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Decay of Coherence and Entanglement of a Superposition State for a Continuous Variable System in an Arbitrary Heat Bath

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Abstract

We consider the case of a pair of particles initially in a superposition state corresponding to a separated pair of wave packets. We calculate *exactly* the time development of this non-Gaussian state due to interaction with an *arbitrary* heat bath. We find that coherence decays continuously, as expected. We then investigate entanglement and find that at a finite time the system becomes separable (not entangled). Thus, we see that entanglement sudden death is also prevalent in continuous variable systems which should raise concern for the designers of entangled systems.

For continuous variable systems "entanglement sudden death" [1], that is, complete termination of entanglement after a finite time interval, has been demonstrated for the special case of a pair of particles in a Gaussian state[2]. Those authors use a master equation and the necessary and sufficient criterion for separability of such states developed by Duan et al. [3]. Here, we present a more general model by considering the case of a widely separated pair of particles initially in a superposition state corresponding to a displaced pair of wave packets. We use a method that allows us to calculate *exactly* the time development due to interaction with an *arbitrary* linear passive heat bath [4]. We find first of all that coherence, defined as the relative amplitude of the interference pattern, decays continuously but very rapidly. Next we consider entanglement and find that after a finite time the system becomes separable, showing that "sudden death" of entanglement occurs for this system as well.

The method is based on the general prescription described in an earlier publication [4] in which a system is put in an initial state by a measurement applied to the equilibrium state and after a finite time is sampled by a second measurement. A key formula is the expression for the Wigner characteristic function given in Eq. (6.5) of [4]. For a two particle system, this formula takes the form:

$$\tilde{W}(Q_1, P_1; Q_2, P_2) = \frac{\langle f^\dagger(1)e^{-i(x_1(t)P_1+m\dot{x}_1(t)Q_1+x_2(t)P_2+m\dot{x}_2(t)Q_2)/\hbar}f(1)\rangle}{\langle f^\dagger(1)f(1)\rangle}, \quad (1)$$

where the initial measurement is described by

$$f(1) = f(x_1(0) - x_1, x_2(0) - x_2), \quad (2)$$

in which $f(x_1, x_2)$ is the c-number function describing the initial measurement while $x_1(t)$ and $x_2(t)$ are the time-dependent Heisenberg operators corresponding to the displacement of either particle:

$$x_j(t) = e^{iHt/\hbar}x_j(0)e^{-iHt/\hbar}. \quad (3)$$

Finally, in this formula the brackets indicate expectation with respect to the state of the system in equilibrium at temperature T ,

$$\langle O \rangle = \frac{\text{Tr} \left\{ O e^{\frac{H}{kT}} \right\}}{\text{Tr} \left\{ e^{\frac{H}{kT}} \right\}}. \quad (4)$$

Here we emphasize that in Eqs. (3) and (4) H is the Hamiltonian operator for the *system* of the pair of particles interacting with the heat bath.

In order to evaluate this formula we make the key assumptions that particles are linear oscillators coupled to a linear passive heat bath and that within the bath the particles are *widely separated* so that we may ignore bath-induced interactions. These assumptions imply that $x_1(t)$ and $x_2(t)$ independently undergo quantum Brownian motion. We can now repeat the discussion leading to Eq. (6.43) of our earlier publication,[4] to obtain

$$\begin{aligned}
& \langle f^\dagger(1) e^{-i(x_1(t)P_1 + m\dot{x}_1(t)Q_1 + x_2(t)P_2 + m\dot{x}_2(t)Q_2)/\hbar} f(1) \rangle \\
&= \exp\left\{-\sum_{n=1}^2 \frac{\langle x^2 \rangle (P_n^2 - K_n^2) + m^2 \langle \dot{x}^2 \rangle Q_n^2}{2\hbar^2}\right\} \\
&\quad \times \int_{-\infty}^{\infty} dx'_1 \int_{-\infty}^{\infty} dx'_2 f^\dagger\left(x'_1 + \frac{L_1}{2}, x'_2 + \frac{L_2}{2}\right) f\left(x'_1 - \frac{L_1}{2}, x'_2 - \frac{L_2}{2}\right) \\
&\quad \times \frac{1}{2\pi \langle x^2 \rangle} \exp\left\{-\sum_{n=1}^2 \frac{(x_n + x'_n)^2}{2 \langle x^2 \rangle} - i(x_n + x'_n) \frac{K_n}{\hbar}\right\}, \tag{5}
\end{aligned}$$

where $\langle x^2 \rangle$ and $\langle \dot{x}^2 \rangle$ are the mean squares of the displacement and velocity, the same for either particle, and we have introduced

$$K_n = \frac{cP_n + m\dot{c}Q_n}{\langle x^2 \rangle}, \quad L_n = GP_n + m\dot{G}Q_n. \tag{6}$$

Here $G = G(t)$ is the Green function and $c = c(t) \equiv \frac{1}{2}\langle x(t)x(0) + x(0)x(t) \rangle$ is the correlation function, again the same for either particle.

These expressions are valid for any measurement function. We now specialize to the case of a pair of particles initially in a superposition state corresponding to a separated pair of wave packets, with measurement function of the form:

$$\begin{aligned}
f(x_1, x_2) &= \frac{1}{\sqrt{4\pi\sigma^2(1 + e^{-d^2/4\sigma^2})}} \left[\exp\left\{-\frac{(x_1 - d/2)^2 + (x_2 + d/2)^2}{4\sigma^2}\right\} \right. \\
&\quad \left. + \exp\left\{-\frac{(x_1 + d/2)^2 + (x_2 - d/2)^2}{4\sigma^2}\right\} \right]. \tag{7}
\end{aligned}$$

Here we emphasize that the wave packet separation d is arbitrary and should not be confused with the separation of the particles in the bath, which is large.

With this measurement function the integrals in the expression (5) are standard Gaussian

[4]. Putting the result in the expression (1) for the Wigner characteristic function we find

$$\begin{aligned}
& \tilde{W}(Q_1, P_1; Q_2, P_2; t) \\
= & \exp \left\{ - \sum_{n=1}^2 \frac{\langle x^2 \rangle \left(P_n^2 - \frac{\langle x^2 \rangle K_n^2}{\langle x^2 \rangle + \sigma^2} \right) + m^2 \langle \dot{x}^2 \rangle Q_n^2 + \frac{\hbar^2}{4\sigma^2} L_n^2}{2\hbar^2} \right\} \\
& \times \frac{\cos \frac{\langle x^2 \rangle (K_1 - K_2) d}{2\hbar(\langle x^2 \rangle + \sigma^2)} + \exp \left\{ - \frac{\langle x^2 \rangle d^2}{4\sigma^2(\langle x^2 \rangle + \sigma^2)} \right\} \cosh \left\{ \frac{(L_1 - L_2) d}{4\sigma^2} \right\}}{1 + \exp \left\{ - \frac{\langle x^2 \rangle d^2}{4\sigma^2(\langle x^2 \rangle + \sigma^2)} \right\}}. \tag{8}
\end{aligned}$$

where in order to center the state at the origin we have put $x_1 = x_2 = 0$.

This expression becomes simpler in the free particle limit : $\langle x^2 \rangle \rightarrow \infty$. In this limit

$$\begin{aligned}
& \tilde{W}(Q_1, P_1; Q_2, P_2; t) \\
= & \exp \left\{ - \frac{A_{11}(P_1^2 + P_2^2) + 2A_{12}(Q_1 P_1 + Q_2 P_2) + A_{22}(Q_1^2 + Q_2^2)}{2\hbar^2} \right\} \\
& \times \frac{\cos \frac{(P_1 - P_2) d}{2\hbar} + \exp \left\{ - \frac{d^2}{4\sigma^2} \right\} \cosh \left\{ \frac{[G(P_1 - P_2) + m\dot{G}(Q_1 - Q_2)] d}{4\sigma^2} \right\}}{1 + \exp \left\{ - \frac{d^2}{4\sigma^2} \right\}}, \tag{9}
\end{aligned}$$

in which we have introduced

$$\begin{aligned}
A_{11} &= \sigma^2 + s + \frac{\hbar^2 G^2}{4\sigma^2}, \\
A_{12} &= \frac{m\dot{s}}{2} + \frac{\hbar^2 m\dot{G}G}{4\sigma^2}, \\
A_{22} &= m^2 \langle \dot{x}^2 \rangle + \frac{\hbar^2 m^2 \dot{G}^2}{4\sigma^2}. \tag{10}
\end{aligned}$$

In these expressions $s = 2(\langle x^2 \rangle - c) = \langle (x(t) - x(0))^2 \rangle$ is the mean square displacement and as above G is the Green function.

The Wigner function is the inverse Fourier transform of the Wigner characteristic function:

$$\begin{aligned}
W(q_1, p_1; q_2, p_2; t) &= \frac{1}{2(1 - e^{-d^2/4\sigma^2})} \left[W_0(q_1 - \frac{d}{2}, p_1; t) W_0(q_2 + \frac{d}{2}, p_2; t) \right. \\
&+ W_0(q_1 + \frac{d}{2}, p_1; t) W_0(q_2 - \frac{d}{2}, p_2; t) \\
&\left. + 2e^{-A(t)} W_0(q_1, p_1; t) W_0(q_2, p_2; t) \cos \Phi(q_1 - q_2, p_1 - p_2; t) \right]. \tag{11}
\end{aligned}$$

Here W_0 is the Wigner function for a single particle wave packet,

$$W_0(q, p; t) = \frac{1}{2\pi \sqrt{A_{11}A_{22} - A_{12}^2}} \exp \left\{ - \frac{A_{22}q^2 - 2A_{12}qp + A_{11}p^2}{2(A_{11}A_{22} - A_{12}^2)} \right\}, \tag{12}$$

while the phase Φ is given by

$$\Phi(q, p; t) = \frac{(GA_{22} - m\dot{G}A_{12})q + (m\dot{G}A_{11} - GA_{12})p}{A_{11}A_{22} - A_{12}^2} \frac{\hbar d}{4\sigma^2} \quad (13)$$

and the quantity A by

$$A(t) = \frac{(A_{11} - \frac{\hbar^2 G^2}{4\sigma^2})(A_{22} - \frac{\hbar^2 m^2 \dot{G}^2}{4\sigma^2}) - (A_{12} - \frac{\hbar^2 m G \dot{G}}{4\sigma^2})^2}{A_{11}A_{22} - A_{12}^2} \frac{d^2}{4\sigma^2}. \quad (14)$$

We note that each of the first two terms in brackets in the expression (8) for the Wigner function corresponds to the product of independently propagating packets. We call these the direct terms. The third term is an interference term. We emphasize that we have assumed that the particles are widely separated within the bath so there is no coupling between them. The presence of this interference term is therefore a purely quantum mechanical phenomenon.

The Wigner function is a quasiprobability distribution, not directly observable. A physical observable is the probability distribution, obtained by integrating over the momentum variables:

$$\begin{aligned} P(q_1, q_2; t) = & \frac{1}{2(1 - e^{-d^2/4\sigma^2})} \left[P_0(q_1 - \frac{d}{2}, t)P_0(q_2 + \frac{d}{2}, t) \right. \\ & + P_0(q_1 + \frac{d}{2}, t)P_0(q_2 - \frac{d}{2}, t) \\ & \left. + 2a(t) \exp\left\{-\frac{d^2}{4A_{11}}\right\} P_0(q_1; t)P_0(q_2; t) \cos\left\{\frac{\hbar G d (q_1 - q_2)}{4A_{11}\sigma^2}\right\} \right]. \quad (15) \end{aligned}$$

Again, the first two terms are direct terms corresponding to independently propagating wave packets with

$$P_0(q; t) = \frac{1}{\sqrt{2\pi A_{11}}} \exp\left\{-\frac{q^2}{2A_{11}}\right\}, \quad (16)$$

the probability distribution for a single wave packet centered at the origin. The third term is an interference term. Viewed in the (q_1, q_2) plane, the direct terms are seen as a pair of peaks centered at $(q_1, q_2) = (\frac{d}{2}, -\frac{d}{2})$ and $(q_1, q_2) = (-\frac{d}{2}, \frac{d}{2})$ and spreading in time as the width A_{11} increases. The interference term is seen as a spreading peak centered at the origin and modulated by the cosine term. The quantity $a(t)$ is the ratio of the geometric mean of the direct term to the factor multiplying the cosine in the interference term and is therefore a measure of the visibility of the interference. We find

$$a(t) = \exp\left\{-\frac{s(t)}{\sigma^2 + s(t) + \frac{\hbar^2 G^2(t)}{4\sigma^2}} \frac{d^2}{4\sigma^2}\right\}. \quad (17)$$

This quantity is initially unity and, for d large, diminishes rapidly to a very small asymptotic value. This is the familiar phenomenon of decoherence of a superposition state. But nevertheless interference is present for all times, albeit with a small amplitude. Our point here is that there is no sudden death of coherence as indicated by the presence of the interference term.

We turn now to the question of entanglement. A two-particle state described by a density matrix ρ is said to be separable (not entangled) if and only if ρ can be expressed in the form

$$\rho = \sum_j p_j \rho_j(1) \rho_j(2), \quad (18)$$

in which $\rho_j(1)$ and $\rho_j(2)$ are projection operators into states of particles 1 and 2, respectively, and the p_j are positive. In our case we seek to express the density matrix elements in the form

$$\langle x'_1, x'_2 | \rho | x_1, x_2 \rangle = \int d^2\alpha_1 \int d^2\alpha_2 P(\alpha_1, \alpha_2) \phi_{\alpha_1}(x'_1) \phi_{\alpha_1}^*(x_1) \phi_{\alpha_2}(x'_2) \phi_{\alpha_2}^*(x_2), \quad (19)$$

where the ϕ 's are what we might call strong form coherent wave functions:

$$\phi_\alpha(x) = (2\pi\sigma_0^2)^{-1/4} \exp \left\{ -\frac{1-i\delta_0}{4\sigma_0^2} (x - \bar{x})^2 + \frac{i\bar{p}x}{\hbar} - i\frac{\bar{x}\bar{p}}{2\hbar} \right\}, \quad (20)$$

with the state labelled with the complex number

$$\alpha = \frac{1-i\delta_0}{2\sigma_0} \bar{x} + i\frac{\sigma_0}{\hbar} \bar{p}, \quad d^2\alpha = \frac{d\bar{x}d\bar{p}}{2\hbar}. \quad (21)$$

This is clearly of the form (18) with the sum replaced by an integral, so if this expansion exists and $P(\alpha_1, \alpha_2)$ is everywhere positive the state is separable. The expression (19) is reminiscent of the Glauber-Sudarshan P -representation, [6] but in that representation the ϕ 's are coherent states, which are expressed in terms of the ground state of an oscillator, or equivalently a minimum uncertainty state, [5] shifted in position and momentum. If in the wave function (20) we set the parameter δ_0 equal to zero we have such a coherent wave function. On the other hand, if δ_0 is not zero, the wave function (20) minimizes the strong form of the uncertainty relation: [7, 8]

$$\langle (x - \bar{x})^2 \rangle \langle (p - \bar{p})^2 \rangle - \left\langle \frac{(x - \bar{x})(p - \bar{p}) + (p - \bar{p})(x - \bar{x})}{2} \right\rangle^2 \geq \frac{\hbar^2}{4}. \quad (22)$$

It is not difficult to show that wave function (20) satisfies this as an equality.

Next, we recall the relation between the Wigner characteristic function and the density function matrix elements:

$$\tilde{W}(Q_1, P_1; Q_2, P_2) = \int dq_1 \int dq_2 e^{-i(q_1 P_1 + q_2 P_2)/\hbar} \left\langle q_1 - \frac{Q_1}{2}, q_2 - \frac{Q_2}{2} \middle| \rho \middle| q_1 + \frac{Q_1}{2}, q_2 + \frac{Q_2}{2} \right\rangle. \quad (23)$$

Using the expansion (19) of the density matrix elements, this becomes

$$\begin{aligned} \tilde{W}(Q_1, P_1; Q_2, P_2) &= \int d^2\alpha_1 \int d^2\alpha_2 P(\alpha_1, \alpha_2) \int dq_1 \phi_{\alpha_1}(q_1 - \frac{Q_1}{2}) \phi_{\alpha_1}^*(q_1 + \frac{Q_1}{2}) e^{-iq_1 P_1/\hbar} \\ &\quad \times \int dq_2 \phi_{\alpha_2}(q_2 - \frac{Q_2}{2}) \phi_{\alpha_2}^*(q_2 + \frac{Q_2}{2}) e^{-iq_2 P_2/\hbar}. \end{aligned} \quad (24)$$

With the explicit form (20) of the coherent state, we see that

$$\int dq \phi_{\alpha}(q - \frac{Q}{2}) \phi_{\alpha}^*(q + \frac{Q}{2}) e^{-iqP/\hbar} = e^{-i\frac{\bar{p}Q + \bar{x}P}{\hbar}} \exp \left\{ -\frac{\sigma_0^2 P^2}{2\hbar^2} - \frac{\delta_0 QP}{2\hbar} - \frac{(1 + \delta_0^2) Q^2}{8\sigma_0^2} \right\}. \quad (25)$$

Therefore, the expression (24) can be written

$$\begin{aligned} &\int d^2\alpha_1 \int d^2\alpha_2 P(\alpha_1, \alpha_2) e^{-i(\bar{p}_1 Q_1 + \bar{x}_1 P_1 + \bar{p}_2 Q_2 + \bar{x}_2 P_2)/\hbar} \\ &= \exp \left\{ \sum_{j=1,2} \frac{\sigma_0^2 P_j^2 + \hbar \delta_0 Q_j P_j + \frac{\hbar^2(1+\delta_0^2)}{4\sigma_0^2} Q_j^2}{2\hbar^2} \right\} \tilde{W}(Q_1, P_1; Q_2, P_2). \end{aligned} \quad (26)$$

This is just the Fourier transform of the P -function, which will exist if the inverse transform exists. From an inspection of the Wigner characteristic function (9) for our superposition state, we see that convergence of the inverse transform will be dominated by the exponential factors and will therefore exist if the quadratic form

$$\begin{pmatrix} P & Q \end{pmatrix} \begin{pmatrix} A_{11} - \sigma_0^2 & A_{12} - \frac{\hbar\delta_0}{2} \\ A_{12} - \frac{\hbar\delta_0}{2} & A_{22} - \frac{\hbar^2(1+\delta_0^2)}{4\sigma_0^2} \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} \quad (27)$$

is positive definite. Since the parameters σ_0 and δ_0 are arbitrary we can first choose δ_0 to make this quadratic form diagonal and then choose σ_0 to minimize the product of the diagonal elements. The corresponding optimum values are

$$(\delta_0)_{\text{opt}} = \frac{2A_{12}}{\hbar}, \quad (\sigma_0^2)_{\text{opt}} = \sqrt{\frac{(\hbar^2 + 4A_{12}^2) A_{11}}{4A_{22}}}. \quad (28)$$

With this choice we find for the diagonal elements that the diagonal elements of the quadratic

form (27) are given by

$$\begin{aligned}\tilde{A}_{11} &= \sqrt{\frac{A_{11}}{A_{22}}} \left(\sqrt{A_{11}A_{22}} - \sqrt{A_{12}^2 + \frac{\hbar^2}{4}} \right), \\ \tilde{A}_{22} &= \sqrt{\frac{A_{22}}{A_{11}}} \left(\sqrt{A_{11}A_{22}} - \sqrt{A_{12}^2 + \frac{\hbar^2}{4}} \right).\end{aligned}\quad (29)$$

It is not difficult to see that these are positive at all times. Thus the expansion (19) exists at all times.

Next we consider the positivity of $P(\alpha_1, \alpha_2)$. With the optimum values (28) of the parameters in (26) we form the inverse Fourier transform. The integrals are all standard Gaussian and the result can be written in the form

$$\begin{aligned}P(\alpha_1, \alpha_2) &= \frac{\hbar^2 \exp \left\{ -\frac{\bar{p}_1^2 + \bar{p}_2^2}{2A_{22}} - \frac{\bar{x}_1^2 + \bar{x}_2^2}{4\tilde{A}_{11}} - \left(1 - \frac{\hbar^2 G^2}{4\sigma^2 \tilde{A}_{11}} - \frac{\hbar^2 \sigma^2 m^2 \dot{G}^2}{4\tilde{A}_{22}} \right) \frac{d^2}{4\sigma^2} \right\}}{\pi^2 \tilde{A}_{11} \tilde{A}_{22} (1 + \exp \left\{ -\frac{d^2}{4\sigma^2} \right\})} \\ &\left[\exp \left\{ \left(1 - \frac{\hbar^2 G^2}{4\sigma^2 \tilde{A}_{11}} - \frac{\hbar^2 \sigma^2 m^2 \dot{G}^2}{4\tilde{A}_{22}} - \frac{\sigma^2}{\tilde{A}_{11}} \right) \frac{d^2}{4\sigma^2} \right\} \cosh \frac{(\bar{x}_1 - \bar{x}_2) d}{2\tilde{A}_{11}} \right. \\ &\quad \left. + \cos \left(\frac{\hbar G d (\bar{x}_1 - \bar{x}_2)}{4\sigma^2 \tilde{A}_{11}} + \frac{\hbar m \dot{G} d (\bar{p}_1 - \bar{p}_2)}{4\sigma^2 \tilde{A}_{22}} \right) \right].\end{aligned}\quad (30)$$

The first line in Eq. (30) is a positive factor, so $P(\alpha_1, \alpha_2)$ is positive if the remaining factor is positive. Clearly this will be the case for all $(\bar{x}_1, \bar{p}_1, \bar{x}_2, \bar{p}_2)$ if and only if

$$C(t) \equiv 1 - \frac{\hbar^2 G^2}{4\sigma^2 \tilde{A}_{11}} - \frac{\hbar^2 m^2 \dot{G}^2}{4\sigma^2 \tilde{A}_{22}} - \frac{\sigma^2}{\tilde{A}_{11}} > 0. \quad (31)$$

At short times $G(t) \cong t/m$ and $s(t) \cong \langle \dot{x}^2 \rangle t^2$. With the expressions (10) for A_{11} , A_{12} and A_{22} and these in turn in the expressions (29) for \tilde{A}_{11} and \tilde{A}_{22} we find

$$C(0) = -\frac{\sqrt{U} + 1}{U - \sqrt{U}} = -\frac{1 + (1 + 4\sigma^2/\bar{\lambda}^2)^{-1/2}}{(1 + 4\sigma^2/\bar{\lambda}^2)^{1/2} - 1}, \quad (32)$$

where $\bar{\lambda} = \hbar/m\sqrt{\langle \dot{x}^2 \rangle}$ is the deBroglie wavelength. Not surprisingly $C(0)$ is always negative, since the initial state is formed with a projection operator (7) corresponding to a necessarily entangled state.

At very long times, the behavior of $G(t)$ and $s(t)$ depends upon the bath parameters.[9]

As an illustration we consider the Ohmic model for which at long times

$$\begin{aligned}
 G(t) &\sim \zeta^{-1}, \\
 s(t) &\sim \frac{2\hbar}{\pi\zeta} \log \frac{\zeta t}{m}, \quad T = 0, \\
 s(t) &\sim \frac{2kT}{\zeta} t, \quad T > 0,
 \end{aligned} \tag{33}$$

where ζ is the Ohmic friction constant. With this it is easy to see that for this Ohmic case at long times $C(t) \sim 1$. Clearly there must be an intermediate time at which $C(t)$ changes sign and the state becomes separable. For example, in Fig. 1 we plot $C(t)$ versus γt for the single relaxation time (τ) model [4] at zero temperature, where $\tau = \gamma^{-1}/6$ and γ is the Ohmic relaxation time. There we see that the change of sign occurs at $\gamma t \approx 6$. In general, most other bath models (that is, models with colored noise [9]) show similar behavior.

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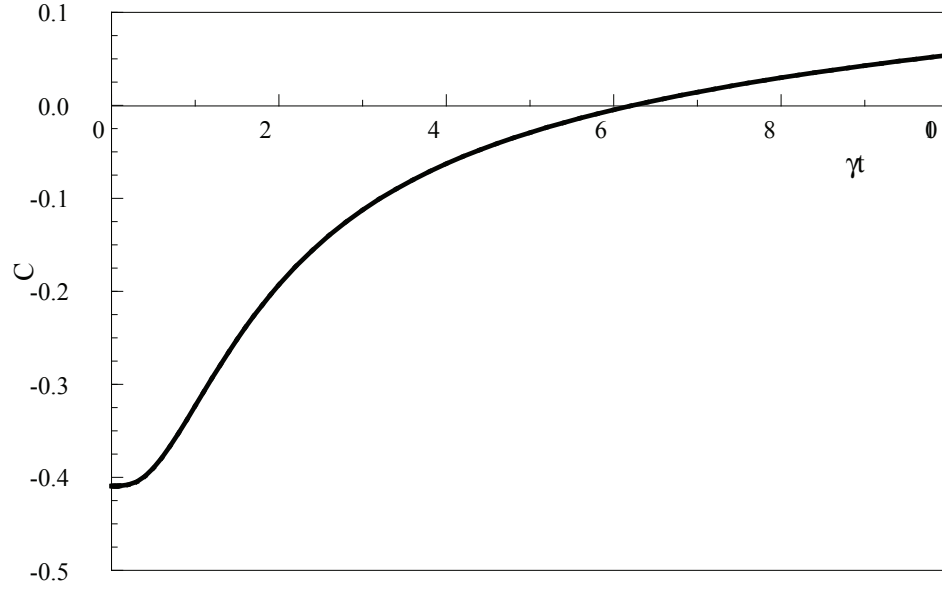


FIG. 1: $C(t)$ versus γt for the single relaxation time (τ) model at zero temperature, where $\tau = \gamma^{-1}/6$ and γ is the Ohmic relaxation time. We note that separability occurs for $\gamma t \gtrsim 6$.