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The (nonrelativistic) Abraham-Lorentz equation of motion for a point electron, while suffering from runaway solutions and an acausal response to external forces, is compatible with the optical theorem. We show that a non-relativistic theory of radiative reaction that allows for a finite charge distribution is not only causal and free of runaway solutions, but is also consistent with the optical theorem and the standard formulas for the Rayleigh and Thomson scattering cross sections.

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The nonrelativistic theory of radiative reaction leading to the Abraham-Lorentz (AL) equation [1], while exhibiting such notorious features as runaway solutions and preacceleration, is nevertheless consistent with the optical theorem and the Rayleigh scattering cross section. One approach to the resolution of the problems besetting that theory is based on the quantum Langevin equation describing dissipative quantum systems [2], together with the assumption that the electron is not a point particle but is described by a form factor with a very high cutoff frequency [3, 4]. The classical, nonrelativistic equation of motion for an electron in this theory is free of preacceleration and runaway difficulties [5], and we show in this Brief Report that it is also consistent with the optical theorem and the Rayleigh and Thomson cross sections.

We first recall some basic aspects of the linear response of an electron, described as a rigid spherically symmetric charge distribution $\rho(\mathbf{r})$ centered at $\mathbf{R}(t)$, to an applied, sufficiently small electric field

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0 e^{\mathrm{i}(\mathbf{k}_0\cdot\mathbf{r}-\omega t)} \quad (|\mathbf{k}_0| = k = \omega/c). \tag{1}$$

The force exerted on the electron by this field is

$$\begin{aligned}\mathbf{F}(t) &= \int d^3r\rho(\mathbf{r}-\mathbf{R}(t))\mathbf{E}(\mathbf{r},t)\\ &= \int d^3r\rho(\mathbf{r})\mathbf{E}(\mathbf{r}+\mathbf{R}(t),t)\\ &\cong \int d^3r\rho(\mathbf{r})e^{\mathrm{i}\mathbf{k}_0\cdot\mathbf{r}}\mathbf{E}_0 e^{-\mathrm{i}\omega t}\\ &= ef(\mathbf{k}_0)\mathbf{E}_0 e^{-\mathrm{i}\omega t},\end{aligned} \tag{2}$$

where e and $f(\mathbf{k}_0)$ are the electron charge and form factor, respectively. We have made the dipole approximation $\mathbf{k}_0 \cdot \mathbf{R}(t) \ll 1$: this is equivalent to saying that the size of the dipole associated with the electron displacement is small with respect to the wavelength of the incident radiation. This, however, does not impose any limitation on the size of the charge distribution and therefore on $f(\mathbf{k})$. Although the exact form of $f(\mathbf{k})$ is not known, on physical grounds it can be assumed that it is unity up to some large cutoff value ($|\mathbf{k}| \lesssim \Omega/c$) roughly given by the inverse of the charge distribution size, after which it falls rapidly to zero. The dominant contribution to the interaction comes therefore from wavelengths larger than the electron radius, consistent with our nonrelativistic treatment and the dipole approximation.

In terms of the Fourier transforms $\tilde{\mathbf{R}}(\omega)$ and $\tilde{\mathbf{F}}(\omega)$ of the electron displacement and the applied force, respectively, the linear response of the electron to the applied field is expressed as

$$\tilde{\mathbf{R}}(\omega) = \alpha(\omega)\tilde{\mathbf{F}}(\omega) = ef(\mathbf{k}_0)\alpha(\omega)\mathbf{E}_0. \tag{3}$$

The function $\alpha(\omega)$ is determined by the equation of motion for the electron in the presence of the applied field and any additional forces acting upon it. Its calculation with radiative reaction can be far from trivial [6], but it is well known that, with or without radiative reaction, it must satisfy certain basic conditions:

$$\alpha(\omega) = \alpha^*(-\omega), \tag{4}$$

and, as a function of complex frequency ζ,

$$\alpha(\zeta) \text{ is analytic for } \mathrm{Im}\,\zeta > 0. \tag{5}$$

The first condition, the "crossing relation," is simply the requirement that the induced dipole moment is real. The second is a direct consequence of causality and implies the familiar Kramers-Kronig relations between the real and imaginary parts of the polarizability [1].

Scattering theory provides a further constraint in terms of the *optical theorem* [7] relating the total scattering cross section and the forward scattering amplitude:

$$\sigma_t = \frac{4\pi}{k}\mathrm{Im}[\hat{\mathbf{e}}_0^* \cdot \mathbf{f}(\mathbf{k}_0, \mathbf{k}_0)]. \tag{6}$$

This is just the requirement of energy conservation, or, in quantum theory, the conservation of probability. In our case, to calculate $\mathbf{f}(\mathbf{k}_0, \mathbf{k}_0)$ it is sufficient to consider the electric field emitted by the electron in the radiation zone [7]: for a confined current density $\mathbf{j}(\mathbf{r}, \omega)$ the field is

$$\mathbf{E}(\mathbf{r}, \omega) = k^2[\mathbf{p}(\mathbf{k}_s, \omega) - (\hat{\mathbf{r}} \cdot \mathbf{p}(\mathbf{k}_s, \omega))\hat{\mathbf{r}}]\frac{e^{\mathrm{i}kr}}{r}, \tag{7a}$$

$$\mathbf{p}(\mathbf{k}_s, \omega) = \frac{\mathrm{i}}{\omega}\int \mathbf{j}(\mathbf{r}', \omega)e^{-\mathrm{i}\mathbf{k}_s\cdot\mathbf{r}'}d^3r' = \frac{\mathrm{i}}{\omega}\tilde{\mathbf{j}}(-\mathbf{k}_s, \omega), \tag{7b}$$

where $\tilde{\mathbf{j}}(\mathbf{k}, \omega)$ is the space-time Fourier transform of the current distribution and $\mathbf{k}_s = k\mathbf{r}/r$ ($|\mathbf{k}_s| = |\mathbf{k}_0|$) is the wavevector in the direction of observation. Writing

$$\mathbf{j}(\mathbf{r}, t) = \dot{\mathbf{R}}(t)\rho(\mathbf{r} - \mathbf{R}(t)), \tag{8}$$

and again making the dipole approximation $\mathbf{k}_s\cdot\mathbf{R}(t) \ll 1$, we have [8]

$$\tilde{\mathbf{j}}(-\mathbf{k}_s, \omega) = -\mathrm{i}\omega ef(-\mathbf{k}_s)\tilde{\mathbf{R}}(\omega) = -\mathrm{i}\omega ef^*(\mathbf{k}_s)\tilde{\mathbf{R}}(\omega) \tag{9}$$

and

$$\mathbf{E}(\mathbf{r}, \omega) = ef^*(\mathbf{k}_s)k^2[\tilde{\mathbf{R}}(\omega) - \hat{\mathbf{r}} \cdot \tilde{\mathbf{R}}(\omega)\hat{\mathbf{r}}]\frac{e^{ikr}}{r}. \tag{10}$$

With $\tilde{\mathbf{R}}(\omega)$ given by (3), we identify the scattering amplitude

$$\mathbf{f}(\mathbf{k}_s, \mathbf{k}_0) = f^*(\mathbf{k}_s)f(\mathbf{k}_0)e^2\alpha(\omega)(k^2\hat{\mathbf{e}}_0 - \mathbf{k}_s \cdot \hat{\mathbf{e}}_0\mathbf{k}_s), \tag{11}$$

where $\hat{\mathbf{e}}_0 = \mathbf{E}_0/E_0 = \tilde{\mathbf{R}}/\tilde{R}$ is the unit polarization vector of the incident field. The forward scattering amplitude ($\mathbf{k}_s$ orthogonal to $\hat{\mathbf{e}}_0$) is therefore

$$\mathbf{f}(\mathbf{k}_0, \mathbf{k}_0) = k^2|f(\mathbf{k}_0)|^2e^2\alpha(\omega)\hat{\mathbf{e}}_0. \tag{12}$$

The total scattering cross section σ_t, obtained by integration over all scattered solid angles, is

$$\sigma_t = \int d\Omega_s|\mathbf{f}(\mathbf{k}_s, \mathbf{k}_0)|^2 = \frac{8\pi}{3}k^4|f(\mathbf{k}_0)|^4e^4|\alpha(\omega)|^2 \tag{13}$$

where spherical symmetry implies $f(\mathbf{k}_s) = f(\mathbf{k}_0)$. Hence, from (6) we have

$$\mathrm{Im}[\alpha(\omega)] = \frac{2e^2\omega^3}{3c^3}|\alpha(\omega)|^2|f(\mathbf{k}_0)|^2, \tag{14}$$

which is an equivalent statement of the optical theorem [6].

In the AL theory of radiative reaction [1] a dipole oscillator subject to a restoring force $-M\omega_0^2\mathbf{R}$ and an external electric field $\mathbf{E}$ is described nonrelativistically by the equation of motion

$$\ddot{\mathbf{R}} + \omega_0^2\mathbf{R} = \frac{e}{M}[\mathbf{E} + \mathbf{E}_{RR}], \tag{15}$$

where M and ω_0 are respectively the (observed) electron mass and the resonance frequency. $\mathbf{E}_{RR} = (2e/3c^3)\,\dddot{\mathbf{R}}$ is the radiative reaction field. This implies

$$\alpha(\omega) = \frac{1}{M}\frac{1}{\omega_0^2 - \omega^2 - \mathrm{i}\omega^3\tau_e}, \quad \text{(AL)} \tag{16}$$

where $\tau_e = 2e^2/3Mc^3$ is on the order of the time for light to travel a distance equal to the classical electron radius, $r_0 = e^2/Mc^2$. This expression for $\alpha(\omega)$ obviously satisfies the crossing relation (4), and it is also seen from (14) that the optical theorem is satisfied with $|f(\mathbf{k})| = 1$, which is the form factor for a point-like electron. It also follows from (13) that, with $|f(\mathbf{k})| = 1$, we have

$$\sigma_t = \frac{2}{3\pi N^2}|n(\omega) - 1|^2\left(\frac{\omega}{c}\right)^4, \tag{17}$$

recovering the familiar, experimentally measured, Rayleigh scattering cross section for a dilute gas of isotropic, point-like scatterers: the gas refractive index $n(\omega)$ is given in this case by $n(\omega) \approx 1 + 2\pi Ne^2\alpha(\omega)$, where N is the particle number density [9].

However, the result (16) of the AL theory violates the causality requirement that $\alpha(\omega)$ be analytic in the upper half of the complex frequency plane. Additionally, as is well known, the equation of motion (15), from which (16) follows, exhibits runaway solutions as a consequence of the "non-Newtonian" dependence of $\mathbf{E}_{RR}$ on the third derivative of $\mathbf{R}$.

The alternative approach to radiative reaction cited earlier [3, 4] (FO) is based on the quantum theory of dissipation in which a particle is coupled to a "bath" of harmonic oscillators, so that it experiences a Langevin force together with a dissipative force due to the back reaction of the oscillators on the particle. In the case of interest here the bath oscillators are associated in the usual way with the electromagnetic field, the Langevin force is due to the fluctuating electric field, and the back reaction results in the radiative damping force. The semiclassical equation of motion for an electric dipole oscillator with bare mass m is [3, 4]

$$m\ddot{\mathbf{R}}(t) + \int_{-\infty}^{t} dt' \mu(t-t')\dot{\mathbf{R}}(t') + K\mathbf{R}(t) = \mathbf{F}(t), \quad (18)$$

which follows by taking an expectation value, so that the Langevin force, having zero mean value, does not appear. $\mathbf{F}(t)$ is the expectation value of the externally applied force; the linearity of the system implies that (18) describes the classical system. The constant K characterizes a harmonic restoring force, while the function

$$\tilde{\mu}(\omega) = \int_0^{\infty} \mu(t) e^{\mathrm{i}\omega t} \mathrm{d}t \quad (19)$$

is a *positive-real function* [10] and can be calculated exactly once the form factor of the electron is provided [10–13]. A possible choice is [3, 11, 13]

$$|f(\mathbf{k})|^2 = \frac{\Omega^2}{\Omega^2 + c^2 k^2}, \quad (20)$$

for which [3, 13, 14]

$$\tilde{\mu}(\omega) = \frac{2e^2}{3c^3} \frac{\Omega^2 \omega}{\omega + \mathrm{i}\Omega}, \quad (21)$$

which gives the function $\mu(t)$ [3, 4]:

$$\mu(t) = M\Omega^2 \tau_e [2\delta(t) - \Omega e^{-\Omega t}], \quad (22)$$

where the delta function represents the memory-less Markovian part and the second term in brackets results in non-Markovian effects. M is again the observed mass of the particle and is defined here by

$$M = m + \frac{2e^2 \Omega}{3c^3}, \text{ or } m = M(1 - \tau_e \Omega). \quad (23)$$

Various authors have connected the existence of runaway solutions of the AL equation with a negative bare mass and the point-electron assumption [3, 7, 15]. This is the case when $\Omega > \tau_e^{-1}$ (a point-like electron is recovered in the $\Omega \to \infty$ limit). Therefore, in the large-cutoff limit, we take $\Omega = \tau_e^{-1}$. When the external force is due to an external electric field $\mathbf{E}(\mathbf{r}, t)$, Eqs. (18), (20), and (22) lead to [3, 13]

$$\alpha(\omega) = \frac{1}{M} \frac{1 - \mathrm{i}\omega\tau_e}{\omega_0^2 - \omega^2 - \mathrm{i}\gamma\omega} \quad \text{(FO)} \quad (24)$$

in the large-cutoff limit, where $\omega_0^2 = K/M$ and we have defined $\gamma = \omega_0^2 \tau_e$ [13]. The equation of motion leading to the polarizability (24) is [3]

$$M\left[\ddot{\mathbf{R}}(t) + \gamma\dot{\mathbf{R}}(t) + \omega_0^2 \mathbf{R}(t)\right] = \mathbf{F}(t) + \tau_e \dot{\mathbf{F}}(t). \quad (25)$$

As already noted in [3], Eliezer [16] wrote a similar equation for the simpler case of a free particle as one possible alternative to the Abraham-Lorentz equation [see his equation (9)]. Thus, the second and third terms on the left side of (25) did not appear. Also, on the right side of (25), he replaced $\mathbf{F}(t)$ by $e\mathbf{E}(t)$ whereas the correct expression is given by our equation (2). Similar remarks apply to the results of Landau and Lifshitz [17], who derived its free-particle version ($\omega_0 \to 0$) from the AL equation using an order reduction scheme (see also [18]).

Eq. (24) obviously satisfies the crossing relation as well as the requirement from causality that it be analytic in the upper half of the complex frequency plane. From (24) it also follows that

$$\mathrm{Im}[\alpha(\omega)] = \frac{2e^2\omega^3}{3c^3} |\alpha(\omega)|^2 \frac{1}{1 + \omega^2 \tau_e^2}. \quad (26)$$

Since $\Omega = \tau_e^{-1}$, the last factor on the right-hand side is $|f(\mathbf{k}_0)|^2$. Therefore the optical theorem in the form (14) for Rayleigh scattering is satisfied identically.

The difference between equation (24) and the result (16) of the AL theory leads to different predictions for the Rayleigh cross section (17). Figure 1 compares the real and imaginary parts of these two expressions for $\alpha(\omega)$ for a particular value of ω_0. It can be seen, however, that, because τ_e is so small, significant differences appear only at extremely high frequencies at which the relativistic effects occur.

More generally, without specifying the form of $f(\mathbf{k})$, the Fourier transform of (18) gives the general expression

$$\alpha(\omega) = \frac{1}{-m\omega^2 - \mathrm{i}\omega\tilde{\mu}(\omega) + K}, \quad (27)$$

with

$$\mathrm{Re}[\tilde{\mu}(\omega)] = \frac{2e^2\omega^2}{3c^3} |f(\mathbf{k}_0)|^2. \quad (28)$$

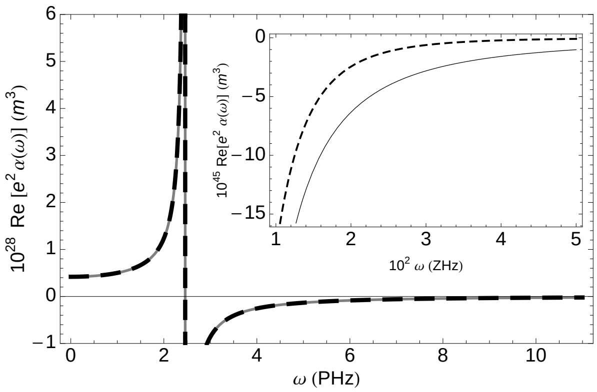
It follows in general, therefore, that the optical theorem is satisfied regardless of the specific choice for the form factor $f(\mathbf{k})$.

For free electrons ($\omega_0, \gamma \to 0$) the polarizability (24) becomes

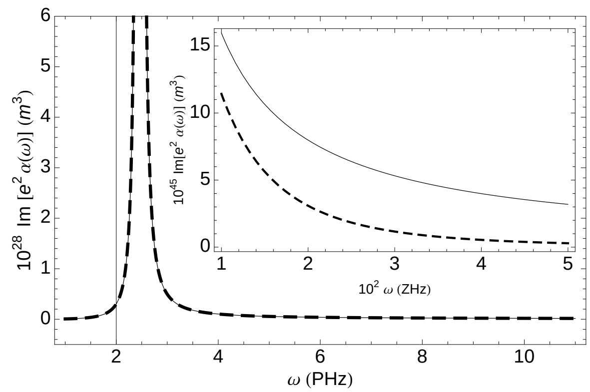
$$\alpha(\omega) = -\frac{1}{M\omega^2}(1 - \mathrm{i}\omega\tau_e). \quad (29)$$

The real part of $\alpha(\omega)$ implies for frequencies $\omega \ll \tau_e^{-1}$ the familiar formula $n_R^2(\omega) = 1 - \omega_p^2/\omega^2$ for the real part of the refractive index, where $\omega_p = (4\pi N e^2/M)^{1/2}$ is the plasma frequency. The imaginary part of $\alpha(\omega)$ implies for $n_R(\omega) \cong 1$ an imaginary part $n_I(\omega) \cong \omega_p^2 \tau_e / 2\omega$ of the refractive index, and therefore a power extinction coefficient

$$a(\omega) = \frac{2\omega}{c} n_I(\omega) = N\sigma(\omega), \quad (30)$$



(a)



(b)

FIG. 1: Real (a) and imaginary (b) parts of the polarizabilities $e^2\alpha(\omega)$ implied by equations (24) (solid curves) and (16) (dashed curves). The electron charge and mass are assumed and the oscillator's resonant frequency is set to hydrogen's first optical resonance, $\omega_0 \approx 2.45 \times 10^{15}$ Hz. The insets show that significant differences between the two theories appear only at very large frequencies.

where

$$\sigma_{\mathrm{Th}}(\omega) = \frac{8\pi}{3}\left(\frac{e^2}{Mc^2}\right)^2 \tag{31}$$

is the Thomson cross section. As noted by Eliezer, the AL equation results in a cross section differing from $\sigma_{\mathrm{Th}}(\omega)$ by a factor $(1+\omega^2\tau_e^2)^{-1}$.

We conclude that, unlike the AL theory, the approach to radiative reaction presented in Reference [3] results in a polarizability that is consistent with all three basic physical requirements referred to in this paper, namely causality, the crossing relation, and the optical theorem. In addition, although the FO polarizability differs from the polarizability that follows from the AL equation, it is nevertheless consistent with the familiar expressions for both the Rayleigh and Thomson scattering cross sections.

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