3-21-1988

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G. W. Ford
University of Michigan, Ann Arbor

J. T. Lewis
Dublin Institute for Advanced Studies

R. F. O'Connell
Louisiana State University

Recommended Citation

Ford, G., Lewis, J., & O'Connell, R. (1988). Dissipative quantum tunneling: quantum Langevin equation approach. *Physics Letters A, 128* (1-2), 29-34. https://doi.org/10.1016/0375-9601(88)91037-7

DISSIPATIVE QUANTUM TUNNELING: QUANTUM LANGEVIN EQUATION APPROACH

G.W. FORD
Department of Physics, The University of Michigan, Ann Arbor, MI 48109, USA

J.T. LEWIS
School of Theoretical Physics, Dublin Institute for Advanced Studies, Dublin 4, Ireland

and

R.F. O'CONNELL
Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803, USA

Received 4 August 1987; revised manuscript received 18 January 1988; accepted for publication 19 January 1988
Communicated by J.P. Vigier

The quantum Langevin equation is used as the basis for a discussion of dissipative quantum tunneling. A general analysis, including strong coupling and non-markovian (memory) effects, is given for the case of tunneling through a parabolic barrier at zero temperature in the presence of linear passive dissipation. It is shown that dissipation *always* decreases the tunneling rate below the barrier and increases transmission above the barrier. As a particular application, the case of the resistively shunted Josephson junction is considered. Simple closed form expressions for the tunneling rate and for the noise power spectrum are obtained and compared with results in the literature.

Dissipative quantum tunneling, the study of which was pioneered by Caldeira and Leggett, has so far been generally treated by techniques involving path integrals [1]. Our purpose here is to show how the quantum Langevin equation can be used to discuss these and related problems for a quantum dissipative system. An advantage of the approach using the quantum Langevin equation, aside from its simplicity, is that it is easy to incorporate non-markovian (memory) and strong coupling effects. In a few short steps we obtain an exact and general result, which we then compare with results in the literature.

In our earlier work in this area, we used the quantum Langevin equation to treat the case of a quantum oscillator in a blackbody radiation heat bath [2]. More recently, we described the form of this equation for an arbitrary external potential and for an arbitrary heat bath [3]. There, too, we showed how this general form can be derived from the independent oscillator (IO) model, in which the bath consists of an infinite number of particles, each coupled to the given particle with a spring. We also showed that many other apparently different models are equivalent to this model (or truncated versions of it). However, as we there argued, the quantum Langevin equation is a model-independent macroscopic description of a quantum particle (which need not itself be macroscopic) interacting with a heat bath.

The quantum Langevin equation has the form

$$m\ddot{x}+\int_{-\infty}^{t} \mathrm{d}t_1\, \mu(t-t_1)\dot{x}(t_1)+U'(x)=F(t)\,, \qquad (1)$$

where the dot and prime denote, respectively, the derivative with respect to t and x. This is the Heisenberg equation of motion for the coordinate operator x of a particle of mass m in a potential $U(x)$. More generally, x may be interpreted as a generalized displacement operator, by which we mean an operator such that, for any c-number $f(t)$, a term $-xf(t)$

added to the hamiltonian of the system of "particle" plus heat bath results in an added (generalized force) term $f(t)$ on the right-hand side of eq. (1).

In the quantum Langevin equation (1), the coupling with the heat bath is described by two terms: an operator-valued random force $F(t)$ with mean zero, and a mean force characterized by a memory function $\mu(t)$. The (symmetric) autocorrelation of $F(t)$ is

$$\tfrac{1}{2}\langle F(t)F(t')+F(t')F(t)\rangle = \frac{1}{\pi}\int_0^\infty d\omega\, \mathrm{Re}\{\tilde{\mu}(\omega+i0^+)\} \times \hbar\omega \coth(\hbar\omega/2kT)\cos[\omega(t-t')] , \quad (2)$$

and the nonequal-time commutator of $F(t)$ is

$$[F(t),F(t')] = \frac{2}{\pi i}\int_0^\infty d\omega\, \mathrm{Re}\{\tilde{\mu}(\omega+i0^+)\}\hbar\omega \sin[\omega(t-t')] . \quad (3)$$

In these expressions

$$\tilde{\mu}(z)=\int_0^\infty dt\, e^{izt}\mu(t) , \quad \mathrm{Im}\, z>0 \quad (4)$$

is the Fourier transform of the memory function $\mu(t)$. (By convention, the memory function vanishes for negative times.) Finally, $F(t)$ has the gaussian property: correlations of an odd number of factors of F vanish, those of an even number of factors are equal to the sum of products of pair correlations (autocorrelations), the sum being over all pairings with the order of the factors preserved within each pair.

It is clear from the above description that the coupling to the heat bath is characterized by the function $\mu(z)$. Now this function has three important mathematical properties which follow in turn from three corresponding general physical principles. The first of these, as we see from (4), is that $\mu(z)$ is analytic in the upper half-plane Im $z>0$. This is a consequence of causality; the mean force exerted by the heat bath on the particle depends only upon the *past* motion of the particle. The second property is that the boundary value of $\mu(z)$ on the real axis has everywhere a positive real part:

$$\mathrm{Re}\{\tilde{\mu}(\omega+i0^+)\}\geqslant 0 , \quad -\infty<\omega<\infty . \quad (5)$$

This, as we showed in ref. [3], is a consequence of the second law of thermodynamics. The third property is the reality condition: $\mu(\omega+i0^+)=\mu(-\omega+i0^+)^*$, which follows from the fact that x is a hermitian operator. Thus Re$\{\mu(\omega+i0^+)\}$ is an even function of ω. Such functions of a complex variable, analytic in the upper half-plane and with real part a positive, even distribution on the real axis, are termed positive real functions. They form a very restricted class of functions of a complex variable, with many remarkable properties. Important among these is the general representation in the upper half-plane (the Stieltjes inversion theorem):

$$\tilde{\mu}(z)=-icz+\frac{2iz}{\pi}\int_0^\infty d\omega \frac{\mathrm{Re}\{\tilde{\mu}(\omega+i0^+)\}}{z^2-\omega^2} , \quad (6)$$

where c is a positive constant. Thus the real positive *distribution* Re$\{\tilde{\mu}(\omega+i0^+)\}$ characterizes the function, except for the constant c, which in our case can be absorbed into the particle mass (beware, this is not mass renormalization). As an illustrative example, for the IO model $\tilde{\mu}(z)$ takes the form [3]:

$$\tilde{\mu}(z)=\sum_j m_j\omega_j^2 \frac{iz}{z^2-\omega_j^2} , \quad (7)$$

where m_j is the mass and ω_j is the natural frequency of the jth bath oscillator.

As a simple application of this formalism, consider the fluctuations (noise) in the displacement of a linear oscillator. There the external potential is of the form

$$U(x)=\tfrac{1}{2}m\omega_0^2x^2 , \quad (8)$$

and the quantum Langevin equation takes the form

$$m\ddot{x}+\int_0^\infty dt'\, \mu(t-t')\dot{x}(t')+m\omega_0^2x=F(t) . \quad (9)$$

Forming the Fourier transform we can write

$$\tilde{x}(\omega)=\alpha(\omega)\tilde{F}(\omega) , \quad (10)$$

where

$$\alpha(\omega)=[-m\omega^2+m\omega_0^2-i\omega\tilde{\mu}(\omega)]^{-1} \quad (11)$$

is the susceptibility. With this it is a straightforward calculation using (2) to show that

$$\tfrac{1}{2}\langle x(t)x(t')+x(t')x(t)\rangle = \int_0^\infty d\omega\,[x^2]_\omega \cos[\omega(t-t')]\,, \quad (12)$$

where $[x^2]_\omega$, the power spectrum of the fluctuations of x, is given by

$$[x^2]_\omega = \frac{\hbar}{\pi}\coth(\hbar\omega/2kT)\,\mathrm{Im}\{\alpha(\omega)\}\,. \quad (13)$$

This is, of course, a well-known result [4].

As a rather different type of application, we now consider the effect of dissipation on quantum tunneling at zero temperature through a parabolic barrier. Consider therefore an external potential of the form

$$U(x) = -\tfrac{1}{2}m\Omega_0^2 x^2\,, \quad (14)$$

i.e., an inverted oscillator potential. In the absence of dissipation, we have an exact expression for the zero-temperature transmission coefficient [5]:

$$D_0 = \frac{1}{1+\exp(-2\pi E/\hbar\Omega_0)}\,, \quad (15)$$

where E is the particle energy measured from the top of the barrier. Thus, $E>0$ corresponds to transmission above the barrier and $E>0$ corresponds to tunneling. This formula applies for energies near the top of any barrier whose dependence on x near the maximum is quadratic [5].

In the presence of dissipation, we can describe the motion by the quantum Langevin equation (1), which with the external potential (14) corresponds to a susceptibility of the form

$$\alpha(\omega) = [-m\omega^2 - m\Omega_0^2 - i\omega\tilde{\mu}(\omega)]^{-1}\,. \quad (16)$$

We now use the fact that the heat bath can be represented by an IO model of coupled oscillators. We stress that we are not saying that the bath is in fact an assembly of coupled oscillators. Rather, to the extent that system can be described by the Langevin equation, the bath is indistinguishable from an oscillator bath.

With the potential (14) we see that the system of particle plus oscillator bath is again an assembly of coupled oscillators with, however, one spring having a negative spring constant. The normal mode frequencies of this coupled system are the poles of the susceptibility $\alpha(\omega)$; the normal mode frequencies of the bath are the zeros of $\alpha(\omega)$, i.e., the poles of $\tilde{\mu}(z)$ [2]. The bath mode frequencies are all real. This is the passivity condition and can be seen clearly using the explicit form (7) for $\mu(\omega)$. For the coupled system, however, there is one isolated imaginary normal mode frequency corresponding to a pole of the susceptibility (16) at a point $z=i\Omega$, $\Omega>0$, in the upper half-plane; all other normal mode frequencies are real. This isolated normal mode can be interpreted as corresponding to a parabolic barrier in the coupled system. The corresponding tunneling rate will be given by (15) with Ω replacing Ω_0.

The equation determining Ω is therefore

$$[\alpha(i\Omega)]^{-1} = m\Omega^2 - m\Omega_0^2 + \Omega\tilde{\mu}(i\Omega) = 0\,. \quad (17)$$

Using the representation (6), we can write this in the form

$$\Omega^2 + \frac{2\Omega^2}{m\pi}\int_0^\infty d\omega \frac{\mathrm{Re}\{\tilde{\mu}(\omega+i0^+)\}}{\Omega^2+\omega^2} = \Omega_0^2\,. \quad (18)$$

With the positivity condition (5) one sees readily that the left hand side of this equation is a monotonically increasing function of Ω, and therefore there will always be exactly one solution. Moreover, the solution Ω will always be less than Ω_0, so that the effect of dissipation is always to reduce the transmission coefficient for a given (negative) energy below the barrier and to increase the transmission for an energy above the barrier.

As an illustration of the ideas we have presented, we consider here their application to the problem of macroscopic quantum tunneling in resistively shunted Josephson junctions [6,7]. There has been considerable interest in this problem in connection with theory and experiment [1,8]. Our interest here is to show how some of the principle theoretical results follow simply from the quantum Langevin approach. For an ideal junction the current is given by the Josephson equation [7]:

$$I = I_C \sin\phi\,, \quad (19)$$

where ϕ is the phase difference of the superconducting wave function across the junction and I_C is the critical current. The voltage across the junction is

$$V=\frac{\hbar}{2e}\dot{\phi}\,, \tag{20}$$

where $\hbar/2e$ is the quantum of flux. A real junction can be viewed as a capacitance C and a shunt resistance R in parallel with an ideal junction [6,7]. The current is then the sum of the ideal junction current, given by (19), the current through the capacitor, $\dot{Q}=C\dot{V}$, and the current through the resistor, $I=V/R$. The junction voltage is still given by (20). The basic equation of motion of the junction can therefore be written

$$\left(\frac{\hbar}{2e}\right)^2 C\ddot{\phi}+\left(\frac{\hbar}{2e}\right)^2\frac{\dot{\phi}}{R}+\frac{\hbar}{2e}I_C\sin\phi$$

$$=\frac{\hbar}{2e}I+F(t)\,. \tag{21}$$

This is of the form of a quantum Langevin equation (1) with mass and friction constant

$$m=(\hbar/2e)^2C\,,\quad \zeta=(\hbar/2e)^2/R\,, \tag{22}$$

and with potential

$$U(\phi)=-\frac{\hbar}{2e}(I\phi+I_C\cos\phi)\,. \tag{23}$$

We have written the equation in the form (21) since only then is ϕ interpretable as a (generalized) displacement. To verify this we consider the effect of adding a term $-f(t)\phi$ to the system hamiltonian or, equivalently, a term $f(t)\phi$ to the system lagrangian. This then becomes

$$L=\tfrac{1}{2}CV^2+...+f(t)\phi$$

$$=\tfrac{1}{2}m\dot{\phi}^2+...+f(t)\phi\,, \tag{24}$$

where "..." represents the unknown or unspecified part of the lagrangian corresponding to the heat bath and the many-body effects resulting in the Josephson current, and where in the second form we have used (20) and the expression (22) for the mass. In this second form it is clear that the effect of the added term would be to add a term $f(t)$ to the right-hand side of the basic equation (21). Thus, ϕ is a displacement in the sense of our discussion following (1) above.

With this we can immediately adapt the results for the Langevin equation described above. From (2), it follows that the correlation of the random force in (21) is

$$\tfrac{1}{2}\langle F(t)F(0)+F(0)F(t)\rangle$$

$$=\frac{\hbar^3}{4\pi e^2R}\int_0^\infty d\omega\,\omega\coth(\hbar\omega/2kT)\cos\omega t\,. \tag{25}$$

Near a local minimum the potential (23) is of the form

$$U(\phi)\approx\frac{\hbar}{4e}(I_C^2-I^2)^{1/2}\Delta\phi^2\,, \tag{26}$$

where $\Delta\phi=\phi-\arcsin(I/I_C)$. This is a linear oscillator potential, so we can apply the result (13) to obtain the power spectrum of the phase fluctuations:

$$[\Delta\phi^2]_\omega=\frac{4e^2}{\pi C\hbar}\coth(\hbar\omega/2kT)\frac{\omega\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}\,, \tag{27}$$

where

$$\gamma=\frac{1}{RC}\,,\quad \omega_0^2=\frac{2e}{C\hbar}(I_C^2-I^2)^{1/2}\,. \tag{28}$$

The mean square derivation of the phase is therefore

$$\langle\Delta\phi^2\rangle$$

$$=\frac{4e^2}{\pi C\hbar}\int_0^\infty d\omega\coth(\hbar\omega/2kT)\frac{\omega\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}\,. \tag{29}$$

In the limit of large shunt resistance (weak coupling limit), $\gamma\ll\omega_0$, this becomes

$$\langle\Delta\phi^2\rangle=\frac{2e^2}{C\hbar\omega_0}\coth(\hbar\omega_0/2kT)\,. \tag{30}$$

This weak coupling limit corresponds to the expression for the phase fluctuations obtained long ago by Josephson [9]. The power spectrum of the voltage fluctuations is readily obtained from (27) using (20);

$$[V^2]_\omega = \frac{\hbar}{\pi C}\coth(\hbar\omega/2kT)\,\frac{\omega^3\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}\,. \tag{31}$$

Note that this becomes very small at low frequencies.

A more interesting question concerns the quantum tunneling near a local maximum of the potential (23), where

$$U(\phi)\approx -\frac{\hbar}{4e}(I_C^2-I^2)^{1/2}[\phi+\arcsin(I/I_C)-\pi]^2\,. \tag{32}$$

From a comparison with (14), it is clear that our resistively shunted Josephson junction is reduced, near a local maximum, to a special case of the parabolic barrier problem, with

$$\Omega_0^2=\frac{2e}{C\hbar}(I_C^2-I^2)^{1/2}\,, \tag{33}$$

and corresponding to a frequency-independent friction constant,

$$\mathrm{Re}\{\mu(\omega+\mathrm{i}0^+)\}=\zeta=m\gamma\,. \tag{34}$$

The integral in (18) is then elementary and the equation becomes

$$\Omega^2+\gamma\Omega=\Omega_0^2\,. \tag{35}$$

The solution is

$$\Omega=(\Omega_0^2+\tfrac{1}{4}\gamma^2)^{1/2}-\tfrac{1}{2}\gamma\,. \tag{36}$$

There is also a negative root obtained by changing the sign of the square root. This, however, corresponds to a point in the "unphysical sheet" reached by analytically continuing $\alpha(z)$ across the real axis into the lower half-plane and does not have immediate physical significance. We should perhaps also remark that there are no imaginary roots corresponding to the infinitely many normal modes arising out of the bath. This is because these modes are continuously distributed so that the real axis becomes a "branch cut" of $\alpha(z)$.

If in (15) we replace Ω_0 by Ω as given in (36), we find for the transmission coefficient

$$D=[1+\exp(-2\pi E/\hbar\Omega)]^{-1}$$
$$=(1+\exp\{-2\pi E[(1+\gamma^2/4\Omega_0^2)^{1/2}+\gamma/2\Omega_0]/\hbar\Omega_0\})^{-1}\,. \tag{37}$$

Here we see explicitly that for any fixed (negative) energy E below the barrier the transmission coefficient is decreased, $D\leqslant D_0$. Indeed, D is a monotonically decreasing function of γ. On the other hand, for energies above the barrier (positive E) the transmission is increased.

In the limit of tunneling well below the barrier, $E\ll-\hbar\Omega_0$, and weak coupling to the heat bath, $\gamma\ll\Omega_0$, the transmission coefficient (37) becomes

$$D\approx\exp\left(\frac{2\pi E(1+\gamma/2\Omega_0)}{\hbar\Omega_0}\right)=D_0\exp\left(\frac{\pi E\gamma}{\hbar\Omega_0^2}\right). \tag{38}$$

We can write this in another way if we introduce the barrier width, w, writing

$$E=-\tfrac{1}{2}m\Omega_0^2(\tfrac{1}{2}w)^2\,. \tag{39}$$

Then (38) becomes

$$D=D_0\exp(-\tfrac{1}{8}\pi\zeta w^2/\hbar)\,, \tag{40}$$

where $\zeta=m\gamma$ is the friction coefficient. This is similar to a result obtained by Caldeira and Leggett using path integration methods [1]; their WKB exponent, B say, is larger than our exponent by a factor ≈ 1.2, which is consistent with the fact that they are calculating an upper limit on B. However, there is a distinct difference between the *form* of the results for the case of strong coupling. In all cases, Caldeira and Leggett express their results in the form $D=D_0\exp(-B)$, where B depends on the strength of the coupling. By contrast, in our case, we note from (37) that this form might not ensue for γ values greater than Ω_0. We should perhaps stress that the result (37) is *exact* for all values of the friction constant, subject only to the assumption that the tunneling is elastic.

In summary, we have used the quantum Langevin equation as the basis for a discussion of dissipative quantum tunneling. We feel that the approach is simple and physically appealing. In essence, we use techniques well-known in the analysis of non-dissipative problems which we supplement with two powerful results, obtained from our work on the quantum Langevin equation, viz. (a) that the real part of the memory function is always positive (see (5)) which led to the very general conclusion (see (18)) that, even in the case of non-markovian interactions, the effect of dissipation at zero temperature is always to reduce the transmission coefficient, and (b) that the poles of the susceptibility give the normal-mode frequencies (see (17)), which lead to the all-important equation (18) for Ω.

This research was partially supported by the U.S. Office of Naval Research under Contract No. N00014-86-K-0002 and by the National Science Foundation Grant No. INT-8504402.

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