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Weierstrass Points on Gorenstein Curves.

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A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

by

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Abstract

This thesis generalizes the classical definition of a Weierstrass point to integral projective Gorenstein curves. For \( X \) an integral projective Gorenstein curve of arithmetic genus \( g \) at least two, pick \( P \in X \), and call the sheaf of dualizing differentials \( \omega \). For a proper closed subscheme \( Z \) of \( X \) with support \( P \) and ideal sheaf \( I \), define the degree of \( Z \) to be \( \dim_\mathcal{O}_X 0_p/I_p \). Call \( Z \) 1-special if \( \dim_\mathcal{O}_X \text{Hom}(I, \mathcal{O}_X) > 1 \).

In a manner analogous to the classical construction, one defines the wronskian of \( X \) to be some element \( \alpha \) of \( \mathbb{P}(H^0(X, \omega^{1/2}(g^2+g))) \). One defines the Weierstrass weight \( W(P) \) to be \( \text{ord}_p \alpha \), and one calls \( P \) a Weierstrass point of \( X \) if \( W(P) > 0 \). Then

Theorem 1. The following statements are equivalent for \( P \in X \).

1. \( W(P) > 0 \).
2. There is a nonzero \( \sigma \in H^0(X, \omega) \) satisfying \( \text{ord}_p \sigma \geq g \).
3. There is a 1-special subscheme with support \( P \) and length equal to \( g \).
4. There is a 1-special subscheme with support \( P \) and length at most \( g \).

Consider the following statements about \( P \in X \).

A. There is a morphism \( \phi: X \rightarrow \mathbb{P}^1 \) of degree at most \( g \) satisfying \( \phi^{-1}(\phi(P)) = \{ P \} \).

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(B) There is a principal l-special subscheme with support P and degree at most g.

(C) P is a Weierstrass point.

If P is nonsingular (A) \Rightarrow (B) \Rightarrow (C). If P is singular, then one only has (A) \Rightarrow (B) \Rightarrow (C). But, there is the following.

Theorem 2. Let $P \in X_{\text{sing}}$ satisfy $\chi_P = 1$.

(1) If P is a cusp then P satisfies (A).
(2) If P is a node then P satisfies (C). P does not satisfy (B) if and only if

(i) X is of genus three.
(ii) Let $\theta: Y \to X$ be the normalization at P with $\theta^{-1}(P) = \{Q,R\}$.

Then Q and R are Weierstrass points of Y.
Introduction

Let $X$ be a nonsingular complex projective curve of genus $g$. For $P \in X$, consider the following question. For which positive integers $n$ does there exist a rational function on $X$ with a pole of order $n$ at $P$ and no other pole? The answer is no for exactly $g$ positive integers $m_1, \ldots, m_g$, and the sequence $m_1, \ldots, m_g$ is called the Weierstrass gap sequence for $P$. The Weierstrass weight of $P$, $W(P)$, is then defined by $W(P) = \sum (m_i - 1)$. We then have the following classical theorem.

**Theorem.** The following statements are equivalent for $P \in X$.

1. $W(P) > 0$.
2. There is a $\sigma \in \mathcal{H}^0(X, \Omega)$ satisfying $\text{ord}_P \sigma \geq g$.
3. $1(\sigma P) > 1$.
4. $1(nP) > 1$ for some $0 < n \leq g$.

A point $P$ of $X$ is called a Weierstrass Point if it satisfies the conditions of the theorem above.

In chapter 1 the classical theory of Weierstrass points is reviewed. Included is the following result which we show later does not generalize to Gorenstein curves.

**Theorem.** The following statements are equivalent.

1. $P$ is a Weierstrass point.
2. There is a morphism $\phi: X \to \mathbb{P}^1$ of degree at most $g$ satisfying $\phi^{-1}(\phi(P)) = \{P\}$.
We wish to generalize the notion of Weierstrass points to Gorenstein curves. To this end we note that S. Diaz proved in [D,A2.1] that a node on an irreducible stable curve is the limit of at least \( g(g-1) \) Weierstrass points on nearby nonsingular curves.

In chapter 2 we construct the wronskian. Let \( X \) be an integral projective complex Gorenstein curve of arithmetic genus \( g \). Then the wronskian of \( X \) is an equivalence class of elements of \( H^0(X, \omega^N) \) where \( N = 1 + \ldots + g \). For \( P \in X \), the Weierstrass weight, \( W(P) \), is then the order of the vanishing at \( P \) of the wronskian. \( W(P) \) has the following properties.

1. \( W(P) \geq \delta_P g(g-1) \), where \( \delta_P = \dim \tilde{\mathcal{O}}_P / \mathcal{O}_P \).
2. \( W(P) = g^3 - g \).

We note that if \( P \) is a node, then \( \delta_P = 1 \), so (1) generalizes Diaz's result. Furthermore, (1) shows that if \( X \) has arithmetic genus at least two, then every singular point of \( X \) has positive Weierstrass weight.

Since the limit of a family of divisors, as a family of smooth curves degenerates to a singular curve, is in general only a subscheme, (see [K]), we are led to consider proper closed subschemes and in particular 1-special subschemes at the end of chapter 2.

In chapter 3 we begin with our main theorem.
Theorem. Let X be an integral projective complex Gorenstein curve of arithmetic genus g. The following statements are equivalent for P \subseteq X.

1. \( W(P) > 0 \)
2. There is a \( \sigma \in H^0(X, \omega) \) satisfying \( \text{ord}_P \sigma \geq g \).
3. There is a 1-special subscheme of X of degree g and support P.
4. There is a 1-special subscheme of X of degree at most g and support P.

As in the classical case, P is a Weierstrass point if it satisfies the conditions of the theorem.

If P \subseteq X_{\text{reg}} , then, as in the classical case, we have

1. \( W(P) \leq \frac{1}{2}g(g-1) \),
2. \( W(P) = \frac{1}{2}g(g-1) \) if and only if there is a degree two morphism \( \phi : X \to \mathbb{P}^1 \) satisfying \( \phi^{-1}(\phi(P)) = \{P\} \).

Consider the following statements about P \subseteq X.

(A) There is a morphism \( \phi : X \to \mathbb{P}^1 \) of degree at most g satisfying \( \phi^{-1}(\phi(P)) = \{P\} \).
(B) X has a principal 1-special subscheme with support P and degree at most g.
(C) P is a Weierstrass point of X.

If P \subseteq X_{\text{reg}} , then (A) \( \Rightarrow \) (B) \( \Rightarrow \) (C). However, if P \subseteq X_{\text{sing}} , then we only have (A) \( \Rightarrow \) (B) \( \Rightarrow \) (C), and we provide counterexamples to each reverse implication. But
there is the following result.

**Theorem.** Let $X$ be a quasi-hyperelliptic curve of arithmetical genus at least two. ($X$ is a quasi-hyperelliptic curve if there is a morphism of degree two $\theta: X \to \mathbb{P}^1$.) Then the following statements are equivalent.

1. $P$ is a Weierstrass point.
2. Every degree two morphism $\phi: X \to \mathbb{P}^1$ satisfies $\phi^{-1}(\phi(P)) = \{P\}$. 
Chapter 1. The Classical Theory

The goal of this thesis is to generalize the concept of a Weierstrass point to Gorenstein curves. In order to facilitate this generalization, chapter one will be a review of the classical theory on nonsingular curves.

In this chapter $X$ is a nonsingular projective curve over $\mathbb{C}$ with sheaf of algebraic functions $\mathcal{O}_X$ and field of rational functions $K(X)$. The term "$P$ is a point of $X" shall mean that $P$ is a closed point of $X$. For a sheaf $F$ over $X$, $F_P$ denotes the stalk of $F$ at $P$. Finally, $m_P$ denotes the maximal ideal of $\mathcal{O}_P$.

Definition: For $P \in X$ let $t$ be a rational function which generates $m_P$ in $\mathcal{O}_P$. Then any nonzero rational function $f$ can be written as $f = t^nh$, where $n$ is a uniquely defined integer and $h$ is a unit in $\mathcal{O}_P$. One then defines $\text{ord}_P f = n$.

Definition: More generally, let $L$ be an invertible sheaf and let $t$ generate $L_P$. Then given any nonzero $\sigma$ in $H^0(X, L)$, $\sigma = f_t$ for some nonzero $f$ in $\mathcal{O}_P$. One then defines $\text{ord}_P \sigma = \text{ord}_P f$. Note that for all $P$ in $X$, $\text{ord}_P \sigma$ is a nonnegative integer.

Definition: Let $D = \sum n_P P$ be a divisor (i.e. an element of the free abelian group generated by the points of $X$).
Define the invertible sheaf $L(D)$ as follows. Let $t_p$ generate $m_p$ in $\mathcal{O}_p$. For an open subset $U$ of $X$ put
\[ \Gamma(U, L(D)) = \bigcap_{P \in U} \mathcal{O}_p. \]
Here all of the $\mathcal{O}_p$ are considered to be subsets of $K(X)$.

**Definition:** The **degree of** $L(D)$, denoted $\deg L(D)$, is defined by
\[ \deg L(D) = \deg D = \Sigma n_p. \]

**Definition:** Given a nonzero rational function $f$ on $X$, the **divisor of** $f$ denoted $(f)$ is defined by
\[ (f) = \Sigma (\text{ord}_p f)P. \]

**Definition:** Two divisors $D$ and $E$ on $X$ are **linearly equivalent** if
\[ D = E + (f) \]
for some nonzero $f \in K(X)$.

**Proposition 1.1.**

1. Given any nonzero $f \in K(X)$, $\deg(f) = 0$.
2. If $D$ and $E$ are linearly equivalent divisors, then $\deg D = \deg E$.

**Proof:**
1. [H, II, 6.4 (b)]
2. This follows immediately from (1).

**Proposition 1.2.** Suppose that $\sigma$ is a nonzero element of $H^0(X, L(D))$. Then
\[ \Sigma \text{ord}_p \sigma = \deg D. \]
Proposition 1.3. The set of all isomorphism classes of invertible sheaves on $X$ forms a group under the operation $\otimes$. This group is called the Picard group of $X$, and is denoted $\text{Pic} X$.

Proof: \[H, \text{II}, 6.12\]

Proposition 1.4. The correspondence $D \mapsto L(D)$ induces an isomorphism from the group of divisors modulo linear equivalence onto the group $\text{Pic} X$.

Proof: \[H, \text{II}, 6.14, 6.14.1\]

We shall denote the sheaf of regular differentials on $X$ by $\Omega = \Omega^*_X$. $\Omega$ is an invertible sheaf. Indeed, if for $P \in X$ it generates $m_P$ in $\mathcal{O}_P$, then $\Omega_P = \mathcal{O}_P dt$.

Definition: Suppose that $\eta$ is the generic point of $X$. One defines the set of rational differentials on $X$, denoted $\mathcal{D}_X$, by $\mathcal{D}_X = \Omega^*_X, \eta$.

Definition: A canonical divisor on $X$ is any divisor $K$ such that $L(K) \cong \mathcal{O}$.

Definition: A divisor $D = \sum n_P P$ is effective if $n_P \geq 0$ for all $P$.

Definition: For a divisor $D$, put $l(D) = \dim_c H^0(X, L(D))$.

Definition: The (arithmetic) genus $g$ of $X$ is defined by $g = \dim H^1(X, \mathcal{O}_X)$.

Theorem 1.1. (Riemann-Roch) Let $X$ have genus $g$ and let $K$ be a canonical divisor. Then for any divisor $D$ on $X$.

$$l(D) - l(K-D) = \deg D + 1 - g.$$
Proposition 1.5. Let \( X \) have genus \( g \) and let \( K \) be a canonical divisor.

\begin{enumerate}
  
  \item \( l(K) = g \).
  
  \item \( \deg K = 2g - 2 \).
\end{enumerate}

Proof: \((1)\) [H,IV,1.1].

\( (2) \) This follows from \((1)\) and the Riemann-Roch theorem.

Proposition 1.6. \( l(D) \) is positive if and only if \( D \) is linearly equivalent to an effective divisor. In particular, if \( \deg D < 0 \), then \( l(D) = 0 \).

Proof: [H,IV,1.2]

Definition: For a sheaf \( F \) over \( X \) the support of \( F \), denoted \( \text{supp } F \), is defined by \( \text{supp } F = \{ P \in X : F_P \neq 0 \} \).

The next result follows easily from elementary facts about sheaf cohomology.

Proposition 1.7. Let \( F \) be a sheaf over \( X \) with finite support \( P_1, \ldots, P_n \). Then

\begin{enumerate}
  
  \item \( H^0(X,F) = F_{P_1} \oplus \cdots \oplus F_{P_n} \).
  
  \item \( H^1(X,F) = 0 \).
\end{enumerate}

Proposition 1.8. (Noether Reduction theorem) For \( P \in X \) and \( D \) a divisor on \( X \).

\begin{enumerate}
  
  \item \( l(D) \leq l(D + P) \leq l(D) + 1 \).
  
  \item \( l(D) = l(D + P) \) if and only if \( l(K - D) = l(K - (D + P)) + 1 \).
\end{enumerate}

Proof: The exact sequence of sheaves

\[ 0 \to L(D) \to L(D + P) \to K \to 0 \]

induces an exact cohomology sequence

\[ 0 \to H^0(X,L(D)) \to H^0(X,L(D + P)) \to H^0(X,K). \]
(1) will now follow if it is established that $H^0(X,K) = \mathbb{C}$.
Suppose that $t$ generates $m_p$ in $0_p$. Put $D = \Sigma n_Q Q$, and
$n_p = -j$. Then $K$ has support $P$, and

$$H^0(X,K) = K_p = t^j 0_p / t^{j+1} 0_p = 0_p / t 0_p = \mathbb{C}.$$ 

To establish (2), consider the two equations below
given by the Riemann-Roch theorem.

(i) $l(D+P) - l(K - (D+P)) = \deg(D+P) + 1 - g,$
(ii) $l(D) - l(K-D) = \deg D + 1 - g.$

Subtracting (ii) from (i) yeilds

$$(l(D+P) - l(D)) + (l(K-D) - l(K - (D+P))) = 1.$$ 

The above equation and (1) together imply (2).

**Proposition 1.9.** Suppose that $D$ is a divisor on $X$ of
degree greater than $2g - 2$. Then

(1) $l(D) = \deg D + 1 - g.$
(2) For $P \in X$, $l(D+P) = l(D) + 1$.

**Proof:** Let $K$ be a canonical divisor. By proposition 1.5,
$\deg(K-D) < 0$, and so by proposition 1.6, $l(K-D) = 0$. (1)
now follows from the Riemann-Roch theorem. (2) follows
immediately from (1).

q.e.d.

Let $X$ have genus $g$ at least one. For $P \in X$ consider
the sequence of divisors

$P, 2P, \ldots, iP, \ldots$.

By the Noether reduction theorem for all positive $i,$
1 \leq l(iP) \leq l((i+1)P) \leq l(iP) + 1.

By proposition 1.9, \( l((2g-1)P) = g \), and for all \( n \geq 2g-1 \), \( l((n+1)P) = l(nP) + 1 \). It follows that there are exactly \( g \) positive integers \( 1 \leq m_1 < \ldots < m_g \leq 2g-1 \) such that \( l(m_iP) = l((m_i - 1)P) \).

**Definition:** The sequence \( m_1, \ldots, m_g \) defined above is called the **Weierstrass gap sequence** (or **gap sequence**) for \( P \). The \( m_i \)'s are called the **Weierstrass gaps** (or **gaps**) for \( P \).

**Proposition 1.10.** Given \( P \in X \), \( n \) is not a gap for \( P \) if and only if there exists \( f \in \Gamma(X-P, 0_X) \) satisfying \( \text{ord}_P f = -n \).

Consequently, the set of nongaps for \( P \) forms a semigroup under addition.

**Proof:** Assume \( n \) is not a gap for \( P \). Then there exists \( f \in H^0(X, L(nP)) - H^0(X, L((n-1)P)) \). \( f \in H^0(X, L(nP)) \) implies \( f \in \Gamma(X-P, 0_X) \), and, considered as a rational function, \( \text{ord}_P f \geq -n \). On the other hand, \( f \notin H^0(X, L((n-1)P)) \) and \( f \in \Gamma(X-P, 0_X) \) together imply \( \text{ord}_P f < -(n-1) \). It follows that \( f \in \Gamma(X-P, 0_X) \) and \( \text{ord}_P f = -n \).

Conversely, let \( f \in \Gamma(X-P, 0_X) \) satisfy \( \text{ord}_P f = -n \). It follows that \( f \in H^0(X, L(nP)) - H^0(X, L((n-1)P)) \). Thus, \( 1(nP) > l((n-1)P) \) and so \( n \) is not a gap for \( P \).

Since for all nongaps \( n \) and \( m \) there exists \( f_n \) and \( f_m \in \Gamma(X-P, 0_X) \), satisfying \( \text{ord}_P f_n = -n \) and \( \text{ord}_P f_m = -m \), setting \( h = f_n \cdot f_m \), it follows that \( h \in \Gamma(X-P, 0_X) \) and \( \text{ord}_P h = -(n+m) \). As a result, the set of nongaps forms
Proposition 1.11. The following statements are equivalent for $P \in X$.

1. $m$ is a Weierstrass gap for $P$.
2. There exists $\sigma \in H^0(X, \Omega)$ with $\text{ord}_P \sigma = m-1$.

Proof: Let $K = \sum n_Q Q$ be a canonical divisor. First note that there exists $f \in H^0(X, \mathcal{L}(K))$ such that, when considered as a rational function, $\text{ord}_P f = i$, if and only if there exists $\sigma \in H^0(X, \Omega)$ satisfying $\text{ord}_P \sigma = i + n_P$. The Noether reduction theorem implies that $m$ is a gap for $P$ if and only if $\mathcal{L}(K - (m-1)P) = \mathcal{L}(K - mP) + 1$. By the proof of proposition 1.10, the above equation holds if and only if there exists $f \in H^0(X, \mathcal{L}(K))$ satisfying $\text{ord}_P f = -n_P + m-1$. By the remark at the start of the proof, this is equivalent to the existence of a $\sigma$ in $H^0(X, \Omega)$ with $\text{ord}_P \sigma = m-1$.

Definition: Suppose that $f$ and $t$ are rational functions on $X$ with $t$ nonconstant. Define $f'(t)$ by $df = f'(t)dt$. Define $f^{(0)}(t) = f(t)$, and define $f^{(k)}(t)$ recursively by the rule $df^{(k-1)} = f^{(k)}(t)dt$.

Definition: Let $t$ be as above. Suppose that $f_1, \ldots, f_g$ are rational functions. Define the wronskian of $f_1, \ldots, f_g$ with respect to $t$, denoted $W_t(f_1, \ldots, f_g)$, by $W_t(f_1, \ldots, f_g) = \det[f_j^{(i-1)}(t)]$, for $1 \leq i, j \leq g$. 
Remark: For $P \in X$ suppose $t$ generates $m_P$ in $\mathcal{O}_P$. Then for $f \in \mathcal{O}_P$, $df = f'(t)dt \in \mathcal{O}_P$. Since $dt$ generates $\mathcal{O}_P$, it follows that $f'(t) \in \mathcal{O}_P$. Thus, if $f_1, \ldots, f_g \in \mathcal{O}_P$, $W_t(f_1, \ldots, f_g) \in \mathcal{O}_P$.

The following basic properties of the wronskian are well-known.

**Proposition 1.12.** Assume $f_1, \ldots, f_g$ are linearly independent over $\mathbb{C}$.

1. $W_t(f_1, \ldots, f_g)$ is a nonzero element of $K(X)$.
2. Assume $h_1, \ldots, h_g$ generates the same subspace of $K(X)$ as $f_1, \ldots, f_g$ with for all $i$ $h_i = \sum b_{ij} f_j, b_{ij} \in \mathbb{C}$. Then $W_t(h_1, \ldots, h_g) = \det[b_{ij}]W_t(f_1, \ldots, f_g)$. So the wronskian of the second basis is a nonzero constant multiple of the wronskian of the first basis.
3. For $P \in X$ assume $t$ generates $m_P$ in $\mathcal{O}_P$, and that for all $t$, $\text{ord}_P f_i = n_i$ where $n_1 < \ldots < n_g$. Then $W_t(f_1, \ldots, f_g) = \sum (n_i - (i-1))$.
4. Suppose that $\sigma_1, \ldots, \sigma_g$ is a basis for $H^0(X, \mathcal{O})$ and that $z$ is a nonconstant rational function. Assume that for all $i$

\[ \sigma_i = f_i dt = s_i dz. \]

Put $N = 1 + \ldots + g$. Then

\[ W_t(f_1, \ldots, f_g) = z'(t)^N W_z(s_1, \ldots, s_g). \]

5. With the same notation as above, note that

\[ W_t(f_1, \ldots, f_g) dt^\mathcal{O}N = z'(t)^N W_z(s_1, \ldots, s_g) dt^\mathcal{O}N = W_z(s_1, \ldots, s_g) dz^\mathcal{O}N. \]
Now suppose that $X$ has nonzero genus $g$, and that $\sigma_1, \ldots, \sigma_g$ is a fixed basis for $H^0(X,\Omega)$. Put $F = \Omega^N$, $N = 1 + \ldots + g$. Define $\alpha \in H^0(X,F)$ as follows. Pick a non-constant rational function $t$ and write $\sigma_i = f_i \, dt$ for $i = 1, \ldots, g$. Then

$$\alpha = W_t(f_1, \ldots, f_g) \, dt^N.$$ 

By proposition 1.12, (5), $\alpha$ is independent of the choice of $t$. Further, given $P \in X$ let $t$ generate $m_P$ in $0_P$. Then $dt^N$ is a generator of $F_P$, and $W_t(f_1, \ldots, f_g) \in 0_P$. Thus for all $P$, $\alpha \in F_P$ and so $\alpha \in H^0(X,F)$.

If $\tau_1, \ldots, \tau_g$ is another basis for $H^0(X,\Omega)$ with $\tau_i = h_i \, dt$ for $i = 1, \ldots, g$, then by proposition 1.12, (2), for some nonzero $b \in \mathbb{C}$, $W_t(h_1, \ldots, h_g) = b W_t(f_1, \ldots, f_g)$. So changing the basis of $H^0(X,\Omega)$ changes $\alpha$ by a nonzero constant multiple.

**Definition:** Let $\alpha$ be as above. One calls the equivalence class of $\alpha$ in $\mathbb{P}(H^0(X,\Omega^N))$ the **Wronskian of $X$**.

**Definition:** Let $X$ have nonzero genus $g$ and let $N$ and $\alpha$ be as above. For $P \in X$ the **Weierstrass weight** of $P$, denoted $W(P)$, is defined by $W(P) = \text{ord}_P \alpha$. Note that $W(P)$ is independent of the choice of $\alpha$. If $g = 0$, define $W(P) = 0$ for all $P \in X$.

**Proposition 1.13.** Suppose that $X$ has genus $g$.

1. $W(P) \geq 0$ for all $P \in X$. 

(2) \[ \sum_{P \in X} W(P) = (g+1)g(g-1). \]

(3) Suppose \( g \) is positive. For \( P \in X \) let \( m_1, \ldots, m_g \) be the gap sequence for \( P \). Then \( W(P) = \sum (m_i - i) \).

Proof: Clearly (1) and (2) hold if \( g = 0 \). So we assume \( g > 0 \).

(1) Let \( \alpha \) represent the wronskian and let \( t \) generate \( m_P \) in \( \mathcal{O}_P \). Put \( N = 1 + \ldots + g \). Then \( \alpha = W_t(f_1, \ldots, f_g) dt^{\mathcal{O}_N} \). Since \( dt^{\mathcal{O}_N} \) generates \( (\mathcal{O}^{\mathcal{O}_N})_P, W(P) = \text{ord}_P W_t(f_1, \ldots, f_g) \). Thus, since \( W_t(f_1, \ldots, f_g) \in \mathcal{O}_P, W(P) > 0 \).

(2) By proposition 1.2, \( \sum_{P \in X} W(P) = \deg \mathcal{O}^{\mathcal{O}_N} = N \deg \mathcal{O} = (g+1)g(g-1) \).

(3) By proposition 1.11, if \( m_i \) is a gap for \( P \), then there exists \( \sigma_i \in H^0(X, \Omega) \) satisfying \( \text{ord}_P \sigma_i = m_i - 1 \). Suppose that \( \sigma_1, \ldots, \sigma_g \in H^0(X, \Omega) \) satisfy \( \text{ord}_P \sigma_i = m_i - 1 \) for \( i = 1, \ldots, g \). Clearly \( \sigma_1, \ldots, \sigma_g \) form a basis for \( H^0(X, \Omega) \). Therefore it follows from proposition 1.12, (3) that \( W(P) = \sum [(m_i - 1) - (i-1)] = \sum (m_i - i) \).

Proposition 1.14. The following statements are equivalent

(1) \( X \) has genus 0.

(2) For all \( P \in X, \lambda(P) > 1 \).

(3) For some \( P \in X, \lambda(P) > 1 \).

(4) \( X = \mathbb{P}^1 \).

Proof:

(1) \( \Rightarrow \) (2) follows from the Riemann-Roch Theorem applied to \( D = P \).
(2) \( \Rightarrow \) (3) is clear.

(3) \( \Rightarrow \) (4): Suppose \( P \) satisfies \( l(P) > 1 \). Then there exists a nonconstant rational function \( f \) on \( X \) with a pole of order one at \( P \) and no other pole. Thus, \( f \) defines a morphism \( \phi: X \rightarrow \mathbb{P}^1 \) of degree one. Since \( \phi \) is of degree one, it is an isomorphism.

(4) \( \Rightarrow \) (1) is clear.

Remarks 1.1: Suppose that \( V \) is an \( n \)-dimensional vector space of rational functions on \( X \) and \( U \) is an \( n \)-dimensional vector space of rational differentials on \( X \). Pick \( P \in X \).

(1) There is a basis \( f_1, \ldots, f_g \) of \( V \) which satisfies \( \text{ord}_P f_1 < \ldots < \text{ord}_P f_g \).

(2) Every nonzero \( f \in V \) satisfies \( \text{ord}_P f = \text{ord}_P f_i \) for some \( i \).

(3) There is a basis \( \tau_1, \ldots, \tau_g \) of \( U \) satisfying \( \text{ord}_P \tau_1 < \ldots < \text{ord}_P \tau_g \).

(4) Every nonzero \( \tau \in U \) satisfies \( \text{ord}_P \tau = \text{ord}_P \tau_i \) for some \( i \).

Theorem 1.2. Suppose \( X \) has genus \( g \). The following statements are equivalent for \( P \in X \).

(1) \( W(P) > 0 \).

(2) There is a nonzero \( \sigma \in H^0(X, \Omega) \) satisfying \( \text{ord}_P \sigma > g \).

(3) \( l(gP) > 1 \).

(4) \( l(nP) > 1 \) for some \( n < g \).
Proof: The theorem is trivial for $g = 0$. Thus, we assume $g \geq 1$.

(1) $\Rightarrow$ (2): Let $\sigma_1, \ldots, \sigma_g$ be a basis for $H^0(X, \Omega)$ satisfying $0 \leq \text{ord}_P\sigma_1 < \ldots < \text{ord}_P\sigma_g$. Since $W(P) = \sum[\text{ord}_P\sigma_i - (i-1)]$, $W(P) > 0$ implies $\text{ord}_P\sigma_i > i-1$ for some $i$, and this in turn implies $\text{ord}_P\sigma_g \geq g$.

(2) $\Rightarrow$ (3): Suppose that $K = \sum n_i Q$ is a canonical divisor and that $\sigma \in H^0(X, \Omega)$ satisfies $\text{ord}_P\sigma \geq g$. Since $l(K) = \Omega$, there is an $f \in H^0(X, L(K))$ which when considered as a rational function satisfies $\text{ord}_P f \geq g - n_P$. This implies that $f \in H^0(X, L(K-gP))$. As a result, $l(K-gP) > 0$, and so the Riemann-Roch theorem implies $l(gP) > 1$.

(3) $\Rightarrow$ (4): This is clear.

(4) $\Rightarrow$ (1): Suppose $l(nP) > 1$ for some $n \leq g$. Then, by the Riemann-Roch theorem $l(K-nP) \geq g + 1 - n = d$. Let $f_1, \ldots, f_d$ be linearly independent elements of $H^0(X, L(K-nP))$ satisfying $\text{ord}_P f_1 < \ldots < \text{ord}_P f_d$. Since $f_i \in H^0(X, L(K-nP))$, $\text{ord}_P f_i \geq n-n_P$, and hence $\text{ord}_P f_d \geq g-n_P$. As above this implies there is a $\sigma \in H^0(X, \Omega)$ satisfying $\text{ord}_P \sigma \geq g$. Now let $\sigma_1, \ldots, \sigma_g$ be a basis for $H^0(X, \Omega)$ satisfying $\text{ord}_P \sigma_1 < \ldots < \text{ord}_P \sigma_g$. By remark 1.1, (4), $\text{ord}_P \sigma = \text{ord}_P \sigma_j$ for some $j$. Therefore, $W(P) = \sum[\text{ord}_P \sigma_i - (i-1)] \geq \text{ord}_P \sigma_j - (j-1) > 0$.

Definition: $P \in X$ is a Weierstrass point of $X$ if $P$ satisfies one and hence all of the conditions in theorem 1.2.
Remark 1.2: Note that by proposition 1.13, if \( g = 0 \) or \( g = 1 \), then \( X \) has no Weierstrass points. For \( g = 2 \), let \( P \) be a Weierstrass point of \( X \). By proposition 1.14, 1 is a gap for \( P \). Thus \( P \) has gap sequence 1, \( b \) where \( b \) must be greater than 2. On the other hand, \( b \leq 2(2) - 1 = 3 \). Thus, 1,3 is the gap sequence for \( P \), and so \( W(P) = (1-1) + (3-2) = 1 \). It follows that every nonsingular curve of genus 2 has six Weierstrass points, each of weight 1.

Definition: A curve \( X \) of genus greater than one is called hyperelliptic if there exists a morphism \( \phi : X \to \mathbb{P}' \) of degree two.

Proposition 1.15. Let \( X \) have genus \( g \) and suppose that there is a morphism of degree two \( \phi : X \to \mathbb{P}' \). Then there are exactly \( 2g + 2 \) points of \( X \), \( P_i \) for \( i = 1, \ldots, 2g + 2 \), such that \( \phi^{-1}(\phi(P_i)) = \{ P_i \} \).

Proof: This follows from the Riemann-Hurwitz formula (cf. [G-H, p,253]).

Definition: Let \( X \) have genus greater than one. We call a point \( P \) of \( X \) a hyperelliptic point if \( 1(2P) > 1 \).

Proposition 1.16. Let \( P \) be a hyperelliptic point of \( X \). Then \( W(P) = \frac{1}{2}g(g-1) \).

Proof: Since \( 1(2P) > 1 \), there is an \( f \in H^0(X, \mathcal{L}(2P)) \) satisfying \( 0 > \text{ord}_P f \geq -2 \). Since \( X \not\cong \mathbb{P}^1 \), it follows that \( \text{ord}_P f \neq -1 \), and so \( \text{ord}_P f = -2 \). Therefore, by
proposition 1.10, 2 is a nongap for \( P \) and hence all even positive integers are nongaps for \( P \). As a result, 
1, 3, ..., 2g-1 is the gap sequence for \( P \). Thus, 
\[ W(P) = \sum ((2i-1) - (i-1)) = \frac{1}{2}g(g-1). \]

**Proposition 1.17.** Let \( X \) have genus \( g \) greater than one. Then the following statements are equivalent.

(1) \( X \) is hyperelliptic.

(2) \( X \) has exactly \( 2g+2 \) hyperelliptic points.

(3) \( X \) has at least one hyperelliptic point.

**Proof:** (1) \( \Rightarrow \) (2): Suppose \( \phi : X \to \mathbb{P}^1 \) is a morphism of degree 2. By proposition 1.15 there is a set of \( 2g+2 \) points \( P_1, \ldots, P_{2g+2} \) satisfying \( \phi^{-1}(\phi(P_i)) = \{ P_i \} \), 
\( i = 1, \ldots, 2g+2 \). Fix an index \( i \). By performing a projective change of coordinates on \( \mathbb{P}^1 \) if necessary, we may assume \( \phi(P_i) = \infty \), put \( \theta = \phi|_{X-P_i} \). Then we may think of \( \theta \) as a rational function, and as such \( \theta \in \Gamma(X-P_i, \mathcal{O}_X) \) and \( \text{ord}_P \theta = -2 \). Thus \( \lambda(2P_i) > 1 \). It follows that \( X \) has at least \( 2g+2 \) hyperelliptic points \( P_1, \ldots, P_{2g+2} \). On the other hand, by propositions 1.16 and 1.13, (2),
\[ \sum_{i=1}^{2g+2} W(P_i) = (g+1)g(g-1) = \sum_{P \in X} W(P). \]

Thus by proposition 1.16, \( P_1, \ldots, P_{2g+2} \) are all of the hyperelliptic points of \( X \).

(2) \( \Rightarrow \) (3): This is clear.
(3) \( (l) \): Let \( P \in X \) satisfy \( l(2P) > 1 \). It follows that there is an \( f \in \Gamma(X-P, O_X) \) satisfying \( \text{ord}_P f = -2 \). So \( f \) defines a morphism \( \phi : X-P \rightarrow \mathbb{C} \) of degree 2. Since \( X \) is nonsingular, \( \phi \) can be extended to a morphism of degree 2, \( \phi : X \rightarrow \mathbb{P}^1 \).

**Corollary 1.1:** Suppose \( X \) is hyperelliptic and \( \phi : X \rightarrow \mathbb{P}^1 \) is a morphism of degree 2. Then \( \phi^{-1}(\phi(P)) = \{P\} \) if and only if \( P \) is a hyperelliptic point.

**Proof:** The corollary follows immediately from the proof of proposition 1.17.

**Corollary 1.2:** Let \( X \) be hyperelliptic. Then \( P \) is a Weierstrass point of \( X \) if and only if \( P \) is a hyperelliptic point of \( X \).

**Proof:** The corollary follows from proposition 1.16, 1.17, and

\[
\sum_{P \in X} W(P) = (g+1)g(g-1).
\]

**Remarks 1.3:** By remark 1.2, all curves of genus 2 are hyperelliptic. For all \( g \geq 3 \) there exist both hyperelliptic and nonhyperelliptic curves of genus \( g \). (For details see [G-H, pp. 253-259]).

**Definition:** Suppose that \( K \) is a canonical divisor. A divisor \( D \) is called special if \( l(K-D) > 0 \).

**Theorem 1.1.** (Clifford) Let \( D \) be an effective special divisor on the curve \( X \). Then
\[ l(D) \leq \frac{1}{2} \deg(D) + 1. \]

Furthermore equality occurs only if \( D = 0, \ D = K, \) or \( X \) is hyperelliptic.

Proof: [H, IV, 5.4]

Corollary 1.3. Let \( X \) be a nonhyperelliptic curve of genus \( g \) greater than one. Then for all \( P \in X, \)

\[ W(P) \leq \frac{1}{2} (g-1)(g-2) + 1. \]

Proof: Let \( m_1, \ldots, m_g \) be the gap sequence for \( P \). By proposition 1.14, \( m_1 - 1 = 0 \). Further, \( m_g \leq 2g - 1 \), and so

\[ m_{g-g} \leq g-1 = (g-2) + 1. \]

Now for all \( 0 < n < m_g \), \( l(K-nP) \geq l(K-(m_g-1)P) \). Since \( m_g \) a gap implies \( l(K-(m_g-1)P) > 0 \), it follows that \( nP \) is an effective special divisor for \( 0 < n < m_g \).

Assume that for some \( i \) with \( 1 < i < g \), \( m_i > 2i-2 \).

Then there are at most \( i-1 \) gaps less than \( 2i-2 \), and so

\[ l((2i-2)P) \geq 1 + ((2i-2) - (i-1)) = i. \]

On the other hand, \( 2i-2 \leq m_i < m_g \), and so by Clifford's theorem

\[ l((2i-2)P) < i. \]

From the above contradiction, we conclude \( m_i \leq 2i-2 \) for \( 2 \leq i < g \). Thus

\[ W(P) = \sum_{i=1}^{g} (m_i-i) \leq \sum_{i=2}^{g} ((2i-2)-i) + (g-2) + 1 = \sum_{i=1}^{g} (i-2) + 1 = \frac{1}{2} (g-1)(g-2) + 1. \]
Corollary 1.4. Let $X$ have genus greater than one. Then $X$ is hyperelliptic if and only if for some $P \in X$,
\[ W(P) = \frac{1}{2}g(g-1). \]
Proof: The corollary follows from proposition 1.16 and corollary 1.3.

Example 1.1. Thinking of $\mathbb{P}^2$ as $\text{Proj } \mathbb{C}[x,y,z]$, let $X$ be the nonsingular plane curve of genus 3 defined by the equation $x^4 = y^4 + z^4$. Let $P$ be the point of $X$ with homogeneous coordinates $[i,0,1]$. Then $P$ has gap sequence $1,2,5$ (for details see [F-K, VII.3.8]). Thus, $W(P) = 2$, which shows that the upper bound for $W(P)$ obtained in corollary 1.3 is as small as possible.

The final result of this chapter shows the relation between Weierstrass points of $X$ and morphisms from $X$ to $\mathbb{P}^1$. We shall see that this result does not generalize to Gorenstein curves.

Theorem 1.4. Let $X$ have genus $g$ at least 2. The following statements are equivalent.

(1) There is a morphism $\phi: X \to \mathbb{P}^1$ of degree at most $g$ such that $\phi^{-1}(\phi(P)) = \{P\}$.

(2) $P$ is a Weierstrass point of $X$.
Proof: (1) $\Rightarrow$ (2): Set $\phi|_{X-P} = \overline{\phi}$. Since $\overline{\phi}(X-P) = \mathbb{P}^1 - \phi(P)$ and since by performing a projective change of coordinates on $\mathbb{P}^1$ if necessary, we may assume that $\phi(P) = \infty$, we may think of $\overline{\phi}$ as an element of $\Gamma(X-P, \mathcal{O}_X)$.
with $\text{ord}_P \overline{\phi} = -\deg \phi \geq -g$. Therefore $\overline{\phi}$ is a nonconstant element of $H^0(X, L(gP))$ and so $l(gP) > 1$. As a result, $P$ is a Weierstrass point.

(2) $\Rightarrow$ (1): Since $P$ is a Weierstrass point, there is an $f \in \Gamma(X-P, \mathcal{O}_X)$ satisfying $0 > \text{ord}_P f \geq -g$. Put $\text{ord}_P f = -n$. Then $f$ defines a morphism $\phi: X \to \mathbb{P}^1$ of degree equal to $n$, and $n \leq g$. 
Chapter 2. Singular Curves

Section 1. Basic Properties

In this chapter $X$ is an integral projective curve over $\mathbb{C}$. $X_{\text{sing}}$ is the subset of singular points of $X$, and $X_{\text{reg}}$ is the subset of nonsingular points of $X$, where, as in chapter 1, point means closed point. For $P \in X$, $\mathcal{O}_P$ is the local ring at $P$, and $m_P$ is the maximal ideal of $\mathcal{O}_P$. For $P \in X_{\text{sing}}$, $\tilde{\mathcal{O}}_P$ is the integral closure of $\mathcal{O}_P$, and $c_P$ is the conductor of $\tilde{\mathcal{O}}_P$ in $\mathcal{O}_P$. Finally, $\pi: \tilde{X} \to X$ denotes the normalization of $X$.

Remarks 2.1. The following particulars about singular curves can be found in [S_1, Chapter 4].

1. $X_{\text{sing}}$ is a finite set.
2. Suppose that $K(X)$ is the field of rational functions on $X$. Then $\pi$ induces an isomorphism $K(\tilde{X}) \cong K(X)$.
3. The induced map of sheaves $\mathcal{O}_X \to \pi_*\tilde{\mathcal{O}}_X$ is an injection.
4. Suppose that $P \in X_{\text{sing}}$. Then, considering all rings below as subsets of $K(X)$,

$$\tilde{\mathcal{O}}_P = \bigcap_{R \in \pi^{-1}(P)} \mathcal{O}_Q.$$ 

5. $f \in m_P$ implies that $f \in m_R$ for all $R \in \pi^{-1}(P)$. In particular, $f \in m_P$ implies that $\text{ord}_R f > 0$ for all $R \in \pi^{-1}(P)$.

Definition: The arithmetic genus $g$ of $X$ is defined by $g = \dim H^1(X, \mathcal{O}_X)$. 

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Theorem 2.1. (Serre duality) $X$ has a dualizing sheaf; that is, there is a coherent sheaf $\omega_X = \omega$ over $X$ such that for all coherent sheaves $F$ over $X$,

$$H^1(X,F) = \text{Hom}(F,\omega_X)'$$

(If $V$ is a complex vector space, we denote its dual by $V'$.)

Proof: The proof of the theorem can be found in [H,III,7].

The following construction of $\omega_X$ comes from [S,IV, section 3]. For $P \in X$, we define $\omega_P$ as follows.

$$\omega_P = \{ \sigma \in D_X^* : \sum_{Q \in \pi^{-1}(P)} \text{res}_Q \sigma f = 0 \text{ for all } f \in \mathcal{O}_P \}$$

In particular, for $P \in X_{\text{reg}}'$

$$\omega_{X,P} = \Omega_{X,P} = \Omega_{X,\pi^{-1}(P)}$$

Corollary 2.1. $\dim \mathcal{C} H^1(X,\mathcal{O}_X) = \dim \mathcal{C} H^0(X,\omega_X)$.\n
Proof: $H^1(X,\mathcal{O}_X) = \text{Hom}(\mathcal{O}_X,\omega_X)'$ by Serre duality, and $\text{Hom}(\mathcal{O}_X,\omega_X) = H^0(X,\omega_X)$.

Definition: Let $D$ be a noetherian local domain with integral closure $\tilde{D}$, and let $c$ be the conductor of $\tilde{D}$ in $D$. Set $\dim \mathcal{C} (\tilde{D}/D) = \delta_D$, and $\dim \mathcal{C} (\tilde{D}/c) = n_D$. One calls $D$ a Gorenstein local ring if $n_D = 2\delta_D$. In particular, if $D$ is integrally closed, then $D$ is Gorenstein.

For $P \in X$, we write $n_P = n_{\mathcal{O}_P}$, $\delta_P = \delta_{\mathcal{O}_P}$, and $\delta_X = \delta = \sum_{P \in X} \delta_P$. We note that $\delta_P = 0$ for all $P \in X_{\text{reg}}'$, and that $\delta_P$ is finite for all $P \in X_{\text{sing}}$. By remark 2.1,(1), $\delta_X$ is also finite.
Definition: X is called a Gorenstein curve if \( \mathcal{O}_p \) is a Gorenstein local ring for all \( p \in X \). In particular, non-singular curves are Gorenstein.

Proposition 2.1. Let \( X \) have arithmetic genus \( g \). Then

\[
g = \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X) + \delta_X.
\]

Proof: [H, Ex. 1.8, p. 298]

Corollary 2.2. Let \( X \) be a singular curve. Then the arithmetic genus of \( X \) is positive. Consequently \( \mathbb{P}^1 \) is the only curve of arithmetic genus 0.

Proof: This is immediate from proposition 2.1.

Proposition 2.2. For all \( p \in \text{Sing} \), \( \delta_p + 1 \leq n_p \leq 2\delta_p \).

Furthermore, \( n_p = 2\delta_p \) if and only if \( \omega_p \) is a free rank one \( \mathcal{O}_p \)-module.

Proof: [S, p. 80]

Corollary 2.3. \( X \) is Gorenstein if and only if \( \omega \) is an invertible sheaf.

Proof: This is immediate from proposition 2.2.

Corollary 2.4. Suppose that \( X \) is a singular curve such that for all \( p \in \text{Sing} \), \( \delta_p = 1 \). Then \( X \) is Gorenstein.

Proof: Since \( \delta_p = 1 \) and \( \delta_p + 1 \leq n_p \leq 2\delta_p \), the corollary follows.

Proposition 2.3. If \( X \) is a complete intersection in \( \mathbb{P}^n \), some \( n \), or if \( X \) can be imbedded in a smooth surface, then \( X \) is Gorenstein. In particular, plane curves are Gorenstein.
Proposition 2.4. We think of the sets below as subsets of $D_X$. Then for $P \in X$, $(\pi_* \Omega_X^-)_P \subseteq \omega_X, P$.

Proof: $(\pi_* \Omega_X^-)_P = \sum_{R \in \pi^{-1}(P)} \Omega_{X, R}$.

Thus, given a nonzero $\sigma \in (\pi_* \Omega_X^-)_P$, $\text{ord}_R \sigma \geq 0$ for all $R \in \pi^{-1}(P)$. Since $0_P \subseteq 0_R$ for all $R \in \pi^{-1}(P)$, $\text{ord}_R f \sigma \geq 0$ for all nonzero $f \in 0_P$. As a result, $\text{res}_R f \sigma = 0$ for all $f \in 0_P$, and so for all $f \in 0_P$, $\sum_{R \in \pi^{-1}(P)} \text{res}_R \sigma = 0$.

Thus, $\sigma \in \omega_P$, from which it follows that $(\pi_* \Omega_X^-)_P \subseteq \omega_P$.

Proposition 2.5. Suppose that for $P \in X_{\text{sing}}$, $\partial_P$ is Gorenstein. Then $C_P$ is the annihilator of $\omega_P / (\pi_* \Omega_X^-)_P$.

Proof: $[S_1, p. 80]$

Corollary 2.5. Let $P$ and $\omega_P$ be as above. Let $\sigma$ generate $\omega_P$ over $\partial_P$, and let $h$ generate $C_P$ in $\partial_P$. Then for all $R \in \pi^{-1}(P)$,

$$\text{ord}_R \sigma = -\text{ord}_R h.$$

Proof: By proposition 2.5, $h \sigma \in (\pi_* \Omega_X^-)_P$, and so $h \sigma \in \Omega_{X, R}$ for all $R \in \pi^{-1}(P)$. Thus, $\text{ord}_R h \geq -\text{ord}_R \sigma$ for all $R \in \pi^{-1}(P)$. Conversely, let $\tau \in D_X$ be such that $\tau$ generates $\Omega_{X, R}$ for all $R \in \pi^{-1}(P)$. (Let $u \in K(X)$ satisfy $\text{ord}_R u = 1$ for all $R \in \pi^{-1}(P)$. Then we may take $\tau = du$.) By definition, $\tau \in (\pi_* \Omega_X^-)_P$. Let $\tau = b \sigma$. Then $-\text{ord}_R b = \text{ord}_R \sigma$ for all $R \in \pi^{-1}(P)$. But, by proposition
2.5, $b \in c_P$, and so $\text{ord}_R b \geq \text{ord}_R h$ for all $R \in \pi^{-1}(P)$. The corollary now follows.

**Remark 2.2.** Let $P, \tau$, and $\sigma$ be as above. Write $\tau/h = u\sigma$. By proposition 2.5, $u$ is a unit in $\hat{\mathcal{O}}_P$. Thus, $\tau/h$ is in $\omega_P$ and $\tau/h$ is a generator of $\omega_P$ if and only if $u$ is a unit in $\mathcal{O}_P$.

**Proposition 2.6.** Given $P \in \text{X}_{\text{sing}}$ and a nonzero $f$ in $\mathcal{O}_P$,

$$\dim_\mathcal{O} \mathcal{O}_P/f\mathcal{O}_P = \dim_\mathcal{O} \mathcal{O}_P/f\mathcal{O}_P.$$

**Proof:** This result follows from the isomorphism

$$\mathcal{O}_P/\mathcal{O}_P \cong f\mathcal{O}_P/\mathcal{O}_P,$$

and the two exact sequences of $\mathcal{O}_P$-modules,

$$0 \to \mathcal{O}_P/f\mathcal{O}_P + \mathcal{O}_P/f\mathcal{O}_P + \mathcal{O}_P/f\mathcal{O}_P + \mathcal{O}_P/f\mathcal{O}_P + 0$$

**Definition:** For a nonzero $f \in \mathcal{O}_P$, define

$$\text{ord}_P f = \dim_\mathcal{O} \mathcal{O}_P/f\mathcal{O}_P.$$

**Definition:** Let $L$ be an invertible sheaf over $X$, and let $\sigma$ be a nonzero element of $\mathcal{H}^0(X, L)$. For $P \in X$, we define $\text{ord}_P \sigma$ as follows. Let $\tau$ generate $L_P$ over $\mathcal{O}_P$. Then $\sigma = f\tau$ for some $f \in \mathcal{O}_P$. We define

$$\text{ord}_P \sigma = \text{ord}_P f.$$

Note that the definition is independent of the choice of generator.

**Remark 2.3.**

(1) It follows from proposition 2.6 that
\[ \text{ord}_P f = \dim_{\mathcal{O}_P} \tilde{\mathcal{O}}_P/f\tilde{\mathcal{O}}_P = \sum_{R \in \pi^{-1}(P)} (\text{ord}_R f). \]

(2) It follows from (1) that if \( X \) is nonsingular, then the new and old definitions of \( \text{ord}_P f \) agree.

(3) For \( f, h \in \mathcal{O}_P \), \( \text{ord}_P fh = \text{ord}_P f + \text{ord}_P h \).

(4) It follows from (1) that
\[ \text{ord}_P g = \sum_{R \in \pi^{-1}(P)} (\text{ord}_R g). \]

(5) It follows from (4) that if \( X \) is nonsingular, then the new and old definitions of \( \text{ord}_P g \) agree.

**Proposition 2.7.** Let \( I \) be a proper \( \mathcal{O}_P \)-ideal satisfying \( \dim_{\mathcal{O}_P} \mathcal{O}_P/I = n \). Then for all integers \( m \) such that \( 0 < m \leq n \), there is an \( \mathcal{O}_P \)-ideal \( J \) such that \( J \not\subset I \) and \( \dim_{\mathcal{O}_P} (\mathcal{O}_P/J) = m \).

**Proof:** It suffices to demonstrate the proposition for \( m = n-1 \). Let \( J \) be an ideal satisfying
\[ J \not\subset I \] and \( \dim_{\mathcal{O}_P} (\mathcal{O}_P/J) \) is maximal.

Let \( f \in J/I \). Then (1) implies \( J = I + (f) \). We claim that for all \( h \in m_P \), \( hf \in I \). If not, then \( hf \not\in I \) and (1) together imply that \( J = I +(hf) \). Since \( f \in J \), there is an \( s \in \mathcal{O}_P \) such that \( f-shf \in I \). Thus \( f \not\in I \) implies \( 1-sh \) is not a unit of \( \mathcal{O}_P \). Since \( \mathcal{O}_P \) is a local ring, this in turn implies \( sh \) and hence \( h \) is a unit of \( \mathcal{O}_P \). The last statement contradicts the fact that \( h \in m_P \).

Now let \( k \in J \). Then there is a \( q \in \mathcal{O}_P \) such that \( k-qf \in I \). On the other hand, \( q \in \mathcal{O}_P \) implies that \( q = b + q_1 \)
for some \( b \in \mathcal{C} \) and \( q_1 \in m_P \). The claim implies that \( q_1 f \in I \), and so \( k-bf \in I \) for some \( b \in \mathcal{C} \). This implies \( \dim_{\mathcal{C}}(J/I) = 1 \), so \( \dim_{\mathcal{C}}(O_P/J) = n-1 \).

**Remarks 2.4.** Suppose \( P \in X_{\text{sing}} \) satisfies \( \delta_P = 1 \). It follows from corollary 2.1 that \( \dim_{\mathcal{C}}(O_P/c_P) = 1 \). As a result, \( m_P = c_P \) and \( O_P = \mathcal{C} + c_P \).

Further, if \( h \) generates \( c_P \) in \( O_P \), then by remark 2.1, (5), \( \text{ord}_Qh > 0 \) for all \( Q \in \pi^{-1}(P) \). Since
\[
\sum_{Q \in \pi^{-1}(P)} \text{ord}_Q h = \text{ord}_h = \dim_{\mathcal{C}}(O_P/c_P) = 2,
\]
it follows that there are only two possibilities for \( P \).

1. \( \pi^{-1}(P) = \{Q\} \) and \( \text{ord}_Q h = 2 \).
2. \( \pi^{-1}(P) = \{Q_1, Q_2\} \) and \( \text{ord}_{Q_1} h = \text{ord}_{Q_2} h = 1 \).

We note that if \( h \in O_P \) satisfies \( \text{ord}_h = 2 \), then \( h \) generates \( c_P \) in \( O_P \).

**Definition:** We call \( P \in X_{\text{sing}} \) cuspidal if \( \pi^{-1}(P) \) is a one point set. If, in addition \( \delta_P = 1 \), then we call \( P \) a cusp.

**Definition:** We call \( P \in X_{\text{sing}} \) a node if \( \delta_P = 1 \), and \( \pi^{-1}(P) \) is a two point set.

**Proposition 2.8.** Suppose \( P \in X_{\text{sing}} \) satisfies \( \delta_P = 1 \) and suppose \( \tau \in D_X^- \) generates \( \Omega_{X,Q}^- \) for all \( Q \in \pi^{-1}(P) \). Then \( \sigma \in D_X^- \) is a generator of \( \omega_P \) if and only if
\[
\sum_{Q \in \pi^{-1}(P)} \text{res}_Q \sigma = 0,
\]
and \( \sigma = \tau/h \) where \( h \) generates \( c_P \) in \( O_P \).
Proof: If \( \sigma \) generates \( \omega_P \), then \( \sigma \in \omega_P \) and so 
\[
\sum_{Q \in \pi^{-1}(P)} \text{res}_Q \sigma = 0.
\]
Further, \( \sigma = \tau/h \) where \( h \) generates \( c_P \) in \( \tilde{\mathcal{O}}_P \) by remark 2.2.

Conversely, let \( \sigma \in \mathcal{D}_X \) satisfy 
\[
\sum_{Q \in \pi^{-1}(P)} \text{res}_Q \sigma = 0,
\]
and \( \sigma = \tau/h \) where \( h \) generates \( c_P \) in \( \tilde{\mathcal{O}}_P \). Pick \( f \in \mathcal{O}_P \). Then by remark 2.4, \( f = b + s \) where \( b \in \mathcal{O} \) and \( s \in c_P \). Thus,
\[
\sum_{Q \in \pi^{-1}(P)} \text{res}_Q f \sigma = b \sum_{Q \in \pi^{-1}(P)} \text{res}_Q \sigma + \sum_{Q \in \pi^{-1}(P)} \text{res}_Q s \sigma.
\]
Therefore, to show that \( \sigma \in \omega_P \), it suffices to show 
\[
\text{res}_Q s \sigma = 0 \quad \text{for all} \quad Q \in \pi^{-1}(P).
\]
Since \( s \in c_P \), \( (s/h) \in \tilde{\mathcal{O}}_Q \) for all \( Q \in \pi^{-1}(P) \). It follows that \( s \sigma = (s/h) \tau \in \tilde{\Omega}_{X, Q} \), and so \( \text{res}_Q s \sigma = 0 \).

To see that \( \sigma \) generates \( \omega_P \), we note that
\[
c_P \sigma = h \tilde{\mathcal{O}}_P \sigma = h \tilde{\mathcal{O}}_P (\tau/h) = \tilde{\mathcal{O}}_P \tau = (\pi \times \tilde{\Omega}_X)_P
\]
Suppose that \( \gamma \) generates \( \omega_P \) and that \( \sigma = k \gamma \) for \( k \in \mathcal{O}_P \). Then by proposition 2.5,
\[
c_P \gamma = c_P \sigma = c_P k \gamma.
\]
Thus \( k \) is a unit in \( \mathcal{O}_P \), and so \( \sigma \) generates \( \omega_P \).

Corollary 2.6.

(1) If \( P \) is a cusp with \( \pi^{-1}(P) = \{Q\} \), if \( \alpha \) generates \( \tilde{\Omega}_{X, Q} \) and if \( f \in \mathcal{K}(X) \) satisfies \( \text{ord}_Q f = 2 \), then \( \alpha/f \) generates \( \omega_P \).

(2) If \( P \) is a node with \( \pi^{-1}(P) = \{Q_1, Q_2\} \), if \( \alpha \) generates
for $i = 1, 2$, and if $f$ is a rational function satisfying $\text{ord}_{Q_i} f = 1$ for $i = 1, 2$, then $\alpha/f$ generates $\omega_p$.

Proof: The corollary follows immediately from the proposition.

The following is a standard result about valuations.

Proposition 2.9. Suppose that $P_1, \ldots, P_m$ are points of a nonsingular curve $X$, and that $n_1, \ldots, n_m$ are arbitrary integers. Then there exists $f \in K(X)$ which satisfies $\text{ord}_{P_i} f = n_i$ for $i = 1, \ldots, m$.

Section 2. The Wronskian

For the rest of this chapter $X$ is an integral projective Gorenstein curve of arithmetic genus $g$. Since $g = 0$ implies $X \cong \mathbb{P}^1$ by corollary 2.2, we assume that $g$ is positive. We now establish that the wronskian of $X$ can be constructed in a manner analogous to the nonsingular construction.

Notation: Put $M = l + \ldots + g - 1$, and $N = M + g$.

Definition: Let $\sigma, \sigma_1, \ldots, \sigma_g$ be nonzero rational differentials on $X$. We define the wronskian of $\sigma_1, \ldots, \sigma_g$ with respect to $\sigma$, denoted $W_\sigma(\sigma_1, \ldots, \sigma_g)$, by

$$W_\sigma(\sigma_1, \ldots, \sigma_g) = \det[A_{ij}]$$

for $1 \leq i, j \leq g$,

where the $A_{ij}$ are rational functions defined by

$$\sigma_j = A_{1j} \sigma$$

for $j = 1, \ldots, g$. 

and for \( i > 1 \), the \( A_{i,j} \) are defined recursively by the formula

\[
dA_{i-1,j} = A_{i,j}^\sigma \quad \text{for} \quad i = 1, \ldots, g.
\]

We note that \( W_{\sigma}(\sigma_1, \ldots, \sigma_g) \in K(X) \).

**Proposition 2.10.** Suppose that \( t \) is a nonconstant rational function, and that \( \sigma, \sigma_1, \ldots, \sigma_g \) are nonzero rational differentials. Put \( \sigma = dt/k \), and for \( i = 1, \ldots, g \),

\[
\sigma_i = f_i \sigma \quad \text{where} \quad k, f_1, \ldots, f_g \in K(X).
\]

1. \( W_{\sigma}(\sigma_1, \ldots, \sigma_g) = k^M W_t(f_1, \ldots, f_g) \).

Assume now that \( \sigma_1, \ldots, \sigma_g \) are linearly independent over \( \mathfrak{C} \). Then

2. \( W_{\sigma}(\sigma_1, \ldots, \sigma_g) \neq 0 \).

Now suppose that \( \tau_1, \ldots, \tau_g \) are rational differentials which span the same vector space as \( \sigma_1, \ldots, \sigma_g \) and that

\[
\tau_i = \sum b_{ij} \sigma_j \quad \text{for} \quad i = 1, \ldots, g \quad \text{and} \quad b_{ij} \in \mathfrak{C}.
\]

Then

3. \( W_{\sigma}(\tau_1, \ldots, \tau_g) = \det(b_{ij})W_{\sigma}(\sigma_1, \ldots, \sigma_g) \).

Finally, suppose that \( \tau \) is a nonzero rational differential with \( \tau = s\sigma \) for \( s \in K(x) \). Then

4. \( W_{\sigma}(\sigma_1, \ldots, \sigma_g) = s^N W_{\tau}(\tau_1, \ldots, \tau_g) \).

**Proof:**

1. Let \( A_{i,j}, \ 1 \leq i, j \leq g \), be as in the definition of

\[
W_{\sigma}(\sigma_1, \ldots, \sigma_g).
\]

Note that

\[
k^M W_t(f_1, \ldots, f_g) = \det(k^{i-1} f_j(i-1)(t)).
\]

Thus, to establish (1), note that the following is easily established by induction on \( i \).
where the $b_{im}$ are rational functions independent of $j$. It follows from basic properties of determinants that

$$W_{\sigma_1, \ldots, \sigma_g} = \det(k_{i j}^{-1} f_i^{-1})(t) = k_{W_t}^{M_g}(f_1, \ldots, f_g).$$

(2) This follows from (1) and proposition 1.12, (1).
(3) This follows from (2) and proposition 1.12, (2).
(4) For $i=1, \ldots, g$, put $\sigma_i = h_i^{\tau}$, $h_i \in K(X)$. Note that $\tau = s \sigma = (s/k)dt$, and that

$$W_t(sh_1, \ldots, sh_g) = s^g W_t(f_1, \ldots, f_g).$$

Therefore,

$$W_{\sigma_1, \ldots, \sigma_g} = \det(k_{i j}^{-1} f_i^{-1})(t) = k_{W_t}^{M_g}(h_1, \ldots, h_g) = s^g(k/s)^{W_t}(h_1, \ldots, h_g) = s^N k_{W_t}^{M_g}(h_1, \ldots, h_g) = s^N W_t(\sigma_1, \ldots, \sigma_g).$$

q.e.d.

Let $\sigma_1, \ldots, \sigma_g$ be a fixed basis for $H^0(X, \omega)$. Put $F = \omega^{\otimes N}$. We define $\alpha \in H^0(X, F)$ as follows. Suppose that \{U_i; i \in I\} is an open cover of $X$ such that for all $i \in I$, $\Gamma(U_i, \omega)$ is a free rank one $\Gamma(U_i, \partial_X)$-module with generator $\tau_i$. For all $i, j \in I$ write $\tau_j = h_{ij}^{\tau_i}$ (here we think of $\tau_i$ and $\tau_j$ as elements of $\Gamma(U_i \cap U_j, \omega)$). Now for all $i \in I$ define $\alpha_i \in \Gamma(U_i, F)$ by the rule

$$\alpha_i = W_{\tau_i}(\sigma_1, \ldots, \sigma_g)^{\tau_i^{\otimes N}}.$$

Note that by proposition 2.10, (4), for all $i, j \in I$,

$$W_{\tau_i}(\sigma_1, \ldots, \sigma_g) = h_{ij}^{N \tau_j}(\sigma_1, \ldots, \sigma_g).$$
Therefore, since $h_{ij}$ is a unit in $\Gamma(U_i \cap U_j, \mathcal{O}_X)$, as an element of $\Gamma(U_i \cap U_j, F)$,

$$W_{\tau_i}(\sigma_1, \ldots, \sigma_g) \tau_i \otimes N = h_{ij} W_{\tau_j}(\sigma_1, \ldots, \sigma_g) \tau_j \otimes N = W_{\tau_j}(\sigma_1, \ldots, \sigma_g)(h_{ij} \tau_i) \otimes N (h_{ij} \text{ is a unit in} \Gamma(U_i \cap U_j, \mathcal{O}_X)).$$

So for all $i, j \in I$, $a_i |_{U_i \cap U_j} = a_j |_{U_i \cap U_j}$.

Hence the $\alpha_i$'s patch to give a section $\alpha \in H^0(X, F)$. We note the following.

1. For all $P \in X$, if $\sigma$ generates $\omega_P$, then $\alpha_P = W_\sigma(\sigma_1, \ldots, \sigma_g) \otimes N$.

2. By proposition 2.10, (3), changing the basis of $H^0(X, \omega)$ changes $\alpha$ by a nonzero constant multiple.

Definition: The wronskian of $X$ is defined to be the equivalence class of $\alpha$ in $\mathcal{P}(H^0(X, \omega \otimes N))$.

Definition: For $P \in X$ let $\sigma$ generate $\omega_P$, write $\alpha_P = f_{\sigma} \otimes N$, $f \in O_P$, where $\alpha$ represents the wronskian of $X$.

The Weierstrass weight of $P$, denoted $W(P)$, is defined by $W(P) = \text{ord}_P f$. Note that $W(P)$ is independent of the choice of $\alpha$ or $\sigma$.

Remarks 2.5.

1. $W(P) \geq 0$ for all $P \in X$.

2. For $P \in X_{\text{reg}}$, suppose that $t$ generates $m_P$, and suppose that for $i = 1, \ldots, g$, $\sigma = f_i dt$. Since $dt$ generates $\omega_P$, it follows from proposition 2.10, (1) that
\[ a_P = W_t(f_1, \ldots, f_g) dt. \]

In particular, if \( X \) is nonsingular, the new definitions of the wronskian and of \( W(P) \) agree with the classical definitions.

**Proposition 2.11.** For \( P \in X \) suppose that \( t \in K(X) \) satisfies \( \text{ord}_Q t = 1 \) for all \( Q \in \pi^{-1}(P) \), and that \( h \) generates \( C_P \) in \( \tilde{O}_P \). Let \( \sigma_1, \ldots, \sigma_g \) be a basis for \( H^0(X, \omega) \). For \( i = 1, \ldots, g \), write \( \sigma_i = f_i (dt/h) \). Then

\[
W(P) = \sum_{Q \in \pi^{-1}(P)} \text{ord}_Q \left( h^{\text{ord}_Q h} W_t(f_1, \ldots, f_g) \right).
\]

**Proof:** By remark 2.2, any generator of \( \omega_P \) has the form \( \sigma = dt/h' \) where \( h' \) generates \( C_P \) in \( \tilde{O}_P \). Thus, \( h' = vh \) where \( v \) is a unit in \( \tilde{O}_P \). Therefore, for \( i = 1, \ldots, g \), \( \sigma_i = (vf_i)(dt/h') \). As a result,

\[
W(P) = \text{ord}_P \omega_0 (\sigma_1, \ldots, \sigma_g) = \\
\text{ord}_P (h')^{\text{ord}_P h} W_t(vf_1, \ldots, vf_g) \text{ (by proposition 2.10.(1))} = \\
N \left( \sum_{Q \in \pi^{-1}(P)} \text{ord}_Q \omega_0 \right) + \sum_{Q \in \pi^{-1}(P)} \text{ord}_Q h^{\text{ord}_Q h} W_t(f_1, \ldots, f_g). 
\]

Thus, it suffices to show that \( \text{ord}_Q v = 0 \) for all \( Q \in \pi^{-1}(P) \).

Since \( v \) is a unit in \( \tilde{O}_P \), \( v \) is a unit in \( O_Q \) for all \( Q \in \pi^{-1}(P) \). Hence the result follows.

**Corollary 2.7.** For all \( P \in X \), \( W(P) \geq \delta_P g(g-1) \).

**Proof:** Since \( t, f_1, \ldots, f_g \in O_P \), \( W_t(f_1, \ldots, f_g) \in O_P \).

By remark 2.3,
Theorem 2.2. If $X$ has arithmetic genus at least two and $P \in X_{\text{sing}}$, then $W(P) > 0$.

Proof: The theorem follows immediately from corollary 2.7.

Example 2.1. Suppose that $X$ is a rational curve of arithmetic genus two such that $X_{\text{sing}}$ consists of 2 nodes $P_1$ and $P_2$. Let $\pi : \mathbb{P}^1 \to X$ be the normalization of $X$. Note that, by performing a projective change of coordinates on $\mathbb{P}^1$ if necessary, we may assume

$$\pi^{-1}(P_1) = \{0, \infty\} \text{ and } \pi^{-1}(P_2) = \{1, b\}, \text{ } b \in \mathbb{C} \setminus \{0, 1\}.$$

Using $\mathcal{C}(T)$ as the rational function field, write $\sigma = \frac{dT}{T}$, and $\tau = \frac{dT}{(T-1)(T-b)}$. Note that $\sigma = \frac{dT}{(T^2+1)}$. Thus, by corollary 2.6, (2), $\sigma$ generates $\omega_{P_1}$, and $\tau$ generates $\omega_{P_2}$. Since $\sigma \in \Omega_{\mathbb{P}^1, S}$ for all $S \in \mathbb{P}^1 \setminus \{0, \infty\}$ and $\tau \in \Omega_{\mathbb{P}^1, Q}$ for all $Q \in \mathbb{P}^1 \setminus \{1, b\}$, it follows that $\sigma$ and $\tau$ are elements of $H^0(X, \omega)$. Since they are linearly independent, they form a basis for $H^0(X, \omega)$.

$dT$ generates $\omega_R$ for all $R \in X_{\text{reg}}$. So,

$$W(R) = \text{ord}_P W_T(1/T, 1/(T-1)(T-b))$$

$$= \begin{vmatrix} \frac{1}{T} & \frac{1}{(T-1)(T-b)} \\ -\frac{1}{T^2} & \frac{b+1-2T}{(T-1)^2(T-b)^2} \end{vmatrix}$$
We conclude that for \( P \in X_{\text{reg}} \),
\[
W(R) = 1 \text{ if } R = \left( \pm \sqrt{b} \right),
\]
\[
W(R) = 0 \text{ otherwise},
\]
For \( P^1 \), \( \sigma \) generates \( \omega_{P_1} \) and \( \tau = \left( \frac{T}{T-1} \right) (T-b)\sigma \). By proposition 2.10, (1),
\[
W_0(\sigma, \tau) + TW_T(1, \frac{T}{T-1} (T-b)) \]
\[
= \begin{vmatrix} 1 & \frac{T}{T-1} (T-b) \\ 0 & \frac{(b-T^2)}{(T-1)^2 (T-b)^2} \end{vmatrix}
\]
\[
= T \frac{(b-T^2)}{(T-1)^2 (T-b)^2}.
\]
Thus,
\[
W(P_1) = \text{ord}_0 W_0(\sigma, \tau) + \text{ord}_0 W_0(\sigma, \tau) = \text{ord}_0 \left( \frac{T(b-T^2)}{(T-1)^2 (T-b)^2} \right) + \text{ord}_\infty \left( \frac{T(b-T^2)}{(T-1)^2 (T-b)^2} \right) = 1 + 1 = 2.
\]
In a similar manner one may show that \( W(P_2) = 2 \). Thus, for any rational curve \( X \) of arithmetic genus 2 with two nodes \( P_1, P_2 \), \( W(P_i) = 2 \) for \( i = 1, 2 \), and \( X \) has two nonsingular points with Weierstrass weight one.

**Proposition 2.12.** Let \( P \) be cuspidal. Then \( W(P) \geq \delta_p g (g-1) + g-1 \). In particular, if \( P \) is a cusp, then \( W(P) \geq g^2 - 1 \).

**Proof:** Let \( Q = \pi^{-1}(P) \), let \( t \) generate \( m_Q \) in \( O_Q \), and let \( h \) generate \( c_p \) in \( \hat{O}_P \). Suppose that \( \sigma_1, \ldots, \sigma_g \) is a basis for \( H^0(X, \omega) \). For \( i = 1, \ldots, g \) write \( \sigma_i = f_i dt/h \). By remark 1.1, (1) we may assume that
ord_{Q f_i} = n_i, and \( n_1 < \ldots < n_g \).

By proposition 2.11,
\[
W(P) = \text{ord}_{Q h} W_t (f_1, \ldots, f_g) = \delta_p g (g-1) + \text{ord}_{Q t} (f_1, \ldots, f_g).
\]

Therefore, it suffices to show that \( \text{ord}_{Q W_t} (f_1, \ldots, f_g) \geq g-1 \).

Since \( P \) is cuspidal, for all \( f \in m_P \), \( \text{ord}_P f > 1 \). Hence \( n_i > i \) for all \( i \geq 2 \). As a result, proposition 1.12, (3) implies
\[
\text{ord}_{Q W_t} (f_1, \ldots, f_g) = \Sigma (n_i - (i-1)) \geq g-1.
\]

**Definition:** For \( P \in X_{\text{sing}} \), the partial normalization at \( P \) is defined to be the birational map \( \theta : Y \to X \) such that,

1. \( \theta|\theta^{-1}(X-P) \) is an isomorphism onto \( X-P \),
2. \( \pi^{-1}(P) \subseteq Y_{\text{reg}} \).

**Remarks 2.6.**

1. We note that \( Y \) can be constructed as follows. Let \( U \) be a copy of \( X-P \), and let \( S \) be a copy of \( \pi^{-1}(P) \). We denote elements of \( U \) and \( S \) as follows. If \( R \in X-P \), then we denote the corresponding point of \( U \) by \( R' \), and if \( Q \in \pi^{-1}(P) \), then we denote the corresponding point of \( S \) by \( Q' \). Then
   
   (i) \( U \cup S \) is the set of closed points of \( Y \),
   
   (ii) \( \mathcal{O}_{R'} = \mathcal{O}_R \) for all \( R' \in U \),
   
   (iii) \( \mathcal{O}_{Q'} = \mathcal{O}_Q \) for all \( Q' \in S \).

2. \( Y_{\text{sing}} = \{ R' ; \ R \in X_{\text{sing-P}} \} \), \( \delta_{R'} = \delta_R \), and \( n_{R'} = n_R \). Thus, if \( X \) is a Gorenstein curve, then \( Y \) is a Gorenstein curve.
The following is a commutative diagram.

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{\theta} & X \\
Y & \xrightarrow{\pi_1} & \pi \downarrow & X \\
\end{array}
\]

The morphism \( \pi_1: X \rightarrow Y \) is the normalization of \( Y \).

**Proposition 2.13.** For \( P e X_{\operatorname{sing}} \), let \( \theta: Y \rightarrow X \) be the partial normalization at \( P \). Then \( H^0(Y, \omega_Y) \subseteq H^0(X, \omega_X) \).

**Proof:** Suppose that \( \sigma \in H^0(Y, \omega_Y) \) and \( A e X \). By the remarks above for all \( f e R_A \),

\[
\sum_{Q \in \pi^{-1}(A)} R_{\pi^{-1}(Q)} f_{\sigma} = \sum_{Q \in \pi^{-1}(A)} R_{\pi^{-1}(Q)} f_{\sigma}.
\]

Since \( \theta_{A} \leq \hat{O}_{A} \leq O_{Q} \) for all \( Q e \pi^{-1}(A) \), and since \( \sigma \in H^0(Y, \omega_Y) \), \( \Sigma_{Q} \operatorname{res}_{R} f_{\sigma} = 0 \) for all \( Q e \pi^{-1}(A) \).

Hence for all \( f e R_{A} \), \( \Sigma_{Q} \operatorname{res}_{R} f_{\sigma} = 0 \). Thus, \( \sigma \in \Omega_{A} \) for all \( A e X \), so \( \sigma e H^0(X, \omega_X) \).

**Proposition 2.14.** With \( Y \) as above, suppose that \( Y \) has arithmetic genus \( g' \). Then \( g' = g - \delta_{P} \).

**Proof:** By proposition 2.1,

\[
g' = \dim H^1(X, \omega_X) + \sum_{Q e Y_{\operatorname{sing}}} \delta_{Q}.
\]

But by remark 2.6, (2),

\[
\delta_{P} = \sum_{R e X_{\operatorname{sing}}} \delta_{R} - \sum_{Q e Y_{\operatorname{sing}}} \delta_{Q}
\]

Since
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\[ g = \dim H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) + \sum_{R \in X_{\text{sing}}} \varepsilon_R \]

It follows that \( g' = g - \varepsilon_p \).

**Proposition 2.15.** Let \( P \) be a node of \( X \) with \( \theta: Y \to X \) the partial normalization at \( P \). Put \( \theta^{-1}(P) = \{Q_1, Q_2\} \).

Then

\[ W(P) = g(g-1) + W(Q_1) + W(Q_2). \]

**Proof:** Let \( \pi_1: \tilde{X} \to Y \) be the normalization of \( Y \) with \( \pi_1^{-1}(Q_i) = \{R_i\} \) for \( i = 1, 2 \). Suppose that \( \sigma_1, \ldots, \sigma_{g-1} \) is a basis for \( H^0(Y, \omega_Y) \) and that \( \sigma \in H^0(X, \omega_X) \) is a generator of \( \omega_P \). By corollary 2.5, (2), we have \( \text{ord}_{R_i} \sigma = -1 \) for \( i = 1, 2 \). Thus \( \sigma \notin \Omega_{\tilde{X}, R_i} = \omega_{\tilde{Y}, Q_i} \) for \( i = 1, 2 \). Therefore \( \sigma \notin H^0(Y, \omega_Y) \), and so \( \sigma, \sigma_1, \ldots, \sigma_{g-1} \) is a basis for \( H^0(X, \omega_X) \).

For \( i = 1, 2 \), let \( t \in K(X) \) satisfy \( \text{ord}_{R_i} t = 1 \). Then \( dt \) is a generator of \( \omega_{Q_i} \). By corollary 2.6, (2), \( \sigma = dt/h \) where \( h \) generates \( \mathcal{O}_P \) in \( \tilde{\mathcal{O}}_P \). Put \( M = 1 + \ldots + g-1 \), and \( \sigma_i = f_i \sigma \), for \( i = 1, \ldots, g-1 \). By proposition 2.11,

\[ W(P) = \sum \text{ord}_{R_i} h^M \omega_t(1, f_1, \ldots, f_{g-1}) = g(g-1) + \text{ord}_{R_1} \omega_t(1, f_1, \ldots, f_{g-1}) + \text{ord}_{R_2} \omega_t(1, f_1, \ldots, f_{g-1}). \]

So it suffices to show that

\[ W(Q_i) = \text{ord}_{R_1} \omega_t(1, f_1, \ldots, f_{g-1}). \]

We may assume that for \( i = 1, \ldots, g-1 \) \( \text{ord}_{R_1} f_i = n_i \) with \( n_1 < \ldots < n_{g-1} \). For \( t = 1, \ldots, g-1 \), put \( h_i = f_i/h \).
Since $dt$ generates $\omega_{X,Q_1}$,

$$W(Q_1) = \text{ord}_{R_1} W_t(h_1, \ldots, h_{g-1}) =$$

$$\sum_{i=1}^{g-1} (\text{ord}_{R_1} f_i - \text{ord}_{R_1} h_{-i+1}) = \sum_{i=1}^{g-1} (n_i - 1 - (i-1)).$$

On the other hand, (recall that $df = f'dt$),

$$W_t(1, f_1', \ldots, f_{g-1}') = W_t(f_1', \ldots, f_{g-1})$$

Since for all $i \text{ord}_{R_1} f_i' = n_i - 1$,

$$\text{ord}_{R_1} W_t(1, f_1', \ldots, f_{g-1}') = \text{ord}_{R_1} W_t(f_1', \ldots, f_{g-1}') =$$

$$\sum_{i=1}^{g-1} (n_i - 1 - (i-1)) = W(Q_1).$$

**Proposition 2.16.** Suppose that $P$ is a cusp with partial normalization at $P \otimes : Y \to X$ and with $\theta^{-1}(P) = \{Q\}$.

If $X$ has arithmetic genus at least one then

$$W(P) = (g+1)(g-1) + W(Q).$$

**Proof:** Suppose that $\sigma_1 \in H^0(Y, \omega_Y)$ generates $\omega_{Y_1,Q}$, and that $\sigma_1', \ldots, \sigma_{g-1}$ is a basis for $H^0(Y, \omega_Y)$ satisfying

$$\text{ord}_{Q_1} \sigma_1 < \ldots < \text{ord}_{Q_1} \sigma_{g-1}.$$ For $i=1, \ldots, g-1$, put $\sigma_i = f_i \sigma_1$.

Now suppose that $\tau \in H^0(X, \omega_X)$ generates $\omega_{X,P}$ and put $\sigma_1 = \tau_1$. Let $\pi_1: \tilde{X} \to Y$ be the normalization of $Y$ with $\pi_1^{-1}(Q) = R$. By corollary 2.5, (1), $\text{ord}_R h = 2$, and so

$$\text{ord}_R \tau = -2.$$ Thus $\tau \not\in \Omega^1_{X,R} \omega_Y Q$. Therefore, $\sigma_1', \ldots, \sigma_{g-1}$, $\tau$ is a basis for $H^0(X, \omega_X)$. It follows that for $t$ a generator of $m_Q$. 

$$W(P) = g(g-1) + \text{ord}_R W_t (l, f_1 h, \ldots, f_{g-1} h) =$$
$$g(g-1) + \text{ord}_R h + \text{ord}_R (l/h, f_1, \ldots, f_{g-1}) =$$
$$g(g-1) + 2g - \text{ord}_R h + \Sigma (\text{ord}_R f_i - i) =$$
$$g(g-1) + 2(g-1) + \Sigma (\text{ord}_R f_i - (i-1)) - (g-1) = (g+1)(g-1) + W(Q).$$

Section 3: Subschemes

In this section $X$ is an integral projective Gorenstein curve of arithmetic genus $g$ with $\omega$ as sheaf of dualizing differentials. Let $Z$ be a proper closed subscheme of $X$. As a topological space, $Z$ consists of a finite number of closed points. The ideal sheaf $I$ of $Z$ is then defined by the exact sequence of $\mathcal{O}_X$-modules.

$$0 \to I \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

In fact the above exact sequence gives a one to one correspondence between closed subschemes $Z$ of $X$ and coherent sheaves of $\mathcal{O}_X$-ideals $I$. (for details see $[H, II.5.9]$). The pair $(Z, I)$ shall represent a subscheme $Z$ and its associated ideal sheaf $I$.

**Proposition 2.17.** Let $F$ be a subsheaf of the constant sheaf with constant value $K(X)$. If $F$ is of finite type, then $F$ is coherent.

**Proof:** $[S_2, p.232]$

**Proposition 2.18.** Any proper closed subscheme of $X$ is given by:

(1) A finite set of points $P_1, \ldots, P_d$ of $X$.  

(2) For $1 \leq j \leq d$, a proper ideal $I_j$ of $\mathcal{O}_{P_j}$.

Proof: We define the ideal sheaf $I$ as follows:

$\mathcal{O}_P = \mathcal{O}_P$ if $P \neq P_j$

$I_{P_j} = I_j$ for $1 \leq j \leq d$.

Clearly $I$ is an ideal sheaf. Put $V = X - \{P_1, \ldots, P_d\}$.

Then for $P \in V$, $l \in \Gamma(V, \mathcal{O})$ generates $\mathcal{O}_P$. For $1 \leq j \leq d$, let $I_j = (f_{j1}, \ldots, f_{jm_j})$, and let $U_j$ be an open neighborhood of $P_j$ satisfying,

(i) $P_i \notin U_j$ for $i \neq j$,

(ii) $f_{j1}, \ldots, f_{jm_j} \in \mathcal{O}_P$ for all $P \in U_j$,

(iii) $f_{j1}$ is a unit of $\mathcal{O}_P$ for all $P \in U_j - P_j$.

Then $f_{j1}, \ldots, f_{jm_j} \in \Gamma(U_j, \mathcal{O})$, and $f_{j1}, \ldots, f_{jm_j}$ generate $\mathcal{O}_P$ for all $P \in U_j$. It follows that $I$ is of finite type, and so $I$ is coherent by proposition 2.17. Therefore $I$ defines a proper closed subscheme $Z$ of $X$.

Definition: Suppose that $(Z, I)$ is a proper closed subscheme of $X$. The degree of $Z$, denoted $d(Z)$, is defined by

$$d(Z) = \sum_{P \in Z} \dim(\mathcal{O}_P/I_P).$$

Since $Z$ has finite point set, $d(Z)$ is finite.

Remark 2.7. If $X$ is a nonsingular curve, then $I$ is an invertible sheaf. In fact, $I \cong L(-D)$ for some effective divisor $D$ on $X$, and $d(Z) = \deg D$. 
We now establish a generalization of the Riemann-Roch theorem for proper closed subschemes of \( X \). Given a proper closed subscheme \((Z, I)\) of \( X \), there exists an exact sequence of \( \mathcal{O}_X \)-modules

\[
0 \to I \to \mathcal{O}_X \to \mathcal{O}_X/I \to 0
\]

Since \( \omega \) is an invertible sheaf, tensoring the above exact sequence with \( \omega \) yields the exact sequence

\[
0 \to \mathcal{O}_\omega \to \omega \to (\mathcal{O}_X/I) \mathcal{O}_\omega \to 0
\]

From the above exact sequence we get the long exact cohomology sequence

\[
(*) \quad 0 \to H^0(X, \mathcal{O}_\omega) \to H^0(X, \omega) \to H^0(X, (\mathcal{O}_X/I) \mathcal{O}_\omega) \to H^1(X, \mathcal{O}_\omega) \to H^1(X, \omega) \to H^1(X, (\mathcal{O}_X/I) \mathcal{O}_\omega).
\]

We note the following about (*)..

(1) \( \mathcal{O}_X/I \) is a sheaf with finite support. Since \( \omega \) is invertible, it follows that \((\mathcal{O}_X/I) \mathcal{O}_\omega \simeq \mathcal{O}_X/I\). Thus, by proposition 1.7,

\[
H^0(X, (\mathcal{O}_X/I) \mathcal{O}_\omega) = \mathcal{C}^d(Z),
\]

\[
H^1(X, (\mathcal{O}_X/I) \mathcal{O}_\omega) = 0.
\]

(2) \( \dim H^0(X, \omega) = g \), and \( H^1(X, \omega) = \mathcal{C} \).

(3) By Serre duality, \( H^1(X, \mathcal{O}_\omega) \) is dual to \( \text{Hom}(I \mathcal{O}_\omega, \omega) \).

Since \( \omega \) is invertible,

\[
\text{Hom}(I \mathcal{O}_\omega, \omega) \simeq \text{Hom}(I, 0).
\]

Combining the above results yeilds a version of the Riemann-Roch theorem for proper closed subschemes of \( X \).
Theorem 2.3. Let \((Z,I)\) be a proper closed subscheme of \(X\). Then
\[
\dim \mathcal{H}om(I,\mathcal{O}) - \dim \mathcal{H}om^0(X,I\mathcal{O}_\omega) = d(Z) + 1 - g.
\]

Remarks 2.8.

(1) If \(X\) is a nonsingular curve and \(D\) is an effective divisor on \(X\), then \(L(-D)\) is an ideal sheaf. Applying theorem 2.3 to \(I = L(-D)\) yields the classical Riemann-Roch theorem for effective divisors.

(2) More generally, suppose that \(D\) is an effective divisor on \(X\) with support in \(X_{\text{reg}}\). Applying theorem 2.3 to \(I = L(-D)\) yields the Riemann-Roch theorem for effective divisors on Gorenstein curves. (cf. [R, p.177])

(3) The elements of \(\mathcal{H}om(I,\mathcal{O}_X)\) can be viewed as rational functions. That is, all homomorphisms \(\phi: I \to \mathcal{O}_X\) are multiplication by some element of \(K(X)\). In fact, \(\mathcal{H}om(I,\mathcal{O}_X) = \bigcap_{P \in X} \mathcal{H}om_{\mathcal{O}_P}(I_P, \mathcal{O}_P)\), where for all \(P \in X\), \(\mathcal{H}om_{\mathcal{O}_P}(I_P, \mathcal{O}_P)\) is an \(\mathcal{O}_P\)-submodule of \(K(X)\). Furthermore, identifying \(\mathcal{O}\) with the constant rational functions, \(\mathcal{O} \subseteq \mathcal{H}om(I,\mathcal{O}_X)\).

Proposition 2.19. Let \(Y\) be an integral projective curve with \(P \in Y_{\text{Sing}}\). Then
\[
\mathcal{H}om_{\mathcal{O}_P}(c_P, \mathcal{O}_P) = \mathcal{O}_P.
\]

Proof: Let \(h\) generate \(c_P\) in \(\mathcal{O}_P\). Clearly, \(\mathcal{O}_P \subseteq \mathcal{H}om_{\mathcal{O}_P}(c_P, \mathcal{O}_P)\). Pick \(f \in \mathcal{H}om_{\mathcal{O}_P}(c_P, \mathcal{O}_P)\). Then
\[ fh\hat{\mathcal{O}}_P = f\mathcal{O}_P \subseteq \mathcal{O}_P. \]

In particular, \( fh \in \mathcal{O}_P \). But, \( fh \in \mathcal{O}_P \) and \((fh)\hat{\mathcal{O}}_P \subseteq \mathcal{O}_P\) together imply that \( fh\mathcal{C}_P = h\hat{\mathcal{O}}_P \). Thus, \( f \in \hat{\mathcal{O}}_P \) and so the proposition follows.

**Corollary 2.8.** Let \( Y \) and \( P \) be as above. Suppose that \( I_P \) is an ideal sheaf of \( Y \) with support \( P \) such that \( c_P \subseteq I_P \). Then \( \text{Hom}(I, \mathcal{O}_Y) = \mathcal{C} \).

**Proof:** Since \( \mathcal{C} \subseteq \text{Hom}(I, \mathcal{O}_Y) \), it suffices to show that if \( f \in \text{Hom}(I, \mathcal{O}_Y) \), then \( f \) is constant. Pick \( f \in \text{Hom}(I, \mathcal{O}_Y) \).

Since \( I \) has support \( P \), for all \( Q \neq P \) in \( Y \),

\[ f \in \text{Hom}_{\mathcal{O}_Q}(I_Q, \mathcal{O}_Q) = \text{Hom}_{\mathcal{O}_Q}(0_Q, \mathcal{O}_Q) = \mathcal{O}_Q. \]

Since \( c_P \subseteq I_P \), proposition 2.19 implies

\[ f \in \text{Hom}_{\mathcal{O}_P}(I_P, \mathcal{O}_P) \subseteq \text{Hom}_{\mathcal{O}_P}(c_P, \mathcal{O}_P) = \hat{\mathcal{O}}_P. \]

Then

\[ f \in \bigcap_{Q \neq P} \mathcal{O}_Q \bigcap \hat{\mathcal{O}}_P \subseteq \bigcap_{R \in \hat{Y}} \mathcal{O}_R. \]

Thus \( f \) is a global regular function on the normalization of \( Y \) and therefore is constant.

q.e.d.

The following is due to Kleiman [K, 1.1, p.4].

**Definition:** Let \((Z, I)\) be a proper closed subscheme of an integral projective curve \( Y \). Then \( Z \) is \( r \)-special if \( \dim \text{Hom}(I, \mathcal{O}_Y) > r \).

**Remarks 2.9.** Suppose that \((Z, I)\) is a proper closed
subscheme of a Gorenstein curve $\mathbf{X}$.

1. If there is a nonconstant rational function $f \in \text{Hom}(I, \mathcal{O}_X)$, then $Z$ is 1-special.

2. If $d(Z) > g + r - 1$, then by theorem 2.3, is $r$-special.

In particular, if $d(Z) > g$, then $Z$ is 1-special.

Proposition 2.20. Let $Y$ be an integral projective curve. For $P \in Y$ let $(Z, I)$ be the subscheme of degree one and support $P$. If $Y$ has nonzero arithmetic genus, then $Z$ is not 1-special.

Proof: We note that $d(Z) = 1$ implies $I_P = m_P$. If $P \in Y_{\text{sing}}$, then $m_P \supseteq c_P$, so $Z$ is not 1-special by corollary 2.8.

For $P \in Y_{\text{reg}}$, let $t$ generate $m_P$. Pick a nonzero $f \in \text{Hom}(I, \mathcal{O}_Y)$. Since $I$ has support $P$, $f \in \mathcal{O}_Q$ for all $Q \neq P$. Thus $f \in \Gamma(Y - P, \mathcal{O}_Y)$, and so $f$ defines a morphism $\phi: Y - P \to \mathbb{P}^1$. Since $P$ is a nonsingular point, $\bar{\phi}$ extends to a morphism $\phi: Y \to \mathbb{P}^1$.

On the other hand, $f \in \text{Hom}(I, \mathcal{O}_Y)$ implies that $ft \in \mathcal{O}_P$, which in turn implies that $\text{ord}_{\pi^{-1}(P)} f \geq -1$, where $\pi: \tilde{Y} \to Y$ is the normalization of $Y$. As a result, if $\phi$ is nonconstant, then $\phi$ has degree one. But this implies that $Y = \mathbb{P}^1$, which contradicts $g > 0$. Therefore $\phi$ is constant, which implies $f$ is constant. Thus $Z$ is not 1-special.
Corollary 2.9. Let X be a Gorenstein curve. Then $\omega_X$ has no base points.

Proof: for $P \in X$, let $(Z,I)$ be the subscheme of degree one and support $P$. By proposition 2.20, $\dim_{\mathcal{C}} \text{Hom}(I, O_X) = 1$. Therefore, by theorem 2.3, $\dim_{\mathcal{C}} H^0(X, I \omega) = g-1$. As a result, there is a $\sigma \in H^0(X, \omega)$ such that $\sigma \notin H^0(X, I \omega)$. Thus, there is a $\sigma \in H^0(X, \omega)$ such that $\sigma \notin m_P \omega_P$, and so $P$ is not a base point of $\omega$.

Proposition 2.21. $W(P) = (g+1)g(g-1)$.

Proof: Suppose that $\alpha$ represents the wronskian of $X$, and put $N = 1 + \ldots + g$. Then

$$\sum_{P \in X} W(P) = \sum_{P \in X} \text{ord}_P \alpha = \dim_{\mathcal{C}} H^0(X, \omega^N) = N \dim_{\mathcal{C}} H^0(X, \omega) = N(2g-2) = (g+1)g(g-1).$$
Chapter 3. Main Results

Section 1. The Main Theorem

We begin this chapter with a generalization of theorem 1.2 to Gorenstein curves.

**Theorem 3.1.** Suppose that X is an integral projective Gorenstein curve of arithmetic genus g. Then the following statements are equivalent for \( P \in X \).

1. \( W(P) > 0 \)
2. There is a nonzero \( \sigma \in H^0(X, \omega) \) satisfying \( \text{ord}_P \sigma \geq g \).
3. There is a \( 1 \)-special subscheme of \( X \) with support \( P \) and degree equal to \( g \).
4. There is a \( 1 \)-special subscheme of \( X \) with support \( P \) and degree at most \( g \).

**Proof:** We note that if \( g = 0 \), then none of the four conditions above hold. Thus we assume that \( g \) is positive.

1. \( \Rightarrow \) (2): Assume \( P \in X_{\text{reg}} \). Suppose that \( \sigma_1, \ldots, \sigma_g \) is a basis for \( H^0(X, \omega) \) satisfying \( \text{ord}_P \sigma_1 < \ldots < \text{ord}_P \sigma_g \).

Then

\[
W(P) = \sum (\text{ord}_P \sigma - (i-1)).
\]

Thus, \( W(P) > 0 \) implies that \( \text{ord}_P \sigma_i > i - 1 \) for some \( i \).

This in turn implies \( \text{ord}_P \sigma_i \geq g \).

Assume \( P \in X_{\text{sing}} \). Suppose that \( Q \in \pi^{-1}(P) \), where \( \pi: \tilde{X} \to X \) is the normalization of \( X \). Pick a \( Q \in \pi^{-1}(P) \) and a basis \( \sigma_1, \ldots, \sigma_g \) of \( H^0(X, \omega) \) which satisfies
ord_{Q_1} < ... < ord_{Q_g}. We note that by proposition 2.21, \( W(P) > 0 \) implies \( g > 1 \).

If \( P \) is cuspidal, then for all \( \tau \in m_P \omega_P \), \( \text{ord}_P \tau \geq 2 \). Therefore, \( \text{ord}_P \sigma \geq 2 \), and so \( \text{ord}_{Q_i} \sigma_i \geq i \) for \( i = 2, \ldots, g \).

In particular,
\[
\text{ord}_{Q_g} \sigma \geq g.
\]

If \( P \) is not cuspidal, let \( Q \not\in R \in \pi^{-1}(P) \). Pick a generator \( \tau \) of \( \omega_P \) and put \( \sigma = f \tau \). Since \( \text{ord}_P f \geq g - 1 \geq 1 \), \( f \in m_P \).

Therefore, by remark 2.1, (5), \( \text{ord}_R f \geq 1 \). Thus,
\[
\text{ord}_P \sigma = \text{ord}_P f \geq \text{ord}_Q f + \text{ord}_R f \geq g.
\]

(2) \( \Rightarrow \) (3): Let \( \tau \) generate \( \omega_P \) and suppose that \( \sigma = f \tau \in H^0(X, \omega) \) satisfies \( \text{ord}_P \sigma \geq g \). It follows that \( \sum_{Q \in \pi^{-1}(P)} \sigma \geq g \), and so
\[
\geq \sum_{Q \in \pi^{-1}(P)} \text{ord}_Q f = \dim (\mathcal{O}_P/f\mathcal{O}_P) = \dim (\mathcal{O}_P/f\mathcal{O}_P)
\]
(by proposition 2.6).

As a result, proposition 2.7 implies that there is an \( \mathcal{O}_P \)-ideal \( J \) such that \( f \in J \) and \( \dim (\mathcal{O}_P/f\mathcal{O}_P) = g \). Call \( I \) the ideal sheaf with support \( P \) defined by \( I_P = J \). By proposition 2.18, \( I \) is coherent. Let \( Z \) be the subscheme defined by \( I \). Clearly \( d(Z) = g \), and \( \sigma \in H^0(X, I \mathcal{O}_X) \).

Thus, \( H^0(X, I \mathcal{O}_X) \neq 0 \). By theorem 2.3 therefore, \( \dim (\text{Hom}(I, \mathcal{O}_X)) \geq 2 \), and so \( Z \) is 1-special.

(3) \( \Rightarrow \) (4): This is clear.
(4) \( \bullet \) (1): We note that proposition 2.20 implies \( g > 1 \).

By corollary 2.7 therefore, \( W(P) > 0 \) for all \( P \in X_{\text{Sing}} \).

Hence we assume \( P \in X_{\text{reg}} \).

Suppose that \( t \) generates \( m_P \) and that \( (Z, I) \) is a \( 1 \)-special subscheme of \( X \) with support \( P \) satisfying \( d(Z) = m \leq g \). By theorem 2.3,

\[
\dim \mathcal{H}^0(X, I \mathcal{O}_W) \geq g + 1 - m
\]

Since \( P \in X_{\text{reg}} \), \( I = t^m P \). Put \( d = g + 1 - m \). Suppose that \( \tau_1, \ldots, \tau_d \) are nearly independent elements of \( \mathcal{H}^0(X, I \mathcal{O}_W) \) satisfying \( \text{ord}_P \tau_1 < \ldots < \text{ord}_P \tau_d \). Since \( \tau_1 \in t^m \mathcal{O}_P \), it follows that \( \text{ord}_P \tau_1 \geq m \) and hence that \( \text{ord}_P \tau_d \geq g \).

Now suppose that \( \sigma_1, \ldots, \sigma_g \) is a basis for \( \mathcal{H}^0(X, \mathcal{O}_W) \) satisfying \( \sigma_i = \tau_d \) for some \( i \) and \( \text{ord}_P \sigma_1 < \ldots < \text{ord}_P \sigma_g \).

Thus,

\[
W(P) = \sum (\text{ord}_P \sigma_i - (j-1)) \geq \text{ord}_P (\tau_d) - i > 0
\]

Definition: We say that \( P \) is a Weierstrass point of \( X \) if \( P \) satisfies any one and hence all four of the conditions of theorem 3.1.

Remarks 3.1.

(1) If \( X \) is nonsingular, then there is a one to one correspondence between proper closed subschemes \( Z \) or \( X \) with support \( P \) and degree \( n \) and divisors \( nP \). Thus, theorem 3.1 reduces to theorem 1.2.

(2) If \( g \) is 0 or 1, then by proposition 2.21 none of the points of \( X \) are Weierstrass points.
(3) If $g > 1$, then by corollary 2.7 all of the singular points of $X$ are Weierstrass points.

**Example 3.1.** Let $X$ be a rational curve of arithmetic genus $g$ with $X_{\text{sing}}$ consisting of $g$ cusps $P_1, \ldots, P_g$.

By proposition 2.12, $W(P_i) \geq (g+1)(g-1)$ for $i = 1, \ldots, g$.

Since $W(P) \geq 0$ for all $P \in X$, and since $\sum_{P \in X} W(P) = (g+1)g(g-1)$ it follows that

$$W(P_i) = (g+1)(g-1) \text{ for } i = 1, \ldots, g,$$

and

$$W(R) = 0 \text{ for all } R \in X_{\text{reg}}.$$  

**Proposition 3.1.** For all positive integers $g$ there exist Gorenstein curves of arithmetic genus $g$ with no nonsingular Weierstrass points.

**Proof:** This follows from example 3.1.

**Example 3.2.** Let $\tilde{X}$ be a nonsingular curve of genus at least one, and let $Q_1$ and $Q_2$ be elements of $\tilde{X}$ which are not Weierstrass points. Identify $Q_1$ and $Q_2$ to make a node $P$, and call the resulting Gorenstein curve $X$. Let $g$ denote the arithmetic genus of $X$. We note that $X_{\text{sing}} = \{P\}$. By proposition 2.16,

$$W(P) = g(g-1) + W(Q_1) + W(Q_2) = g(g-1).$$

Since $g > 1$, $g(g-1) < (g-1)g(g+1)$. It follows that $X$ has nonsingular Weierstrass Points. In particular, for all $g \geq 2$, there exist singular Gorenstein curves of arithmetic genus $g$ with nonsingular Weierstrass points.
We now consider nonsingular Weierstrass points on Gorenstein curves. In this case we will show that the theory is much the same as the classical theory.

**Definition:** Let \((Z,I)\) be a proper closed subscheme of \(X\). Put:

\[
\begin{align*}
1(Z) &= \dim \mathbb{C}\text{Hom}(I, \mathcal{O}_X) \\
i(Z) &= \dim \mathbb{C}\text{H}^0(X, I\otimes \omega).
\end{align*}
\]

**Remark 3.2.** For \(P \in X\) let \(t\) generate \(m_P\). Then if \((Z,I)\) is the subscheme of degree \(j\) and support \(\{P\}\), \(I_P = t^j \omega_P\). Thus, for all \(j\) there is exactly one subscheme with support \(\{P\}\) and degree equal to \(j\).

**Proposition 3.2.** For \(P \in X\) put \((Z_j, I_j)\) equal to the subscheme with support \(P\) and degree \(j\). Then, with \(\pi: \tilde{X} \rightarrow X\) the normalization of \(X\),

1. For all \(j \geq 2g - 1\), \(i(Z_j) = 0\),
2. \(1(Z_j) \leq 1(Z_{j+1}) \leq 1(Z_j) + 1\),
3. \(1(Z_j) = 1(Z_{j+1})\) if and only if \(i(Z_{j+1}) + 1 = i(Z_j)\),
4. \(1(Z_j) + 1 = 1(Z_{j+1})\) if and only if there is an \(f \in \Gamma(X-P, \mathcal{O}_X)\) satisfying \(\text{ord}_P f = -(j-1)\),
5. \(i(Z_{j+1}) + 1 = i(Z_j)\) if and only if there is a \(\sigma \in \mathbb{H}^0(X, \omega)\) satisfying \(\text{ord}_P \sigma = j\).

**Proof:** Let \(t\) generate \(m_P\).

1. Given a nonzero \(\sigma \in \mathbb{H}^0(X, I_j \otimes \omega)\), then \(\sigma \in t^j \omega_P\) and so \(\text{ord}_P \sigma \geq j\). Since \(\text{ord}_P \sigma \leq 2g - 2\), (1) now follows.
(2) This follows from the long exact cohomology sequence derived from the exact sequence

\[
0 \to \text{Hom}(I_j, O_X) \to \text{Hom}(I_{j+1}, O_X) \to H \to 0
\]

where \( H \) is the sheaf with support \( P \) defined by

\[
H_p = t^j O_p / t^{j+1} O_p.
\]

(3) This now follows from (2) and theorem 2.3.

(4) We note that \( \text{Hom}_P(I_j^p, O_p) = t^{-j} O_p \). (4) now follows easily from this observation.

(5) We note that \( (I_j^p \omega)_p = t^j \omega_p \). (5) now follows easily from this observation.

Remark 3.3. It follows that for \( P \in X_{\text{reg}} \) there are \( g \) integers

\[
l \leq m_1 < \ldots < m_g \leq 2g-1
\]

such that \( l(Z_{m_i}) = l(Z_{m_{i-1}}) \) (here \( Z_0 = X \)).

Definition: The sequence \( m_1, \ldots, m_g \) defined above is called the Weierstrass gap sequence (or gap sequence) for \( P \). The \( m_i \) are called the Weierstrass gaps (or gaps) for \( P \).

The following proposition follows easily from proposition 3.1.

Proposition 3.3. Suppose that \( P \in X_{\text{reg}} \) has Weierstrass gap sequence \( m_1, \ldots, m_g \). Then

(1) The nongaps for \( P \) form a semigroup under addition.

(2) \( W(P) = \Sigma (m_i - i) \).
Proposition 3.4. For $P \in \mathcal{X}_{\text{reg}}$, suppose that $1 < b_1 < \ldots < b_{g-1}$ are the first $g-1$ nongaps for $P$. Then

$$
\sum_{j=1}^{g-1} b_j \geq g(g-1).
$$

Furthermore, equality holds if and only if $b_1 = 2$.

Proof: Since the nongaps for $P$ form a semigroup, the proof of this proposition is the same as the proof in [F-K, III.5.7].

Corollary 3.1. For $P \in \mathcal{X}_{\text{reg}}$, $W(P) \leq \frac{1}{2}g(g-1)$, with equality if and only if $b_1 = 2$.

Proof:

$$
W(P) = \sum_{i=1}^{g-1} (m_i - i)^{-1} = \sum_{i=1}^{g-1} b_i - \sum_{i=1}^{g-1} i.
$$

Thus,

$$
W(P) \leq (2g-1)g - g(g-1) - \frac{1}{2}g(g+1) = \frac{1}{2}g(g-1).
$$

Moreover, we have equality if and only if $b_1 = 2$ by proposition 3.4.

Section 2. Principal Subschemes

In this section we define principal subschemes. We demonstrate that (non-principal) subschemes are essential to the theory of Weierstrass points on Gorenstein curves; and we prove that theorem 1.4 does not generalize to Gorenstein curves.
Definition: We call a proper closed subscheme \((Z,I)\) of \(X\) principal if \(I\) is an invertible sheaf.

Proposition 3.5. Let \(P\) be a point on a Gorenstein curve \(X\) of arithmetic genus \(g\). Then

1. \(P\) is a Weierstrass point of \(X\) if and only if there is an ideal \(I_p\) of \(O_P\) satisfying \(\dim F(O_P/I_p) \leq g\) and such that for some nonconstant \(f \in \Gamma(X-P, O_X)\), \(f I_p \subseteq O_p\).

2. There is a principal \(1\)-special subscheme with support \(P\) and degree at most \(g\) if and only if there is an \(h \in O_p\) satisfying \(0 < \text{ord}_P h \leq g\), and an \(f \in \Gamma(X-P, O_X)\) such that \(fh \in O_p\).

3. There is a morphism \(\phi: X \to \mathbb{P}^1\) of degree at most \(g\) satisfying \(\phi^{-1}(\phi(P)) = \{P\}\) if and only if there is an \(h \in O_p\), satisfying \(0 < \text{ord}_P h \leq g\), and an \(f \in \Gamma(X-P, O_X)\) such that \(fh\) is a unit of \(O_p\).

Proof:

1. and 2.: For \((Z,I)\) a subscheme with support \(P\), we note that \(f \in \text{Hom}(I, O_X)\) if and only if \(f \in \Gamma(X-P, O_X)\) and \(f I_p \subseteq O_p\). Both (1) and (2) follow from this statement and proposition 2.18.

3.: Suppose that \(\phi: X \to \mathbb{P}^1\) is a morphism of degree \(d \leq g\) satisfying \(\phi^{-1}(\phi(P)) = \{P\}\). By performing a projective change of coordinates on \(\mathbb{P}^1\) if necessary, we may assume that \(\phi(P) = \infty\). Put \(f = \phi\mid_{X-P}\). Then it
follows that \( f \in \Gamma(X-P, O_X) \) and \( f^{-1} \in O_p \). Put \( h = f^{-1} \).

Then by the construction of \( f \), \( 0 < \text{ord}_p h \leq g \), and \( fh \) is a unit in \( O_p \).

Conversely, assume that there is an \( h \in O_p \) satisfying \( 0 < \text{ord}_p h \leq g \), and an \( f \in \Gamma(X-P, O_X) \) such that \( fh \) is a unit in \( O_p \). Note that this implies that \( f^{-1} \in O_p \) and that \( \text{ord}_p f^{-1} = d \leq g \). Clearly \( f \) gives rise to a morphism \( \varphi: X-P \to \mathcal{C} \) of degree \( d \) with associated field homomorphism \( \theta: \mathcal{C}(T) \to K(X) \) defined by \( \theta(T) = f \).

\( \varphi \) will extend to the required morphism \( \Phi: X \to \mathbb{P}^1 \) if it can be established that

\[
\theta(\mathbb{C}[1/T](1/T)) \subseteq O_p
\]

Since \( \theta(1/T) = f^{-1} \), and \( f^{-1} \in O_p \), \( \theta(\mathbb{C}[1/T]) \subseteq O_p \).

Thus it suffices to show that for all \( a \in \mathbb{C} - 0 \),

\[
\theta((1-a/T)^{-1}) = (1-af^{-1})^{-1} \in O_p
\]

Clearly \( f^{-1} \in m_p \), and so for some positive integer \( n \), \( (f^{-1})^n \in \mathbb{C}_p \). Further, \( f^{-1} \in m_p \) implies that \( f^{-1} \in m_R \) for all \( R \in \pi^{-1}(P) \).

Thus, \( (1-af^{-1})^{-1} \) is a unit in \( O_R \) for all \( R \in \pi^{-1}(P) \). In particular, \( (1-af^{-1})^{-1} \) is a unit of \( O_{\tilde{P}} \). Since

\[
1 - (af^{-1})^n = (1-af^{-1})(1+(af^{-1}) + \ldots + (af^{-1})^{n-1}),
\]

\[\begin{align*}
(1-af^{-1})^{-1}(1-(af^{-1})^n) &= 1 + (af^{-1}) + \ldots + (af^{-1})^{n-1},
\end{align*}\]

and so
(1-af^{-1})^{-1} = (1-af^{-1})^{-1}(af^{-1})^n + 1 + (af^{-1}) + ... + (af^{-1})^{n-1}.

Now 1 + (af^{-1}) + ... + (af^{-1})^{n-1} \in \mathcal{O}_P. Further, (f^{-1})^n \in \mathcal{O}_P

and \( (1-af^{-1})^{-1} \in \mathcal{O}_P \) imply that \( (1-af^{-1})^{-1}(af^{-1})^n \in \mathcal{O}_P \).

Therefore, \( (1-af^{-1})^{-1} \in \mathcal{O}_P \).

**Corollary 3.2.** There exists a morphism \( \phi: X \to \mathbb{P}^1 \) of degree \( d \) satisfying \( \phi^{-1}(\phi(P)) = \{P\} \), if and only if there exists an \( f \in \mathcal{O}(X-P, \mathcal{O}_X) \) such that \( f^{-1} \in \mathcal{O}_P \), and \( \text{ord}_P f^{-1} = d \).

**Proof:** This follows from the proof of proposition 3.5, (3).

**Proposition 3.6.** Let \( X \) be a Gorenstein curve of arithmetic genus \( g \). Consider the following statements about \( P \in X \).

(A) There is a nonconstant morphism \( \phi: X \to \mathbb{P}^1 \) of degree at most \( g \) satisfying \( \phi^{-1}(\phi(P)) = \{P\} \).

(B) There is a principal 1-special subscheme with support \( P \) and degree at most \( g \).

(C) \( P \) is a Weierstrass point of \( X \).

Then

(i) For all \( P \in X \), (A) \( \Rightarrow \) (B) \( \Rightarrow \) (C)

(ii) For \( P \in X_{\text{reg}} \), (A) \( \Leftrightarrow \) (B) \( \Leftrightarrow \) (C).

**Proof:**

(i) Since (B) \( \Rightarrow \) (C) is clear, it suffices to show
(A) $\Rightarrow$ (B).

Suppose that $\phi: X \to \mathbb{P}^1$ is a morphism of degree at most $g$ satisfying $\phi^{-1}(\phi(P)) = \{P\}$. By proposition 3.5, (3) there is an $h \in \mathcal{O}_P$ such that $0 < \text{ord}_P h \leq g$, and such that for some $f \in \Gamma(X-P, \mathcal{O}_X)$, $hf$ is a unit in $\mathcal{O}_P$. Let $(Z, I)$ be the principal subscheme with support $P$ defined by $h$. Then $d(Z) = \text{ord}_P f \leq g$, and since $f \in \text{Hom}(I, \mathcal{O}_X)$, $Z$ is 1-special.

(ii) It suffices to show that for $P \in X_{\text{reg}}$, (C) $\Rightarrow$ (A). Suppose there is a 1-special subscheme $(Z, I)$ with support $P$ and degree at most $g$. Pick a nonconstant element $f$ of $\text{Hom}(I, \mathcal{O}_X)$. Then $f \not\in \mathcal{O}_P$. Since $P$ is nonsingular, $\mathcal{O}_P$ is a discrete valuation ring. Thus $f \not\in \mathcal{O}_P$ implies that $f^{-1} \in \mathcal{O}_P$. Furthermore, $f^* I_P \subseteq \mathcal{O}_P$ implies $\text{ord}_P f^{-1} \leq d(Z) \leq g$. Consequently, (ii) follows from proposition 3.6, (3).

q.e.d.

We shall now establish that in general, (C) does not imply (B), and (B) does not imply (A). Thus, theorem 1.4 does not generalize to arbitrary Gorenstein curves, and non-principal subschemes are an essential part of the theory.

Example 3.3. This is a counterexample to (B) $\Rightarrow$ (A). Let $Y$ be a nonsingular hyperelliptic curve of genus 3, and let $Q \in Y$ be a Weierstrass point. Let $h$ be an element
of \( \Gamma (Y-Q, O_Q) \) satisfying \( \text{ord}_Q h = -2 \). We note that if
\( f \in \Gamma (Y-Q, O_Q) \) satisfies \( \text{ord}_Q f \geq -6 \), then
\[
f = a_1 + a_2 h + a_3 h^2 + a_4 h^3 \quad \text{with} \quad a_i \in \mathbb{C} \quad \text{for} \quad i = 1, 2, 3, 4.
\]
Finally, suppose \( t \in K(Y) \) satisfies
\[
\text{ord}_Q t = 1 \quad \text{and} \quad t^2 h = 1 + bt + kt^2 \quad \text{where} \quad b \in \mathbb{C} - \{0\} \quad \text{and} \quad k \in O_Q.
\]
(Note that if \( s \in K(X) \) satisfies \( \text{ord}_Q s = 1 \) and \( \text{ord}_Q (s^2 h - 1) > 1 \), then we may put \( t = s + s^2 \).)

We now construct a Gorenstein curve \( X \) of arithmetic genus 7 and normalization \( Y \) as follows. \( X_{\text{sing}} = \{P\} \)
with \( P \) cuspidal, \( X-P \cong Y-Q \), and
\[
0_p = \mathbb{C} + t^3 C + t^3 / h C + t^6 C + t^8 O_Q.
\]
We note that \( c_p = t^8 O_Q \). Thus, \( n_p = 8 \) and \( \delta_p = 4 \),
so \( X \) is Gorenstein.

Let \( (Z, I) \) be the principal subscheme with support \( P \) defined by \( t^3 / h \). Then \( \text{ord}_Q t^3 / h = 5 \) implies \( d(Z) = 5 \).
Since \( h \in \Gamma (Y-Q, O_Y) = \Gamma (X-P, O_X) \) and \( h(t^3 / h) = t^3 e \in O_P \),
h is a nonconstant element of \( \text{Hom}(I, O_X) \). Therefore, \( Z \) is 1-special and so \( P \) satisfies (B).

By proposition 3.5,(3), to show that \( P \) does not satisfy (A), it suffices to show that there does not exist a nonconstant \( f \in \Gamma (X-P, O_X) \) which satisfies
\( \text{ord}_Q f \geq -7 \) and \( f^{-1} \in O_P \). Assume that such an \( f \) exists.

We note that
\[
\text{ord}_Q f = -2, -4, -7 \implies f^{-1} \notin O_P,
\]
It follows that \( \text{ord}_Q f = -6 \), and so we may assume
\[
f = h^3 + c_2 h^2 + c_1 h + c_0 \quad \text{with} \quad c_i \in \mathfrak{C} \quad \text{for} \quad i = 0, 1, 2.
\]
But,
\[
h = (1 + bt + kt^2)/t^2,
\]
so
\[
f = (1 + 3b^3 t + k^2 t^2)/t^6 \quad \text{with} \quad k^2 \in \mathfrak{O}_Q.
\]
As a result,
\[
f^{-1} = t^6(1 - 3b^3 t + k^2 t^2) \quad \text{with} \quad k^2 \in \mathfrak{O}_Q.
\]
On the other hand, \( f^{-1} \in \mathfrak{O}_P \) implies that
\[
(f^{-1} - t^6) \in \mathfrak{O}_P. \quad \text{But} \quad b \neq 0 \implies -3b^3 \neq 0, \quad \text{and so} \quad \text{ord}_Q (f^{-1} - t^6) = 7. \quad \text{This is impossible for an element of} \ \mathfrak{O}_P, \quad \text{so no such} \ f \ \text{exists.}
\]

Example 3.4. This is a counterexample to \((C) \Rightarrow (B)\).
Suppose that \( Y \) is a Gorenstein curve of arithmetic genus two with \( Q_1 \) and \( Q_2 \) nonsingular Weierstrass points of \( Y \). We now construct a Gorenstein curve \( X \) from \( Y \) by identifying \( Q_1 \) and \( Q_2 \) to make a node \( P \). That is, \( X \) is a curve of arithmetic genus 3 with \( P \in X \) a node such that \( \theta : Y \to X \) is the partial normalization at \( P \) and \( \theta^{-1}(P) = \{Q_1, Q_2\} \). Since \( Y \) has arithmetic genus two and \( Q_1 \) and \( Q_2 \) are Weierstrass points of \( Y \), for \( i = 1, 2 \) there is a \( \sigma_i \in \mathcal{H}^0(Y, \omega_Y) \) satisfying
Suppose \( \tau \in H^0(X, \omega_X) \) generates \( \omega_P \). Write \( \sigma_i = f_i^{\tau} \) where \( f_i \in \mathcal{O}_P \). It follows that \( \tau, \sigma_1, \sigma_2 \) is a basis for \( H^0(X, \omega_X) \), and that
\[
\text{ord}_{Q_i} f_i = 3 \quad \text{and} \quad \text{ord}_{Q_j} f_i = 1 \quad \text{for} \quad i \neq j.
\]
Thus, given \( \sigma \in H^0(X, \omega_X) \) with \( \sigma = f^{\tau} \) for \( f \in \mathcal{O}_P \),
there are four possibilities.

1. \( \text{ord}_{Q_1} f = 0 \) and \( \text{ord}_{Q_2} f = 0 \)
2. \( \text{ord}_{Q_1} f = 1 \) and \( \text{ord}_{Q_2} f = 1 \)
   (*)
3. \( \text{ord}_{Q_1} f = 3 \) and \( \text{ord}_{Q_2} f = 1 \)
4. \( \text{ord}_{Q_1} f = 1 \) and \( \text{ord}_{Q_2} f = 3 \)

For \( h \in \mathcal{M}_P \) let \( I \) be the invertible ideal sheaf with support \( P \) defined by \( h \).

Assume \( \text{ord}_P h = 2 \). Clearly this implies
\[ \text{ord}_{Q_1} h = \text{ord}_{Q_2} h = 1. \]
Thus, given a nonzero \( \sigma \in H^0(X, I \otimes \omega_X) \) such that
\( \sigma = hk^{\tau} \) for \( k \in \mathcal{O}_P \), since \( k \in \mathcal{M}_P \) implies \( \text{ord}_{Q_i} k \geq 1 \)
for \( i = 1, 2 \), the conditions (*) imply that \( k \) is a unit
in \( \mathcal{O}_P \). Hence, \( \dim \mathcal{H}^0(X, I \otimes \omega_X) = 1 \). Therefore theorem
2.3 implies that \( h \) does not define a 1-special subscheme.

Assume \( \text{ord}_P h = 3 \). We note that without loss of
generality we may further assume
ord_{Q_1} h = 2 \quad \text{and} \quad \operatorname{ord}_{Q_2} h = 1.

Pick \sigma \in H^0(X, I\omega_X) and write \sigma = qh^\tau \quad \text{for} \quad q \in \mathcal{O}_P.

Clearly the conditions (*) imply that \( q \in \mathcal{O}_P \) only if \( q = 0 \). Thus, \( H^0(X, I\omega_X) = 0 \), and so theorem 2.3 implies that \( h \) does not define a 1-special subscheme.

As a result, \( X \) does not have a principal 1-special subscheme with degree at most 3 and support \( P \). So \( P \) does not satisfy (B). Since \( P \) is a singular point, \( P \) does satisfy (C).

We shall now establish that for \( \delta_P = 1 \), the example above is the only counterexample to (C) \implies (B).

**Theorem 3.2.** Suppose that \( X \) has arithmetic genus \( g \) greater than one, that \( P \in X_{\text{sing}} \) satisfies \( \delta_P = 1 \), and that \( \theta : Y \to X \) is the partial normalization at \( P \).

1. If \( P \) is a cusp then there is a morphism \( \phi : X \to \mathbb{P}^1 \) of degree at most \( g \) satisfying \( \phi^{-1}(\phi(P)) = \{P\} \).

2. If \( P \) is a node with \( \theta^{-1}(P) = \{Q_1, Q_2\} \) and \( Q_1 \) and \( Q_2 \) are not both Weierstrass points of \( Y \), then there is a morphism \( \phi : X \to \mathbb{P}^1 \) of degree at most \( g \) satisfying \( \phi^{-1}(\phi(P)) = \{P\} \).

3. If \( P \) is a node, then there is a principal 1-special subscheme with support \( P \) and degree at most \( g \), unless \( g = 3 \) and \( \theta^{-1}(P) \) consists of two Weierstrass points of \( Y \).

**Proof:**

1. \( P \) is a cusp with \( \theta^{-1}(P) = \{Q\} \). Since \( Y \) has
arithmetic genus $g-1$, by theorem 2.3 there is a nonconstant $h \in \Gamma(Y-Q, \mathcal{O}_Y) = \Gamma(X-P, \mathcal{O}_X)$ satisfying 

$$-1 > \operatorname{ord}_Q h \geq -g.$$ 

Since $\operatorname{ord}_Q 1/h \geq 2$ and $P$ is a cusp, $1/h \in \mathcal{O}_P$. By proposition 3.5, (3), (1) now follows.

(2) Assume $Q_1$ is not a Weierstrass point of $Y$. Call $(Z,I)$ the principal subscheme of $Y$ with support 

$\{Q_1, Q_2\}$ of degree $g$ defined by 

$$\dim \mathcal{O}_{Q_1}/I_{Q_1} = g-1 \text{ and } \dim \mathcal{O}_{Q_2}/I_{Q_2} = 1.$$ 

(Note that such a $Z$ exists by proposition 2.9.) Since $\dim \mathcal{O}_Y/I = g$, by theorem 2.3 there is a nonconstant $h \in \operatorname{Hom}(I, \mathcal{O}_Y)$. If $\operatorname{ord}_{Q_1} h \geq 0$, then the definition of $I$ implies $\operatorname{ord}_{Q_1} h = -1$. Since $g-1 \neq 0$, this is impossible. Thus $\operatorname{ord}_{Q_1} h < 0$. If $\operatorname{ord}_{Q_2} h \geq 0$, then the definition of $I$ implies that $0 > \operatorname{ord}_{Q_1} h \geq -g$. But this contradicts the fact that $Q_1$ is not a Weierstrass point of $Y$. So, for $i = 1,2$, $\operatorname{ord}_{Q_i} h < 0$. Since $P$ is a node, this implies that $1/h \in \mathcal{O}_P$. (2) now follows from proposition 3.5, (3).

(3) We note that (2) and example 3.4 together imply that (3) is true for $g \leq 3$. As a result, we assume that $g \geq 4$ and that $\theta^{-1}(P) = \{Q_1, Q_2\}$ with $Q_1$ and $Q_2$ both Weierstrass points of $Y$. Suppose that $\tau$ generates $\omega_P$, and that $\sigma_1, \ldots, \sigma_g$ is a basis for $H^0(X, \omega)$ with $\sigma_i = f_i \tau$ for $i = 1, \ldots, g$ satisfying
\[ \text{ord}_{Q_1} f_1 < \ldots < \text{ord}_{Q_1} f_g \text{ and } \text{ord}_{Q_2} f_i \neq \text{ord}_{Q_2} f_j \]

for \( i \neq j \).

Note that \( Q_1 \) a Weierstrass point implies that \( \text{ord}_{Q_1} f_g > g \).

Case (1) Assume there is a \( \sigma \in H^0(X, \omega) \) with \( \sigma = s_{\pi} \) satisfying

\[ \text{ord}_{Q_1} s \geq g \text{ and } \text{ord}_{Q_2} s > 1. \]

Pick \( f \in O_p \) satisfying \( \text{ord}_{Q_1} f = g-1 \) and \( \text{ord}_{Q_2} f = 1 \).

(By proposition 2.9 such an \( f \) exists.) Let \((Z,I)\) be the principal subscheme with support \( P \) defined by \( f \). Then \( d(Z) = g \) and \( \sigma \in H^0(X, I\otimes \omega) \). By theorem 2.3, \( Z \) is 1-special.

Case (2) Assume \( \text{ord}_{Q_1} f_{g-1} > g \). Then for some \( a, b \in \mathbb{C}, \)

\[ \sigma = a \sigma^g + b \sigma^{g-1} \text{ satisfies } \sigma = s_{\pi} \text{ and } \]

\[ \text{ord}_{Q_1} s \geq g \text{ and } \text{ord}_{Q_2} s > 1. \]

Thus, we have reduced to case 1.

Case (3) Assume \( \text{ord}_{Q_1} f_{g-1} = g-1 \). By case (1) we may assume \( \text{ord}_{Q_2} f_g = 1 \). Then for some \( a, b \in \mathbb{C}, \sigma = a \sigma^g \]

+ \( b \sigma^{g-1} \) satisfies \( \sigma = s_{\pi} \) and

\[ \text{ord}_{Q_1} s = g-1 \text{ and } \text{ord}_{Q_2} s = 1 \]

Let \((Z,I)\) be the principal subscheme with support \( P \) defined by \( s \). Then \( d(Z) = g \) and \( \sigma \in H^0(X, I\otimes \omega) \). So \( Z \) is 1-special by theorem 2.3.
Case (4) We have reduced to

\[ \text{ord}^f_i = i - 1 \text{ for } 1 < i < g - 1, \text{ and } \text{ord}^f_g = 1. \]

Note that by reversing the roles of \( Q_1 \) and \( Q_2 \) we may assume that for some \( j \)
\[ \text{ord}^{P_2} f_j = 2. \]
Let \( \sigma = f_{\tau} \) satisfy

\[ \text{ord}^{P_2} f_j = 2 \text{ and } \text{ord}^{P_1} f \text{ is maximal.} \]

Let \((Z, I)\) be the principal subscheme with support \( P \) defined by \( f \).
Then \( d(Z) = 2 + (i - 1) = i + 1. \) By remark 1.1, (2) \( \text{ord}^{P_1} f = \text{ord}^{P_1} f_i \) for some \( i, \) and by our assumptions \( i \neq g. \) Therefore, \( \sigma, \sigma_{i+1}, \ldots, \sigma_{g-1} \)
are linearly independent elements of \( H^0(X, I\Omega) \), so \( \dim_{\mathbb{C}} H^0(X, I\Omega) \geq g - i. \) By theorem 2.3, \( Z \) is \( 1 \)-special.

Example 3.5. Let \( Y \) be a nonsingular hyperelliptic curve of arithmetic genus \( 4n - 1. \) Let \( Q_1 \) and \( Q_2 \) be Weierstrass points of \( Y. \) By corollary 1.2, for \( i = 1, 2 \) there is an \( f_i \in \Gamma(Y - Q_i, O_Y) \) satisfying \( \text{ord}^{Q_i} f_i = -2. \)

Let \( X \) be the genus \( 4n \) curve defined by \( P \in \text{sing} \) is a node and \( \theta: Y \to X \) is the partial normalization at \( P \) with \( \theta^{-1}(P) = \{Q_1, Q_2\}. \) Put \( f = (f_1 f_2)^n. \) Then
\[ f \in \Gamma(X - P, O_X), \ f^{-1} \in O_P, \text{ and } \text{ord}_p f = 4n. \] By proposition 3.5, (3) there is a morphism \( \phi: X \to \mathbb{P}^1 \) of degree equal to \( 4n \) satisfying \( \phi^{-1}(\phi(P)) = \{P\}. \) Therefore there exist nodes \( P \) with \( \theta^{-1}(P) \) consisting of two Weierstrass points for which (A) holds.
Example 3.6. We now construct a Gorenstein curve $X$ of arithmetic genus 5, with $ReX$ a node which does not satisfy (A). Thus the node of example 3.4 is not the only node which does not satisfy (A).

We note that $Y$ below is in fact constructed in [F-K, VII.3.4]. Let $Y$ be the nonsingular curve of genus four defined by the equation

$$w^3 = z(z-1)(z-\lambda_1)^2(z-\lambda_2)^2(z-\lambda_3)^2,$$

where for $i = 1, 2, 3$ the $\lambda_i$'s are distinct elements of $\mathbb{C} - 0$, and $z$ and $w$ are nonconstant rational functions.

We note that the equation above defines a morphism $\phi: Y \to \mathbb{P}^1$ with $\phi^{-1}(0) = P$ and $\phi^{-1}(1) = Q$ as two points of total ramification for $\phi$. Thus,

$$\text{ord}_Q z = 0 \quad \text{and} \quad \text{ord}_P z = 3,$$

$$\text{ord}_Q (z-\lambda_1) = 3 \quad \text{and} \quad \text{ord}_P (z-\lambda_1) = 0,$$

$$\text{ord}_Q w = 2 \quad \text{and} \quad \text{ord}_P w = 1,$$

$$\text{ord}_Q dz = 2 \quad \text{and} \quad \text{ord}_P dz = 2,$$

and

$$\text{ord}_Q (z-\lambda_i) = 0 \quad \text{and} \quad \text{ord}_P (z-\lambda_i) = 0 \quad \text{for} \quad i \neq 1.$$

Put,

$$\sigma_1 = (z-\lambda_1)^2(z-\lambda_2)(z-\lambda_3)dz/w^2,$$

$$\sigma_2 = (z-\lambda_1)dz/w,$$

$$\sigma_3 = z(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)dz/w^2,$$

$$\sigma_4 = zdz/w.$$

We note that $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$, are elements of $H^0(Y, \omega_Y)$. 

(For details see [F-K, VII.3.4]. Furthermore,
\[
\begin{align*}
\text{ord}_{Q_1} & = 4 \quad \text{and} \quad \text{ord}_{P_1} = 0, \\
\text{ord}_{Q_2} & = 3 \quad \text{and} \quad \text{ord}_{P_2} = 1, \\
\text{ord}_{Q_3} & = 1 \quad \text{and} \quad \text{ord}_{P_3} = 3, \quad \text{and} \\
\text{ord}_{Q_4} & = 0 \quad \text{and} \quad \text{ord}_{P_4} = 4.
\end{align*}
\]
Since \( \text{ord}_{P_1} < \ldots < \text{ord}_{P_4} \), it follows that \( \sigma_1, \sigma_2, \sigma_3, \sigma_4 \), form a basis for \( H^0(Y, \omega_Y) \).

We now identify \( P \) and \( Q \) to form a node \( R \) on a curve \( X \) of arithmetic genus 5.

Suppose that \( \tau \in H^0(X, \omega_X) \) generates \( \omega_P \). Then \( \tau, \sigma_1, \sigma_2, \sigma_3, \sigma_4 \) form a basis for \( H^0(X, \omega_X) \). For \( i = 1, 2, 3, 4 \) write \( \sigma_i = f_i \tau \). Then

\[
\begin{align*}
\text{ord}_{Q_1} & = 5 \quad \text{and} \quad \text{ord}_{P_1} = 1, \\
\text{ord}_{Q_2} & = 4 \quad \text{and} \quad \text{ord}_{P_2} = 2, \\
\text{ord}_{Q_3} & = 2 \quad \text{and} \quad \text{ord}_{P_3} = 4, \quad \text{and} \\
\text{ord}_{Q_4} & = 1 \quad \text{and} \quad \text{ord}_{P_4} = 5.
\end{align*}
\]

Assume there is a morphism \( \psi: X \to \mathbb{P}^1 \) of degree at most 5 satisfying \( \psi^{-1}(\psi(R)) = \{R\} \). By proposition 3.5, (3), there is an \( h \in \Gamma(X-R, 0_X) \) satisfying \( h^{-1} \in 0_{R'} \), \( \text{ord}_P h < 0, \text{ord}_Q h < 0 \), and \( -5 < \text{ord}_P h + \text{ord}_Q h \). Put \( f = h^{-1}, \text{ord}_P f = a, \) and \( \text{ord}_Q f = b \). We also note that \( f \) defines a principal \( l \)-special subscheme of \( X \) with support \( P \) of degree \( a+b \). Call this subscheme \( (Z, I) \).

By theorem 2.3, \( \dim H^0(X, I_0 \omega) \geq 6 - (a+b) \).
Case (1) \((a,b) = (1,1)\),

This case is impossible since \(Y\) is not hyperelliptic.

Case (2) \((a,b) = (2,1)\).

Then \(\dim H^0(X, I_\omega_X) \geq 3\). On the other hand suppose that \(\sigma = k_T\) is a nonzero element of \(H^0(X, I_\omega_X)\). Then (*) implies either

\[ \text{ord}_P k = 2 \quad \text{and} \quad \text{ord}_Q k = 1 \quad \text{or} \quad \text{ord}_P k = 4 \quad \text{and} \quad \text{ord}_Q k = 2. \]

Thus \(\dim H^0(X, I_\omega_X) \leq 2\), and so \((a,b) \neq (2,1)\).

Case (3) \((a,b) = (3,1)\).

Then \(\dim H^0(X, I_\omega_X) \geq 2\). On the other hand suppose that \(\sigma = k_T\) is a nonzero element of \(H^0(X, I_\omega_X)\). Then (*) implies

\[ \text{ord}_P k = 4 \quad \text{and} \quad \text{ord}_Q k = 2. \]

Therefore, \(\dim H^0(X, I_\omega_X) \leq 1\), and so \((a,b) \neq (3,1)\).

Case (4) \((a,b) = (1,2), (1,3), \text{ or } (2,2)\).

Using methods similar to those of cases (2) and (3), it follows that this case is impossible.

Case (5) \((a,b) = (4,1) \text{ or } (1,4)\).

In this case (*) implies that \(\dim H^0(X, I_\omega_X) \leq 1\), and so by theorem 2.3 and our assumptions on \(f\),

\[ \dim \text{Hom} (I, \mathcal{O}_X) = 2. \]

On the other hand, (*) implies that \(P\) and \(Q\) both have gap sequences equal to \(1,2,4,5\). Thus there are \(k_P, k_Q \in \Gamma(X-R, \mathcal{O}_X)\) satisfying
ord_p k_P = -3, ord_p k_Q = 0, ord_Q k_P = 0, and ord_Q k_Q = -3.

Since one of k_P f or k_Q f is in O_R, it follows that one of k_P or k_Q is in \text{Hom}(I, O_Y). But, by our assumptions, \( f^{-1} \in \text{Hom}(I, O_X) \). But this implies that \( \text{dim}_I \text{Hom}(I, O_X) = 3 \).

Thus, \((a, b) \neq (4, 1)\) or \((1, 4)\).

Since no such \( f \) can be defined, \( R \) cannot satisfy (A).

For the final result of this section, we consider a partial converse to (A) = (B).

**Proposition 3.7.** Let \( Y \) be an integral projective curve of arithmetic genus at least two such that for some \( P \in Y \) there is a principal 1-special subscheme \((Z, I)\) with support \( P \) and degree equal to two. Then there is a morphism of degree two \( \phi: Y \to \mathbb{P}^1 \) satisfying \( \phi^{-1}(\phi(P)) = \{P\} \).

**Proof:** Write \( I_P = f O_P \). Then there is a nonconstant \( h \in \Gamma(Y-P, O_Y) \) which satisfies \( hf \in O_P \). Clearly, \( 0 \leq \text{ord}_P fh < \text{ord}_P f = 2 \). It follows that \( \text{ord}_P fh = 0 \) or 1. But \( fh \in O_P \) and \( P \in Y\text{sing} \) imply that \( \text{ord}_P fh \neq 1 \).

Thus, \( \text{ord}_P fh = 0 \), and so \( fh \) is a unit in \( O_P \). Using the proof of proposition 3.5, (3), one shows that this implies that there is a morphism \( \phi: Y \to \mathbb{P}^1 \) of degree two satisfying \( \phi^{-1}(\phi(P)) = \{P\} \).
Section 3: Quasi-Hyperelliptic Curves

In this section we consider quasi-hyperelliptic curves. We prove that all quasi-hyperelliptic curves are Gorenstein; and we prove that a Gorenstein curve is quasi-hyperelliptic if and only if it has a degree two 1-special proper closed subscheme.

Definition: An integral projective curve $Y$ is quasi-hyperelliptic if there exists a morphism of degree two $\phi: Y \to \mathbb{P}^1$.

A point $P$ of $Y$ is called a quasi-hyperelliptic point if there exists a morphism of degree two $\psi: Y \to \mathbb{P}^1$ satisfying $\psi^{-1}(\psi(P)) = \{P\}$.

Definition: Let $Y$ be an integral projective curve and let $P \in Y_{\text{sing}}$. We call $P$ a singularity of type I, if for some $f \in \mathcal{O}_P$ and some positive integer $n$

$$\mathcal{O}_P = \mathcal{O} + f\mathcal{O} + \ldots + f^{n-1}\mathcal{O} + f^n\mathcal{O}_P.$$  

Now let $P$ be a type I singularity, and let $\pi: \tilde{Y} \to Y$ be the normalization of $Y$. If $\pi^{-1}(P)$ consists of two points, we call $P$ a singularity of type IA. If $\pi^{-1}(P)$ consists of one point, we call $P$ a singularity of type IB.

Proposition 3.8. Suppose that for some $f \in \mathcal{O}_P$,

$$\mathcal{O}_P = \mathcal{O} + f\mathcal{O} + \ldots + f^{n-1}\mathcal{O} + f^n\mathcal{O}_P.$$
Then $P$ is a Gorenstein singularity if and only if $\text{ord}_p f = 2$. Consequently, all type I Gorenstein singularities are either of type IA or type IB.

Proof: The first statement follows from

$$n_p = (\text{ord}_p f)n \text{ and } \delta_p = 2n.$$ 

Since $\text{ord}_p f = 2$ implies that the preimage of $P$ in the normalization consists of at most two points, the second statement follows from the first.

Remark 3.4. If $P \in Y_{\text{sing}}$ satisfies $\delta_p = 1$, then

$$\hat{\mathcal{O}}_P = \mathfrak{C} + f \hat{\mathcal{O}}_P,$$

where $f$ generates $c_P$ in $\mathfrak{C}_P$. Thus, $P$ is a singularity of type IA or type IB.

Proposition 3.9. $P$ is a type I Gorenstein singularity if and only if for some $f \in \mathfrak{C}_P$, $\text{ord}_p f = 2$.

Proof: If $P$ is a type I Gorenstein singularity then for some $f \in \mathfrak{C}_P$,

$$\mathcal{O}_P = \mathfrak{C} + f \mathfrak{C} + \ldots + f^{\delta_p - 1} \mathfrak{C} + f^{\delta_p} \hat{\mathcal{O}}_P.$$ 

By proposition 3.8, $\text{ord}_p f = 2$.

Conversely, suppose that there is an $f \in \mathfrak{C}_P$ satisfying $\text{ord}_p f = 2$. Let $h$ generate $c_P$ in $\hat{\mathcal{O}}_P$. Call $d$ the smallest integer such that $f^d \in c_P$. Then for some $R \subseteq \pi^{-1}(P)$, (here $\pi$ is the normalization),

$$\text{ord}_R f^{d-1} < \text{ord}_R h.$$ 

Thus, $\text{ord}_R f^j < \text{ord}_R h$ for $0 \leq j \leq d-1$. As a result, the images of $1, f, \ldots, f^{d-1}$ in $\mathcal{O}_P/c_P$ are linearly independent. Thus,

$$n_p - \delta_p = \text{dim}_\psi \mathcal{O}_P/c_P \geq d.$$
By proposition 2.2, \( n_P \leq 2\delta_P \), so \( \delta_P \geq d \). On the other hand let

\[ A = \zeta + f\zeta + \ldots + f^{d-1}\zeta + f^d \zeta. \]

By proposition 3.8, it suffices to show that \( A = 0_P \).

Since \( f^d \in c_P \), \( A \leq 0_P \), and so \( \delta_A \geq \delta_P \). But \( \delta_A = d \leq \delta_P \).

Therefore \( \delta_A = \delta_P \), so \( A = 0_P \).

**Proposition 3.10.** Let \( Y \) be an integral projective curve such that there is a degree two morphism \( \phi: Y \to \mathbb{P}^1 \).

Then for all \( P \in Y_{\text{sing}} \), \( \phi^{-1}(\phi(P)) = \{P\} \).

**Proof:** Pick \( P \in Y_{\text{sing}} \). By performing a projective change of coordinates on \( \mathbb{P}^1 \) if necessary, we may assume that \( \phi(P) = 0 \). Let \( \theta: \zeta(T) \to K(Y) \) be the associated field homomorphism with \( \theta(T) = h \). Since \( \phi(P) = 0 \), \( h \in m_P \). Thus \( \text{ord}_P h \geq 2 \). Since \( \phi \) has degree 2, it follows that \( \text{ord}_P h = 2 \), and hence that \( h \notin m_R \) for all \( R \in Y - P \). Thus, \( \phi^{-1}(\phi(P)) = \{P\} \).

**Corollary 3.3.** Let \( Y \) be a quasi-hyperelliptic curve. Then \( Y \) is Gorenstein.

**Proof:** Pick \( P \in Y_{\text{sing}} \). Let \( \phi: Y \to \mathbb{P}^1 \) be a morphism of degree two. By proposition 3.10, \( \phi^{-1}(\phi(P)) = \{P\} \). But this implies that there is an \( h \in \Gamma(Y - P, \mathcal{O}_Y) \) satisfying \( h^{-1} \in 0_P \) and \( \text{ord}_P h^{-1} = 2 \). The corollary now follows from proposition 3.9.

**Corollary 3.4.** Let \( X \) be a quasi-hyperelliptic curve
and let $P \in X_{\text{sing}}$. Then $P$ is a singularity of type IA or type IB.

**Proof:** The corollary follows from corollary 3.3 and proposition 3.8.

**Remarks 3.5.** We assume $P$ is a type I singularity on a Gorenstein curve $X$ with

$$\mathcal{O}_P = \mathfrak{c} + f\mathfrak{c} + \ldots + f^{\delta P-1}\mathfrak{c} + f^{\delta P}\mathfrak{o}_P$$

for some $f \in \mathcal{O}_P$. Then there is a sequence of morphisms

$$Y_0 \to Y_1 \to \ldots \to Y_{\delta P}$$

of Gorenstein curves, and for $i \geq 1$, a sequence of points $P_i$ such that

1. $Y_{\delta P} = X$,
2. $Y_0 \to X$ is the partial normalization at $P$,
3. $P_{\delta P} = P$,
4. $P_i \to P_{i+1}$
5. $\mathcal{O}_P = \mathfrak{c} + f\mathfrak{c} + \ldots + f^{i-1}\mathfrak{c} + f^i\mathfrak{o}_P$.

In particular, for $i = 1, \ldots, \delta P$, the point $P_i$ is a type I singularity. Finally we note that if $I$ is the ideal sheaf of $X$ with support $P$ defined by

$$I_P = \{ h \in \mathcal{O}_P; \text{ord}_P h \geq 2i + 2 \},$$

where $0 \leq i \leq \delta P - 1$, then $Y_i$ is the blowing-up of $X$ with respect to $I$. 
Proposition 3.11. Suppose that $X$ is a Gorenstein curve of arithmetic genus $g$, and that $P \in X_{\text{sing}}$ is a type I singularity. Put $Y = Y_{\delta P^{-1}}$ and $Q = P_{\delta P^{-1}}$, where $Y_{\delta P^{-1}}$ and $P_{\delta P^{-1}}$ are as in remarks 3.5. Assume $\delta P > 1$.

Let

$$W(P) = \delta P g (g-1) + b,$$
$$W(Q) = (\delta P^{-1}) (g-1) (g-2) + c.$$

Then

1. if $P$ is of type IA, then $b = c$,
2. if $P$ is of type IB, then $b = c + g-1$.

Proof: Let $\pi: \tilde{X} \to X$ be the normalization of $X$. Suppose that $t \in K(X)$ satisfies $\text{ord}_R t = 1$ for all $R \in \pi^{-1}(P)$. Let $\sigma_1, \ldots, \sigma_{g-1}$ be a basis for $H^0(Y, \omega_Y)$. Let $f$ be as in remarks 3.5, and for $i = 1, \ldots, g-1$ write

$$\sigma_i = k_i (dt/f^\delta P^{-1})$$
with $k_i \in O_{\tilde{Q}}$. Let $\tau \in H^0(X, \omega_X)$ generate $\omega_{X, P}$, and write $\sigma_i = f_i \tau$ for $f_i \in O_P$ and $i = 1, \ldots, g-1$. Then $\tau, \sigma_1, \ldots, \sigma_{g-1}$ is a basis for $H^0(X, \omega_X)$. We note that $\tau = h(dt/f^\delta P)$ where $h$ is a unit in $O_{\tilde{P}}$, and that $f_i = fh^{-1}k_i$. By remarks 3.5 and proposition 2.11,

$$W(P) = \delta P g (g-1) + \sum_{R \in \pi^{-1}(P)} \text{ord}_R w_t (h^{-1}, h^{-1}f_1, \ldots, h^{-1}f_{g-1}),$$

$$W(Q) = (\delta P^{-1}) (g-1) (g-2) + \sum_{R \in \pi^{-1}(P)} \text{ord}_R w_t (k_1, \ldots, k_{g-1}).$$

We now fix $R \in \pi^{-1}(P)$ and assume $\text{ord}_R k_1 < \ldots < \text{ord}_R k_{g-1}$.
Then
\[
\text{ord}_RW_t(h^{-1}, h^{-1}f_1, \ldots, h^{-1}f_{g-1}) = \\
\text{ord}_RW_t(l, f_1, \ldots, f_{g-1})(h \text{ is a unit in } O_P) = \\
\text{ord}_RW_t(l, h^{-1}f_1, \ldots, h^{-1}f_{g-1}) = \\
g^{-1} \sum_{i=1}^{g-1} (\text{ord}_R(h^{-1}f_i) - i) = \\
(g-1)\text{ord}_Rf + \sum_{i=1}^{g-1} (\text{ord}_Rk_i - (i-1)) - (g-1) = \\
(g-1)(\text{ord}_Rf-1) + \text{ord}_RW_t(k_1, \ldots, k_{g-1}).
\]

If \( P \) is of type IA, then \( \text{ord}_Rf = 1 \) for all \( R \in \pi^{-1}(P) \).

If \( P \) is of type IB, then \( \text{ord}_Rf = 2 \) for \( \pi^{-1}(P) = \{R\} \).

As a result, equation (*) establishes the proposition.

**Corollary 3.5.** Let \( P \) be a type IA singularity with \( \theta: X_1 \to X \) the partial normalization at \( P \) and \( \theta^{-1}(P) = \{Q_1, Q_2\} \). Then
\[
W(P) = \delta_P g(g-1) + W(Q_1) + W(Q_2).
\]

Proof: The corollary follows from propositions 2.16 and 3.11 and induction on \( \delta_P \).

**Corollary 3.6.** Let \( P \) be a type IB singularity with \( \theta: X_1 \to X \) the partial normalization at \( P \) and \( \theta^{-1}(P) = \{Q\} \). Then
\[
W(P) = \delta_P g(g-1) + \frac{1}{2}(2g - \delta_P - 1) \delta_P + W(Q).
\]

Proof: The corollary follows from propositions 2.16 and 3.11 and induction on \( \delta_P \).
Proposition 3.12. Let $X$ be a quasi-hyperelliptic curve of arithmetic genus $g$. The following statements are equivalent for $P \in X_{\text{reg}}$:

1. There is an $h \in \Gamma (X - P, 0_X)$ satisfying $\text{ord}_P h = -2$.
2. $W(P) = \frac{1}{2}g(g-1)$
3. $P$ is a quasi-hyperelliptic point.

Proof:
1) $\Rightarrow$ (2): This follows from corollary 3.1.
2) $\Rightarrow$ (3): This follows from proposition 3.5,(3).

Proposition 3.13. Let $X$ be a quasi-hyperelliptic curve of arithmetic genus $g$, and let $P \in X_{\text{sing}}$ be a singularity of type IB. Then

$$W(P) = \left(\delta_P + \frac{1}{2}\right)g(g-1)$$

Proof: Let $\theta: Y \rightarrow X$ be the partial normalization at $P$ with $\theta^{-1}(P) = \{Q\}$. Let $\phi: X \rightarrow \mathbb{P}^1$ be a morphism of degree two, and put $\psi = \phi \circ \theta$. Then $\psi: Y \rightarrow \mathbb{P}^1$ is a morphism of degree two and $\psi^{-1}(\psi(Q)) = \{Q\}$. By proposition 3.12, therefore, $W(Q) = \frac{1}{2}(g-\delta_P)(g-\delta_P-1)$. By corollary 3.6,

$$W(P) = \delta_P g(g-1) + \frac{1}{2}(2g - \delta_P - 1)\delta_P + \frac{1}{2}(g-\delta_P)(g-\delta_P-1) =$$

$$\left(\delta_P g + \frac{1}{2}\delta_P + \frac{1}{2}(g-\delta_P)\right)(g-1) = (\delta_P + \frac{1}{2})g(g-1).$$

Remark 3.6. Let $\phi: X \rightarrow \mathbb{P}^1$ be a morphism of degree two, let $\pi: \tilde{X} \rightarrow X$ be the normalization of $X$, and let $\psi = \phi \circ \pi$. 


Then $X$ and $\tilde{X}$ are quasi-hyperelliptic. Let $A_1, \ldots, A_d$ be the points of $X$ which satisfy $\psi^{-1}(\psi(A_i)) = \{A_i\}$ for $i = 1, \ldots, d$. Put $V = \{\pi(A_1), \ldots, \pi(A_d)\}$, put $V_1$ equal to the set of type IB Weierstrass points of $X$, and put $V_2$ equal to the set of nonsingular quasi-hyperelliptic points of $X$. Clearly, the fact that $P$ is a nonsingular quasi-hyperelliptic point of $X$ implies that $\pi^{-1}(P)$ is a nonsingular quasi-hyperelliptic point of $\tilde{X}$. Thus, by propositions 3.10 and 3.12, $V = V_1 \cup V_2$.

**Proposition 3.14.** Let $X$ be a quasi-hyperelliptic curve of arithmetic genus $g$ at least two.

1. If $P \in X_{\text{reg}}$ is a Weierstrass point then $W(P) = \frac{1}{2}g(g-1)$.

2. If $R \in X_{\text{sing}}$ is a singularity of type IA, then $W(R) = \delta_R g(g-1)$.

**Proof:** Let $\pi: \tilde{X} \to X$ be the normalization of $X$, and let $g'$ be the genus of $\tilde{X}$. Let $\phi: X \hookrightarrow \mathbb{P}^1$ be a morphism of degree two, and let $\psi = \phi \circ \pi$. Then $\psi: \tilde{X} \to \mathbb{P}^1$ is a morphism of degree two. By proposition 1.15 there are $2g' + 2$ points of $X$, $A_i$ for $i = 1, \ldots, 2g' + 2$ such that $\psi^{-1}(\psi(A_i)) = \{A_i\}$. Put $\pi(A_i) = P_i$, and assume that $P_1', \ldots, P_n$ are the singularities of type IB on $X$. Note that $\phi^{-1}(\phi(P_i)) = \{P_i\}$ for $i = 1, \ldots, 2g' + 2$. By propositions 3.12 and 3.13 therefore,
Let \( R_1, \ldots, R_m \) be the singularities of type IA on \( X \).

By corollary 3.5, for \( i = 1, \ldots, m \), \( W(R_i) \geq \delta_{R_i} g(g-1) \).

Put \( V = \{ P_1, \ldots, P_{2g'+2}, R_1, \ldots, R_m \} \). Then

\[
\sum_{Q \in V} W(Q) \geq (\sum_{R_i} \delta_{R_i} + \sum_{P_j} g' + 1)g(g-1)
\]

But, by proposition 2.1,

\[
g = \sum_{R_i} \delta_{R_i} + \sum_{P_j} g'
\]

Thus,

\[
(\ast) \quad \sum_{Q \in V} W(Q) \geq (g+1)g(g-1)
\]

Clearly equality holds in (\ast) only if \( W(R_i) = \delta_{R_i} g(g-1) \) for \( i = 1, \ldots, m \). Thus (2) is established. Furthermore, \( P \in X \) is a Weierstrass point of \( X \) if and only if \( P \in V \). Hence \( P \in X_{\text{reg}} \) is a Weierstrass point if and only if \( P = P_i \) for \( n+1 \leq i \leq 2g'+2 \). As a result, proposition 3.12 establishes (1).

**Corollary 3.7.** Let \( X \) be a quasi-hyperelliptic curve of arithmetic genus at least two. Then the following statements are equivalent for \( P \in X \).

1. For all morphisms of degree two \( \phi : X \to \mathbb{P}^1 \), \( \phi^{-1}(\phi(P)) = \{P\} \).
2. For some morphism of degree two \( \phi : X \to \mathbb{P}^1 \), \( \phi^{-1}(\phi(P)) = \{P\} \).
3. \( P \) is a Weierstrass point of \( X \).

**Proof:** This is an easy consequence of propositions 3.10
and 3.14.

Remark 3.6. The fact that \( X \) is quasi-hyperelliptic implies that \( X \) is Gorenstein is proved in the last section of \([R]\).

Proposition 3.15. Let \( Y \) be an integral projective curve of arithmetic genus at least two such that \( Y \) has a principal 1-special subscheme of degree two \((Z,I)\). Then \( Y \) is quasi-hyperelliptic. In particular, \( Y \) is Gorenstein.

Proof: If the support of \( P \) is one point, then \( Y \) is quasi-hyperelliptic by proposition 3.7. Thus we assume that the support of \( Z \) is two points \( P \) and \( Q \). Since \( d(Z) = 2 \), \( I_P = m_P \) and \( I_Q = m_Q \). Since \( m_P \) and \( m_Q \) are principal ideals, \( P \) and \( Q \) are nonsingular points. Let \( h \) be a nonconstant element of \( \text{Hom}(I,\mathcal{O}_X) \). Then \( h \) defines a morphism of degree one or two \( \overline{\phi} : Y - \{P,Q\} \to \mathbb{P}^1 \).

Since \( P \) and \( Q \) are nonsingular, \( \overline{\phi} \) extends to a morphism \( \phi : Y \to \mathbb{P}^1 \). Since \( Y \neq \mathbb{P}^1 \), it follows that \( \phi \) has degree two, and so \( Y \) is quasi-hyperelliptic. The last statement follows from corollary 3.3.

Proposition 3.16. Let \( X \) be a Gorenstein curve of arithmetic genus \( g \) at least two and let \( P \in X_{\text{Sing}} \). Then for all \( Q \in X - P \), the degree two subscheme with support \( \{P,Q\}, (Z,I) \), is not 1-special.

Proof: Pick \( h \in \text{Hom}(I,\mathcal{O}_X) \). It suffices to show that \( h \)
is constant. Note that $d(Z) = 2$ implies that $I_P = m_P$ and $I_Q = m_Q$. By proposition 2.19, since $h m_P \subseteq 0_P$, $h \in 0_P$. If $Q \in X_{\text{sing}}$, then again by proposition 2.19, $h \in 0_Q$.

Thus, $h \in \cap_{R \in X} \tilde{0}_R$, and so $h$ is constant.

If $Q \in X_{\text{reg}}$, then $h m_Q \subseteq 0_Q$ implies that $\text{ord}_Q h^{-1} = 0$ or 1. If $\text{ord}_Q h^{-1} = 0$, then again $h \in \bigcap_{R \in X} \tilde{0}_R$, and so $h$ is constant. Therefore, to finish the proof, it suffices to show that $\text{ord}_Q h^{-1} \neq 1$.

Assume that $\text{ord}_Q h^{-1} = 1$. Let $\theta : X_1 \to X$ be the normalization at $P$. Then $h \in \Gamma(X - \{P, Q\}, 0_X)$, $h \in 0_P$, and $\text{ord}_Q h^{-1} = 1$ imply that $h$ defines a degree one map $\phi : X_1 \to \mathbb{P}^1$. Thus $X_1 \cong \mathbb{P}^1$. By performing a projective change of coordinates on $\mathbb{P}^1$ if necessary, we may assume that $\theta(0) = Q$, $\theta(1) = P$, and $\theta(\infty) \neq P$ or $Q$. Let $a = 1$ if $P$ is cuspidal, and let $a$ satisfy $a \neq 1$ and $\theta(a) = P$ if $P$ is not cuspidal. Put $\tau = d\tau/(T-1)(T-a)$. Then $\tau \in H^0(X, \omega)$, and since $\delta_P = g \geq 2$, $\tau$ does not generate $\omega_P$. Then $\sigma = (1-a/T)\tau$ satisfies $\text{ord}_1 \sigma = -1$, and $\text{ord}_b \sigma = 0$ for all $b \neq 1$ in $\theta^{-1}(P)$. Thus, $\sum_{b \in \theta^{-1}(P)} \text{res}_b \sigma \neq 0$, and so $\sigma \notin \omega_P$.

On the other hand, since $\text{ord}_Q h^{-1} = 1$, we may assume that $h = 1/T$. Since $d(Z) = 2$, by theorem 2.3, $\dim H^0(X, I\theta \omega) = g-1$. Thus $\alpha \in H^0(X, \theta \omega)$ is in $H^0(X, I\theta \omega)$ if
and only if \( a \) is not a generator of \( \omega_p \). As a result, 
\[
\tau \in H^0(X, \mathcal{I}_Z \omega_X),
\]
and so \( \tau \in I_F \omega_p \). But this implies \( \sigma \in \omega_p \).

This contradiction shows that \( \text{ord}_a h^{-1} \neq 1 \).

**Proposition 3.17.** Let \( X \) be a Gorenstein curve of arithmetic genus \( g \) at least two. If \( X \) has a degree two \( 1 \)-special subscheme \((Z, \mathcal{I})\), then \( Z \) is principal and \( X \) is quasi-hyperelliptic.

**Proof:** Let \((Z, \mathcal{I})\) be a degree two \( 1 \)-special subscheme. By proposition 3.15, it suffices to show that \( Z \) is principal. If the support of \( Z \) is contained in \( X_{\text{reg}} \), then clearly \( Z \) is principal. Thus we assume that \( P \in X_{\text{sing}} \) is in the support of \( Z \). By proposition 3.16 therefore, the support of \( Z \) equals \( P \).

Let \( h \) be a nonconstant element of \( \text{Hom}(\mathcal{I}, \mathcal{O}_X) \), and let \( \pi: \tilde{X} \rightarrow X \) be the normalization of \( X \). Since \( h \) is nonconstant \( \text{ord}_A h < 0 \) for some \( A \in \pi^{-1}(P) \). Pick \( \sigma \in H^0(X, \mathcal{I} \omega) \) satisfying \( \text{ord}_A \sigma \) is minimal. By theorem 2.3 \( \dim H^0(X, \mathcal{I} \omega) = g-1 \), and so \( \alpha \in H^0(X, \omega) \) is not in \( I_F \omega_p \) if and only if \( \alpha \) generates \( \omega_p \). Since \( \sigma \in I_p \omega_p \), \( h \sigma \in \omega_p \) and hence \( h\sigma \in H^0(X, \omega) \). But \( \text{ord}_A h\sigma < \text{ord}_A \sigma \), and so by the definition of \( \sigma \), \( h\sigma \notin I_F \omega_p \). Thus \( h \sigma \) generates \( \omega_p \). Let \( \sigma = fh \sigma \). Then \( f \in \mathcal{O}_p \) and \( fh = 1 \). Thus given \( k \in I_p \), \( k = (hk)f \). Consequently, \( I_p = f\mathcal{O}_p \), and so \( Z \) is principal.
We now summarize our results on quasi-hyperelliptic curves in the following theorem.

**Theorem 3.3.** Let $Y$ be an integral projective curve. Then the following statements are equivalent.

1. $Y$ is quasi-hyperelliptic.
2. $Y$ is Gorenstein and quasi-hyperelliptic.
3. $Y$ has a principal 1-special subscheme of degree two.
4. $Y$ is Gorenstein and $Y$ has a 1-special subscheme of degree two.

We end this section with one more result on type I singularities.

**Proposition 3.18.** Let $X$ be an integral projective curve and let $P \in X$ be a Gorenstein singularity satisfying $\delta_P = 2$. Then $P$ is a singularity of type I.

**Proof:** Let $\pi : \tilde{X} \to X$ be the normalization of $X$. Let $h$ generate $c_P$ in $\tilde{\mathcal{O}}_P$. Then $\text{ord}_P h = 4$. Let $m_P = f \mathcal{O} + c_P$ where $\text{ord}_R f \leq \text{ord}_R h$ for all $R \in \pi^{-1}(P)$. So $\text{ord}_P f = 1, 2, \text{or } 3$. Since $P$ is a singular point, $\text{ord}_P f \neq 1$.

Let $\tau$ generate $\omega_P$. If $\text{ord}_P f = 3$, then $\sum_{R \in \pi^{-1}(P)} \text{res}_R f \tau \neq 0$. Consequently, $\text{ord}_P f \neq 3$, and so $\text{ord}_P f = 2$. By proposition 3.9 therefore, $P$ is a type I singularity.

**Section 4. Examples**

**Example 3.7.** This is an example of an integral projective curve which has a degree two 1-special subscheme
and which is not quasi-hyperelliptic. Let $X$ be the rational curve of arithmetic genus three defined by

$$X_{\text{sing}} = \{ P \}, \text{ and}$$

$$0_P = \mathcal{O} + T^3 \mathcal{O}[T](T)$$

where $\mathcal{O}(T)$ is the field of rational function for $\mathbb{P}^1$.

Since $P$ is not a Gorenstein singularity, $X$ is not Gorenstein. In particular, $X$ is not quasi-hyperelliptic by corollary 3.3. Let $(Z,I)$ be the subscheme with support $P$ defined by $I_P = T^4 \mathcal{O}[T](T)$. Then $d(Z) = 2$. Since, $1/T \in \Gamma(X-P,0_X)$ and $1/T I_P \subseteq 0_P$, $Z$ is $1$-special.

Example 3.8. Gorenstein curves of arithmetic genus two.

We suppose that $X$ is a Gorenstein curve of arithmetic genus two. By theorems 3.1 and 3.3, $X$ is quasi-hyperelliptic.

Case (1) $X$ is nonsingular. In this case $X$ has six Weierstrass points each of weight 1.

Case (2) $X_{\text{sing}} = \{ P \}$ where $P$ is a node. By proposition 2.15, $W(P) = 2$. Consequently, $X$ has four nonsingular Weierstrass points each of weight 1.

Case (3) $X_{\text{sing}} = \{ Q \}$ where $Q$ is a cusp. By proposition 2.16, $W(P) = 3$. Consequently, $X$ has three nonsingular Weierstrass points each of weight 1.

Case (4) $X_{\text{sing}} = \{ P_1, P_2 \}$ where $P_1$ and $P_2$ are both nodes. By proposition 2.15, $W(P_1) = W(P_2) = 2$. Consequently, $X$
has two nonsingular Weierstrass points each of weight 1.

Case (5) \( X_{\text{sing}} = \{P, Q\} \) where \( P \) is a node and \( Q \) is a cusp. By proposition 2.15, \( W(P) = 2 \), and by proposition 2.16, \( W(Q) = 3 \). Consequently, \( X \) has one nonsingular Weierstrass point of weight one.

Case (6) \( X_{\text{sing}} = \{Q_1, Q_2\} \) where \( Q_1 \) and \( Q_2 \) are cusps. By proposition 2.16, \( W(Q_1) = W(Q_2) = 3 \). Consequently, \( X \) has no nonsingular Weierstrass points.

Case (7) \( X_{\text{sing}} = \{R\} \), where \( \delta_R = 2 \) and \( R \) is a singularity of type IA. By proposition 3.14, \( W(R) = 4 \). Consequently, \( X \) has two nonsingular Weierstrass points each of weight 1.

Case (8) \( X_{\text{sing}} = \{A\} \), where \( \delta_A = 2 \) and \( A \) is a singularity of type IB. By proposition 3.13, \( W(A) = 5 \). Consequently, \( X \) has one nonsingular Weierstrass point of weight 1.

Example 3.9. Let \( X \) be the rational nodal curve of arithmetic genus three such that \( X_{\text{sing}} = \{P_1, P_2, P_3\} \), and the normalization \( \pi: \mathbb{P}^1 \to X \) satisfies

\[
\pi^{-1}(P_1) = \{i, -i\}, \quad \pi^{-1}(P_2) = \{0, \infty\}, \quad \text{and} \quad \pi^{-1}(P_3) = \{1, -1\}.
\]

Let \( \theta: X_1 \to X \) be the partial normalization at \( P_1 \), and for \( i = 2, 3 \), put \( \theta(Q_i) = P_i \). Let \( \pi_1: \mathbb{P}^1 \to X_1 \) be the
normalization of \( X_1 \). Note that \( \pi_1^{-1}(Q_2) = 0 \), and 
\( \pi_1^{-1}(Q_3) = \{1, -1\} \). Therefore, the formula in example 2.1 implies that \( \pi_1(i) \) and \( \pi_1(-i) \) are nonsingular Weierstrass points of \( X_1 \). Thus, by proposition 2.15, 
\( W(P_1) = 8 \). Furthermore, \( P_1 \) is the exceptional node described in theorem 3.2, (3), so \( X \) does not have a principal l-special subscheme with support \( P_1 \) and degree at most three.

Let \( \alpha \) be the automorphism of \( \mathbb{P}^1 \) defined by \( T \rightarrow iT \), and let \( \beta \) be the automorphism of \( \mathbb{P}^1 \) defined by \( T \rightarrow i(T-1)/(T+1) \). Then 
\[
\alpha(\{1,-1\}) = \{i,-i\}, \quad \alpha(\{0,\infty\}) = \{0,\infty\}, \\
\alpha(\{i,-i\}) = \{1,-1\}, \quad \beta(\{0,\infty\}) = \{i,-i\}, \\
\beta(\{i,-i\}) = \{1,-1\}, \quad \text{and} \quad \beta(\{1,-1\}) = \{0,\infty\}.
\]

Thus, by performing a suitable change of coordinates on \( \mathbb{P}^1 \), we conclude that for \( i = 1,2,3 \), \( W(P_i) = 8 \), and there does not exist a principal l-special subscheme with support \( P_i \) of degree at most three.

Moreover, since \( W(P_1) + W(P_2) + W(P_3) = 24 \), \( X \) has no nonsingular Weierstrass points. Thus there does not exist a morphism \( \phi: X \rightarrow \mathbb{P}^1 \) of degree at most three such that for some \( P \in X \), \( \phi^{-1}(\phi(P)) = \{P\} \).

**Example 3.10.** In this example we construct for all \( g \) at least equal to two a rational quasi-hyperelliptic curve of arithmetic genus \( g \) that has exactly two
Weierstrass points, one singular and one nonsingular.

Suppose that $X$ is the rational quasi-hyperelliptic curve of arithmetic genus $g$ at least two defined by

$$X_{\text{sing}} = \{P\},$$

and

$$0_P = \phi + T^2\phi + \ldots + T^{2g-2}\phi + T^{2g}\phi[T](T)$$

where $\phi(T)$ is the field of rational functions for $X$.

We note that there is a commutative diagram

$$\begin{array}{ccc}
P^1 & \xrightarrow{\phi} & P^1 \\
\pi \downarrow & & \downarrow \\
X & \xrightarrow{\phi} & P^1 \\
\end{array}$$

where $\pi$ is the normalization, and $\phi$ is defined by $T \mapsto T^2$. Since $\phi$ has degree two, $X$ is quasi-hyperelliptic.

By corollary 3.7, $P$ and $\pi(\infty) = Q$ are the only Weierstrass points of $X$. Furthermore,

$$W(P) = (g + \frac{1}{2})g(g-1) \quad \text{and} \quad W(Q) = \frac{1}{2}g(g-1).$$

At this point we conjecture that all Gorenstein curves of arithmetic genus at least two have at least two Weierstrass points. We note that the conjecture is true for nonsingular curves, and for curves with more than one singularity. We further note that the conjecture is true for a curve $X$ satisfying $X_{\text{sing}}$ consists of one point $P$, and the preimage of $P$ in the normalization contains at most two points. For then it is easy to show that $W(P) \leq \delta_p g(g-1) + (g-1)^2$. 
Since \( g_p(g-1) + (g-1)^2 < (g+1)(g)(g-1) \), \( X \) has a Weierstrass point other than \( P \).

**Example 3.11.** In this example we construct a cuspidal singularity \( P \) which does not satisfy condition (B) of proposition 3.6.

Suppose that \( X_1 \) is a Gorenstein curve of arithmetic genus two such that \( X_1 \) has a nonsingular Weierstrass point \( Q \). Let \( h \in \mathbb{K}(X_1) \) satisfy

\[
h \in (X_1 - Q, O_Q) \quad \text{and} \quad \text{ord}_Q h^{-1} = 2, \quad \text{and let} \quad t \in \mathbb{K}(X_1),
\]

satisfy \( \text{ord}_Q t = 1 \). Now let \( X \) be the Gorenstein curve of arithmetic genus 5 with \( P \in X_{\text{sing}} \) defined by

\[
\theta : X_1 \rightarrow X \quad \text{is the partial normalization at} \quad P, \quad \text{and}
\]

\[
0_p = \mathfrak{C} + th^{-1} \mathfrak{C} + (1+t)h^{-2} \mathfrak{C} + h^{-3} O_Q
\]

Then there does not exist a principal \( l \)-special subscheme of \( X \) with support \( P \) and degree at most 5.

Assume that \( (Z, I) \) is such a subscheme. Put \( I_p = f_0 \). Note that for all \( k \in \mathbb{N} \), \( \text{ord}_p k \neq 1, 2, \) or 5. Therefore, \( \text{ord}_p f = 3 \) or 4. Thus, \( d(Z) \) is 3 or 4.

If \( d(Z) = 3 \), then,

\[
f = ath^{-1} + b(1+t)h^{-2} + rh^{-3} \quad \text{for} \quad a, b \in \mathbb{C}, \quad a \neq 0,
\]

and \( r \in 0_Q \). If \( s \in \text{Hom}(I, 0_X) \), then \( s = c_1 + c_2 h \) for \( c_1, c_2 \in \mathbb{C} \). Then \( sf \in 0_p \) implies \( c_2 at + k_1 h^{-1} \in 0_p \) for some \( k_1 \in 0_Q \). But this implies that \( c_2 = 0 \). Thus, \( \text{Hom}(I, 0_X) = \mathbb{C} \), and so there does not exist a principal \( l \)-special subscheme of \( X \) with support \( P \) of degree 3.
If $d(Z) = 4$, then

$$f = b(1+t)h^{-2} + rh^{-3},$$

for $b \in \mathbb{C} - 0$, and $r \in \mathcal{O}_Q$.

If $s \in \text{Hom}(I, \mathcal{O}_X)$, then $s = c_1 + c_2 h^{-1} + c_3 h^{-2}$ for $c_1, c_2, c_3 \in \mathbb{C}$. Again, $sf \in \mathcal{O}_P$ implies that $c_3 b(1+t) + k_2 h^{-1} \in \mathcal{O}_P$ for some $k_2 \in \mathcal{O}_Q$. This implies that $c_3 = 0$, which in turn implies that $sf = c_2 bh^{-1} + k_3 th^{-1}$ for some $k_3 \in \mathcal{O}_Q$. But this implies that $c_2 = 0$. Thus, \( \text{Hom}(I, \mathcal{O}_X) = \emptyset \), and so there does not exist a principal $1$-special subscheme of $X$ with support $P$ of degree $4$. 
Bibliography


[D] Diaz, S., Exceptional Weierstrass points and the divisor on moduli space that they define, preprint, 1983.


[K] Kleiman, S., r-Special subschemes and an argument of Severi's, Advances in Math. 22(1976), 1-23.


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