \subseteq B$. Thus $BS^1B \subseteq B$, so $B$ is a bi-ideal of $S$.

2) Let $A = C \cup CS^1C$. Then $A$ is a bi-ideal since $AS^1A = CS^1C \cup CS^1(CS^1C) \cup (CS^1C)S^1C \cup (CS^1C)S^1(CS^1C) \subseteq CS^1C \subseteq A$.

If $B$ is a bi-ideal of $S$ and $C \subseteq B$, then $A \subseteq B$ since $CS^1C \subseteq BS^1B \subseteq B$. So by part 1, $B(C) = A$. $\diamond$

**Theorem:** If $S$ is regular, then $a \preceq_H b$ if and only if $a \in B(b)$.

**Proof:** It suffices to show that for each $b$ in $S$, $\{b\} \cup bS^1b = bS^1 \cap S^1b$. So fix $b \in S$. Then there exists $x \in S$ such that $bxb = b$. Let $e = bx$ and $f = xb$. Then $bS^1 = bxbS^1 \subseteq bxs^1 = eS^1 \subseteq bS^1$, so $bS^1 = eS^1$. Similarly, $S^1b = S^1f$. Then $bS^1 \cap S^1b = eS^1 \cap S^1f = eS^1f$, and $eS^1f = bxs^1xb \subseteq bS^1b = bxbS^1xb \subseteq bxs^1xb = eS^1f$. So $bS^1b = eS^1f = bS^1 \cap S^1b$. $\diamond$
CHAPTER 2: MIN SETS

The sets of elements which are minimal with respect to Green’s quasiorders are closely related to the various types of minimal ideals. The characterizations of the minimal sets in terms of minimal ideals are useful in studying the relationships between the minimal elements of a subsemigroup and those elements of the subsemigroup which lie in the corresponding minimal set of the supersemigroup.

§2.1 CHARACTERIZATIONS

For any semigroup $S$, define:

$R_{min}(S) = \{ a \in S : \text{if } b \leq_R a, \text{ then } bRa \}$;

$L_{min}(S) = \{ a \in S : \text{if } b \leq_L a, \text{ then } bLa \}$;

$H_{min}(S) = \{ a \in S : \text{if } b \leq_H a, \text{ then } bHa \}$; and

$J_{min}(S) = \{ a \in S : \text{if } b \leq_J a, \text{ then } bJa \}$.

Notice that if $S$ is normal, these four sets are equal.

Also, if $x \in R_{min}(S)$, then the $R$ class of $x$, $R_x(S)$, is contained in $R_{min}(S)$.

Similar statements hold for $L$, $H$, and $J$.

**Theorem:** Let $S$ be a semigroup.

1) $J_{min}(S) \neq \emptyset$ if and only if $S$ has a minimal ideal $M(S)$. If so, $J_{min}(S) = M(S)$.

2) $H_{min}(S) \neq \emptyset$ if and only if $S$ has a completely simple minimal ideal $M(S)$. If so, $H_{min}(S) = J_{min}(S) = M(S)$.

3) $R_{min}(S) \neq \emptyset$ if and only if $S$ has a minimal right ideal. If so, $R_{min}(S)$ is the union of all of the minimal right ideals of $S$.

4) $L_{min}(S) \neq \emptyset$ if and only if $S$ has a minimal left ideal. If so, $L_{min}(S)$ is the
union of all of the minimal left ideals of \( S \).

**Proof:** 1) First suppose \( J_{\text{min}}(S) \neq \emptyset \).

Then \( J_{\text{min}}(S) \) is an ideal of \( S \); Let \( x \in J_{\text{min}}(S) \) and let \( a, b \in S^1 \). Suppose \( y \leq_J axb \). Then there exist elements \( c, d \in S^1 \) such that \( y = caxbd \). So \( y \leq_J x \) and hence \( yJx \). Thus there exist \( p, q \in S^1 \) such that \( x = pyq \). Then \( axb \leq_J y \). Thus \( axbJy \) and so \( axb \in J_{\text{min}}(S) \).

Also, \( J_{\text{min}}(S) \) is minimal: Let \( \emptyset \neq A \subseteq J_{\text{min}}(S) \) such that \( A \) is an ideal of \( S \). Let \( x \in J_{\text{min}}(S) \) and \( a \in A \). Then \( axa \leq_J x \). Thus \( xJaxa \) and so there exist \( b, c \in S^1 \) such that \( x = baxac \). Hence \( x \in A \), so \( J_{\text{min}}(S) = A \).

On the other hand, suppose \( S \) has a minimal ideal \( M(S) \). Show that \( M(S) \subseteq J_{\text{min}}(S) \): Let \( x \in M(S) \). Suppose \( y \leq_J x \). Then there exist \( a, b \in S^1 \) such that \( y = axb \in M(S) \). Now \( S^1yS^1 = M(S) \). Hence, \( x \in S^1yS^1 \), so \( x \leq_J y \) and \( xJy \). Thus \( x \in J_{\text{min}}(S) \).

Result: If \( J_{\text{min}}(S) \neq \emptyset \), then \( J_{\text{min}}(S) = M(S) \).

2) Recall \( M(S) \) is completely simple if and only if it contains a primitive idempotent.

First suppose \( H_{\text{min}}(S) \neq \emptyset \).

If \( I \) is any ideal of \( S \), \( H_{\text{min}}(S) \subseteq I \): Let \( a \in H_{\text{min}}(S) \), and \( x \in I \). Then \( axa \leq_H a \Rightarrow axaHa \Rightarrow a \leq_H axa \). So \( a \in I \) since \( x \in I \). Thus \( H_{\text{min}}(S) \subseteq I \).

Thus, \( \emptyset \neq H_{\text{min}}(S) \subseteq \bigcap\{ I : I \text{ is an ideal of } S \} = M(S) \).

Further, \( M(S) \) is completely simple: Let \( a \in H_{\text{min}}(S) \). Then \( a^2 \leq_H a \Rightarrow a^2 \in H_a \), and so \( H_a \) is a group. Let \( e \) be the identity of \( H_a \). Since \( eHa \), then \( e \in H_{\text{min}}(S) \). To see that \( e \) is primitive, suppose \( f \in E(S) \) and \( f = ef = fe \). Then \( fHe \), so \( f = e \). Thus \( e \) is primitive, and \( e \in H_{\text{min}}(S) \subseteq M(S) \).

On the other hand, suppose \( S \) has a completely simple minimal ideal \( M(S) \). Then \( M(S) \) must contain a primitive idempotent \( e \). Suppose \( x \leq_H e \). Then there exist \( t_1, t_2 \in S^1 \) such that \( x = t_1e = et_2 \). But then \( x \in eSe = He \), so \( xHe \). Thus
\[ e \in H_{\text{min}}(S) \neq \emptyset. \]

Result: If \( H_{\text{min}}(S) \neq \emptyset \) and \( S \) contains a completely simple minimal ideal \( M(S) \), then we have seen that \( H_{\text{min}}(S) \subseteq M(S) \). To see that \( M(S) \subseteq H_{\text{min}}(S) \), let \( a \in M(S) \). Since \( H_a \) is a group, there exists \( f \in E(S) \cap H_a \). Since \( f \) is in \( M(S) \), which is completely simple, \( f \) is primitive. As in the preceding paragraph, \( f \in H_{\text{min}}(S) \), so \( a \in H_a = H_f \subseteq H_{\text{min}}(S) \). Therefore, \( M(S) \subseteq H_{\text{min}}(S) \), and so \( M(S) = H_{\text{min}}(S) \).

3) First assume \( R_{\text{min}}(S) \neq \emptyset \).

Then \( R_{\text{min}}(S) \) is a right ideal of \( S \): Let \( x \in R_{\text{min}}(S) \) and \( a \in S \). Suppose \( y \leq_R xa \). Then there exists \( b \) such that \( xab = y \), so there exists \( c \) such that \( x = yc \). Then \( xa \leq_R y \). Thus \( xaRy \).

Now let \( x \in R_{\text{min}}(S) \). Since \( R_{\text{min}}(S) \) is a right ideal, \( xS^1 \subseteq R_{\text{min}}(S) \). If \( p \in xS^1 \), there exists \( r \in S \) such that \( p = xr \). Then \( p \leq_R x \Rightarrow pRx \), so \( pS^1 = xS^1 \). Hence \( xS^1 \) is minimal, so \( R_{\text{min}}(S) \) is contained in the union of all of the minimal right ideals.

On the other hand, let \( A \) be a minimal right ideal and let \( a \in A \). Notice \( aS^1 = A \). If \( b \leq_R a \), then there exists \( x \) such that \( b = ax \in aS^1 = A \). Since \( b \in A \), \( bS^1 = A = aS^1 \), so \( bRa \). Thus \( a \in R_{\text{min}}(S) \).

4) The proof of part 4 corresponds to the proof of part 3. \( \diamond \)

**Corollary 1:** Let \( S \) be a semigroup. If \( R_{\text{min}}(S) \neq \emptyset \) and \( L_{\text{min}}(S) \neq \emptyset \), then \( S \) contains a completely simple minimal ideal \( M(S) = R_{\text{min}}(S) = L_{\text{min}}(S) = H_{\text{min}}(S) = J_{\text{min}}(S) \).

**Example:** It is possible to have \( R_{\text{min}}(S) \neq \emptyset \) and \( L_{\text{min}}(S) \neq \emptyset \). For example, consider the Baer-Levi semigroup

\[ S = \{ f : \mathbb{N} \to \mathbb{N} \text{ such that } f \text{ is } 1-1 \text{ and } \mathbb{N} \setminus f(\mathbb{N}) \text{ is infinite} \}. \]

One can show that \( S = Sf \) for each \( f \), so \( S \) is a single \( L \) class. Thus \( L_{\text{min}}(S) \)
Also, it is straightforward to show that \( f \leq_R g \) if and only if either \( f = g \) or both \( f(\mathbb{N}) \subseteq g(\mathbb{N}) \) and \( g(\mathbb{N}) \setminus f(\mathbb{N}) \) is infinite. As a result, \( Rmin(S) = \emptyset \).

Notice that in this semigroup, \( Jmin(S) = S \).

**Corollary 2:** Let \( E(S) \) be the set of idempotents in a semigroup \( S \). If \( E(S) = \emptyset \), then \( Hmin(S) = \emptyset \).

**Corollary 3:** If \( S \) contains a zero element \( 0 \), then \( \{0\} = Rmin(S) = Lmin(S) = Hmin(S) = Jmin(S) \).

**Corollary 4:** If \( S \) is a group, \( S = Jmin(S) = Hmin(S) = Rmin(S) = Lmin(S) \).

We have seen that if \( Hmin(S) \neq \emptyset \) or if \( Rmin(S) \cap Lmin(S) \neq \emptyset \), then \( Rmin(S) = Lmin(S) = Hmin(S) = Jmin(S) \).

Recall that the bicyclic semigroup is simple but not completely simple. Also, \( Rmin(B) = Lmin(B) = Hmin(B) = 0 \), but \( Jmin(B) = B \).

Also recall that in the Baer-Levi example, \( Lmin(S) = Jmin(S) = S \), but \( Rmin(S) \) is empty. If \( S \) is a semigroup in which \( Lmin(S) \) is not empty, then can the set \( Jmin(S) \) be empty?

### §2.2 SUBSEMIGROUPS

In this section, we consider the question: If \( T \) is a subsemigroup of \( S \), what is the relationship between \( Hmin(T) \) and \( T \cap Hmin(S) \)? Similar questions concerning the other orders are also studied.

In the first three examples below, the multiplication is commutative, so the orders under consideration are equivalent. Also, \( T \) is a subsemigroup of \( S \) in each case.
Example 1: Let $S = I_u \times \{0,1\}$ and $T = I_u \times \{1\}$. Here $\text{Min}(S) = \{(0,0)\}$, $\text{Min}(T) = \{(0,1)\}$, and $T \cap \text{Min}(S) = \emptyset$. So $T \cap \text{Min}(S) \subset \text{Min}(T)$.

Example 2: Let $S = (\mathbb{R}\setminus \{0\}, \cdot)$, a group, and let $T = (\mathbb{N}, \cdot)$. Here $\text{Min}(S) = S$, $\text{Min}(T) = \emptyset$ since in $T$, $n \leq m$ if and only if $m$ divides $n$. So $\text{Min}(T) \subset T \cap \text{Min}(S) = T$.

Example 3: Let $S = (\mathbb{N}, \ast)$, where $a \ast b = \text{min}\{a, b\}$. Let $T = \{1, 3, 5\}$. Then $\text{Min}(S) = \{1\}$, $\text{Min}(T) = \{1\}$, and so $\text{Min}(T) = \text{Min}(S) = \text{Min}(S) \cap T$. Notice this $T$ is not an ideal of $S$. We could have chosen $T = \{1, 2, 3, 4\}$, which is an ideal of $S$.

Example 4: Let $S$ be the affine triangle semigroup

$$\left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} : x, y \in \mathbb{R}, 0 \leq x, 0 \leq y, x + y \leq 1 \right\}.$$ 

As usual, we will denote $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ by $(x, y)$. Notice that $\text{Lmin}(S) = \{(0,y) : 0 \leq y \leq 1\}$.

Let $T = \{(x,0) : 0 \leq x \leq 1\}$, a subsemigroup of $S$. Then $S \setminus T$ is a right ideal of $S$. Also, $\text{Lmin}(T) = \{(0,0)\}$. Hence $\text{Lmin}(T) = T \cap \text{Lmin}(S) = \{(0,0)\}$.

Theorem: Let $S$ be a semigroup.

1) If $T$ is an ideal of $S$ and $\text{Jmin}(S) \neq \emptyset$, then $\text{Jmin}(T) \neq \emptyset$ and $\text{Jmin}(S) = \text{Jmin}(T)$.

2) If $T$ is an ideal of $S$ and $\text{Hmin}(S) \neq \emptyset$, then $\text{Hmin}(T) \neq \emptyset$ and $\text{Hmin}(S) = \text{Hmin}(T)$.

3) If $T$ is a subsemigroup of $S$ and $S \setminus T$ is a left ideal of $S$ and if $\text{Rmin}(S) \neq \emptyset$, then $\text{Rmin}(T) \neq \emptyset$ and $T \cap \text{Rmin}(S) \subseteq \text{Rmin}(T)$.

4) If $T$ is a subsemigroup of $S$ and $S \setminus T$ is a right ideal of $S$ and if $\text{Lmin}(S) \neq \emptyset$, then $\text{Lmin}(T) \neq \emptyset$ and $T \cap \text{Lmin}(S) \subseteq \text{Lmin}(T)$. 
Notes: The previous example 3 shows that $T$ does not necessarily need to be an ideal of $S$ to have $\text{Min}(T) = T \cap \text{Min}(S)$. Example 4 illustrates part 4 of the theorem.

Proof: 1) and 2) Now $\text{M}(S) \subseteq T$ and $\text{M}(S)$ is an ideal of $T$. Also, $\text{M}(S)$ is simple. Thus $\text{M}(T) = \text{M}(S)$.

3) Let $x \in T \cap \text{Rmin}(S)$ and suppose $y \in T$ such that $y \leq_{R_T} x$. Then $y \leq_{R_S} x$, and so $y R_S x$. Thus there exists $b \in S^1$ such that $x = yb$. Since $x \in T$ and $S \setminus T$ is a left ideal of $S$, $b \in T^1$. Thus $x R_T y$, and so $x \in \text{Rmin}(T)$.

4) The proof of part 4 corresponds to the proof of part 3. ◊
CHAPTER 3: MAX SETS

The sets of elements which are maximal with respect to Green's quasiorders are the Green's classes of the identity in monoids. Hence for a non-monoid, the $H$ maximal set is an extension of the concept of the group of units in a monoid.

The sets of nonmaximal elements are characterized as ideals or unions of bi-ideals. Examples illustrate that the sets of maximal elements do not necessarily satisfy any inclusion relationships with each other. The ideal structure of the complement of a subsemigroup in its supersemigroup has implications on the relationships between the maximal elements of the subsemigroup and those elements of the subsemigroup which also lie in the corresponding maximal set of the supersemigroup. A complete characterization is given for the maximal sets of a product semigroup in terms of the maximal sets of the component semigroups.

§3.1 CHARACTERIZATIONS

For any semigroup $S$, define:

$R_{\max}(S) = \{a \in S : \text{if } a \preceq_R b, \text{ then } aRb\}$;

$L_{\max}(S) = \{a \in S : \text{if } a \preceq_L b, \text{ then } aLb\}$;

$H_{\max}(S) = \{a \in S : \text{if } a \preceq_H b, \text{ then } aHb\}$; and

$J_{\max}(S) = \{a \in S : \text{if } a \preceq_J b, \text{ then } aJb\}$.

In McCharen's 1969 Ph.D. dissertation, he introduces this concept of $R$, $L$, and $J$ maximal elements. His dissertation is devoted primarily to topological results.

Notice that if $S$ is normal, these four sets are equal.

Also, if $x \in R_{\max}(S)$, then the $R$ class of $x, R_x(S)$, is contained in $R_{\max}(S)$. Similar statements hold for $L, H$, and $J$. 

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Theorem 1: If $S$ is a monoid, then:

1) $J_{\text{max}}(S) = J_1(S)$;
2) $H_{\text{max}}(S) = H_1(S)$;
3) $R_{\text{max}}(S) = R_1(S)$; and
4) $L_{\text{max}}(S) = L_1(S)$.

Proof: First notice that for each of the four quasiorders, $x \leq 1$ for each $x \in S$. Thus $1 \in \text{Max}(S)$ and so the class of 1 is contained in $\text{Max}(S)$. Further, if $x \in \text{Max}(S)$, then $x$ is related to 1. That is, $x$ is in the class of 1. $\Diamond$

Corollary 1: If $S$ is a group, then $S = J_{\text{max}}(S) = H_{\text{max}}(S) = R_{\text{max}}(S) = L_{\text{max}}(S)$.

Theorem 2: If $H_{\text{max}}(S)$ is a subsemigroup of $S$, it is a union of groups.

Proof: Let $x \in H_{\text{max}}(S)$. Since $H_{\text{max}}(S)$ is a subsemigroup, $x^2 \in H_{\text{max}}(S)$. But then $x^2 \leq_H x$, so $x^2Hx$. By a result of Green (1951), then $H_x$ is a group. $\Diamond$

Recall that if $S$ is a commutative monoid, then $S \setminus H_1(S)$ is an ideal of $S$. We think of $H_{\text{max}}(S)$ as a generalization of the concept of $H_1(S)$, and the next theorem shows that if $S$ is commutative, then $S \setminus \text{Max}(S)$ is an ideal of $S$.

Theorem 3: Let $S$ be a semigroup. Then:

1) $S \setminus J_{\text{max}}(S)$ is an ideal of $S$;
2) $S \setminus R_{\text{max}}(S)$ is a right ideal of $S$;
3) $S \setminus L_{\text{max}}(S)$ is a left ideal of $S$; and
4) $S \setminus H_{\text{max}}(S)$ is a union of bi-ideals of $S$.

Proof: 1) Let $ab \in J_{\text{max}}(S)$. Then $ab \leq_J a$ and $ab \leq_J b$, so $a, b \in J_{\text{max}}(S)$. 

2) Let \( ab \in R_{\text{max}}(S) \). Since \( ab \leq_R a \), we have \( a \in R_{\text{max}}(S) \).

Therefore, \((S \setminus R_{\text{max}}(S)) \cdot S \subseteq (S \setminus R_{\text{max}}(S))\).

3) The proof of part 3 corresponds to the proof of part 2.

4) Let \( x, y \in S \) such that \( xyx \in H_{\text{max}}(S) \). Since \( xyx \leq_H x \), then \( x \in H_{\text{max}}(S) \).

Thus if \( x \notin H_{\text{max}}(S) \), then the bi-ideal \( xSx \subseteq S \setminus H_{\text{max}}(S) \). Since \( S \) is regular, \( x \in xSx \). We therefore have \( S \setminus H_{\text{max}}(S) = \bigcup \{xSx : x \in S \setminus H_{\text{max}}(S)\} \). ♦

**Corollary 2:** If \( S \) is a normal semigroup, then \( S \setminus \text{Max}(S) \) is an ideal of \( S \).

**Example 1:** Let \( B \) be the bicyclic semigroup (the monoid generated by \( \{p, q\} \) subject to the relation \( qp = e \)):

\[
\begin{array}{ccccccc}
  e & q & q^2 & q^3 & q^4 & \cdots \\
p & pq & pq^2 & pq^3 & pq^4 & \cdots \\
p^2 & p^2q & p^2q^2 & p^2q^3 & p^2q^4 & \cdots \\
p^3 & p^3q & p^3q^2 & p^3q^3 & p^3q^4 & \cdots \\
p^4 & p^4q & p^4q^2 & p^4q^3 & p^4q^4 & \cdots \\
\end{array}
\]

Notice that \( B \setminus H_1(B) \) is neither an ideal of \( B \) nor a bi-ideal of \( B \) since \( qp = e \in H_1(B) \). However, \( B \setminus H_1(B) = pBp \cup qBq \).

Also, \( R_1(B) \) is the top row, and \( B \setminus R_1(B) \) is a right ideal of \( B \). Similarly, \( L_1(B) \) is the left column, and \( B \setminus L_1(B) \) is a left ideal of \( B \).

**Example 2:** Note that \( S \setminus \text{Max}(S) \) needn’t be a maximal proper ideal of \( S \). For example, let \( S \) be the three element semilattice \( \{e, f, 0\} \) with multiplication given by

\[
\begin{array}{c|ccc}
  \cdot & e & f & 0 \\
  e & e & 0 & 0 \\
f & 0 & f & 0 \\
 0 & 0 & 0 & 0 \\
\end{array}
\]

Then \( \text{Max}(S) = \{e, f\} \), but \( \{f, 0\} \) is an ideal of \( S \).
Example 3: If S is not commutative, $S \setminus Hmax(S)$ need not be an ideal of S. For example, let S be $\{a, x, b\}$ with multiplication given by:

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Then S is a semigroup, $Hmax(S) = \{x, b\} = Lmax(S)$, and $\{a\}$ is not an ideal of S but is a left ideal of S. Notice $\{a\}$ is a bi-ideal of S. Also $Jmax(S) = Rmax(S) = \{x\}$, and $\{a, b\}$ is an ideal of S.

The following theorem of McCharen establishes the relationship between maximal elements and maximal proper ideals. Recall that $J_0(A)$ is the union of all ideals contained in $A$, $L_0(A)$ is the union of all left ideals contained in $A$, and $R_0(A)$ is the union of all right ideals contained in $A$.

Theorem 4: Let S be a semigroup and $a \in S \setminus M(S)$. Then
1) $a \in Jmax(S)$ if and only if $J_0(S \setminus \{a\})$ is a maximal proper ideal.
2) $a \in Lmax(S)$ if and only if $L_0(S \setminus \{a\})$ is a maximal proper left ideal.
3) $a \in Rmax(S)$ if and only if $R_0(S \setminus \{a\})$ is a maximal proper right ideal.

Proof: See Theorem 0.6 of McCharen's 1969 dissertation.

§3.2 RELATIONSHIPS AMONG THE MAX SETS

In this section, we consider the possible relationships between the following sets: $Rmax(S)$, $Lmax(S)$, $Hmax(S)$, $Jmax(S)$, and $Rmax(S) \cap Lmax(S)$.

Example 1: In the bicyclic semigroup $B$, all four max sets are nonempty, and $Rmax(B) \subset Jmax(B)$;
\[ L_{\text{max}}(B) \subset J_{\text{max}}(B); \]
\[ H_{\text{max}}(B) \subset R_{\text{max}}(B); \]
\[ H_{\text{max}}(B) \subset L_{\text{max}}(B); \text{ and} \]
\[ H_{\text{max}}(B) \subset J_{\text{max}}(B). \]

**Example 2:** Let \( S = \{a, b, c\} \) with multiplication given by
\[
\begin{array}{ccc}
  . & a & b & c \\
  a & a & a & a \\
  b & b & b & b \\
  c & b & b & b \\
\end{array}
\]

Then \( R_{\text{max}}(S) = H_{\text{max}}(S) = \{a, c\} \), and \( L_{\text{max}}(S) = J_{\text{max}}(S) = \{c\} \). So \( L_{\text{max}}(S) = J_{\text{max}}(S) \subset R_{\text{max}}(S) = H_{\text{max}}(S) \).

**Note:** Dually, there is a semigroup \( S \) in which \( R_{\text{max}}(S) = J_{\text{max}}(S) \subset L_{\text{max}}(S) = H_{\text{max}}(S) \).

**Result:** Among the four max sets, \( R_{\text{max}}, L_{\text{max}}, H_{\text{max}}, \) and \( J_{\text{max}} \), any one can be strictly contained in any other.

Notice that for any semigroup \( S \), \( R_{\text{max}}(S) \cap L_{\text{max}}(S) \subsetneq R_{\text{max}}(S) \), and also \( R_{\text{max}}(S) \cap L_{\text{max}}(S) \subsetneq L_{\text{max}}(S) \). In the bicyclic semigroup, these containments are strict. Also, for any semigroup \( S \), \( R_{\text{max}}(S) \cap L_{\text{max}}(S) \subsetneq H_{\text{max}}(S) \). In example 2 above, \( R_{\text{max}}(S) \cap L_{\text{max}}(S) \subset H_{\text{max}}(S) \).

If \( S \) is a monoid, \( R_1 \cap L_1 = H_1 \subsetneq J_1 \).

Two interesting related questions are still open. Is there a semigroup \( S \) in which \( R_{\text{max}}(S) \) and \( L_{\text{max}}(S) \) are not empty but \( R_{\text{max}}(S) \cap L_{\text{max}}(S) \) is empty? How can \( R_{\text{max}}(S) \cap L_{\text{max}}(S) \) relate to \( J_{\text{max}}(S) \)?
§3.3 SUBSEMIGROUPS

In this section, we consider the question: If T is a subsemigroup of S, what is the relationship between $H_{\text{max}}(T)$ and $T \cap H_{\text{max}}(S)$? Similar questions concerning the other quasiorders are also studied.

In the first three examples below, the semigroups are commutative, so the four max sets are equal. In each case, T is a subsemigroup of S.

**Example 1:** $T \cap \text{Max}(S) \subseteq \text{Max}(T)$:

Let $S = (\mathbb{N}, +)$ and $T = \{2, 3, 4, \ldots\}$, $+$. Then $\text{Max}(T) = \{2, 3\}$, $\text{Max}(S) = \{1\}$, and $T \cap \text{Max}(S) = \emptyset$.

**Example 2:** $\text{Max}(T) = T \cap \text{Max}(S)$:

Let $S = \{2, 3, 4, \ldots\}$, $+$ and $T = \{2, 4, 6, \ldots\}$, $+$. Then $\text{Max}(T) = \{2\}$, $\text{Max}(S) = \{2, 3\}$, and $T \cap \text{Max}(S) = \{2\}$.

**Example 3:** $\text{Max}(T) \subset T \cap \text{Max}(S)$:

Let $S = (\mathbb{Z}, +)$ and $T = (\mathbb{N}, +)$. Then $\text{Max}(T) = \{1\}$, $\text{Max}(S) = S$, and $T \cap \text{Max}(S) = T$.

**Example 4:** $T \cap L_{\text{max}}(S) = L_{\text{max}}(T)$:

Let $S$ be the affine triangle semigroup and $T = \{(x, 0) : 0 \leq x \leq 1\} \subseteq S$. We have seen that $S \setminus T$ is a right ideal of $S$. Notice that $1_S = (1,0)$, and $L_1(S) = \{(1,0)\} = L_1(T)$.

**Example 5:** Let $S$ be the three element semigroup $\{a,b,c\}$ with multiplication given by
\[
\begin{array}{ccc}
  & a & b & c \\
\hline
  a & a & a & a \\
b & b & b & b \\
c & b & b & b \\
\end{array}
\]

Let \( T = \{a\} \), a subsemigroup of \( S \). Then \( S \setminus T = \{b, c\} \) is a right ideal of \( S \). Also, \( \text{Lmax}(T) = \{a\} \), but in \( S \), \( aLb <_L c \), so \( \text{Lmax}(S) = \{c\} \). Hence \( T \cap \text{Lmax}(S) = \emptyset \subset \text{Lmax}(T) \).

**Lemma:** Let \( S \) be a semigroup with subsemigroup \( T \). Let \( a, b \in T \).

1) If \( S \setminus T \) is a left ideal of \( S \), then \( a \leq_{RT} b \) if and only if \( a \leq_{Rs} b \).

2) If \( S \setminus T \) is a right ideal of \( S \), then \( a \leq_{LT} b \) if and only if \( a \leq_{Ls} b \).

3) If \( S \setminus T \) is an ideal of \( S \), then \( a \leq_{HT} b \) if and only if \( a \leq_{Hs} b \), and \( a \leq_{JT} b \) if and only if \( a \leq_{Js} b \).

**Proof:**

1) If \( a \leq_{RT} b \), then there exists \( x \in T^1 \) such that \( a = bx \). Since \( x \in S^1 \), \( a \leq_{Rs} b \).

Now suppose \( a \leq_{Rs} b \). Then for some \( c \in S^1 \), \( a = bc \). Since \( a \in T \) and \( S \setminus T \) is a left ideal, \( c \not\in S \setminus T \). So \( c \in T^1 \) and hence \( a \leq_{RT} b \). \( \Diamond \)

2) The proof of 2) is similar to the proof of 1).

3) Suppose \( S \setminus T \) is an ideal of \( S \). Of course \( a \leq_{HT} b \) implies \( a \leq_{Hs} b \).

Suppose \( a \leq_{Hs} b \). Then \( a \leq_{Rs} b \) and \( a \leq_{Ls} b \). By 1) and 2), \( a \leq_{RT} b \) and \( a \leq_{LT} b \), so \( a \leq_{HT} b \).

We also know \( a \leq_{JT} b \) implies \( a \leq_{Js} b \) since \( T \subseteq S \).

Suppose \( a \leq_{Js} b \). Then there exist \( c, d \in S^1 \) such that \( a = cbd \). Since \( a \in T \) and \( S \setminus T \) is an ideal, \( c, d \not\in S \setminus T \), so \( c, d \in T^1 \). Hence \( a \leq_{JT} b \). \( \Diamond \)

**Theorem:** Let \( S \) be a semigroup and let \( T \) be a subsemigroup of \( S \).

1) If \( S \setminus T \) is a left ideal of \( S \), then \( T \cap \text{Rmax}(S) \subseteq \text{Rmax}(T) \).
2) If $S \setminus T$ is a right ideal of $S$, then $T \cap L_{\max}(S) \subseteq L_{\max}(T)$.

3) If $S \setminus T$ is an ideal of $S$, then $T \cap J_{\max}(S) = J_{\max}(T)$; $T \cap H_{\max}(S) = H_{\max}(T)$; $T \cap R_{\max}(S) = R_{\max}(T)$; and $T \cap L_{\max}(S) = L_{\max}(T)$.

Note: The preceding example 2 does not satisfy the condition in part 3 of the theorem, yet it does satisfy the conclusion.

Proof: 1) Let $x \in T \cap R_{\max}(S)$ and suppose $y \in T$ such that $x \leq R_T y$. Then $x \leq R_S y$, so $x R_S y$. Thus $y \leq R_S x$, and so by the lemma $y \leq R_T x$. Hence $x R_T y$.

2) The proof of this part is analogous to that for part 1.

3) The proof that $T \cap J_{\max}(S) \subseteq J_{\max}(T)$ is similar to the proof in part 1.

Let $x \in J_{\max}(T)$. If $y \in S$ and $x \leq J_S y$, then by the lemma $x \leq J_T y$, so $x J_T y$ and hence $x J_S y$. So $x \in J_{\max}(S)$.

The proofs for $H, R,$ and $L$ are similar to the proof for $J$.$\diamond$

The answer to the following question is still unknown. Suppose $S$ is a semigroup with a subsemigroup $T$ such that $S \setminus T$ is a right ideal and $T \cap L_{\max}(S)$ is not empty. Then must $T \cap L_{\max}(S) = L_{\max}(T)$?

§3.4 PRODUCTS

In this section we consider the relationships between the two sets $Max(S \times T)$ and $Max(S) \times Max(T)$ for semigroups $S$ and $T$ and for the $R, L,$ and $H$ quasiorders.

Except in the examples, $S$ and $T$ are arbitrary semigroups. The $Max$ sets are with respect to the $R, L,$ and $H$ quasiorders. The proofs are given for $R$, and the proofs for $L$ and $H$ are similar.
Notation: By \( S^2 \), we mean \( \{ab : a, b \in S\} \).

**Lemma 1:** The inclusion \( S \setminus S^2 \subseteq \text{Max}(S) \) holds. That is, \( S \setminus \text{Max}(S) \subseteq S^2 \).

**Proof:** Suppose \( x \notin R\text{max}(S) \). Then for some \( y \in S \), \( x \leq_R y \) and \( y \not\leq_R x \). So there exists \( a \in S^1 \) such that \( x = ya \). Since \( x \neq y \), \( a \neq 1 \), so \( x \in S^2 \).

**Example 1:** Let \( I_u \) be the usual thread semigroup. Notice that \( (I_u)^2 = I_u \) and \( \text{Max}(I_u) = \{1\} \subset (I_u)^2 \).

**Example 2:** Let \( S = [0, \frac{1}{2}] \). Then \( S^2 = [0, \frac{1}{4}] \) and \( \text{Max}(S) = (\frac{1}{4}, \frac{1}{2}] \). So \( S \) is the disjoint union of \( S^2 \) and \( \text{Max}(S) \).

**Example 3:** Let \( T = \{0, a, x, f\} \) with multiplication given by

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Then \( T^2 = \{0, x, f\} \) and \( \text{Max}(T) = \{a, x, f\} \), so \( \emptyset \neq T \setminus T^2 \subseteq \text{Max}(T) \).

**Lemma 2:** If \( x \in T \setminus T^2 \), then \( S \times \{x\} \subseteq \text{Max}(S \times T) \) for \( R \), \( L \), and \( H \).

**Proof:** First notice that \( (S \times T)^2 = S^2 \times T^2 \). Then since \( (S \times T)^2 \) and \( S \times (T \setminus T^2) \) are disjoint, \( S \times (T \setminus T^2) \subseteq (S \times T) \setminus (S \times T)^2 \subseteq \text{Max}(S \times T) \) by lemma 1.

**Theorem 1:** For \( R \), \( L \), and \( H \), \( \text{Max}(S) \times \text{Max}(T) \subseteq \text{Max}(S \times T) \).

**Proof:** Let \( x \in R\text{max}(S) \) and \( y \in R\text{max}(T) \) and \( (a, b) \in S \times T \) such that
(x, y) \leq_{R_{S \times T}} (a, b). \text{ Show that } (x, y) R_{S \times T} (a, b).

There exists \((p, q) \in (S \times T)^1 \subseteq S^1 \times T^1\) such that \((x, y) = (a, b)(p, q)\). Since \(p \in S^1\) and \(q \in T^1\), then \(x = ap\) and \(y = bq\) imply that \(x \leq_{R_S} a\) and \(y \leq_{R_T} b\). Thus \(xR_SA\) and \(yR_TB\) and so there exist \(s \in S^1\) and \(t \in T^1\) such that \(a = xs\) and \(b = yt\).

Case 1: Assume \(s \in S\) and \(t \in T\). Then \((s, t) \in S \times T \subseteq (S \times T)^1\). So \((a, b) = (x, y)(s, t)\) and hence \((a, b) R_{S \times T} (x, y)\).

Case 2: Assume \(s = 1_{S^1}\) and \(t = 1_{T^1}\). Then \((a, b) R_{S \times T} (x, y)\) since \(x = a\) and \(y = b\).

Case 3: It is left to assume \(s = 1_{S^1} \notin S\) and \(t \neq 1_{T^1}\). In this case, \(x = a\).

If \((p, q) = 1_{(S \times T)^1}\) then \((a, b) = (x, y)\), so \((a, b) R_{S \times T} (x, y)\).

So assume \((p, q) \neq 1_{(S \times T)^1}\). Since \(1_{S^1} \notin S\), then \(p \neq 1_{S^1}\). Since \(x = ap = xp\) and \(b = yt\), we have \((a, b) = (x, b) = (xp, yt) = (x, y)(p, t)\) where \(p \neq 1_{S^1}\) and \(t \neq 1_{T^1}\).

Then \((a, b) \leq_{R_{S \times T}} (x, y)\) since \((p, t) \in S \times T \subseteq (S \times T)^1\), so \((a, b) R_{S \times T} (x, y)\).\)

**Theorem 2:** For semigroups \(S\) and \(T\),

\[ \text{Max}(S \times T) = (\text{Max}(S) \times \text{Max}(T)) \bigcup (S \times (T \setminus T^2)) \bigcup ((S \setminus S^2) \times T). \]

Hence, \(\text{Max}(S \times T) \subseteq (\text{Max}(S) \times T) \bigcup (S \times \text{Max}(T))\).

**Proof:** Let \(A = S \times (T \setminus T^2)\) and \(B = (S \setminus S^2) \times T\). By lemma 2 and theorem 1, \((\text{Max}(S) \times \text{Max}(T)) \bigcup A \bigcup B \subseteq \text{Max}(S \times T)\).

Let \((x, y) \in R_{\text{Max}}(S \times T)\).

If \(x \notin S^2\), then \((x, y) \in B\). Similarly, if \(y \notin T^2\), then \((x, y) \in A\). So it suffices to assume that \(x \in S^2\) and \(y \in T^2\) and show that \((x, y) \in R_{\text{Max}}(S \times R_{\text{Max}}(T))\).

Let \(a \in S\) such that \(x \leq_{R_S} a\) and show that \(xR_SA\). Let \(b \in S^1\) such that \(x = ab\).

If \(b = 1_S \notin S\), then \(x = a\), so \(xR_SA\).

If \(b \in S\), then since \(y \in T^2\), there exist \(p, q \in T\) such that \(y = pq\). We have \((x, y) = (a, p)(b, q)\), which implies \((x, y) \leq_{R_{S \times T}} (a, p)\) since \((b, q) \in S \times T \subseteq (S \times T)^1\). Because \((x, y) \in R_{\text{Max}}(S \times T)\), then \((x, y) R_{S \times T} (a, p)\). So there exists
$(s,t) \in (S \times T)^1 \subseteq S^1 \times T^1$ such that $(a,p) = (x,y)(s,t)$. That is, $a = xs$ and $s \in S^1$. Hence $xRsa$.

Therefore $x \in Rmax(S)$. Similarly, $y \in Rmax(T)$, so $(x,y) \in Rmax(S) \times Rmax(T)$.$\diamond$

As a result, if $S = S^2$ and $T = T^2$, then $\text{Max}(S \times T) = (\text{Max}(S) \times \text{Max}(T))$. Hence, if $S$ and $T$ are monoids, then $\text{Max}(S \times T) = (\text{Max}(S) \times \text{Max}(T))$.

**Example 4:** Let $S = [0, \frac{1}{2}] \subseteq I_u$ and let $T$ be the semigroup $\{0, a, x, f\}$ of example 3. Then

- $\text{Max}(I_u \times S) = I_u \times \left(\frac{1}{4}, \frac{1}{2}\right)$
- $\text{Max}(I_u \times T) = (I_u \times a) \bigcup \{(1,x),(1,f)\}$
- $\text{Max}(S \times T) = \left(\frac{1}{4}, \frac{1}{2}\right) \times (S \times \{a\})$
- $\text{Max}(T \times T) = (\{a, x, f\} \times \{a, x, f\}) \bigcup (\{a\} \times T) \bigcup (T \times \{a\})$
CHAPTER 4: MAX SETS AND MIN SETS

If the $J$ maximal and $J$ minimal sets meet, then each must equal the entire semigroup. However, the $H$ maximal and the $H$ minimal sets may intersect non-trivially, in which case the intersection is a union of groups.

§4.1 INTERSECTIONS

If $S$ is a group, then for each of the four quasiorders, $Max(S) = Min(S) = S$. Under what conditions can we have $Max(S) \cap Min(S) \neq \emptyset$?

**Theorem 1**: Let $S$ be a semigroup. Then $Jmax(S) \cap Jmin(S) \neq \emptyset$ if and only if $Jmax(S) = Jmin(S) = S$.

**Proof**: Assume $Jmax(S) \cap Jmin(S) \neq \emptyset$ and fix $x \in Jmax(S) \cap Jmin(S)$. For any $y \in S$, $yxy \leq x$, so $yxy \in Jmin(S)$. Since $y \leq yxy$, then $yxyy$. Hence $xJy$, so $S$ is a single $J$ class.

**Corollary**: If $S$ is normal, then for each of the four quasiorders $Max(S) \cap Min(S) \neq \emptyset$ if and only if $Max(S) = Min(S) = S$.

**Example 1**: Let $S = \{a, b\}$ under left trivial multiplication. Notice that $S$ is not a group.

Also notice that $aJb$, $H = \triangle S = R$, and $aLb$. Hence $S = Max(S) = Min(S)$ for each of the four quasiorders.

**Example 2**: Let $S$ be the three element semigroup $\{a, b, c\}$ with multiplication given by
Then $R_{max}(S) = \{a, c\} = H_{max}(S)$ and $R_{min}(S) = \{a, b\} = H_{min}(S)$. So $\emptyset \neq R_{max}(S) \cap R_{min}(S) \neq S$ and $\emptyset \neq H_{max}(S) \cap H_{min}(S) \neq S$.

**Theorem 2:** Let $S$ be a semigroup. If $H_{max}(S) \cap H_{min}(S) \neq \emptyset$, then this intersection is a union of groups.

**Proof:** Let $x \in H_{max}(S) \cap H_{min}(S)$. Since $x \leq x^2$, then $xHx^2$, so $H_x(S)$ is a group. ♦

Notice that the following statements hold for each of the four quasiorders:

1) If $0 \in S$ and $Max(S) \cap Min(S) \neq \emptyset$, then $Min(S) = Max(S) = S = \{0\}$.

2) If $1 \in S$ and $Max(S) \cap Min(S) \neq \emptyset$, then $Min(S) = Max(S) = S = \text{the class of } 1$.

**Example 3:** We conclude this section with an example of a commutative semigroup in which $\{Max(S), Min(S)\}$ is a partition of $S$.

Let $S$ consist of a zero, $0$, and at least one other element, together with zero multiplication (For all $x, y \in S$, $xy = 0$). Then $Min(S) = \{0\}$ and $Max(S) = S \setminus \{0\}$.

### §4.2 SUBGROUPS

Let $S$ be a semigroup with a subgroup $T$. Recall that $T$ equals each of its eight extreme sets.

If $a, b \in T$, then $aJ_S b$, $aH_S b$, $aR_S b$, and $aL_S b$, so $T$ is contained in a single class for each of the four quasiorders in $S$. Thus if $T$ meets any one of the eight extreme
sets of $S$, then $T$ is contained in that set.
As the $H$ maximal set of a monoid, the group of units has the property that every element of the semigroup lies under an element of the group of units in the $H$ ordering. We extend this idea to non-monoids and the $H$ maximal set in general. This is useful in characterizing homomorphisms which preserve the sets of maximal elements.

§5.1 CHARACTERIZATION

A semigroup $S$ is said to be $H$ paved if for all $x \in S$ there exists $y \in Hmax(S)$ such that $x \leq_H y$. Define $J$ paved, $R$ paved, and $L$ paved semigroups similarly. By a paved semigroup we mean one which is $J$ paved, $H$ paved, $R$ paved, and $L$ paved.

Theorem: Every finite semigroup is paved.

Proof: Let $S$ be a finite semigroup. Assume $S$ is not $H$ paved. Then there is an element $x \in S$ such that for all $y \in S$, $x \leq_H y \Rightarrow y \notin Hmax(S)$. Since $x \notin Hmax(S)$, there exists $x \leq_H x_1$ and $x$ is not $H$ related to $x_1$. Since $x_1 \notin Hmax(S)$, there is an element $x_2 \in S$ such that $x \leq_H x_1 \leq_H x_2$ and $x_1$ is not $H$ related to $x_2$. Continuing, we get an infinite sequence of distinct elements of $S$, a contradiction. Hence $S$ is $H$ paved.

Similar proofs hold for $J, R,$ and $L$. ◊

Also notice that every monoid is paved.
§5.2 HOMOMORPHISMS

Recall that homomorphisms preserve each of the four quasiorders. We now address the question: Which homomorphisms preserve the sets of maximal elements?

**Theorem:** Let $S$ be an $H$ paved semigroup and $\phi : S \to T$ a surmorphism, where $T$ is a semigroup with trivial $H$ classes. Then $Hmax(T) \subseteq \phi(Hmax(S))$.

Corresponding statements hold for $J$, $R$, and $L$.

**Proof:** Let $t \in Hmax(T)$. Let $x \in S$ such that $\phi(x) = t$. There is an element $y \in S$ such that $y \in Hmax(S)$ and $x \leq_H y$. Since $t = \phi(x) \leq_H \phi(y)$, then $tH\phi(y)$. Hence $t = \phi(y) \in \phi(Hmax(S))$.

The proofs for $J$, $R$, and $L$ are analogous to the proof for $H$. ♦

**Example 1:** In this example, $S$ and $T$ are paved commutative semigroups and $\phi : S \to T$ is a surmorphism such that $Max(T) \subseteq \phi(Max(S))$.

Let $S = \{(x,y) \in I_u \times I_u : xy = 0\}$. Define $\phi : S \to I_u$ by $\phi(0,y) = 0$ for each $y \in I_u$ and $\phi(x,0) = x$ for each $x \in I_u$. Notice that $\phi$ is a surmorphism. Also, $Max(S) = \{(1,0),(0,1)\}$, $\phi(Max(S)) = \{0,1\}$, and $Max(I_u) = \{1\}$.

**Example 2:** In this example, $S$ and $T$ are paved semigroups and $\phi : S \to T$ is a surmorphism such that $\phi(Max(S)) \subseteq Max(T)$.

Let $T$ be a group with identity $e$ and at least one other element. Let $S = T \cup \{1\}$, where $1$ is a super identity for $S$. Let $\phi : S \to T$ be defined by $\phi(1) = e$ and $\phi(x) = x$ for $x \neq 1$. Then $\{e\} = \phi(Max(S)) \subseteq Max(T) = T$.

**Example 3:** The homomorphic image of a paved semigroup needn’t be paved.

For example, let $S$ and $T$ be the illustrated semilattices:
Define $\phi : S \to T$ by $\phi(a_n) = x_n = \phi(x_n)$ for each $n$. Then $\phi$ is a surmorphism.

Notice that $S$ is paved but $\text{Max}(T) = \emptyset$. 
PART 2: THE TRANSLATIONAL HULL

The set of $H$ maximal elements fits nicely into the theory of the translational hull of a semigroup. In considering the translational degree of a semilattice, this question arose: Is the degree of a subsemilattice always less than or equal to the degree of its supersemilattice? We provide examples to illustrate that the answer is no, and we give conditions under which the answer is yes.

CHAPTER 6: THE TRANSLATIONAL HULL
AND THE CONDITION $S = ESE$

The concept of the translational hull of a semigroup is reviewed. If an $H$ paved semigroup has a separating set or a determination set, then the set of $H$ maximal elements is also such a set.

If $S$ is an $H$ paved semigroup and the set of $H$ maximal elements is a union of groups, then $S = ESE$. In general, however, the condition that $S$ be $H$ paved and the condition that $S = ESE$ are independent.

§6.1 THE TRANSLATIONAL HULL OF A SEMIGROUP

Much work has been done on the translational hull and its applications to ideal extensions. For an algebraic treatment of the translational hull, see Clifford and Preston, volume I.

Another good reference is Chapter 4 of Carruth, Hildebrant, and Koch, 1986. The terminology and notation herein presented is consistent with this reference.

Let $S$ be a semigroup. A left translation of $S$ is a map $\lambda : S \rightarrow S$ such that
\( \lambda(xy) = (\lambda x)y. \) We call \( \rho : S \to S \) a right translation of \( S \) if \( (xy)\rho = x(y\rho) \). A bitranslation \( \omega \) of \( S \) is a pair \( \omega = (\lambda, \rho) \) consisting of a left translation \( \lambda \) and a right translation \( \rho \) satisfying \( x(\lambda y) = (x\rho)y \) for all \( x, y \in S \). For \( a \in S \), the inner bitranslation \( \omega_a \) is the pair \( (\lambda_a, \rho_a) \) where \( \lambda_a(x) = ax \) and \( (x)\rho_a = xa \). The maps \( \lambda_a \) and \( \rho_a \) are called inner left and right translations.

A semigroup \( S \) is reductive if and only if whenever \( xa = xb \) for every \( x \in S \) then \( a = b \) and whenever \( ax = bx \) for every \( x \in S \) then \( a = b \). Also, \( S \) is weakly reductive if and only if whenever \( xa = xb \) and \( ax = bx \) for all \( x \in S \), then \( a = b \).

In terms of translations, \( S \) is reductive if \( a \neq b \Rightarrow \lambda_a \neq \lambda_b \) and \( \rho_a \neq \rho_b \). Also, \( S \) is weakly reductive if \( a \neq b \Rightarrow \lambda_a \neq \lambda_b \) or \( \rho_a \neq \rho_b \).

A subset \( A \) of \( S \) is called a separating set for \( S \) if and only if whenever \( a \neq b \) in \( S \), there are elements \( x, y \in A \) such that \( xa \neq xb \) and \( ay \neq by \). The subset \( A \) is called a determination set for \( S \) if and only if whenever \( a \neq b \) in \( S \), there is an element \( x \in A \) such that \( xa \neq xb \) or \( ax \neq bx \).

In terms of translations, \( A \) is a separating set if for all \( a, b \in S \), \( \lambda_a|_A = \lambda_b|_A \) implies \( a = b \) and also \( \rho_a|_A = \rho_b|_A \) implies \( a = b \). The subset \( A \) is a determination set if \( \omega_a|_A = \omega_b|_A \) implies \( a = b \).

Notice that \( S \) is reductive if and only if \( S \) contains a separating set, and \( S \) is weakly reductive if and only if \( S \) contains a determination set.

**Theorem 1**: Let \( S \) be an \( H \) paved semigroup. Then

1) If \( S \) is reductive then \( H\text{max}(S) \) is a separating set for \( S \).

2) If \( S \) is weakly reductive, then \( H\text{max}(S) \) is a determination set for \( S \).

**Proof**: 1) Let \( a, b \in S \) such that \( a \neq b \). Since \( S \) is reductive, there are elements \( z, w \in S \) such that \( az \neq bz \) and \( wa \neq wb \). Since \( S \) is \( H \) paved, there are elements \( x, y \in H\text{max}(S) \) such that \( z \leq_H x \) and \( w \leq_H y \). Let \( t, r \in S^1 \) such that \( z = xt \) and \( w = ry \). Then \( axt \neq bxt \) and \( rya \neq ryb \), so \( ax \neq bx \) and \( ya \neq yb \). Thus \( H\text{max}(S) \) is a separating set for \( S \).
2) Again let \( a \neq b \) in \( S \). If \( S \) is weakly reductive, then there is an element \( z \in S \) such that \( az \neq bz \) or \( za \neq zb \). Since \( S \) is \( H \) paved, there exists \( z \in H_{\text{max}}(S) \) such that \( z \leq_H x \). Let \( t, r \in S^1 \) such that \( z = xt = rx \). Then \( axt \neq bxt \) or \( rxa \neq rxb \).

Therefore, \( ax \neq bx \) or \( xa \neq xb \), so \( H_{\text{max}}(S) \) is a determination set for \( S \). ♦

By definition, the \textit{translational hull} of a semigroup \( S \) is the set of all bitranslations on \( S \) and is denoted \( \Omega(S) \).

\textbf{Theorem 2:} (Clifford and Preston) If \( S \) is a weakly reductive semigroup, the map \( \Psi : S \rightarrow \Omega(S) \) defined by \( \Psi(a) = \omega_a \) is an isomorphism of \( S \) onto an ideal of \( \Omega(S) \).

\section{6.2 THE CONDITION \( S = ES = SE \)}

The condition \( S = ES = SE \) appears in many of the theorems in the recent paper \textit{The Translational Degree of a Semigroup} by J. A. Hildebrant. In this section we describe the relationships between the condition that \( S \) is \( H \) paved and the condition that \( S = ESE \).

\textbf{Theorem:} Let \( S \) be an \( H \) paved semigroup. If \( H_{\text{max}}(S) \) is a union of groups, then \( S = \bigcup \{ eSe : e \in E(S) \} = ESE \).

\textbf{Proof:} Let \( x \in S \) and \( a \in H_{\text{max}}(S) \) such that \( x \leq_H a \). Since \( H_a \) is a group, there is an idempotent \( e \) such that \( eHa \). Now \( x = ay = za \) for some \( y, z \in S^1 \), so \( x = eay = zae \). That is, \( x = ex = xe \in eSc \). ♦

\textbf{Example 1:} Let \( S = \{a, b, c\} \) with multiplication given by
In this semigroup, \( H_{\text{max}}(S) = \{a, c\} \) is not a union of groups.

**Corollary 1:** Let \( S \) be an \( H \) paved semigroup. If \( H_{\text{max}}(S) \) is a subsemigroup of \( S \), then \( S = ES = SE \).

**Corollary 2:** Let \( S \) be a semigroup. If \( S \) is \( n \)-divisible and \( H \) paved, then \( S = ES = SE \).

The following examples illustrate that the condition that \( S \) is \( H \) paved and the condition \( S = ESE \) are independent.

**Example 2:** Let \( S \) be the following semilattice:

\[
\begin{array}{ccc}
   & \nearrow & \\
   & 2 & \\
   S & 4 & \\
   & \nearrow & \\
   & 3 & \\
   & \nearrow & \\
    & 1 & \\
\end{array}
\]

Then \( E(S) = S \) and \( H_{\text{max}}(S) = \{a_0\} \). Hence \( S = ESE \) but \( S \) is not \( H \) paved.

**Example 3:** Let \( S \) be \( \{0, a\} \) under \( 0 \) multiplication (all products are \( 0 \)). Then \( SE = \{0\} \neq S \), but \( H_{\text{max}}(S) = \{a\} \) and \( 0 \leq_H a \), so \( S \) is \( H \) paved.
CHAPTER 7: THE TRANSLATIONAL DEGREE
OF A FINITE SEMILATTICE

The translational degree of a semilattice is reviewed. The examples given lead us to the theorem that if a subsemilattice has the same maximal set as its supersemilattice, then the degree of the subsemilattice is no larger than the degree of the supersemilattice.

§7.1 THE TRANSLATIONAL DEGREE

If $S$ is weakly reductive, we have noted that $S$ is isomorphic to an ideal of $\Omega(S)$ by the map $a \rightarrow \omega_a$, the inner bitranslation defined by $a$. The translational degree of $S$ is defined to be $\text{deg}(S) = \#\Omega(S) - \#S = \#\{\omega \in \Omega(S) : \omega \neq \omega_a \text{ for any } a \in S\}$. This concept was first defined in the 1984 paper by Hildebrant. It provides a measure of the relative size of the semigroup and its translational hull.

Restricting our attention to finite semilattices, theorem 5.13 of the 1976 paper by Hildebrant, Lawson, and Yeager gives us the following result.

Theorem: Let $S$ be a finite semilattice. Let $\text{Max}(S) = \{e_1, e_2, ..., e_n\}$. Let $K = \{x \in e_1S \times e_2S \times ... \times e_nS : \pi_j(x) \cdot e_k = \pi_k(x) \cdot e_j \text{ for all } j, k = 1, 2, ..., n\}$.

Then $K$ is a subsemigroup of $\Pi e_iS$ and $K$ is isomorphic to $\Omega(S)$. Further, $\omega_a$ in $\Omega(S)$ corresponds to $(e_1a, e_2a, ..., e_na)$ in $K$.

Example 1: Let $S_1$ be the illustrated semilattice.

\[
\begin{array}{cccc}
a & b & c & d \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
S_1 & \downarrow & \downarrow & \downarrow & \\
\end{array}
\]
Notice that $aS_1 = \{a, ab, abc, 0\}; bS_1 = \{b, ab, bc, abc, bcd, 0\}; cS_1 = \{c, bc, cd, abc, bcd, 0\}$; and $dS_1 = \{d, cd, bcd, 0\}$.

Then $K = \{(a,b,c,d),(a,b,c,cd),(a,b,be,bed),(a,ab,abc,0),(ab,b,c,d),
(ab,b,c,cd),(ab,b,be,bcd),(ab,ab,abc,0),(abc,be,c,d),(abc,be,c,cd),
(abc,be,be,bcd),(abc,abe,abe,0),(0,bcd,cd,d),(0,bcd,cd,ed),(0,bed,bed,bcd),
(0,0,0,0)\}$

Thus $deg(S) = 16 - 10 = 6$.

**Example 2:** By a similar computation, we find the degrees of the following semilattices.

$deg(S_2) = 20 - 11 = 9.$

$deg(S_3) = 25 - 12 = 13.$
\[ \text{deg}(S_4) = 16 - 7 = 9. \] This degree can also be found using the product theorem in Hildebrant, 1984.

\[ \begin{array}{c}
S_4 \\
\downarrow \\
S_5 \\
\downarrow \\
\downarrow \\
\downarrow \\
\end{array} \]

\[ \text{deg}(S_5) = 16 - 8 = 8. \]

\subsection{Subsemilattices}

Throughout this section, \( S \) and \( T \) will be finite semilattices such that \( S \) is a subsemilattice of \( T \). Notice that semilattices are reductive. We address the question: under what conditions is \( \text{deg}(S) \leq \text{deg}(T) \)?

In the examples presented in the preceding section, we have

\[ S_4 \hookrightarrow S_5 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow S_3. \]

Notice that \( S_5 \) is a subsemilattice of \( S_1 \), but \( \text{deg}(S_5) = 8 \) and \( \text{deg}(S_1) = 6 \). Thus we need some additional condition to ensure that \( \text{deg}(S) \leq \text{deg}(T) \). Since \( \text{Max}(S_5) \nsubseteq \text{Max}(S_1) \), we might consider the condition \( \text{Max}(S) \subseteq \text{Max}(T) \). However, \( \text{deg}(S_4) \nsubseteq \text{deg}(S_1) \) and \( \text{Max}(S_4) \nsubseteq \text{Max}(S_1) \). This leads us to the following theorem.

\[ \textbf{Theorem:} \text{ If } S \text{ and } T \text{ are finite semilattices such that } S \text{ is a subsemilattice of } T \text{ and } \text{Max}(S) = \text{Max}(T), \text{ then } \text{deg}(S) \leq \text{deg}(T). \]

\[ \textbf{Proof:} \text{ We use the theorem in the previous section. Let } \text{Max}(S) = \text{Max}(T) = \{e_1, e_2, ..., e_n\}. \text{ Let } A = e_1S \times e_2S \times ... \times e_nS \text{ and } B = e_1T \times e_2T \times ... \times e_nT. \text{ Then } A \subseteq B. \]

\[ \text{If } t \in \Omega(S), \text{ then } t \text{ corresponds to } (t_1, t_2, ..., t_n) \in A \text{ such that } e_i t_j = e_j t_i \text{ for } i, j = 1, 2, ..., n. \text{ Since } A \subseteq B, \text{ then } t \in \Omega(T). \text{ Now } S \hookrightarrow \Omega(S) \hookrightarrow \Omega(T) \text{ and } T \hookrightarrow \Omega(T). \text{ If we show that } \Omega(S) \cap T = S, \text{ then } \Omega(S) \setminus S \subseteq \Omega(T) \setminus T, \text{ so } \text{deg}(S) \leq \text{deg}(T). \]
Let \( x \in T \) such that \((x e_1, x e_2, \ldots x e_n)\) corresponds to an element of \( \Omega(S) \). Show \( x \in S \). For all \( j \), \( x e_j \in e_j S \subseteq S \). Since \( T \) is paved, for some \( k \) between 1 and \( n \), \( x \leq_{HT} e_k \). Then \( x e_k = x \) is an element of \( S \). Hence, \( \Omega(S) \cap T = S \).
PART 3: TOPOLOGICAL RESULTS AND BOUNDARIES

We now turn our attention to topological semigroups. We provide an example to illustrate that the set of nonmaximal elements of a compact connected semigroup can be extended in more than one way.

For certain topological notions of boundary, it is known that the group of units of a topological monoid must lie entirely in the boundary. We show that this result does not extend to the set of $H$ maximal elements of a non-monoid.

CHAPTER 8: EXTREME SETS AND COMPACT SEMIGROUPS

In a compact semigroup, the sets of minimal elements are nonempty. Every compact semigroup is paved, but not every paved semigroup is compact.

It is known that if two compact connected monoids have topologically isomorphic sets of non-units, then the monoids are topologically isomorphic. We present an example which illustrates two compact connected semigroups which are not topologically isomorphic but which have topologically isomorphic sets of nonmaximal elements.

§8.1 MIN SETS

Theorem: If $S$ is a compact semigroup, then $\text{Min}(S)$ is nonempty for each of the four quasiorders.

Proof: By Theorem 1.29 on page 28 of volume I of Carruth, Hildebrant, and Koch, every compact semigroup contains a minimal right ideal and a minimal left ideal. This theorem then follows from our characterization of min sets and our
corollary that if \( R_{\text{min}}(S) \neq \emptyset \) and \( L_{\text{min}}(S) \neq \emptyset \) then each of the four min sets equals the minimal ideal. \( \diamond \)

§8.2

**TOPOLOGICAL PAVED SEMIGROUPS**

**Theorem:** Every compact semigroup is paved.

**Proof:** Let \( S \) be a compact semigroup. We will show that \( S \) is \( R \)-paved. The proofs for \( L \), \( H \), and \( J \) are similar.

Let \( x \in S \) and let \( A = \{ y \in S : x \leq_{R} y \} \). Then \( A \) is not empty since \( x \in A \). We will show that every chain in \( A \) has an upper bound. Then by Zorn’s Lemma, \( A \) has a maximal element.

Let \( C \) be a chain \( y_{1} \leq_{R} y_{2} \leq_{R} y_{3} \leq_{R} y_{4} \leq_{R} \ldots \) in \( A \). Since \( S \) is compact, the sequence \( \{y_{n}\} \) clusters to a point \( y \in S \). Let \( \{y_{n}\} \) be a subsequence converging to \( y \).

Each \( y_{n,i} \leq_{R} y \): Fix \( i \). For each \( j > i \), there is an element \( a_{j} \in S \) such that \( y_{n,i} = y_{n,i}a_{j} \). Taking subsequences if necessary, \( \{a_{j}\} \) converges to \( a \in S \). Then \( y_{n,i} = y_{n,i}a_{j} \Rightarrow y_{a} \). Hence \( y_{n,i} = ya \), and so \( y_{n,i} \leq_{R} y \).

Now for \( y_{n} \in C \), there is an \( i \) such that \( y_{n} \leq_{R} y_{n,i} \), so \( y_{n} \leq_{R} y \). Thus \( y \) is an upper bound for the chain \( C \).

Let \( a \) be the maximal element in \( A \). Then \( x \leq_{R} a \). If \( a \leq_{R} b \), then \( b \in A \), so \( aRb \). Hence, \( a \in R_{\text{max}}(S) \). Thus \( S \) is \( R \)-paved. \( \diamond \)

**Example:** Not every paved semigroup is compact. For example, let \( S \) be the semilattice:
with the discrete topology. Then $S$ is paved and not compact.

McCharen's 1969 Ph.D. dissertation includes many topological results involving $R$, $L$, and $J$ maximal elements. In his first chapter, he often considers semigroups $S$ satisfying the conditions that $S$ is a continuum and $S^2 = S$. In his third chapter, he often assumes that $S$ is a continuum satisfying $S = ESE$. He notes that if $S$ is compact, then $S$ is what we call $R$, $L$, and $J$ paved.

Three interesting results from his first chapter concern compact semigroups. He shows that if $S$ is compact, then $J_{\text{max}}(S) \subseteq L_{\text{max}}(S)$. If $S$ also satisfies $S^2 = S$, then $L_{\text{max}}(S) \subseteq ES$ and $S = SES$.

§8.3 EXTENSIONS OF THE SET OF NONMAXIMAL ELEMENTS

It is well known that if $S$ is a monoid which is not a group, then $S\setminus H_S(1)$ is an ideal of $S$. If $S$ is also compact and connected, then $S\setminus H_S(1)$ is also dense in $S$. See volume I of the book by Carruth, Hildebrant, and Koch.

In the 1973 paper by Hildebrant and Lawson, it is shown that if $S$ is a topological monoid with dense ideal $I$, then the Bohr compactification of $I$ is topologically isomorphic to the Bohr compactification of $S$. By the uniqueness of the Bohr compactification, we can then conclude the following result: If $S$ and $T$ are compact connected monoids such that $S\setminus H_S(1)$ is topologically isomorphic to $T\setminus H_T(1)$, then
S is topologically isomorphic to T. This result was determined by C. Eberhart and J. Selden prior to the work of Hildebrant and Lawson, but was first published in the 1973 paper by Hildebrant and Lawson.

We now ask to what extent can we replace the groups of units by the sets of \( H \) maximal elements.

**Example 1:** Recall from section 3.1 that \( S \backslash H_{\text{max}}(S) \) need not be an ideal of \( S \). For example, let \( S \) be \( \{a, x, b\} \) with multiplication given by:

\[
\begin{array}{c|ccc}
& a & x & b \\
\hline
a & a & a & b \\
x & a & x & b \\
b & a & a & b \\
\end{array}
\]

Then \( H_{\text{max}}(S) = \{x, b\} \), and \( \{a\} \) is not an ideal of \( S \).

If we endow \( S \) with the discrete topology, then \( S \) is compact.

**Example 2:** Even if \( S \backslash H_{\text{max}}(S) \) is an ideal, it need not be dense. For example, let \( S \) be the interval \( [0, \frac{1}{2}] \). Then \( H_{\text{max}}(S) = (\frac{1}{4}, 1] \). Notice that this semigroup is compact and connected.

The following example shows that extensions of the set of nonmaximal elements need not be unique. Notice that the semigroup \( T \) is a compact connected semigroup and that \( T \backslash H_{\text{max}}(T) \) is not dense in \( T \).

**Example 3:** Let \( S \) be the min interval; that is, the semigroup \( ([0,1], \star) \), where \( x \star y = \min\{x, y\} \). Let \( T \) be the semigroup on \( [0,2] \) with multiplication given by \( x \star y = \min\{x', y'\} \), where \( a' = a \) if \( a \in [0, 1] \) and \( a' = 2 - a \) if \( a \in [1, 2] \). Notice that this multiplication is associative, continuous, and commutative.

Notice that \( S \) is a monoid and \( \text{Max}(S) = \{1\} \). In \( T \), \( r \leq_H t \) if and only if \( r = t \) or \( t \in [r, 2 - r] \). Hence, \( \text{Max}(T) = [1,2] \).
Now $S$ and $T$ are compact, connected semigroups. Both $S \setminus H_{max}(S)$ and $T \setminus H_{max}(T)$ equal the interval $[0,1)$ under min multiplication. However, $S$ and $T$ are not isomorphic since $S$ is a monoid and $T$ is not.

The results in the 1973 paper by Hildebrant and Lawson have the following corollary.

**Theorem:** Let $S$ and $T$ be compact semigroups such that $S \setminus H_{max}(S)$ is a dense ideal of $S$, $T \setminus H_{max}(T)$ is a dense ideal of $T$, and $S \setminus H_{max}(S)$ is topologically isomorphic to $T \setminus H_{max}(T)$. Then $S$ is topologically isomorphic to $T$. 
CHAPTER 9: BOUNDARIES OF COMPACT CONNECTED METRIC SEMIGROUPS

Five topological notions of boundary are compared with the set of $H$ maximal elements of a topological semigroup. We find no topological concept of boundary which must always contain the set of $H$ maximal elements. Hence, the fact that the group of units must be contained in the topological boundary of a topological monoid does not extend to the set of $H$ maximal elements of a non-monoid.

§9.1 NOTIONS OF BOUNDARY

We now describe various topological notions of boundary. Unless otherwise noted, $S$ is an arbitrary compact connected metric semigroup.

1. Cohomological Boundary: Let $\partial_c(S)$ be the set of points $p$ in $S$ which satisfy the following condition: For each open neighborhood $U$ of $p$ there exists an open neighborhood $V$ of $p$, $V \subseteq U$, such that the inclusion map $i : S\setminus V \to S$ induces an isomorphism $i^* : H^*(S) \to H^*(S\setminus V)$.

Such points are called marginal points by Hofmann and Mostert and edge points by Carruth, Hildebrant, and Koch (Volume II).

A slightly different concept of peripheral points appears in two 1970 papers by Lawson and Madison.

2. Maximum Distance from $M(S)$:

Let $S$ be a compact metric semigroup. Then there is a subinvariant metric $d$ on $S$ compatible with the topology of $S$. That is, there is a metric $d$ satisfying $d(ax, ay) \leq d(x, y)$ and $d(xa, ya) \leq d(x, y)$ for all $a, x, y \in S$. For a proof of the existence of such a metric, see Hofmann and Mostert.
Recall that every compact semigroup has a closed, completely simple minimal ideal M(S). (See Hofmann and Mostert.) Hence if S is compact, then for all four quasiorders, $\text{Min}(S) = M(S)$.

Since M(S) is compact, then for each $x \in S$, $d(x, M(S)) = \min \{d(x, y) : y \in M(S)\}$. (By definition, $d(x, M(S)) = \inf \{d(x, y) : y \in M(S)\} = \lim_{n \to \infty} d(x, y_n)$ for some sequence $\{y_n\} \subseteq M(S)$. Since M(S) is compact, $y_n \xrightarrow{f} y_0 \in M(S)$. Then $d(x, M(S)) = d(x, y_0)$.)

Define $\lambda : S \to [0, \infty)$ by $\lambda(x) = d(x, M(S))$. Then $\lambda$ is a continuous function on a compact set, so $\lambda(S)$ is compact. Let $r$ be the maximum value in $\lambda(S)$. Define $\partial_d(S) = \{x \in S : d(x, M(S)) = r\}$

**3: Maximum Diameter:** Recall that the diameter of a nonempty set A is $\delta(A) = \sup \{d(x, y) : x, y \in A\}$. (See Dugundji.)

If S is a compact metric semigroup, there exist $x, y \in S$ such that $\delta(S) = d(x, y)$.

Define $\partial_\delta(S) = \{x : \exists y \in S \text{ such that } d(x, y) = \delta(S)\}$.

**4. Usual Boundary in $\mathbb{R}^n$:** If S can be embedded into $\mathbb{R}^n$ and not into $\mathbb{R}^{n-1}$, let $\partial(S) = S \cap (\mathbb{R}^n \setminus S)^\prime$, the usual notion of boundary. (Here, $\prime$ indicates closure.) Of course, not all of the semigroups we are considering can be so embedded.

**5. Extreme Points of an Affine Semigroup:** We refer to the 1959 paper by Cohen and Collins for these concepts.

Let $X$ be a topological vector space over $\mathbb{R}$, and let $S$ be a convex subset of $X$. Let $m$ be a multiplication on $S$ which is continuous with respect to the topology inherited from $X$. Then $S$ is called an *affine semigroup* if for all $x, y, z \in S$ and for all $\alpha$ such that $0 \leq \alpha \leq 1$, we have
\[(ax + (1 - \alpha)y)z = \alpha(xz) + (1 - \alpha)(yz),\]

\[z(ax + (1 - \alpha)y) = \alpha(zz) + (1 - \alpha)(zy).\]

Let S be an affine semigroup. An extreme point of S is an element $x$ which is interior to no line segment of S. That is, if $x = \alpha y + (1 - \alpha)z$ and $0 < \alpha < 1$, $y, z \in S$, then $x = y = z$.

Denote the set of all extreme points of S by $\text{ext}(S)$.

**Example:** The following example illustrates that the boundary sets associated with notions 2 and 3 may depend upon the particular subinvariant metric employed.

Let $S = \{(-1,1] \times [-1,1]\} \subset \mathbb{R} \times \mathbb{R}$ under co-ordinate multiplication. Let $d_1$ be the usual metric on $\mathbb{R} \times \mathbb{R}$ restricted to S. Let $d_2(x, y) = \min\{d_1(x, y), 1\}$. Then $d_1$ is a subinvariant metric compatible with the topology on S. Also, $d_2$ is a metric equivalent to $d_1$. (See Murdeshwar.)

To see that $d_2$ is subinvariant, let $x, y, a \in S$. Since $d_1$ is subinvariant, we have $d_1(ax, ay) \leq d_1(x, y)$. If $1 \leq d_1(ax, ay)$, then $d_2(ax, ay) = 1 = d_2(x, y)$. If $d_1(ax, ay) \leq 1 \leq d_1(x, y)$, then $d_2(ax, ay) = d_1(ax, ay) \leq 1 = d_2(x, y)$. Otherwise, $d_1(ax, ay) \leq d_2(x, y) \leq 1$, in which case $d_2(ax, ay) \leq d_2(x, y)$. Hence $d_2$ is subinvariant.

Notion 2: Notice that $M(S) = (0,0)$, $\tau_1 = \sqrt{2}$, and $\tau_2 = 1$. Then $\partial _{d_1}(S)$ is the set $\{(1,1), (1,-1), (-1,1), (-1,-1)\}$, and $\partial _{d_2}(S)$ is the complement of the open unit disk in S.

Notion 3: Here, $\delta_1(S) = 2\sqrt{2}$, and $\delta_2(S) = 1$. Hence, we have $\partial _{\delta_1}(S) = S$ and $\partial _{\delta_2}(S) = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$.

So even though $d_1$ and $d_2$ are equivalent subinvariant metrics for S, $\partial _{d_1}(S) \neq \partial _{d_2}(S)$ and $\partial _{\delta_1}(S) \neq \partial _{\delta_2}(S)$.

It is considered desirable for notions of boundary to have the property that if S is a monoid, then $H_1(S)$ is contained in the boundary of S. In Carruth, Hildebrant, and Koch, Volume II, Corollary 1.8 states that if S is a compact connected monoid.
which is not a group, then $H_1(S) \subseteq \partial_c(S)$. A theorem by Wendel appearing in the 1959 paper by Cohen and Collins says that if $S$ is a compact affine topological monoid and $X$ is locally convex, then $H_1(S) \subseteq ext(S)$.

**Theorem:** If $S$ is a compact metric monoid with $0$, then $H_i(S) \subseteq \partial_d(S)$.

**Proof:** First notice that $1$ is in $\partial_d(S)$: Let $a \in \partial_d(S)$. Then $d(a,0) = d(a \cdot 1, a \cdot 0) \leq d(1,0)$, so $1 \in \partial_d(S)$. Now if $xx^{-1} = x^{-1}x = 1$, then $d(1,0) = d(xx^{-1},0x^{-1}) \leq d(x,0)$, so $x \in \partial_d(S)$ as well.

Notice that the affine triangle example in the following section shows that in the above theorem, equality need not hold.

Now suppose $S$ is a monoid. Under what conditions must $H_1(S) \subseteq \partial_d(S)$? If $S$ can be embedded into $\mathbb{R}^n$, must $H_1(S) \subseteq \partial(S)$?

§9.2 **COMPARISONS**

In this section, we compare the max sets with the various topological notions of boundary described in the preceding section.

**Example 1:** Let $S$ be the unit disk with the usual metric $d(z,w) = |z - w|$. Then $Max(S) = H_1(S) = \{z : |z| = 1\} = \partial(S)$. Since $M(S) = \{0\}$, then $\partial_d(S) = \partial(S)$. Notice that $S$ is affine and that $ext(S) = \partial(S)$. Also, $\delta(S) = 2$, so we have $\partial_d(S) = \partial(S) = \partial_e(S) = Max(S) = \partial_c(S) = ext(S)$.

**Example 2:** Let $S$ be $I_u$. Then $Max(S) = \partial_d(S) = \{1\} \subset \{0,1\} = \partial(S) = \partial_e(S) = \partial_c(S) = ext(S)$.

**Example 3:** Let $S$ be the unit circle in the complex plane. Then $S = Max(S) =$...
\[ \partial(S) \text{ and } \partial_c(S) = 0. \] Since \( S \) is not convex, it is not affine. If \( d \) is the usual metric, \( \partial_d(S) = \partial_c(S) = S. \) So \( \partial_c(S) \subset Max(S) = \partial(S) = \partial_d(S) = \partial_c(S). \)

**Example 4:** Let \( S = [0, \frac{1}{2}] \subset I_u. \) Then \( Max(S) = (\frac{1}{4}, \frac{1}{2}], \) and \( \partial_d(S) = \{ \frac{1}{2} \} \subset \partial_c(S) = \partial(S) = ext(S) = \{0, \frac{1}{2}\}. \)

**Example 5:** Let \( T \) be the affine triangle; i.e., the set \( \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \geq 0, y \geq 0, \text{ and } x + y < 1\} \) with multiplication given by \( (x, y)(a, b) = (xa, xb + y). \)

It is straightforward to show that \( T \) is affine and \( ext(T) = \{(0, 0), (0, 1), (1, 0)\}. \) Also, \( \{1, 0\} = Hmax(T) = \partial_d(S) \subset \partial_c(S) = \{1, 0, (0, 1)\} \subset ext(S) \subset \partial(S) = \partial_c(S). \)

**Example 6:** Let \( S = \{\{(x, y) : 0 < x < 1 \text{ and } |y| < \frac{3}{4}(1 - x)\}, \cdot \} \subset \mathbb{R} \times \mathbb{R}. \) Then \( \partial_d = \{(1, 0)\} \) and \( \partial_c = \{(0, \pm \frac{3}{4})\}. \)

Now \( (1, 0) \in Max(S) \) and \( (x, y) \in Max(S) \) whenever \( 0 \leq x < 1 \) and \( \frac{9}{16}(1 - x) < |y| \leq \frac{3}{4}(1 - x). \) To see this, fix such \( x \) and \( y \) and suppose there are elements \( (a, b), (c, d) \in S \) such that \((x, y) = (a, b)(c, d). \) Then \( x = ac, y = bd, |b| \leq \frac{3}{4}(1 - a) \) and \( |d| \leq \frac{3}{4}(1 - c). \) Then

\[
\frac{9}{16}(1 - x) < |y| = |bd| \leq \frac{9}{16}(1 - a - c + x)
\]

so \( a + c < 2x. \) But \( a \geq x \) and \( c \geq x, \) so \( a + c \geq 2x, \) a contradiction. Hence no such \((a, b), (c, d)\) exist.

Now \( S \) is affine and \( ext(S) = \{(0, \frac{3}{4}), (0, -\frac{3}{4}), (1, 0)\}. \) Since \( (0, 0) \) is not in \( Max(S), \) then \( Max(S) \) and \( \partial(S) = \partial_c(S) \) do not compare. Also, \( \partial_d(S) \) and \( \partial_c(S) \) do not compare. Each of \( \partial_d(S) \) and \( \partial_c(S) \) is properly contained in \( ext(S), \) and \( ext(S) \) is properly contained in each of \( Max(S) \) and \( \partial(S) = \partial_c(S). \)

Examples 4 and 5 show that \( Hmax(S) \) need not compare with any of the sets \( \partial_c(S), \partial_d(S), \partial_c(S), \partial(S), \) or \( ext(S). \) Hence, the fact that the group of units must be contained in the boundary of a topological monoid does not extend to the idea
that $H_{\text{max}}(S)$ must be contained in the boundary of a topological semigroup in general.

In example 6 we see that $\partial_d(S)$ and $\partial_e(S)$ need not compare. Also, examples 3 and 5 show that $\partial_c(S)$ and $\partial_d(S)$ do not compare and that $\partial_c(S)$ and $\partial_e(S)$ do not compare.

In all of the examples, $\partial_c(S) \subseteq \partial(S)$, $\partial_d(S) \subseteq \partial(S)$, and $\partial_e(S) \subseteq \partial(S)$. Notice that each of these three inclusions is strict in one of the above examples 3 and 6. In all of the affine examples, $\partial_d(S) \subseteq \text{ext}(S)$, $\partial_e(S) \subseteq \text{ext}(S)$, $\text{ext}(S) \subseteq \partial(S)$, and $\text{ext}(S) \subseteq \partial_c(S)$. All four of these inclusions are strict in example 6. Must these seven inclusions always hold?
PART 4: OTHER RESULTS

In the chapter on divisibility, we describe the extent to which the divisibility properties of a semigroup influence divisibility properties of the sets of minimal and maximal elements. We also present a partial solution to the compact divisible embedding conjecture.

For a regular semigroup, the set of elements which are maximal with respect to the Nambooripad partial order is defined and is compared with the $H$, $R$, and $L$ maximal sets.

CHAPTER 10: DIVISIBILITY

The divisibility theorems for minimal sets state that the $H$ minimal set of $S$ is the same type of semigroup as $S$ if $S$ is a divisible commutative semigroup or if $S$ is a uniquely divisible semigroup. The results for maximal sets are even better. Any divisibility property of a semigroup is reflected in the set of $H$ maximal elements.

The compact divisible embedding conjecture states that every compact semigroup can be embedded into a compact divisible semigroup. We extend a partial solution by Hildebrant and Lawson.

§10.1 MIN SETS

We now give some conditions under which $Min(S)$ mimics the divisibility properties of $S$. Suggestions by D. R. Brown and J. W. Stepp have been incorporated into this section. It is still unknown whether or not $Min(S)$ is divisible whenever $S$ is divisible and $Min(S) \neq \emptyset$.
Recall that a semigroup $S$ is divisible if for each element $x$ in $S$ and for each natural number $n$, there exists an element $y$ in $S$ such that $y^n = x$. If each such $y$ is unique, then $S$ is said to be uniquely divisible.

**Theorem 1:** Let $S$ be a divisible commutative semigroup such that $Min(S)$ is not empty. Then $Min(S)$ is divisible.

**Proof:** Since $S$ is commutative, $Min(S) = M(S)$ is a group. Let $e$ be the identity of $M(S)$.

Let $x \in Min(S)$ and $n \in \mathbb{N}$. Then there is a $y \in S$ such that $y^n = x$. But $(ey)^n = e^ny^n = ex = x$. Since $ey \in M(S)$, then $M(S)$ is divisible. $\diamond$

**Corollary:** If $S$ is a uniquely divisible commutative semigroup and $Min(S)$ is not empty, then $Min(S)$ is also a uniquely divisible commutative semigroup.

**Theorem 2:** Let $S$ be a uniquely divisible semigroup such that $Hmin(S)$ is not empty. Then $Hmin(S)$ is uniquely divisible.

**Proof:** Let $x \in M(S)$ and $n \in \mathbb{N}$. Then $(x^{\frac{1}{n}})^{n+1} = x \cdot x^{\frac{1}{n}} = x^{\frac{1}{n}} \cdot x$. So $(x^{\frac{1}{n}})^{n+1} \leq H x$, and hence $(x^{\frac{1}{n}})^{n+1} H x$. Let $e \in H(x)$, and let $x^{-1}$ be the inverse of $x$ in $H(e)$. Then $ex^{\frac{1}{n}} = x^{-1}xx^{\frac{1}{n}} = x^{-1}(x^{\frac{1}{n}})^{n+1} \subseteq H_x \cdot H_x = H_x = H(e)$. Hence $ex^{\frac{1}{n}} e = ex^{\frac{1}{n}}$, and so $(ex^{\frac{1}{n}})^n = e(x^{\frac{1}{n}})^n = ex = x$. Thus $ex^{\frac{1}{n}} = x^{\frac{1}{n}} \in H(e)$. $\diamond$

§10.2 MAX SETS

Since we think of multiplication in a semigroup $S$ as movement downward and we think of the max sets at the top of $S$, we would expect that divisibility of $S$ implies divisibility of $Max(S)$. Indeed, the theorems in this section show this is the case.
Theorem 1: Let $S$ be a semigroup. For any of the four quasiorders, if $x \in S$ and $x^n \in \text{Max}(S)$ for some $n \in \mathbb{N}$, then $x \in \text{Max}(S)$.

Proof: Assume $x \in S$ and $x^n \in \text{Max}(S)$. Since $x^n \leq x$, then $x$ and $x^n$ are in the same class. Thus $x \in \text{Max}(S)$. ◇

Corollary 1: Let $S$ be an $n$-divisible semigroup. For any of the four quasiorders, if $\text{Max}(S) \neq \emptyset$, then $\text{Max}(S)$ is $n$-divisible.

Corollary 2: Let $S$ be a divisible semigroup. Then for any of the four quasiorders, if $\text{Max}(S) \neq \emptyset$, then $\text{Max}(S)$ is divisible.

Example 1: If $\text{Max}(S)$ is divisible, $S$ need not be divisible. For example, let $S = \{0, a, e\}$ with multiplication given by

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Then $\text{Max}(S) = \{e\}$ is divisible, but $S$ is not.

Theorem 2: If $S$ is $n$-divisible for some $n \geq 2$, then $H\text{Max}(S)$ is a union of groups, each of which is $n$-divisible.

Proof: Let $a \in H\text{Max}(S)$. There is an element $b \in S$ such that $b^n = a$. Since $n \geq 2$, $a \leq_H b$ and $a \leq_H b^2$. So $bHab^2$ and hence $H_a = H_b$ is a group. ◇

Example 2: Let $S$ be the subsemigroup of the unit disk consisting of all points with polar coordinates $(r, \theta)$ where $0 \leq r < 1$ and $\theta \in \{0, \frac{2\pi}{3}, \frac{4\pi}{3}\}$. Then $S$ is a (compact) 2-divisible semigroup which is not divisible. Notice that $\text{Max}(S)$ consists of the three points with $r = 1$, the group of units of $S$. 
§10.3 A COMPACT DIVISIBLE EMBEDDING THEOREM

An unsolved problem first proposed by R. J. Koch appears in the 1969 paper by Brown and Friedberg as follows:

"If $S$ is a (finite dimensional) compact Abelian semigroup, then $S$ is isomorphically embeddable in a (finite dimensional) compact Abelian divisible semigroup."

The theorem presented herein is a partial solution to this problem. Other partial solutions have been published in the 1972 paper by Hildebrant and Lawson, the 1976 paper by Hildebrant, Lawson, and Yeager, and the 1976 paper by Hildebrant.

Recall the following definitions.

By a $udc$ we mean a uniquely divisible commutative semigroup.

A semigroup $S$ is power cancellative if $\forall x, y \in S$ and $\forall n \in \mathbb{N}$, $x^n = y^n \Rightarrow x = y$.

If $S$ is a semigroup and $A \subseteq S$, then by $A_n$ we mean $\{a^n : a \in A\}$. A semigroup $S$ is a power ideal semigroup if $S_n$ is an ideal of $S \ \forall n \in \mathbb{N}$.

If $S$ is a semigroup and $B \subseteq S$, we call $B$ a separating set for $S$ if $\forall x, y \in S$, $x \neq y \Rightarrow \exists b, c \in B$ such that $bx \neq by$ and $xc \neq yc$. We say a semigroup $S$ is reductive if $xa = xb \ \forall x \Rightarrow a = b$, and if $ax - bx \ \forall x \Rightarrow a = b$.

If a semigroup $S$ is locally compact, we let $\Omega(S)$ be the set of all continuous bitranslations of $S$. Also, $\pi : S \to \Omega(S)$ is the natural map $\pi(a) = \omega_a$, the inner bitranslation defined by $a$.

**Lemma 1:** Let $S$ be a compact reductive semigroup, $\omega_\alpha$ a net of bitranslations of $S$, and $\omega$ a bitranslation of $S$. The following are equivalent:

a) $\omega_\alpha \to \omega$ in $\Omega(S)$

b) $\omega_\alpha$ converges continuously to $\omega$

c) $\omega_\alpha$ converges pointwise to $\omega$

**Proof:** See volume II of Carruth, Hildebrant, and Koch, Chapter 4, Corollary 4.7.
Recall that if $S$ is locally compact, $\pi$ is continuous. Further, $\pi$ is an embedding whenever $S$ is compact and reductive.

If $S$ is a locally compact semigroup and $I$ is a closed ideal of $S$, let $I^* = \{\omega \in \Omega(S) : \omega I \cup I \omega \subseteq I\}$. Then $I^*$ is a closed subsemigroup of $\Omega(S)$ containing $\pi(S)$ and $I_{\Omega(S)}$.

**Lemma 2:** Let $S$ be a compact, power cancellative, power ideal abelian semigroup. Then there exists a compact udc subsemigroup $T$ of $\Omega(S)$ such that $\pi : S \rightarrow \Omega(S)$ is an embedding of $S$ into $T$.

**Proof:** Let $T = \bigcap\{S_n : n \in \mathbb{N}\}$. $T$ is a compact subsemigroup of $\Omega(S)$ containing $\pi(S)$. Since $\Omega(S)$ is power cancellative and $\pi$ is an embedding, it remains to show that $T$ is divisible. (A power cancellative divisible semigroup is always uniquely divisible.)

For the details of this proof, see volume II of Carruth, Hildebrant, and Koch, Chapter 4, Theorem 4.18. ~

**Example:** Let $S$ be the set $\{(x, y) | x, y \in ([0, \frac{1}{2}] \cup \{1\}) \subset \mathbb{R}\} \setminus \{(1, 1)\}$ together with the operation of multiplication. Then $S$ is a subsemigroup of $I_u \times I_u$, where $I_u$ is the usual interval $[0, 1] \subset \mathbb{R}$ under multiplication.

Notice that Lemma 2 does not apply to this example. The semigroup given is not a power ideal semigroup: Since $\left(\frac{1}{2}, \frac{1}{2}\right) \times (1, \frac{1}{4}) = \left(\frac{1}{2}, \frac{1}{4}\right) \notin S_2$ but $(1, \frac{1}{4}) \in S_2$, then $S_2$ is not an ideal of $S$.

Notice $S$ is a union of three connected components satisfying the conditions of the following theorem. That $S$ satisfies the conclusion of the theorem is evident since $I_u \times I_u$ is a compact udc.

Let $S$ be a locally connected semigroup with $0$ having connected components $A_i$ such that each $A_i$ is a compact, abelian, power cancellative, power ideal semigroup containing $0_i$. 
For each \(i\) and \(j\), \(\exists k_{ij}\) such that \(A_i A_j \subseteq A_{k_{ij}}\). We can define a multiplication * on \(\{A_i\}\) by \(A_i * A_j = A_{k_{ij}}\). This forms a semigroup.

If \(\{A_i\}, *\) is a semilattice, then it is abelian, so \(S\) itself is abelian.

**Theorem:** Let \(S\) be a locally connected semigroup containing a zero \(0_S\) and having connected components \(A_i\) such that:

i) Each \(A_i\) is a compact, abelian, power cancellative, power ideal semigroup containing a zero \(0_i\).

ii) \(\{A_i\}, *\) is a semilattice.

iii) The component \(A_0\) containing \(0_S\) also contains a set \(B\) which is a separating set for \(S\).

iv) If \(x \in A_j \geq A_i\), then \(x(A_i)_n \subseteq (A_i)_n \forall n \in \mathbb{N}\).

Then \(S\) can be embedded into a compact uniquely divisible commutative semigroup \(T\).

**Proof:** By lemma 2, \(\forall i \exists a\) a compact udc \(T_i \subseteq \Omega(A_i)\) such that \(\pi_i : A_i \to \Omega(A_i)\) embeds \(A_i\) into \(T_i\). Let \(T = \prod T_i\), a compact udc.

Define \(\Phi : \Omega(S) \to \prod \Omega(A_i)\) as follows. Each \(A_i\) is connected and each \(\omega \in \Omega(S)\) is continuous, so for each \(i\) and for each \(\omega\) there is a \(j\) such that \(\omega(A_i) \subseteq A_j\). Define \((\Phi\omega)_i = \omega|_{A_i}\); if \(\omega(A_i) \subseteq A_i\) and \((\Phi\omega)_i = \omega_0|_{A_i}\); if \(\omega(A_i) \cap A_i = \emptyset\).

Let \(\pi : S \to \Omega(S) : x \mapsto \omega_x\). We will show:

1) \(\Phi\pi\) is one-to-one;

2) \(\Phi\pi\) is continuous;

3) \(\Phi\pi\) is a homomorphism; and

4) \(\Phi\pi(S) \subseteq T\).

1) \(\Phi\pi\) is one-to-one:

If \(x, y \in S\) and \(x \neq y\), then \(\exists b \in B\) such that \(xb \neq yb\). That is, \(\omega_x b \neq \omega_y b\).

Since \((\Phi\omega_x)_0(b) = xb \neq yb = (\Phi\omega_y)_0(b)\), then \(\Phi\pi(x) \neq \Phi\pi(y)\).
2) $\Phi \pi$ is continuous:

Since $S$ is locally connected, each $A_i$ is open. Let $x_\alpha \to x$ in $S$. Fix $j$ such that $x \in A_j$. Without loss of generality, $x_\alpha \in A_j \forall \alpha$. Fix any $i$.

If $A_i \subseteq A_j$, then $(\Phi \pi(x_\alpha))_i = \omega_{x_\alpha} |_{A_i} \to \omega_x |_{A_i} = (\Phi \pi(x))_i$ by lemma 1.

If, on the other hand, $A_i \not\subseteq A_j$, then $(\Phi \pi(x_\alpha))_i = \omega_{0_i}$, which converges under $\alpha$ to $\omega_{0_i} = (\Phi \pi(x))_i$ since $x_\alpha A_i \in A_j A_i \not\subseteq A_i$.

3) $\Phi \pi$ is a homomorphism:

Let $x, y \in S$ and let $a \in A_i$ for some fixed $i$. Suppose $x \in A_x$ and $y \in A_y$.

Case 1: $xya \in A_i$. Then $A_x A_y A_i \subseteq A_i$, so $A_i \subseteq A_y$. Thus $ya \in A_i$. Then we have $(\Phi \omega_{xy})_i(a) = (xy)a = x(ya) = (\Phi \omega_x)_i(\Phi \omega_y)_i(a)$.

Case 2: $xya \not\in A_i$. If $yA_i \not\subseteq A_i$, then $(\Phi \omega_{xy})_i(a) = 0_i = (\Phi \omega_x)_i(0_i)$, which equals $(\Phi \omega_x)_i(\Phi \omega_y)_i(a)$. If $xA_i \not\subseteq A_i$, then $(\Phi \omega_{xy})_i(a) = 0_i = (\Phi \omega_x)_i((\Phi \omega_y)_i(a))$.

In any case, $(\Phi \omega_x)_i(\Phi \omega_y)_i = (\Phi \omega_{xy})_i$, so $\Phi \pi(x) \Phi \pi(y) = \Phi \pi(xy)$.

4) $\Phi \pi(S) \subseteq T$:

Fix $x \in A_z$ and fix $i$ and show that $(\Phi \omega_x)_i \in T_i$. Recall $T_i = \cap((A)_i)^*$ where $I^* = \{\omega \in \Omega(A_i) : \omega \cap I \omega \subseteq I\}$. (See the proof of lemma 2.)

If $A_i \subseteq A_z$, then $z A_i \subseteq A_i$, so $(\Phi \omega_x)_i = \omega_x |_{A_i}$. For each $n$, $\omega_x |_{A_i}((A)_i)^*$ equals $x((A)_i)^* \subseteq A_i$ by condition iv in the hypotheses of the theorem. So $\omega_x |_{A_i} \in T_i$; that is, $(\Phi \pi(x))_i \in T_i$.

If $A_i \not\subseteq A_z$, then $(\Phi \omega_x)_i = \omega_{0_i} |_{A_i} = \pi_i(0_i) \in T_i$. ◇
CHAPTER 11: THE NAMBOORIPAD PARTIAL ORDER
ON REGULAR SEMIGROUPS

We now consider regular algebraic semigroups. The Nambooripad partial order is reviewed. We prove that the set of $N$ maximal elements contains the union of the sets of $H$, $R$, and $L$ maximal elements.

§11.1 CHARACTERIZATION

The definition of this partial order on regular semigroups is attributed to Nambooripad. The theorems in this first section appear in the paper by Mitsch.

Let $S$ be a regular semigroup; that is, for each $x \in S$, $x \in xSx$. For $a,b \in S$ we define $a \leq_N b$ if and only if $a \in bS$ and there exists $e \in E(S)$ such that $eRa$ and $a = eb$.

In the usual order on idempotents, $e$ is below $f$ if and only if $e = ef = fe$. Notice that for idempotents $e$ and $f$, $e$ is below $f$ if and only if $e \leq_N f$.

Theorem 1: The following are equivalent:

1) $a \leq_N b$;
2) $a = eb = bf$ for some $e, f \in E(S)$;
3) $a = axb = bxa = axa$ for each $x \in S$ such that $b = bxb$;
4) $a = xb = by$ and $a = xa$ for some $x, y \in S$;
5) $a = eb = bx$ for some $x \in S$ and $e \in E(S)$.

Proof: We show $4 \Rightarrow 5 \Rightarrow 2 \Rightarrow 4$ and $2 \Rightarrow 3 \Rightarrow 1 \Rightarrow 5$.

4 $\Rightarrow$ 5: Let $a = aa'a$ and $e = aa'x$. Then $e^2 = aa'xaa'x = aa'aa'x = aa'x = e$
E(S). Also, \( eb = aa'xb = aa'a = a \).

5 \( \Rightarrow \) 2: Let \( a = aa'a \) and \( f = xa'a \). Then \( a = eb \), so \( a = ea \). Thus \( f^2 = xa'axa'a = xa'ebx = xa'ea = xa'a = f \in E(S) \). Also, \( bf = bx'a = aa'a = a \).

2 \( \Rightarrow \) 4 is clear.

2 \( \Rightarrow \) 3: Assume \( a = eb = bf \) and \( a = aa'a \). Notice \( a = ea = af \). Let \( y = fa'e \). Then \( ayb = af a'eb = aa'a = a \) and similarly \( a = bya = aya \). If \( b = bxb \), then \( axa = aybxy = ayby = aya = a, axb = aybx = aby = a, \) and \( bx = a \).

3 \( \Rightarrow \) 1: Since \( a = axa \), then \( ax \in E(S) \) and \( axRa \). Let \( e = ax \). Then \( a = axb = eb \) and \( a = bxa \in bS \).

1 \( \Rightarrow \) 5 is clear. ∆

There are many more equivalent statements in the paper by Mitsch.

A useful fact is that if \( a = xb = by = xa \), then \( ay = a \) as well: \( ay = xby = xa = a \). Indeed, all of the left-right dual statements are also equivalent.

**Theorem 2:** The relation \( \leq_N \) is a partial order.

**Proof:** To see it is reflexive, we use 2). For all \( a \in S \) \( a = aa'a \) and \( aa', a'a \in E(S) \).

For the proof of antisymmetry, we use 3). Suppose
\[
\begin{align*}
    a &= axb = bxa = axa \\
    b &= bya = ayb = byb \\
    a &= aya
\end{align*}
\]
Then \( a = axb = ax(byb) = (axb)y = aby = b \).

To show \( \leq_N \) is transitive, we use 4). Suppose \( a = xb = by = xa \) and \( b = pc = cq = pb \). Then \( a = xpc = cqa \) and \( xpa = xpyb = xby = xa = a \). ∆

Notice by 2) that if \( e = ef = fe \) then \( e \leq_N f \).

If \( a \) and \( b \) satisfy description 4), we say \( a \leq_M b \), since this is Mitsch's order. Mitsch shows that his order is a partial order on every semigroup, whether or not it is regular.
Notice the partial order is trivial whenever the semigroup is cancellative.

Example: In the bicyclic semigroup, \( p^n q^m \leq_N p^aq^b \) if and only if \( n \geq a, m \geq b, \) and \( n + b = m + a. \) To see this, we use 2).

\[ \Rightarrow: \] If \( p^n q^m = p^c q^a p^d q^b \) then \( n = c + a \vee c - \varepsilon = a + b \vee d - b = a \vee c, \) so \( n \geq a, \) and \( m = a \vee c - a + b = b \vee d - d + d = b \vee d, \) so \( m \geq b. \) Also, \( n + b = a + b \vee d = a + m. \)

\[ \Leftarrow: \] Assume \( n = a + x, \) \( m = b + y, \) and \( n + b = m + a. \) Then \( p^n q^m = p^n q^{a+y} p^a q^b = p^a q^b p^{b+x} q^m. \) But \( n = m = b + a = a + y \) and \( m = b + x, \) so \( p^n q^{a+y} \) and \( p^{b+x} q^m \) are idempotents. Hence \( p^n q^m \leq_N p^a q^b. \)

§11.2 THE \( N \) MAXIMAL SET

Let \( S \) be a regular semigroup. We define \( Nmax(S) = \{ x : x \leq_N y \Rightarrow x = y \} \) since \( \leq_N \) is a partial order.

Notice that if \( a \leq_N b \) then \( a \leq_H b, a \leq_R b, a \leq_L b, \) and \( a \leq_J b. \)

Example 1: \( Nmax(\mathcal{B}) = \{ e, q, q^2, q^3, \ldots \} \cup \{ p, p^2, p^3, \ldots \}. \)

Theorem (Nambooripad, 1980): If \( x \leq_N y \) and \( xRy, \) then \( x = y. \)

Proof: If \( x = py = px = yq = xq \) and \( y = xr, \) then \( y = xr = pxr = py = x. \) \( \Diamond \)

Corollary 1: If \( x \leq_N y \) and \( xHy, \) then \( x = y; \) and if \( x \leq_N y \) and \( xLy, \) then \( x = y. \)

Corollary 2: The set \( Nmax(S) \) contains \( Hmax(S) \cup Rmax(S) \cup Lmax(S). \)

Proof: If \( x \in Rmax(S) \) and \( x \leq_N y, \) then \( x \leq_R y, \) so \( xRy. \) By the theorem, \( x = y. \) Hence \( x \in Nmax(S). \)
Similar proofs hold for $H$ and $L$.\hfill\diamond

**Corollary 3:** If $aRb$ and $a \neq b$, then $a \|_N b$.

If $aHb$ and $a \neq b$, then $a \|_N b$.

If $aLb$ and $a \neq b$, then $a \|_N b$.

**Example 2:** In the bicyclic semigroup, $p \in N_{max}(B)$ and $pRpq$, but $pq \not\in N_{max}(B)$.

**Example 3:** This example was inspired by the Croisot semigroup; the semigroup used here may be replaced by the Croisot semigroup and all statements still hold.

Let $S = \{ f : \mathbb{N} \to \mathbb{N} : f \text{ is a function} \}$.

This semigroup is regular: Let $f \in S$ and find $g \in S$ so that $f = fgf$. Let $f(\mathbb{N}) = \{a_1, a_2, a_3, \ldots \}$, and for each $n \in \mathbb{N}$, choose $b_n \in f^{-1}(a_n)$. Define $g(a_n) = b_n$ and $g(t) = t$ if $t \not\in f(\mathbb{N})$. If $f(x) = a_r$, then $fgf(x) = fg(a_r) = f(b_r) = a_r$, so $f = fgf$.

The right quasiorder is given by $f \leq_R g$ if and only if $f(\mathbb{N}) \subseteq g(\mathbb{N})$:

If $f \leq_R g$, then $f = gh$ for some $h$. So $f(\mathbb{N}) = gh(\mathbb{N}) \subseteq g(\mathbb{N})$.

Now suppose $f(\mathbb{N}) \subseteq g(\mathbb{N})$. Define $h$ as follows. For $n \in \mathbb{N}$, $f(n) \in g(\mathbb{N})$, so there exists $m_n \in \mathbb{N}$ such that $f(n) = g(m_n)$. Define $h(n) = m_n$. Then $f = gh$.

The set $R_{max}(S) = \{ f : f \text{ is onto} \}$:

If $f$ is not onto, there exists $a \not\in f(\mathbb{N})$. Define $g$ by $g(1) = a$ and $g(n + 1) = f(n)$ for $n \geq 1$. Then $g \in S$ and $f \leq_R g$ and $f$ is not $R$ related to $g$. Hence, $f \not\in R_{max}(S)$.

If $f$ is onto and $f \leq_R g$, then $\mathbb{N} = f(\mathbb{N}) \subseteq g(\mathbb{N})$. Thus $g$ is onto, so $fRg$. Hence $f \in R_{max}(S)$.

The left quasiorder is given by $f \leq_L g$ if and only if $g(n) = g(m) \Rightarrow f(n) = f(m)$ for all $n, m \in \mathbb{N}$:

If $f \leq_L g$, then there exists $h \in S$ such that $f = hg$. So whenever $g(n) = g(m)$,
then \( f(n) = hg(n) = hg(m) = f(m) \).

Suppose \( f, g \in S \) such that \( g(n) = g(m) \Rightarrow f(n) = f(m) \). Define \( h \) as follows.

Let \( A = g(N) = \{a_1, a_2, a_3, \ldots \} \) and let \( B = N \setminus A = \{b_1, b_2, b_3, \ldots \} \). (Note that \( B \) may be empty.) Let \( h(a_i) = f(g^{-1}(a_i)) \) and \( h(b_i) = b_i \). Then \( h \) is well-defined since if \( n, m \in g^{-1}(a_i) \), then \( g(n) = g(m) \), so \( f(n) = f(m) \). Also notice that \( f = hg \). Thus \( f \leq_L g \).

The set \( L_{\max}(S) = \{ f : f \text{ is one-to-one} \} \):

If \( f \) is not one-to-one. There are elements \( p, q \in N \) such that \( p \neq q \) and \( f(p) = f(q) \). Define \( g \) by \( g(p) = 1 \) and \( g(n) = f(n) = 1 \) for \( n \neq p \). Then \( g(n) = g(m) \Rightarrow f(n) = f(m) \), but \( f(p) = f(q) \) and \( g(p) \neq g(q) \). Hence \( f \leq_L g \) and \( g \not\leq_L f \), so \( f \not\in L_{\max}(S) \).

On the other hand, suppose \( f \) is one-to-one and \( f \leq_L g \). If \( f(n) = f(m) \), then \( n = m \), so \( g(n) = g(m) \). Hence \( g \leq_L f \) so \( gLf \). Therefore, \( f \in L_{\max}(S) \).

Also, \( H_{\max}(S) = R_{\max}(S) \cap L_{\max}(S) \):

If \( f \) is one-to-one and onto, then \( f^{-1} \). Since \( f^{-1} \in S \), then \( f \in H(1_S) \). Conversely, if \( f \in H(1_S) \), then \( f f^{-1} = 1_S = f^{-1} f \), so \( f \) is one-to-one and onto.

The Nambooripad quasiorder is given by \( f \leq_N g \) if and only if \( f(N) \subseteq g(N) \), \( g(n) = g(m) \Rightarrow f(n) = f(m) \) for \( n \) and \( m \) in \( N \), and \( g^{-1}(N) \cap f^{-1}(m) \neq \emptyset \) for all \( m \in f(N) \). That is, \( f \leq_N g \) if and only if \( g(n) = g(m) \Rightarrow f(n) = f(m) \) and for all \( n \) there exists \( m \) such that \( f(n) = g(m) = f(m) \):

\[ \Rightarrow: \text{ Let } h_1, h_2 \in S \text{ such that } f = h_1 g = h_1 f = gh_2 = fh_1. \text{ Then } g(n) = g(m) \Rightarrow f(n) = h_1 g(n) = h_1 g(m) = f(m), \text{ and for each } n \in N, f(n) = f(h_2(n)) = g(h_2(n)) \]

\[ \Leftarrow: \text{ To define } h_2, \text{ choose } h_2(n) \in f^{-1}(f(n)) \cap g^{-1}(f(n)) \text{ for each } n. \text{ Then } h_2 \in S, \text{ and } f = fh_2 = gh_2. \text{ Since } f \leq_L g, \text{ there is a function } h_1 \in S \text{ such that } f = h_1 g. \text{ Thus } f \leq_N g. \]

Finally, we show that \( N_{\max}(S) = \{ f : f \text{ is one-to-one or onto} \} = R_{\max}(S) \cup L_{\max}(S) \):
First let $f$ be one-to-one and $f \leq_N g$. Now for all $n$ there exists $m$ such that $f(n) = g(m) = f(m)$. Since $f$ is one-to-one, $m = n$, so $f = g$.

Now let $f$ be onto and $f \leq_N g$. Let $a \in \mathbb{N}$. Since $f$ is onto, there is an $n \in \mathbb{N}$ such that $f(n) = g(a)$. For some $m \in \mathbb{N}$, $f(n) = g(a) = g(m) = f(m)$. Then $g(a) = g(m) \Rightarrow f(a) = f(m) \Rightarrow f(a) = g(a)$. That is, $f = g$.

On the other hand, if $f$ is not one-to-one and not onto, let $a, b, t \in \mathbb{N}$ such that $f(a) = f(b)$, $a \neq b$, and $t$ is in $\mathbb{N} \setminus f(\mathbb{N})$. Define $g : \mathbb{N} \rightarrow \mathbb{N}$ by $g(a) = t$ and $g(n) = f(n)$ for $n \neq a$. Then $g \in S$, $f \leq_N g$ and $f \neq g$. 

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