Homological width and Turaev genus

Adam Lowrance
Louisiana State University and Agricultural and Mechanical College

Follow this and additional works at: https://repository.lsu.edu/gradschool_dissertations

Part of the Applied Mathematics Commons

Recommended Citation
https://repository.lsu.edu/gradschool_dissertations/3797

This Dissertation is brought to you for free and open access by the Graduate School at LSU Scholarly Repository. It has been accepted for inclusion in LSU Doctoral Dissertations by an authorized graduate school editor of LSU Scholarly Repository. For more information, please contact gradetd@lsu.edu.
HOMOLOGICAL WIDTH AND TURAEV GENUS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

Adam Lowrance
B.A. in Math., Amherst College, 2004
M.S., Louisiana State University, 2006
December 2009
Acknowledgments

First, I would like to thank my advisor Dr. Scott Baldridge for his guidance and encouragement. His dedication has been invaluable to my education. I would like to express my gratitude to Dr. Oliver Dasbach for his continuous support, ideas and insight. I am indebted to Dr. Neal Stoltzfus for introducing me to research mathematics and for many helpful conversations throughout my time as a graduate student. Finally, I would like to thank my friends and family for their support and encouragement.
# Table of Contents

Acknowledgments ................................................................. ii

List of Tables ................................................................. v

List of Figures ................................................................. vi

Abstract ................................................................. viii

Chapter 1: Introduction ........................................................... 1

Chapter 2: Links and Ribbon Graphs .............................................. 4
  2.1 Ribbon Graphs .......................................................... 4
  2.2 Link Diagrams and Ribbon Graphs ....................................... 5
  2.3 The Turaev Surface and Duality .......................................... 8
  2.4 The Tait Graph .......................................................... 12
  2.5 The Turaev Surface and Crossing Changes ............................. 15
  2.6 Spanning Quasi-trees .................................................. 17
  2.7 The Turaev Surface and Reidemeister Moves .......................... 21

Chapter 3: Khovanov Homology .................................................... 24
  3.1 The Jones Polynomial .................................................. 24
    3.1.1 The Kauffman Bracket Approach ................................ 24
    3.1.2 A Spanning Tree Expansion ...................................... 26
  3.2 Constructing $Kh(L)$ .................................................. 28
    3.2.1 Vertices of the Cube .............................................. 28
    3.2.2 Edges of the Cube ................................................. 29
    3.2.3 The Homology of the Cube ...................................... 30
  3.3 Reduced Khovanov Homology ............................................. 32
  3.4 Khovanov Homology and Spanning Trees ............................... 34
  3.5 The Long Exact Sequence and Quasi-alternating Links ............... 37

Chapter 4: Twisting Links ..................................................... 41
  4.1 Khovanov Width and Twisting Links .................................. 41
  4.2 Turaev Genus and Twisting Links .................................... 49

Chapter 5: Applications to Closed 3-braids .................................. 51
  5.1 Torus Links .......................................................... 51
5.2 Khovanov Width of Closed 3-braids ............................................. 55
5.3 Turaev Genus of Closed 3-braids ............................................. 64

Chapter 6: Knot Floer Homology ......................................................... 67
  6.1 The Alexander Polynomial ....................................................... 67
  6.2 Construction of $\hat{HFK}(K)$ .................................................. 70
  6.3 From Decorated Knot Diagrams to Heegaard Diagrams ................. 74
  6.4 Heegaard Diagrams and Spanning Trees .................................... 75
  6.5 Knot Floer Width ................................................................. 76

Chapter 7: Conclusion ................................................................. 84

References ................................................................. 86

Vita ................................................................. 90
List of Tables

2.1 Type I Reidemeister move and the all $A$ and all $B$ Kauffman states . . . . . . 22
2.2 Type II Reidemeister move and the all $A$ and all $B$ Kauffman states . . . . . . 22
2.3 Type III Reidemeister move and the all $A$ and all $B$ Kauffman states . . . . . . 23
List of Figures

2.1 Basic examples of ribbon graphs ................................. 6
2.2 The Reidemeister moves ......................................... 6
2.3 The $A$-smoothing and $B$-smoothing of a crossing ............. 7
2.4 Construction of $A$ and $B$ for $10_{124}$ ............................. 8
2.5 A saddle in the Turaev surface of $s$ ............................... 9
2.6 The Turaev surface of $10_{124}$ .................................... 9
2.7 Dual embeddings of $A$ and $B$ .................................... 11
2.8 The Tait graphs of $10_{124}$ ....................................... 12
2.9 The sign of an oriented crossing .................................. 13
2.10 Changing the smoothing at a crossing ............................ 14
2.11 The bouquet of a Tait graph ..................................... 16
2.12 The spanning quasi-tree correspondence .......................... 20
3.1 The Kauffman resolution tree for the trefoil ...................... 26
3.2 The twisted unknots for the trefoil ................................. 27
3.3 The cube of resolutions ........................................... 31
3.4 Smoothing an unoriented crossing .................................. 37
3.5 Smoothing an oriented crossing .................................... 37
4.1 A link diagram twisted by a rational tangle .......................... 41
4.2 Resolutions for the tangle $C(2)$ .................................... 43
4.3 Resolutions for the one component tangle $C(−2)$ .................... 44
4.4 Resolutions for the two component tangle $C(−2)$ .................... 45
4.5 Constructing a rational tangle using $C(2)$ and $C(−2)$ ............ 47
4.6 Adding twists to a crossing ....................................... 47
5.1 Coloring convention to compute signature .......................... 57
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.2</td>
<td>Calculating the signature from an alternating diagram</td>
<td>58</td>
</tr>
<tr>
<td>5.3</td>
<td>A transformation of the closure of $(\sigma_1\sigma_2)^3\sigma_2^{-4}$ into $L(6,n,1)$</td>
<td>60</td>
</tr>
<tr>
<td>6.1</td>
<td>A connected Kauffman state of $4_1$</td>
<td>69</td>
</tr>
<tr>
<td>6.2</td>
<td>The local contribution to $\Delta_L(t)$</td>
<td>69</td>
</tr>
<tr>
<td>6.3</td>
<td>Connected Kauffman states and spanning trees</td>
<td>70</td>
</tr>
<tr>
<td>6.4</td>
<td>The $\alpha$ and $\beta$ curves near a crossing</td>
<td>74</td>
</tr>
<tr>
<td>6.5</td>
<td>A Heegaard diagram for $4_1$</td>
<td>75</td>
</tr>
<tr>
<td>6.6</td>
<td>An intersection point of $T_\alpha \cap T_\beta$</td>
<td>76</td>
</tr>
<tr>
<td>6.7</td>
<td>The local Alexander filtration level</td>
<td>76</td>
</tr>
<tr>
<td>6.8</td>
<td>The local Maslov grading</td>
<td>77</td>
</tr>
<tr>
<td>6.9</td>
<td>The edge labels for the Tait graph</td>
<td>77</td>
</tr>
</tbody>
</table>
Abstract

Khovanov homology and knot Floer homology are generalizations of the Jones polynomial and the Alexander polynomial respectively. They are bigraded $\mathbb{Z}$-modules, and their underlying polynomials are recovered by taking the graded Euler characteristic. The two homologies share many characteristics, however their relationship has yet to be fully understood. In both Khovanov homology and knot Floer homology, the two gradings can be combined into a single diagonal grading. Homological width is a measure of the support of the homology with respect to the diagonal grading. In this thesis, we show that the homological width of Khovanov homology and knot Floer homology have a common upper bound.

Every link diagram has an associated Turaev surface, a certain Heegaard surface in $S^3$ on which the knot has an alternating projection. The Turaev genus of a knot is the minimum genus of a Turaev surface where the minimum is taken over all diagrams of the knot. Turaev introduced this surface in order to prove a conjecture about the span of the Jones polynomial. Previously, it has been shown that Turaev genus gives an upper bound for the homological width of Khovanov homology. Since Khovanov homology is a generalization of the Jones polynomial, one might expect that Turaev genus and Khovanov homology are related.

In this thesis, we show that Turaev genus also gives an upper bound for the homological width of knot Floer homology, giving the first known relationship between the Alexander polynomial and the Turaev surface. It is also more evidence towards a relationship between Khovanov homology and knot Floer homology.

In addition, we construct infinite families of links whose Khovanov homology have the same homological width. Using this construction, we compute the Khovanov width of all closed 3-braids.
Khovanov homology and reduced Khovanov homology are generalizations of the Jones polynomial introduced by Khovanov [Kho00]. Knot Floer homology is a generalization of the Alexander polynomial introduced by Ozsváth and Szabó [OS04b] and independently by Rasmussen [Ras03]. Many theorems that are true for Khovanov homology are also true for knot Floer homology. For instance, both Khovanov homology and knot Floer homology can be generated by a spanning tree complex (see [Weh08], [CK09a], and [OS03a]). Also, both homologies can be generated by cube of resolution complexes (see [BN02] and [OS07]). In addition, one can construct concordance invariants from Khovanov homology and knot Floer homology (see [Ras04] and [OS03b]). Despite the many connections between the two theories, the underlying reason for the relationship between Khovanov homology and knot Floer homology has not yet been discovered.

Knot Floer homology, Khovanov homology, and reduced Khovanov homology are all bigraded $\mathbb{Z}$-modules, and for each homology, the two gradings can be combined into a single diagonal grading. The homological width of a bigraded $\mathbb{Z}$-module is a measure of the support of that $\mathbb{Z}$-module with respect to the single diagonal grading. The homological width of (reduced) Khovanov homology is called (reduced) Khovanov width, while the homological width of knot Floer homology is called knot Floer width. Their precise definitions are given in Sections 3.2 and 6.2. For all known examples, reduced Khovanov width and knot Floer width are equal. In general, how reduced Khovanov width relates to knot Floer width is unknown. In this thesis, we prove that reduced Khovanov width and knot Floer width have a common upper bound, giving one more connection between the two theories.

The upper bound on reduced Khovanov width and knot Floer width arises from a construction called the Turaev surface. The Turaev surface is a certain Heegaard surface in $S^3$ on
which the link has an alternating projection. The precise construction of the Turaev surface is given in Section 2.3. The Turaev genus of a link is the minimum genus of a Turaev surface where the minimum is taken over all diagrams of the link. We prove the following theorem in Section 6.5.

**Main Theorem 1.** Let $K$ be a knot. Suppose $g_T(K)$ is the Turaev genus of $K$ and $w_{\widehat{HF}}(K)$ is the knot Floer width of $K$. Then

$$w_{\widehat{HF}}(K) \leq g_T(K) + 1.$$  

Manturov [Man05] and Champanerkar, Kofman, and Stoltzfus [CKS07] proved that Turaev genus gives the same upper bound on reduced Khovanov width, i.e. that $w_{\widehat{Kh}}(K) \leq g_T(K) + 1$, where $w_{\widehat{Kh}}(K)$ denotes the reduced Khovanov width of $K$. Since Turaev originally developed the Turaev surface to prove a conjecture about the span of the Jones polynomial, it is natural that the Turaev surface has applications to Khovanov homology. Main Theorem 1 is somewhat unexpected because until this point, there has been no known connection between the Turaev surface and the Alexander polynomial.

If a knot is alternating, then its reduced Khovanov homology and knot Floer homology are supported on only one diagonal grading (see [Lee02] and [OS05a]). Suppose $D$ is a link diagram and that $D$ is width-preserving at $x$ (a technical condition defined in Section 4.1). The crossing $x$ can be replaced with an alternating rational tangle $\tau$ to form the diagram $D_\tau$. See Figure 4.1 for an example. The following theorem is proved in Section 4.1.

**Main Theorem 2.** Let $D$ be a link diagram with crossing $x$. Suppose $D$ is width-preserving at $x$. Let $D_\tau$ be the diagram obtained by replacing the crossing $x$ with the alternating rational tangle $\tau$. Then the Khovanov width of $D$ is equal to the Khovanov width of $D_\tau$.

Corollary 3.10 immediately implies that the previous theorem is also true for reduced Khovanov width. The author expects that an analogous result holds for knot Floer homology.

Using Main Theorem 2, one can compute the Khovanov width of closed 3-braids. Murasugi [Mur74] classifies all closed 3-braids up to conjugation, as stated in Theorem 5.5, which says
that every closed 3-braid is of the form $h^n A$, where $h = (\sigma_1 \sigma_2)^3$ is a full twist and $A$ is specified in the theorem. For $n \neq 0$, we say that the closure of $h^n A$ has cancellation if the braid word for $A$ contains a $\sigma_i^\varepsilon$ for $i = 1, 2$ where $\text{sign}(\varepsilon) \neq \text{sign}(n)$. The following theorem is proved in Section 5.2.

**Main Theorem 3.** Let $L$ be a closed 3-braid of the form $h^n A$, as in Theorem 5.5, where $h = (\sigma_1 \sigma_2)^3$ and $n \neq 0$. Then

$$w_{Kh}(L) = \begin{cases} 
|n| + 2 & \text{if } L \text{ has no cancellation or} \\
|n| + 1 & \text{if } L \text{ is the closure of } h^\pm 1 \sigma_2^m \text{ where } m > 3, \\
|n| + 1 & \text{otherwise.}
\end{cases}$$

In Section 5.3, we also compute the Turaev genus of all closed 3-braids up to an additive error of at most one.

**Main Theorem 4.** If $L$ is a closed 3-braid, then

$$0 \leq g_T(L) - (w_{Kh}(L) - 1) \leq 1,$$

where $g_T(L)$ is the Turaev genus of $L$ and $w_{Kh}(L)$ is the reduced Khovanov width of $L$.

This thesis is organized as follows. In Chapter 2, we introduce ribbon graphs and the Turaev surface. Chapter 3 is a review of Khovanov homology. In Chapter 4, we show how to construct an infinite family of links with the same Khovanov width. Several applications to closed 3-braids are given in Chapter 5. In Chapter 6, we review knot Floer homology and prove that Turaev genus gives an upper bound on knot Floer width. Finally, in Chapter 7, we give possible directions for future research.
Chapter 2
Links and Ribbon Graphs

In this chapter, we will explore the relationship between link diagrams and ribbon graphs. Informally, an oriented ribbon graph is a multi-graph (loops and multiple edges are allowed) such that the edges at each vertex are cyclically ordered, as discussed in Tutte’s book on graph theory [Tut01]. Two oriented ribbon graphs are isomorphic if there is a graph isomorphism from one to the other that preserves the cyclic order of the edges. Any ribbon graph \( G \) can be properly embedded on an oriented surface \( \Sigma \) such that the cyclic orientation of the edges at each vertex is preserved and such that the faces of \( G \), defined to be the connected components of \( \Sigma \setminus G \), are a collection of 2-cells. The genus of \( G \) is defined to be the genus of \( \Sigma \).

For the definitions and properties of ribbon graphs, we follow [DFK+08] and [LZ04]. Most of the theorems proved in this chapter appear in either [CKS07] or [DFK+08]. Ribbon graphs were originally introduced in the 1891 article of L. Heffter [Hef91].

2.1 Ribbon Graphs

The formal definition of a ribbon graph is equivalent to the intuition given above, however it highlights the combinatorial structure.

Definition 2.1. A connected oriented ribbon graph is a triple \( G = (\sigma_0, \sigma_1, \sigma_2) \) of permutations of a finite set \( B = \{1, \ldots, 2n\} \) of half edges, satisfying:

- \( \sigma_1 \) is a fixed point free involution, i.e. \( \sigma_1(\sigma_1(b)) = b, \sigma_1(b) \neq b \) for all \( b \in B \),
- \( \sigma_0(\sigma_1(\sigma_2(b))) = b \), and
- the group generated by \( \langle \sigma_0, \sigma_1 \rangle \) acts transitively on \( B \).

An oriented ribbon graph is a disjoint union of connected oriented ribbon graphs. Throughout this text, we consider only oriented ribbon graphs, and thus use the term ribbon graph, keeping
the orientation implicit. The second condition implies that the ribbon graph \( G \) is determined from any two of the permutations \( \sigma_0, \sigma_1, \) and \( \sigma_2. \)

Let \( G = (\sigma_0, \sigma_1, \sigma_2) \) be an oriented ribbon graph. The orbits of \( \sigma_0 \) form the vertex set, the orbits of \( \sigma_1 \) form the edge set, and the orbits of \( \sigma_2 \) form the face set. We denote the cardinality of the vertex set, edge set, and face set of \( G \) by \( V(G), E(G) \) and \( F(G) \) respectively. Since \( \sigma_1 \) is a fixed point free involution, each of its orbits has cardinality two, and the two elements of each orbit correspond to an edge of the underlying graph of \( G. \) An edge connects the vertices in whose orbit its two half edges lie. The embedding of \( G \) onto the surface \( \Sigma \) is encoded in the orbits of \( \sigma_2 \) as follows. The faces of \( G \) correspond to the orbits of \( \sigma_2 \) and are oriented so that \( \sigma_0 \) cyclically rotates the half edges meeting at a vertex in the rotation direction determined by the orientation of \( \Sigma. \)

**Definition 2.2.** The *genus* \( g(G) \) of a ribbon graph \( G \) with \( k \) components is determined by its Euler characteristic: 
\[
2k - 2g(G) = V(G) - E(G) + F(G).
\]

**Example 2.3.** Let the set of half edges be \( B = \{1, 2, 3, 4\}. \) Define the ribbon graph \( G = (\sigma_0, \sigma_1, \sigma_2) \) by \( \sigma_0 = (1, 3, 4, 2), \sigma_1 = (1, 3)(2, 4) \) and \( \sigma_2 = (1, 4)(2)(3). \) The genus of \( G \) is 0. See Figure 2.1 for an embedding of \( G. \)

**Example 2.4.** Again, let the set of half edges be \( B = \{1, 2, 3, 4\}. \) Define the ribbon graph \( G' = (\sigma'_0, \sigma'_1, \sigma'_2) \) by \( \sigma'_0 = (1, 2, 3, 4), \sigma'_1 = (1, 3)(2, 4) \) and \( \sigma'_2 = (1, 2, 3, 4). \) The genus of \( G' \) is 1. See Figure 2.1 for an embedding of \( G'. \) Observe that the underlying graph of \( G' \) is isomorphic to the underlying graph of \( G \) from Example 2.3, but \( G \) and \( G' \) are not isomorphic as ribbon graphs.

### 2.2 Link Diagrams and Ribbon Graphs

A *k-component link* is a subset of \( S^3 \) consisting of \( k \) disjoint embedded circles. A link with one component is called a *knot*. Two links \( L_1 \) and \( L_2 \) are equivalent if there is an orientation preserving homeomorphism \( h : S^3 \to S^3 \) such that \( h(L_1) = L_2. \) A *link diagram* is a planar projection of the link with only transversal double points (called *crossings*) together with
FIGURE 2.1. The diagram on the left is the ribbon graph $G$ from Example 2.3, while the diagram on the right is the ribbon graph $G'$ from Example 2.4. Both diagrams are depicted in two ways: on top, as a graph together with the cyclic orientation of the edges at vertices, and on bottom, as a graph together with a cellular embedding on a surface.

data that indicates which strand goes over the other strand at each crossing. The associated projection of a link diagram is the projection of the link without the data indicating over and under crossings. Two link diagrams represent the same link if one can be transformed into the other through a sequence of Reidemeister moves as in Figure 2.2.

![Reidemeister Moves](image)

FIGURE 2.2. The Reidemeister moves.

Each crossing of a link diagram $D$ has an $A$-smoothing and a $B$-smoothing, as depicted in Figure 2.3. An embedded arc called the trace of the crossing connects the two arcs of the smoothing. A Kauffman state $s$ is a choice of an $A$-smoothing or $B$-smoothing at each crossing.
of the diagram. In [Kau87], Kauffman gives an equation to calculate the Jones polynomial as a weighted sum over all possible Kauffman states for a diagram.

![Diagram of A and B smoothing of a crossing](image)

**FIGURE 2.3.** The $A$-smoothing and $B$-smoothing of a crossing.

By smoothing crossings according to a Kauffman state $s$, one obtains a collection of circles in the plane connected by traces of the crossings. These circles will eventually become the vertices of the ribbon graph $G_s$. Orient each circle clockwise or counter-clockwise according to whether the circle is inside an odd or even number of circles, respectively.

The edges of $G_s$ correspond to the crossings in the diagram $D$. Label the endpoints of the traces of the crossings 1 through $2c$ where $c$ is the number of crossings in $D$. The set of half edges $B = \{1, \ldots, 2c\}$ is the set of endpoints of the traces. Define $\sigma_0$ to permute each endpoint to the next endpoint along the circle according to the orientation of the circle on which it lies. Define $\sigma_1$ to be the permutation that takes one endpoint of a trace to the other endpoint of the same trace. The permutations $\sigma_0$ and $\sigma_1$ determine $\sigma_2$; therefore we define the oriented ribbon graphs $G_s$ to be $(\sigma_0, \sigma_1, \sigma_2)$. If $s$ is the state consisting of only $A$-smoothings, then we denote the associated ribbon graph by $A$. Similarly, if $s$ is the state consisting of only $B$-smoothings, then we denote the associated ribbon graph by $B$.

**Example 2.5.** Let $D$ be the diagram for the knot 10$_{124}$ appearing in Rolfsen’s table. Figure 2.4 depicts 10$_{124}$, its all $A$-smoothing, and its all $B$-smoothing. From Figure 2.4, one can construct $A = (\sigma_0, \sigma_1, \sigma_2)$ and $B = (\sigma'_0, \sigma'_1, \sigma'_2)$. We have

\[
\sigma_0 = (1, 2)(3, 11, 6, 7, 8, 9, 10, 12, 5, 4)(13, 14, 15, 20, 19, 18, 17, 16),
\]

\[
\sigma'_0 = (1, 6, 3, 11, 12, 15, 20, 2)(4, 13)(5, 14)(7, 16)(8, 17)(9, 18)(10, 19),
\]
FIGURE 2.4. The knot $10_{124}$ and its all $A$-smoothing and all $B$-smoothing (with the endpoints of the traces labeled).

and $\sigma_1 = \sigma_1'$ is the involution that permutes $k$ with $k + 10$ for $k = 1, \ldots, 10$. The genus of both $A$ and $B$ is one. Figure 2.7 shows $A$ and $B$ as graphs with a cyclic order on their edges at each vertex and as graphs embedded on a genus one surface.

## 2.3 The Turaev Surface and Duality

Let $D$ be a link diagram and $\Gamma$ be the planar projection of the link, i.e. $\Gamma$ is the diagram $D$ without the over and under information at each crossing. Let $s$ be a Kauffman state of $D$, and denote the dual state, the state where every crossing is smoothed in the opposite way of $s$, by $\hat{s}$. Turaev [Tur87] constructs a surface $\Sigma^s_D$ on which the ribbon graphs $\mathcal{G}_s$ and $\mathcal{G}_{\hat{s}}$ embed (see also Cromwell [Cro04]).

Consider $\Gamma$ as a subset of $\mathbb{R}^2$ which is a subset of $\mathbb{R}^3$. Begin the construction of the surface $\Sigma^s_D$ with $\Gamma \times [-1, 1]$. This is a surface with boundary and codimension one singularities corresponding to the double points of $\Gamma$. Replace a neighborhood of a singularity with a saddle, as in Figure 2.5, such that the boundary circles in $\mathbb{R}^2 \times 1$ are the state circles of $s$ and the boundary circles in $\mathbb{R}^2 \times -1$ are the state circles of $\hat{s}$. If each state circle is capped off with a
disk, then we obtain the closed oriented surface \( \Sigma_D^s \), called the \textit{Turaev surface of } s. If \( s \) is the all A state or the all B state, then the surface is called the \textit{Turaev surface of } D, and is denoted \( \Sigma_D \). Figure 2.6 shows an example of the Turaev surface for the diagram of \( 10_{124} \) appearing in Figure 2.4.

![FIGURE 2.5](image)

**FIGURE 2.5.** Near double point of \( \Gamma \), there is a saddle in \( \Sigma_D^s \).

![FIGURE 2.6](image)

**FIGURE 2.6.** The Turaev surface of \( 10_{124} \) before the disks corresponding to state circles are added. The blue boundary components are the circles of the all A-smoothing, while the red boundary components are the circles of the all B-smoothing.

The following lemma from [DFK+08] establishes a connection between the Turaev surface \( \Sigma_D^s \) and the ribbon graphs \( G_s \) and \( \hat{G}_s \).

**Lemma 2.6** (Dasbach - Futer - Kalfagianni - Lin - Stoltzfus). \textit{The oriented ribbon graphs } \( G_s \) \textit{and } \( \hat{G}_s \) \textit{can both be embedded in } \( \Sigma_D^s \), \textit{the Turaev surface for } s. \textit{Moreover, } \( G_s \) \textit{and } \( \hat{G}_s \) \textit{are}
dual on $\Sigma^*_D$, i.e. the vertices of one correspond to the faces of the other, and the edges of one correspond to the edges of the other.

Proof. The Turaev surface $\Sigma^*_D$ is a cell complex, whose 1-skeleton is the planar projection $\Gamma$ and whose 2-cells correspond to the Kauffman state circles of $s$ and $\hat{s}$. The 2-cells can be two-colored in a chessboard fashion, with the 2-cells corresponding to the $s$ state circles colored white and the 2-cells corresponding to the $\hat{s}$ state circles colored black.

Embed the ribbon graph $G_s$ in $\Sigma^*_D$ as follows. Embed a vertex of $G_s$ into each of the white 2-cells. If two white 2-cells meet at a crossing, then connect the corresponding vertices with an edge that is a gradient flowline through the saddle corresponding to that crossing. Since each saddle corresponds to a crossing of $D$, the edges correspond to the traces of the crossings, and the cyclic order of the edges around each vertex is correct.

Similarly, one can embed $G_{\hat{s}}$ in $\Sigma^*_D$ by placing vertices into the black 2-cells and connecting vertices with edges that are gradient flowlines through the saddles. Embedded on each saddle in $\Sigma^*_D$ are two transversely intersecting edges, one of $G_s$ and one of $G_{\hat{s}}$. Also, every face of $G_s$ corresponds to a black 2-cell of $\Sigma^*_D$, which in turn corresponds to a vertex of $G_{\hat{s}}$, and vice versa. Therefore $G_s$ and $G_{\hat{s}}$ are dual on $\Sigma^*_D$. \hfill \Box

Example 2.7. We return to the ribbon graphs for $10_{124}$. The ribbon graphs $A$ and $B$ are displayed in Figure 2.7. Also, $A$ and $B$ have dual embeddings on the torus.

Corollary 2.8. The genera of $\Sigma^*_D$, $G_s$, and $G_{\hat{s}}$ are all equal.

Corollary 2.8 implies that the genus of $\Sigma_D$ is determined by

$$2 - 2g(\Sigma_D) = V(A) - c(D) + V(B), \quad (2.1)$$

where $c(D)$ is the number of crossings of $D$. In addition to being the number of vertices in $A$, the quantity $V(A)$ is also the number of state circles in the all $A$-smoothing of $D$. Similarly, $V(B)$ is the number of state circles in the all $B$-smoothing of $D$. This leads to a natural link invariant:
FIGURE 2.7. The two ribbon graphs $A$ and $B$ for $10_{124}$ are depicted with black and white vertices respectively. Also pictured is their dual toroidal embedding.

Definition 2.9. Let $L$ be a link. The *Turaev genus* of $L$ is defined as

$$g_T(L) = \{ g(\Sigma_D) \mid D \text{ is a diagram of } L \}.$$ 

The following theorem from [DFK+08] shows that Turaev genus gives an obstruction to a link being alternating.

**Theorem 2.10** (Dasbach - Futer - Kalfagianni - Lin - Stoltzfus). A link $L$ has Turaev genus 0 if and only if it is alternating.

Proof. Let $D$ be a diagram of $L$, and let $\Gamma$ be its associated projection. If $D$ is alternating, then the state circles coming from the all $A$-smoothing and all $B$-smoothing correspond to the faces of $\Gamma$. Therefore $\Sigma_D$ is a sphere, and $g_T(L) = 0$.

In general, the state circles of the all $A$-smoothing and the all $B$-smoothing have disjoint projections to $\Sigma_D$, and thus near each double point of $\Gamma$, the surface is identical to the chess-
board coloring of an alternating diagram in the plane. If $D$ has two consecutive over-crossings, then the surface $\Sigma_D$ has one half twist between the crossings. Thus $L$ has an alternating projection to $\Sigma_D$. If $g_T(L) = 0$, then there exists a diagram $D$ such that $\Sigma_D$ is a sphere. But $L$ has an alternating projection to $\Sigma_D$, a sphere, and hence $L$ is alternating.

2.4 The Tait Graph

In this section, we show how to associate a planar graph, called the Tait graph $G$, to a link diagram $D$. Let $\Gamma$ be the projection associated to $D$. Color the faces of $\Gamma$ white and black in a chessboard fashion, i.e. so that if two faces are incident to the same edge, then they are different colors. Define a graph $G$ as follows. The vertices of $G$ correspond to the black faces of $\Gamma$. The edges of $G$ correspond to the crossings of $D$. Each edge is incident to the vertices that correspond to the black faces near the double point corresponding to that edge. An edge in $G$ is called an $A$-edge (respectively a $B$-edge) if the $A$-smoothing (respectively the $B$-smoothing) separates the black faces.

The white faces of $\Gamma$ also give rise to a graph $G^*$ in a similar manner. The vertices of $G^*$ correspond to the white faces, and the edges in $G^*$ correspond to the crossings of $D$. An edge in $G^*$ is an $A$-edge (respectively a $B$-edge) if and only if the edge in $G$ corresponding to the same crossing of $D$ is a $B$-edge (respectively an $A$-edge). By construction, the graph $G^*$ is the planar dual of $G$.

![Figure 2.8](image-url)

**FIGURE 2.8.** The Tait graphs of $10_{124}$. In the black graph, edges 1 and 2 are $A$-edges and edges 3 – 10 are $B$-edges. Conversely, in the white graph edges 1 and 2 are $B$-edges while edges 3 – 10 are $A$-edges.
If $D$ is an oriented link diagram, then each crossing is either positive or negative, as in Figure 2.9. If a crossing is positive, then label its associated edges in $G$ and $G^*$ as positive. Similarly, if a crossing is negative, then label its associated edges in $G$ and $G^*$ as negative. For any subgraph $H$ of $G$ or $G^*$, let $E^+_A(H)$ denote the number of edges in $H$ that are both $A$-edges and positive. Similarly define $E^-_A(H)$, $E^+_B(H)$, and $E^-_B(H)$. Also, let $E^+(H)$ denote the number of positive edges in $H$ and $E^-(H)$ denote the number of negative edges in $H$. Similarly, let $E_A(H)$ be the number of $A$-edges in $H$ and $E_B(H)$ be the number of $B$-edges in $H$. We alert the reader that in the literature $A$-edges are sometimes called negative edges and $B$-edges are called positive edges. Since we have a different notion of positive and negative, we use the $A$ and $B$ notation instead.

The Tait graphs $G$ and $G^*$ are special cases of the ribbon graphs $G_s$ and $G_{\tilde{s}}$. Let $s$ be the Kauffman state whose state circles correspond to the black faces in the chessboard coloring. The state $s$ can be constructed as follows: if an edge in $G$ is an $A$-edge (respectively a $B$-edge), then choose the $A$-smoothing (respectively the $B$-smoothing) for that crossing. The ribbon graph $G_s$ is the Tait graph $G$, and the ribbon graph $G_{\tilde{s}}$ is the Tait graph $G^*$.

Next, we examine what happens to the ribbon graph $G_s$ as the choice of $A$-smoothing or $B$-smoothing is changed at one crossing for the state $s$. By iteratively changing the choice of smoothing at a sequence of crossings, the Tait graph can be transformed into either $A$ or $B$. Consequently, one can compute the genus of the Turaev surface of $D$ using only the Tait graph $G$.
Changing the smoothing of one crossing corresponds to merging two state circles together or splitting one state circle into two. Therefore, in the Tait graph, either two vertices are merged into one or one vertex is split into two. Figure 2.10 shows how changing the smoothing at a crossing effects the state circles and the ribbon graph.

Equation 2.1 implies that to compute the genus of the Turaev surface \( \Sigma_D \) it suffices to know the number of crossings in \( D \) and the number of vertices in \( \mathcal{A} \) and \( \mathcal{B} \). The following is an algorithm to count the state circles of the all \( A \)-smoothing (and thus \( V(\mathcal{A}) \)). The algorithm is given by performing a sequence of operations on the Tait graph \( G \).

**Step 1.** Remove all \( A \)-edges from \( G \). If two vertices of \( G \) are the endpoints of an \( A \)-edge, then their corresponding circles are separated by an \( A \)-smoothing (see Figure 2.3). Thus choosing an \( A \)-smoothing for that crossing does not change the number of state circles, and so each \( A \)-edge in \( G \) can be removed.

**Step 2.** Contract all non-loop positive edges. If in the resulting graph there exists a \( B \)-edge whose endpoints are distinct vertices, then the circles corresponding to these vertices are joined by an \( A \)-smoothing (see Figure 2.10). Thus choosing an \( A \)-smoothing for that crossing decreases the number of circles by one; likewise, contracting the edge decreases the number
of vertices by one. Either the resulting graph contains a non-loop $B$-edge or all remaining edges are loops. If the graph contains a non-loop $B$-edge, then repeat this step. Otherwise, the resulting graph is a collection of vertices and loops, and is called the $A$-bouquet of $G$.

**Step 3.** Count vertices and loops. Each vertex in the $A$-bouquet of $G$ corresponds to a state circle in the all $A$ Kauffman state, and each loop in the $A$-bouquet of $G$ corresponds to the trace of crossing (after choosing the $B$-smoothing) between a state circle and itself. Changing to an $A$-smoothing at that crossing splits that state circle into two circles (see Figure 2.10). Therefore, each loop also corresponds to a state circle in the all $A$ Kauffman state. Hence, $V(A)$ is the number of vertices plus the number of loops in the $A$-bouquet of $G$.

In order to calculate $V(B)$, the algorithm is modified as follows. Since we are calculating the number of state circles in the all $B$ Kauffman state, in Step 1, $B$-edges are deleted. Also, non-loop $A$-edges are contracted in Step 2. Then $V(B)$ is equal to the number of vertices plus the number of loops in the $B$-bouquet of $G$. Figure 2.11 shows this algorithm for the Tait graphs of the $10_{124}$ knot. Note that the algorithm does not depend on the choice of Tait graph; one may start with either $G$ or $G^*$ and compute $V(A)$ and $V(B)$ in this way. The above algorithm immediately implies the following theorem.

**Proposition 2.11.** Let $D$ be a diagram for a link $L \subset S^3$, and let $\Sigma_D$ be the Turaev surface of $D$. Let $G$ and $G^*$ be the Tait graphs of $D$. Let $V$ be the number of vertices and loops in the $A$-bouquet of $G$, $E$ be the number of edges in $G$ (or $G^*$), and $F$ be the number of vertices and loops in the $B$-bouquet of $G^*$. Then

$$2 - 2g(\Sigma_D) = V - E + F.$$ 

\[\square\]

### 2.5 The Turaev Surface and Crossing Changes

This section is dedicated to understanding the behavior of the Turaev surface under a crossing change. Changing a crossing in a diagram corresponds to changing an $A$-edge in one Tait graph...
FIGURE 2.11. Constructing the $A$-bouquet from the black Tait graph of $10_{124}$ (top), and the $B$-bouquet from the white Tait graph of $10_{124}$ (bottom). Counting vertices and loops of the bouquet gives $V(A) = 3$ and $V(B) = 7$ and thus $g(\Sigma_D) = 1$.

to a $B$-edge and changing a $B$-edge in the other Tait graph to an $A$-edge. A useful notion in understanding the behavior of the Turaev surface is that of $A$-cycles and $B$-cycles. An edge $e$ in the Tait graph is said to be in an $A$-cycle (respectively in a $B$-cycle) if there is a cycle in the graph containing $e$ that is comprised entirely of $A$-edges (respectively $B$-edges).

Let $D$ be a diagram of the link $L$, and let $D'$ be the diagram obtained from $D$ by a single crossing change. Let $G$ and $G^*$ be the Tait graphs of $D$. Let $\Sigma_D$ and $\Sigma_{D'}$ be the two Turaev surfaces. Suppose that $e_A$ and $e_B$ are the edges in the Tait graphs that are associated to the crossing that is changed. Assume that $e_A$ is an $A$-edge. Since $e_B$ is dual to $e_A$, it follows that $e_B$ is a $B$-edge. The crossing change causes $e_A$ to switch to a $B$-edge and $e_B$ to switch to an $A$-edge.

**Theorem 2.12.** Let $D$ be a diagram of a link $L$ and $D'$ be the diagram obtained from $D$ by a single crossing change. Suppose $e_A$ and $e_B$ are the edges in the Tait graphs $G$ and $G^*$ of $D$ associated to the crossing that is changed. Then the genus of the Turaev surface under a crossing change behaves as follows:

1. If $e_A$ is in an $A$-cycle and $e_B$ is in a $B$-cycle, then $g(\Sigma_{D'}) = g(\Sigma_D) + 1$. 

16
2. If $e_A$ is in an $A$-cycle and $e_B$ is not in any $B$-cycle, then $g(\Sigma_{D'}) = g(\Sigma_D)$.

3. If $e_A$ is not in any $A$-cycle and $e_B$ is in a $B$-cycle, then $g(\Sigma_{D'}) = g(\Sigma_D)$.

4. If $e_A$ is not in any $A$-cycle and $e_B$ is not in any $B$-cycle, then $g(\Sigma_{D'}) = g(\Sigma_D) - 1$.

Proof. If $e_A$ is not an edge in $G$, then relabel $G$ and $G^*$ so that it is. In order to compute $V(\mathbb{A})$, the algorithm of Proposition 2.11 states that all $B$-edges are removed from $G$. Since a crossing change switches $e_A$ to a $B$-edge, after the crossing change this edge will be deleted. If $e_A$ is in an $A$-cycle, then this decreases the number of loops in the $A$-bouquet of $G$ by one, and thus $V(\mathbb{A})$ decreases by one. If $e_A$ is not in any $A$-cycle, then this increases the number of vertices in the $A$-bouquet of $G$ by one, and thus $V(\mathbb{A})$ increases by one.

Similarly, in order to compute $V(\mathbb{B})$ all $A$-edges are deleted from $G^*$, and after the crossing change, the edge corresponding to $e_B$ will be deleted. If $e_B$ is in a $B$-cycle, then this decreases the number of loops in the $B$-bouquet of $G^*$ by one, and thus $V(\mathbb{B})$ decreases by one. If $e_B$ is not in any $B$-cycle, this increases the number of vertices in the $B$-bouquet of $G^*$ by one, and thus $V(\mathbb{B})$ increases by one.

The number of edges $E$ is equal to the number of crossings in the diagram, which remains the same under a crossing change. These conditions determine the behavior of the Euler characteristic, and thus the genus of $\Sigma_D$ under a crossing change. □

2.6 Spanning Quasi-trees

For any graph, a spanning tree is a connected, spanning subgraph with no cycles. If the graph is embedded in the plane, then an equivalent definition for a spanning tree is a connected, spanning subgraph with one face. One can generalize the notion of spanning tree of planar graphs to spanning quasi-trees of ribbon graphs. Informally, a spanning quasi-tree of a ribbon graph is a connected, spanning ribbon subgraph with only one face.

Let $G = (\sigma_0, \sigma_1, \sigma_2)$ be a ribbon graph where the set of half edges is $\mathcal{B}$. Let $H = (\sigma_0', \sigma_1', \sigma_2')$ be a ribbon graph where the set of half edges $\mathcal{B}'$ is a subset of $\mathcal{B}$. For each $b' \in \mathcal{B}'$ define $f(b') = \min\{k \in \mathbb{Z}_+ \mid \sigma_0^k(b') \in \mathcal{B}'\}$. 

17
Definition 2.13. A ribbon subgraph of $G = (\sigma_0, \sigma_1, \sigma_2)$ with half edges $B$ is a triple $H = (\sigma'_0, \sigma'_1, \sigma'_2)$ of permutations on the set of half edges $B'$ satisfying:

- $B' \subset B$,
- $\sigma'_1(b') = \sigma_1(b')$ for all $b' \in B'$, and
- $\sigma'_0(b') = \sigma_0^{f(b')}(b')$ for all $b' \in B'$.

Furthermore a spanning quasi-tree $T$ of $G$ is a connected ribbon subgraph of $G$ such that $V(T) = V(G)$ and $F(T) = 1$.

Informally, we think of a ribbon subgraph $H$ of $G$ as a subgraph of $G$ where the cyclic orientation of the edges of $G$ induce the cyclic orientation of the edges of $H$. The ribbon graphs $H$ and $G$ do not necessarily have cellular embeddings onto the same surface. In general, the genus of $H$ is less than or equal to the genus of $G$. Moreover, if the genus of $G$ is zero, then the set of spanning quasi-trees is the set of spanning trees.

The following theorem is implicit in [CKS07]. However, we give a different proof.

Theorem 2.14 (Champanerkar - Kofman - Stoltzfus). Let $D$ be a link diagram, and let $s$ and $s'$ be Kauffman states of $D$. The set of spanning quasi-trees of $G_s$ is in one-to-one correspondence with the set of spanning quasi-trees of $G_{s'}$.

Proof. Suppose $s$ and $s'$ are Kauffman states that differ only at one crossing $x$ of $D$. Furthermore, suppose that $s'$ has one less state circle than $s$, i.e. changing the smoothing at $c$ of $s$ merges two state circles of $s$ into one state circle of $s'$. It is enough to show that the spanning quasi-trees of $G_s$ and $G_{s'}$ are in one-to-one correspondence.

Suppose $a_1, b_1 \in B$ are the two half edges associated to the crossing $c$. Let $T = (\sigma_0, \sigma_1, \sigma_2)$ be a spanning quasi-tree of $G_s$ with half edges $B$. A map from the spanning quasi-trees of $G_s$ to the spanning quasi-trees of $G_{s'}$ is determined by where it sends the edges of each spanning quasi-tree. Since the edges of $G_s$ and $G_{s'}$ are in one-to-one correspondence with the crossings, we identify the two sets. Let $e$ be the edge (in either $G_s$ or $G_{s'}$) that corresponds to the crossing
x. The map is then the identity on the set of edges minus e. If the spanning quasi-tree T of G contains the edge e, then the corresponding spanning quasi-tree T' of G' does not. Conversely, if T does not contain the edge e, then T' does. Since the vertices incident to e are merged together, T' is spanning. It remains to show that $F(T') = 1$.

Suppose T contains the edge e. The endpoints of e are two vertices v and w, which correspond to cycles in $\sigma_0$:

$$v \leftrightarrow (a_1, \ldots, a_k) \text{ and } w \leftrightarrow (b_1, \ldots, b_l).$$

Write the corresponding spanning quasi-tree of G' as $T' = (\sigma'_0, \sigma'_1, \sigma'_2)$. The set of half edges of T' is $B' = B \setminus \{a_1, b_1\}$. For each $b' \in B'$, set $\sigma'_1(b') = \sigma_1(b)$. All cycles of $\sigma'_0$ are the same as cycles of $\sigma_0$ except those two corresponding to the vertices v and w of G. Replace the cycles corresponding to v and w by $(a_2, \ldots, a_k, b_2, \ldots, b_l)$ (as in Figure 2.10).

It remains to show that $\sigma'_2$ has only one orbit. Recall that for any ribbon graph $\sigma_0(\sigma_1(\sigma_2(b))) = b$. Therefore, $\sigma'_2(a_2) = \sigma'_1(b_1) = \sigma_1(b_l) = \sigma_2(\sigma_2(a_2))$. Also, $\sigma'_2(b_2) = \sigma'_1(a_k) = \sigma_1(a_k) = \sigma_2(\sigma_2(b_2))$. For any other $b' \in B'$ such that $b' \neq a_2$ or $b_2$, we have $\sigma'_2(b') = \sigma_2(b')$. Thus the one orbit of $\sigma_2$ is pared down by removing $a_1$ and $b_1$, resulting in $\sigma'_2$. Therefore $\sigma'_2$ has only one orbit.

Suppose T does not contain the edge e. The endpoints of e are two vertices v and w, which correspond to cycles in $\sigma_0$:

$$v \leftrightarrow (a_2, \ldots, a_k) \text{ and } w \leftrightarrow (b_2, \ldots, b_l).$$

Write the corresponding spanning quasi-tree of G' as $T' = (\sigma'_0, \sigma'_1, \sigma'_2)$. The set of half edges of T' is $B' = B \cup \{a_1, b_1\}$. For each $b' \in B'$, set $\sigma'_1(b') = \sigma_1(b)$. All cycles of $\sigma'_0$ are the same as cycles of $\sigma_0$ except those two corresponding to the vertices v and w of G. Replace the cycles corresponding to v and w by one cycle $(a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l)$, thus merging the two vertices into one.

It remains to show that $\sigma'_2$ has one orbit. Observe that $\sigma'_2(a_2) = b_1$ and $\sigma'_2(b_1) = \sigma_1(a_k) = \sigma_2(a_2)$. Also, $\sigma'_2(b_2) = a_1$ and $\sigma'_2(a_1) = \sigma_1(b_l) = \sigma_2(b_2)$. For any $b' \in B'$ such that $b' \neq a_1, a_2, b_1$,
or \(b_2\), we have \(\sigma'_2(b') = \sigma_2(b')\). Thus the one orbit of \(\sigma_2\) is extended by inserting \(a_1\) and \(b_1\) in the appropriate places, resulting in \(\sigma'_2\). Therefore \(\sigma'_2\) has only one orbit.

Hence, the map from the spanning quasi-trees of \(G_s\) to the spanning quasi-trees of \(G_{s'}\) is well defined. It is clear that the map is injective. Also, the map must be surjective since for any spanning quasi-tree of \(G_{s'}\), the above process can be reversed to obtain a spanning quasi-tree of \(G_s\).

\[\Box\]

**Remark 2.15.** Let \(T\) be a spanning quasi-tree of \(s\) and \(T'\) its associated spanning quasi-tree of \(s'\) as in the proof of Theorem 2.14. Since \(s'\) contains one less state circle than \(s\), it follows that \(V(T) = V(T') + 1\). If \(T\) contains the edge associated to the crossing \(x\), then \(T'\) does not. Hence, \(E(T) = E(T') + 1\), and thus \(g(T) = g(T')\). If \(T\) does not contain the edge associated to \(x\), then \(T'\) does. Hence, \(E(T) = E(T') - 1\), and thus \(g(T) = g(T') - 1\).

**Example 2.16.** Let \(D\) be the standard diagram of the left handed trefoil. Label the crossings of \(D\) as in Figure 2.12. Let \(s\) be the all \(A\) Kauffman state. Let \(s'\) be the Kauffman state given by \(A\)-smoothings at crossings 1 and 2, and a \(B\)-smoothing at crossing 3. The two ribbon graphs \(G_s\) and \(G_{s'}\) are pictured to the immediate right of the knot diagram. There are two spanning quasi-trees of \(G_s\) that contain edge 3 and one spanning quasi-tree that does not. Also, there is one spanning quasi-tree of \(G_{s'}\) that contains edge 3 and two that do not.

![Figure 2.12](image-url)  

**FIGURE 2.12.** The spanning quasi-tree correspondence for two Kauffman states of the left handed trefoil.

For any two Kauffman states, there is a one-to-one correspondence between the spanning quasi-trees of their associated ribbon graphs. Also, the Tait graphs are the ribbon graphs for
two special Kauffman states. Since the Tait graphs are embedded in the plane, all of their spanning quasi-trees are genus zero (and thus actual spanning trees). We will further examine the correspondence between spanning trees of the Tait graph and spanning quasi-trees of either \(A\) or \(B\). Since the vertex set of any spanning quasi-tree is the entire vertex set of the ribbon graph, it suffices to define the correspondence on edges alone.

Let \(D\) be a link diagram, \(G\) its Tait graph, and \(A\) and \(B\) the all \(A\) and all \(B\) ribbon graphs respectively. Denote the set of spanning trees of \(G\) by \(\mathcal{T}(G)\) and denote the sets of spanning quasi-trees of \(A\) and \(B\) by \(\mathcal{Q}(A)\) and \(\mathcal{Q}(B)\) respectively. Define maps \(\psi_A : \mathcal{T}(G) \rightarrow \mathcal{Q}(A)\) and \(\psi_B : \mathcal{T}(G) \rightarrow \mathcal{Q}(B)\) as follows. Let \(T \in \mathcal{T}(G)\). An \(A\)-edge of \(G\) is in the spanning quasi-tree \(\psi_A(T)\) if and only if it is in \(T\), and a \(B\)-edge of \(G\) is in the spanning quasi-tree \(\psi_A(T)\) if and only if it is in \(G \setminus T\). Similarly, an \(A\)-edge of \(G\) is in the spanning quasi-tree \(\psi_B(T)\) if and only if it is in \(G \setminus T\), and a \(B\)-edge of \(G\) is in the spanning quasi-tree \(\psi_B(T)\) if and only if it is in \(T\). Theorem 2.14 implies that \(\psi_A\) and \(\psi_B\) are well defined bijections.

**Proposition 2.17** (Champanerkar - Kofman - Stoltzfus). For any spanning tree \(T \in \mathcal{T}(G)\), the genera of \(\psi_A(T)\) and \(\psi_B(T)\) are determined by

\[
\begin{align*}
g(\psi_A(T)) + E_B(T) &= \frac{V(G) + E_B(G) - V(A)}{2} \\
g(\psi_B(T)) + E_A(T) &= \frac{V(G) + E_A(G) - V(B)}{2}.
\end{align*}
\]

### 2.7 The Turaev Surface and Reidemeister Moves

In this section, we examine the behavior of the Turaev surface \(\Sigma_D\) under Reidemeister moves on \(D\). If \(D\) is changed by a Type I Reidemeister move, as in Table 2.1, then there is one more state circle in the all \(A\)-smoothing and the same number of state circles in the all \(B\)-smoothing. Since this move increases the number of crossings by 1, it follows that the genus of the Turaev surface has the same genus before and after a Type I Reidemeister move.

Suppose the diagram \(D\) is changed by a Type II Reidemeister move, as in Table 2.2. In this move, one segment \(l_1\) of \(D\) is pulled over another segment \(l_2\) of \(D\). If \(l_1\) and \(l_2\) are in distinct state circles of the all \(A\)-smoothing (or of the all \(B\)-smoothing), then performing the Type II
Reidemeister move merges those two state circles into one circle. Conversely, if \( l_1 \) and \( l_2 \) are in the same state circle of the all \( A \)-smoothing (or of the all \( B \)-smoothing), then performing the Type II Reidemeister move splits that state circle into two. If pulling \( l_1 \) over \( l_2 \) splits both an all \( A \) state circle and an all \( B \) state circle into two circles, then the genus of the Turaev surface remains the same. If pulling \( l_1 \) over \( l_2 \) splits an all \( A \) state circle into two and merges two all \( B \) state circles into one (or vice versa), then the genus of the Turaev surface increases by one. Finally, if pulling \( l_1 \) over \( l_2 \) merges two state circles in both the all \( A \)-smoothing and the all \( B \)-smoothing, then the genus of the Turaev surface increases by two.

Suppose the diagram \( D \) is changed by a Type III Reidemeister move, as in Table 2.3. The state circles of the all \( A \)-smoothing are unaltered by the pictured move. However, the state circles of the all \( B \)-smoothing are changed. Of course, if one examines the mirror image of this Type III Reidemeister move, the state circles from the all \( A \)-smoothing are changed while
the state circles from the all $B$-smoothing remain the same. There are three line segments in the all $B$-smoothing that appear in the local picture. If these three segments are in three different state circles of the all $B$-smoothing before the move, then they will all be part of the same state circle of the all $B$-smoothing after the move. Therefore, the genus of the Turaev surface will increase by one. If the three segments are in the same all $B$ state circle before the move, then they will be in three separate all $B$ state circles after the move, and the genus of the Turaev surface will decrease by one. If the three segments are in two distinct all $B$ state circles before the move, then they will again be in two distinct all $B$ state circles after the move, and the genus of the Turaev surface is the same.

**TABLE 2.3.** Type III Reidemeister move and the all $A$ and all $B$ Kauffman states

<table>
<thead>
<tr>
<th>$L$</th>
<th>$A$</th>
<th>$B$</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="diagram" /></td>
<td><img src="image2" alt="diagram" /></td>
<td><img src="image3" alt="diagram" /></td>
</tr>
<tr>
<td><img src="image4" alt="diagram" /></td>
<td><img src="image5" alt="diagram" /></td>
<td><img src="image6" alt="diagram" /></td>
</tr>
</tbody>
</table>
Chapter 3
Khovanov Homology

For an oriented link $L \subset S^3$, the Khovanov homology of $L$, denoted $Kh(L)$, is a bigraded $\mathbb{Z}$-module that was defined by Khovanov in [Kho00]. The group $Kh(L)$ is equipped with a homological grading $i$ and a Jones (or polynomial) grading $j$ and is written

$$Kh(L) = \bigoplus_{i,j} Kh^{i,j}(L).$$

Khovanov homology is a generalization of the Jones polynomial; the Jones polynomial of a link $V_L(q)$ is recovered by taking the graded Euler characteristic of the group $Kh(L)$:

$$(q + q^{-1})V_L(q) = \sum_{i,j} (-1)^i \text{rank}(Kh^{i,j}(L)) \cdot q^j.$$

Khovanov homology has many important applications to knot theory. Bar-Natan [BN02] was the first to show that Khovanov homology is a stronger invariant than the Jones polynomial by showing that two knots with the same Jones polynomial have different Khovanov homology. Jacob Rasmussen [Ras04] used Khovanov homology to give the first combinatorial proof of the Milnor conjecture on the slice genus of torus knots. Lehnard Ng [Ng05] established an upper bound on the Thurston-Bennequin number of a Legendrian link using the Khovanov homology of the underlying topological link type.

3.1 The Jones Polynomial

The Jones polynomial of a link $L$, denoted $V_L(q)$, is a Laurent polynomial introduced by Vaughan Jones [Jon85]. The Jones polynomial can be defined via an approach discovered by Louis Kauffman [Kau87].

3.1.1 The Kauffman Bracket Approach

The following is a scaled version of the Kauffman bracket.
Definition 3.1. The Kauffman bracket $\langle D \rangle$ of a link diagram $D$ is defined by the following three relations:

1. $\langle \bigcirc \bigcirc \rangle = 1$.

2. $\langle D \sqcup \bigcirc \rangle = (q + q^{-1}) \langle D \rangle$.

3. $\langle \twisted \rangle = \langle \bigcirc \bigcirc \rangle - q \langle \twisted \rangle$, where the three links agree outside of the pictured neighborhood.

The Kauffman bracket of a link diagram is a Laurent polynomial in the variable $q$. It can be shown the Kauffman bracket is invariant under Type II and III Reidemeister moves. Moreover, it behaves predictably under Type I Reidemeister moves; more specifically, we have

\[ \langle \twisted \twisted \rangle = q^{-1} \langle \twisted \twisted \rangle, \quad \langle \bigcirc \bigcirc \bigcirc \rangle = -q^{2} \langle \twisted \twisted \rangle. \]

If $D$ is a link diagram, let $n_+ = n_+(D)$ denote the number of positive crossings of the diagram and $n_- = n_-(D)$ denote the number of negative crossings of the diagram, where positive and negative crossings are as in Figure 2.9.

The following statement asserts a polynomial obtained from the Kauffman bracket is a link invariant. We also take it to be the definition of the Jones polynomial.

**Theorem 3.2 (Kauffman).** The Jones polynomial of an oriented link $L$ is a link invariant and is defined by

\[ V_L(q) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle, \]

where $D$ is any diagram of $L$.

Recall that a Kauffman state is a choice of an $A$-smoothing or $B$-smoothing for each crossing of a link diagram. Each Kauffman state gives a collection of circles in the plane, and the Kauffman bracket of such a diagram is straightforward to compute. Let $S(D)$ denote the set of Kauffman states for a link diagram $D$. If $s$ is a Kauffman state, then define the height of $s$, denoted $h(s)$, to be the number of $B$-smoothings of $s$. Also, let $|s|$ denote the number of circles.
obtained by smoothing $D$ according to $s$. Then the Kauffman bracket of $D$ can be written as

$$\langle D \rangle = \sum_{s \in S(D)} (-q)^{-h(s)}(q + q^{-1})^{|s| - 1}. \quad (3.1)$$

### 3.1.2 A Spanning Tree Expansion

In [Thi87], Thistlethwaite gives an expansion of the Jones polynomial as a weighted sum over the set of spanning trees of the Tait graph. Instead of smoothing every crossing of a diagram to get the complete set of Kauffman states, one may stop whenever the diagram is obviously the unknot. These obvious unknots correspond to spanning trees, and their Kauffman brackets are easily computable.

Let $D$ be a link diagram with its crossings numbered. The Kauffman bracket of $D$ can be expressed as a state sum over all Kauffman states (see Equation 3.1). The Kauffman states are the leaves in a resolution tree of the diagram. The root of the tree is the diagram $D$, and the children of a diagram are the $A$-smoothing and $B$-smoothing at the appropriate crossing. Figure 3.1 shows an example for the trefoil.

![Figure 3.1. A resolution tree for the trefoil. The leaves are the set of Kauffman states.](image)

If a knot diagram can be transformed into the crossingless diagram of the unknot using only type I Reidemeister moves, then the diagram is called a twisted unknot. The Kauffman
bracket of a twisted unknot $D'$ is given by the formula

$$
\langle D' \rangle = (-1)^{n_-(D')} q^{2n_-(D') - n_+(D')} (q + q^{-1}).
$$

(3.2)

Let $T(D)$ be the set of diagrams in the resolution tree of $D$ that are twisted unknots. Figure 3.2 shows $T(D)$ for the trefoil. Similar to Equation 3.1, one can calculate the Kauffman bracket of a diagram in terms of the Kauffman brackets of its twisted unknots:

$$
\langle D \rangle = \sum_{D' \in T(D)} (-q)^{-h(D')} \langle D' \rangle,
$$

(3.3)

where $h(D')$ is the number of $B$ smoothings needed to obtain $D'$ from $D$.

A *connected Kauffman state* is Kauffman state whose associated smoothing has only one component. Each twisted unknot in the resolution tree has exactly one descendent that is a connected Kauffman state. The set of connected Kauffman states of $D$ are in one-to-one correspondence with set of spanning trees of the Tait graph of $D$. Let $s$ be a connected Kauffman state. An edge $e$ of the Tait graph is in the associated spanning tree if and only if the smoothing of $s$ at the crossing corresponding to $e$ joins the two local faces corresponding to vertices of that Tait graph. Hence the set $T(D)$ can be considered as the set of spanning trees of the Tait graph, and Equation 3.3 is the spanning tree expansion of the Kauffman bracket.
3.2 Constructing $Kh(L)$

We follow Bar-Natan’s “cube of resolutions” construction of Khovanov homology [BN02]. Instead of associating a monomial to each Kauffman state, Khovanov associates a graded vector space. Each Kauffman state corresponds to a vertex in the cube. The chain complex generating $Kh(L)$ is the direct sum of the vector spaces associated to each vertex in the cube. There is an edge from a Kauffman state $s$ to a Kauffman state $s'$ if $s$ and $s'$ differ at one crossing, and $s$ is an $A$-smoothing at that crossing while $s'$ is a $B$-smoothing at that crossing. The differential in the chain complex is based on the edges of the cube. For a schematic of this construction, see Figure 3.3.

3.2.1 Vertices of the Cube

Let $D$ be the diagram of some link $L$. We associate a bigraded $\mathbb{Z}$-module to each Kauffman state. Let $V$ be the free $\mathbb{Z}$-module generated by two elements $v_+$ and $v_-$. Endow $V$ with a homological grading $i$ and a Jones grading $j$ such that $i(v_+) = i(v_-) = 0$, $j(v_+) = 1$, and $j(v_-) = -1$.

For any $\mathbb{Z}$-module $M$ with homological and Jones gradings, let $M[s]$ denote the module $M$ with its homological grading shifted $s$ degrees. Thus homological degree $r$ of $M$ corresponds to the homological degree $r - s$ of $M[s]$. Similarly, let $M\{s\}$ denote the module $M$ with its Jones grading shifted $s$ degrees.

Let $\mathcal{X}$ be the set of crossings of $D$. A vertex in the cube of resolutions is a Kauffman state, thought of as an element $\alpha \in \{A,B\}^{\mathcal{X}}$. For each vertex $\alpha$, define the height of $\alpha$ to be the number of $B$-smoothings in $\alpha$. To each vertex $\alpha$ in the cube, we associate the vector space $V_{\alpha}(D) := V^{\otimes k}\{r\}$, where $k$ is the number of circles in the smoothing corresponding to $\alpha$ and $r$ is the height of $\alpha$. Let $n_+$ and $n_-$ be the number of positive and negative crossings in $D$ respectively. Define the chain complex $CKh(D)$ to be the direct sum of the $V_\alpha$’s over all vertices shifted by $-n_-$ in the homological degree and by $n_+ - 2n_-$ in the Jones degree. More
specifically, define

\[ CKh(D) = \bigoplus_{\alpha \in \{A, B\}^X} V_{\alpha}(D)[−n−]\{n+ − 2n−\}. \]

### 3.2.2 Edges of the Cube

The differential in this chain complex is a sum of maps that are associated to each edge \( \xi \) of the cube. Label the edges of the cube by sequences in \( \{A, B, *\}^X \) such that the sequence for an edge has a * for the crossing where the vertices it connects are different and is identical to the vertices it connects otherwise. Define the height of an edge \( |\xi| \) to be the height of its tail, i.e., the number of \( B \)-smoothings in the sequence associated to the edge. The edge maps are denoted \( d_\xi \), and hence the differential of the complex is defined as \( d^r := \sum_{|\xi|=r} (-1)^\xi d_\xi. \)

We will define the edge maps \( d_\xi \) so that each square in the cube commutes. Given that all the squares in the cube of resolutions commute, if one multiplies some edge maps by \(-1\) so that every square has an odd number of negative edge maps, then each square must anti-commute, thus ensuring that \( d \circ d = 0 \). This can be done by multiplying \( d_\xi \) by \((-1)^\xi\), which is defined to be \(-1\) to the number of \( B \)-smoothings before the * in \( \xi \). In Figure 3.3, the negative edge maps are indicated by the small circles on their tails.

The edge maps \( d_\xi \) are defined to make the cube commutative (when taken without signs). In the complex \( CKh(D) \) each state circle is associated to a copy of the vector space \( V \). For any edge \( \xi \), the vertex at the head can be obtained from the vertex at the tail by either merging two state circles or splitting a single state circle into two. The edge maps are defined to be the identity on any of the state circles that are identical in the vertices at the head and tail of that edge. If two state circles are merged, then we define a multiplication map \( m : V \otimes V \to V \), and if one state is split into two, then we define a comultiplication map \( \Delta : V \to V \otimes V \).
maps $m$ and $\Delta$ are defined by

$$(\bigcirc \bigcirc \longrightarrow \bigcirc \bigcirc) \longrightarrow (V \otimes V \xrightarrow{m} V) \quad m : \begin{cases} v_+ \otimes v_- \mapsto v_- \quad v_+ \otimes v_+ \mapsto v_+ \\ v_- \otimes v_- \mapsto v_- \quad v_- \otimes v_- \mapsto 0 \end{cases} \quad (3.4)$$

$$(\bigcirc \bigcirc \longrightarrow \bigcirc \bigcirc) \longrightarrow (V \Delta V \otimes V) \quad \Delta : \begin{cases} v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ v_- \mapsto v_- \otimes v_- \end{cases} \quad (3.5)$$

Figure 3.3 shows the cube of resolutions for the left-handed trefoil. Inside the box for each vertex is the Kauffman state given as both a diagram and the choice of A or B smoothings for each crossing and also the vector space associated to that Kauffman state. Each chain group is the direct sum of the $V_\alpha$’s for all the vertices in one column. Here we use the notation $C^r$ for the $r$th chain group of the unshifted $CKh$ complex. The differential is defined as a sum of the edge maps along a column. Also, recall that edges with a small circle on the tail have negative signs in the differential. One can check that all squares in the cube have an odd number of negative edges.

3.2.3 The Homology of the Cube

**Definition 3.3.** Let $D$ be a diagram of a link $L$. The homology of the complex $CKh(D)$ will be equivalently denoted by $Kh(D)$ or $Kh(L)$ and is called the **Khovanov homology of $L$**. Khovanov homology has a homological grading $i$ and a Jones grading $j$ and is written

$$Kh(L) = \bigoplus_{i,j} Kh^{i,j}(L).$$

**Theorem 3.4 (Khovanov).** The Khovanov homology of a link $L$ is a link invariant. Moreover, the graded Euler characteristic gives the Jones polynomial:

$$(q + q^{-1})V_L(q) = \sum_{i,j} (-1)^i \text{rank}(Kh^{i,j}(L)) \cdot q^j.$$

Khovanov homology tends to be supported on slope two lines with respect to the bigrading. In order to capture this behavior, we define an auxiliary grading called the $\delta$-grading by

$$\delta(x) = j(x) - 2i(x).$$
If $Kh(L)$ is decomposed using the $\delta$-grading, then it is written

$$Kh(L) = \bigoplus_\delta K h^\delta(L).$$

For a given link $L$, either $Kh(L)$ is supported in all even $\delta$-gradings or in all odd $\delta$-gradings.

**Definition 3.5.** If $\delta_{\text{min}}$ is the minimum $\delta$-grading where $Kh(L)$ is nontrivial and $\delta_{\text{max}}$ is the maximum $\delta$-grading where $Kh(L)$ is nontrivial, then $Kh(L)$ is said to be $[\delta_{\text{min}}, \delta_{\text{max}}]$-thick.

The **Khovanov width of $L$** is defined by

$$w_{Kh}(L) = \frac{1}{2}(\delta_{\text{max}} - \delta_{\text{min}}) + 1.$$

If $\mathbb{F}$ is a field, then let $Kh(L; \mathbb{F})$ denote $Kh(L) \otimes \mathbb{F}$ and $w_{Kh}(L; \mathbb{F})$ denote the width of $Kh(L; \mathbb{F})$.

Let $L_1$ and $L_2$ be oriented links, and let $C$ be a component of $L_1$. Denote by $l$ the linking number of $C$ with its complement $L_1 - C$. Let $L'_1$ be the link $L_1$ with the orientation of $C$ reversed. Denote the mirror image of $L_1$ by $\overline{L_1}$ and the disjoint union of $L_1$ and $L_2$ by $L_1 \sqcup L_2$.

The following proposition was proved by Khovanov in [Kho00].
Proposition 3.6 (Khovanov). For \( i, j \in \mathbb{Z} \) there are isomorphisms

\[
Kh^{i,j}(L_1') \cong Kh^{i+2l,j+2l}(L_1),
\]

\[
Kh^{i,j}(\overline{L}_1; \mathbb{Q}) \cong Kh^{-i,-j}(L_1; \mathbb{Q})
\]

\[
Tor(Kh^{i,j}(\overline{L}_1)) \cong Tor(Kh^{1-i,-j}(L_1)), \text{ and}
\]

\[
Kh^{i,j}(L_1 \sqcup L_2) \cong \bigoplus_{k,m \in \mathbb{Z}} (Kh^{k,m}(L_1) \otimes Kh^{i-k,j-m}(L_2)) \oplus \bigoplus_{k,m \in \mathbb{Z}} Tor_1(Kh^{k,m}(L_1), Kh^{i-k+1,j-m}(L_2))
\]

Let \( D \) be a diagram for \( L_1 \) and \( D' \) be the diagram \( D \) with the component \( C \) reversed. Denote the number of negative crossings in \( D \) by \( n_-(D) \), where the sign of a crossing is as in Figure 2.9. Set \( s = n_-(D) - n_-(D') \). Then Proposition 3.6 implies

\[
Kh^\delta(D') \cong Kh^{\delta+s}(D), \text{ and}
\]

\[
Kh^\delta(\overline{L}_1; \mathbb{Q}) \cong Kh^{-\delta}(L_1; \mathbb{Q}).
\]

### 3.3 Reduced Khovanov Homology

Khovanov [Kho03] introduced a variant of his original construction known as reduced Khovanov homology. The reduced Khovanov homology, denoted \( \widehat{Kh}(L) \), is similar to \( Kh(L) \) except its graded Euler characteristic is \( V_L(q) \) instead of \( (q + q^{-1})V_L(q) \).

In the complex \( CKh(D) \), one associates the vector space \( V = \langle v_-, v_+ \rangle \) to each state circle. Fix a point \( p \) on the diagram \( D \) away from the crossings. For each vertex, instead of associating the vector space \( V \) to the state circle with the point \( p \), either associate \( \langle v_+ \rangle \) or \( \langle v_- \rangle \) to the circle containing \( p \). In the former case, we obtain the complex \( CKh_+(D) \) and in the latter case, we obtain the complex \( CKh_-(D) \). This gives a decomposition of the entire complex: \( CKh(D) = CKh_+(D) \oplus CKh_-(D) \). In fact, \( CKh_+(D) \) is a subcomplex of \( CKh(D) \).

**Definition 3.7.** Let \( D \) be a diagram of the link \( L \) and \( p \) be some some point on \( D \) away from the crossings. Suppose the point \( p \) is on the component \( C \) of \( L \). The reduced Khovanov homology of \( L \), denoted \( \widehat{Kh}(L, C) \), is the homology of the complex \( CKh_+(D) \).
Theorem 3.8 (Khovanov). The reduced Khovanov homology of a link $L$ depends only on the link $L$ and a choice of a component $C$. In particular, if $L$ is a knot, then the reduced Khovanov homology is a knot invariant. Moreover, the graded Euler characteristic gives the Jones polynomial:

$$V_L(q) = \sum_{i,j} (-1)^i \text{rank}(\tilde{Kh}^{i,j}(L,C)) \cdot q^j.$$ 

Similar to Khovanov homology, reduced Khovanov homology has a homological grading $i_{\tilde{Kh}}$, a Jones grading $j_{\tilde{Kh}}$, and a $\delta$-grading $\delta_{\tilde{Kh}} = j_{\tilde{Kh}} - 2i_{\tilde{Kh}}$. We will sometimes denote the gradings of Khovanov with the subscript $Kh$ and the gradings of reduced Khovanov homology with the subscript $\tilde{Kh}$ to avoid confusion. If it is clear from context which invariant we are discussing, then we will omit the subscripts. As with Khovanov homology, if $\tilde{\delta}_{\text{min}}$ is the minimum $\delta$-grading where $\tilde{Kh}(L,C)$ is nontrivial and $\tilde{\delta}_{\text{max}}$ is the maximum $\delta$-grading where $\tilde{Kh}(L,C)$ is nontrivial, then we say that $\tilde{Kh}(L,C)$ is $[\tilde{\delta}_{\text{min}}, \tilde{\delta}_{\text{max}}]$-thick. The reduced Khovanov width is defined as $w_{\tilde{Kh}}(L) = \frac{1}{2}(\tilde{\delta}_{\text{max}} - \tilde{\delta}_{\text{min}}) + 1$. Asaeda and Przytycki [AP04] show that Khovanov and reduced Khovanov homology are related by a long exact sequence.

Proposition 3.9 (Asaeda-Przytycki). There is a long exact sequence relating the reduced and unreduced versions of Khovanov homology:

$$\cdots \rightarrow \tilde{Kh}^{i,j+1}(L,C) \rightarrow Kh^{i,j}(L) \rightarrow \tilde{Kh}^{i,j-1}(L,C) \rightarrow \tilde{Kh}^{i+1,j+1}(L,C) \rightarrow \cdots$$

Corollary 3.10. Let $L$ be a link with marked component $C$. Then $Kh(L)$ is $[\delta_{\text{min}}, \delta_{\text{max}}]$-thick if and only if $\tilde{Kh}(L,C)$ is $[\delta_{\text{min}} + 1, \delta_{\text{max}} - 1]$-thick. Hence $w_{Kh}(L) - 1 = w_{\tilde{Kh}}(L)$.

Proof. The long exact sequence of Theorem 3.9 can be rewritten with respect to the $\delta$-grading as

$$\cdots \rightarrow \tilde{Kh}^{\delta+1}(L,C) \rightarrow Kh^{\delta}(L) \rightarrow \tilde{Kh}^{\delta-1}(L,C) \rightarrow \tilde{Kh}^{\delta-1}(L,C) \rightarrow \cdots.$$ 

Suppose $Kh(L)$ is $[\delta_{\text{min}}, \delta_{\text{max}}]$-thick. Therefore $\tilde{Kh}^{\delta}(L,C) = 0$ for $\delta > \delta_{\text{max}} + 1$ and for $\delta < \delta_{\text{min}} - 1$. 

33
Suppose $\widetilde{Kh}_{\delta_{\max} + 1} (L, C)$ is nontrivial. Then for some $i$ and $j$ where $j - 2i = \delta_{\max} + 1$, the group $\widetilde{Kh}^{i,j} (L, C)$ is nontrivial. By repeatedly applying the long exact sequence of Proposition 3.9, one sees that $\widetilde{Kh}^{i+k,j+2k} (L, C)$ is nontrivial for all $k \geq 0$. However, the group $\widetilde{Kh}_{\delta_{\max} + 1} (L, C)$ is finitely generated. Hence $\widetilde{Kh}_{\delta_{\max} + 1} (L, C)$ is trivial. Similarly, one can show that $\widetilde{Kh}_{\delta_{\min} - 1} (L, C)$ is also trivial.

The long exact sequence also implies that $\widetilde{Kh}_{\delta_{\max} - 1} (L, C)$ and $\widetilde{Kh}_{\delta_{\min} + 1} (L, C)$ are nontrivial. Thus $\widetilde{Kh}(L, C)$ is $[\delta_{\min} + 1, \delta_{\max} - 1]$-thick.

Suppose $\widetilde{Kh}(L, C)$ is $[\delta_{\min} + 1, \delta_{\max} - 1]$-thick. Similar to the case above, if either $Kh_{\delta_{\min}} (L)$ or $Kh_{\delta_{\max}} (L)$ are trivial, then one can show that $\widetilde{Kh}_{\delta_{\min} + 1} (L, C)$ or $\widetilde{Kh}_{\delta_{\max} - 1} (L, C)$ respectively are infinitely generated. Hence $Kh(L)$ is $[\delta_{\min}, \delta_{\max}]$-thick.

Corollary 3.10 implies that if $C$ and $C'$ are two components of $L$, then $\widetilde{Kh}(L, C)$ is $[\tilde{\delta}_{\min}, \tilde{\delta}_{\max}]$-thick if and only if $\widetilde{Kh}(L, C')$ is $[\tilde{\delta}_{\min}, \tilde{\delta}_{\max}]$-thick. Hence, the notation $w_{\widetilde{Kh}}(L)$ is unambiguous.

### 3.4 Khovanov Homology and Spanning Trees

To construct Khovanov homology, graded vector spaces are associated to each of the Kauffman states and these vector spaces form a chain complex whose homology gives us the desired invariant. The cube of resolutions is a generalization of the Kauffman state sum from Equation 3.1. As shown in Equation 3.3, one can also write the Kauffman bracket as a sum over the spanning trees of the Tait graph. Wehrli [Weh08] and Champanerkar and Kofman [CK09a] generalize the spanning tree expansion of the Kauffman bracket to a spanning tree model for Khovanov homology. Essentially, the spanning tree model for Khovanov homology says that the cube of resolutions complex retracts onto a complex generated by twisted unknots. As discussed in Section 3.1.2, the set of twisted unknots is in one-to-one correspondence with the set of spanning trees of the Tait graph.

Define the spanning tree complex for Khovanov homology as

$$CKh_{\text{tree}}(D) = \bigoplus_{T \in T(G)} \mathbb{Z}[T_+, T_-]$$
and define the spanning tree complex for reduced Khovanov homology as

$$\widetilde{CKh}_{\text{tree}}(D) = \bigoplus_{T \in T(G)} \mathbb{Z}[T].$$

**Proposition 3.11** (Wehrli, Champanerkar - Kofman). *Let D be a diagram for a link L.*

1. There exists a spanning tree complex $CKh_{\text{tree}}(D)$ whose homology is $Kh(L)$.

2. There exists a spanning tree complex $\widetilde{CKh}_{\text{tree}}(D)$ whose homology is $\widetilde{Kh}(L, C)$, where $C$ is some component of $L$.

The gradings of the Khovanov complex and the reduced Khovanov complex are related by

$$i_{Kh}(T_+) = i_{\widetilde{Kh}}(T) = i_{Kh}(T)$$

and

$$j_{Kh}(T_+) - 1 = j_{\widetilde{Kh}}(T) = j_{Kh}(T_-) + 1,$$

for any tree $T \in T(G)$.

Recall that edges in the Tait graph are either $A$-edges or $B$-edges and also, are either positive or negative edges. Choose the Tait graphs $G$ and $G^*$ so that $E_B(G) \geq E_B(G^*)$. The $\delta$-grading corresponding to a spanning tree $T$ in $\widetilde{CKh}_{\text{tree}}(D)$ is

$$\delta_{\widetilde{Kh}}(T) = 2E_B(T) + \frac{1}{2}(E^+(G) - E^-(G) - E_B(G) + E_A(G) - 2(V(G) - 1)). \quad (3.6)$$

If a link is alternating, then all of the edges in the Tait graph $G$ are $B$-edges. Therefore, for any two spanning trees $T, T' \in T(G)$, we have $\delta_{\widetilde{Kh}}(T) = \delta_{\widetilde{Kh}}(T')$. Since the differential $d$ of the chain complex lowers $\delta$-grading by two, it follows that the differential in the reduced Khovanov spanning tree complex coming from an alternating diagram is zero. Furthermore, Lee [Lee02] uses the Gordon-Litherland [GL78] formula for the signature of a link to show that if $T \in T(G)$ where $G$ is the Tait graph for some alternating diagram, then $\delta(T) = -\sigma(L)$.

**Theorem 3.12** (Lee). *Let L be an nonsplit alternating link. Then $\widetilde{Kh}(L)$ is entirely supported in the $\delta = -\sigma(L)$ grading.*
Theorems 3.8 and 3.12 imply that the reduced Khovanov homology of a nonsplit alternating link is entirely determined by its Jones polynomial and signature.

Manturov [Man05] and Champanerkar, Kofman, and Stoltzfus [CKS07] proved that Turaev genus gives an upper bound for reduced Khovanov width.

**Proposition 3.13** (Manturov, Champanerkar - Kofman - Stoltzfus). Let $L$ be a link. Then

$$\overline{w_{Kh}}(L) \leq g_T(L) + 1.$$  

**Proof.** Recall that there is a bijection $\psi_A$ from spanning trees of $G$ to spanning quasi-trees of the all $A$ ribbon graph $A$. Proposition 2.17 says that the sum of number of $B$-edges in the spanning tree $T$ and the genus of the spanning quasi-tree $\psi_A(T)$ is a constant depending only on the link diagram. More specifically, we have

$$g(\psi_A(T)) + E_B(T) = \frac{V(G) + E_B(G) - V(A)}{2}.$$

By Equation 3.6, the $\delta$-grading of a spanning tree $T$ is twice the number of $B$-edges in the tree plus some constant depending only on the diagram. Therefore, two spanning trees have the same $\delta$-grading if and only if their associated spanning quasi-trees have the same genus.

There is a complex that generates reduced Khovanov homology where the generators are in one-to-one correspondence with spanning quasi-trees of $A$, and the $\delta$-grading of the spanning quasi-tree is twice its genus. The minimum genus spanning quasi-tree of $A$ has genus 0, and the maximum genus spanning quasi-tree of $A$ has genus $g(\Sigma_D)$. Let $\delta_{\min}(D)$ and $\delta_{\max}(D)$ be the minimum and maximum $\delta$-gradings for any tree $T$ in $T(G)$. Since the grading of any spanning quasi-tree is twice its genus, we have $\delta_{\max}(D) - \delta_{\min}(D) = 2g(\Sigma_D)$. The support of the homology is necessarily no greater than the support of the complex that generates it. Therefore, we obtain the inequality $\overline{w_{Kh}}(L) - 1 \leq \frac{1}{2}(\delta_{\max}(D) - \delta_{\min}(D))$ which proves the result. \[\square\]
3.5 The Long Exact Sequence and Quasi-alternating Links

Let $D$ be a link diagram, and let $D_A$ and $D_B$ be two link diagrams that differ from $D$ in a neighborhood of a crossing $x$ as in Figure 3.4 and are otherwise identical to $D$.

There is a short exact sequence between the three cube of resolution complexes of the links with diagram $D$, $D_A$, and $D_B$:

$$0 \to CKh(D_B) \xrightarrow{f} CKh(D) \xrightarrow{g} CKh(D_A) \to 0.$$  

The map $f$ maps the complex $CKh(D)$ to the subcomplex of $CKh(D)$ generated by Kauffman states with a $B$-smoothing at crossing $x$ via the identity map. The map $g$ is the identity on the vector spaces associated to Kauffman states of $D$ with an $A$-smoothing at $x$ and 0 otherwise. One can check that $f$ and $g$ are chain maps and that $\text{im}(f) = \text{ker}(g)$.

The short exact sequence on chain complexes induces a long exact sequence of the homology groups. Since the gradings on Khovanov homology depend on the number of positive and negative crossings of the diagram, our diagrams need to be oriented. Let $D_+, D_-, D_v,$ and $D_h$ be diagrams of links that agree outside a neighborhood of a crossing $x$ as in Figure 3.5. Choose some orientation on $D_h$, and define $e = n_-(D_h) - n_-(D_+)$. There are long exact sequences...
relating the Khovanov homology of these links. Khovanov [Kho00] implicitly describes these sequences, and Viro [Vir04] explicitly states both sequences. The graded versions are taken from Rasmussen [Ras05] and Manolescu - Oszváth [MO07].

**Theorem 3.14** (Khovanov). There are long exact sequences

\[ \cdots \to Kh^{i-e-1,j-3e-2}(D_h) \to Kh^{i,j}(D_+)^{\delta_v} \to Kh^{i,j-1}(D_v) \to Kh^{i-e,j-3e-2}(D_h) \to \cdots \]

and

\[ \cdots \to Kh^{i+1,j}(D_v) \to Kh^{i,j}(D_-) \to Kh^{i-e+1,j-3e+2}(D_h) \to Kh^{i+1,j+1}(D_v) \to \cdots . \]

When only the \( \delta = j - 2i \) grading is considered, the long exact sequences become

\[ \cdots \to Kh^{\delta-e}(D_h) \xrightarrow{f_{\delta-e}} Kh^{\delta}(D_+) \xrightarrow{g_{\delta}} Kh^{\delta-1}(D_v) \xrightarrow{h_{\delta-1}^{\delta-e}} Kh^{\delta-e-2}(D_h) \to \cdots \]

and

\[ \cdots \to Kh^{\delta+1}(D_v) \xrightarrow{f_{\delta+1}} Kh^{\delta}(D_-) \xrightarrow{g_{\delta}} Kh^{\delta-e}(D_h) \xrightarrow{h_{\delta}^{\delta-e}} Kh^{\delta-1}(D_v) \to \cdots . \]

There are versions of these long exact sequences where Khovanov homology is replaced with reduced Khovanov homology. In the reduced sequences, the gradings are identical to the unreduced sequences.

Theorem 3.14 directly implies the following corollary:

**Corollary 3.15.** Let \( D_+, D_-, D_v \) and \( D_h \) be as in Figure 3.5. Suppose \( Kh(D_v) \) is \([v_{\min}, v_{\max}]\)-thick and \( Kh(D_h) \) is \([h_{\min}, h_{\max}]\)-thick. Then \( Kh(D_+) \) is \([\delta_{\min}^+, \delta_{\max}^+]\)-thick, and \( Kh(D_-) \) is \([\delta_{\min}^-, \delta_{\max}^-]\)-thick, where

\[
\delta_{\min}^+ = \begin{cases} 
\min\{v_{\min} + 1, h_{\min} + e\} & \text{if } v_{\min} \neq h_{\min} + e + 1 \\
v_{\min} + 1 & \text{if } v_{\min} = h_{\min} + e + 1 \text{ and } h_{\min}^{v_{\min}} \text{ is surjective} \\
v_{\min} - 1 & \text{if } v_{\min} = h_{\min} + e + 1 \text{ and } h_{\min}^{v_{\min}} \text{ is not surjective,}
\end{cases}
\]

38
Recall that the determinant of a link \( L \), denoted \( \det(L) \), is defined equivalently as \( \det(L) = |V_L(-1)| \) or as \( \det(L) = |\Delta_L(-1)| \), where \( V_L \) denotes the Jones polynomial and \( \Delta_L \) denotes the Alexander polynomial.

**Definition 3.16.** Let \( L, L_A, \) and \( L_B \) be links with diagrams \( D, D_A, \) and \( D_B \) as in Figure 3.4. The set \( Q \) of quasi-alternating links is defined by

- The unknot is in \( Q \);

- If the link \( L \) has a diagram with a crossing \( x \) such that

  1. both of the links \( L_A \) and \( L_B \) are in \( Q \),

  2. \( \det(L) = \det(L_A) + \det(L_B) \),

then \( L \) is in \( Q \). We say that \( D \) is quasi-alternating at \( x \).

Every alternating link is also quasi-alternating. Manolescu and Ozsváth [MO07] generalize Theorem 3.12 to quasi-alternating links.
**Theorem 3.17** (Manolescu - Ozsváth). *Let L be a quasi-alternating link. Then $\tilde{Kh}(L)$ is entirely supported in the $\delta = -\sigma(L)$ grading.*

**Proof.** Using the Gordon - Litherland [GL78] formula for computing signature, one can show that the long exact sequences of Theorem 3.14 become

$$
\cdots \rightarrow \tilde{Kh}^{\delta - \sigma(L_B)}(L_B) \rightarrow \tilde{Kh}^{\delta - \sigma(L)}(L) \rightarrow \tilde{Kh}^{\delta - \sigma(L_A)}(L_A) \rightarrow \tilde{Kh}^{\delta - \sigma(L_B) - 2}(L_B) \rightarrow \cdots
$$

The unknot satisfies the conclusions of the theorem. If $L_A$ and $L_B$ satisfy the conditions of the theorem, then the long exact sequence implies that $L$ does as well. \[\square\]
Chapter 4
Twisting Links

In this chapter, we construct infinite classes of links with the same Khovanov width. One can replace a crossing in a link diagram with any rational tangle to obtain a new link. If the original diagram satisfies certain conditions, then the new link will have the same Khovanov width. Similarly, the genus of the Turaev surface does not change under this operation.

4.1 Khovanov Width and Twisting Links

Let \( \tau = C(a_1, \ldots, a_m) \) be a rational tangle, and let \( D \) be a link diagram with a distinguished crossing \( x \). Suppose the slopes of the arcs near \( x \) are \( \pm 1 \).

**Definition 4.1.** Define \( D \) twisted at \( x \) by \( \tau \) to be the diagram obtained by removing \( x \) and inserting \( \tau \) such that a neighborhood of the rightmost crossing or topmost crossing of \( \tau \) in \( D_\tau \) looks exactly like a neighborhood of \( x \) in \( D \). The resulting link diagram is denoted \( D_\tau \).

The main result of this section, Main Theorem 2, is a generalization of a proposition proved by Champanerkar and Kofman in [CK09b].
Proposition 4.2 (Champanerkar-Kofman). Let $D$ be a link diagram with crossing $x$, and let $\tau$ be an alternating rational tangle such that $D$ is twisted at $x$ by $\tau$. If $D$ is quasi-alternating at $x$, then $D_\tau$ is quasi-alternating at each crossing of $\tau$.

Recall that Theorem 3.17 states that all quasi-alternating links have reduced Khovanov width one (and thus Khovanov width two). The previous proposition implies that if one twists a diagram by a rational tangle at a quasi-alternating crossing, then the resulting link has reduced Khovanov width one.

Let $D$ be a diagram with crossing $x$. Resolve $D$ at the crossing $x$ to obtain diagrams $D_v$ and $D_h$. Suppose $\tilde{Kh}(D_v)$ is $[v_{\text{min}}, v_{\text{max}}]$-thick and $Kh(D_h)$ is $[h_{\text{min}}, h_{\text{max}}]$-thick. As before, set $e = n_-(D_h) - n_-(D_+)$, where $D_+$ is the same diagram as $D$ except if the crossing $x$ in $D$ is negative, then it is changed to positive in $D_+$.

Definition 4.3. The diagram $D$ is said to be width-preserving at $x$ if either of the following conditions hold.

- If $x$ is a positive crossing in $D$, then both $v_{\text{min}} \neq h_{\text{min}} + e + 1$ and $v_{\text{max}} \neq h_{\text{max}} + e + 1$.
- If $x$ is a negative crossing in $D$, then both $v_{\text{min}} \neq h_{\text{min}} + e - 1$ and $v_{\text{max}} \neq h_{\text{max}} + e - 1$.

Proposition 4.4. Let $D$ be a link diagram with crossing $x$. If $D$ is quasi-alternating at $x$, then $D$ is width-preserving at $x$.

Proof. Suppose $D$ is quasi-alternating at $x$. Let $D_v$ and $D_h$ be the two resolutions of $D$ at $x$. Since $D$ is quasi-alternating at $x$, it follows that $D_v$ and $D_h$ are also quasi-alternating. Theorem 3.17 implies that $\tilde{Kh}(D)$, $\tilde{Kh}(D_v)$ and $\tilde{Kh}(D_h)$ are each supported entirely in one $\delta$-grading. Suppose $\tilde{Kh}(D_v)$ is supported in $\delta$-grading $v$ and $\tilde{Kh}(D_h)$ is supported in $\delta$-grading $h$. Corollary 3.9 implies that $Kh(D_v)$ is $[v - 1, v + 1]$-thick and $Kh(D_h)$ is $[h - 1, h + 1]$-thick. Let $e = n_-(D_h) - n_-(D_+)$ where $D_+$ is the same diagram as $D$ except if $x$ is negative in $D$, then it is changed to positive in $D_+$. Since $\det(D) = \det(D_v) + \det(D_h)$, it follows that the nontrivial parts of $\tilde{Kh}(D)$, $\tilde{Kh}(D_v)$ and $\tilde{Kh}(D_h)$ lie in three consecutive spots in the long
exact sequence of Theorem 3.14 such that $\widetilde{Kh}(D_v)$ and $\widetilde{Kh}(D_h)$ are not adjacent. Therefore, if $x$ is positive, then $v = h + e - 1$, and if $x$ is negative, then $v = h + e + 1$. The result follows directly. $\square$

**Lemma 4.5.** Let $D$ be an oriented link diagram with crossing $x$, and let $\tau$ be an alternating rational tangle with exactly two crossings $x_0$ and $x_1$. Let $D_{\tau}$ be $D$ twisted at $x$ by $\tau$. If $D$ is width-preserving at $x$, then for any orientation, $D_{\tau}$ is width-preserving at $x_0$ and $x_1$. Moreover, $w_{Kh}(D) = w_{Kh}(D_{\tau})$.

**Proof.** There are two ways to twist $D$ at $c$, either horizontally or vertically. Let $\tau_1 = C(2)$ and $\tau_2 = C(-2)$.

For each case, it is only necessary to prove the result for one choice of orientations on $D$ and $D_{\tau}$. Proposition 3.6 implies the result for all other choices of orientations on $D$ and $D_{\tau}$.

Let $D_v$ and $D_h$ be the diagrams obtained by resolving $D$ at $x$, and let $D^i_v$ and $D^i_h$ be the diagrams obtained by resolving $D_{\tau}$ at the crossing $x_i$ for $i = 0, 1$. Suppose $Kh(D_v)$ and $Kh(D_h)$ are $[v_{\min}, v_{\max}]$-thick and $[h_{\min}, h_{\max}]$-thick respectively. Let $e = n_-(D_h) - n_-(D_+)$ where $D_+$ is the same diagram as $D$ except if the crossing $x$ is negative in $D$, then it is changed to positive in $D_+$. Similarly set $e_i = n_-(D^i_h) - n_-(D^i_+)$ where $D^i_+$ is the same diagram as $D_{\tau}$ except if the crossing $x_i$ is negative in $D_{\tau}$, then it is changed to positive in $D^i_+$.

![FIGURE 4.2. The resolutions for $x$ positive and $\tau = C(2)$.](image)

Suppose $x$ is positive. Choose the orientation on $D_{\tau_1}$ given in Figure 4.2. Also, Figure 4.2 shows the resolutions $D^0_v$ and $D^0_h$. 43
Observe that \( x_i \) is positive in \( D_{\tau_1} \) for \( i = 0, 1 \). Corollary 3.15 implies that \( Kh(D) \) is \([\alpha, \beta]\)-thick where \( \alpha = \min \{v_{\min} + 1, h_{\min} + e\} \) and \( \beta = \max \{v_{\max} + 1, h_{\max} + e\} \). The diagrams \( D^i_v \) and \( D \) represent the same link, and the diagrams \( D_h \) and \( D^i_h \) represent the same link. Therefore, \( Kh(D^i_v) \) is \([\alpha, \beta]\)-thick and \( Kh(D^i_h) \) is \([h_{\min}, h_{\max}]\)-thick. The diagram \( D^i_h \) is the same as the diagram \( D_h \) except \( D^i_h \) has one additional negative Reidemeister I twist, and hence

\[
n_-(D^i_h) = n_-(D_h) + 1.
\]

Since the diagrams \( D \) and \( D^i_v \) are identical, \( n_-(D) = n_-(D^i_v) \). Thus \( e_i = e + 1 \). Since \( D \) is width-preserving, it follows that \( v_{\min} \neq h_{\min} + e + 1 \) and \( v_{\max} \neq h_{\max} + e + 1 \). Therefore,

\[
h_{\min} + e_i + 1 = h_{\min} + e + 2 \neq \alpha,
\]

and

\[
h_{\max} + e_i + 1 = h_{\max} + e + 2 \neq \beta.
\]

Hence \( D_{\tau_1} \) is width-preserving at \( x_i \). Also, Corollary 3.15 implies that \( Kh(D_{\tau_1}) \) is \([\alpha + 1, \beta + 1]\)-thick, and thus \( w_{Kh}(D) = w_{Kh}(D_{\tau_1}) \).

![Figure 4.3](image)

**FIGURE 4.3.** The resolutions for \( x \) positive, \( \tau = C(-2) \), and with the depicted strands of \( D \) in the same component.

The possible orientations of \( D_{\tau_2} \) depend on whether the strands forming the crossing \( x \) are in the same component of \( D \) or different components of \( D \). Suppose they are in the same component. Choose the orientation on \( D_{\tau_2} \) given in Figure 4.3. Also, Figure 4.3 shows the resolutions \( D^0_v \) and \( D^0_h \).

Observe that \( x_i \) is positive in \( D_{\tau_2} \) for \( i = 0, 1 \). With suitably chosen orientations, we have

\[
n_-(D_v) = n_-(D) = n_-(D^i_h), \tag{4.1}
\]

and

\[
n_-(D^i_v) = n_-(D_h). \tag{4.2}
\]

44
The diagram $D^i_v$ is the same as $D_v$ except $D^i_v$ has one component reversed and an additional positive Reidemeister I twist. Therefore, Proposition 3.6 implies that $Kh(D^i_v)$ is $[v_{\min} - e, v_{\max} - e]$-thick. Also, equations 4.1 and 4.2 imply that $e_i = -e$. The diagram $D^i_h$ is identical to $D_v$.

Therefore, $Kh(D^i_h)$ is $[\alpha, \beta]$-thick where $\alpha = \min\{v_{\min} + 1, h_{\min} + e\}$ and $\beta = \max\{v_{\max} + 1, h_{\max} + e\}$. Since $D$ is width-preserving at $x$, we have $v_{\min} \neq h_{\min} + e + 1$ and $v_{\max} \neq h_{\max} + e + 1$. Therefore,

$$\alpha + e_i + 1 = \min\{v_{\min} + 1, h_{\min} + e\} - e + 1 = \min\{v_{\min} - e + 2, h_{\min} + 1\} \neq v_{\min} - e,$$

and

$$\beta + e_i + 1 = \max\{v_{\max} + 1, h_{\max} + e\} - e + 1 = \max\{v_{\max} - e + 2, h_{\max} + 1\} \neq v_{\max} - e.$$

Thus $D_{\tau_2}$ is width-preserving at $x_i$. Moreover, Corollary 3.15 implies that $Kh(D_{\tau_2})$ is $[\alpha - e, \beta - e]$-thick, and hence $w_{Kh}(D) = w_{Kh}(D_{\tau_2})$.

![Diagram](image_url)

**FIGURE 4.4.** The resolutions for $x$ positive, $\tau = C(-2)$, and with the depicted strands of $D$ in different components.

Suppose the strands that form the crossing $x$ are in different components of the link. Choose the orientation on $D_{\tau_2}$ given in Figure 4.4. Also, Figure 4.4 shows the resolutions $D^0_v$ and $D^0_h$.

Observe that $x_i$ is a negative crossing in $D_{\tau_2}$ for $i = 0, 1$. Orient $D^i_h$ so that it represents the same oriented link as $D_v$. With a suitably chosen orientation on $D_h$, we have

$$n_-(D) = n_-(D_v) = n_-(D^i_h),$$

(4.3)

and

$$n_-(D_h) + 1 = n_-(D^i_v) = n_-(D^i_v).$$

(4.4)
Equations 4.3 and 4.4 imply that $e_i = -e - 1$. The diagram $D^i_v$ is the same as $D$ except $D^i_v$ has one component reversed. Equations 4.3 and 4.4 along with Proposition 3.6 imply that $Kh(D^i_v)$ is $[\alpha - e - 1, \beta - e - 1]$-thick where $\alpha = \min\{v_{\min} + 1, h_{\min} + e\}$ and $\beta = \max\{v_{\max} + 1, h_{\max} + e\}$. Since $D^i_h$ and $D_v$ represent the same oriented link, it follows that $Kh(D^i_h)$ is $[v_{\min}, v_{\max}]$-thick. Since $D$ is width-preserving at $x$, we have $v_{\min} \neq h_{\min} + e + 1$ and $v_{\max} \neq h_{\max} + e + 1$. Therefore,

$$\alpha - e - 1 = \min\{v_{\min} - e, h_{\min} - 1\} \neq v_{\min} - e - 2 = v_{\min} + e_i - 1,$$

and

$$\beta - e - 1 = \max\{v_{\max} - e, h_{\max} - 1\} \neq v_{\max} - e - 2 = v_{\max} + e_i - 1.$$

Thus $D_{\tau_2}$ is width-preserving at $x_i$. Moreover, Corollary 3.15 implies that $Kh(D_{\tau_2})$ is $[\alpha - e - 1, \beta - e - 1]$-thick, and hence $w_{Kh}(D) = w_{Kh}(D_{\tau_2})$.

The case where $x$ is a negative crossing in $D$ is proved similarly. $\blacksquare$

**Main Theorem 2.** Let $D$ be a link diagram with crossing $x$, $\tau$ be an alternating rational tangle, and $D_\tau$ be the diagram $D$ twisted at $x$ by $\tau$. If $D$ is width-preserving at $x$, then $w_{Kh}(D) = w_{Kh}(D_\tau)$.

**Proof.** Let $\tau = C(a_1, \ldots, a_m)$. Since $\tau$ is alternating, either $a_i > 0$ for all $i$ or $a_i < 0$ for all $i$. Suppose $a_i > 0$ for all $i$. Beginning with the diagram $D$ and the crossing $x$, one can alternate twisting the diagram by $C(2)$ and $C(-2)$. Replacing the appropriate crossings $m$ times results in the diagram $D_{\tau'}$ where $\tau' = C(2, 1, \ldots, 1)$. Lemma 4.5 implies that each crossing in $D_{\tau'}$ is width-preserving, and $w_{Kh}(D) = w_{Kh}(D_{\tau'})$.

Replace crossings corresponding to the $m$-th term in $\tau'$ by $C(2)$ until the resulting diagram is obtained by twisting $D$ by $C(2, 1, \ldots, 1, a_m)$ at $x$. Next, replace crossings corresponding to the $(m - 1)$-st term in $C(2, 1, \ldots, 1, a_m)$ with $C(-2)$ until the resulting diagram is obtained by twisting $D$ by $C(2, 1, \ldots, 1, a_{m-1}, a_m)$ at $x$. Continue replacing crossings in the tangle by either $C(2)$ or $C(-2)$ until the resulting diagram is obtained by twisting $D$ by $C(a_1, \ldots, a_m)$.
at $x$. Since at each step, the only tangles used are $C(2)$ and $C(-2)$, Lemma 4.5 implies that $w_{Kh}(D) = w_{Kh}(D_\tau)$. The case where each $a_i < 0$ is proved similarly.

\[ \text{Figure 4.5. The inductive process of Main Theorem 2. At each step, the circled crossing is replaced with either } C(2) \text{ or } C(-2). \]

Remark 4.6. Watson [Wat09] proves that $w_{Kh}(D_\tau)$ is bounded by $w_{Kh}(D_v)$ and $w_{Kh}(D_h)$. By assuming that $D$ is width-preserving at $x$, we are able to strengthen the result and calculate $w_{Kh}(D_\tau)$.

Suppose $D$ is an oriented link diagram with crossing $x$. If $D$ is twisted at $x$ by $\tau_n = C(n)$ as in Figure 4.6, then the assumptions of Main Theorem 2 can be relaxed and a slightly stronger result holds. The following technical result is needed to compute the Khovanov width of closed 3-braids.

\[ \text{Figure 4.6. For } n > 0, \text{ twist } D_+ \text{ by } C(n) \text{ and twist } D_- \text{ by } C(-n). \text{ Then choose the above orientations for } D_{C(n)} \text{ and } D_{C(-n)}. \]

**Proposition 4.7.** Suppose $D$ is an oriented diagram with crossing $x$. Suppose $D$ is twisted at $x$ by $\tau_n = C(n)$ as in Figure 4.6. Let $D_v$ and $D_h$ be the two resolutions of $D$ at $x$. Suppose
$Kh(D_v)$ is $[v_{\min}, v_{\max}]$-thick and $Kh(D_h)$ is $[h_{\min}, h_{\max}]$-thick. Let $\alpha_{\pm} = \min\{v_{\min} \pm 1, h_{\min} + e\}$ and $\beta_{\pm} = \max\{v_{\max} \pm 1, h_{\max} + e\}$.

1. Let $n > 0$. Suppose that $v_{\min} \neq h_{\min} + e + 1$. If $v_{\max} = h_{\max} + e + 1$, then suppose that there exist integers $i$ and $j$ such that $j - 2i = v_{\max}$, $Kh^{i,j}(D_v)$ is nontrivial, and $Kh^{k,l}(D_h)$ is trivial for all $k$ whenever $l \leq j - 3e - 1$. Then $Kh(D_{r_n})$ is $[n + \alpha_{+}, n + \beta_{+}]$-thick.

2. Let $n < 0$. Suppose that $v_{\max} \neq h_{\max} + e - 1$. If $v_{\min} = h_{\min} + e - 1$, then suppose that there exist integers $i$ and $j$ such that $j - 2i = v_{\min}$, $Kh^{i,j}(D_v)$ is nontrivial, and $Kh^{k,l}(D_h)$ is trivial for all $k$ whenever $l \geq j - 3e - 1$. Then $Kh(D_{r_n})$ is $[n + \alpha_{-}, n + \beta_{-}]$-thick.

Proof. Let $n > 0$. Since $D$ is twisted at $x$ by $\tau_n$ as in Figure 4.6, it follows that $x$ is a positive crossing. If both $v_{\min} \neq h_{\min} + e + 1$ and $v_{\max} \neq h_{\max} + e + 1$, then $D$ is width-preserving at $x$. It follows from the proof of Main Theorem 2 that $Kh(D_{r_n})$ is $[n + \alpha_{+}, n + \beta_{+}]$-thick.

Suppose $v_{\min} \neq h_{\min} + e + 1$ and $v_{\max} = h_{\max} + e + 1$. Thus there exist integers $i$ and $j$ such that $j - 2i = v_{\max}$, $Kh^{i,j}(D_v)$ is nontrivial, and $Kh^{k,l}(D_h)$ is trivial for all $k$ and for all $l \leq j - 3e - 1$. Since $v_{\min} \neq h_{\min} + e + 1$, it follows that the minimum $\delta$-grading where $Kh(D_{r_n})$ is nontrivial is $n + \alpha_{+}$. We show, by induction on $n$, that $Kh^{i,j+n}(D_{r_n}) \cong Kh^{i,j}(D_v)$. This implies that the maximum $\delta$-grading supporting $Kh(D_{r_n})$ is $n + \beta_{+}$.

If $n = 1$, then the long exact sequence of Theorem 3.14 looks like

$$0 \to Kh^{i,j+1}(D) \to Kh^{i,j}(D_v) \to Kh^{i,e,j-3e-1}(D_h) \to \cdots .$$

By hypothesis, $Kh^{i,e,j-3e-1}(D_h)$ is trivial, and hence $Kh^{i,j+1}(D) \cong Kh^{i,j}(D_v)$.

Suppose, by way of induction, that $Kh^{i,j+n}(D_{r_n}) \cong Kh^{i,j}(D_v)$. Resolve $D_{r_{n+1}}$ at any crossing in $\tau_{n+1}$ to obtain diagrams $D_v'$ and $D_h'$. Let $e_{n+1} = n_{-}(D_v') - n_{-}(D_{r_{n+1}})$. Since $n_{-}(D_h') = n_{-}(D_h) + n$ and $n_{-}(D_{r_{n+1}}) = n_{-}(D)$, it follows that $e_{n+1} = e + n$. Observe that $D_v'$ and $D_{r_n}$ are the same diagram, and $D_h'$ and $D_h$ are diagrams for the same link. Hence the long exact sequence of Theorem 3.14 looks like

$$0 \to Kh^{i,j+n+1}(D_{r_{n+1}}) \to Kh^{i,j+n}(D_{r_n}) \to Kh^{i,e,n,j-3e-3n-1}(D_h) \to \cdots .$$
Since \( j - 3e - 3n - 1 \leq j - 3e - 1 \), it follows that \( Kh^{j-e-n,j-3e-3n-1}(D_h) \) is trivial. Thus \( Kh^{j+n+1}(D_{r_n+1}) \cong Kh^{j+n}(D_{r_n}) \cong Kh^{j}(D_v) \). Therefore \( Kh(D_{r_n}) \) is \([n + \alpha_+, n + \beta_+]\)-thick.

The case where \( n < 0 \) is proved in a similar fashion using the second sequence from Theorem 3.14.

**4.2 Turaev Genus and Twisting Links**

Recall that every link diagram \( D \) has an associated Turaev surface \( \Sigma_D \). Champanerkar and Kofman [CK09b] implicitly state that twisting a link diagram by a rational tangle does not change the genus of the Turaev surface. Here we give an explicit proof.

**Proposition 4.8.** Let \( D \) be a link diagram with crossing \( x \), and let \( \tau \) be an alternating rational tangle such that \( D \) is twisted by \( \tau \) at \( x \). Then \( g(\Sigma_{D_\tau}) = g(\Sigma_D) \).

**Proof.** Suppose \( \tau = C(a_1, \ldots, a_m) \), where \( \text{sign}(a_i) = \text{sign}(a_j) \) for all \( i \) and \( j \). Let \( a = \sum_{i=1}^{m} |a_i| \).

The all \( A \)-smoothing of \( D \) is the same as the all \( A \)-smoothing of \( D_\tau \), except \( D_\tau \) has an additional \( k \) circles. Similarly, the all \( B \)-smoothing of \( D \) is the same as the all \( B \)-smoothing of \( D_\tau \), except \( D_\tau \) has an additional \( l \) circles. Since \( \tau \) is alternating, it follows that \( k + l = a - 1 \). Also, \( c(D_\tau) = c(D) + a - 1 \). Therefore,

\[
\begin{align*}
g(\Sigma_D) & = \frac{2 - V(\mathbb{A}(D)) - V(\mathbb{B}(D)) + c(D)}{2} \\
& = \frac{2 - (V(\mathbb{A}(D_\tau)) + V(\mathbb{B}(D_\tau))) - (a - 1)) + c(D_\tau) - (a - 1)}{2} \\
& = \frac{2 - V(\mathbb{A}(D_\tau)) - V(\mathbb{B}(D_\tau)) + c(D_\tau)}{2} \\
& = g(\Sigma_{D_\tau}).
\end{align*}
\]

In the case where \( D \) is the closure of a braid, there is a particularly nice version of Proposition 4.8. Let \( w = w(\sigma_1, \sigma^{-1}_1, \ldots, \sigma_{n-1}, \sigma^{-1}_{n-1}) \in B_n \) be a word in the braid group, and let \( D \) be the link diagram obtained from taking the closure of \( w \). Suppose \( w' \) is word in \( B_n \) obtained by replacing \( \sigma_i \) in \( w \) with \( \sigma_i^k \) where \( k > 0 \) or by replacing \( \sigma_i^{-1} \) in \( w \) with \( \sigma_i^{-k} \) where \( k < 0 \). Let \( D' \) be the link diagram obtained by taking the braid closure of \( w' \).
Corollary 4.9. Let $D$ and $D'$ be link diagrams obtained from the closures of the braids $w$ and $w'$ respectively. Then $g(\Sigma_D) = g(\Sigma_{D'})$. 
Chapter 5
Applications to Closed 3-braids

Closed 3-braids are a rich class of links in which computation of invariants are possible. In [BM93], Birman and Menasco classify the link types of closed 3-braids. Several papers (see [Sch26], [Mur74], and [Gar69]) have algorithms that determine when two 3-braids are conjugate in $B_3$. In this chapter, we will be interested in Murasugi’s solution to the conjugacy problem.

5.1 Torus Links

Let $T(p,q)$ denote the $(p,q)$ torus link. In this section, we determine the Turaev genus and Khovanov width of $T(3,q)$. Turner [Tur08] and Stošić [Sto09] give formulas for the rational Khovanov homology of $T(3,q)$. The following theorem specifies the support of $Kh(T(3,q); \mathbb{Q})$ for $q \geq 3$. If $q \leq -3$, one can deduce the support from this theorem and the fact that $T(3,-q)$ is the mirror of $T(3,q)$.

**Theorem 5.1** (Stošić, Turner). Suppose $n \geq 1$.

1. The group $Kh(T(3,3n); \mathbb{Q})$ is $[4n - 3, 6n - 1]$-thick. Thus
   \[ w_{Kh}(T(3,3n); \mathbb{Q}) = n + 2. \]

2. The group $Kh(T(3,3n+1); \mathbb{Q})$ is $[4n - 1, 6n + 1]$-thick. Thus
   \[ w_{Kh}(T(3,3n+1); \mathbb{Q}) = n + 2. \]

3. The group $Kh(T(3,3n+2); \mathbb{Q})$ is $[4n + 1, 6n + 3]$-thick. Thus
   \[ w_{Kh}(T(3,3n+2); \mathbb{Q}) = n + 2. \]

The following lemma gives several normal forms for braids in $B_3$ whose closures are torus links. We will use these normal forms to compute the Turaev genus of a $(3,q)$ torus link as well as the Turaev genus of many closed 3-braids.
Lemma 5.2. Let $B_3$ be the braid group on three strands. Then for any $n > 1$, we have

$$(\sigma_1 \sigma_2)^3 = \sigma_2^2 \sigma_1^2 \sigma_2,$$

$$(\sigma_1 \sigma_2)^4 = \sigma_2^2 \sigma_1^3 \sigma_2 \sigma_1,$$

$$(\sigma_1 \sigma_2)^5 = \sigma_2^3 \sigma_1 \sigma_2^2 \sigma_1^2,$$

$$(\sigma_1 \sigma_2)^{3n} = \sigma_2^3 \sigma_1^4 \sigma_2^4 \cdots \sigma_1^4 \sigma_2^4 \sigma_1^3 \sigma_2 \sigma_1^{n+1} \sigma_2,$$

$$(\sigma_1 \sigma_2)^{3n+1} = \sigma_2^3 \sigma_1^4 \sigma_2^4 \cdots \sigma_1^4 \sigma_2^4 \sigma_1^3 \sigma_2 \sigma_1^{n+2} \sigma_2 \sigma_1,$$

$$(\sigma_1 \sigma_2)^{3n+2} = \sigma_2^3 \sigma_1^4 \sigma_2^4 \cdots \sigma_1^4 \sigma_2^4 \sigma_1^3 \sigma_2 \sigma_1^{n+1}.$$

Proof. Observe

$$(\sigma_1 \sigma_2)^3 = \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2$$

$= \sigma_1^2 \sigma_2^2 \sigma_1^2 \sigma_2,$

$$(\sigma_1 \sigma_2)^4 = \sigma_2^2 \sigma_1^3 \sigma_2 \sigma_1 \sigma_2$$

$= \sigma_1 \sigma_2^2 \sigma_1^3 \sigma_2^2 \sigma_1,$

and

$$(\sigma_1 \sigma_2)^5 = \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2$$

$= \sigma_1 \sigma_2^3 \sigma_1 \sigma_2^2 \sigma_1 \sigma_2 \sigma_1 \sigma_2$.

The braid relation directly implies the following two relations:

$$\sigma_1^k \sigma_2 \sigma_1 = \sigma_2 \sigma_1^k \sigma_2,$$

and

$$\sigma_1 \sigma_2^k \sigma_1 = \sigma_2^k \sigma_1 \sigma_2,$$

for $k > 0$. These relations will be used to prove the last three equations in the lemma.

For $n > 1$, we prove that

$$(\sigma_1 \sigma_2)^{3n} = \sigma_2^3 \sigma_1^4 \sigma_2^4 \cdots \sigma_1^4 \sigma_2^4 \sigma_1^3 \sigma_2 \sigma_1^{n+1} \sigma_2.$$
by induction. Let \( n = 2 \). Then

\[
(\sigma_1 \sigma_2)^6 = \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 = \\
\sigma_1^3 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 = \\
\sigma_1^3 \sigma_2 \sigma_1^3 \sigma_2 \sigma_1^3 \sigma_2.
\]

Suppose, by way of induction, that

\[
(\sigma_1 \sigma_2)^{3n} = \sigma_1^3 \sigma_2 \sigma_1^4 \sigma_2 \cdots \sigma_1^4 \sigma_2 \sigma_1^3 \sigma_2 \sigma_1^{n+1} \sigma_2.
\]

Then

\[
(\sigma_1 \sigma_2)^{3(n+1)} = \sigma_1^3 \sigma_2 \sigma_1^4 \sigma_2 \cdots \sigma_1^4 \sigma_2 \sigma_1^3 \sigma_2 \sigma_1^{n+1} \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 = \\
\sigma_1^3 \sigma_2 \sigma_1^4 \sigma_2 \cdots \sigma_1^4 \sigma_2 \sigma_1^3 \sigma_2 \sigma_1^{n+2} \sigma_2 = \\
\sigma_1^3 \sigma_2 \sigma_1^4 \sigma_2 \cdots \sigma_1^4 \sigma_2 \sigma_1^3 \sigma_2 \sigma_1^{n+2} \sigma_2.
\]

Hence, for all \( n > 1 \),

\[
(\sigma_1 \sigma_2)^{3n} = \sigma_1^3 \sigma_2 \sigma_1^4 \sigma_2 \cdots \sigma_1^4 \sigma_2 \sigma_1^3 \sigma_2 \sigma_1^{n+1} \sigma_2. \tag{5.1}
\]

Equation 5.1 implies

\[
(\sigma_1 \sigma_2)^{3n+1} = \sigma_1^3 \sigma_2 \sigma_1^4 \sigma_2 \cdots \sigma_1^4 \sigma_2 \sigma_1^3 \sigma_2 \sigma_1^{n+1} \sigma_2 \sigma_1 \sigma_2 = \\
\sigma_1^3 \sigma_2 \sigma_1^4 \sigma_2 \cdots \sigma_1^4 \sigma_2 \sigma_1^3 \sigma_2 \sigma_1^{n+2} \sigma_2 \sigma_1.
\]

53
Furthermore,

\[(\sigma_1 \sigma_2)^{3n+2} = \sigma_1^3 \sigma_2^3 \sigma_2^4 \cdots \sigma_1^4 \sigma_2^3 \sigma_2^n \sigma_1^{n+2} \sigma_2 \sigma_1 \sigma_2\]
\[= \sigma_1^3 \sigma_2^3 \sigma_2^4 \cdots \sigma_1^4 \sigma_2^3 \sigma_2 \sigma_1 \sigma_2 \sigma_1^2 \sigma_2 \sigma_1^{n+1} \sigma_1 \sigma_2\]
\[= \sigma_1^3 \sigma_2^3 \sigma_2^4 \cdots \sigma_1^4 \sigma_2^3 \sigma_2 \sigma_1^2 \sigma_2 \sigma_1^{n+1} \sigma_1 \sigma_2\]
\[= \sigma_1^3 \sigma_2^3 \sigma_2^4 \cdots \sigma_1^4 \sigma_2^3 \sigma_2 \sigma_1^2 \sigma_2 \sigma_1^{n+1} \sigma_1 \sigma_2\]

□

Abe and Kishimoto [AK08] have independently calculated the Turaev genus for the \((3,q)\)-torus links. We give diagrams in closed braid form that minimize Turaev genus, while they have a different approach.

**Proposition 5.3.** Suppose \(q > 0\). The Turaev genus of \(T(3, q)\) and \(T(3, -q)\) is \([q/3]\).

**Proof.** Let \(D\) be the diagram obtained by taking the closure of the normal form for \((\sigma_1 \sigma_2)^q\) given in Lemma 5.2. Thus \(D\) is a diagram for \(T(3, q)\) and is the closure of a braid in the form

\[\sigma_1^{a_1} \sigma_2^{b_1} \cdots \sigma_1^{a_s} \sigma_2^{b_s} \sigma_1^{a_{s+1}}\]

where \(s = [q/3] + 1\), both \(a_i > 0\) and \(b_i > 0\) for all \(1 \leq i \leq s\), and \(a_{s+1} \geq 0\). Let \(D'\) be the diagram obtained by taking the closure of the braid \((\sigma_1 \sigma_2)^s\). Corollary 4.9 implies that \(g(\Sigma_D) = g(\Sigma_{D'})\). Since \(c(D') = 2[q/3] + 2\), the number of circles in the all \(A\)-smoothing of \(D'\) is 3, and the number of circles in the all \(B\)-smoothing of \(D'\) is 1, it follows that \(g(\Sigma_{D'}) = [q/3]\). Proposition 3.13 and Theorem 5.1 imply that the Turaev genus of \(T(3, q)\) is greater than or equal to \([q/3]\). Therefore, \(g_T(T(3, q)) = [q/3]\). The genera of the Turaev surfaces for a diagram and its mirror are equal, and hence \(g_T(T(3, -q)) = [q/3]\). □

The next corollary establishes the support of the Khovanov homology (with \(\mathbb{Z}\)-coefficients) of the \((3, q)\) torus links. It follows directly from Theorem 5.1, Proposition 5.3, and Proposition 3.13.
Corollary 5.4. Suppose $n \geq 1$.

1. The group $Kh(T(3, 3n))$ is $[4n - 3, 6n - 1]$-thick and the group $Kh(T(3, -3n))$ is $[-6n + 1, -4n + 3]$-thick. Therefore

$$w_{Kh}(T(3, 3n)) = w_{Kh}(T(3, -3n)) = n + 2.$$ 

2. The group $Kh(T(3, 3n + 1))$ is $[4n - 1, 6n + 1]$-thick and the group $Kh(T(3, -3n - 1))$ is $[-6n - 1, -4n + 1]$-thick. Therefore

$$w_{Kh}(T(3, 3n + 1)) = w_{Kh}(T(3, -3n - 1)) = n + 2.$$ 

3. The group $Kh(T(3, 3n + 2))$ is $[4n + 1, 6n + 3]$-thick and the group $Kh(T(3, -3n - 2))$ is $[-6n - 3, -4n - 1]$-thick. Therefore

$$w_{Kh}(T(3, 3n + 2)) = w_{Kh}(T(3, -3n - 2)) = n + 2.$$ 

5.2 Khovanov Width of Closed 3-braids

In this section, we determine the Khovanov width of closed 3-braids based upon Murasugi’s classification of closed 3-braids up to conjugation. Murasugi [Mur74] proves the following:

Theorem 5.5 (Murasugi). Let $w \in B_3$ be a braid on three strands, and let $h = (\sigma_1 \sigma_2)^3$ be a full twist. Let $n \in \mathbb{Z}$. Then $w$ is conjugate to exactly one of the following:

1. $h^n \sigma_1^{p_1} \sigma_2^{-q_1} \cdots \sigma_1^{p_s} \sigma_2^{-q_s}$ where $p_i, q_i$ and $s$ are positive integers.

2. $h^n \sigma_2^m$ where $m \in \mathbb{Z}$.

3. $h^n \sigma_1^m \sigma_2^{-1}$, where $m \in \{-1, -2, -3\}$.

Let $L$ be a closed 3-braid. Theorem 5.5 says, in effect, that $L$ is the closure of a braid of the form $h^n A$. For $n \neq 0$, we say that $L$ has cancellation if the braid word for $A$ contains a $\sigma_i^c$ for $i = 1, 2$ where $\text{sign}(\varepsilon) \neq \text{sign}(n)$. Besides two infinite family of braids, we prove that $w_{Kh}(L) = |n| + 2$ if there is no cancellation and $w_{Kh}(L) = |n| + 1$ if there is cancellation.
The following several propositions establish the support of $Kh(L)$. The proofs require the computation of Khovanov homology for a few specific links. We represent the rational Khovanov homology as a Poincare polynomial $P(L)$, a Laurent polynomial in the variables $q$ and $t$ such that the coefficient of $q^it^j$ is the dimension of $Kh^{i,j}(L; \mathbb{Q})$. These computations were taken from KnotInfo [CL09].

**Proposition 5.6.** Suppose $n > 0$ and $k \geq 0$. Let $D$ be the closure of the braid $(\sigma_1\sigma_2)^{3n}\sigma_1^k\sigma_2^{-1}$, and let $D'$ be the closure of $(\sigma_1\sigma_2)^{-3n}\sigma_1^{-k}\sigma_2$. Then $Kh(D)$ is $[4n + k - 2, 6n + k - 2]$-thick and $Kh(D')$ is $[-6n - k + 2, -4n - k + 2]$-thick.

**Proof.** Observe that $(\sigma_1\sigma_2)^{3n}\sigma_1^k\sigma_2^{-1} = (\sigma_1\sigma_2)^{3n-1}\sigma_1^{k+1}$ for $n > 0$. Let $D_+$ be the closure of the braid $(\sigma_1\sigma_2)^{3n-1}\sigma_1$. Resolve the crossing given by the last $\sigma_1$ to obtain two link diagrams $D_v$ and $D_h$. Then $D_v$ is a diagram for $T(3, 3n - 1)$, and $D_h$ is a diagram for the unknot. By Corollary 5.4, $Kh(D_v)$ is $[4n - 3, 6n - 3]$-thick. Since $D_h$ is the unknot, $Kh(D_h)$ is $[-1, 1]$-thick. Recall that $e = n_-(D_h) - n_-(D_+)$. The diagram $D_h$ has $4n - 1$ negative crossings, while the diagram $D_+$ has no negative crossings. Thus $e = 4n - 1$.

If $n \neq 2$, then $D_+$ is width-preserving. If $n = 2$, then the Poincare polynomial of $D_+ = T(3, 5)$ is

$$P(T(3, 5)) = q^7 + q^9 + q^{11}t^2 + q^{15}t^3 + q^{13}t^4 + q^{15}t^4 + q^{17}t^5 + q^{17}t^6 + q^{19}t^5 + q^{21}t^7.$$  

Therefore, $Kh^{0,9}(D_v)$ is nontrivial. Moreover, $Kh^{i,j}(D_h) = 0$ for all $i$ if $j \leq 9 - 3e - 1 = -13$. Therefore, for $n > 0$, Proposition 4.7 implies that $Kh(D)$ is $[4n + k - 2, 6n + k - 2]$-thick. The proof for $D'$ is similar. \qed

**Proposition 5.7.** Let $D$ be the closure of the braid $(\sigma_1\sigma_2)^{3n}\sigma_1^{a_1}\sigma_2^{-b_1}\cdots\sigma_1^{a_k}\sigma_2^{-b_k}$, where each $a_i, b_i > 0$. Let $a = \sum_{i=1}^k a_i$ and $b = \sum_{i=1}^k b_i$. If $n > 0$, then $Kh(D)$ is $[4n + a - b - 1, 6n + a - b - 1]$-thick. If $n < 0$, then $Kh(D)$ is $[6n + a - b + 1, 4n + a - b + 1]$-thick. Hence, if $n \neq 0$, then $w_{Kh}(D) = |n| + 1$.  

56
Proof. Suppose \( n > 0 \). We proceed by induction on \( b \). Suppose \( b = 1 \). Let \( D_1 \) be the closure of the braid \((\sigma_1\sigma_2)^3\sigma_1^a\sigma_2^{-1}\). Proposition 5.6 states that \( Kh(D_1) \) is supported in the band 
\([4n + a - 2, 6n + a - 2]\). Since \((\sigma_1\sigma_2)^3\) is in the center of \( B_3 \), it follows that \( D_1 \) represents the same link as \( D'_1 \), the closure of \((\sigma_1\sigma_2)^3\sigma_1^a\sigma_2^{-1}\sigma_1^{a-a_1}\).

If \( D_b \) is the closure of the braid \((\sigma_1\sigma_2)^3\sigma_1^{p_i}\sigma_2^{-q_i}\cdots\sigma_1^{p_j}\sigma_2^{-q_j}\) where each \( p_i, q_i > 0 \), \( \sum_{i=1}^{j} p_i = a \) and \( \sum_{i=1}^{j} q_i = b \), then by way of induction, suppose \( Kh(D_b) \) is \([4n + a - b - 1, 6n + a - b - 1]\)-thick. Let \( D_{b+1} \) be the closure of the braid \((\sigma_1\sigma_2)^3\sigma_1^{p'_i}\sigma_2^{-q'_i}\cdots\sigma_1^{p'_l}\sigma_2^{-q'_l}\), where \( p'_i, q'_i > 0 \), \( \sum_{i=1}^{l} p'_i = a \), and \( \sum_{i=1}^{l} q'_i = b + 1 \). Resolve \( D_{b+1} \) at the crossing corresponding to the last \( \sigma_2^{-1} \) to obtain diagrams \( D_v \) and \( D_h \). By the inductive hypothesis, \( Kh(D_v) \) is \([4n + a - b - 1, 6n + a - b - 1]\)-thick. Let \( m \) be the number of negative crossings in the alternating part of \( D_h \). The alternating part of \( D_h \) has \( a + b \) crossings.

Also, \( D_h \) is a non-alternating diagram for an alternating link \( L \). Hence, Theorem 3.17 implies that \( Kh(L) \) is \([-\sigma(L) - 1, -\sigma(L) + 1]\)-thick. One can calculate the signature of an alternating link from any alternating diagram by a result of Gordon and Litherland [GL78]. Color the regions of the alternating diagram in a checkerboard fashion so that near each crossing it looks like Figure 5.1.

![Figure 5.1](image-url)  
**FIGURE 5.1.** Color the alternating diagram in a checkerboard fashion such that a neighborhood of each crossing appears as above.

Then the signature is given by

\[ \sigma(L) = \#(\text{black regions}) - \#(\text{positive crossings}) - 1. \]

There is an alternating diagram representing \( L \) that has \( b + 2 \) black regions and \( a + b - m \) positive crossings (see Figure 5.2). Therefore, \( \sigma(L) = m - a + 3 \), and hence \( Kh(D'_h) \) is \([a - m - 4, a - m - 2]\)-thick. Since there are \( 4n \) negative crossings in the full twist part of \( D_h \) and
$m$ negative crossings in the alternating part of $D_h$, it follows that $n_-(D_h) = 4n + m$. Let $D_+$ be the closure of the braid $(\sigma_1 \sigma_2)^{3n} \sigma_1^{\ell} \sigma_2^{-d_1} \cdots \sigma_1^{\ell} \sigma_2^{-d_l+1}$. Then $n_-(D_+) = b$, and thus $e = n_-(D_h) - n_-(D_+) = 4n + m - b$. For $n > 0$,

\[ 4n + a - b - 1 \neq (a - m - 4) + (4n + m - b) - 1 \]

\[ 6n + a - b - 1 \neq (a - m - 2) + (4n + m - b) - 1. \]

Therefore, Theorem 3.15 implies that $Kh(D_{b+1})$ is $[4n + a - b - 2, 6n + a - b - 2]$-thick. The proof for $n < 0$ is similar. □

\[ \text{FIGURE 5.2. The closure of the braid } (\sigma_1 \sigma_2)^3 \sigma_1^{\ell} \sigma_2^{-3} \sigma_1^{3} \sigma_2^{-2} \text{ together with its resolution and an alternating diagram of its resolution. There are 5 black regions and 2 negative crossings in the alternating diagram.} \]

**Proposition 5.8.** Let $D$ be the closure of the braid $(\sigma_1 \sigma_2)^{3n} \sigma_2^{-m}$.

1. If $n > 0$ and $m \geq 0$, then $Kh(D)$ is $[4n + m - 3, 6n + m - 1]$-thick and $w_{Kh}(D) = n + 2$.

2. If $n < 0$ and $m \leq 0$, then $Kh(D)$ is $[6n + m + 1, 4n + m + 3]$-thick and $w_{Kh}(D) = -n + 2$. 

58
3. If \( n = 1 \) and \( m < -3 \), then \( \text{Kh}(D) \) is \([m + 3, m + 7]\)-thick and \( w_{\text{Kh}}(D) = 3 \).

4. If \( n = -1 \) and \( m > 3 \), then \( \text{Kh}(D) \) is \([m - 7, m - 3]\)-thick and \( w_{\text{Kh}}(D) = 3 \).

5. If both \( n = 1 \) and \( -3 \leq m < 0 \) or both \( n > 1 \) and \( m < 0 \), then \( \text{Kh}(D) \) is \([4n + m - 1, 6n + m - 1]\)-thick and \( w_{\text{Kh}}(D) = n + 1 \).

6. If both \( n = -1 \) and \( 0 < m \leq 3 \) or both \( n < -1 \) and \( m > 0 \), then \( \text{Kh}(D) \) is \([6n + m + 1, 4n + m + 1]\)-thick and \( w_{\text{Kh}}(D) = -n + 1 \).

**Proof.** We prove statements (1), (3), and (5). Statements (2), (4), and (6) are proved similarly.

(1). Suppose \( n > 0 \) and \( m \geq 0 \). Let \( D_\pm \) be the closure of the braid \((\sigma_1 \sigma_2)^{3n} \sigma_2 \). Resolve \( D_\pm \) at the crossing corresponding to the last \( \sigma_2 \) to obtain diagrams \( D_\text{v} \) and \( D_\text{h} \). Then \( D_\text{v} \) is a diagram for \( T(3, 3n) \). By Corollary 5.4, \( \text{Kh}(D_\text{v}) \) is \([4n - 3, 6n - 1]\)-thick. Also, \( D_\text{h} \) is the two component unlink, and hence \( \text{Kh}(D_\text{h}) \) is \([-2, 2]\)-thick. The diagram \( D_\text{h} \) has \( 4n \) negative crossings, and the diagram \( D_\pm \) has no negative crossings. Thus \( e = 4n \).

Observe that \( 4n - 3 \neq -2 + e + 1 \) and \( 6n - 1 = 2 + e + 1 \) when \( n = 2 \). If \( n = 2 \), then \( D_\text{v} \) is \( T(3, 6) \), and

\[
P(T(3, 6)) = q^9 + q^{11} + q^{13} t^2 + q^{17} t^3 + q^{15} t^4 + q^{17} t^4 + q^{19} t^5 + q^{19} t^6 + q^{21} t^7 + q^{21} t^8 + q^{23} t^7 + 3q^{23} t^8 + 2q^{25} t^8.
\]

Therefore \( \text{Kh}^{0,11}(D_\text{v}) \) is nontrivial. Also, \( \text{Kh}^{i,j}(D_\text{h}) = 0 \) for all \( i \) if \( j \leq 11 - 3e - 1 = -14 \). Hence, Theorem 4.7 implies that \( \text{Kh}(D) \) is \([4n + m - 3, 6n + m - 1]\)-thick.

(3). Suppose \( n = 1 \) and \( m < -3 \). Let \( D_- \) be the closure of \((\sigma_1 \sigma_2)^{3} \sigma_2^{-5} \). Resolve \( D_- \) at the crossing corresponding to the last \( \sigma_2^{-1} \) to obtain diagrams \( D_\text{v} \) and \( D_\text{h} \). The diagram \( D_\text{h} \) is a diagram for the two component unlink, and hence \( \text{Kh}(D_\text{h}) \) is \([-2, 2]\)-thick. The diagram \( D_\text{v} \) is a diagram for the link \( L(6, n, 1) \) in Thistlethwaite’s link table (see Figure 5.3). The Poincare polynomial for \( L(6, n, 1) \) is given by

\[
P(L(6, n, 1)) = 2q^{-1} + 3q + q^3 + qt + q^5 t^2 + q^7 t^4 + q^9 t^4.
\]
Therefore $Kh(D_v)$ is $[-1, 3]$-thick. The diagram $D_h$ has 4 negative crossings while the diagram $D_+$ also has 4 negative crossings. Thus $e = 0$. Since $-1 \neq -2 + e - 1$ and $3 \neq 2 + e - 1$, Theorem 4.7 implies that $Kh(D)$ is $[m + 3, m + 7]$-thick.

(5). If $n = 1$ and $-3 \leq m < 0$, then Baldwin [Bal08] has shown that $D$ is quasi-alternating. Therefore, Theorem 3.17 implies that $Kh(D)$ is $[-\sigma(L) - 1, -\sigma(L) + 1]$-thick, where $L$ is the link type of $D$. A straightforward calculation of signature gives the desired result.

Suppose $n > 1$ and $m < 0$. Observe that $(\sigma_1 \sigma_2)^{3n} \sigma_2^{-1} = (\sigma_1 \sigma_2)^{3n-1} \sigma_1$. Let $D_+$ be the closure of the braid $(\sigma_1 \sigma_2)^{3n-1} \sigma_1$. Resolve $D_+$ at the crossing corresponding to the last $\sigma_1$ to obtain diagrams $D_v$ and $D_h$. Since $D_v$ is a diagram for $T(3, 3n - 1)$, it follows that $Kh(D_v)$ is $[4n - 3, 6n - 3]$-thick. Since $D_h$ is a diagram for the unknot, it follows that $Kh(D_h)$ is $[-1, 1]$-thick. The diagram $D_h$ has $4n - 1$ negative crossings, and $D_v$ has no negative crossings. Thus $e = 4n - 1$.

If $n = 2$, then $6n - 3 = 1 + e + 1$, and the long exact sequence of Theorem 3.14 looks like

$$0 \rightarrow Kh^{0,10}(D_+) \rightarrow Kh^{0,9}(D_v) \rightarrow Kh^{-7,-13}(D_h) \rightarrow \cdots.$$ 

Since $Kh^{-7,-13}(D_h) = 0$ and $Kh^{0,9}(D_v)$ is nontrivial, it follows that $Kh^{0,10}(D_+)$ is nontrivial. Since $4n - 3 \neq -1 + e + 1$ and $6n - 3 \neq 1 + e + 1$ for $n > 2$, Corollary 3.15 implies that $Kh(D_+)$ is $[4n - 2, 6n - 2]$-thick.

Let $D_-$ be the closure of $(\sigma_1 \sigma_2)^{3n-1} \sigma_1 \sigma_2^{-1}$. Resolve $D_-$ at the crossing given by the last $\sigma_2^{-1}$ to obtain diagrams $D_v$ and $D_h$. The diagram $D_v$ is the closure of the braid $(\sigma_1 \sigma_2)^{3n-1} \sigma_1$, and hence $Kh(D_v)$ is $[4n - 2, 6n - 2]$-thick. The link $D_h$ is a diagram for the two component
unlink, and thus $Kh(D_h)$ is $[-2, 2]$-thick. The diagram $D_h$ has $4n - 1$ negative crossings, and $D_+$ has no negative crossings. Thus $e = 4n - 1$.

For $n > 1$, we have $4n - 2 \neq -2 + e - 1$ and $6n - 2 \neq 2 + e - 1$. Therefore, Theorem 4.7 implies that $Kh(D)$ is $[4n + m - 1, 6n + m - 1]$-thick. □

**Proposition 5.9.** Let $D$ be the closure of the braid $(\sigma_1\sigma_2)^{3n}\sigma_1^m\sigma_2^{-1}$, where $m \in \{-1, -2, -3\}$.

1. If $n > 0$, then $Kh(D)$ is $[4n + m - 2, 6n + m - 2]$-thick, and $w_{Kh}(D) = n + 1$.

2. If $n < 0$, then $Kh(D)$ is $[6n + m, 4n + m + 2]$-thick, and $w_{Kh}(D) = -n + 2$.

**Proof.** (1). Suppose $n > 0$. If $m = -1$, then $D$ is a diagram for $T(3, 3n - 1)$, and the result follows.

Let $m = -2$. Then, up to conjugation in $B_3$, we have

\[
(\sigma_1\sigma_2)^{3n}\sigma_1^{-2}\sigma_2^{-1} = (\sigma_1\sigma_2)^{3n-1}\sigma_1^{-1} = (\sigma_2\sigma_1)^{3n-2}\sigma_2 = (\sigma_1\sigma_2)^{3n-2}\sigma_1.
\]

If $D'$ is the closure of the braid $(\sigma_1\sigma_2)^{3n-2}\sigma_1$, then $D$ and $D'$ represent the same link. Resolve $D'$ at the crossing corresponding to the final $\sigma_1$ to obtain diagrams $D_v$ and $D_h$. Then $D_v$ is a diagram for $T(3, 3n - 2)$, and $D_h$ is a diagram for the unknot. Hence $Kh(D_v)$ is $[4n - 5, 6n - 5]$-thick, and $Kh(D_h)$ is $[-1, 1]$-thick. The diagram $D_h$ has $4n - 3$ negative crossings, and the diagram $D'$ has none. Thus $e = 4n - 3$.

Observe that $4n - 5 \neq -1 + e + 1$, and $6n - 5 = 1 + e + 1$ when $n = 2$. If $n = 2$, the long exact sequence of Theorem 3.14 looks like

\[
0 \rightarrow Kh^{0,8}(D') \rightarrow Kh^{0,7}(D_v) \rightarrow Kh^{-5,-9}(D_h) \rightarrow \cdots
\]

Since $Kh^{-5,-9}(D_h) = 0$ and $Kh^{0,7}(L_v)$ is nontrivial, it follows that $Kh^{0,8}(L)$ is nontrivial. Hence Theorem 3.15 implies that $Kh(D')$ is $[4n - 4, 6n - 4]$-thick.
Let $m = -3$. Then, up to conjugation in $B_3$, we have

$$(\sigma_1\sigma_2)^{3n}\sigma_1^{-3}\sigma_2^{-1} = (\sigma_1\sigma_2)^{3n-1}\sigma_1^{-2} = (\sigma_2\sigma_1)^{3n-3}\sigma_2\sigma_1\sigma_2^{-1} = (\sigma_1\sigma_2)^{3n-2}. \tag{2}$$

Hence $D$ is a diagram for $T(3, 3n - 2)$, and the result follows.

(2). Let $n < 0$. If $m = -1$, then $D$ is a diagram for $T(3, 3n - 1)$, and the result follows.

Let $m = -2$. Then $D$ is the closure of $(\sigma_1\sigma_2)^{3n-1}\sigma_1^{-1}$. Resolve $D$ at the crossing corresponding to the last $\sigma_1^{-1}$ to obtain diagrams $D_v$ and $D_h$. Then $D_v$ is a diagram for $T(3, 3n - 1)$, and hence $Kh(D_v)$ is $[6n - 1, 4n + 1]$-thick. Also, $D_h$ is a diagram for the unknot, and hence $Kh(D_h)$ is $[-1, 1]$-thick. The diagram $D_h$ has $-2n - 1$ negative crossings, and the diagram $D_+$ has $-6n + 2$ negative crossings. Thus $e = 4n - 1$.

Observe that $4n + 1 \neq 1 + e - 1$, and $6n - 1 = -1 + e - 1$ if $n = -1$. If $n = -1$, the long exact sequence of Theorem 3.14 looks like

$$\cdots \to Kh^{5,9}(D_h) \to Kh^{0,-7}(D_v) \to Kh^{0,-8}(D) \to 0.$$ 

Since $Kh^{5,9}(D_h) = 0$ and $Kh^{0,-7}(D_v)$ is nontrivial, it follows that $Kh^{0,-8}(D)$ is nontrivial. Thus $Kh(D)$ is $[6n - 2, 4n]$-thick.

Let $m = -3$. Then, up to conjugacy in $B_3$, we have

$$(\sigma_1\sigma_2)^{3n}\sigma_1^{-3}\sigma_2^{-1} = (\sigma_1\sigma_2)^{3n-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-2} = (\sigma_1\sigma_2)^{3n-2}. \tag{2}$$

In this case $D$ is a diagram for $T(3, 3n - 2)$, and the result follows.

Propositions 5.7, 5.8 and 5.9 directly imply Main Theorem 3 which states the Khovanov width of closed 3-braids.
Main Theorem 3. Let $L$ be a closed 3-braid of the form $h^n A$, as in Theorem 5.5, where $h = (\sigma_1 \sigma_2)^3$ and $n \neq 0$. Then

$$w_{Kh}(L) = \begin{cases} |n| + 2 & \text{if } L \text{ has no cancellation or} \\ |n| + 1 & \text{if } L \text{ is the closure of } h^\pm \sigma_2^\mp m \text{ where } m > 3, \\ |n| + 1 & \text{otherwise.} \end{cases}$$

Remark 5.10. If $n = 0$, then $L$ is a (possibly split) alternating link, and thus $w_{Kh}(L)$ can be deduced from Theorem 3.17 and Proposition 3.6.

Baldwin [Bal08] classifies quasi-alternating closed 3-braids.

Proposition 5.11 (Baldwin). Let $L$ be a closed 3-braid and let $h = (\sigma_1 \sigma_2)^3$.

- If $L$ is the closure of the braid $h^n \sigma_1^{a_1} \sigma_2^{b_2} \cdots \sigma_1^{a_k} \sigma_2^{-b_k}$, where each $a_i, b_i > 0$, then $L$ is quasi-alternating if and only if $n \in \{-1, 0, 1\}$.

- If $L$ is the closure of the braid $h^n \sigma_2^m$, then $L$ is quasi-alternating if and only if either $n = 1$ and $m \in \{-1, -2, -3\}$ or $n = -1$ and $m \in \{1, 2, 3\}$.

- If $L$ is the closure of the braid $h^n \sigma_1^m \sigma_2^{-1}$, where $m \in \{-1, -2, -3\}$, then $L$ is quasi-alternating if and only if $n \in \{0, 1\}$.

Using the spectral sequence from reduced Khovanov homology of a link to the Heegaard Floer homology of the branched double cover of that link, Baldwin shows the following corollary. This corollary is also a consequence of Main Theorem 3 and Proposition 5.11.

Corollary 5.12 (Baldwin). Let $L$ be a closed 3-braid. Then $L$ is quasi-alternating if and only if $w_{Kh}(L) = 1$.

Remark 5.13. Shumakovitch has shown that the $9_{46}$ and $10_{140}$ knots (both closed 4-braids) have reduced Khovanov width one, but they are not quasi-alternating. One can use either of these knots to generate infinite families of counterexamples to Corollary 5.12 for braids with index greater than 3.
5.3 Turaev Genus of Closed 3-braids

Combining Lemma 5.2 with Corollary 4.9 gives a useful tool to compute the Turaev genus of closed 3-braids. By using the lower bound given by Proposition 3.13, the Turaev genus of closed 3-braids can be calculated up to an additive error of at most 1.

Proposition 5.14. Let $L$ be the link type of the closure of $(\sigma_1 \sigma_2)^{3n} \sigma_1^{a_1} \sigma_2^{-b_1} \cdots \sigma_1^{a_k} \sigma_2^{-b_k}$, where each $a_i, b_i > 0$ and $n \neq 0$. Then $|n| - 1 \leq g_T(L) \leq |n|$.

Proof. Suppose $n > 0$. We have

$$(\sigma_1 \sigma_2)^{3n} \sigma_1^{a_1} \sigma_2^{-b_1} \cdots \sigma_1^{a_k} \sigma_2^{-b_k} = (\sigma_1 \sigma_2)^{3n-1} \sigma_1^{a_1+1} \sigma_2^{-b_1} \cdots \sigma_1^{a_k} \sigma_2^{-b_k+1}.$$ 

If $b_k > 1$, let $D$ be the closure of the braid $(\sigma_1 \sigma_2)^n(\sigma_1 \sigma_2^{-1})^k$ and if $b_k = 1$, let $D$ be the closure of the braid $(\sigma_1 \sigma_2)^n(\sigma_1 \sigma_2^{-1})^{k-1}$. By applying the normal form of Lemma 5.2 to $(\sigma_1 \sigma_2)^{3n-1}$ and then using Corollary 4.9, it follows that $g_T(L) \leq g_T(D)$. A straightforward calculation shows that $g_T(D) = n$. Since $w_{Kh}(L) = n + 1$ and $w_{Kh}(L) - 2 \leq g_T(L)$, we have $n - 1 \leq g_T(L)$. The case where $n < 0$ is similar.

Proposition 5.15. Let $L$ be the link type of the closure of $(\sigma_1 \sigma_2)^{3n} \sigma_2^m$, where $n \neq 0$.

1. If $L$ has no cancellation, then $g_T(L) = |n|$.

2. If $L$ has cancellation and $|n| > 1$, then $|n| - 1 \leq g_T(L) \leq |n|$.

3. If either both $n = 1$ and $-3 \leq m < 0$ or both $n = -1$ and $0 < m \leq 3$, then $g_T(L) = 0$.

4. If either both $n = 1$ and $m < -3$ or both $n = -1$ and $m > 3$. Then $g_T(L) = 1$.

Proof. (1). If $L$ has no cancellation, then either both $n > 0$ and $m \geq 0$ or $n < 0$ and $m \leq 0$. Corollary 4.9 implies that $g_T(L) \leq g_T(T(3, 3n)) = |n|$. Since $w_{Kh}(L) = |n| + 2$, it follows that $g_T(L) = |n|$.

(2). Suppose that $L$ has cancellation and $n > 1$. Then $m < 0$ and

$$(\sigma_1 \sigma_2)^{3n} \sigma_2^m = (\sigma_1 \sigma_2)^{3n-1} \sigma_1 \sigma_2^{(m+1)}.$$ 

64
If $m \leq -1$, let $D$ be the closure of $(\sigma_1 \sigma_2)^n \sigma_1 \sigma_2^{-1}$, and if $m = -1$, let $D$ be the closure $(\sigma_1 \sigma_2)^n$. Lemma 5.2 and Corollary 4.9 imply that $g_T(L) \leq g(\Sigma_D)$. A straightforward calculation shows that $g(\Sigma_D) = n$. Since $w_{Kh}(L) = n + 1$, it follows that $n - 1 \leq g_T(L)$. The case where $n < -1$ and $m > 0$ is similar.

(3). Suppose $n = 1$ and $-3 \leq m < 0$. As noted in Baldwin’s paper [Bal08], we have

$$(\sigma_1 \sigma_2)^3 \sigma_2^m = \sigma_1 \sigma_2^2 \sigma_1 \sigma_2^{-1} \sigma_2^m.$$ 

By canceling the $\sigma_2^m$ with the final $\sigma_2$, one obtains a diagram for $L$ with 5 crossings or less. Therefore $L$ is alternating and $g_T(L) = 0$. The case where $n = -1$ and $0 < m \leq 3$ is similar.

(4). Suppose $n = 1$ and $m < -3$. Then $L$ can be represented by the closure of $\sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1}$. Let $D$ be the closure of the braid $\sigma_1 \sigma_2 \sigma_1 \sigma_2^{-1}$. By Corollary 4.9, we have $g_T(L) \leq g(\Sigma_D)$, and a straightforward calculation shows that $g(\Sigma_D) = 1$. Since $w_{Kh}(L) = 3$, it follows that $g_T(L) = 1$. The case where $n = -1$ and $m > 3$ is similar.

Proposition 5.16. Let $L$ be the link type of the closure of $(\sigma_1 \sigma_2)^m \sigma_1 \sigma_2^{-1}$ where $m \in \{-1, -2, -3\}$. If $n > 0$, then $g_T(L) = n - 1$ and if $n < 0$, then $g_T(L) = |n|$.

Proof. Let $n > 0$. Using the forms in the proof of Proposition 5.9 and the reductions of Lemma 5.2 and Corollary 4.9, one sees that $g_T(L) \leq g(\Sigma_D)$ where $D$ is the closure of $(\sigma_1 \sigma_2)^n$. A straightforward calculation shows that $g(\Sigma_D) = n - 1$. Since $w_{Kh}(L) = n + 1$, it follows that $g_T(L) = n - 1$.

Let $n < 0$. Using the forms in the proof of Proposition 5.9 and the reductions of Lemma 5.2 and Corollary 4.9, one sees that $g_T(L) \leq g(\Sigma_D')$ where $D'$ is the closure of $(\sigma_1 \sigma_2)^{n+1}$. A straightforward calculation shows that $g(\Sigma_D') = |n|$. Since $w_{Kh}(L) = |n| + 2$, it follows that $g_T(L) = |n|$.

Propositions 5.14, 5.15, and 5.16 imply Main Theorem 4.

Main Theorem 4. Let $L$ be a closed 3-braid. Then

$$0 \leq g_T(L) - (w_{Kh}(L) - 1) \leq 1.$$
Remark 5.17. Both the lower bound and upper bound of the above inequality are achieved by closed 3-braids. For example, the links in Proposition 5.16 achieve the lower bound while the links in Proposition 5.15 part (4) achieve the upper bound. There are also closed 3-braids (see Proposition 5.14) where it is unknown whether the lower bound or upper bound is achieved.
Chapter 6
Knot Floer Homology

For an oriented knot $K \subset S^3$, the knot Floer homology of $K$, denoted $\widehat{HF}(K)$, is a powerful knot invariant defined by Ozsváth and Szabó [OS04b] and independently by Rasmussen [Ras03]. The group $\widehat{HF}(K)$ is equipped with a homological grading $M$ (called the Maslov grading) and a polynomial grading $A$ (called the Alexander grading), and is written

$$\widehat{HF}(K) = \bigoplus_{A,M} \widehat{HF}_M(K, A).$$

Knot Floer homology is a generalization of the Alexander polynomial; the Alexander polynomial of a knot $\Delta_K(t)$ is recovered by taking the graded Euler characteristic of the group $\widehat{HF}(K)$:

$$\Delta_K(t) = \sum_{A,M} (-1)^M \text{rank}(\widehat{HF}_M(K, A)) \cdot t^A.$$

Knot Floer homology has provided new insight into several classical problems. Ozsváth and Szabó showed the $\widehat{HF}(K)$ detects the genus of a knot [OS04a]. New Legendrian link invariants have been found using knot Floer homology [OST08]. One can also use knot Floer homology to determine if a knot is fibered (see [OS05a], [Ghi08], and [Ni07]).

6.1 The Alexander Polynomial

Alexander [Ale28] introduced a polynomial knot invariant now known as the Alexander polynomial. Let $K$ be a knot in $S^3$, and let $X_\infty$ denote the infinite cyclic cover of $S^3 \setminus K$. Since $X_\infty$ has a covering transformation $t$, one can consider $H_1(X_\infty)$ as a $\mathbb{Z}[t, t^{-1}]$-module. The Alexander polynomial is defined to be the generator of a certain ideal of $H_1(X_\infty)$.

One way to define the Alexander polynomial is through a skein relation. Let $L_+, L_-$, and $L_v$ be links with diagrams $D_+, D_-$, and $D_v$ as in Figure 3.5.

Definition 6.1. For an oriented link $L$, the symmetrized Alexander polynomial $\Delta_L(t)$ is defined by
1. \( \Delta_{\text{unknot}}(t) = 1 \), and

2. \( \Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{-1/2} - t^{1/2}) \Delta_{L_v}(t) \).

An alternate construction of the Alexander polynomial involves a state sum over connected Kauffman states (much like Equation 3.3). In this formulation, it is convenient to use a different, but equivalent, definition of connected Kauffman state. Recall that a connected Kauffman state was previously defined to be a choice of \( A \)-smoothing or \( B \)-smoothing for each crossing of a link diagram such that the resulting collection of circles has only one component. Our first task is to make the alternate definition and show it is equivalent to the previous one.

Let \( D \) be a diagram of the link \( L \) with associated projection \( \Gamma \). Decorate \( \Gamma \) by choosing a distinguished \( \varepsilon \), and let \( Q \) and \( R \) denote the two faces of \( \Gamma \) incident to the distinguished edge.

**Definition 6.2.** A **connected Kauffman state** for a decorated link diagram \( D \) with distinguished edge \( \varepsilon \) is a bijection between the vertices of \( \Gamma \) and all faces of \( \Gamma \) other than \( Q \) and \( R \). The set of all connected Kauffman states for the decorated link diagram \( D \) is denoted \( S_c(D, \varepsilon) \).

Given a link diagram \( D \), we depict a connected Kauffman state by placing a dot in one of the four local faces near each crossing such that each face other than \( Q \) and \( R \) contains exactly one dot. A connected Kauffman state gives a Kauffman state (a choice of \( A \)-smoothing or \( B \)-smoothing at each crossing) of the link diagram by smoothing each crossing so that the local face containing the dot is joined with the local face that is opposite the crossing. The resulting Kauffman state consists of only one state circle, hence the name connected Kauffman state. See Figure 6.1.

Next, we show how to obtain the Alexander polynomial from a weighted sum over connected Kauffman states. Let \( D \) be a link diagram and \( \mathcal{X} \) the set of crossings of \( D \). For each crossing \( x \in \mathcal{X} \) and connected Kauffman state \( s \), define \( x(s) \) to be one of the monomials in Figure 6.2 depending on where the dot is placed around that crossing.

The following is Kauffman’s state sum expansion for the Alexander polynomial [Kau83].
FIGURE 6.1. **Left.** A connected Kauffman state represented by dots. **Right.** The same connected Kauffman state represented by smoothing each crossing.

\[ -t^{-1/2} \quad \begin{array}{c|c}
1 & 1 \\
\hline
\frac{1}{2} & \frac{1}{2} \\
\end{array} \quad -t^{1/2} \]

\[ \frac{1}{2} \quad \begin{array}{c|c}
t & t \\
\end{array} \]

FIGURE 6.2. Each dot in a connected Kauffman state is assigned one of the above monomials.

**Proposition 6.3** (Kauffman). Let $D$ be a diagram of the knot $K$, and let $S_c(D, \varepsilon)$ be the set of connected Kauffman states where $\varepsilon$ is a distinguished edge in $D$. Then

\[
\Delta_K(t) = \sum_{s \in S_c(D, \varepsilon)} \prod_{x \in \mathcal{X}} x(s).
\]

Let $G$ and $G^*$ be the Tait graphs of $D$. Each connected Kauffman state gives rise to a pair of spanning trees $T \subset G$ and $T^* \subset G^*$ as follows. Every crossing of $D$ corresponds to an edge of $G$ and an edge of $G^*$. If a dot in the connected Kauffman state is placed in one of the two local faces corresponding to vertices in $G$, then $T$ contains the edge associated to that crossing. Similarly, if a dot is placed in one of the two local faces corresponding to vertices in $G^*$, then $T^*$ contains the edge associated to that crossing. Furthermore, each of the spanning trees $T$ and $T^*$ has a distinguished vertex, called the root, corresponding to either the face $R$ or $Q$. The spanning trees are oriented so that each edge is pointing away from the root. Therefore, the set $S_c(D, \varepsilon)$ is in one-to-one correspondence with pairs of oriented, rooted spanning trees $T \subset G$ and $T^* \subset G^*$ satisfying the condition that each crossing of $D$ has exactly one associated edge in $T \cup T^*$. By placing a dot near the crossing associated to each edge in the local face corresponding to the head of that edge, one recovers a connected Kauffman state from $T$. 

69
and $T^*$. We will often identify a connected Kauffman state $s \in S_c(D, \varepsilon)$ with a pair of such spanning trees, and we write $s = (T, T^*)$. See Figure 6.3 for an example of the spanning trees associated to a Kauffman state.

![Figure 6.3](image)

**FIGURE 6.3.** A connected Kauffman state corresponds to a pair of spanning trees in the Tait graphs. Solid arcs indicate edges that are in the spanning tree, while dashed arcs indicate edges in the Tait graph but not in the spanning tree.

### 6.2 Construction of $\widehat{HFK}(K)$

The construction of knot Floer homology is much more geometric in flavor than the construction of Khovanov homology presented in Section 3.2. The chain complex generating $\widehat{HFK}(K)$ comes from a special Heegaard diagram associated to the knot $K$, and the differential in the complex counts pseudo-holomorphic disks in a symplectic manifold associated to the Heegaard splitting.

Let $U_\alpha$ and $U_\beta$ be genus $g$-handlebodies with boundary $\Sigma$. Then $U_\alpha \cup_\Sigma U_\beta$ is a closed 3-manifold $Y$. The decomposition of $Y$ into two handlebodies glued together along their boundary is called a Heegaard splitting of $Y$. Every closed 3-manifold has a Heegaard splitting.

One can encode a Heegaard splitting by a certain collection of curves on a surface. Let $\Sigma$ be a surface of genus $g$. Suppose $\alpha = (\alpha_1, \ldots, \alpha_g)$ is a $g$-tuple of pairwise disjoint, homologically linearly independent simple closed curves on $\Sigma$. Similarly, suppose that $\beta = (\beta_1, \ldots, \beta_g)$ is a $g$-tuple of pairwise disjoint, homologically linearly independent simple closed curves on $\Sigma$. These curves will be known as the $\alpha$-curves and $\beta$-curves of the Heegaard diagram. On the interior of $\Sigma$, attach disks along the $\alpha$ curves to obtain a handlebody $U_\alpha$, and on the exterior
of $\Sigma$, attach disks along the $\beta$ curves to obtain a handlebody $U_\beta$. The triple $(\Sigma, \alpha, \beta)$ is called a Heegaard diagram of the 3-manifold $Y = U_\alpha \cup_\Sigma U_\beta$.

**Definition 6.4.** A Heegaard diagram $(\Sigma, \alpha, \beta)$ is said to be subordinate to the knot $K \subset S^3$ if $(\Sigma, \alpha, \beta)$ is a Heegaard diagram of $S^3$ and $K$ is supported entirely inside $U_\beta$ such that $K$ intersects the attaching disk for $\beta_1$ once transversely and $K$ is disjoint from all the attaching disks for $\beta_j$ where $j > 1$.

Since $K$ is supported inside $U_\beta$ and is disjoint from every attaching disk other than $\beta_1$, it follows that $K$ is isotopic to a knot lying on $\Sigma$ that intersects $\beta_1$ once transversely and is disjoint from all other $\beta$-circles. After isotoping $K$ onto $\Sigma$, fix a small interval in $K$ containing the intersection point $K \cap \beta_1$. Choose the interval to be small enough so that it does not intersect any $\alpha$-curves. Let $z$ and $w$ be the boundary points of the interval chosen so that the orientation of $K$ goes from $z$ to $w$. Observe that $z$ and $w$ are points in $\Sigma - \alpha_1 - \cdots - \alpha_g - \beta_1 - \cdots - \beta_g$. The data $(\Sigma, \alpha, \beta, w, z)$ is called a doubly pointed Heegaard diagram, and it uniquely determines a topological knot class.

Let $S_n$ denote the symmetric group on a set of order $n$. For a surface $\Sigma$, denote the $g$-fold symmetric product of $\Sigma$ by

$$\text{Sym}^g(\Sigma) = \Sigma^g / S_g.$$ 

Let $(\Sigma, \alpha, \beta, w, z)$ be a doubly pointed Heegaard diagram, and define two $g$-dimensional tori $T_\alpha, T_\beta \subset \text{Sym}^g(\Sigma)$ by

$$T_\alpha = \alpha_1 \times \cdots \alpha_g \quad \text{and} \quad T_\beta = \beta_1 \times \cdots \beta_g.$$ 

Define $\widetilde{CFK}(\Sigma, \alpha, \beta, w, z)$ to be the free $\mathbb{Z}$-module generated by the intersection points $T_\alpha \cap T_\beta$. Let $\mathbb{D}$ be the unit disk in $\mathbb{C}$, and label two arcs in the boundary $e_1$ and $e_2$ such that $e_1 = \{z \in \partial \mathbb{D} \mid \text{Re}(z) \geq 0\}$ and $e_2 = \{z \in \partial \mathbb{D} \mid \text{Re}(z) \leq 0\}$.

**Definition 6.5.** Let $x$ and $y$ be intersection points in $T_\alpha \cap T_\beta$. A Whitney disk connecting $x$ and $y$ is a continuous map $u : \mathbb{D} \to \text{Sym}^g(\Sigma)$ satisfying $u(-i) = x, u(i) = y, u(e_1) \subset T_\alpha,$
and \( u(e_2) \subset T_\beta \). Denote the set of homotopy classes of Whitney disks connecting \( x \) and \( y \) by \( \pi_2(x, y) \).

It is often beneficial to study the “shadow” of a Whitney disk \( \phi \in \pi_2(x, y) \) in \( \Sigma \) for intersection points \( x, y \in T_\alpha \cap T_\beta \). Let \( p \in \Sigma \) be any point that is in the complement of the \( \alpha \) and \( \beta \) curves. Define

\[
    n_p : \pi_2(x, y) \to \mathbb{Z}
\]

to be the algebraic intersection number

\[
    n_p(\phi) = \#\phi^{-1}(\{p\} \times \text{Sym}^{g-1}(\Sigma)).
\]

Since \( \{p\} \times \text{Sym}^{g-1}(\Sigma) \) is disjoint from \( T_\alpha \cup T_\beta \), the map \( n_p \) is well-defined.

Endow \( \text{Sym}^g(\Sigma) \) with a symplectic structure \( \omega \). One may choose a compatible almost complex structure \( J \) on \( \text{Sym}^g(\Sigma) \) so that the moduli spaces of \( J \)-holomorphic Whitney disks are Gromov-compact manifolds. Let \( \mathcal{M}_\phi \) denote the set of \( J \)-holomorphic Whitney disks in the equivalence class \( \phi \), and define the Maslov index of \( \phi \), denoted \( \mu(\phi) \), to be the formal dimension of \( \mathcal{M}_\phi \). The moduli space \( \mathcal{M}_\phi \) admits an \( \mathbb{R} \)-action corresponding to complex automorphisms of the unit disk that preserve \( -i \) and \( i \). Let \( \widehat{\mathcal{M}}_\phi = \mathcal{M}_\phi / \mathbb{R} \) denote the quotient of the moduli space by the \( \mathbb{R} \)-action. If \( \mu(\phi) = 1 \), then the dimension of \( \widehat{\mathcal{M}}_\phi \) is zero, and hence it is a collection of signed points. Define \( c(\phi) \) to be the signed count of points in \( \widehat{\mathcal{M}}_\phi \) if \( \mu(\phi) = 1 \) and zero otherwise.

Define a map \( \partial : \widehat{CFK}(\Sigma, \alpha, \beta, w, z) \to \widehat{CFK}(\Sigma, \alpha, \beta, w, z) \) by

\[
    \partial(x) = \sum_{y \in T_\alpha \cap T_\beta} \sum_{\{\phi \in \pi_2(x, y) | \mu(\phi) = 1, n_z(\phi) = n_w(\phi) = 0\}} c(\phi) \cdot y.
\]

**Theorem 6.6** (Ozsváth - Szabó, Rasmussen). Let \( (\Sigma, \alpha, \beta, w, z) \) be a doubly pointed Heegaard diagram subordinate to the knot \( K \subset S^3 \). Then \( \widehat{CFK}(\Sigma, \alpha, \beta, w, z); \partial \) is a chain complex whose homology \( \widehat{HFK}(K) \) is a knot invariant, i.e. it is independent of the choice of doubly pointed Heegaard diagram and of the almost complex structure.
In light of this theorem, we will sometimes denote the complex $\mathbb{Z}$-module $\hat{CFK}(\Sigma, \alpha, \beta, w, z)$ by $\hat{CFK}(K)$ or $\hat{CFK}(D)$ if $D$ is a diagram of $K$.

There are two gradings on $\hat{CFK}(K)$ corresponding to functions:

$$M, A : T_\alpha \cap T_\beta \to \mathbb{Z}.$$  

Both gradings are first defined relatively, so that given two generators, we define the difference in grading between them. Once the relative grading is established, the additive indeterminacy is removed. For $x, y \in T_\alpha \cap T_\beta$, define

$$M(x) - M(y) = \mu(\phi) - 2n_w(\phi),$$

where $\phi \in \pi_2(x, y)$. The additive indeterminancy of $M$ is removed using the Heegaard Floer homology of $S^3$. Specifically, $\widehat{CF}(S^3)$ is a chain complex generated by the same intersection points of $T_\alpha \cap T_\beta$, its differential decreases the relative grading by one, and its homology is $\mathbb{Z}$ supported in a single grading. The Maslov grading on $\hat{CFK}(K)$ is determined by declaring that the homology of $\widehat{CF}(S^3)$ is supported in grading zero.

For $x, y \in T_\alpha \cap T_\beta$ define

$$A(x) - A(y) = n_z(\phi) - n_w(\phi),$$

where $\phi$ is Whitney disk connecting $x$ to $y$. By [OS04b], $A(x) - A(y)$ is independent of the choice of $\phi$. Ozsváth and Szabó show that

$$\sum_{x \in T_\alpha \cap T_\beta} (-1)^{M(x)}t^{A(x)} = t^c \Delta_K(t),$$

where $\Delta_K(t)$ is the symmetrized Alexander polynomial of $K$ and $c$ is some integer. The additive indeterminancy is removed by setting $c = 0$.

Knot Floer homology is supported on a finite number of slope one lines with respect to the Maslov - Alexander bigrading. In order to capture this behavior, we define an auxiliary grading called the $\delta$-grading by

$$\delta(x) = A(x) - M(x).$$
**Definition 6.7.** Let $\delta_{\text{min}}$ be the minimum $\delta$-grading where $\widehat{HFK}(K)$ is nontrivial, and let $\delta_{\text{max}}$ be the maximum $\delta$-grading where $\widehat{HFK}(K)$ is nontrivial. Define the knot Floer width of $K$ as

$$w_{\widehat{HFK}}(K) = \delta_{\text{max}} - \delta_{\text{min}} + 1.$$ 

**6.3 From Decorated Knot Diagrams to Heegaard Diagrams**

Let $D$ be the diagram of a knot $K$ in $S^3$, and let $\Gamma$ be the associated projection, thought of as a 4-regular graph embedded in the plane. Choose a distinguished edge $\varepsilon$ in $\Gamma$ that is incident with the unbounded face. The knot diagram $D$ together with a choice of a distinguished edge is called a decorated knot diagram. Given a decorated knot diagram, Ozsváth and Szabó [OS03a] construct a Heegaard diagram subordinate to the knot $K \subset S^3$.

Let $\Sigma$ be the boundary of a regular neighborhood of $\Gamma$ in $S^3$. Thus $\Sigma$ is a genus $c(D) + 1$, where $c(D)$ is the number of crossings of $D$. To each face of $\Gamma$, excluding the unbounded region, we associate an $\alpha$-curve that follows the boundary of the face. To each crossing in $D$, we associate a $\beta$ curve as depicted in Figure 6.4. In addition, we choose $\beta_1$ to be a meridian of $K$ situated around the distinguished edge $\varepsilon$. Finally, choose two points $z$ and $w$ in a neighborhood of $\beta_1$ but on opposite sides.

![Figure 6.4](image.png)

**FIGURE 6.4.** The $\alpha$ curves trace out the faces of the projection $\Gamma$, while the $\beta$ curves are modeled after the crossings of $D$.

Each crossing is contained in four (not necessarily distinct) faces of $\Gamma$. In a neighborhood of a crossing, there are at most four points of intersection between the $\alpha$ curves and the $\beta$ curve.
associated to that crossing. If one of the faces of \( \Gamma \) near a crossing is the unbounded face, then there are fewer than four intersection points. Figure 6.5 shows the construction the Heegaard diagram for the figure eight knot.

![Figure 6.5](image)

**FIGURE 6.5.** **Left.** A Heegaard diagram for the figure eight knot. **Right.** The knot can be isotoped to \( \Sigma \) such that the only \( \beta \) curve it intersects is \( \beta_1 \).

### 6.4 Heegaard Diagrams and Spanning Trees

Let \((\Sigma, \alpha, \beta, w, z)\) be a doubly pointed Heegaard diagram constructed from a decorated projection \( D \). The set of generators \( T_\alpha \cap T_\beta \) of \( \widehat{CFK}(K) \) are in one-to-one correspondence with connected Kauffman states. An intersection point in \( T_\alpha \cap T_\beta \) can be represented by a \((c+1)\)-tuple of intersection points of the \( \alpha \) and \( \beta \) curves on \( \Sigma \) such that each \( \alpha \) and each \( \beta \) curve contains exactly one intersection point. Each crossing of \( D \) corresponds to a \( \beta \) curve on \( \Sigma \), and the choice of one of the four local faces at that crossing corresponds to choosing an \( \alpha \) curve (which each correspond to a face of the knot projection) that intersects that \( \beta \) curve. Because \( \beta_1 \) intersects only one \( \alpha \) curve, that intersection point must be part of the \((c+1)\)-tuple, and therefore that \( \alpha \) curve is not assigned to any of the vertices. Figure 6.6 shows an example of a Kauffman state for the figure eight knot.

Ozsváth and Szabó give a combinatorial way to compute two gradings \( A : S \to \mathbb{Z} \) and \( M : S \to \mathbb{Z} \) for a connected Kauffman state. For each vertex in \( \Gamma \), the choice of a local face determines the local contribution to both the Maslov and the Alexander gradings as shown in Figures 6.7 and 6.8. The Maslov grading is defined to be the sum of all local Maslov
FIGURE 6.6. The intersection points between the $\alpha$ and $\beta$ curves correspond to the connected Kauffman state in Figure 6.1.

FIGURE 6.7. The local Alexander filtration level contributions, and the Alexander filtration level is the sum of all local Alexander contributions. Compare this with the weights assigned to connected Kauffman states in the expansion of the Alexander polynomial (Figure 6.2).

Ozsváth and Szabó [OS03a] prove that if a knot is alternating, then the complex $\hat{CFK}(D)$ constructed from a decorated knot diagram has only one $\delta$-grading, which is determined by the signature of the knot.

**Theorem 6.8** (Ozsváth - Szabó). Let $K$ be an alternating knot. Then $\hat{HF}(K)$ is entirely supported in the $\delta = -\sigma(K)/2$ grading.

Recall an edge in the Tait graph can either be an $A$-edge or a $B$-edge, and it can also be positive or negative. Label each edge $A_+, A_-, B_+, B_-$ according to these four choices. See Figure 6.9.

### 6.5 Knot Floer Width

In this section, we examine the $\delta$-gradings of the spanning trees in the complex $\hat{CFK}(D)$. Let $\delta_{\min}(D)$ be the minimum $\delta$-grading in the spanning tree complex $\hat{CFK}(D)$, and let $\delta_{\max}(D)$
be the maximum $\delta$-grading in the spanning tree complex $\widehat{CFK}(D)$. Define the width of the complex $\widehat{CFK}(D)$ by
\[ w(\widehat{CFK}(D)) = \delta_{\text{max}}(D) - \delta_{\text{min}}(D) + 1. \]

In the notation above, we have omitted all references to the distinguished edge $\varepsilon$ since the connected Kauffman state complex depends on the choice of marked edge, but the width does not.

**Proposition 6.9.** Let $D$ be an oriented knot diagram, and let $\varepsilon$ and $\varepsilon'$ be marked edges in $\Gamma$. The width of the complex $\widehat{CFK}(D)$ does not depend on the choice of the distinguished edge.

**Proof.** Consider the Kauffman state $s = (T, T^*)$ as a pair of rooted spanning trees in the Tait graphs $G$ and $G^*$. The dots of the state $s$ can be recovered as follows. Let $e$ be a directed edge in either $T$ or $T^*$. Then $e$ has an associated crossing $x$ in $D$ and the head and tail of $e$ lie in two different local faces around $x$. The local face of $x$ that contains the dot for $s$ is the face that contains the head of $e$. Changing the distinguished edge in $D$ corresponds to (possibly) changing the root in $T$ or $T^*$. This implies that the direction of the edge $e$ may change. However, the local difference between the Alexander and Maslov grading does not depend on the endpoint of $e$ chosen. Notice if $e$ is labeled $B_+$, then the local difference is $\frac{1}{2}$,
regardless of where the head (ie. the dot in the Kauffman state) is. Similarly, if \( e \) is marked \( A_- \), then the local difference is \( -\frac{1}{2} \), and if \( e \) is marked \( A_+ \) or \( B_- \), then the local difference is zero (see Figures 6.7, 6.8, and 6.9). Thus the \( \delta \)-grading remains unchanged. Hence, the width of the complex \( \widehat{CFK}(D) \) does not depend on the choice of the distinguished edge. \( \square \)

A Kauffman state \( s \in Sc(D, \varepsilon) \) is said to be on the maximal diagonal if \( \delta(s) = \delta_{\text{max}}(D) \) and on the minimal diagonal if \( \delta(s) = \delta_{\text{min}}(D) \). Define a map \( \eta : Sc(D, \varepsilon) \rightarrow \mathbb{Z} \) by setting \( \eta(s) = E_B^+(T) + E_B^+(T^*) - E_A^-(T) - E_A^-(T^*) \). From the proof of Proposition 6.9, the calculation of the local difference for each edge implies that

\[
\delta(s) = \frac{1}{2} \eta(s).
\]

Therefore \( s \) is on the maximal diagonal if \( \eta(s) \) is maximized and on the minimal diagonal if \( \eta(s) \) is minimized. Moreover,

\[
w(\widehat{CFK}(D)) = \frac{1}{2}(\max\{\eta(s) | s \in Sc(D, \varepsilon)\} - \min\{\eta(s) | s \in Sc(D, \varepsilon)\}) + 1.
\]

The width of the complex \( \widehat{CFK}(D, \varepsilon) \) behaves predictably under a crossing change. Before this behavior can be described, a lemma is needed.

**Lemma 6.10.** Let \( D \) be a diagram with marked edge \( \varepsilon \) for the knot \( K \), and let \( e \) be an edge in either of the checkerboard graphs \( G \) or \( G^* \).

1. If \( e \) is in an \( A \) (or \( B \)) cycle, then there exists a state \( s \in Sc(D, \varepsilon) \) on the minimal (or maximal) diagonal that does not contain \( e \).

2. If \( e \) is an \( A \) (or \( B \)) edge and is not in an \( A \) (or \( B \)) cycle, then every state \( s \in Sc(D, \varepsilon) \) on the maximal (minimal) diagonal must contain \( e \).

**Proof.** Only the statements for \( B \)-edges are proved; the proofs for the \( A \)-edges are analogous. Without loss of generality, suppose that \( e \) is a \( B \)-edge in \( G \).

(1) Suppose \( e \) is in a \( B \)-cycle \( \gamma \) and suppose all states on the maximal diagonal contain \( e \). Let \( s \) be a state on the maximal diagonal consisting of the two spanning trees \( T \subset G \) and
$T^* \subset G^*$. Since $T$ contains the edge $e$, there exists some other edge $e_0$ in $\gamma$ not contained in $T$. The graph obtained by adding the edge $e_0$ to $T$ contains a unique cycle $\tau$.

Suppose the edge $e$ is contained in this unique cycle. Then form a new state $s_0$ consisting of two new spanning trees $T_0$ and $T_0^*$, where $T_0$ is the spanning tree obtained by adding $e_0$ and deleting $e$ in $T$, and $T_0^*$ is the spanning tree obtained by deleting the dual of $e_0$ and adding the dual of $e$ in $T^*$.

To show that $s_0$ is on the maximal diagonal, it is enough to show that $\eta(s) = \eta(s_0)$. Since $e$ and $e_0$ are in a $B$-cycle, both are $B$-edges, and their duals are $A$-edges. Deleting a $B$-edge from $T$ and adding its dual $A$-edge to $T^*$ results in a net decrease of $\eta(s)$ by one, since this corresponds to removing a $B_+$ edge from $T$ and replacing it with an $A_+$ edge in $T^*$ or removing a $B_-$ edge from $T$ and replacing it with a $A_-$ edge in $T^*$. Likewise, deleting an $A$-edge from $T$ and adding its dual $B$-edge to $T^*$ results in a net increase of $\eta(s)$ by one. To construct $s_0$, first a $B$-edge is removed from $T$ and its dual $A$-edge is inserted into $T^*$. Then an $A$-edge is removed from $T^*$ and its dual $B$-edge is inserted into $T$. Thus $\eta(s) = \eta(s_0)$, and $s_0$, a state not containing the edge $e$, is on the maximal diagonal.

Now suppose the edge $e$ is not contained the cycle $\tau$. Thus $\tau \neq \gamma$, and there is some edge $e_1$ in $\tau$ not contained in $\gamma$. Construct a new state $s_1$ by deleting $e_1$ from $T$, replacing it with its dual in $T^*$, inserting $e_0$ into $T$, and deleting its dual from $T^*$. Notice that if $e_1$ is an $A$-edge, then two $A$-edges were deleted and two $B$-edges were inserted in the construction of $s_1$. Thus $\eta(s_1) = \eta(s) + 2$, contradicting the fact that $s$ is on the maximal diagonal. Hence $e_1$ must be a $B$-edge, and the construction of $s_1$ simultaneously exchanges an $A$-edge for a $B$-edge and a $B$-edge for an $A$-edge. Therefore $\eta(s_1) = \eta(s)$, and $s_1$ is again on the maximal diagonal.

Iterate this process as follows: continue by choosing a new edge in $\gamma$ not in $s_1$ (and thus this edge is also not in $s$). Adding this new edge to $s_1$ forms a unique cycle. If $e$ is contained in this unique cycle, the process ends as described above. If $e$ is not contained in this unique cycle, then some edge not in $\gamma$ can be removed, resulting in a state still on the maximal diagonal. Since $\gamma$ contains only a finite number of edges, in a finite number of steps, the edge
$e$ must be contained in the unique cycle. Therefore there is a state on the maximal diagonal not containing the edge $e$.

(2) Suppose $e$ is a $B$-edge and is not in a $B$-cycle. Also, suppose that there exists a state $s$ on the maximal diagonal, consisting of spanning trees $T \subset G$ and $T^* \subset G^*$, not containing the edge $e$. Then consider the subgraph of $G$ obtained by adding the edge $e$. There is a unique cycle in this subgraph, and since $e$ is not in a $B$-cycle, this cycle contains an $A$-edge $e_0$. Let $s_0$ be the state obtained by deleting $e_0$ and adding $e$ in $T$ and adding the dual of $e_0$ and deleting the dual of $e$ in $T^*$. Both of these switches adds a $B$-edge and deletes an $A$-edge. Thus $\eta(s_0) = \eta(s) + 2$ and this contradicts the fact that $s$ is on the maximal diagonal. Hence all states on the maximal diagonal must contain $e$. \hfill \Box

With the previous lemma established, the behavior of the width of $\hat{CFK}(D)$ under a crossing change can now be determined. Let $D$ be a diagram for the knot $K$ with marked edge $\varepsilon$, and let $D_0$ be the diagram obtained from $D$ by a single crossing change. Let $G$ and $G^*$ be the Tait graphs for $D$ and $G_0$ and $G_0^*$ be the Tait graphs for $D_0$. The crossing in $D$ has an associated $A$-edge $e_A$ and an associated $B$-edge $e_B$ in the Tait graphs. These two edges are dual to each other. Moreover, the crossing change switches $e_A$ to a $B$-edge and $e_B$ to an $A$-edge.

**Theorem 6.11.** Let $D$ be a diagram of a knot $K$ and $D_0$ be the diagram obtained from $D$ by a single crossing change. Suppose $e_A$ (an $A$-edge) and $e_B$ (a $B$-edge) are the edges in the Tait graphs $G$ and $G^*$ of $D$ associated to the crossing that is changed. Then the width of $\hat{CFK}(D)$ under a crossing change behaves as follows:

1. $|w(\hat{CFK}(D)) - w(\hat{CFK}(D_0))| \leq 1$.

2. If $e_A$ is in an $A$-cycle and $e_B$ is in $B$-cycle, then $w(\hat{CFK}(D_0)) = w(\hat{CFK}(D)) + 1$.

3. If $e_A$ is not in any $A$-cycle and $e_B$ is in a $B$-cycle, then $w(\hat{CFK}(D_0)) = w(\hat{CFK}(D))$.

4. If $e_A$ is an $A$-cycle and $e_B$ is not in any $B$-cycle, then $w(\hat{CFK}(D_0)) = w(\hat{CFK}(D))$. 80
5. If $e_A$ is not in any $A$-cycle and $e_-$ is not in any $B$-cycle, then $w(\hat{CFK}(D_0)) = w(\hat{CFK}(D)) - 1$.

Proof. (1) Let $s = (T, T^*)$ be a Kauffman state for $D$. The edges $e_A$ and $e_B$ are dual in the Tait graphs. Thus exactly one of them is an edge in either $T$ or $T^*$. The crossing change corresponds to changing this edge and no others in $T$ or $T^*$. Label the new Kauffman state for $D_0$ by $s_0 = (T_0, T_0^*)$.

Suppose $e_B$ is marked $B_+$. Then $e_A$ is marked $A_+$, and after the crossing change, $e_B$ switches to an $A_-$-edge and $e_A$ switches to a $B_-$ edge (see Figure 6.9). Thus if $e_B$ is in either $T$ or $T^*$, it follows that $\eta(s) = \eta(s_0) + 2$ and $\delta(s) = \delta(s_0) + 1$. If $e_B$ is not in either $T$ or $T^*$, then $e_A$ must be in either $T$ or $T^*$. Then $\eta(s) = \eta(s_0)$ and $\delta(s) = \delta(s_0)$. This implies that $\delta_{\max}(D)$ and $\delta_{\min}(D)$ either decrease by one or remain the same. The case where $e_B$ is marked $B_-$ is analogous. Therefore, $|w(\hat{CFK}(D)) - w(\hat{CFK}(D_0))| \leq 1$.

(2) Suppose $e_A$ is in an $A$-cycle and $e_B$ is in a $B$-cycle. Then by Lemma 6.10, there are states $s_{\max}$ and $s_{\min}$ in $S_c(D, \varepsilon)$ such that $s_{\max}$ is on the maximal diagonal and does not contain $e_B$ and $s_{\min}$ is on the minimal diagonal and does not contain $e_A$. Since $e_A$ and $e_B$ are dual and $s_{\max}$ does not contain $e_B$, it follows that $s_{\max}$ contains $e_A$. Similarly, $s_{\min}$ contains $e_B$. Let $s'_{\max}$ and $s'_{\min}$ be the states after the crossing change with the same edges as $s_{\max}$ and $s_{\min}$ respectively.

If $e_B$ is marked $B_+$, then $e_A$ is marked $A_+$. After the crossing change, $e_B$ is switched to a $A_-$ edge, and $e_A$ is switched to a $B_-$ edge. It follows that $\eta(s'_{\max}) = \eta(s_{\max}) + 2$ and $\eta(s'_{\min}) = \eta(s_{\min})$. In this case, the crossing change induces an increase in $\delta_{\max}(D)$ by one and no change in $\delta_{\min}(D)$. Similarly, if $e_B$ is marked $B_-$ and $e_A$ is marked $A_-$, then the crossing change induces no change in $\delta_{\max}(D)$ and a decrease in $\delta_{\min}(D)$ by one. Therefore, $w(\hat{CFK}(D_0)) = w(\hat{CFK}(D)) + 1$.

(3) Suppose $e_A$ is not in any $A$-cycle and $e_B$ is in a $B$-cycle. As before, there is a state $s_{\max}$ in $S_c(D, \varepsilon)$ on the maximal diagonal not containing $e_B$. Now, however, every state on
the minimal diagonal must contain the edge $e_A$. Hence $s_{\text{max}}$, as well as every state on the minimal diagonal contains the edge $e_A$. So, if $e_B$ is marked $B_-$ and $e_A$ is marked $A_-$, then both $\delta_{\text{max}}(D)$ and $\delta_{\text{min}}(D)$ are increased by one under a crossing change. If $e_B$ is marked $B_+$ and $e_A$ is marked $A_+$, then the crossing change does not alter $\eta(s_{\text{max}})$ or $\eta(s)$ for $s$ any state on the minimal diagonal. If another state $s'_{\text{max}}$ on the maximal diagonal contains the edge $e_B$, then the crossing change decreases $\eta(s'_{\text{max}})$ by two. Therefore $\delta_{\text{max}}(D)$ and $\delta_{\text{min}}(D)$ are unchanged. Thus $w(\hat{CFK}(D_0)) = w(\hat{CFK}(D))$.

(4) Suppose $e_A$ is in an $A$-cycle and $e_B$ is not in any $B$-cycle. This case is completely analogous to the case above. If $e_B$ is marked $B_-$ and $e_A$ is marked $A_-$, then both $\delta_{\text{max}}(D)$ and $\delta_{\text{min}}(D)$ remain unchanged, and if $e_B$ is marked $B_+$ and $e_A$ is marked $A_+$, then both $\delta_{\text{max}}(D)$ and $\delta_{\text{min}}(D)$ are decreased by one. Therefore, $w(\hat{CFK}(D_0)) = w(\hat{CFK}(D))$.

(5) Suppose $e_A$ is not in any $A$-cycle and that $e_B$ is not in any $B$-cycle. Then all states on the maximal diagonal contain $e_B$ and all states on the minimal diagonal contain $e_A$. If $e_B$ is marked $B_-$ and $e_A$ is marked $A_-$, then $\delta_{\text{max}}(D)$ remains unchanged and $\delta_{\text{min}}(D)$ is increased by one. If $e_B$ is marked $B_+$ and $e_A$ is marked $A_+$, then $\delta_{\text{max}}(D)$ is decreased by one and $\delta_{\text{min}}(D)$ remains unchanged. Thus $w(\hat{CFK}(D_0)) = w(\hat{CFK}(D)) - 1$. \hfill \Box

Theorems 2.12 and 6.11 show that the genus of the Turaev surface and the width of $\hat{CFK}(D)$ behave the same under crossing changes.

**Theorem 6.12.** Let $D$ be a diagram for a knot $K \subset S^3$ and $\Sigma_D$ be the Turaev surface for $D$. Then $w(\hat{CFK}(D)) = g(\Sigma_D) + 1$.

**Proof.** Let $D$ be a diagram for $K$. If $D$ is an alternating diagram, then the connected Kauffman states are supported in one $\delta$-grading, and $w(\hat{CFK}(D)) = 1$. Also, if $D$ is an alternating diagram, then $\Sigma_D$ is a sphere, and hence the result holds for alternating knots. Since any knot diagram can be obtained from an alternating diagram through a sequence of crossing changes, Theorem 2.12 and Theorem 6.11 imply the result. \hfill \Box
The following theorem is the main result of [Low08].

**Main Theorem 1.** Let $K$ be a knot in $S^3$. Then

$$w_{\widehat{HF}}(K) \leq g_T(K) + 1.$$

**Proof.** Denote the connected Kauffman state complex by $\widehat{CFK}(D)$ where $D$ is a diagram of $K$.

$$w_{\widehat{HF}}(K) \leq \min\{w(\widehat{CFK}(D)) \mid D \text{ is a diagram of } K\}$$

$$= \min\{g(\Sigma_D) + 1 \mid D \text{ is a diagram of } K\}$$

$$= g_T(K) + 1.$$

Comparing Main Theorem 1 with Proposition 3.13, one sees that reduced Khovanov width and knot Floer width have a common bound.
Chapter 7
Conclusion

We conclude this thesis by giving possible avenues for future research. Proposition 3.13 and Main Theorem 1 state that reduced Khovanov width and knot Floer width are both bounded from above by Turaev genus plus one. However, it is still unclear whether a direct relationship between reduced Khovanov width and knot Floer width exists. Using computations of Khovanov homology from KnotInfo [CL09] and the computations of knot Floer homology from Baldwin and Gillam [BG06], one can conclude that reduced Khovanov width and knot Floer width are equal for knots with twelve or fewer crossings. Proposition 3.10 and Corollary 5.4 give the reduced Khovanov width of \((3, q)\) torus knots. Ozsváth and Szabó [OS05b] describe a way to compute the knot Floer homology of \((3, q)\) torus knots. Using these two computations, one can conclude that the reduced Khovanov width and knot Floer width of the \((3, q)\) torus knots are equal. In addition, Manolescu and Ozsváth [MO07] prove that the reduced Khovanov width and knot Floer width of quasi-alternating links are both equal to one. These computations indicate a potential relationship between reduced Khovanov width and knot Floer width.

Theorems 3.12 and 6.8 state that if \(K\) is an alternating knot, then \(\tilde{\mathcal{K}h}(K)\) is entirely supported in the \(\delta = -\sigma(K)\) grading, and \(\widehat{HF}K(K)\) is entirely supported in the \(\delta = -\sigma(K)/2\) grading. Let \(K\) be a knot (not necessarily alternating). Suppose \(\tilde{\delta}_{\text{min}}\) is the minimum \(\delta\)-grading where \(\tilde{\mathcal{K}h}(K)\) is supported and \(\tilde{\delta}_{\text{max}}\) is the maximum \(\delta\)-grading where \(\tilde{\mathcal{K}h}(K)\) is supported. Furthermore, suppose \(\hat{\delta}_{\text{min}}\) is the minimum \(\delta\)-grading where \(\widehat{HF}K(K)\) is supported and \(\hat{\delta}_{\text{max}}\) is the maximum \(\delta\)-grading where \(\widehat{HF}K(K)\) is supported. For all known examples, we have the inequalities

\[
\tilde{\delta}_{\text{min}} \leq -\sigma(K) \leq \tilde{\delta}_{\text{max}} \quad \text{and} \quad \hat{\delta}_{\text{min}} \leq -\sigma(K)/2 \leq \hat{\delta}_{\text{max}}.
\]
If the above inequalities hold in general, then one can construct a lower bound for homological width from the signature and the concordance invariants $s$ and $\tau$ mentioned in the introduction.

Main Theorem 2 states that in certain cases, one can replace a crossing in a link diagram with an alternating rational tangle and the Khovanov width does not change. The main tool used in proving this theorem is the long exact sequence in Khovanov homology. Manolescu [Man07] proved that there is a similar long exact sequence in knot Floer homology. The long exact sequence in Khovanov homology contains more information about gradings than Manolescu’s long exact sequence in knot Floer homology. In the time since Manolescu developed his sequence, knot Floer homology has been shown to have a combinatorial description [MOS09]. It may be possible to reconstruct the long exact sequence in knot Floer homology using the combinatorial description and then prove an analogue of Main Theorem 2 for knot Floer homology. If an analogue to Main Theorem 2 exists, then one could likely prove that the reduced Khovanov width and knot Floer width of closed 3-braids are equal.

Finally, Turaev genus is not yet a well understood invariant. We know that it relates to the span of the Jones polynomial, Khovanov width, and knot Floer width; however, it potentially has connections with other knot and link invariants. In addition, the behavior of Turaev genus under common knot theory constructions is unknown. For instance, it is unknown whether $g_T(K_1 \# K_2) = g_T(K_1) + g_T(K_2)$, where $K_1 \# K_2$ denotes the connect sum of two knots $K_1$ and $K_2$. It is also unknown how Turaev genus behaves when a knot is cabled. One approach to understanding the behavior of Turaev genus under these constructions is to find a definition of Turaev genus that does not depend on link diagrams. If such a definition exists, then one can use 3-dimensional arguments to prove results about Turaev genus.
References


Vita

Adam Lowrance was born in July 1982, in Baton Rouge, Louisiana. He finished his undergraduate studies at Amherst College in May 2004. In August 2004, he came to Louisiana State University to pursue graduate studies in mathematics. He earned a master of science degree in mathematics from Louisiana State University in May 2006. He is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in December 2009.