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The Graphs and Matroids Whose Only Odd Circuits Are Small

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THE GRAPHS AND MATROIDS WHOSE ONLY ODD CIRCUITS ARE SMALL

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Abstract

This thesis is motivated by a graph-theoretical result of Maffray, which states that a 2-connected graph with no odd cycles exceeding length 3 is bipartite, is isomorphic to K_4, or is a collection of triangles glued together along a common edge. We first prove that a connected simple binary matroid M has no odd circuits other than triangles if and only if M is affine, M is $M(K_4)$ or F_7, or M is the cycle matroid of a graph consisting of a collection of triangles glued together along a common edge. This result implies that a 2-connected loopless graph G has no odd bonds of size at least five if and only if G is Eulerian or G is a subdivision of either K_4 or the graph that is obtained from a cycle of parallel pairs by deleting a single edge. The main theorem of the dissertation extends Maffray's theorem to n-connected graphs with no odd cycles exceeding size $2n-1$. To prove this, we first prove the special cases when $n = 3$ and $n = 4$. The proof of the theorem is competed with an argument that treats all $n \geq 5$.

Chapter 1 Binary Matroids with No Odd Circuits Exceeding Size Three

It is a well known result from graph theory that a graph is bipartite if and only if it has no odd cycles. For each $n \geq 1$, let $K'_{2,n}$ be the graph that is obtained from $K_{2,n}$ by adding an edge joining the vertices in the two-vertex class (see Figure 1.1). In 1992, Maffray [7, Theorem 2] proved the following result.

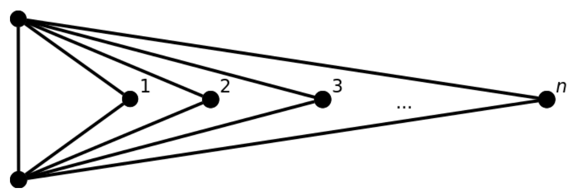


Figure 1.1: $K'_{2,n}$

Theorem 1.0.1. *A 2-connected simple graph G has no odd cycles of length exceeding three if and only if*

(i) G is bipartite;

(ii) $G \cong K_4$; or

(iii) $G \cong K'_{2,n}$ for some $n \geq 1$.

There is a long history of generalizing results for graphs to binary matroids (see, for example, [4, 12] or, more recently, [9, Section 15.4]). We shall continue this tradition by proving a generalization of Maffray's result. A circuit in a matroid is *even* if it has even cardinality; otherwise, it is *odd*. A *triangle* is a 3-element circuit. A binary matroid is *affine* if all of its circuits are even. Hence the cycle matroid, $M(G)$, of a graph G is affine if and only if G is bipartite. The following is the main theorem of this chapter [10].

Theorem 1.0.2. *A connected simple binary matroid M has no odd circuits other than triangles if and only if* *(i) M is affine;*

(ii) $M \cong M(K_4)$ or F_7; or

(iii) $M \cong M(K'_{2,n})$ for some $n \geq 1$.

The terminology used here will follow Oxley [9]. Binary affine matroids have several attractive characterizations. Indeed, Welsh [13] proved that the link between bipartite and Eulerian graphs via duality extends to binary matroids. His result is the equivalence of the first two parts of the next theorem (see, for example, [9, Theorem 9.4.1]). The equivalence of the first and third parts was proved independently by Brylawski [2] and Heron [5].

Theorem 1.0.3. *The following are equivalent for a binary matroid M.*

(i) M is affine;

(ii) M is loopless and its simplification is isomorphic to a restriction of $AG(r-1, 2)$ for some $r \geq 1$;

(iii) $E(M)$ can be partitioned into cocircuits.

Recall that a *bond* of a graph is a minimal edge cut. The next result follows immediately by applying our Theorem 1.0.2 to the bond matroid of a graph, that is, to the dual of its cycle matroid.

Corollary 1.0.4. *A 2-connected loopless graph G has no odd bonds of size exceeding three if and only if*

(i) G is Eulerian; or

(ii) G is a subdivision of either K_4 or the graph that is obtained from an n-edge cycle for some $n \geq 2$ by adding an edge in parallel to all but one of the edges.

Another straightforward consequence of Theorems 1.0.2 and 1.0.3 is the following.

Corollary 1.0.5. *Let M be a connected cosimple binary matroid of rank at least four. Then M has no odd circuits of size exceeding three if and only if M is affine.*

We shall implement the use of the following two lemmas in the proof of Theorem 1.0.2.

Lemma 1.0.6. *A simple binary matroid having an even circuit meeting a triangle T in a single element has an odd circuit of size exceeding three.*

Proof. From among even circuits that meet T in a single element, choose C to have minimum cardinality. As M is binary, $C\Delta T$ is the disjoint union of k circuits for some $k \geq 1$. As $|C\Delta T| = |C| + 1$, if $k = 1$, then the lemma holds. Thus we may assume that $k \geq 2$. Since each circuit contained in $C\Delta T$ must contain an element of $T - C$, we deduce that $k \leq 2$, so $k = 2$. Thus, as $C\Delta T$ has odd cardinality, it is the disjoint union of an odd circuit and an even circuit, C_0, each of which meets T in a single element. As $|C_0| < |C|$, the choice of C is contradicted. □

Our second lemma is more general than we need to prove the theorem. For an integer n exceeding one, let $M_1, M_2, \ldots, M_n$ be matroids such that $E(M_i) \cap E(M_j) = \{p\}$ for all distinct i and j in $\{1, 2, \ldots, n\}$, and $\{p\}$ is not a component of any M_k. The *parallel connection* $P(M_1, M_2, \ldots, M_n)$ is the matroid with ground set $E(M_1) \cup E(M_2) \cup \cdots \cup E(M_n)$ whose set of circuits consists of the union of the sets of circuits of $M_1, M_2, \ldots, M_n$ along with, for all distinct elements i and j of $\{1, 2, \ldots, n\}$, all sets of the form $(C_i - p) \cup (C_j - p)$ where C_i is a circuit of M_i containing p, and C_j is a circuit of M_j containing p (see, for example, [9, Proposition 7.1.18]). Thus if $M_k \cong U_{2,3}$ for all k, then $P(M_1, M_2, \ldots, M_n) \cong M(K'_{2,n})$. The element p is called the *basepoint* of the parallel connection.

Lemma 1.0.7. *Let M be a simple connected matroid. Then M has an element p such that the only circuits of M that contain p are triangles if and only if M is isomorphic to $U_{1,1}$ or to $U_{2,k}$ for some $k \geq 3$, or M is the parallel connection with basepoint p of some collection of simple rank-2 matroids each of which contains at least three points.*

Proof. It is straightforward to check that, for each of the matroids listed, the only circuits containing p are triangles. Now assume that the only circuits of M containing p are triangles. We may assume that $r(M) \geq 3$ otherwise the result certainly holds. As M is connected, each of its elements is in some circuit with p. By hypothesis, this circuit must be a triangle. Thus, in M/p, every element is in a non-trivial parallel class. If every component of M/p has rank one, then it follows by a result of Brylawski [1] (see also [9, Theorem 7.1.16]) that M is a parallel connection as asserted. Therefore we may assume that M/p has a component of rank exceeding one. Thus M/p has a circuit D of size exceeding two and, as $D \cup p$ is not a circuit of M, we deduce that D is a circuit of M. Similarly, $(D-d) \cup d'$ is a circuit of M where d is some element of D, and d' is parallel to d in M/p. Thus $\mathrm{cl}_M(D-d)$ contains $\{d, d'\}$ and so contains p. Then $r_{M/p}(D-d) < |D-d|$; a contradiction. □

We are now ready to prove Theorem 1.0.2.

Proof of Theorem 1.0.2. It is easily checked that $M(K_4)$, F_7, and each $M(K'_{2,n})$ are binary having no odd circuits of size greater than three. For the converse, assume that M has no odd circuits of size greater than three. Suppose M is not affine. If $r(M) = 3$, then clearly M is isomorphic to $M(K'_{2,2})$, $M(K_4)$, or F_7. Thus we may assume that $r(M) \geq 4$. First we show the following.

1.0.3.1. *If T_0 is a triangle of M and C is a circuit that meets but is not equal to T_0, then $|C| \leq 4$ and $M|(T_0 \cup C) \cong M(K'_{2,2})$.*

This is certainly true if C is a triangle, so we assume that $|C| \geq 4$. By Lemma 1.0.6, $|C \cap T_0| = 2$. Then $C \Delta T_0$ is a circuit of M of cardinality $|C|-1$. Thus $|C| = 4$ and $C \Delta T_0$ is a triangle T_1 meeting T_0 in a single element. Hence $M|(T_0 \cup C) = M|(T_0 \cup T_1) \cong M(K'_{2,2})$, and (1.0.3.1) holds.

As M is not affine, it contains a triangle T. As M is connected, it follows by (1.0.3.1) that M has a triangle T' that meets T in a single element, say f.

1.0.3.2. *For each g not in* $\mathrm{cl}(T \cup T')$*, there is a triangle that contains* $\{g, f\}$*.*

As M is connected, it has a circuit D that contains g and meets $T \cup T'$. Without loss of generality, we may assume that D meets T. By (1.0.3.1), $M|(D \cup T) \cong M(K'_{2,2})$. Thus M has a triangle T'' that contains g and meets T in a single element, h. We may assume that $h \neq f$ otherwise (1.0.3.2) holds. Then T'' meets the 4-element circuit $(T \cup T') - f$ in a single element; a contradiction to Lemma 1.0.6. We deduce that (1.0.3.2) holds.

We may assume that M has a circuit C' that contains f and is not a triangle otherwise the result follows by Lemma 1.0.7. By Lemma 1.0.6, C' meets each triangle containing f in two elements. Moreover, by (1.0.3.1), $|C'| = 4$. Hence M has at most three triangles containing f. But, as $r(M) \geq 4$, it follows that $r(M) = 4$, and M has exactly two elements not in $\mathrm{cl}(T \cup T')$, these elements being contained in a common triangle with f.

If $T \cup T'$ is a flat of M, then $M \cong M(K'_{2,3})$. Thus we may assume that $\mathrm{cl}(T \cup T') - (T \cup T')$ contains an element h. Then $M|(T \cup T' \cup h) \cong M(K_4)$, so $T \cup T' \cup h$ contains a 4-circuit D' containing $\{f, h\}$. By (1.0.3.2), M has a triangle that meets D' in $\{f\}$. This contradiction to Lemma 1.0.6 completes the proof of the theorem. $\square$

Chapter 2 Graphs with No Odd Cycles Exceeding Size Five

From here, we explored possible extensions of Theorem 1.0.1 and Theorem 1.0.2. Initially we proved a purely graph-theoretical extension of Theorem 1.0.1. Subsequently, we extended this proof to binary matroids. This extension does not appear in this dissertation.

Theorem 2.0.1. *A 3-connected simple graph G has no odd cycles of length exceeding five if and only if*

(i) G is bipartite;

(ii) G is a graph on six or fewer vertices; or

(iii) $G \cong K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some $n \geq 4$ where $K'_{3,n}$, $K''_{3,n}$, and $K'''_{3,n}$ are shown below in Figure 2.1.

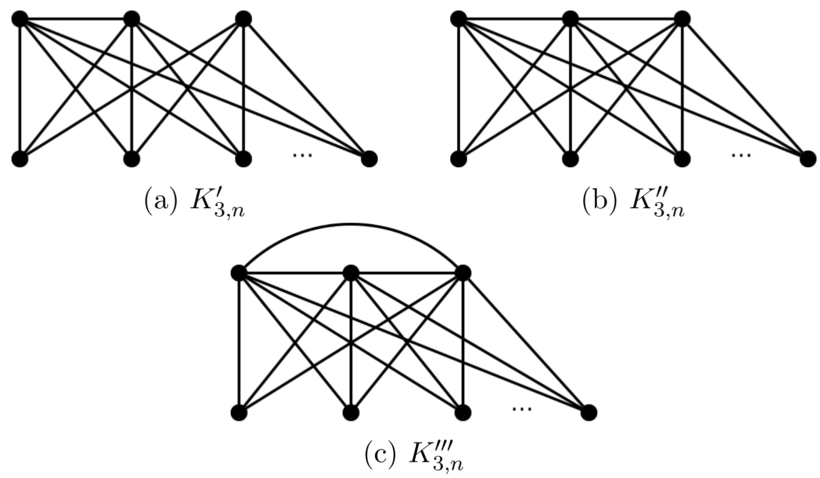
Note that $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ can be viewed as n copies of K_4 identified at a common triangle with 1, 2 or 3 edges left in respectively.

For the proof of Theorem 2.0.1, we will need the following theorem of Menger [6].

Theorem 2.0.2. *Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimum number of vertices separating A from B in G is equal to the maximum of disjoint $A - B$ paths in G.*

Proof of Theorem 2.0.1. It is easily checked that the graphs mentioned in (i), (ii), and (iii) have no odd cycles of length exceeding five. Now assume G is a 3-connected graph with a 5-cycle and $|V(G)| > 6$. Select a 5-cycle, C, with vertex set $V(C) = \{v_1, v_2, v_3, v_4, v_5\}$. For all i in $1, 2, 3, 4, 5$, let e_i be the edge $\{v_i, v_{i+1}\}$ where $v_6 = v_1$ as in Figure 2.2

Since $|V(G)| > 6$, there is a vertex in $V(G) - V(C)$; call it v_0. Since G is 3-connected, by Theorem 2.0.2, there are three paths from v_0 to $V(C)$ where v_0 is the only common



(a) $K'_{3,n}$ (b) $K''_{3,n}$

(c) $K'''_{3,n}$

Figure 2.1: $K'_{3,n}$, $K''_{3,n}$, and $K'''_{3,n}$

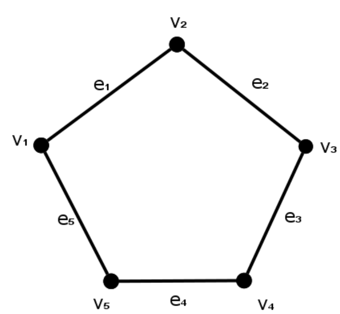
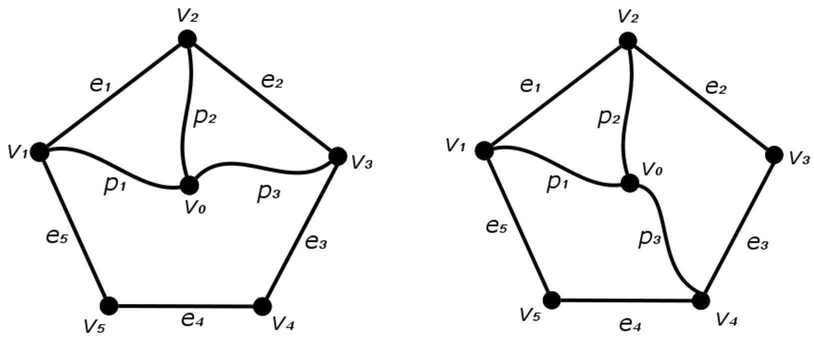


Figure 2.2: 5-cycle

vertex of any two of the three paths. Call the paths p_1, p_2 and p_3. By symmetry, we have one of the configurations shown in Figure 2.3.

Let us first consider the case shown in Figure 2.3a. We will use G_1 to denote such a graph. All cycles in G_1 must have even length or length 3 or 5, since G_1 is a subgraph of G. Consider the cycles $A = \{p_1, p_2, e_2, e_3, e_4, e_5\}$, $B = \{p_1, p_3, e_3, e_4, e_5\}$ and $C = \{p_2, p_3, e_3, e_4, e_5, e_1\}$ where, for example, A consists of all of the edges of each of p_1 and p_2 along with the edges e_2, e_3, e_4, and e_5. Let $|p_i|$ denote the number of edges in the path p_i and let $|A|$ be the number of edges in cycle A. If we sum the lengths of theses three cycles, we get $2|p_1| + 2|p_2| + 2|p_3| + 11$. Thus at least one of these cycles has odd length. Thus we have a cycle of length five as each $|p_i|$ is positive. Since $|A| \geq 6$ and $|C| \geq 6$,



(a) G_1 (b) G_2

Figure 2.3: 5-cycle path configurations

we see that the $|B| = 5$ and so $|p_1| = 1 = |p_3|$. As $|A| > 5$, it is even. Thus the cycle $\{p_1, p_2, e_1\}$ must be odd of length equal to 3 or 5. Thus $|p_2| \in \{1, 3\}$.

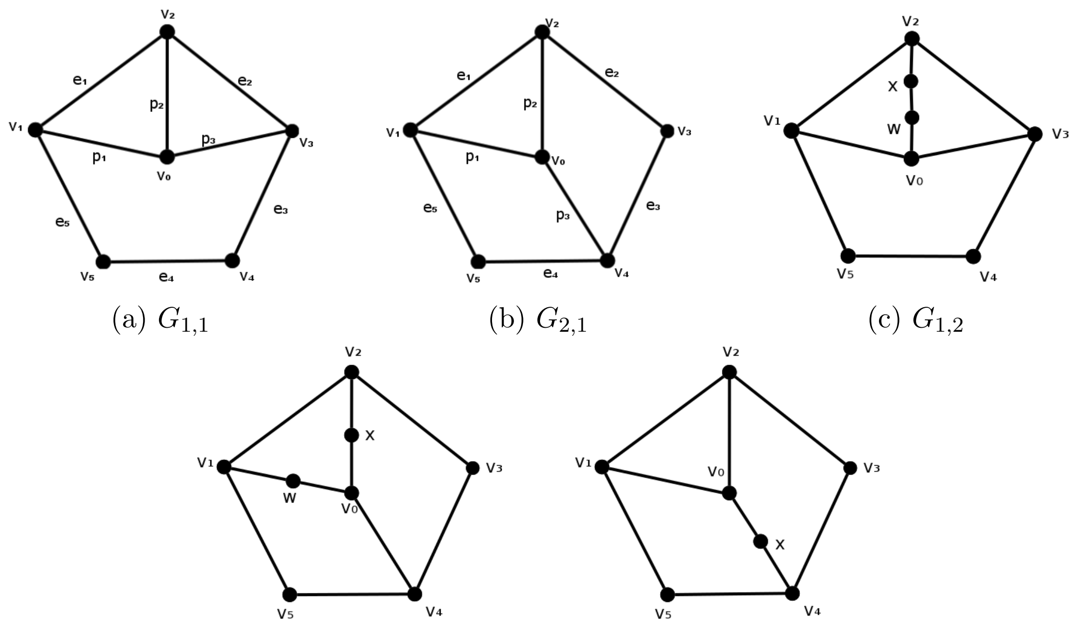
We conclude that if we have a graph of the form G_1, we are guaranteed one of the substructures in Figure 2.4a or Figure 2.4c in the graph G.

Let us consider now the case pictured in Figure 2.3b. Again, p_1, p_2 and p_3 are paths that share v_0, but are otherwise disjoint. We will call this G_2. Consider the cycles $A = \{p_1, p_2, e_2, e_3, e_4, e_5\}$, $B = \{p_1, p_3, e_3, e_2, e_1\}$, and $C = \{p_2, p_3, e_3, e_2\}$. The sum of their lengths is $2|p_1| + 2|p_2| + 2|p_3| + 9$. By a similar argument as before, exactly one of $|B|$ and $|C|$ has length 5, so either $|p_1|$ and $|p_3|$ have the same cardinality, or $|p_2|$ and $|p_3|$ have the same cardinality.

If $|p_1|$ and $|p_3|$ have the same cardinality, then by the cycles $\{p_1, p_3, e_4, e_5\}$ and $\{p_1, p_3, e_3, e_2, e_1\}$, we deduce that $|p_1| = |p_3| = 1$. Similarly, by considering $\{p_2, p_3, e_4, e_5, e_1\}$ and $\{p_1, p_2, e_2, e_3, e_4, e_5\}$, we see that $|p_2| = 1$.

Now, if $|p_2|$ and $|p_3|$ have different cardinalities, by $\{p_2, p_3, e_3, e_2\}$ and $\{p_2, p_3, e_4, e_5, e_1\}$ one of p_2 or p_3 has length 1 and the other has length 2. If $|p_2| = 2$, then, by cycles $\{p_1, p_2, e_1\}$ and $\{p_1, p_2, e_3, e_2, e_1\}$, we deduce that $|p_1| = 2$. If $|p_3| = 2$, then by cycles $\{p_1, p_3, e_3, e_2, e_1\}$ and $\{p_1, p_3, e_4, e_5\}$, it follows that $|p_1| = 1$.

We conclude that G_2 must be one of the graphs among Figure 2.4b, Figure 2.4d and Figure 2.4e below.



(a) $G_{1,1}$ (b) $G_{2,1}$ (c) $G_{1,2}$

(d) $G_{2,2}$ (e) $G_{2,3}$

Figure 2.4: Path length possibilities for G

Next we note the following fact.

2.0.2.1. *Let u and v be vertices of G such that G contains an even-lengthed path p_e and an odd-lengthed path p_o joining u and v. If $|p_e| \geq 6$ and $|p_o| \geq 5$, then G has no path p that joins u and v and is internally disjoint from both p_e and p_o.*

If such a p existed, we could examine the cycles $\{p_e, p\}$ and $\{p_o, p\}$ These paths have opposite parity and have length greater than five, contradicting our choice of G.

Relabel the graph in Figure 2.4c as G_3.

2.0.2.2. *G does not have G_3 as a subgraph.*

Assume the contrary. As G is 3-connected, by Theorem 2.0.2, there is at least one path of G between w and $V(C) = V(G_3) \setminus N(w) = (v_3, v_4, v_5, v_1, v_2)$, where $N(v)$ is the set of vertices adjacent to v in G_3, as shown below in Figure 2.5.

By (2.0.2.1), since $(w, x, v_2, v_1, v_5, v_4, v_3)$ is a path of length 6 and $(w, v_0, v_1, v_5, v_4, v_3)$ is a path of length 5, there is no $w - v_3$ path that is internally disjoint from these two paths. By symmetry, there is no $w - v_1$ path that is internally disjoint from $V(C) \cup \{w, x, v_0\}$.

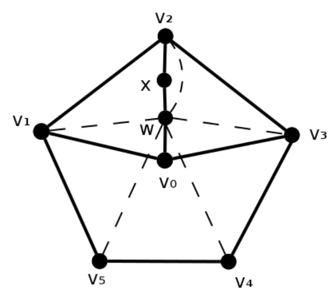


Figure 2.5: One of the dashed paths must exist

Similarly, using the paths $(w, v_0, v_3, v_2, v_1, v_5, v_4)$ of length 6, and $(w, w, x, v_2, v_1, v_5, v_4)$ of length 5 and (2.0.2.1), there is no $w - v_4$ path that is internally disjoint from $V(C) \cup \{w, x, v_0\}$. Again by symmetry, no $w - v_5$ path that is internally disjoint from $V(C) \cup \{w, x, v_0\}$ can exist. By the 6-path $(w, v_0, v_1, v_5, v_4, v_3, v_2)$, we see that $|p|$ must be even. By the 3-path (w, v_0, v_1, v_2), we see that p must be of length two. By symmetry between the cycle C and the cycle with the vertex set $\{v_0, v_1, v_5, v_4, v_3\}$, we deduce that G must have a $x - v_0$ path p' of length two that is internally disjoint from $V(C) \cup \{x, w, v_0\}$

We now know that G has a $w - v_2$ path p that is internally disjoint from $V(C) \cup \{w, x, v_0\}$.

If the $x - v_0$ path p' and the $w - v_2$ path p do not intersect, we create a 7-cycle with vertex set $\{v_1, v_2, y, w, x, z, v_0\}$ where y is the internal vertex on p and z is the internal vertex on p' as shown below in Figure 2.6. If the p and the p' path intersect, we create

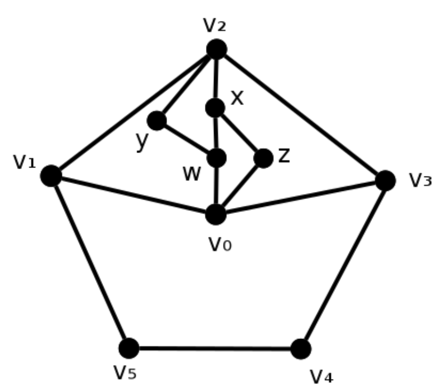


Figure 2.6: Paths p and p' do not intersect

a 7-cycle with vertex set $\{v_2, y, v_0, v_3, v_4, v_5, v_1\}$ where y is the common vertex on p and

P' as shown in Figure 2.7. We conclude that G cannot have G_3 as a subgraph, that is, (2.0.2.2) holds.

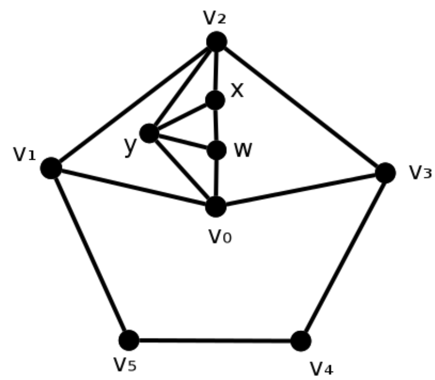


Figure 2.7: Paths p and p' intersect

Relabel the graph shown in Figure 2.4d as G_4. By Theorem 2.0.2, G must have a path from x to to $V(G_4) \setminus N(x) = \{v_3, v_4, v_5, w, v_1\}$ that is internally disjoint from $V(C) \cup \{v_0, x, w\}$ as shown in Figure 2.8.

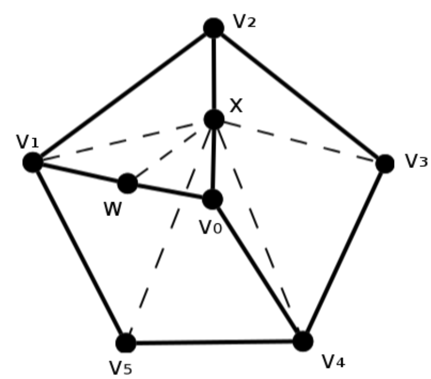


Figure 2.8: G_4

2.0.2.3. *G does not have G_4 as a subgraph.*

Assume the contrary. By (2.0.2.1), using the paths $(x, v_0, w, v_1, v_2, v_3)$ of length 5 and $(x, v_0, w, v_1, v_5, v_4, v_3)$ of length 6, there is no $x - v_3$ path that is internally disjoint from $V(C) \cup \{x, w, v_0\}$. By (2.0.2.1), using the paths $(x, v_0, w, v_1, v_5, v_4)$ and $(x, v_0, w, v_1, v_2, v_3, v_4)$ of length 5 and 6 respectively, there is no $x - v_4$ path that is internally disjoint from $V(C) \cup \{x, w, v_0\}$. Similarly, using the paths $(x, v_0, w, v_1, v_2, v_3, v_4, v_5)$ of length 7 and $(x, v_0, v_4, v_3, v_2, v_1, v_5)$ of length 6 and (2.0.2.1), there is no $x - v_5$

path that is internally disjoint from $V(C) \cup \{x, w, v_0\}$. Likewise, by (2.0.2.1), using the paths $(x, v_0, v_4, v_5, v_1, w)$ and $(x, v_2, v_1, v_5, v_4, v_0, w)$, there is no $x - w$ path that is internally disjoint from $V(C) \cup \{x, w, v_0\}$. Finally, using the paths $(x, v_0, v_4, v_3, v_2, v_1)$ and $(x, v_2, v_3, v_4, v_0, w, v_1)$, there is no $x - v_1$ path that is internally disjoint from $V(C) \cup \{x, w, v_0\}$. We conclude that G_4 cannot be a subgraph of G, that is, (2.0.2.3) holds.

Relabel the graph in Figure 2.4e as G_5.

2.0.2.4. *G does not have G_5 as a subgraph.*

Assume the contrary. By Theorem 2.0.2, there must be a path from x to $V(G_5) \setminus N(x) = (v_1, v_2, v_3, v_5)$ that is internally disjoint from $V(C) \cup \{x, v_0\}$ as shown in Figure 2.9.

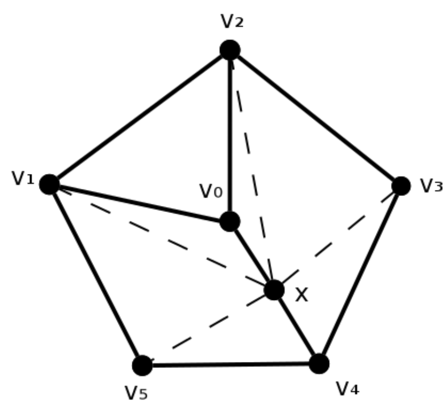


Figure 2.9: One of the dashed lines must exist

By (2.0.2.1), the paths $(x, v_4, v_3, v_2, v_0, v_1)$ and $(x, v_0, v_2, v_3, v_4, v_5, v_1)$ imply there is no $x - v_1$ path that is internally disjoint from $V(C) \cup \{x, v_0\}$. By symmetry, there is no $x - v_2$ path that is internally disjoint from $V(C) \cup \{x, v_0\}$. Similarly, the paths $(x, v_0, v_1, v_5, v_4, v_3)$ and $(x, v_4, v_5, v_1, v_0, v_2, v_3)$ imply there is no $x - v_3$ path that is internally disjoint from $V(C) \cup \{x, v_0\}$. By symmetry, there is no $x - v_5$ path that is internally disjoint from $V(C) \cup \{x, v_0\}$. We conclude that (2.0.2.4) holds.

This eliminates the cases in Figure 2.4 where p_1, p_2, or p_3 has more than one edge. So vertices in G not on the 5-cycle C are of the types 2.4a and 2.4b.

We start with the following observation.

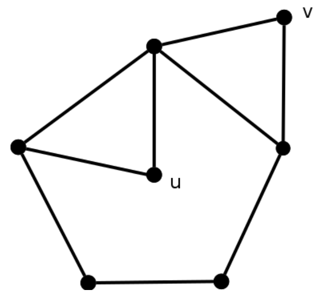


Figure 2.10: Forced 7-cycle

2.0.2.5. *If a 5-cycle in G has two or more distinct edges that belong to triangles whose third vertices avoid $V(C)$ and are distinct, then G has a 7-cycle*

If we follow the 5-cycle along replacing the edges that are common to the triangles with the other edges of those triangles, we get a cycle of length $5 - 1 + 2 - 1 + 2 = 7$ as shown in Figure 2.10.

Since $|V(G)| > 6$, we have more than one vertex not in $V(C)$. Suppose we have at least one vertex of type 2.4a not on C. Since each such vertex creates two triangles off of cycle C and the vertices are distinct, we will always have two edge-disjoint triangles each sharing a single edge with C. So by 2.0.2.5, we may not have graphs of type 2.4a as a subgraph.

We now know that all vertices not in $V(C)$ are of type 2.4b. Furthermore, the triangles that meet the 5-cycle must share the same edge; otherwise, we would create disjoint triangles, and thereby a contradiction of 2.0.2.5.

We are left with subgraphs that look like the following (see Figure 2.11).

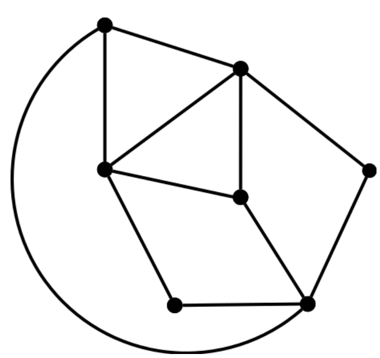


Figure 2.11: Forced configuration

All extra vertices added not on the 5-cycle meet at the same three points. We may add as many as we like.

Now we must check for possible additional edges within this graph without adding a larger odd cycle. In order to be 3-connected, v_3 and v_5 in our construction must have additional edges. By our previous argument, none of these edges can be to any of the vertices outside of the 5-cycle. This leaves v_2 and v_3 as possible neighbors for v_5 and v_1 and v_5 as possible neighbors for v_3.

If there is an edge $\{v_3, v_5\}$, we get the 7-cycle $(v_3, v_5, v_1, v_0, v_4, v_0, v_2)$. The edges $\{v_3, v_1\}$ and $\{v_2, v_5\}$ create no 7-cycles, so these are the desired necessary edges to complete 3-connectivity.

We look at the remaining possible edges. From our previous argument concerning v_0, we know all unknown edges meeting a vertex not in $V(C)$ must join to another vertex not in $V(C)$. Assume G has such an edge $\{v_0, v_0'\}$. This creates a 7-cycle $(v_1, v_0, v_0', v_2, v_3, v_4, v_5)$ as shown in Figure 2.12.

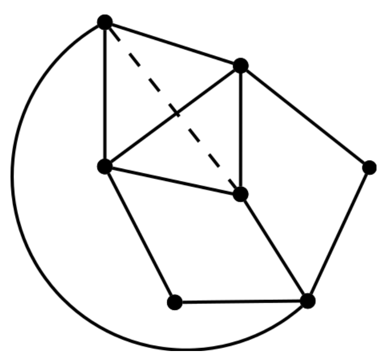


Figure 2.12: Forced 7-cycle

Now we need only look at possible edges from the vertices on the 5-cycle to other vertices on the 5-cycle. Remaining edges not in the graph are $\{v_1, v_4\}, \{v_2, v_4\}$, and $\{v_3, v_5\}$. We have already eliminated $\{v_3, v_5\}$. As $\{v_1, v_4\}$ and $\{v_2, v_4\}$ are symmetric, we only need check the cases where one or both are present. Neither causes a larger odd cycle.

This completes the construction of G. All vertices meet a 5-cycle at the same three vertices. This creates one side of our partition. The other two vertices of the 5-cycle connect to the three vertices. The three-vertex side of the bipartition may have one, two, or three edges between them.

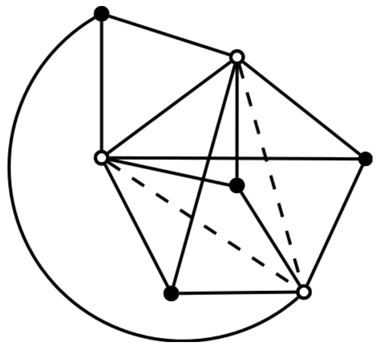


Figure 2.13: $K'_{3,n}$

□

Chapter 3 4-connected Graphs with No Odd Cycles Exceeding Size Seven

Here we extend the size of the possible odd cycles. The proof of the result is strikingly similar to the previous result in Section 1.2. The infinite class of graphs are built from a bipartite graph with the side of the bipartition that has four vertices having at least one edge.

Theorem 3.0.1. *A* 4*-connected simple graph* G *has no odd cycles of length exceeding seven if and only if*

(i) G *is bipartite;*

(ii) G *is a graph on eight or fewer vertices; or*

(iii) for some $n \geq 5$*, the graph* G *is isomorphic to a graph that is obtained from* $K_{4,n}$ *by adding* $1, 2, 3, 4, 5,$ *or* 6 *edges each having both ends in the* 4*-vertex side of the vertex bipartition as in Figure* **??**.

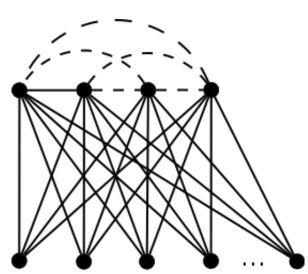


Figure 3.1: $K'_{4,n}$

Proof. We start with the following observation.

3.0.1.1. *If a* 7*-cycle in* G *has two or more distinct edges that belong to triangles whose third vertices avoid* $V(C)$ *and are distinct, then* G *has a* 9*-cycle.*

Follow the 7-cycle along replacing the edges that are common to the triangles with the other edges of those triangles to get a cycle of length $7 - 1 + 2 - 1 + 2 = 9$.

It is easily checked that the graphs mentioned in (i), (ii), and (iii) have no odd cycles of length exceeding exceeding seven. Now assume G is a 4-connected graph with a 7-cycle and $|V(G)| > 8$. Select a 7-cycle, C, with vertex set $V(C) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$. For all i in $\{1, 2, \ldots, 5\}$, let e_i be the edge $\{v_i, v_{i+1}\}$ where $v_8 = v_1$ (see Figure 3.2).

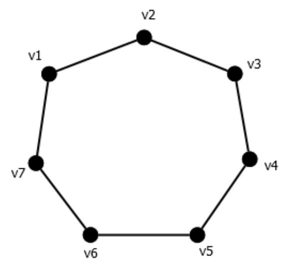


Figure 3.2: 7-cycle

Since $|V(G)| > 8$, there is a vertex in $V(G) - V(C)$; call it v_0. Since G is 4-connected, by Theorem 2.0.2, there are four paths from v_0 to $V(C)$ whose only common vertex is v_0. By symmetry, we have one of the configurations shown in Figure 3.3 where the wavy lines meeting v_0 correspond to paths. These paths are labeled p_1, p_2, p_3, and p_4 reading clockwise from the path p_1 that joins v_0 and v_1.

Consider the case shown in Figure 3.3a. We will use G_1 to denote such a graph. All cycles in G_1 must have even length, or length 3, 5, or 7, as G_1 is a subgraph of G. Consider the cycle $D_{1,2}$ using p_1 and p_2 through $(v_1, v_0, v_2, v_3, v_4, v_5, v_6, v_7)$, that is, $D_{1,2}$ uses the path p_1 from v_1 to v_0, the path p_2 from v_0 to v_2, and then the edges $\{v_i, v_{i+1}\}$ for all i in $\{2, 3, \ldots, 7\}$ where $v_8 = v_1$. Similarly, consider the cycles $D_{1,3}$ using p_1 and p_3 through $(v_1, v_0, v_3, v_4, v_5, v_6, v_7)$, and $D_{2,3}$ using p_2 and p_3 through $(v_2, v_0, v_3, v_4, v_5, v_6, v_7, v_1)$. Then $D_{1,2}, D_{1,3}$, and $D_{2,3}$ have lengths $|p_1|+|p_2|+6$, $|p_1|+|p_3|+5$, and $|p_2|+|p_3|+6$ respectively. If we sum the lengths of these cycles, we get $2|p_1| + 2|p_2| + 2|p_3| + 17$. Hence at least one of the cycles is odd. Thus we have a cycle of length seven as each $|p_i|$ is positive. Since

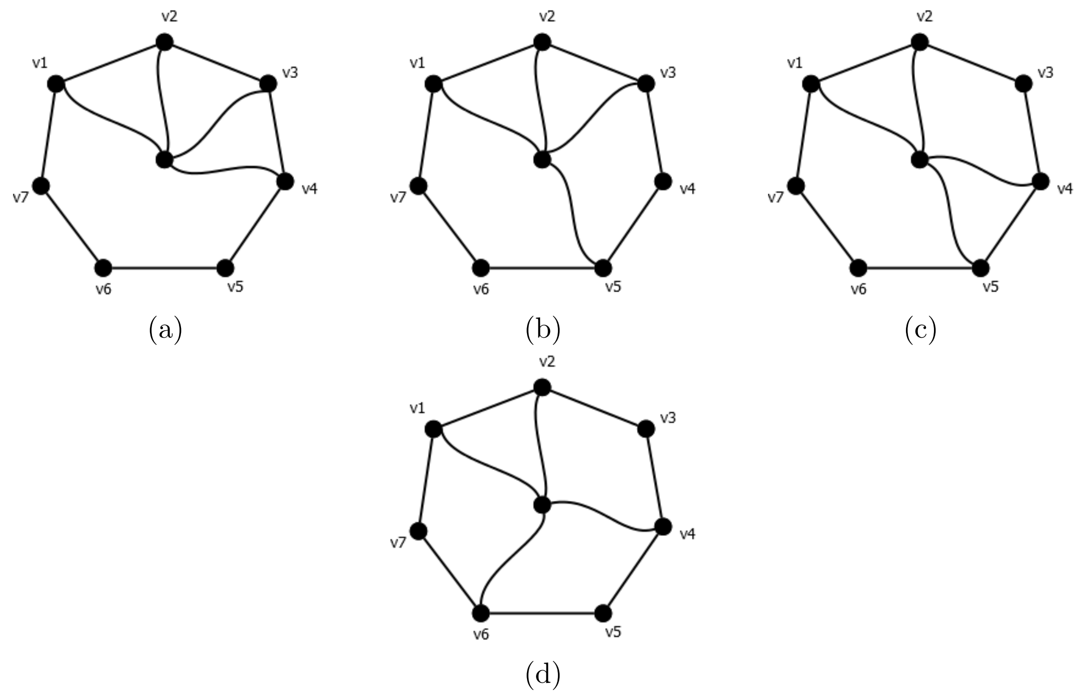


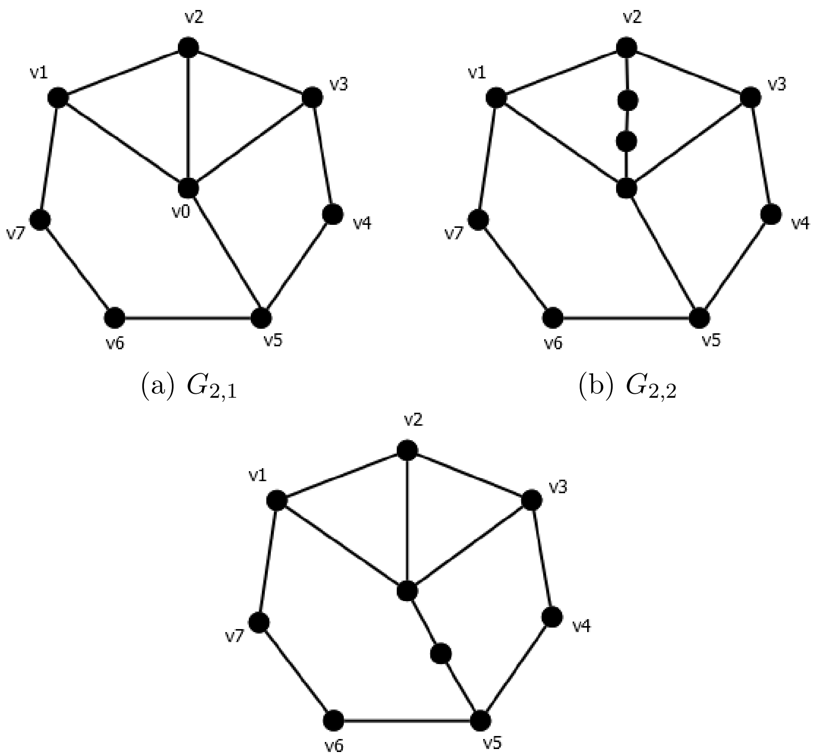
Figure 3.3: Possible path configurations

$D_{1,2}$ and $D_{2,3}$ have lengths exceeding seven, $D_{1,3}$ must have length seven. So $|p_1| = 1$ and $|p_3| = 1$. By symmetry, we see that $|p_2| = 1$ and $|p_4| = 1$. Thus we have the following.

3.0.1.2. *In G_1, each p_i has length* 1.

Now we consider the configuration shown in Figure 3.3b, which we will call G_2. Then G_2 has the same cycles $D_{1,2}$, $D_{1,3}$, and $D_{2,3}$ that were considered in G_1. Hence $|p_1| = 1$ and $|p_3| = 1$. Now we look at the cycle $F_{2,3}$ using p_2 and p_3 through $(v_2, v_0, v_3, v_4, v_5, v_6, v_7, v_1)$ of length $|p_2|+|p_3|+6 = |p_2|+7$, the cycle $F_{2,4}$ using p_2 and p_4 through $(v_2, v_0, v_5, v_6, v_7, v_1)$ of length $|p_2|+|p_4|+4$, and the cycle $F_{3,4}$ using p_3 and p_4 through $(v_3, v_0, v_5, v_6, v_7, v_1, v_2)$ of length $|p_3|+|p_4|+5 = |p_4|+6$. If we sum the lengths of these cycles, we get $2|p_2|+2|p_4|+17$. Thus at least one of the cycles is odd. Since $F_{2,3}$ has length $|p_2| + 7$, we see that this cycle is even, so $|p_2|$ is odd. If $F_{3,4}$ is odd, then $|p_4| = 1$. Since the cycle using p_2 and p_4 through $(v_2, v_0, v_5, v_6, v_7, v_1)$ is even, the cycle using p_2 and p_4 through $(v_2, v_0, v_5, v_4, v_3)$ is odd of length $|p_2| + |p_4| + 3 = |p_2| + 4$. So $|p_2| \in \{1, 3\}$. If $F_{2,4}$ is odd, then $|p_2| + |p_4| + 4 = 7$, so $|p_2| + |p_4| = 3$. Since $|p_2|$ is odd, $|p_2| = 1$ and $|p_4| = 2$. We deduce the following.

3.0.1.3. *G_2 is one of the graphs in Figure 3.4.*



(a) $G_{2,1}$ (b) $G_{2,2}$

(c) $G_{2,3}$

Figure 3.4: Path lengths of G_2

In the configuration in Figure 3.3c, which we will call G_3, we will use the same approach. Consider the cycle $H_{1,2}$ using p_1 and p_2 through $(v_1, v_0, v_2, v_3, v_4, v_5, v_6, v_7)$, the cycle $H_{1,3}$ using p_1 and p_3 through $\{v_1, v_0, v_4, v_5, v_6, v_7\}$, and the cycle $H_{2,3}$ using p_2 and p_3 through $(v_2, v_0, v_4, v_5, v_6, v_7, v_1)$ of lengths $|p_1|+|p_2|+6$, $|p_1|+|p_3|+4$, and $|p_2|+|p_3|+5$ respectively. If we sum the lengths of these cycles, we get $2|p_1| + 2|p_2| + 2|p_3| + 15$. Thus at least one cycle is odd; however, not all cycles are odd, since the first cycle has size larger than 7. Thus, either the second or the third cycle has odd length.

If $H_{1,3}$ is odd, then $|p_1| + |p_3| + 4 = 7$. Thus one of p_1 and p_3 has length 2 and one has length 1. Suppose $|p_1| = 1$ and $|p_3| = 2$. As $H_{1,2}$ has length $|p_2| + 7$, the path p_2 has odd length. As the cycle using p_3 and p_4 through $(v_4, v_0, v_5, v_6, v_7, v_1, v_2, v_3)$ has length $|p_4| + 8$, the path p_4 has even length. The cycle using p_2 and p_4 through $\{v_2, v_0, v_5, v_6, v_7, v_1\}$ has the length $|p_2| + |p_4| + 4$, which is odd. Thus $|p_2| = 1$ and $|p_4| = 2$. We deduce that when $|p_1| = 1$ and $|p_3| = 2$, we get $|p_2| = 1$ and $|p_4| = 2$.

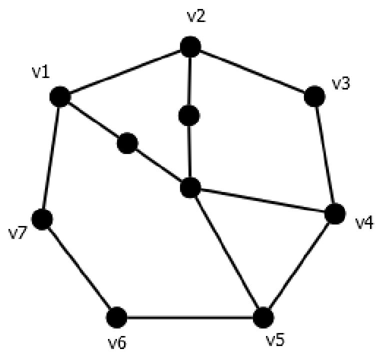
Now suppose $|p_1| = 2$ and $|p_3| = 1$. The cycle $H_{1,2}$ has length $|p_2| + 8$. Thus p_2 has even length. As the cycle using p_3 and p_4 through $(v_4, v_0, v_5, v_6, v_7, v_1, v_2, v_3)$ has length $|p_4| + 7$, the path p_4 has odd length. Again the cycle using p_2 and p_4 through $(v_2, v_0, v_5, v_6, v_7, v_1)$ has length $|p_2| + |p_4| + 4$, which is odd. Thus $|p_2| = 2$ and $|p_4| = 1$; that is, when $|p_1| = 2$ and $|p_1| = 1$, we get $|p_2| = 2$ and $|p_4| = 1$. This case is symmetric to the one noted earlier with $|p_1| = 1 = |p_2|$ and $|p_3| = 2 = |p_4|$.

Next suppose $H_{1,3}$ is even. Then $H_{2,3}$ is odd. Thus $|p_2| + |p_3| + 5 = 7$ so $|p_2| = 1$ and $|p_3| = 1$. The even cycle $H_{1,3}$ has length $|p_1| + 5$, so $|p_1|$ is odd. By the cycle using p_3 and p_4 through $(v_4, v_0, v_5, v_6, v_7, v_1, v_2, v_3)$, which has length $|p_4| + 7$, the path p_4 has odd length. The cycle using p_1 and p_4 through $(v_1, v_0, v_5, v_6, v_7)$ has length $|p_1| + |p_4| + 3 \leq 7$. So, we can have both p_1 and p_4 of length 1, or one of p_1 and p_4 is length 3 and the other is length 1. By the symmetry in G_3 between p_1 and p_4, this yields two additional cases. Summarizing the possibilities for G_3, we have the following.

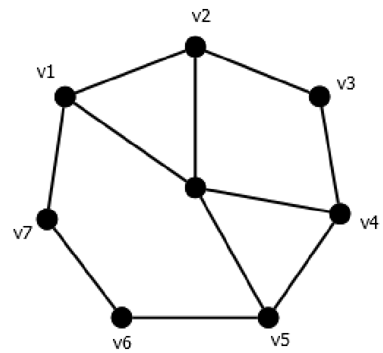
3.0.1.4. *G_3 is one of the three graphs shown in Figure 3.5.*

Next we consider the configuration in Figure 3.3d, which we shall call G_4. Consider the cycle $J_{1,2}$ using p_1 and p_2 through $(v_1, v_0, v_2, v_3, v_4, v_5, v_6, v_7)$ of length $|p_1| + |p_2| + 6$, the cycle $J_{1,3}$ using p_1 and p_3 through $(v_1, v_0, v_4, v_5, v_6, v_7)$ of length $|p_1| + |p_3| + 4$, and the cycle $J_{2,3}$ using p_2 and p_3 through $(v_2, v_0, v_4, v_5, v_6, v_7, v_1)$ of length $|p_2| + |p_3| + 5$. The sum of the lengths of the cycles is $2|p_1| + 2|p_2| + 2|p_3| + 15$. The length of $J_{1,2}$ and the fact that each path is non-empty imply that $J_{1,2}$ has even length and exactly one of the other two cycles is odd.

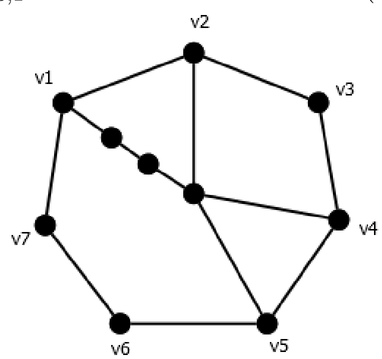
Suppose the length $|p_2| + |p_3| + 5$ of $J_{2,3}$ is odd. Then $|p_2| = 1$ and $|p_3| = 1$. The cycle $J_{1,3}$ of length $|p_1| + |p_3| + 4$ is even by assumption, so $|p_1|$ is odd. Hence the cycle using p_1 and p_3 through $(v_1, v_0, v_4, v_3, v_2)$ of length $|p_1| + |p_3| + 3 = |p_1| + 4$ is odd. Thus $|p_1|$ is 1 or 3. From



(a) $G_{3,1}$



(b) $G_{3,2}$



(c) $G_{3,3}$

Figure 3.5: Path lengths of G_3

the cycle using p_2 and p_4 through $(v_2, v_0, v_6, v_5, v_4, v_3)$ of length $|p_2|+|p_4|+4 = |p_4|+5$, we deduce that $|p_4| \in \{1, 2\}$. The two cycles using p_1 and p_4 and the rim of the outer 7-cycle of lengths $|p_1| + |p_4| + 2$ and $|p_1| + |p_4| + 5$ give us cases with $|p_1| = 1$ and $|p_4| = 1$, with $|p_1| = 1$ and $|p_4| = 2$, and with $|p_1| = 3$ and $|p_4| = 2$.

Suppose $J_{1,3}$ is odd. Then $|p_1| + |p_3| + 4 = 7$, so one of p_1 and p_3 has length 1 and one has length 2. If p_1 has length 2 and p_3 has length 1, then the size of the cycle using p_1 and p_4 through $(v_1, v_0, v_6, v_5, v_4, v_3, v_2)$ is $|p_1| + |p_4| + 5 = |p_4| + 7$. Thus $|p_4|$ is odd. From the cycle $J_{1,2}$ of length $|p_1| + |p_2| + 6 = |p_2| + 8$, we deduce that $|p_2|$ is even. Since $|p_2|$ is even and $|p_4|$ is odd, the cycle using p_2 and p_4 through $(v_2, v_0, v_6, v_5, v_4, v_3)$ has odd length $|p_2| + |p_4| + 4 = 7$. So we get $|p_2| = 2$ and $|p_4| = 1$.

If p_1 has length 1 and p_3 has length 2, then the size of the cycle using p_3 and p_4 through $(v_4, v_0, v_6, v_7, v_1, v_2, v_3)$ is $|p_3| + |p_4| + 5 = |p_4| + 7$. Thus the path p_4 has odd length. From the cycle using p_1 and p_4 through $(v_1, v_0, v_6, v_5, v_4, v_3, v_2)$ of length $|p_1|+|p_4|+5 = |p_4|+6$, we deduce that $|p_4| = 1$. The cycle $J_{1,2}$ has length $|p_1| + |p_2| + 6 = |p_2| + 7$. Thus the path p_2 has odd length. From the cycle using p_2 and p_3 through (v_2, v_0, v_4, v_3) of length $|p_2|+|p_3|+2 = |p_2|+4 \leq 7$, we see that $p_2 \in \{1, 3\}$. Thus $(|p_1|, |p_2|, |p_3|, |p_4|)$ is $(1, 1, 2, 1)$ or $(1, 3, 2, 1)$. These cases are symmetric to those with $(|p_1|, |p_2|, |p_3|, |p_4|)$ equal to $(1, 1, 1, 2)$ or $(3, 1, 1, 2)$, which were identified earlier. We conclude this case by noting that the following holds.

3.0.1.5. *G_4 is one of the four graphs shown in Figure 3.6.*

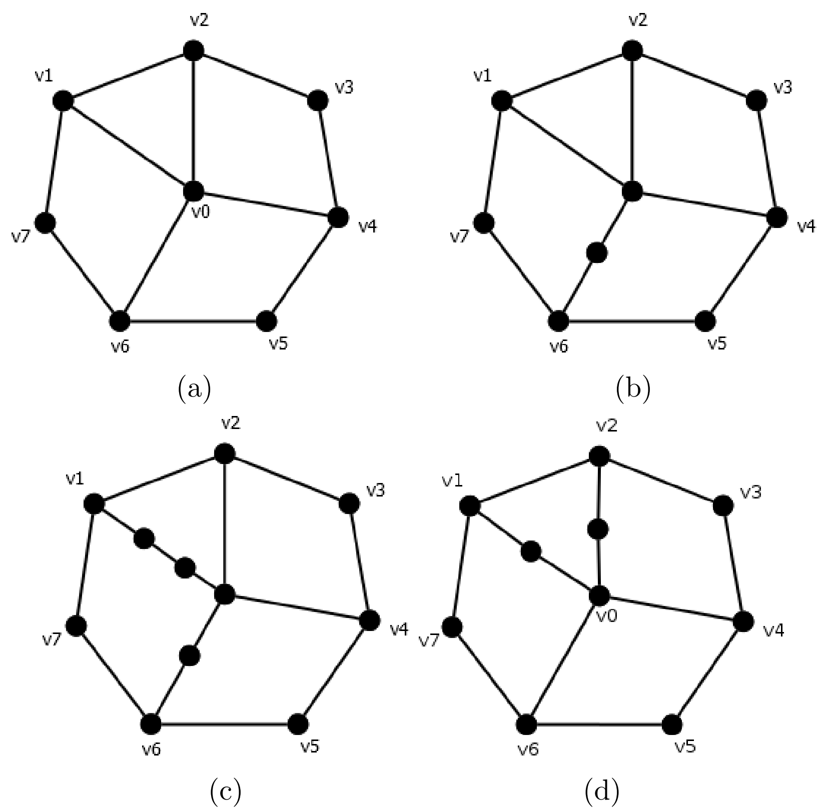


Figure 3.6: Path lengths of G_4

Summarizing our analysis above, we see that we showed that there is a single possibility for G_1, the one in which all of p_1, p_2, p_3, and p_4 have length one. There are three possibilities for each of G_2 and G_3, these being shown in Figures 3.4 and 3.5. Finally, there are four possibilities for G_4, these being shown in Figure 3.6. We will continue our argument by looking first at the graphs above in which some p_i has more than one edge. The following observation plays a key role for much of the rest of the argument.

3.0.1.6. *Let u and v be vertices of G such that G contains an even-length path p_e and an odd-length path p_o joining u and v. If $|p_e| \geq 8$ and $|p_o| \geq 7$, then G has no path p that joins u and v and is internally disjoint from both p_e and p_o.*

If such a p existed, we could examine the cycles $\{p_e, p\}$ and $\{p_o, p\}$, which have opposite parity and have size greater than seven. This contradicts our choice of G.

Recall the graph in Figure 3.4b is $G_{2,2}$.

3.0.1.7. *G does not have $G_{2,2}$ as a subgraph.*

Assume the contrary. By Theorem 2.0.2, there must be two paths from x to $V(G_{2,2}) \setminus N(x) = \{v_0, v_1, v_3, v_4, v_5, v_6, v_7\}$ that have only x in common and that are internally disjoint from $V(C) \cup \{v_2, x, w\}$ (see Figure 3.7).

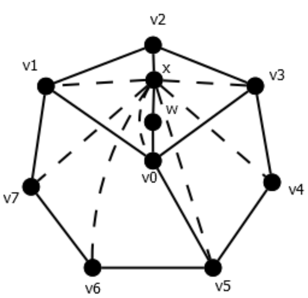


Figure 3.7: Possible paths from x in $G_{2,2}$

By (3.0.1.6), using the paths $(x, v_2, v_1, v_7, v_6, v_5, v_4, v_3)$ of length 7 and $(x, w, v_0, v_1, v_7, v_6, v_5, v_4, v_3)$ of length 8, there is no $x - v_3$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. By (3.0.1.6), using the paths $(x, w, v_0, v_1, v_7, v_6, v_5, v_4)$ and $(x, v_2, v_1, v_7, v_6, v_5, v_0, v_3, v_4)$ of lengths 7 and 8 respectively, there is no $x - v_4$ path that is internally disjoint $V(C) \cup \{v_0, x, w\}$. Again, by (3.0.1.6), using the paths $(x, w, v_0, v_3, v_2, v_1, v_7, v_6, v_5)$ and $(x, v_2, v_3, v_0, v_1, v_7, v_6, v_5)$ of lengths 8 and 7 respectively, there are no $x - v_5$ paths that are internally disjoint from $V(C) \cup \{v_0, x, w\}$. Similarly, we will find no $x - v_6$ paths that are internally disjoint from $V(C) \cup \{v_0, x, w\}$, by (3.0.1.6) using the paths $(x, v_2, v_3, v_4, v_5, v_0, v_1, v_7, v_6)$ and

$(x, w, v_0, v_3, v_2, v_1, v_7, v_6)$. By the paths $(x, v_2, v_3, v_4, v_5, v_0, v_1, v_7)$ and $(x, w, v_0, v_5, v_4, v_3, v_2, v_1, v_7)$, there are no $x - v_7$ paths that are internally disjoint from $V(C) \cup \{v_0, x, w\}$. By the paths $(x, v_2, v_3, v_4, v_5, v_6, v_7, v_1)$ and $(x, w, v_0, v_3, v_4, v_5, v_6, v_7, v_1)$, there are no $x - v_1$ paths that are internally disjoint from $V(C) \cup \{v_0, x, w\}$.

We now know that a path from x to $\{v_0, v_1, v_3, v_4, v_5, v_6, v_7\}$ that is internally disjoint from $V(C) \cup \{v_2, x, w\}$ must end in v_0. As there are at least two such paths that meet only in x, we conclude that G cannot have $G_{2,2}$ as a subgraph, that is, (3.0.1.7) holds.

Recall that the graph in Figure 3.4c is $G_{2,3}$.

3.0.1.8. *G does not have $G_{2,3}$ as a subgraph.*

Assume the contrary. By Theorem 2.0.2, G has a path from x to $V(G_{2,3}) \setminus N(x) = \{v_1, v_2, v_3, v_4, v_6, v_7\}$ that is internally disjoint from $V(C) \cup \{v_0, x\}$ as shown in Figure 3.8.

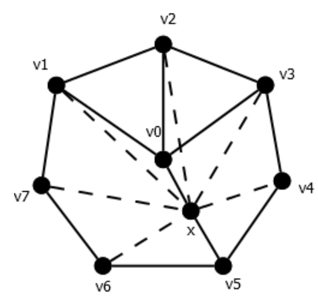


Figure 3.8: Possible paths from x in $G_{2,3}$

By (3.0.1.6), the paths $(x, v_0, v_3, v_4, v_5, v_6, v_7, v_1)$ and $(x, v_0, v_2, v_3, v_4, v_5, v_6, v_7, v_1)$ of lengths 7 and 8 imply there is no $x - v_1$ path internally disjoint from $V(C) \cup \{v_0, x\}$. Again, the paths $(x, v_5, v_6, v_7, v_1, v_0, v_3, v_2)$ and $(x, v_0, v_3, v_4, v_5, v_6, v_7, v_1, v_2)$ of lengths 7 and 8 imply there is no $x - v_2$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$. By paths $(x, v_0, v_2, v_1, v_7, v_6, v_5, v_4, v_3)$ and $(x, v_0, v_1, v_7, v_6, v_5, v_4, v_3)$, there is no $x - v_3$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$. If we consider the paths $(x, v_5, v_6, v_7, v_1, v_0, v_3, v_4)$ and $(x, v_5, v_6, v_7, v_1, v_0, v_2, v_3, v_4)$, then we find that there is no $x - v_4$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$. Again, the paths $(x, v_5, v_4, v_3, v_2, v_1, v_7, v_6)$ and $(x, v_5, v_4, v_3, v_0, v_2, v_1, v_7, v_6)$ imply that there is no $x - v_6$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$.

Similarly, by the paths $(x, v_5, v_4, v_3, v_0, v_2, v_1, v_7)$ and $(x, v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$, there is no $x - v_7$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$. Thus x has no paths to $\{v_1, v_2, v_3, v_4, v_6, v_7\}$ internally disjoint from $V(C) \cup \{v_0, x\}$. Thus (3.0.1.8) holds.

Recall the graph in Figure 3.5a is $G_{3,1}$.

3.0.1.9. *G does not have $G_{3,1}$ as a subgraph.*

Assume the contrary. By Theorem 2.0.2, there must be some path from x to $V(G_{2,3}) \setminus N(x) = \{v_1, v_3, v_4, v_5, v_6, v_7, w\}$ that is internally disjoint from $V(C) \cup \{v_0, x, w\}$ as shown in Figure 3.9.

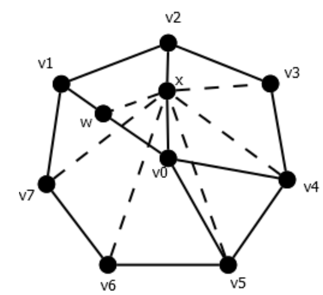


Figure 3.9: Possible paths from x in $G_{3,1}$

By (3.0.1.6), the paths $(x, v_2, v_3, v_4, v_5, v_6, v_7, v_1)$ and $(x, v_2, v_3, v_4, v_0, v_5, v_6, v_7, v_1)$ imply that there is no $x - v_1$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. From the paths $(x, v_2, v_1, v_7, v_6, v_5, v_4, v_3)$ and $(x, v_2, v_1, v_7, v_6, v_5, v_0 v_4, v_3)$, we see that there is no $x - v_3$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Again, the paths $(x, v_0, w, v_1, v_7, v_6, v_5, v_4)$ and $(x, v_0, v_5, v_6, v_7, v_1, v_2, v_3, v_4)$ of lengths 7 and 8 respectively imply there is no $x - v_4$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Similarly, the paths $(x, v_0, w, v_1, v_2, v_3, v_4, v_5)$ and $(x, v_0, v_4, v_3, v_2, v_1, v_7, v_6, v_5)$ imply that there is no $x - v_5$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. By the paths $(x, v_0, v_4, v_3, v_2 v_1, v_7, v_6)$ and $(x, v_0, v_5, v_4, v_3, v_2, v_1, v_7, v_6)$, there is no $x - v_6$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Similarly, the paths $(x, v_2, v_1, w, v_0, v_5, v_6, v_7)$ and $(x, v_2, v_1, w, v_0, v_4, v_5, v_6, v_7)$ imply that there is no $x - v_7$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Finally, the paths $(x, v_0,$

$v_4, v_5, v_6, v_7, v_1, w)$ and $(x, v_2, v_3, v_4, v_5, v_6, v_7, v_1, w)$ imply there is no $x - w$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Thus x has no paths to $\{v_1, v_3, v_4, v_5, v_6, v_7, w\}$ internally disjoint from $V(C) \cup \{v_0, x, w\}$. Thus (3.0.1.9) holds.

Recall the graph in Figure 3.5c is $G_{3,3}$.

3.0.1.10. *G does not have $G_{3,3}$ as a subgraph.*

Assume the contrary. By Theorem 2.0.2, there must be two paths from x to $V(G_{3,3}) \setminus N(x) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ that have only the vertex x in common and that are internally disjoint from $V(C) \cup \{v_0, x, w\}$ (see Figure 3.10).

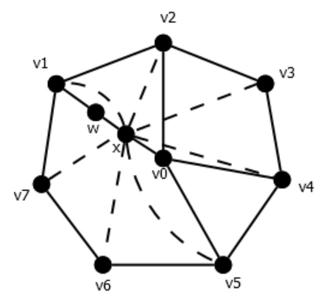


Figure 3.10: Possible paths from x in $G_{3,3}$

By (3.0.1.6), the paths $(x, v_0, v_4, v_5, v_6, v_7, v_1, v_2)$ and $(x, w, v_1, v_7, v_6, v_5, v_4, v_3, v_2)$ of lengths 7 and 8 imply there is no $x - v_2$ path internally disjoint from $V(C) \cup \{v_0, x, w\}$. Similarly, by the paths $(x, w, v_1, v_7, v_6, v_5, v_4, v_3)$ and $(x, v_0, v_4, v_5, v_6, v_7, v_1, v_2, v_3)$ of lengths 7 and 8 respectively, there is no $x - v_3$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. By paths $(x, v_0, v_2, v_1, v_7, v_6, v_5, v_4)$ and $(x, v_0, v_5, v_6, v_7, v_1, v_2, v_3, v_4)$, there is no $x - v_4$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. If we consider the paths $(x, w, v_1, v_2, v_3, v_4, v_0, v_5)$ and $(x, v_0, v_4, v_3, v_2, v_1, v_7, v_6, v_5)$, then we find that there is no $x - v_5$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Again, the paths $(x, v_0, v_4, v_3, v_2, v_1, v_7, v_6)$ and $(x, v_0, v_5, v_4, v_3, v_2, v_1, v_7, v_6)$ imply that there is no $x - v_6$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Similarly, by the paths $(x, v_0, v_2, v_3, v_4, v_5, v_6, v_7)$ and $(x, w, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$, there is no $x - v_7$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$.

This leaves only v_1 in $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ that can be the end of a path from x that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Since there are two such paths that have only the vertex x in common, we conclude that G cannot have $G_{3,3}$ as a subgraph, that is, (3.0.1.10) holds.

Recall the graph in Figure 3.6b is $G_{4,2}$.

3.0.1.11. *G does not have $G_{4,2}$ as a subgraph.*

Assume the contrary. By Theorem 2.0.2, there must be two paths from x to $V(G_{4,2}) \setminus N(x) = \{v_1, v_2, v_3, v_4, v_5, v_7\}$ that have only the vertex x in common and that are internally disjoint from $V(C) \cup \{v_0, x\}$ (see Figure 3.11).

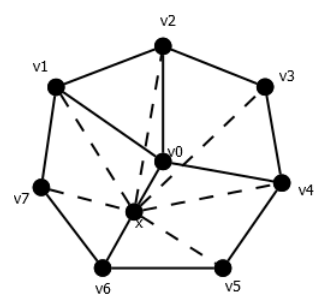


Figure 3.11: Possible paths from x in $G_{4,2}$

By (3.0.1.6), the paths $(x, v_6, v_5, v_4, v_3, v_2, v_0, v_1)$ and $(x, v_0, v_2, v_3, v_4, v_5, v_6, v_7, v_1)$ of lengths 7 and 8 imply there is no $x{-}v_1$ path internally disjoint from $V(C) \cup \{v_0, x\}$. Similarly, by the paths $(x, v_0, v_4, v_5, v_6, v_7, v_1, v_2)$ and $(x, v_0, v_1, v_7, v_6, v_5, v_4, v_3, v_2)$, there is no $x{-}v_2$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$. By paths $(x, v_0, v_1, v_7, v_6, v_5, v_4, v_3)$ and $(x, v_0, v_2, v_1, v_7, v_6, v_5, v_4, v_3)$, there is no $x - v_3$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$. If we consider the paths $(x, v_6, v_7, v_1, v_2, v_3, v_4, v_5)$ and $(x, v_0, v_4, v_3, v_2, v_1, v_7, v_6, v_5)$, then we find that there is no $x - v_5$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$. Again, the paths $(x, v_6, v_5, v_4, v_3, v_2, v_1, v_7)$ and $(x, v_0, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ imply that there is no $x - v_7$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$. The paths $(x, v_0, v_2, v_1, v_7, v_6, v_5, v_4)$ of length 7 and $(x, v_6, v_7, v_1, v_2, v_3, v_4)$ of length 6 imply that any $x - v_4$ path that is internally disjoint from $V(C) \cup \{v_0, x\}$ must have length 1. As the graph G is 4-connected, if $G_{4,2}$ is a subgraph, there can only be one such path from x to v_4. Therefore we do not have the required two paths from x to $V(G_{4,2}) \setminus N(x)$. Thus (3.0.1.11) holds.

Recall the graph in Figure 3.6c is $G_{4,3}$.

3.0.1.12. *G does not have $G_{4,3}$ as a subgraph.*

Assume the contrary. By Theorem 2.0.2, there must be two paths from x to $V(G_{4,3}) \setminus N(x) = \{v_1, v_2, v_3, v_4, v_5, v_7, u, w\}$ that have only the vertex x in common and that are internally disjoint from $V(C) \cup \{v_0, x, u, w\}$ (see Figure 3.12).

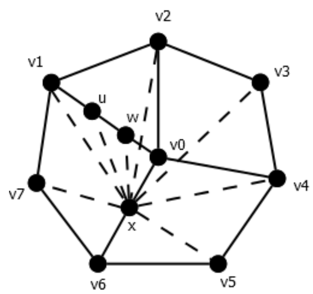


Figure 3.12: Possible paths from x in $G_{4,3}$

By (3.0.1.6), the paths $(x, v_6, v_5, v_4, v_0, w, u, v_1)$ and $(x, v_0, v_2, v_3, v_4, v_5, v_6, v_7, v_1)$ of lengths 7 and 8 imply that there is no $x{-}v_1$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. The paths $(x, v_0, v_4, v_5, v_6, v_7, v_1, v_2)$ and $(x, v_6, v_5, v_4, v_0, w, u, v_1, v_2)$ imply there is no $x{-}v_2$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$; the paths $(x, v_6, v_7, v_1, v_2, v_0, v_4, v_3)$ and $(x, v_0, v_2, v_1, v_7, v_6, v_5, v_4, v_3)$ imply there is no $x{-}v_3$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. Similarly, by the paths $(x, v_0, w, u, v_1, v_2, v_3, v_4)$ and $(x, v_0, w, u, v_1, v_7, v_6, v_5, v_4)$, there is no $x - v_4$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. By the paths $(x, v_6, v_7, v_1, v_2, v_3, v_4, v_5)$ and $(x, v_6, v_7, v_1, u, w, v_0, v_4, v_5)$, there is no $x - v_5$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. Similarly, the paths $(x, v_6, v_5, v_4, v_3, v_2, v_1, v_7)$ and $(x, v_6, v_5, v_4, v_0, w, u, v_1, v_7)$ imply that there is no $x - v_7$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. The paths $(x, v_6, v_5, v_4, v_3, v_2, v_1, u)$ and $(x, v_6, v_5, v_4, v_3, v_2, v_0, w, u)$ imply there is no $x - u$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. Finally, by the paths $(x, v_6, v_5, v_4, v_3, v_2, v_0, w)$ and $(x, v_6, v_5, v_4, v_3, v_2,$

$v_1, u, w)$, there is no $x - w$ path that is internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. Thus x has no paths to $\{v_1, v_2, v_3, v_4, v_5, v_7, u, w\}$ internally disjoint from $V(C) \cup \{v_0, x, u, w\}$. Therefore (3.0.1.12) holds.

Recall the graph in Figure 3.6d is $G_{4,4}$.

3.0.1.13. *G does not have $G_{4,4}$ as a subgraph.*

Assume the contrary. By Theorem 2.0.2, there must be two paths from x to $V(G_{4,4}) \setminus N(x) = \{v_2, v_3, v_4, v_5, v_6, v_7, w\}$ that have only the vertex x in common and that are internally disjoint from $V(C) \cup \{v_0, x, w\}$ (see Figure 3.13).

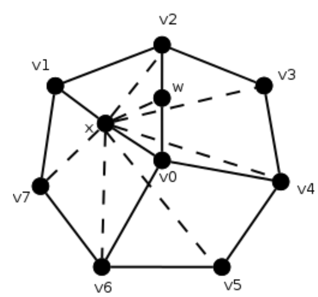
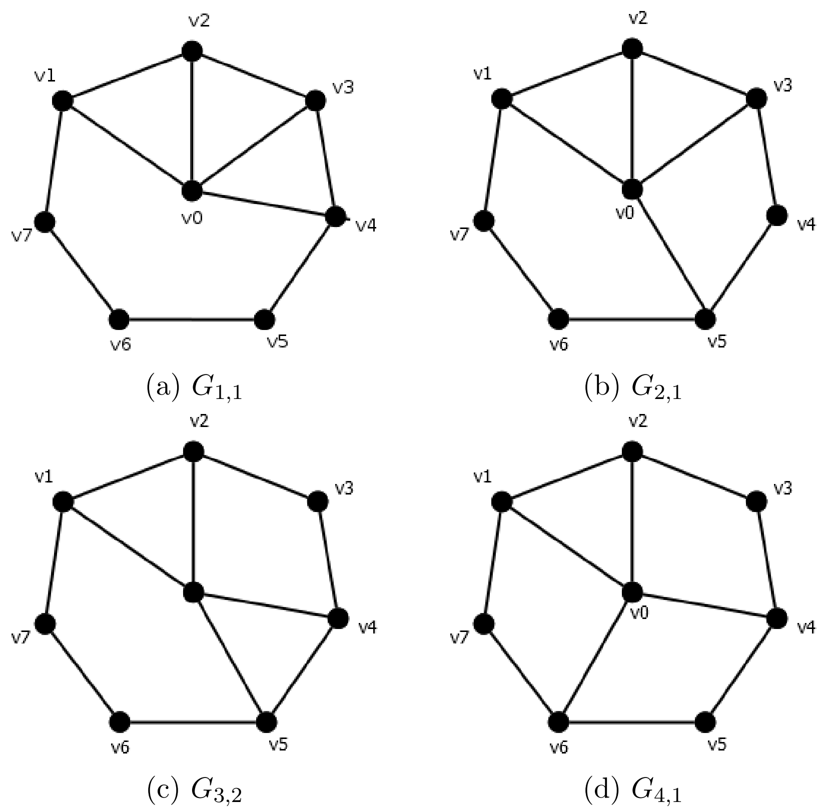


Figure 3.13: Possible paths from x in $G_{4,4}$

By (3.0.1.6), the paths $(x, v_1, v_7, v_6, v_5, v_4, v_3, v_2)$ and $(x, v_1, v_7, v_6, v_5, v_4, v_0, w, v_2)$ of lengths 7 and 8 imply there is no $x - v_2$ path internally disjoint from $V(C) \cup \{v_0, x, w\}$; the paths $(x, v_1, v_7, v_6, v_0, w, v_2, v_3)$ and $(x, v_1, v_2, w, v_0, v_6, v_5, v_4, v_3)$ imply there is no $x - v_3$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. By paths $(x, v_1, v_2, w, v_0, v_6, v_5, v_4)$ and $(x, v_1, v_7, v_6, v_0, w, v_2, v_3, v_4)$, there is no $x - v_4$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. If we consider the paths $(x, v_1, v_2, v_3, v_4, v_0, v_6, v_5)$ and $(x, v_0, v_4, v_3, v_2, v_1, v_7, v_6, v_5)$, then we find that there is no $x - v_5$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Again, the paths $(x, v_1, v_2, v_3, v_4, v_5, v_6, v_7)$ and $(x, v_0, w, v_2, v_3, v_4, v_5, v_6, v_7)$ imply that there is no $x - v_7$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Similarly, by the paths $(x, v_0, v_6, v_5, v_4, v_3, v_2, w)$ and $(x, v_1, v_7, v_6, v_5, v_4, v_3, v_2, w)$, there is no $x - w$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$. Finally, by the paths $(x, v_0, w, v_2, v_3, v_4, v_5, v_6)$ of length 7 and $(x, v_1, v_2, v_3, v_4, v_5, v_6)$ of length 6, any $x - v_6$ path that is internally disjoint from $V(C) \cup \{v_0, x, w\}$ has length 1. Thus there is only one such $x - v_6$ path; a contradiction. Therefore (3.0.1.13) holds.

This eliminates all cases where p_1, p_2, p_3, or p_4 has more than one edge. Thus each vertex in G that is not in the selected 7-cycle is of a type shown in Figure 3.14.



(a) $G_{1,1}$ (b) $G_{2,1}$

(c) $G_{3,2}$ (d) $G_{4,1}$

Figure 3.14: Only single-edge paths

Since $|V(C)| > 8$, we have more than one vertex not in $V(C)$. Suppose we have at least one vertex of type 3.14a, 3.14b, or 3.14c not on C. Since each such vertex creates two triangles sharing a single edge with C, we will always have two edge-disjoint triangles sharing a single edge with C, as every vertex not on C is in at least one triangle with an edge of C. So, by (3.0.1.1), G has no vertices of type 3.14a, 3.14b, or 3.14c. Thus all vertices not on C are of type 3.14d. Furthermore, if we have two such vertices, they must be adjacent to the same four vertices of C by (3.0.1.1) again. We deduce that G has the graph in Figure 3.15 as a subgraph.

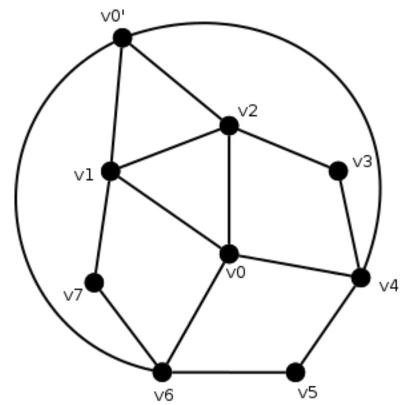


Figure 3.15: A subgraph of G

Now we must check for additional possible edges within G that do not create a larger odd cycle. In order for G to be 4-connected, v_3, v_5, and v_7 in our subgraph must be connected to two additional vertices to have degree four. As $|V(C)| > 8$, v_1, v_2, v_4, and v_6 already have degree at least four. By our previous argument, each of v_3, v_5, and v_7 can only be adjacent to other vertices of C. This leaves v_1, v_5, v_6 and v_7 as possible additional neighbors for v_3, while v_1, v_2, v_3, and v_7 are possible additional neighbors for v_5. Finally, v_2, v_3, v_4, and v_5 are possible additional neighbors for v_7.

If there is an edge $\{v_3, v_5\}$, we get a 9-cycle $(v_5, v_6, v_7, v_1, v_0, v_2, v_0', v_4, v_3)$. By symmetry, $\{v_5, v_7\}$ is not an edge. If there is an edge $\{v_3, v_7\}$, we get a 9-cycle $(v_3, v_4, v_5, v_6, v_0', v_2, v_0, v_1, v_7)$. For v_3, the edges $\{v_1, v_3\}$ and $\{v_6, v_3\}$ remain as possibilities. These edges create no 9-cycles and therefore are the desired necessary edges to complete degree requirements. For v_5, the edges $\{v_1, v_5\}$ and $\{v_2, v_5\}$ remain as possibilities. These edges create no 9-cycles and therefore are the desired necessary edges to complete degree requirements. For v_5, the edges $\{v_2, v_7\}$ and $\{v_4, v_7\}$ remain as possibilities. These edges create no 9-cycles and therefore are the desired necessary edges to complete degree requirements. Our subgraph is now 4-connected.

Now we examine remaining possible edges. From our previous argument, we know all edges that meet vertices in $V(C)$ and not in $V(C)$. If any such vertex v_0 meets an additional edge, this edge must be $\{v_0, v_0'\}$ for some v_0' not in $V(C)$. Assume such an edge exists. This would create a 9-cycle $(v_0, v_2, v_3, v_4, v_5, v_6, v_7, v_1, v_0')$. Thus there is no edge between any vertices outside the 7-cycle C.

The remaining possible edges are $\{v_1, v_4\}$, $\{v_1, v_6\}$, $\{v_2, v_4\}$, $\{v_2, v_6\}$ and $\{v_4, v_6\}$. If we include all such edges, we do not create an odd cycle larger than a 7-cycle. Thus if we include any subset of these edges, we will not create such an odd cycle.

This concludes our construction of G. All vertices not in the 7-cycle C are adjacent to the same four vertices of C. In our construction, these vertices are v_1, v_2, v_4, and v_6. These four vertices form one side of a bipartition. The other three vertices of the 7-cycle are adjacent to all four of these vertices (see Figure 3.16). The subgraph induced by the four-vertex side of the bipartition is any subgraph of K_4 having at least one edge. We conclude that Theorem 3.0.1 holds. □

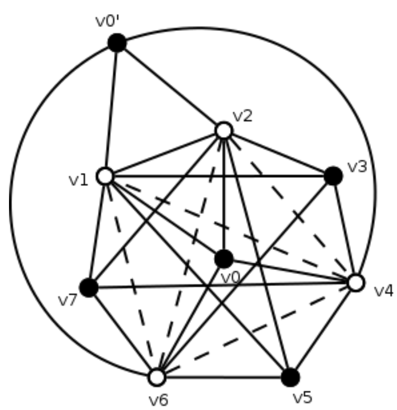


Figure 3.16: $K'_{4,n}$

Chapter 4 n-connected Graphs with No Odd Cycles Exceeding Size $2n-1$

In this chapter, we will generalize the graph results of the earlier chapters.

Theorem 4.0.1. *Suppose $n \geq 2$. Let G be an n-connected simple graph having a cycle of length $2n-1$. Then G has no odd cycle of length exceeding $2n-1$ if and only if*

(i) $|V(G)| \leq 2n$; or

(ii) for some $t \geq n+1$, the graph G is isomorphic to a graph that is obtained from $K_{n,t}$ by adding at least one and at most $\frac{n(n-1)}{2}$ edges each having both ends in the n-vertex side of the vertex bipartition.

Proof. By Theorem 2.2, Theorem 3.0.1, and Theorem 1.0.1, the theorem holds for graphs with $n < 5$. Let $n \geq 5$. If G satisfies (i) or (ii), it is straightforward to check that G has no odd cycles of size exceeding $2n-1$.

Conversely, suppose G has no odd cycles of length exceeding $2n-1$. If $|V(G)| < 2n+1$, there is no larger odd cycle. Assume $|V(G)| \geq 2n+1$. Select a $(2n-1)$-cycle C of G and label its vertices, in order, by $v_1, v_2, \ldots, v_{2n-1}$. Since $|V(G)| > 2n-1$, there is an additional vertex outside of $V(C)$.

We will now take note of the following observations.

4.0.1.1. *Suppose v_a and v_{a+1} are consecutive vertices on the cycle C and there are paths p_a and p_{a+1} from v_a and v_{a+1} to some vertex u not on C such that these paths meet only in u. Then $|p_a|$ and $|p_{a+1}|$ have the same parity.*

Assume not. Then we have a cycle consisting of a path in C from v_{a+1} to v_a having length exceeding one along with the paths p_a and p_{a+1}. This cycle has length $2n-2$ plus the sum of two numbers of opposite parities. So we have an odd cycle of length $2n+1$ or greater. Thus $|p_a|$ and $|p_{a+1}|$ have the same parity.

Next we show the following.

4.0.1.2. *Suppose* v_a *and* v_{a+2} *are two vertices on the cycle* C *that are distance two apart on* C*, and assume there are paths* p_a *and* p_{a+2} *from* v_a *and* v_{a+2} *to some vertex* u *not on* C *such that these two paths meet only in* u*. Then* $|p_a|$ *and* $|p_{a+2}|$ *are either both one or they are have different parities.*

In G, we have a cycle D consisting of a path in C from v_{a+2} to v_a of length $2n-1-2$ along with the paths p_a and p_{a+2}. This cycle has length $2n-1-2+|p_a|+|p_{a+2}|$. If D has odd length, then $|p_a|=1$ and $|p_{a+2}|=1$. If D has even length, then $|p_a|+|p_{a+2}|$ is odd, and the paths p_a and p_{a+2} have opposite parities.

4.0.1.3. *Suppose* G *has distinct vertices* v_0 *and* v_0' *not on* C*. Assume* C *has distinct edges* $\{v_a, v_{a+1}\}$ *and* $\{v_b, v_{b+1}\}$ *such that there are paths* p_a *and* p_{a+1} *from* v_a *and* v_{a+1} *to* v_0 *that meet only in* v_0*, and there are paths* p_b *and* p_{b+1} *from* v_b *and* v_{b+1} *to* v_0' *that meet only in* v_0'*. Assume also that* p_a *and* p_{a+1} *are vertex disjoint from* p_b *and* p_{b+1} *except that* v_a *may equal* v_{b+1}*, or* v_{a+1} *may equal* v_b *but not both. Then* G *has an odd cycle of length exceeding* $2n-1$*.*

By (4.0.1.1), $|p_a|$ and $|p_{a+1}|$ have the same parity, and $|p_b|$ and $|p_{b+1}|$ have the same parity. Thus $|p_a|+|p_{a+1}|=2j$ and $|p_b|+|p_{b+1}|=2k$ for some natural numbers j,k. If we follow the cycle C replacing the edges $\{v_a, v_{a+1}\}$ and $\{v_b, v_{b+1}\}$ with the paths p_a and p_{a+1} and p_b and p_{b+1}, we get a cycle of length $2n-1-2+2j+2k \geq 2n+1$.

4.0.1.4. *Let* v_0 *be a vertex not on* C*. If* v_a*,* v_{a+1}*, and* v_{a+2} *are three consecutive vertices in order on* C *with paths* p_a*,* p_{a+1} *and* p_{a+2} *to* v_0 *that have no other common vertices, then* $|p_a|=1$*,* $|p_{a+2}|=1$ *and* $|p_{a+1}|$ *is odd.*

By (4.0.1.1) $|p_a|$ and $|p_{a+1}|$ have the same parity, and $|p_{a+1}|$ and $|p_{a+2}|$ have the same parity. Thus all three paths have the same parity. Hence, by (4.0.1.2), $|p_a|=1=|p_{a+2}|$. Since $|p_{a+1}|$ has the same parity as $|p_a|$, we deduce that $|p_{a+1}|$ is odd.

4.0.1.5. *Suppose $v_a, v_{a+1}, \ldots, v_{a+t}$ are consecutive vertices of C with $t \geq 3$ and these vertices are joined to some vertex v_0 not on C by paths $p_a, p_{a+1}, \ldots, p_{a+t}$ that meet only in v_0. Then all these paths have length one.*

By (4.0.1.1), all of $|p_a|, |p_{a+1}|, \ldots, |p_{a+t}|$ have the same parity. By (4.0.1.2), it follows that all of $p_a, p_{a+1}, \ldots, p_{a+t}$ have length one.

Finally, we will need the following observation about path lengths within subgraphs of G.

4.0.1.6. *Suppose v_a and v_b are vertices of a subgraph H of G and there are two paths in H from v_a to v_b each of length at least $2n-1$ and of different parities. Then G has no $v_a - v_b$ path disjoint from $H - \{v_a, v_b\}$.*

Assume G has a $v_a - v_b$ path p' disjoint from $H - \{v_a, v_b\}$. Let p_o and p_e be $v_a - v_b$ paths in H of odd and even lengths, respectively, each of length at least $2n-1$. Then G contains cycles of lengths $|p'| + |p_o| \geq 1 + 2n - 1$ and $|p'| + |p_e| \geq 1 + 2n - 1$. Hence we have two cycles whose lengths exceed $2n-1$ and have opposite parities. Thus there is an odd cycle of size greater than $2n-1$. We deduce that no such p' exists.

Choose a vertex v_0 of G that is not in C. By Theorem 2.0.2, there are n paths from v_0 to $V(C)$ whose only common vertex is v_0. Since we have $2n-1$ vertices on C and n paths from v_0 to distinct vertices of C, we will always have at least two consecutive vertices on C that meet distinguished paths from v_0.

Now we focus on showing the following.

4.0.1.7. *None of the distinguished paths from v_0 to C has length exceeding one.*

By (4.0.1.5), if all paths from v_0 to C meet at consecutive vertices, then all paths have length one. Thus we may assume that the distinguished paths do not all meet C at consecutive vertices.

First, suppose G has a fan-like subgraph and at least one additional distinguished path to C from v_0 as shown in Figure 4.1 where $f \geq 4$. The paths from v_0 to the cycle that meet in the string, $v_1, v_2, \ldots, v_f$, of consecutive vertices will all have length one by (4.0.1.5). We assume that neither v_{2n-1} nor v_{f+1} is the end of one of the distinguished paths from v_0.

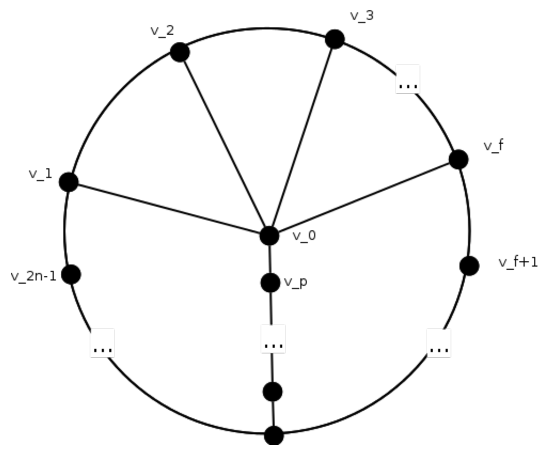


Figure 4.1: Graph with large fan-like subgraph

Assume at least one distinguished path p from v_0 to one of $v_{f+2}, v_{f+3}, \ldots, v_{2n-2}$ has length greater than one. Label the vertex adjacent to v_0 on p as v_p. By Theorem 2.0.2, there are n paths in G from v_p to C having only the vertex v_p in common. Consider the paths $(v_p, v_0, v_{i-1}, v_{i-2}, \ldots, v_i)$ and $(v_p, v_0, v_{i-2}, v_{i-3}, \ldots, v_i)$ where $i \in \{3, 4, \ldots, f+1\}$ and $i - i = 2n-1$. These have lengths $1+1+((2n-1)-1) = 2n$ and $1+1+((2n-1)-2) = 2n-1$. By (4.0.1.6), there is no $v_p - v_i$ path in G disjoint from $V(C) \cup \{v_p, v_0\}$. It follows using symmetry that v_p does not have paths to any of the vertices $v_{2n-1}, v_1, v_2, \ldots, v_{f+1}$ that have no member of $V(C) \cup v_0$ as internal vertices. Since $f \geq 4$, there must be $n-1$ paths from v_p to the remaining vertices of C of which there there are at most $2n-1-6$. Suppose two such paths meet at consecutive vertices on C. The triangle $\{v_0, v_2, v_3\}$ and these two new paths satisfy the hypotheses of (4.0.1.3) as the paths are internally disjoint from C and are disjoint from v_0, v_2, and v_3. Thus G has an odd cycle of length exceeding $2n-1$; a contradiction. We deduce that the distinguished paths from v_p to C do not end at consecutive vertices. This is a contradiction, since we have $n-1$ paths but only $2n-7$ vertices that can be ends of these paths. We deduce that, in this case, (4.0.1.7) holds.

Continuing with the proof of (4.0.1.7), we may now assume the following.

4.0.1.8. *If a distinguished path has length greater than one, then the longest sequence of consecutive vertices of C that are ends of distinguished paths from v_0 has length at most three.*

Now assume that C has three consecutive vertices v_1, v_2, and v_3 that are ends of distinguished paths.

Next we show the following.

4.0.1.9. *If the distinguished paths meet three consecutive vertices v_1, v_2, and v_3 and each of these paths to the consecutive vertices on C has length one, then all other distinguished paths have length one.*

By (4.0.1.4), we know the distinguished paths from v_0 to v_1 and v_3 have length one while the path from v_0 to v_2 has odd length. Suppose the $v_0 - v_2$ path has length one. Then one of the other distinguished paths from v_0 to C has length greater than one. Let v_p be the vertex of this path adjacent to v_0. Then we may use the paths from the previous argument to see that G does not have a path from v_p to v_{2n-1}, v_1, v_2, v_3, or v_4 that is disjoint from $V(C) \cup v_0$. Now, G contains $n-1$ paths from v_p to $V(C)$ avoiding v_0 and having only v_p in common. Again by (4.0.1.3) and using $\{v_0, v_2, v_3\}$, we get an odd cycle of length exceeding $2n-1$ if two of the paths from v_p end in consecutive vertices of C. As there are at most $2n-6$ vertices that are ends of such paths and there are $n-1$ such paths, we obtain a contradiction. We deduce that the $v_0 - v_2$ path has odd length exceeding one or (4.0.1.9) holds.

4.0.1.10. *If a distinguished path has length greater than one and we have three consecutive vertices meeting distinguished paths at v_1, v_2, and v_3, no two of the distinguished paths from v_0 can end in consecutive vertices of C other than those ending in v_1, v_2, v_3.*

Since G is at least 5-connected, there are at least two other distinguished paths from v_0 apart from those that have v_1, v_2, and v_3 as their ends. Suppose two of these additional paths meet C at adjacent vertices, v_a and v_{a+1}, as shown in Figure 4.2.

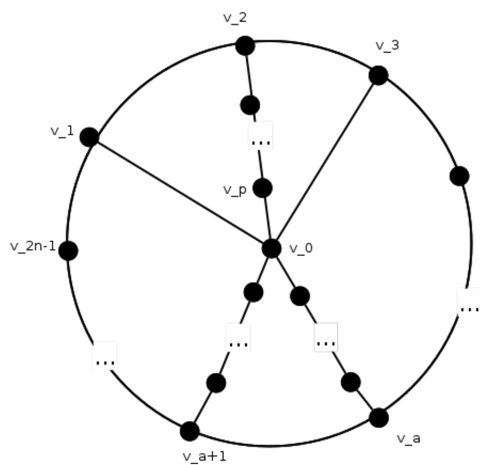


Figure 4.2: Subgraph of G

Consider the vertex on the $v_2 - v_0$ path that is adjacent to v_0. Label this vertex v_p and the distinguished $v_0 - v_2$ path p_2. Label the distinguished paths from v_0 to v_a and v_{a+1} by p_a and p_{a+1}. Moreover, label the portion of p_2 from v_p to v_2 by p_2'. By (4.0.1.2), the lengths of p_a and p_{a+1} have the same parity. By Theorem 2.0.2, v_p has n paths to C that meet only in v_p.

Consider the path consisting of the union of (v_p, v_0), p_{a+1}, and $(v_{a+1}, v_{a+2}, \ldots, v_a)$. Also consider the path consisting of the union of p_2', $(v_2, v_1, v_{2n-1}, \ldots, v_{a+1})$, p_{a+1}, and $(v_0, v_3, v_4, \ldots, v_a)$. These paths have lengths $1 + |p_{a+1}| + ((2n-1)-1) = 2n - 1 + |p_{a+1}|$ and $(|p_2| - 1) + ((2n-1) - 1 - 1) + |p_{a+1}| + 1 = 2n - 3 + |p_{a+1}| + |p_2|$. By (4.0.1.4), $|p_3|$ is odd. By (4.0.1.6), there is no $v_p - v_a$ path internally disjoint from C and v_0. By symmetry, there is no $v_p - v_{a+1}$ path internally disjoint from C and v_0. By using the paths $(v_p, v_0, v_{a+1}, v_a, v_{a-1}, \ldots, v_{a+2})$ and $(v_p, v_0, v_a, v_{a-1}, \ldots, v_{a+2})$ of lengths $2+((2n-1)-1) =$

$2n$ and $2 + ((2n-1)-2) = 2n-1$, we deduce that there is no $v_p - v_{a+2}$ path internally disjoint from C and v_0. By symmetry, there is no $v_p - v_{a-1}$ path internally disjoint from C and v_0.

Let v_q be any internal vertex on p_a, provided $|p_a| > 1$. Label the portion of the path from v_q to p_a as p'_a. Consider the path consisting of the union of (v_p, v_0), p_{a+1}, $(v_{a+1}, v_{a+2}, \ldots, v_a)$, and p'_a. Also consider the path consisting of the union of p'_2, $(v_2, v_1, v_{2n-1}, \ldots, v_{a+1})$, p_{a+1}, $(v_0, v_3, v_4, \ldots, v_a)$ and p'_a. These paths have lengths $1 + |p_{a+1}| + ((2n-1)-1) + |p'_a| = 2n-1+|p_{a+1}|+|p'_a|$ and $(|p_2|-1)+((2n-1)-1-1)+|p_{a+1}|+|p'_a| = |p_2|+2n-3+|p_{a+1}|+|p'_a|$. Since $|p_2|$ is odd, these lengths have different parities and have size greater than $2n-1$. Thus, by (4.0.1.6), there is no $v_p - v_q$ path disjoint from v_0 and C for any v_q in the interior of p_a. By symmetry, there is no $v_p - v_q$ path disjoint from C and v_0 for any v_q in the interior of p_{a+1}. Thus v_p does not have paths internally disjoint from $V(C) \cup v_0$ to any of v_{a-1}, v_a, v_{a+1}, or v_{a+2}. By (4.0.1.3), if the paths from v_p to C internally disjoint from $V(C) \cup v_0$ meet C at two consecutive vertices, we find an odd cycle of length exceeding $2n-1$. Thus the $n-1$ paths from v_p to C that are internally disjoint from $V(C) \cup v_0$ have their ends in vertices of $V(C) \setminus \{v_{a-1}, v_a, v_{a+1}, v_{a+2}\}$ that are not consecutive on C. Since there are only $2n-6$ such vertices, this is a contradiction. We deduce that if there is a distinguished path with path length greater than one, no two of the distinguished paths can end in consecutive vertices of C other than those ending in v_1, v_2, v_3, that is, (4.0.1.10) holds.

4.0.1.11. *If G has consecutive vertices of C meeting distinguished paths at v_1, v_2, and v_3 and if p_2 has length greater than one, then the distinguished paths that do not meetC at v_1, v_2, and v_3 may not have distance two on C, that is distinguished paths may not meet at v_a and v_{a+2}.*

To this end suppose there is at least one pair of distinguished paths from v_0 to C whose ends are a distance two apart on C aside from those ending in v_1, v_2 and v_3. Label these paths by p_a and p_{a+2}, and let their ends on C be v_a and v_{a+2}, respectively.

Call a vertex of C that meets a distinguished path a *distinguished vertex*. Let $\mathscr{P}'$ be the set of distinguished paths from v_0 to C other than those to v_1, v_2, and v_3.

4.0.1.12. *If G has consecutive vertices of C meeting distinguished paths at v_1, v_2, and v_3 and if p_2 has length greater than one, then the distinguished paths in $\mathscr{P}'$ do not all have length one.*

First suppose all paths in $\mathscr{P}'$ have length one as if Figure 4.3. By (4.0.1.4), we know the distinguished paths from v_0 to v_1 and v_3 have length one while the path from v_0 to v_2 has odd length. By (4.0.1.10), there are no consecutive vertices in the $2n-1-5$ vertices of $V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3, v_4\}$ that meet the distinguished paths from v_0. Thus there are $n-3$ paths meeting $2n-6$ vertices no two of which are consecutive. Therefore in the path $(v_4, v_5, \ldots, v_{2n-1})$, the vertices alternate between undistinguished and being distinguished except in exactly one place where there are two consecutive undistinguished vertices.

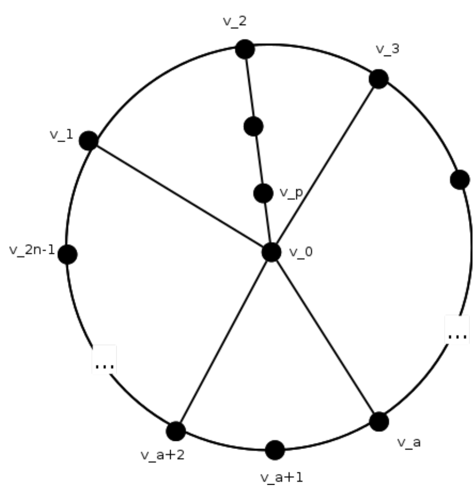


Figure 4.3: Subgraph of G

Let us examine two arbitrary paths in $\mathscr{P}'$ whose endpoints are distance two on C. The path $(v_p, v_0, v_{a+2}, v_{a+3}, \ldots, v_a)$ and the path that consists of the union of p_2' and $(v_2, v_1,$

$v_{2n-1}, \ldots, v_{a+2}, v_0, v_3, v_4, \ldots, v_a)$ have lengths $1+1+(2n-1)-2 = 2n-1$ and $(|p_2|-1)+((2n-1)-1-2)+1+1 = 2n-3+|p_2|$. Thus there are no $v_p - v_a$ paths that are internally disjoint from $V(C) \cup v_0$. By symmetry, there are no $v_p - v_{a+2}$ paths that are internally disjoint from $V(C) \cup v_0$. Consider the path $(v_p, v_0, v_a, v_{a+1}, \ldots, v_{a-1})$ and the path that consists of the union of p_2' and $(v_2, v_1, \ldots, v_a, v_0, v_3, v_4, \ldots v_{a-1})$. These paths have lengths $1+1+(2n-1)-1 = 2n$ and $(|p_2|-1)+((2n-1)-1-1)+1+1 = |p_2|+2n-2$. Thus there are no $v_p - v_{a-1}$ paths internally disjoint from $V(C) \cup v_0$. By symmetry, there are no $v_p - v_{a+3}$ paths internally disjoint from C and v_0. By using the path $(v_p, v_0, v_a, v_{a-1}, \ldots, v_{a+1})$ and the path that is the union of p_2' and $(v_2, v_1, \ldots, v_{a+2}, v_0, v_3, v_4, \ldots v_{a+1})$, whose lengths are $1+1+(2n-1)-1 = 2n$ and $(|p_2|-1)+((2n-1)-1-1)+1+1 = |p_2|+2n-2$, we deduce that there are no $v_p - v_{a+1}$ paths disjoint from C and v_0. By the paths $(v_p, v_0, v_3, v_4, \ldots, v_1)$ and $p_2' \cup (v_2, v_3, \ldots, v_1)$, there are no $v_p - v_1$ paths disjoint from C and v_0. By symmetry, there are no $v_p - v_3$ paths internally disjoint from C and v_0. By the path $(v_p, v_0, v_3, v_2, \ldots, v_4)$ and the path that consists of the union of p_2' and $(v_2, v_1, \ldots, v_4)$, there are no $v_p - v_4$ paths internally disjoint from C and v_0. By symmetry, there are no $v_p - v_{2n-1}$ paths internally disjoint from C and v_0.

By Theorem 2.0.2, there are $n-1$ paths from v_p to C that avoid v_0 and meet C at distinct vertices. We showed above that none of these paths meets $V(C)$ in any member of $\{v_1, v_3, v_4, v_{2n-1}\} \cup \{v_{a-1}, v_a, v_{a+1}, v_{a+2}, v_{a+3}\}$. The union of $\{v_1, v_3, v_4, v_{2n-1}\}$ and the collection of all sets $\{v_{a-1}, v_a, v_{a+1}, v_{a+2}, v_{a+3}\}$ where each of v_a and v_{a+2} meets paths in $\mathscr{P}'$ includes all but at most three vertices of C including v_2. This is a contradiction, since it implies there are at most three distinguished v_p paths, so (4.0.1.12) holds. Note that the extreme case occurs when the consecutive non-distinguished vertices of C isolate a distinguished v_{2n-2} or a v_5, as otherwise all vertices on C meet paired distinguished paths or the previous collection.

4.0.1.13. *If G has consecutive vertices of C meeting distinguished paths at v_1, v_2, and v_3 and if p_2 has length greater than one, then there are no pairs of distinguished paths in $\mathscr{P}'$ that meet at distance two on C where at least one path has length greater than one.*

Suppose there is at least one pair of paths in $\mathscr{P}'$ from v_0 to C that meet C at vertices that are distance two apart and the one of the path lengths is not one as in Figure 4.4.

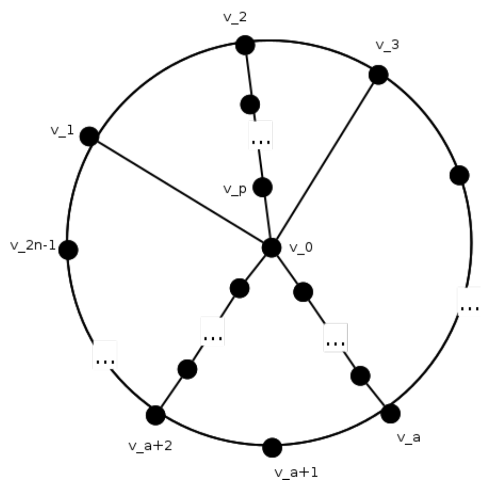


Figure 4.4: Subgraph of G

By (4.0.1.2) these paths have opposite parities, so one is even. Let v_q be the vertex on that path adjacent to v_0. Relabel v_p to be any vertex on the p_2 path interior. Let the even path be p_a with endpoints v_0 and v_a. By symmetry we may assume the distinguished path p_{a+2} from v_0 to v_{a+2} is odd. Let p'_a be the path from v_q to v_a contained in the larger path p_a. Let p'_2 be the path from v_p to v_2 contained in the larger path p_2, and let $p_2 - p'_2$ be the subpath of p_2 from v_0 to v_p. Consider the paths that consist of the union of p'_a , $(v_a, v_{a-1}, \ldots, v_{a+2})$, p_{a+2}, and $p_2 - p'_2$ and the union of p'_a, $(v_a, v_{a-1}, \ldots, v_3, v_0)$, p_{a+2}, $(v_{a+2}, v_{a+3}, \ldots, v_2)$, and p'_2 of lengths $(|p_a| - 1) + ((2n - 1) - 2) + |p_{a+2}| + |p_2 - p'_2| = 2n - 4 + |p_a| + |p_{a+2}| + |p_2 - p'_2|$ and $(|p_a| - 1) + ((2n - 1) - 2 - 1) + 1 + |p_{a+2}| + |p'_2| = 2n - 4 + |p_a| + |p_{a+2}| + |p'_2|$. Since p_2 has odd length, $|p'_2|$ and $|p_2 - p'_2|$ have different parities. Thus, by (4.0.1.6), there is no $v_q - v_p$ path disjoint from C and v_0 where v_p is on the interior of p_2. By the path consisting of the union of (v_q, v_0), p_2, and $(v_2, v_3, \ldots, v_1)$ of length $1 + |p_2| + ((2n - 1) - 1) = 2n - 1 + |p_2|$ and the path $(v_q, v_0, v_3, v_4, \ldots, v_1)$ of length $1 + 1 + ((2n - 1) - 2) = 2n - 1$, there is no path from v_q to v_1 disjoint from v_0 and C. By symmetry, there is no $v_q - v_3$ path disjoint from v_0 and C. The path $(v_q, v_0, v_3, v_4, \ldots, v_2)$ and the path consisting of the union of p'_a, $(v_a, v_{a-1}, \ldots v_3, v_0)$, p_{a+2}, and $(v_{a+2}, v_{a+3}, \ldots v_2)$ have lengths $1 + 1 + (2n-1) - 1 = 2n$ and $(|p_a| - 1) + ((2n-1) - 2 - 1) + 1 + |p_{a+2}| = 2n - 4 + |p_a| + |p_{a+2}|$. Since p_a has even length and p_{a+2} has odd length, the second path is odd and has size greater than or equal to $2n-1$. Thus there are no $v_q - v_2$ paths disjoint from C and v_0. By the path $(v_q, v_0, v_3, v_2, \ldots, v_4)$ and the path consisting of the union of (v_q, v_0), p_2, and $(v_2, v_1, \ldots v_4)$ of lengths $1+1+((2n-1)-1) = 2n$ and $1+|p_2|+((2n-1)-1) = 2n+|p_2|$, there is no $v_q - v_4$ path disjoint from v_0 and C. By symmetry, there is no $v_q - v_{2n-1}$ path disjoint from v_0 and C. Thus, there is no path from v_q to v_{2n-1}, v_1, v_2, v_3 or v_4 not through v_0. By Theorem 2.0.2, there are $n-1$ internally disjoint paths to distinct vertices of $V(C) \setminus \{v_1, v_2, v_3, v_4, v_{2n-1}\}$.. However, by (4.0.1.3) and the paths p_2 and (v_0, v_3), they may not meet C at consecutive vertices. This requires $2n-3$ vertices. Thus we deduce that (4.0.1.13) holds.

Since we have that (4.0.1.10), (4.0.1.12) and (4.0.1.13), if there are still distinguished paths that meet at distance two on C and p_2 has length greater than one, then we are in exactly the case where the only path in $\mathscr{P}'$ with length greater than one meets an unpaired v_{2n-2} or v_5. Without loss of generality, assume the distinguished path meets v_5 and call it p_5. By (4.0.1.2), p_5 and p_3 have opposite parities, so p_5 is even. By the path consisting of the union of p_2, p_5, $(v_5, v_6, \ldots, v_1, v_2)$ of length $(2n-1) - 3 + |p_2| + |p_5|$, there is an odd cycle larger than $2n-1$ and hence we get the following result.

4.0.1.14. *If G has consecutive vertices of C meeting distinguished paths at v_1, v_2, and v_3 and if p_2 has length greater than one, then there are no distinguished paths in $\mathscr{P}'$ that meet at distance two or one on C.*

By (4.0.1.14) and (4.0.1.10), we now know that every two of the $n-3$ paths in $\mathscr{P}'$ meet C at vertices that are at distance 3 or greater. Thus we need at least $3(n-3)-2 = 3n-11$ vertices remaining in C. Since there is no larger string of consecutive vertices that meet distinguished paths, the paths in $\mathscr{P}'$ do not meet v_4 or v_{2n-1}. Thus we have $2n-1-5=2n-6$ vertices that can be endpoints of paths in C. So we get $2n-6 \geq 3n-11$. Hence $n \leq 5$. Since $n \geq 5$, this may only occur if $n=5$. If $n \neq 5$, then all distinguished paths must have length one.

We now address the case where $n=5$. From before, we do not have the case where distinguished paths from v_0 to C meet C in consecutive vertices other than v_1, v_2, and v_3 and paths from v_0 to C must meet C at vertices that are distance three or more. So we get the following configuration in Figure 4.5.

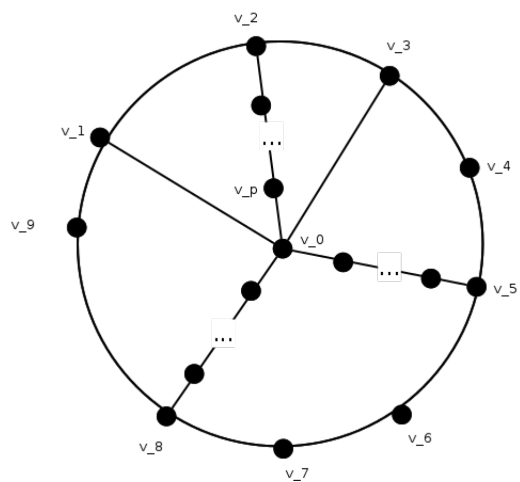


Figure 4.5: Subgraph of G when $n=5$

By (4.0.1.2), the paths p_8 and p_5 from v_0 to v_8 and from v_0 to v_5 are either even or length one, since they are distance two from the edges from v_0 to v_1 and v_0 to v_3, respectively. The length of p_2 is odd and not one.

Suppose at least one of p_8 and p_5 has even length. Without loss of generality, we may assume it is p_8. Let v_q be the vertex on p_8 adjacent to v_0. Label the path from v_q to v_8 contained in p_8 as p_8'. Let v_p be any vertex on the interior of path p_2. Label the path from v_p to v_2 contained in p_2 as p_2'. Consider the path $(v_q, v_0, v_3, v_4, \ldots, v_9, v_1, v_2)$ and the path consisting of the union of p_2', p_8', $(v_8, v_7, v_6, \ldots, v_3, v_0, v_1, v_2)$, and p_2' of lengths $1+1+8+|p_2'| = 10+|p_2'|$ and $|p_8'|+5+1+1+1+|p_2'| = 8+|p_8'|+|p_2'| = 8+|p_8|-1+|p_2'| = 7+|p_8|+|p_2'|$, which have opposite parities since p_8 has even length. By (4.0.1.6), there are no $v_q - v_p$ paths internally disjoint from C and v_0 for any v_p on the interior of p_2. By the path $\{v_q, v_0, v_3, v_4, \ldots, v_2\}$ of length 10 and the path consisting of the union of p_8, and $(v_8, v_7, \ldots, v_3, v_0, v_1, v_2)$ of length $|p_8'| + 8 = |p_8| - 1 + 8 = 7 + |p_8|$, there is no $v_q - v_2$ path internally disjoint from C and v_0. By the path $(v_q, v_0, v_3, v_4, \ldots v_1)$ and the path consisting of the union of (v_q, v_0), p_2, and $(v_2, v_3, v_4, \ldots v_1)$ of lengths $1 + 1 + 7 = 9$ and $1 + |p_2| + 8 = 9 + |p_2|$, there is no $v_q - v_1$ path internally disjoint from C and v_0. By symmetry, there is no $v_q - v_3$ path internally disjoint from C and v_0. By the path $(v_q, v_0, v_3, v_2, \ldots, v_4)$ of length 10 and the path consisting of the union of (v_q, v_0), p_2, and $(v_2, v_1, \ldots, v_4)$ of length $1 + |p_2| + 7 = |p_2| + 8$, there is no $v_q - v_4$ path internally disjoint from C and v_0. By symmetry, there is no $v_q - v_9$ path internally disjoint from C and v_0. By the paths consisting of the union of p_8' and $(v_8, v_9, \ldots v_7)$ and the union of p_8', (v_8, v_9, v_1, v_0), p_2, and $(v_2, v_3, \ldots v_7)$ of lengths $(|p_8| - 1) + 8 = |p_8| + 7$ and $3 + |p_2| + 5 = 8 + |p_2|$, there is no $v_q - v_7$ path internally disjoint from C and v_0, since $|p_2|$ is even and positive. This leaves us with v_6, v_8, and v_5 as vertices where paths from v_q meet C disjoint from v_0. As we need $n - 1 = 4$ such vertices, this contradicts 5-connectivity. Thus neither p_8 nor p_5 is even.

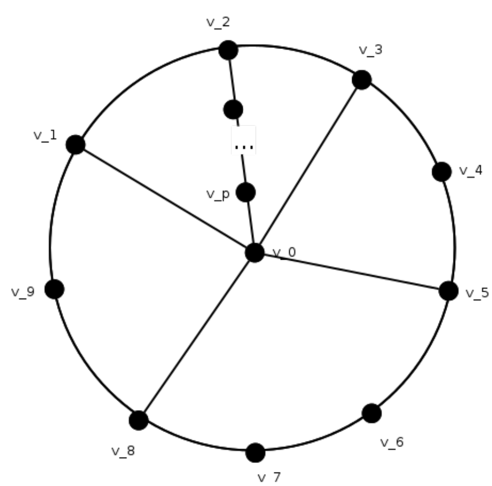


Figure 4.6: Subgraph of G when $n = 5$

Suppose the paths p_8 and p_5 both have length one. Let v_p be the vertex on p_2 that adjacent to v_0. Label the path from v_p to v_2 contained in p_2 as p_2'. By the path consisting of the union of p_2' and $(v_2, v_3, \ldots, v_1)$ and the path $(v_p, v_0, v_8, v_7, \ldots, v_1)$, which have lengths $|p_2'| + 1 + 7 = |p_2| - 1 + 8 = |p_2| + 7$ and 9, there is no $v_p - v_1$ path internally disjoint from C and v_0. By symmetry, there is no $v_p - v_3$ path internally disjoint from C and v_0. By the path $(v_p, v_0, v_3, v_2, \ldots, v_4)$ and the path consisting of the union of p_2' and $(v_2, v_1, \ldots, v_4)$, there is no $v_p - v_4$ path internally disjoint from C and v_0. By symmetry, there is no $v_p - v_9$ path internally disjoint from C and v_0. By the paths $(v_p, v_0, v_8, v_9, \ldots v_7)$ and $(v_p, v_0, v_5, v_4, \ldots, v_7)$, there is no $v_p - v_7$ path internally disjoint from C and v_0. By symmetry, there is no $v_p - v_6$ path internally disjoint from C and v_0. Thus v_p does not have 5 disjoint paths to C not through v_0 as only v_2, v_8, and v_5 remain as possible endpoints of such paths. Thus all distinguished paths in this configuration must have length one. So by (4.0.1.9), (4.0.1.14), and the previous case, this completes the proof of the following

4.0.1.15. *No three or more of the distinguished paths from v_0 meet C at consecutive vertices unless all distinguished paths have length one*

Call a vertex *distinguished* if it meets a distinguished path on C. Call a pair of distinguished vertices that are consecutive on C a *consecutive distinguished pair*. Next we prove the following.

4.0.1.16. *Suppose there is a distinguished path of length greater than one. If there are two distinct consecutive distinguished pairs, then these pairs are disjoint. Moreover, each path in C that contains exactly one vertex in each pair must contain two consecutive undistinguished vertices.*

Assume that (4.0.1.16) fails. It is immediate from (4.0.1.15) that the two consecutive pairs are disjoint and that each path containing exactly one vertex from each pair has length at least two. Let q be such a path. Then q contains a subpath q' whose endpoints are distinguished vertices, whose vertices are alternately distinguished and undistinguished, and such that the neighbors in $V(C) - V(q')$ of the endpoints of q' are distinguished vertices.

Suppose the sets are separated by a single vertex.

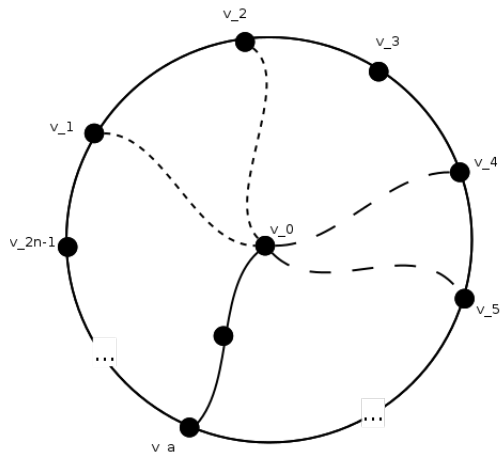


Figure 4.7: Configuration when the pairs of distinguished vertices are distance two on C

Label one consecutive distinguished pair of vertices v_1 and v_2 with distinguished paths p_1 and p_2 from v_0, and the other consecutive distinguished pair of vertices v_4 and v_5 with paths p_4 and p_5 from v_0. Let v_3 be the vertex on C between the pairs. By (4.0.1.1), $|p_1|$ and $|p_2|$ have the same parity, and $|p_4|$ and $|p_5|$ have the same parity. By (4.0.1.2), $|p_2|$ and $|p_4|$ have opposite parities or both have length one. Thus, $|p_1|$ and $|p_4|$ path lengths have opposite parities or both are odd with the inner paths having length one.

4.0.1.17. *The configuration in Figure 4.7 may not occur if distinguished paths have length greater than one.*

First we will show the following.

4.0.1.18. *The configuration in Figure 4.7 may not occur if distinguished consecutive pairs have even length.*

Suppose $|p_2|$ and $|p_4|$ have opposite parities. Thus $|p_1|$ and $|p_4|$ have opposite parities and $|p_2|$ and $|p_5|$ have opposite parities. By the cycle through p_1, p_4 and $V(C) \setminus \{v_2, v_3\}$ of length $((2n-1)-3)+|p_1|+|p_4|$, the sum of the lengths of the paths p_1 and p_4 is three. We deduce that the even path has length two and the odd path has length one. Similarly, by the cycle through p_2, p_5 and $V(C) \setminus \{v_3, v_4\}$ of length $((2n-1)-3)+|p_2|+|p_5|$, the sum of lengths of the paths p_2 and p_5 is three. Again, we deduce that the even path has length two and the odd path has length one. Without loss of generality, let p_1 and p_2 be the even paths as shown in Figure 4.8.

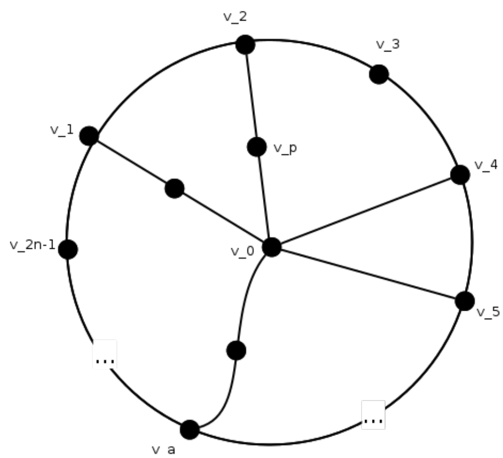


Figure 4.8: Subgraph when distinguished consecutive pair have distance two on C

Label the vertex on the interior of the p_2 path as v_p. Label the vertex on the interior of the p_1 path as v_r. Let v_a be any distinguished vertex in $V(C) \setminus \{v_1, v_2, v_4, v_5\}$ and p_a be distinguished path from v_0 to v_a. By (2.0.2) v_p has $n-2$ distinct paths to C not through v_0 or v_2. By (4.0.1.6), the paths $(v_p, v_2, v_3, v_4, \ldots, v_{2n-1}, v_1)$ and $(v_p, v_2, v_3, v_4, v_0, v_5, v_6, \ldots, v_{2n-1}, v_1)$ of lengths $((2n-1)-1)+1 = 2n-1$ and $((2n-1)-1-1)+1+2 = 2n$ imply that no $v_p - v_1$ path exists that is internally disjoint from $V(C) \cup \{v_0\}$. Again, by (4.0.1.6), the paths $(v_p, v_2, v_3, v_4, \ldots, v_{2n-1}, v_1, v_r)$ and $(v_p, v_2, v_3, v_4, v_0, v_5, v_6, \ldots, v_{2n-1}, v_1, v_r)$ of lengths $((2n-1)-1)+1+1 = 2n$ and $(2n-1)-1-1)+1+2+1 = 2n+1$, no $v_p - v_r$ path exists that is internally disjoint from $V(C) \cup \{v_0\}$. The paths $(v_p, v_2, v_1, v_{2n-1}, \ldots, v_5, v_4, v_3)$ of length $1+((2n-1)-1) = 2n-1$ and $(v_p, v_0, v_4, v_5, \ldots, v_{2n-1}v_1, v_2, v_3)$ of length $2+((2n-1)-1) = 2n$ imply that there is no $v_p - v_3$ path internally disjoint from $V(C) \cup \{v_0\}$. The paths $(v_p, v_0, v_5, v_6, \ldots, v_{2n-1}, v_1, v_2, v_3, v_4)$ and $(v_p, v_0, v_r, v_1, v_{2n-1}, \ldots, v_5, v_4)$ of lengths $1 + 1 + ((2n-1)-1) = 2n$ and $1+2+((2n-1)-3) = 2n-1$ imply that there is no $v_p - v_4$ path that is internally disjoint from $V(C) \cup \{v_0\}$. By (4.0.1.6) and the paths $(v_p, v_0, v_4, v_3, \ldots, v_6, v_5)$ and $(v_p, v_2, v_3, v_4, v_0, v_r, v_1, v_{2n-1}, \ldots, v_6, v_5)$ of lengths $1 + 1 + ((2n-1)-1) = 2n$ and $1 + ((2n-1)-1-1)+1+2 = 2n+1$, no $v_p - v_5$ path exists that is internally disjoint from $V(C) \cup \{v_0\}$. Assume there is a vertex v_q on the interior of p_a. Let p'_a be the subpath of p_a from v_q to v_a and let $p_a - p'_a$ be the subpath of p_a from v_0 to v_q. The paths that consist of a union of $(v_p, v_2, v_1, \ldots, v_4, v_0)$ and $p_a - p'_a$ and a union of $(v_p, v_2, v_1, \ldots, v_5, v_0)$ and $p_a - p'_a$ of lengths $1 + ((2n-1)-2) + 1 + |p_a - p'_a| = 2n - 1 + |p_a - p'_a|$ and $1 + ((2n-1)-3) + 1 + |p_a - p'_a| = 2n - 2 + |p_a - p'_a|$ imply that there is no $v_p - v_q$ path that is internally disjoint from $V(C) \cup \{v_0\}$ for any v_q on the interior of any p_a.

By (4.0.1.3) and the vertices v_0, v_4 and v_5, any path from v_p to v_a implies that there is no path from v_p to v_{a-1} or v_{a+1} that is disjoint from $V(C) \cup v_0$. Since we have $n-2$ paths from v_0 to $V(C) \setminus \{v_1, v_2, v_3, v_4, v_5\}$, a set of size $2n - 1 - 5 = 2n - 6$, without consecutive vertices, this is a contradiction and we may not have the configuration in Figure (4.8), that is (4.0.1.18) holds.

4.0.1.19. *Suppose there is a distinguished path with length greater than one. The configuration in Figure 4.7 may not occur if distinguished consecutive pairs have odd length.*

Suppose $|p_2|$ and $|p_4|$ are both one, that is have the same parity from above. By (4.0.1.1), $|p_1|$ and $|p_5|$ are both odd. The cycle that is a union of p_1, p_5 and $V(C) \setminus \{v_2, v_3, v_4\}$ of length $((2n-1)-4)+|p_1|+|p_5| = 2n-5+|p_1|+|p_5|$ implies that one of p_1 and p_5 has length one and the other has length one or three. Without loss of generality, let $|p_1| \in \{1, 3\}$.

First, assume $|p_1| = 1$. Then there is a distinguished path p_a to vertex v_a on C with length greater than one. Let v_q be the any vertex on on the interior of the p_a path. Let p'_a be the subpath of p_a from v_q to v_0.

By (4.0.1.6) and the cycles consisting of the union of p'_a and $(v_0, v_4, v_5, \ldots, v_2, v_3)$ and the union of p'_a and $(v_0, v_5, v_6 \ldots, v_2, v_3)$ of lengths $|p'_a| + 1 + ((2n-1)-1) = 2n - 1 + |p'_a|$ and $|p'_a| + 1 + ((2n-1)-2) = 2n - 2 + |p'_a|$, no $v_q - v_3$ path internally disjoint from $V(C) \cup v_0$ can exist. By (4.0.1.6) and the cycles consisting of the union of p'_a and $(v_0, v_4, v_5, \ldots, v_1, v_2)$ and the union of p'_a and $(v_0, v_1, v_{2n-1} \ldots, v_3, v_2)$ of lengths $|p'_a| + 1 + ((2n-1)-2) = 2n - 2 + |p'_a|$ and $|p'_a| + 1 + ((2n-1)-1) = 2n - 1 + |p'_a|$, no $v_q - v_2$ path disjoint from $V(C) \cup v_0$

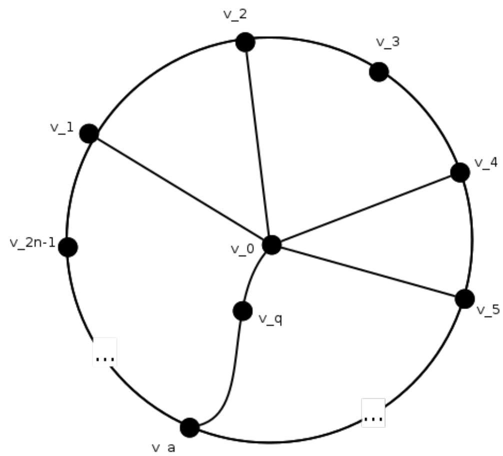


Figure 4.9: Subgraph when the sets of paths are distance two

can exist. By symmetry, no $v_q - v_4$ path internally disjoint from $V(C) \cup v_0$ can exist. By (4.0.1.6) and the cycles consisting of the union of p'_a and $(v_0, v_4, v_3, \ldots, v_6)$ and the union of p'_a and $(v_0, v_5, v_4 \ldots, v_6)$ of lengths $|p'_a| + 1 + ((2n-1)-2) = 2n - 2 + |p'_a|$ and $|p'_a| + 1 + ((2n-1)-1) = 2n - 1 + |p'_a|$, no $v_q - v_6$ path internally disjoint from $V(C) \cup v_0$ can exist. By symmetry, there is no $v_q - v_{2n-1}$ path internally disjoint from $V(C) \cup v_0$. If we include two possible paths from v_q to v_1 and v_5, there are $n-3$ remaining paths from v_q to $V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3, v_4, v_5, v_6\}$. By the triangle $\{v_0, v_4, v_5\}$ and (4.0.1.3), v_q cannot have distinct paths disjoint from v_0 meeting $V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3, v_4, v_5, v_6\}$ at consecutive vertices. Thus, we have $n-3$ non-consecutive paths in $2n - 1 - 7 = 2n - 8$ vertices, and v_q is not n connected. We deduce that $|p_1| \neq 1$.

Now assume that $|p_1| = 3$ as in Figure 4.10. Let v_p be the vertex on the interior of p_1 adjacent to v_0. Let p'_2 be the subpath of p_2 from v_p to v_1. Let v_a be any distinguished vertex not in $\{v_1, v_2, v_4, v_5\}$ and p_a be the distinguished path from v_0 to v_a. Let v_q be any vertex on the interior of p_a. Let p'_a be the subpath of p_a from v_q to v_a. Let v_r be the additional interior vertex on p_1.

By Theorem 2.0.2, the graph $G \setminus \{v_0, v_1\}$ has $n-2$ internally disjoint paths from v_p to C. By (4.0.1.6) and the paths consisting of the union of p'_1 and $(v_1, v_{2n-1}, \ldots, v_3, v_2)$ and the union of p'_1 and $(v_1, v_{2n-1}, \ldots, v_5, v_0, v_4, v_3, v_2)$ of lengths $|p'_1| + 1 + ((2n-1)-1) =$

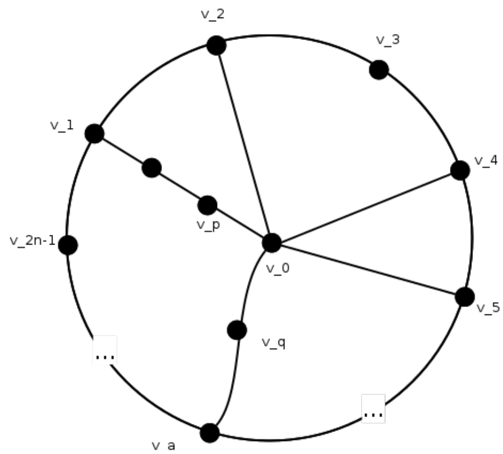


Figure 4.10: Subgraph when the distinguished consecutive pairs are distance two on C

$2n-1+|p_1'| = 2n-2+2 = 2n$ and $|p_1'|+((2n-1)-1-1)+1+1 = 2n-1+|p_1'| = 2n+1$, no $v_q - v_2$ path internally disjoint from $V(C) \cup v_0$ can exist. By (4.0.1.6) and the path consisting of the union of p_1' and $(v_1, v_{2n-1}, \ldots, v_4, v_3)$ and the path $(v_p, v_0, v_2, v_1 \ldots, v_4, v_3)$ of lengths $|p_1'|+((2n-1)-2) = 2n-1+|p_1'| = 2n-1$ and $1+1+((2n-1)-1) = 2n$, no $v_q - v_3$ path internally disjoint from $V(C) \cup v_0$ can exist. The paths $(v_p, v_0, v_2, v_1 \ldots, v_5, v_4)$ and $(v_p, v_5, v_6, \ldots, v_3, v_4)$ of lengths $1+1+((2n-1)-2) = 2n-1$ and $1+1+((2n-1)-1) = 2n$ imply that there is no $v_p - v_4$ path internally disjoint from $V(C) \cup v_0$. The paths $(v_p, v_0, v_4, v_3 \ldots, v_6)$ and $(v_p, v_0, v_5, v_4, \ldots, v_6)$ of lengths $1+1+((2n-1)-2) = 2n-1$ and $1+1+((2n-1)-1) = 2n$ imply that there is no $v_p - v_6$ path internally disjoint from $V(C) \cup v_0$. By the paths consisting of the union of p_1', $(v_1, v_{2n-1}, \ldots, v_3, v_2, v_0)$ and p_a' and the union of p_1', $(v_1, v_{2n-1}, \ldots, v_5, v_0)$ and p_a', of lengths $2+((2n-1)-1)+1+|p_a'| = 2n+1+|p_a'|$ and $2+((2n-1)-4)+1+|p_a'| = 2n-2+|p_a'|$, no $v_p - v_q$ path internally disjoint from $V(C) \cup v_0$ can exist for any v_q on the interior of some p_a. The path consisting of the union of p_1' and $(v_1, v_2 \ldots v_{2n-1})$ and the union p_1' and $(v_1, v_2, v_3, v_4, v_0, v_5, v_6 \ldots v_{2n-1})$ of lengths $|p_1'| + ((2n-1)-1) = 2+2n-2 = 2n$ and $|p_1'| + ((2n-1)-1-1)+1+1 = 2+2n-1 = 2n+1$ imply that there is no $v_p - v_{2n-1}$ path internally disjoint from $V(C) \cup v_0$. Allowing for a possible path to v_5, there are $n-3$ remaining paths from v_p to $V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3, v_4, v_5, v_6\}$. By the triangle $\{v_0, v_4, v_5\}$ and (4.0.1.3), v_p cannot have paths disjoint from v_0 meeting $V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3, v_4, v_5, v_6\}$ at consecutive vertices, otherwise we have a larger odd cycle. Thus, we have $n-3$ non-adjacent paths meeting $V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3, v_4, v_5, v_6\}$ in $2n-1-7 = 2n-8$ vertices . We deduce v_p is not n connected, and (4.0.1.19) holds. . By (4.0.1.19) holds and (4.0.1.18), we deduce that if we have longer path length of a distinguished path, then the two consecutive pairs are not distance two on C, that is (4.0.1.17) holds.

Suppose the distance on C between two consecutive distinguished pairs is greater than two. Label the first adjacent pair of vertices v_1 and v_2 with paths p_1 and p_2 from v_0, and the second adjacent pair of vertices v_a and v_{a+1} with paths p_a and p_{a+1}. As noted before, between v_2 and v_a on C is an alternating sequence of distinguished and undistinguished vertices as shown in Figure 4.11. Let p_k be a distinguished path not in a distinguished consecutive pair from v_0 to v_k between v_2 and v_a.

First we show the following result.

4.0.1.20. *The length of p_2 is not greater than one.*

Suppose p_2 has length greater than one. Let v_p be the vertex on p_2 adjacent to v_0. Let p'_2 as the subpath from v_2 to v_p.

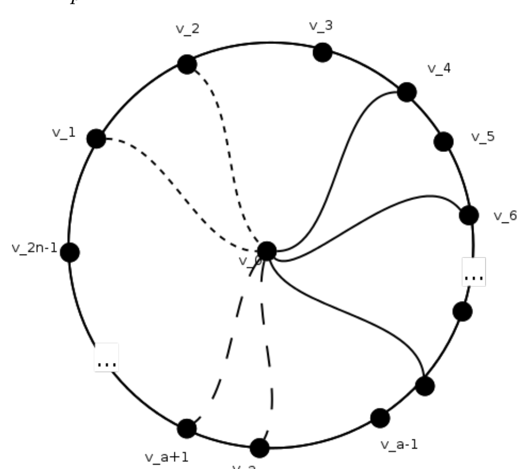


Figure 4.11: Subgraph when the sets of paths are distance two

By (4.0.1.1), p_1 and p_2 have the same parity and p_a and p_{a+1} have the same parity. This is shown in Figure 4.11 with varying path textures. By Theorem 2.0.2, the graph G

has $n-1$ distinct internally disjoint paths from v_p to C not through v_0 . Observe that $|p_a|+|p_{a+1}|$ is even. Let v_q be a vertex on the interior of p_1, and p_1' be the subpath from v_q to v_1 on p_1. By (4.0.1.6), and the paths consisting of the union of p_2', $(v_2, v_3, \ldots, v_{2n-1}, v_1)$, and p_1' and the union of p_2', $(v_2, v_3, \ldots, v_{a-1}, v_a)$, p_a, p_{a+1}, $(v_{a+1}, v_{a+2}, \ldots, v_{2n-1}, v_1)$ and p_1' of lengths $2n-2+|p_1'|+|p_2'|$ and $2n-3+|p_1'|+|p_2'|+|p_a|+|p_{a+1}|$, no v_p-v_q path can exist that internally is disjoint from $V(C)\cup v_0$. By (4.0.1.6), and the paths consisting of the union of p_2', and $(v_2, v_3, \ldots, v_{2n-1}, v_1)$ and the union of p_2', $(v_2, v_3, \ldots, v_{a-1}, v_a)$, p_a, p_{a+1}, and $(v_{a+1}, v_{a+2}, \ldots, v_{2n-1}, v_1)$ of lengths $2n-2+|p_2'|$ and $2n-3+|p_2'|+|p_a|+|p_{a+1}|$, no v_p-v_1 path can exist that is internally disjoint from $V(C)\cup v_0$. Since p_2 has a length greater than one, the length of p_4 has the opposite parity by (4.0.1.2). The paths consisting of the union of p_2' and $(v_2, v_1, \ldots, v_4, v_3)$ and the union of (v_p, v_0), p_4, $(v_4, v_5, \ldots, v_3)$ imply there is no v_p-v_3 path that is internally disjoint from $V(C)\cup v_0$. Label an interior vertex of p_k as v_r, if it exists. Let p_k' as the subpath from v_r to v_p and let p_k-p_k' be the subpath of p_k from v_r to v_0. When p_k has length greater than one, $|p_{k-2}|$ has the opposite parity by (4.0.1.2). Thus $|p_k'|+|p_{k-2}|$ and $|p_k|-|p_k'|$ have opposite parities, since $|p_k'|+|p_{k-2}|+|p_k|-|p_k'|=|p_{k-2}|+|p_k|$. By the paths consisting of the union of p_2', $(v_2, v_3, \ldots, v_{2n-1}, v_1)$, p_1, and p_k-p_k' and the union of p_2', $(v_2, v_3, \ldots, v_{k-2})$, p_{k-2}, p_1 $(v_1, v_{2n-1}, \ldots, v_k)$ and p_k' of lengths $|p_2'|+((2n-1)-1)+|p_1|+|p_k-p_k'| = 2n-2+|p_2'|+|p_1|+|p_k-p_k'|$ and $|p_2'|+((2n-1)-2-1)+|p_{k-2}|+|p_1|+|p_k'| = 2n-4+|p_2'|++p_1|+|p_{k-2}|+|p_k'|$, no v_p-v_r path can exist that is internally disjoint from $V(C)\cup v_0$ for any interior vertex v_r on some p_k with $k\neq 4$. If $k=4$, the paths that consist of the union of p_2', $(v_2, v_1, \ldots, v_{a+1})$, p_{a+1}, p_a, $(v_a, v_{a-1}, \ldots, v_4)$, and p_4' and the union of p_2', $(v_2, v_1, \ldots, v_4)$, and p_4' suffice to show no v_p-v_r path can exist that is internally disjoint from $V(C)\cup v_0$ for any interior vertex v_r on $k=4$. Label an interior vertex of p_a as v_s, if it exists. Let the subpath of p_a from v_s to v_a be p_a'. Since p_a has a length greater than one, p_{a-2} has the opposite parity by (4.0.1.2). By (4.0.1.1), $|p_{a+1}|$ has the same parity as $|p_a|$. Note that since $|p_1|+|p_2|$ is even, $|p_1|+|p_2'|$ is odd. By the paths consisting of the union of p_2', $(v_2, v_3, \ldots, v_{a-2})$, p_{a-2}, p_1, and $(v_1, v_{2n-1}, \ldots, v_a)$ of length $|p_2'| + 2n - 4 + |p_{a-2}| + |p_1| + |p_a'|$ and the union (v_p, v_0), p_{a-2}, $(v_{a-2}, v_{a-3}, \ldots v_a)$ and p_a' of length $1 + 2n - 1 - 2 + |p_{a-2}| + |p_a'|$, no $v_p - v_s$ path can exist that is internally disjoint from $V(C) \cup v_0$, if and interior v_s exists. Let p_{a+1} have length greater than one. Label an interior vertex of p_{a+1} as v_t. Let the subpath of p_{a+1} from v_t to v_a be p_{a+1}'. Note that since $|p_a| + |p_{a+1}'|$ is even, $|p_a| + |p_{a+1}'|$ and $|p_{a+1} - p_{a+1}'|$ have the same parity. By the paths consisting of the union of p_2', $(v_2, v_3, \ldots, v_1)$, p_1, and $p_{a+1} - p_{a+1}'$ and the union of p_2', $(v_2, v_1, \ldots, v_a)$, p_a, p_1, $(v_1, v_{2n-1}, \ldots, v_{a+1})$ and p_{a+1}' with lengths $|p_2'| + ((2n-1)-1) + |p_1| + |p_{a+1} - p_{a+1}'|$ and $|p_2'| + ((2n-1)-2) + +|p_a| + |p_1| + |p_{a+1}'|$, no $v_p - v_t$ path can exist that is internally disjoint from $V(C) \cup v_0$. Let p_b be an additional distinguished path to v_b; that is, $b > a+1$ in our labeling. Label an interior vertex of p_b as v_u. Let the subpath of p_b from v_u to v_b be p_b' and the subpath of p_b from v_u to v_0 be $p_b - p_b'$ By the paths consisting of the union of p_2', $(v_2, v_3, \ldots, v_1)$, p_1 and $p_b - p_b'$ and the union of p_2', $(v_2, v_3, \ldots, v_a)$, p_a, p_{a+1}, $(v_{a+1}, v_{a+2}, \ldots, v_1)$, p_1, and $p_b - p_b'$, no $v_p - v_u$ path can exist that is internally disjoint from $V(C) \cup v_0$ for any interior vertex v_u of p_b. By the paths consisting of the union of (v_p, v_0), p_{a-2}, and $(v_{a-2}, v_{a-3}, \ldots, v_a)$ and p_2', $(v_2, v_3, \ldots v_{a+2})$, p_{a-2}, p_1, and $(v_1, v_{2n-1}, \ldots v_a)$, no $v_p - v_a$ path can exist that is internally disjoint from $V(C) \cup v_0$. By the paths (v_p, v_0), p_a, and $(v_a, v_{a-1}, \ldots v_{a+2})$ and (v_p, v_0), p_{a+1}, and $(v_{a+1}, v_a, \ldots v_{a+2})$, no $v_p - v_{a+2}$ path can exist that is internally disjoint from $V(C) \cup v_0$. By symmetry, no $v_p - v_{a-1}$ path can exist that is internally disjoint from $V(C) \cup v_0$. By the paths consisting of the union of p_2' and $(v_2, v_3, \ldots, v_1)$ and the union of p_2', $(v_2, v_3, \ldots, v_a)$, p_a, p_{a+1}, and $(v_{a+1}, v_{a+2}, \ldots, v_1)$, no $v_p - v_1$ path can exist that is disjoint from $V(C) \cup v_0$. By the paths consisting of the union of p_2' and $(v_2, v_1, \ldots, v_3)$ and p_2', $(v_2, v_1, \ldots, v_{a+1})$, p_{a+1}, p_a, and $(v_a, v_{a-1}, \ldots, v_3)$, no $v_p - v_3$ path can exist that is internally disjoint from $V(C) \cup v_0$. By the paths consisting of the union of p_2',$(v_2, v_3, \ldots, v_a)$, p_a, p_{a+1}, and $(v_{a+1}, v_{a+2}, \ldots, v_{2n-1}$ and the union of (v_p, v_0) p_1, and $(v_1, v_2, \ldots, v_{2n-1})$, no $v_p - v_{2n-1}$ path can exist that is internally disjoint from $V(C) \cup v_0$.

Allowing for possible paths to v_{a+1} and v_2, there are $n-3$ remaining paths from v_p to $V(C)\setminus\{v_{a+2}, v_{a+1}, v_a, v_{a-1}, v_{2n-1}, v_1, v_2, v_3\}$. By the triangle $\{v_0, v_a, v_{a+1}\}$ and (4.0.1.3), v_p cannot have paths meeting C at consecutive vertices in $V(C)\setminus\{v_{a+2}, v_{a+1}, v_a, v_{a-1}, v_{2n-1}, v_1, v_2, v_3\}$. Thus we have $n-3$ paths meeting non-consecutive vertices in $2n-1-8=2n-9$ vertices which is divided in two paths of C. Thus p_2 does not have length greater than one and (4.0.1.20) holds.

The cases where some p_k path has length greater than one and only p_1 has length greater than one are included in the appendix, which completes the proof of (4.0.1.16).

Suppose there is only one consecutive distinguished pair of vertices. Let v_1 and v_2 be the adjacent pair meeting distinguished paths p_1 and p_2. Observe that the vertices $v_3, v_4, \ldots v_{2n-1}$ alternate between undistinguished and distinguished vertices the the first and last being undistinguished as in Figure 4.12.

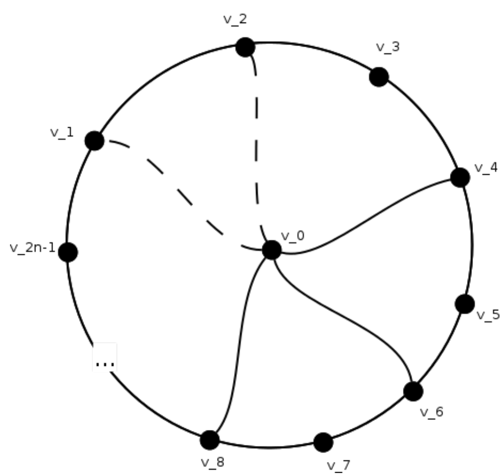


Figure 4.12: Subgraph configuration

Suppose some distinguished path p_a with $a \notin \{1,2\}$ has length greater than one as in Figure 4.13. Let v_a be the vertex on C meeting p_a and let v_p be the vertex adjacent to v_0 on p_a. Let p'_a be the subpath from v_p to v_a.

By Theorem 2.0.2, the graph G has $n-1$ distinct internally disjoint paths from v_p to C not through v_0 . By (4.0.1.1), $|p_1|$ and $|p_2|$ have the same parities, and by (4.0.1.2) p_a and $p_{a\pm 2}$ have opposite parities, since p_a has length greater than one . Let p_2 have length greater

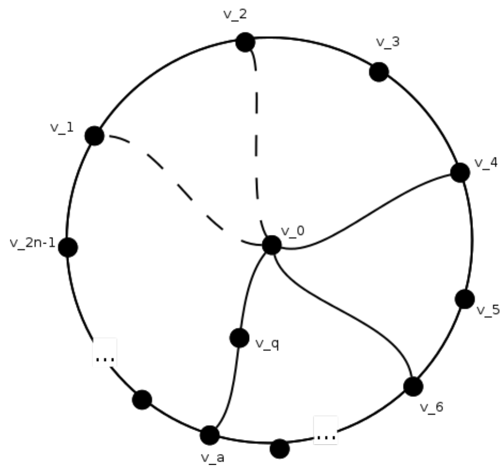


Figure 4.13: Subgraph configuration

than one. Let v_t be a point on the interior of p_2. Let p_2' be the subpath from v_t to v_2. By (4.0.1.6) and the paths consisting of the union of (v_p, v_0), p_1, $v_1, v_{2n-1}, \ldots, v_2)$, and p_2' and the union of p_a' $(v_a, v_{a-1}, \ldots, v_{a+2})$, p_{a+2} and $(p_2 - p_2')$ of lengths $1 + |p_1| + 2n - 1 - 1 + |p_2'|$ and $|p_a'| + 2n - 1 - 2 + |p_{a+2}| + |p_2 - p_2'|$, no $v_p - v_t$ path disjoint from $V(C) \cup v_0$ can exist. By symmetry, no path from v_p to a vertex on the interior of p_1 that is disjoint from $V(C) \cup v_0$ can exist. Let the distinguished path with the smallest distance on C from p_a also has length greater than one. Let such a distinguished path be p_{a+2}. Let v_s be a point on the interior of p_{a+2}. Let p_{a+2}' be the subpath from v_t to v_{a+2}. By (4.0.1.6) and the paths consisting of the union of p_a', $v_a, v_{a-1}, \ldots, v_{a-2})$, and p_{a-2}' and the union of p_a', $(v_a, v_{a-1}, \ldots, v_1)$, p_1, p_2, $(v_2, v_3, \ldots, v_{a-2})$ and p_{a-2}' of lengths $|p_a'| + 2n - 1 - 2 + |p_{a_2}'|$ and $|p_a'| + 2n - 1 - 2 - 1 + |p_1| + |p_2| + |p_{a-2}|$, no $v_p - v_s$ path disjoint from $V(C) \cup v_0$ can exist. By symmetry, no path from v_p to a vertex on the interior of p_{a+2} that is disjoint from $V(C) \cup v_0$ can exist. Let a distinguished path p_b that does not meet C at a or $a \pm 2$ have length greater than one. Let v_b be the vertex on C that meets the path. Let v_q be an internal point on p_b. Let p_b be the subpath from v_q to v_b. By (4.0.1.2) p_b and $p_{b\pm2}$ have opposite parities, since p_b has length greater than one . By the paths consisting of the union of p_a', $(v_a, v_{a-1}, \ldots, v_{a-2})$, p_{a-2}, $p_b - p_b'$ and the union of p_a', $(v_a, v_{a+1}, \ldots, v_{b-2})$, p_{b-2}, p_{a-2}, $(v_{a-2}, v_{a-3}, \ldots, v_b)$ and p_b', and p_b' of lengths $|p_a'| + 2n - 1 - 2 + |p_{a-2}| + |p_b - p_b'|$ and

$|p'_a| + 2n - 1 - 2 - 2 + |p_{b-2}| + |p_{a-2}| + |p'_b|$ where $|p_{b-2}| + |p'_b|$ has the opposite parity as $|p_b - p'_b|$, no $v_p - v_q$ path that is disjoint from $V(C) \cup v_0$ can exist. The paths consisting of the union of (v_p, v_0), p_1, $(v_1, v_{2n-1}, \ldots, v_2)$ and the union of p'_a, $(v_a, v_{a+1}, \ldots v_1)$, p_1, p_{a-2}, $(v_{a-2}, v_{a-3} \ldots v_2$ imply there is no $v_p - v_2$ path that is disjoint from $V(C) \cup v_0$ unless $a = 4$. If $a = 4$, the paths consisting of the union of p'_4, and $(v_4, v_5, \ldots v_2)$ and the union of (v_p, v_0), p_1, and $(v_1, v_{2n-1}, \ldots, v_2)$ suffice. By symmetry, there is no $v_p - v_1$ path that is disjoint from $V(C) \cup v_0$. By the paths that consist of a union of (v_p, v_0), p_1 and $(v_1, v_{2n-1}, \ldots v_3)$ and the union of (v_p, v_0), p_2, and $(v_2, v_1 \ldots, v_3)$, no $v_p - v_3$ path may exist that is disjoint from $V(C) \cup v_0$. By symmetry, no $v_p - v_{2n-1}$ path may exist that is disjoint from $V(C) \cup v_0$. Thus v_p has $n - 1$ remaining paths to $V(C) \setminus \{v_{2n-1}, v_1, v_2, v_3\}$. By (4.0.1.3) and the paths p_1 and p_2 with edge $v_1 - v_2$, v_p paths may not meet adjacent vertices on the remainder of C. Thus we have $n - 1$ non-adjacent vertices into $2n - 1 - 4 = 2n - 5$ remaining vertices of C. Thus one of our single spoke paths cannot have length greater than one and we find the following

4.0.1.21. *All distinguished paths other than the paths that meet C at adjacent vertices have length one.*

Suppose one of the consecutive paths has length greater than one. Let p_2 have length greater than one as in Figure 4.14. Since p_4 has length one, p_2 has an even length by (4.0.1.2).

Let v_p be the point on p_2 that is adjacent to v_0. Let p'_2 be the subpath of p_2 from v_p to v_2.

Since p_2 is even, p_1 is also even by (4.0.1.1). Let v_q be a vertex on the interior of p_1. Let p'_1 be the subpath of p_1 from v_p to v_1. The path consisting of the union of p'_2, $(v_2, v_3, \ldots, v_1)$, and p'_1 and the union of $(v_p, v_0, v_4, v_5, \ldots v_1)$ and p'_1 imply there is no $v_p - v_q$ path that is internally disjoint from $V(C) \cup v_0$. By the paths consisting of the union of p'_2, and $(v_2, v_1, \ldots, v_3)$ and (v_p, v_0), p_1, and $(v_1, v_{2n-1}, \ldots, v_3)$, no $v_p - v_3$ path that is disjoint from

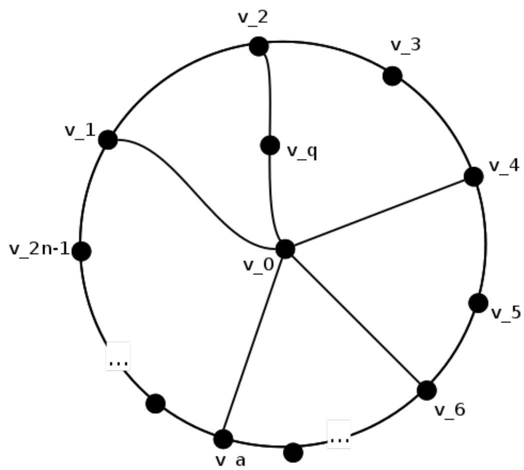


Figure 4.14: Subgraph configuration

$V(C) \cup v_0$ can exist. By the paths $(v_p, v_0, v_{2n-2}, v_{2n-3} \ldots v_{2n-1}$ and the union of (v_p, v_0), p_1, $(v_1, v_2, \ldots v_{2n-1})$, there is no $v_p - v_{2n-1}$ path that is internally disjoint from $V(C) \cup v_0$. Let a be an even number that is not four. By the paths $(v_p, v_0, v_{a-2}, v_{a-3}, \ldots, v_a)$ and the union of p_2', $(v_2, v_3, \ldots, v_{a-2}, v_0)$, p_1, $(v_1, v_{2n-1}, \ldots, v_a)$, there is no $v_p - v_a$ path that is internally disjoint from $V(C) \cup v_0$. Let a be an odd number that is not three. By the paths, $(v_p, v_0, v_{a-1}, v_{a-2}, \ldots, v_a)$ and the union of p_2', $(v_2, v_3, \ldots, v_{a-1}, v_0)$, p_1, $(v_1, v_{2n-1}, \ldots, v_a)$, there is no $v_p - v_a$ path that is internally disjoint from $V(C) \cup v_0$. Thus there are only four possible vertices for v_p paths to meet, and v_p contradicts n-connectivity. Thus p_2 has path length one and by symmetry p_1 has path length one, and we find (4.0.1.7) holds.

Since $|V(C)| > 2n$, we have more than one vertex not in $V(C)$. Label two of these vertices v_0 and v_0'. Each meets C with n paths of length one. Examine all of the consecutive vertices where v_0 paths meet C and the consecutive vertices where v_0' meets C. Suppose there is more than one adjacent pair of vertices where both v_0 and v_0' meet C. There is at least one pair for each v_0 and v_0'. Label the pair for v_0 as v_1 and v_2 and label the pair for v_0' as v_a and v_{a+1}. By the cycle $(v_1, v_0, v_2, v_3, \ldots, v_a, v_0', v_{a+1}, v_{a+2}, \ldots, v_{2n-1})$, there is a cycle of odd length greater than $2n - 1$. Thus there is only one consecutive distinguished pair. This implies we must have configurations of the type if Figure 4.15 with all paths meeting at the same vertices. We may add as many vertices as we like connecting to the same set.

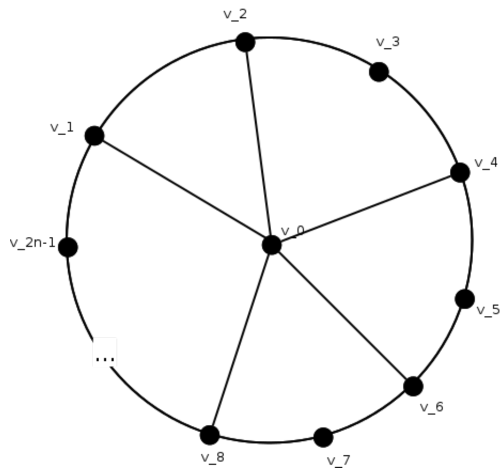


Figure 4.15: Subgraph configuration

Now we must check for additional possible edges within G that do not create a larger odd cycle. Let v_a and v_b be any two odd vertices other than one. From above we know that v_a does not connect to any v_0. Suppose we have an edge from v_a to v_b. Suppose $a > b$. By the $2n+1$-cycle $(v_a, v_{a+1}, \ldots, v_1, v_0, v_2, v_3, \ldots, v_{b-1}, v_0', v_{a-1}, v_{a-2} \ldots, v_b)$, the odd vertices except v_1 are not adjacent. Since they must have degree n this implies they are connected to the even vertices and v_1. These edges create no odd cycles. This fulfills the degree requirement of each vertex. By arranging the graph in a bipartite fashion, with one side of the partition $\{v_1, v_2, v_4, \ldots, v_{2n-2}\}$, we see that any edge between the this side of the partition will create no new additional odd cycles as we may only use the a vertex of the other partition exactly once. Thus we may have as many edges as we like between these vertices, and we reach our desired configuration.

$\square$

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Vita

Kristen Wetzler was born in Arkansas to a long line of educators. She finished her undergraduate degree in mathematics and German at the University of Arkansas, Fayetteville, in 2009. In 2010, Kristen moved to Louisiana to continue her education in mathematics. She has earned her Master of Science degree in mathematics from Louisiana State University and is a degree candidate for Doctor of Philosophy in mathematics, to be awarded in May 2018.