1979

Convergence Theorems for Linear Evolution Equations.

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7927505

ANDERSON, MARY JORGENSEN
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THE LOUISIANA STATE UNIVERSITY AND AGRICULTURAL AND MECHANICAL COL., PH.D., 1979

CONVERGENCE THEOREMS FOR LINEAR EVOLUTION EQUATIONS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
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B.S., Louisiana State University, 1965
M.S., Louisiana State University, 1968
August, 1979

ACKNOWLEDGEMENT

This dissertation was prepared under the direction of Dr. J.R. Dorroh. The author wishes to express her appreciation for the invaluable aid and assistance rendered by Dr. Dorroh in the preparation of this dissertation. She also wishes to express her appreciation to her husband, Dr. E.H. Anderson, without whose help it would never have been written.

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ABSTRACT

The differential equations

$$u'(t) = A(t)u(t), \quad a \leq t \leq b$$

and

$$u'(t) - A(t)u(t) = f(t), \quad a \leq t \leq b$$

where $\{A(t): a \leq t \leq b\}$ is a family of unbounded linear operators in a Banach space X, are studied under the hypotheses that a weak evolution system exists, or that approximate solutions to the differential equation exist.

Given that $\{A_n(\cdot)\}_{n=1}^{\infty}$ is a sequence of strongly measurable functions from [a,b] into B(Y,X) with norm $||A_n(\cdot)||_{Y,X}$ bounded above by an integrable function for each n, and that $\{A_n(\cdot)\}$ converges strongly almost everywhere to $A(\cdot)$, where $U_n(\cdot,\cdot)$ is a proper evolution system generated by $A_n(\cdot)$, $u_n(t)$ represents the solution to the corresponding differential equation for each n, $U(\cdot,\cdot)$ is the proper evolution system generated by $A(\cdot)$ with u(t) the solution to this differential equation, then sufficient conditions are developed under which U_n converges to U, and sufficient conditions are also found for $u_n(t)$ to converge to u(t).

INTRODUCTION

This dissertation is concerned with the differential equations

$$u'(t) = A(t)u(t),\ a \leq t \leq b,$$

and

$$u'(t) - A(t)u(t) = f(t),\ a \leq t \leq b.$$

J.R. Dorroh and R.A. Graff, in a paper entitled Integral equations in Banach spaces [3], demonstrated the existence and uniqueness of an evolution system generated by a family $\{A(t): a \leq t \leq b\}$ of unbounded operators in a Banach space X when a weak evolution system exists, and when approximate solutions to the differential equation exist.

In the two papers entitled Linear evolution equations of "hyperbolic" type [7], and Linear evolution equations of "hyperbolic" type, II [8], Tosio Kato dealt with the subject of evolution systems where the generating family $\{A(t)\}$ meets other conditions: in [7] the hypothesis is that the family $\{A(t)\}$ is stable, and in [8] the hypothesis is weakened to quasi-stability.

In [8] Kato gave several theorems concerning convergence of evolution systems and convergence of solutions, when families of differential equations exist. Here we present some results similar to those of Kato, but under the hypotheses of Dorroh and Graff in [3].

CHAPTER I

PRELIMINARIES

In this chapter, the background material needed in this dissertation is presented. A basic familarity with Banach spaces and semigroups of operators is assumed. General references dealing with these areas are Butzer and Berens [1], Dunford and Schwartz [4], Hille and Phillips [5], Ladas and Lakshmikantham [9], and Yosida [11].

We will employ the following notation and make the following assumptions throughout Chapters 1, 2, and 3. Let X and Y be Banach spaces, with Y densely and continuously included in X. We denote by B(Y,X) the set of all bounded linear operators from Y into X. Let $a < b$, and I denote the interval [a,b]. Let $A:[a,b] \rightarrow B(Y,X)$ be <u>strongly</u> <u>measurable</u>; i.e., $A(\cdot)y$ is a measurable function from [a,b] into X for each $y \in Y$. Let Δ be the subset of I x I defined by $\Delta = \{(t,s):a\leq s\leq t\leq b\}$.

Several different norms are used, so for clarity they will be subscripted as follows: $||\cdot||_X$ is the norm in the Banach space X, $||\cdot||_Y$ is the norm in the Banach space Y, and $||\cdot||_{Y,X}$ denotes the operator norm in B(Y,X). If the meaning is clear from context, the subscript may be omitted. L_1 (X) consists of all measurable functions f defined on [a,b] into X which are such that $||f(t)||_X$ is integrable, with $|||f|||_{1,X} = \int_a^b ||f(t)||_X dt$.

For $f: I \to X$, we write $||f||_{\infty,X} = \text{ess} \sup_{t \in I} ||f(t)||_X$. Also $\bar{\int}_I f(t)dt$ denotes the upper-integral of f, and $\dot{f}(t)$ denotes the derivative of f with respect to t. We say f is <u>absolutely continuous</u> on I if given $\varepsilon > 0$ there is a $\delta > 0$ such that $\Sigma_{i=1}^n ||f(t_i') - f(t_i)||_X < \varepsilon$ for every finite collection $\{[t_i, t_i']: i=1, 2, \ldots, n; t_i, t_i' \in I\}$ of nonoverlapping intervals with $\Sigma_{i=1}^n [t_i' - t_i] < \delta$. It is sometimes necessary to denote the norm used for a particular property; for example, <u>X - absolutely continuous</u> will mean absolutely continuous with respect to the norm of the space X.

The following definitions and statements are taken from Dorroh and Graff [3].

<u>Definition 1.1</u>. An <u>ε - approximate solution</u> of the differential equation

$$u'(t) = A(t)u(t), \; u(a) = y \in Y$$

on the interval [a,b] is a bounded measurable function v from [a,b] into Y which is X - absolutely continuous and X - differentiable a.e. on [a,b], with v(a) = y, and

$$\int_a^b ||v'(t) - A(t)v(t)||_X dt < \varepsilon.$$

If $v'(t) = A(t)v(t)$ a.e. on [a,b], then v is a <u>solution</u> of the differential equation.

<u>Definition 1.2</u>. A <u>weak evolution system generated by</u> $A(\cdot)$ is a bounded function U from Δ into $B(X,X^{**})$ such that $U(a,a,) = I$, the identity function, and such that if

$a < t \leq b$, then $U(t,\cdot)$ is a weak solution of the differential equation

$$Q'(s) = -Q(s)A(s)$$

on the interval $[0,t]$.

Definition 1.3. A proper evolution system is a function U from Δ into $B(X)$ which satisfies

$$U(t,t) = I, \; U(t,s)U(s,r) = U(t,r)$$

for $a \leq r \leq s \leq t \leq b$. A proper evolution system generated by $A(\cdot)$ is a weak evolution system generated by $A(\cdot)$ which is also a proper evolution system. A strongly continuous evolution system is one in which the function $(t,s) \to U(t,s)x$ is jointly continuous from Δ into X for each $x \in X$.

CHAPTER 2

OPERATOR VOLTERRA EQUATIONS

Let $a < b$, and let $\Delta = \{(t,s): a \le s \le t \le b\}$. Let X denote a Banach space, let K denote the collection of all strongly continuous functions from Δ into B(X) and let M denote the collection of all essentially bounded strongly measurable functions from [a,b] into X. Then K is a Banach space under the norm

$$||U||_\infty = \sup_{(t,s)\varepsilon\Delta} ||U(t,s)||_{X,X}$$

and M is a Banach space under the norm

$$||B||_\infty = \operatorname{ess\,sup}_{t\varepsilon[a,b]} ||B(t)||_{X,X}$$

If U, V ε K and B ε M, then we define UBV ε K by

(1) $$(UBV)(t,r) = \int_r^t U(t,s)B(s)V(s,r)ds,$$

where the integral is in the strong operator topology. See [8, Lemma 1, p. 654]; the integrand is strongly measurable by [8, Lemma A4, p. 665].

The following theorem shows that the convolution operation defined above by (1) is associative.

Theorem 2.1. UB (VDW) = (UBV) DW for U, V, W ε K and B, D ε M.

Proof:

$$UB\ (VDW) = \int_\sigma^\tau U(\tau,s)B(s)[\int_\sigma^s V(s,r)D(r)W(r,\sigma)dr]ds$$

$= \int_{\sigma}^{\tau} \int_{\sigma}^{s} U(t,s)B(s)V(s,r)D(r)W(r,\sigma)dr\ ds$

(where $r < s$, and r goes from σ to τ)

$= \int_{\sigma}^{\tau} \int_{r}^{\tau} U(\tau,s)B(s)V(s,r)D(r)W(r,\sigma)ds\ dr$

(by Fubini's Theorem [4])

$= \int_{\sigma}^{\tau} [\int_{r}^{\tau} U(\tau,s)B(s)V(s,r)ds]\ D(r)W(r,\sigma)dr$

$= \int_{\sigma}^{\tau} (UBV)(t,r)D(r)W(r,\sigma)dr$

$= (UBV)\ DW.$ //

If U, $V \in K$ and $B \in M$, then we can consider the following two Volterra integral equations

(2) $W(t,r) = V(t,r) + \int_{r}^{t} U(t,s)B(s)W(s,r)ds$ and

(3) $W(t,r) = V(t,r) + \int_{r}^{t} W(t,s)B(s)U(s,r)ds$

where the solution W is sought in K. These equations can be rewritten in terms of the convolution product we have just defined as

(2') $W = V + UBW$ and

(3') $W = V + WBU$.

It is convenient to define, for $U \in K$ and $B \in M$, the linear operator $\{UB\}$ and $\{BU\}$ from K into K by

(4) $\{UB\}W = UBW$, $\{BU\}W = WBU$.

Thus (2) and (3) can also be rewritten as

(2") $W = V + \{UB\}W$ and

(3") $W = V + \{BU\}W$.

The solutions to these equations can then be found using standard techniques, as shown in Theorem 2.2. First we need the following lemma.

Lemma 2.2.1. Let U, V ε K and B ε M. Then for each n,

$$||\{UB\}^n V||_\infty \le \frac{1}{n!} ||U||_\infty^n \, |||B|||_{1,X}^n \, ||V||_\infty.$$

Also $||\{BU\}^n V||_\infty \le \frac{1}{n!} ||U||_\infty^n \, |||B|||_{1,X}^n \, ||V||_\infty$.

Proof: Let β be an integrable function dominating $||B(\cdot)||$ on [a,b].

$$||\{UB\}V||_\infty = ||\int_r^t U(t,s)B(s)V(s,r)ds||_\infty$$
$$\le ||U||_\infty \, ||V||_\infty \int_r^t \beta(s)ds.$$

(1) $$||\{UB\}^n V||_\infty \le ||U||_\infty^n \, ||V||_\infty \underset{r\le s_1\le\cdots\le s_n\le t}{\int\cdots\int} \beta(s_1)\cdots\beta(s_n)ds_1\cdots ds_n.$$

(2) $$\underset{r\le s_1\le\cdots\le s_n\le t}{\int\cdots\int} \beta(s_1)\cdots\beta(s_n)ds_1\cdots ds_n = \frac{1}{n!}\left[\int_r^t \beta(s)ds\right]^n.$$

This equation may be "seen geometrically" as follows. The region of integration in the first integral is an n-dimensional pyramid. The n-dimensional box $[r,t]^n$ is composed of the n-factorial pyramids obtained by permuting the s_i's; e.g. $\{r \le s_2 \le s_1 \le s_3 \le \cdots \le s_n \le t\}$. Clearly the integrals over these pyramids are equal, and the integral over the box is $[\int_r^t \beta(s)ds]^n$.

Since (1) and (2) are true for any integrable function dominating $||B(\cdot)||_{X,X}$, we have

$$||\{UB\}^n V||_\infty \le \frac{1}{n!} ||U||_\infty^n \, |||B|||_{1,X}^n \, ||V||_\infty.$$

The second inequality in the lemma follows by the same argument. //

Theorem 2.2. The solutions of (2) and (3) are $W = \Sigma_{n=0}^{\infty} \{UB\}^n V$, $W = \Sigma_{n=0}^{\infty} \{BU\}^n V$, respectively, where both of the series converge in operator norm.

Proof:
$$W = V + \{UB\}W$$
$$V = [I - \{UB\}]W$$
$$W = [I - \{UB\}]^{-1}V \text{ is the solution of (2).}$$

Similarly, $W = [I - \{BU\}]^{-1}V$ is the solution of (3). We can formally write Neumann expansions

$$[I - \{UB\}]^{-1} = \Sigma_{n=0}^{\infty} \{UB\}^n,$$
$$[I - \{BU\}]^{-1} = \Sigma_{n=0}^{\infty} \{BU\}^n.$$

From the lemma we have

$$||\{UB\}^n V||_\infty \le \frac{1}{n!} ||U||_\infty^n \, |||B|||_{1,X}^n \, ||V||_\infty,$$

and $||\{BU\}^n V||_\infty \le \frac{1}{n!} ||U||_\infty^n \, |||B|||_{1,X}^n \, ||V||_\infty.$

Therefore $\{UB\}$ and $\{BU\}$ are bounded operators and

$$||\{UB\}^n||, \; ||\{BU\}^n|| \le \frac{1}{n!} ||U||_\infty^n \, |||B|||_{1,X}^n.$$

Then,

$$\Sigma_{n=0}^{\infty} ||\{UB\}^n||, \; \Sigma_{n=0}^{\infty} ||\{BU\}^n|| \le e^{||U||_\infty |||B|||_{1,X}},$$

so both series converge in operator norm. //

We now prove in Theorem 2.3 that under certain conditions, equations (2) and (3) will have the same solutions.

Theorem 2.3. If $U, V, P \in K$, $B \in M$, and $UBP = PBV$, then the equations $W = P + UBW$; $W = P + WBV$ have the same solution.

Proof: Since the solutions to the above equations can be written $[I - \{UB\}]^{-1}P$ and $[I - \{BV\}]^{-1}P$ respectively, we need to show that these are equal. This can be seen as follows:

$$\begin{aligned}[I - \{UB\}]^{-1}P &= P + \{UB\}P + \{UB\}^2P + \cdots \\ &= P + PBV + UB(PBV) + \cdots \\ &= P + PBV + (UBP)BV + \cdots \\ &= P + PBV + (PBV)BV + \cdots \\ &= P + \{BV\}P + \{BV\}^2P + \cdots \\ &= [I - \{BV\}]^{-1}P. \quad //\end{aligned}$$

In particular, the equations $W = U + UBW$, $W = U + WBU$ have the same solution for any $U \in K$ and $B \in M$.

Recall that an element U of K is a <u>proper evolution system</u> if

$$U(t,s)U(s,r) = U(t,r)$$

for (t,s), $(s,r) \in \Delta$, and $U(t,t) = I$ for $t \in [a,b]$.

If, in the equation $W = U + WBU$, U is an evolution system, then the solution $W = [I - \{BU\}]^{-1}U$ is also an evolution system, as shown in the following theorem, which is essentially contained in Theorem 2.12 of [3].

<u>Theorem 2.4</u>. If U is an evolution system, and $B \in M$, then $[I - \{BU\}]^{-1}U$ is an evolution system; thus, $[I - \{UB\}]^{-1}U$ is also an evolution system.

<u>Proof</u>: Let $W = [I - \{BU\}]^{-1}U$, then $W = U + WBU$. It is obvious that $W(t,t) = I$ for $t \in [a,b]$. Let $\tau \in [a,b]$, and define Z on Δ by

$$Z(t,r) = \begin{cases} W(t,r) \text{ if } \tau \le r \le t \le b \\ W(t,\tau)W(\tau,r) \text{ if } a \le r \le \tau \le t \le b \\ W(t,r) \text{ if } a \le r \le t \le \tau. \end{cases}$$

Then Z is strongly continuous from Δ to $B(X)$. We want to show that $Z = U + ZBU$. This will imply that $Z = W$, which will imply that $W(t,\tau)W(\tau,r) = W(t,r)$ for $a \le r \le \tau \le t \le b$. Since τ was arbitrary, this will prove that W is an evolution system. If $a \le r \le t \le \tau$, or $\tau \le r \le t \le b$, then it is clear that $Z(t,r) = [U + ZBU](t,r)$.

If $a \le r \le \tau \le t \le b$, then

$$W(t,\tau) = U(t,\tau) + \int_\tau^t W(t,s)B(s)U(s,\tau)ds,$$

$$W(\tau,r) = U(\tau,r) + \int_r^\tau W(\tau,s)B(s)U(s,r)ds,$$

$$\begin{aligned} W(t,\tau)W(\tau,r) &= U(t,r) + \int_\tau^t W(t,s)B(s)U(s,r)ds \\ &\quad + \int_r^\tau W(t,\tau)W(\tau,s)B(s)U(s,r)ds \\ &= U(t,r) + \int_r^t Z(t,s)B(s)U(s,r)ds. \end{aligned}$$

Thus we have $Z = U + ZBU$, which completes the proof. //

CHAPTER 3

CONVERGENCE THEOREMS

The theorems which follow extend theorems III, IV, Va, VI, and VIa in Kato's paper Linear evolution equations of hyperbolic type, II [8] to the situation where the differential equation has ε - approximate solutions for every $\varepsilon > 0$, instead of under the assumption of quasi-stability.

Throughout this chapter, assume that the differential equation

$$u'(t) = A(t)u(t)$$

has approximate solutions, where $A(\cdot)$ is a strongly measurable function from $[a,b]$ into $B(Y,X)$, and let $U(\cdot,\cdot)$ be a strongly continuous proper evolution system generated by $A(\cdot)$. For conditions under which such evolution systems exist, see [3].

Let $\{A_n(\cdot)\}_{n=1}^{\infty}$ be a sequence of strongly measurable functions from $[a,b]$ into $B(Y,X)$ and $\{\alpha_n\}_{n=1}^{\infty}$ a uniformly integrable sequence of integrable nonnegative functions on $[a,b]$ such that α_n dominates $||A_n(\cdot)||_{Y,X}$ for each n. Also assume that if $y \in Y$, then $\{A_n(\cdot)y\}$ converges a.e. on $[a,b]$ to $A(\cdot)y$, and that α is an integrable nonnegative function on $[a,b]$ which dominates $||A(\cdot)||_{Y,X}$. For each n, let $U_n(\cdot,\cdot)$ be a proper evolution system generated by

$A_n(\cdot)$, where $||U_n(t,s)||_{X,X}$ is uniformly bounded for $(t,s) \in \Delta$ and $n=1, 2, 3, \cdots$.

It was shown by Dorroh and Graff in Theorem 2.15 of [3] that this implies that

(1) $$\lim_{n\to\infty} ||U_n(t,s)x - U(t,s)x||_X = 0$$

for each $x \in X$ and $s \in [a,b]$, uniformly for $s \leq t \leq b$.

Now consider the nonhomogeneous equation

(2) $$u'(t) - A(t)u(t) = f(t), \; u(a) = \phi$$

The "mild solution" of (2) can be formally written

$$u(t) = U(t,a)\phi + \int_a^t U(t,s)f(s)ds.$$

In general, u need not be a solution of (2) in the strict sense. Also we have the equations

$$u_n'(t) - A_n(t)u_n(t) = f_n(t), \; u_n(a) = \phi_n$$

with formal solutions

$$u_n(t) = U_n(t,a)\phi_n + \int_a^t U_n(t,s)f(s)ds.$$

The following theorem gives an estimate of the difference between $u_n(t)$ and $u(t)$.

<u>Theorem 3.1</u>. Let ϕ, $\phi_n \in X$ and f, $f_n \in L_1 (X)$. Then $||u_n(t) - u(t)||_{\infty,X} \leq K[||\phi_n - \phi||_X + |||f_n - f|||_{1,X}] + M||U_n - U||_{\infty,X}$, where K and M depend only on the norms of U_n, ϕ, and f.

<u>Proof</u>: $$u_n(t) - u(t) = U_n(t,a)\phi_n - U(t,a)\phi + \int_a^t [U_n(t,s)f_n(s) - U(t,s)f(s)]ds$$
$$= U_n(t,a)(\phi_n - \phi) + \int_a^t U_n(t,s)[f_n(s) - f(s)]ds + [U_n(t,s) - U(t,s)]\phi + \int_a^t [U_n(t,s) - U(t,s)]f(s)ds,$$

so $$||u_n - u||_{\infty,X} \leq ||U_n||_{\infty,X}[||\phi_n - \phi||_X + |||f_n - f|||_{1,X}] + ||U_n - U||_{\infty,X}[||\phi||_X + |||f|||_{1,X}].$$

Let $K = ||U_n||_{\infty,X}$, and $M = ||\phi||_X + |||f|||_{1,X}$, then the result follows. //

Now we present some conditions under which $u_n(t)$ will converge to $u(t)$.

<u>Theorem 3.2</u>. If in addition $\phi_n \to \phi$ in X and $f_n \to f$ in $L_1(X)$ then $u_n \to u$ in X.

<u>Proof</u>: From Theorem 3.1 we have

$$\begin{aligned} ||u_n(t) - u(t)||_X \leq\ & ||U_n||_{\infty,X}[||\phi_n - \phi||_X + |||f_n - f|||_{1,X}] \\ & + ||[U_n(t,s) - U(t,s)]\phi||_X \\ & + \int_a^t ||U_n(t,s)f(s) - U(t,s)f(s)||_X ds. \end{aligned}$$

By hypothesis, $||\phi_n - \phi||_X \to 0$ and $|||f_n - f|||_{1,X} \to 0$. Also $||[U_n(t,s) - U(t,s)]\phi||_X \to 0$ by (1) above. Since $U_n(t,s)$ is uniformly bounded, we have that $U_n(t,s)f(s)$ forms a uniformly integrable family of functions and so, by a generalization of Lebesque's theorem, $U_n(t,s)f(s)$ converges to $U(t,s)f(s)$ in L_1 norm (see [10] p. 17, 18).

Thus $||u_n(t) - u(t)||_X \to 0$. //

For the rest of the chapter, assume further that the following version of Kato's condition (ii''') of [8] is satisfied.

(ii'''') There is a family $\{S(t): a \leq t \leq b\}$ of isomorphisms of Y onto X, a strongly measurable function $B(t)$ from $[a,b]$ into $B(X)$ with $||B(\cdot)||_{X,X}$ upper-integrable (i.e., bounded above by an integrable function), and a strongly measurable function $\dot{S}$ from $[a,b]$ into $B(Y,X)$ with $||\dot{S}(\cdot)||_{Y,X}$ upper-integrable such that S is a strong indefinite integral of $\dot{S}$, and such that if $y \in Y$, then

$$S(t)A(t)S(t)^{-1}y = A(t)y + B(t)y$$

a.e. on [a,b].

Then it was shown in Theorem 2.13 of [3] that $U(t,s) \in B(Y)$ for each $(t,s) \in \Delta$ and $U(\cdot,\cdot)y$ is jointly continuous from Δ into Y for each $y \in Y$.

The next theorem presents some properties of $S(\cdot)$ and $S(\cdot)^{-1}$ which are needed for Theorem 3.4.

Theorem 3.3. $S(\cdot)$ and $S(\cdot)^{-1}$ are continuous (in fact, absolutely continuous) with $||S(\cdot)||_{Y,X}$ and $||S(\cdot)^{-1}||_{X,Y}$ bounded.

Proof: We have $||\dot{S}(\cdot)||_{Y,X}$ upper-integrable on I (i.e., bounded above by $\beta(t)$, which is integrable) and S equal to an indefinite strong integral of $\dot{S}$. Then

$$||S(t) - S(s)||_{Y,X} \leq \overline{\int_s^t} ||\dot{S}(r)||_{Y,X}dr \leq \int_s^t \beta(r)dr$$

which approaches zero as $|t - s| \to 0$. Thus $S(\cdot)$ is continuous from $[a,b] \to B(Y,X)$ and [a,b] compact implies $||S(\cdot)||$ is bounded. Since $S(\cdot)$ is continuous, we have $S(\cdot)^{-1}$ continuous with $||S(\cdot)^{-1}||_{X,Y}$ bounded.

Suppose $||S(\cdot)^{-1}|| \leq M$. From above we have $||S(t_j) - S(s_j)|| \leq \int_{s_j}^{t_j} \beta(r)dr$, so $\Sigma||S(t_j) - S(s_j)|| \leq \Sigma \int_{s_j}^{t_j} \beta(r)dr$ which approaches zero as $\Sigma(t_j - s_j)$ approaches zero. Then $\Sigma||S(t_j)^{-1}- S(s_j)^{-1}||$

$$\leq \Sigma||S(t_j)^{-1}||\ ||S(s_j) - S(t_j)||\ ||S(s_j)^{-1}||$$

$$\leq M^2\ \Sigma||S(s_j) - S(t_j)||$$

which also approaches zero as $\Sigma(t_j - s_j)$ approaches zero.

Thus $S(\cdot)$ and $S(\cdot)^{-1}$ are absolutely continuous in operator norm. //

Now, define C from [a,b] into B(X) by $C(t) = \dot{S}(t)S(t)^{-1}$. Then C is strongly measurable with $||C(\cdot)||_{X,X}$ upper-integrable. Define W from Δ into B(X) by means of the Volterra-type integral equation

$$W(t,r)x = U(t,r)x + \int_r^t W(t,s)[B(s) + C(s)]U(s,r)xds$$

(see [2] pp. 475-477, [8] pp. 652,653). Then, we have $U(t,s)S(s)^{-1} = S(t)^{-1}W(t,s)$,([3], pp. 31,32).

Let us also suppose that (ii'''') is satisfied by all the A_n, uniformly in n. We will use the obvious notation S_n, B_n, C_n, $U_n(t,s)$, $W_n(t,s)$, etc. The space Y is assumed to be common to all A_n.

Theorem 3.4. In addition to the above assumptions, suppose that

(i) $B_n(t) + C_n(t) \to B(t) + C(t)$ strongly in B(X) for a.e. $t \in I$

(ii) $\lim_{K\to\infty} \int_{\{||B_n+C_n|| \geq K\}} ||B_n(t) + C_n(t)||dt = 0$

(i.e., the collection of functions $\{B_n(t) + C_n(t)\}$ is uniformly integrable)

(iii) $S_n(t) \to S(t)$ strongly in B(Y,X) uniformly in $t \in I$.

Then, $U_n(t,s) \to U(t,s)$ strongly in B(Y), uniformly in $(t,s) \in \Delta$.

Proof: Write U(t,s) in the form $S(t)^{-1}W(t,s)S(s)$ and also $U_n(t,s)$ as $S_n(t)^{-1}W_n(t,s)\ S_n(s)$. Then,

$$||U_n(t,s)y - U(t,s)y||_Y =$$

$$= ||S_n(t)^{-1}W_n(t,s)S_n(s)y - S(t)^{-1}W(t,s)S(s)y||_Y$$

$$\leq ||S_n(t)^{-1}W_n(t,s)S_n(s)y - S_n(t)^{-1}W_n(t,s)S(s)y||_Y$$

$$+ ||S_n(t)^{-1}W_n(t,s)S(s)y - S_n(t)^{-1}W(t,s)S(s)y||_Y$$

$$+ ||S_n(t)^{-1}W(t,s)S(s)y - S(t)^{-1}W(t,s)S(s)y||_Y$$

$$\leq ||S_n(t)^{-1}||_{X,Y}||W_n(t,s)||_{X,X}||S_n(s)y - S(s)y||_X$$

$$+ ||S_n(t)^{-1}||_{X,Y}||W_n(t,s)S(s)y - W(t,s)S(s)y||_X$$

$$+ ||S_n(t)^{-1}W(t,s)S(s)y - S(t)^{-1}W(t,s)S(s)y||_Y.$$

From Theorem 3.3 we have $||S_n(t)^{-1}||_{X,Y}$ is bounded. Using a Volterra-type estimate as in Chapter 2, we have $||W_n(t,s)||_{X,X}$ bounded. Then since $S_n(t) \to S(t)$ strongly in $B(Y,X)$, uniformly in t, we have $||S_n(t)^{-1}||_{X,Y}||W_n(t,s)||_{X,X}||S_n(s)y - S(s)y||_X \to 0$ uniformly in s. Since $S_n(t) \to S(t)$ strongly in $B(Y,X)$, $S_n(t)^{-1} \to S(t)^{-1}$ strongly in $B(X,Y)$, so $||S_n(t)^{-1}W(t,s)S(s)y - S(t)^{-1}W(t,s)S(s)y||_Y \to 0$ uniformly in t.

Thus it is sufficient to show that $||W_n(t,s)x - W(t,s)x||_X \to 0$ uniformly in $(t,s) \in \Delta$.

We have

$$W(t,r)x = U(t,r)x + \int_r^t W(t,s)[B(s) + C(s)]U(s,r)xds,$$

$$W_n(t,r)x = U_n(t,r)x + \int_r^t W_n(t,s)[B_n(s) + C_n(s)]U(s,r)xds,$$

so $W_n(t,r)x - W(t,r)x = U_n(t,r)x - U(t,r)x$

$$+ \int_r^t W_n(t,s)[B_n(s) + C_n(s)][U_n(s,r)x - U(s,r)x]ds$$

$$+ \int_r^t W_n(t,s)\ \{[B_n(s) + C_n(s)]U(s,r)x - [B(s) + C(s)]U(s,r)x\}\ ds$$

$$+ \int_r^t [W_n(t,s) - W(t,s)][B(s) + C(s)]U(s,r)xds.$$

Let $V_n(t,r)x = U_n(t,r)x - U(t,r)x$

$$+ \int_r^t W_n(t,s)[B_n(s) + C_n(s)][U_n(s,r)x - U(s,r)x]ds$$

$$+ \int_r^t W_n(t,s) \{[B_n(s) + C_n(s)]U(s,r)x - [B(s) + C(s)]U(s,r)x\} ds;$$

then, $||V_n(t,r)x||_X \leq ||U_n(t,r)x - U(t,r)x||_X$

$$+ ||W_n(t,s)||_{\infty,X}||U_n(\cdot,\cdot)x - U(\cdot,\cdot)x||_{\infty,X}\left[\int_I ||B_n(s) + C_n(s)||_X ds\right]$$

$$+ ||W_n(t,s)||_{\infty,X} \int_r^t||[B_n(s) + C_n(s)]U(s,r)x - [B(s) + C(s)]U(s,r)x||_X ds.$$

Since $U_n \to U$ strongly in X, uniformly in $(t,s) \in \Delta$, $U_n \to U$ strongly in ∞,X norm, and $B_n(s) + C_n(s) \to B(s) + C(s)$ strongly, with $B_n(s) + C_n(s)$ uniformly integrable, so by the generalized Lebesque theorem [10] $B_n(s) + C_n(s) \to B(s) + C(s)$ in L_1 norm, we then have $||V_n(t,r)x||_X \to 0$ as $n \to \infty$, for every $x \in X$.

Thus we have $W_n(t,r)x - W(t,r)x$ written as

$W_n(t,r)x - W(t,r)x = V_n(t,r)x$

$$+ \int_r^t [W_n(t,s) - W(t,s)][B(s) + C(s)]U(s,r)xds.$$

Let $Z_n(t,r)x = W_n(t,r)x - W(t,r)x$, and $D(s) = B(s) + C(s)$.

Note $|||D|||_1 = \int_I ||D(\cdot)||_{X,X}dt$.

Then, $Z_n(t,r)x = V_n(t,r)x + \int_r^t Z_n(t,s)D(s)U(s,r)xds$

or $\quad V_n(t,r)x = Z_n(t,r)x - \int_r^t Z_n(t,s)D(s)U(s,r)xds$

$$V_n(t,r)x = [I - \{DU\}]Z_n(t,r)x$$

$$Z_n(t,r)x = [I - \{DU\}]^{-1}V_n(t,r)x$$

where $[I - \{DU\}]^{-1} = \Sigma_{j=0}^{\infty} \{DU\}^j$.

So $||Z_n(t,r)x||_X \leq ||V_n(t,r)x||_X[\Sigma_{j=0}^{\infty} ||\{DU\}^j||_\infty]$.

Then by Theorem 2.1 we have

$$||Z_n(t,r)x||_X \leq ||V_n(t,r)x||_X \ [\Sigma_{j=0}^{\infty} \frac{1}{j!} \ |||D|||_1^j \ ||U||_\infty^j]$$

$$= ||V_n(t,r)x||_X \ e^{|||D|||_1||U||_\infty}$$

which approaches zero as $n \to \infty$.

Thus, $||U_n(t,s)y - U(t,s)y||_Y \to 0$. //

Recall that we have the following mild solutions to the nonhomogeneous differential equations:

$$u(t) = U(t,a)\phi + \int_a^t U(t,s)f(s)ds$$

and

$$u_n(t) = U_n(t,a)\phi_n + \int_a^t U_n(t,s)f_n(s)ds,$$

and we also have $U(t,s)S(s)^{-1} = S(t)^{-1}W(t,s)$ which we will use in the form $S(t)U(t,s) = W(t,s)S(s)$. Also, of course, the analogous relationships $S_n(t)U_n(t,s) = W_n(t,s)S_n(s)$ hold.

The following theorem gives an estimate of the difference between $u_n(t)$ and $u(t)$, this time in Y norm.

Theorem 3.5. Let ϕ_n, $\phi \in Y$ and f_n, $f \in L_1(Y)$. Then

$$||u_n - u||_{\infty,Y} \leq K\Big\{||(S - S_n)u||_{\infty,X} + ||(W_n - W)S(0)\phi||_{\infty,X} + |||(W_n - W)Sf|||_{1,X}\Big\}$$
$$+ M\Big\{||S_n||_{\infty,Y,X}\,[||\phi_n - \phi||_Y + |||f_n - f|||_{1,Y}] + ||(S_n(0) - S(0))\phi||_X + |||(S_n - S)f|||_{1,X}\Big\},$$

where K and M depend only on the norms of S_n^{-1} and W_n.

Proof:

$$S_n(t)u_n(t) = S_n(t)U_n(t,0)\phi_n + \int_0^t S_n(t)U_n(t,s)f_n(s)ds$$
$$= W_n(t,0)S_n(0)\phi_n + \int_0^t W_n(t,s)S_n(s)f_n(s)ds.$$

Also, $S(t)u(t) = W(t,0)S(0)\phi + \int_0^t W(t,s)S(s)f(s)ds.$

So, $(S_n(t)u_n(t) - S(t)u(t))$

$$= W_n(t,0)S_n(0)\phi_n + \int_0^t W_n(t,s)S_n(s)f_n(s)ds$$
$$+ W_n(t,0)S_n(0)\phi + \int_0^t W_n(t,s)S_n(s)f(s)ds$$
$$- W_n(t,0)S_n(0)\phi - \int_0^t W_n(t,s)S_n(s)f(s)ds$$
$$+ W_n(t,0)S(0)\phi + \int_0^t W_n(t,s)S(s)f(s)ds$$
$$- W_n(t,0)S(0)\phi - \int_0^t W_n(t,s)S(s)f(s)ds$$
$$- W(t,0)S(0)\phi - \int_0^t W(t,s)S(s)f(s)ds.$$

Then, $||S_n(t)u_n(t) - S(t)u(t)||_X \leq$

$$||W_n||_{\infty,X,X}||S_n||_{\infty,Y,X}||\phi_n - \phi||_Y$$
$$+ ||W_n||_{\infty,X,X}||S_n||_{\infty,Y,X}|||f_n - f|||_{1,Y}$$

$$+ ||W_n||_{\infty,X,X}||S_n(0)\phi - S(0)\phi||_X$$

$$+ ||W_n||_{\infty,X,X}|| \int_0^t [S_n(s)f(s) - S(s)f(s)]ds||_X$$

$$+ ||W_n(t,0)S(0)\phi - W(t,0)S(0)\phi||_X$$

$$+ || \int_0^t [W_n(t,s)S(s)f(s) - W(t,s)S(s)f(s)]ds||_X.$$

We have $u_n(t) - u(t) = S_n(t)^{-1}S_n(t)(u_n(t) - u(t))$,

so $||u_n(t) - u(t)||_{\infty,Y} \leq ||S_n^{-1}||_{\infty,X,Y}||S_n(t)(u_n(t) - u(t))||$

$$\leq ||S_n^{-1}||_{\infty,X,Y}\Big[||S(t)u(t) - S_n(t)u(t)||_X + ||S_n(t)u_n(t) - S(t)u(t)||_X\Big]$$

$$\leq ||S_n^{-1}||_{\infty,X,Y}\Big[||(S - S_n)u||_{\infty,X} + ||(W_n-W)S(0)\phi||_{\infty,X} + |||(W_n-W)Sf|||_{1,X}\Big]$$

$$+ ||S_n^{-1}||_{\infty,X,Y}||W_n||_{\infty,X,Y}\Big\{||S_n||_{\infty,Y,X}[||\phi_n - \phi||_Y] + ||S_n||_{\infty,Y,X}[|||f_n - f|||_{1,Y}] + ||(S_n(0) - S(0))\phi||_X + |||(S_n - S)f|||_{1,X}\Big\}.$$

Then with $K = ||S_n^{-1}||_{\infty,X,Y}$ and $M = ||S_n^{-1}||_{\infty,X,Y}||W_n||_{\infty,X,X}$ the result follows. //

Now we present some conditions under which $u_n(t)$ will converge to $u(t)$ in Y norm.

Theorem 3.6. If, in addition, $\phi_n \to \phi$ in Y and $f_n \to f$ in $L_1(Y)$, then $u_n \to u$ in Y.

Proof: We have S_n converging strongly to S, so $||S(t)u(t) - S_n(t)u(t)||_X$ approaches zero as $n \to \infty$.

Also we have, from Theorem 3.5,

$$||S_n(t)u_n(t) - S(t)u(t)||_X \le ||W_n||_{\infty,X,X}||S_n(0)\phi - S(0)\phi||_X$$
$$+ ||W_n||_{\infty,X,X}||S_n||_{\infty,Y,X}[||\phi_n - \phi||_Y + |||f_n - f|||_{1,Y}]$$
$$+ ||W_n(t,0)S(0)\phi - W(t,0)S(0)\phi||_X$$
$$+ ||W_n||_{\infty,X,X} \int_0^t ||S_n(s)f(s) - S(s)f(s)||_X ds$$
$$+ \int_0^t ||W_n(t,s)S(s)f(s) - W(t,s)S(s)f(s)||_X ds.$$

Since $W_n \to W$ strongly in X (shown in Theorem 3.4) the first three terms in the above expression approach zero as $n \to \infty$.

Then $S_n(s)x$ converges to $S(s)x$ for every $s \in I$, so $\sup_{s\in I} ||S_n(s)x - S(s)x|| \to 0$ and $\sup_{s\in I} ||S_n(s)x|| \le \sup_{s\in I} ||S(s)x|| + \varepsilon$ for every $\varepsilon > 0$. We have $\sup_{s\in I} ||S(s)|| \le M$, which implies

$$||S_n(s)f(s) - S(s)f(s)|| \le 2M||f(s)||.$$

Thus, by Lebesque's dominated convergence theorem ([4],p.151) $\int ||S_n(s)f(s) - S(s)f(s)||ds$ approaches zero as $n \to \infty$.

Since $||W_n(t,s)||$ is also bounded, the same argument gives $\int ||W_n(t,s)S(s)f(s) - W(t,s)S(s)f(s)||ds$ approaches zero as $n \to \infty$.

Thus $||S_n(t)u_n(t) - S(t)u(t)|| \to 0$ as $n \to \infty$. Hence,

$$||u_n(t) - u(t)||_Y \le ||S_n(t)^{-1}||_{X,Y}\ ||S_n(t)(u_n(t) - u(t))||_X$$

$$\le ||S_n(t)^{-1}||_{X,Y}\ ||S(t)u(t) - S_n(t)u(t)||_X$$

$$+ ||S_n(t)u_n(t) - S(t)u(t)||_X$$

which approaches zero as $n \to \infty$. //

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Date of Examination:

July 16, 1979