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ARC REVERSALS IN TOURNAMENTS

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in

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ABSTRACT

In 1964, H.J. Ryser [27] proved that, given any two n -tournaments T, U with the same score (or outdegree) sequence, T can be transformed into (a copy of) U by successively reversing the orientations of appropriately chosen 3-cycles. In 1973, K.B. Reid [22] showed that any n -tournament can be transformed into any other by successively reversing paths of any fixed length k , $1 \leq k \leq n-1$.

Based on these results, a problem is abstracted and examined in Chapters 1 and 2. If D is a digraph and T, U are n -tournaments, T is equivalent to U via D -reversals if T can be transformed into U by successively reversing copies of D , subject to the proviso that if D is not isomorphic to its directional dual D^* , we allow D^* -reversals also (at any stage). An equivalence relation on n -tournaments is thereby obtained, so consider:

THE REVERSAL PROBLEM. Given a digraph D , determine (preferably, characterize) the resulting equivalence classes of n -tournaments.

A proof technique involving the concepts of "the directed difference graph" and "refinement" is developed and applied to a variety of digraphs in connection with this problem, included among which are cycles, generalized 4-cy-

cles, paths, antidirected paths, and "claws." In most cases, characterizations of the equivalence classes in terms of simple tournament parameters are obtained, e.g., Ryser's theorem holds if "3-cycles" is replaced by "4-cycles" or by "5-cycles," and this is best possible, though partial results are obtained for k -cycle-reversals. In addition to these results, new proofs are supplied for those cited previously.

Panconnectivity in tournaments is investigated in (the independent) Chapter 3. A tournament is strongly (resp., weakly) panconnected if it contains paths of all possible lengths greater than two with prescribed initial and terminal vertices (resp., prescribed endvertices). In recent work, C. Thomassen [29] has completely characterized weakly panconnected tournaments. Using this characterization, the following main result is obtained.

THEOREM. An n -tournament T is strongly panconnected provided that $n \geq \max \{ 5q(T)+4, 2q(T)+13 \}$, where $q(T)$ is the maximum difference between the scores of T . Moreover, the bound is best possible whenever $q(T) \geq 3$.

This extends work in [1; 2; 15; and 29] by considerably broadening the class of tournaments known to be strongly panconnected.

A local aspect of panconnectivity is also studied, and some results are utilized in Chapters 1 and 2.

Chapter 1

REVERSALS OF CYCLES

1.1 Preliminaries.

In this section we introduce the terminology, notation, and conventions to be employed throughout this work. We assume some familiarity on the part of the reader with graphs and digraphs (i.e., directed graphs) and their basic properties as can be found, for instance, in [6;11;13]. If G is a graph (digraph), we prefer the terms vertex and edge (respectively, vertex and arc) to others of similar ilk, and we denote by $V(G)$ its vertex-set and by $E(G)$ its edge-set (resp., arc-set); for $x, y \in V(G)$, we use xy (resp., \vec{xy}) to denote the edge incident with x and y (resp., arc from x to y), if it exists in G . Herein, all graphs/digraphs are assumed to be finite (usually with non-empty vertex-sets) with no loops and no multiple edges/arcs.

Let D be a digraph. If $x, y \in V(D)$ and $\vec{xy} \in E(D)$, we say x dominates y (in D , for emphasis) and write $x \rightarrow y$. The dominance relation \rightarrow thusly obtained is always irreflexive (there are no loops in D), and if it is asymmetric, transitive, etc., we say D is, respectively, asymmetric, transitive, etc. A complete (in the sense of comparable), asymmetric digraph T is a tournament, and an n -tournament is one of order n , that is, one having n vertices (and hence $\binom{n}{2}$ arcs). We sometimes use T_n to

denote an n -tournament. Generally speaking, we regard two tournaments (or digraphs) T and U as being the same if they are isomorphic (written $T \cong U$), where by an isomorphism of directed graphs is meant a bijection between their vertex-sets which preserves dominance in both directions. In Figure 1 below are shown all the nonisomorphic tournaments of order $n \leq 4$, several of which have been labelled for future reference. (The double arrows (\Downarrow) in this and subsequent tournament diagrams indicate the orientations of all undrawn arcs.)

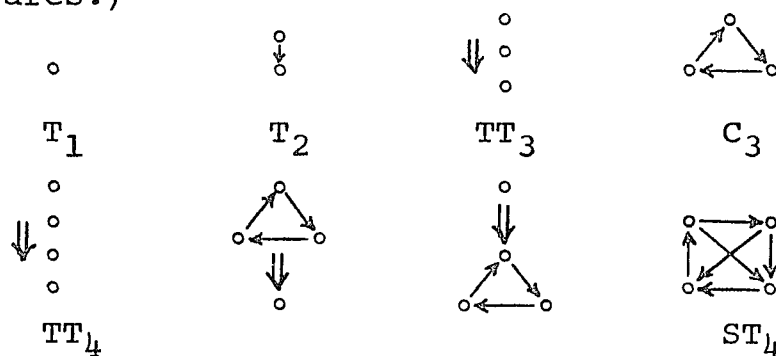


Figure 1. The tournaments of order $n \leq 4$

It is with tournament theory that we shall be primarily concerned, and, as a general reference on the subject, J.W. Moon's monograph [19] is highly recommended, as are survey articles [12; 23; 3] and the appropriate sections of the texts [6; 13].

We borrow the terms walk, closed walk, path, cycle, k-path, k-cycle, Hamiltonian path, and Hamiltonian cycle from [6]. As applied to digraphs, these terms, unless otherwise modified (as in §2.5 and others), will be used in the directed sense. Words involving "path" or "cycle" imply

distinct vertices, and the prefix "k-" in all cases refers to the length, or number of arcs, in the path, cycle, etc.

Two vertices x, y of a digraph (in particular, of a tournament) D are mutually reachable if there are paths (equivalently, walks) in D from x to y and from y to x . Mutual reachability is, clearly, an equivalence relation on D and the resulting equivalence classes are the strong components of D . If D has only one strong component, it is strongly connected or, briefly, strong. The only strong tournaments in Figure 1 are T_1 (the trivial tournament), C_3 , and ST_4 .

Because of their specialized structure, tournaments exhibit many striking path- and cycle-phenomena not found in more general digraphs. For example, it is easy to see that every tournament has a Hamiltonian path, and it is a (non-trivial) fact that each has an odd number of such paths [21]. Moreover, every nontrivial strong tournament is Hamiltonian; indeed (see [17] or [19, Thm. 3]), each vertex x of a strong n -tournament T , $n \geq 3$, is contained in a k -cycle, for $k = 3, 4, \dots, n$; that is, T is vertex-pancyclic.

We employ the following notation for any n -tournament T and $x \in V(T)$:

$$I_T(x) = \{y \in V(T) : y \rightarrow x\}, \quad O_T(x) = \{y \in V(T) : x \rightarrow y\},$$

$$id_T(x) = |I_T(x)|, \quad od_T(x) = |O_T(x)|.$$

These are called, respectively, the inset, outset, indegree, outdegree of the vertex x (in T), and the last of these is often called the score of x . When no confusion can

arise, we shall omit the subscript "T" and write just $I(x)$, $od(x)$, etc., and we shall use analogous notation for arbitrary digraphs. Evidently, $od(x) + id(x) = n - 1$ and $\sum_{x \in V(T)} od(x) = \binom{n}{2} = \sum_{x \in V(T)} id(x)$. The score sequence (synonyms: score vector, score list, outdegree sequence) of T is the n -tuple (s_1, s_2, \dots, s_n) of the scores of the n vertices of T , arranged so that $s_1 \leq s_2 \leq \dots \leq s_n$. For example, the score sequences of TT_3 and ST_4 are, respectively, $(0, 1, 2)$ and $(1, 1, 2, 2)$. Although the score sequence does not distinguish nonisomorphic n -tournaments when $n > 4$ (it does for $n \leq 4$), it is a vitally important tournament parameter, as it completely determines many properties of a tournament T . As a case in point, it determines the number p of strong components s^1, s^2, \dots, s^p of T (hence, strong connectedness), the number of vertices in, and score sequences of, each s^i (when considered as themselves tournaments), and the linear (i.e., transitive) ordering of them, i.e., they may be relabelled, if necessary, so that s^i dominates s^j (in the sense that each vertex of s^i dominates every vertex of s^j) whenever $i < j$. Indeed, these assertions are immediate consequences of the well-known "criteria for a (strong) score sequence" due to G. Landau [16] (see also [13; 19; 12]). We shall have frequent occasion to deal with the following classes of tournaments, all of which are determined entirely by the score sequence parameter:

- 1) transitive tournaments: for each $n \geq 1$, there is

a unique (up to isomorphism) transitive n -tournament, denoted TT_n .

2) strong tournaments: defined above.

3) regular tournaments: an n -tournament is regular if n is odd and all of its vertices have score $\frac{n-1}{2}$.

4) near-regular tournaments: an n -tournament is near-regular (or almost regular) if n is even, half of its vertices have score $\frac{n}{2} - 1$, and half have score $\frac{n}{2}$.

We include in the following paragraphs a list of terminology and notation that applies to directed graphs in general, though the special case $G = T$, a tournament, will be most common in the sequel.

Let D, D_0 , and G be digraphs. If $V(D_0) \subseteq V(G)$ and $E(D_0) \subseteq E(G)$, we say D_0 is a (partial) subdigraph of G and write $D_0 \subseteq G$. A tournament $D_0 \subseteq G$ is a subtournament of G . If $D_0 \subseteq G$ and $D_0 \cong D$, we call D_0 a copy of D in G . Any path or cycle in G may be identified with its "underlying subdigraph" of G , e.g., $P = (V(P), E(P))$ for a path P , whenever it is convenient to do so; it should be clear from context which usage of "P", "path", "cycle", etc., is intended.

Let G be a digraph and $X \subseteq V(G)$. The span of X (in G), denoted $\langle X \rangle_G$, or just $\langle X \rangle$ when G is understood, is the (full) subdigraph of G having vertex-set X and dominance given by $x \rightarrow y$ in $\langle X \rangle$ if and only if $x \rightarrow y$ in G (for all $x, y \in X$). If $D_0 \subseteq G$, we also set $\langle D_0 \rangle = \langle V(D_0) \rangle$, the span of D_0 . In case $D_0 = \langle D_0 \rangle$, we

say D_0 is full, i.e., a full subdigraph of G . The strong components of G are to be regarded as full subdigraphs of G .

Associated with any digraph D is a digraph D^* called the dual (i.e., directional dual or converse) of D which has $V(D^*) = V(D)$ and dominance given by $x \rightarrow y$ in D^* if and only if $y \rightarrow x$ in D (for all vertices x, y). In other words, D^* is obtained from D by reversing the orientations of all arcs of D . Clearly, D, D^* are duals of each other, and if $D \cong D^*$, D is called self-dual. The two unlabelled tournaments of Figure 1 above are duals of each other, and all the rest are self-dual. We have already encountered several "dual" concepts, such as indegree and outdegree, and these and other directional concepts are related by the following well-known, labor-saving device.

THE PRINCIPLE OF DIRECTIONAL DUALITY (see, e.g., [11, p. 200]). Each proposition θ concerning directed graphs is logically equivalent to its dual proposition θ^* , obtained from θ by replacing every concept by its dual concept (nondirectional concepts such as $|V(D)|$ are self-dual).

We shall use such phrases as "Dually,..." and "By duality,..." with reference to the above principle.

The general conventions listed below will be employed throughout the body of this work.

$|\cdot|$ will denote cardinality or absolute value, depending on the argument.

The symbol \square will mark the end of a proof, or (as in the case of some corollaries) that the proof is immediate. Unless the context urgently requires it, we shall not distinguish notationally between singleton $\{x\}$ and the object x itself.

All definitions, examples, remarks, theorems, propositions, etc. are numbered consecutively and jointly within sections of chapters (the x.y.z system); only figures are separately numbered throughout the work (1,2,...). All are cross-referenced by their full numerical titles.

1.2 The reversal problem.

Following the relevant definitions, we state in this section the problem with which we shall be chiefly concerned in this chapter and the next, and give its history.

1.2.1 Definition. If T is a tournament and D is a digraph, a D-reversal in T is a transformation $T \mapsto T^1$ in which the orientation of each arc in a copy $D_0 \subseteq T$ of the digraph D (i.e., $D_0 \cong D$) is reversed, yielding a new tournament T^1 ; that is, $V(T^1) = V(T)$ and $E(T^1) = (E(T) - E(D_0)) \cup E(D_0^*)$.

Note that T must contain some copy of D in order for any D -reversal to be performed, so the only digraphs that need be considered in this connection are asymmetric ones. Also observe that the tournaments T and T^1 are not, in general, isomorphic, but do have the same order. (As we shall see, order may be the only property preserved.)

1.2.2 Definition. Let T, U be n -tournaments and D an asymmetric digraph. We say T is equivalent to U ($T \sim U$) via D -reversals if there is a (possibly null) sequence

$$T = T^0 \rightsquigarrow T^1 \rightsquigarrow \dots \rightsquigarrow T^m \cong U \quad (m \geq 0)$$

of D -reversals beginning with T and ending with a copy of U . This has a natural generalization to a collection \mathcal{D} of asymmetric digraphs: $T \sim U$ via \mathcal{D} -reversals if each transformation $T^{i-1} \rightsquigarrow T^i$ (in the above sequence) is a D_i -reversal for some $D_i \in \mathcal{D}$.

For any collection \mathcal{D} of digraphs, let $\mathcal{D}^* = \{D^* : D \in \mathcal{D}\}$. We call \mathcal{D}^* the dual of \mathcal{D} , and if $\mathcal{D} = \mathcal{D}^*$, we say \mathcal{D} is self-dual.

1.2.3 Proposition. The relation of equivalence via \mathcal{D} -reversals on the class \mathcal{T} of all tournaments is reflexive and transitive. If \mathcal{D} is self-dual, it is symmetric as well and hence an equivalence relation on \mathcal{T} .

Proof: The relation is clearly transitive, and it is reflexive since we allow the null sequence of \mathcal{D} -reversals. In addition, it is symmetric if $\mathcal{D} = \mathcal{D}^*$, as then any D -reversal $T^{i-1} \rightsquigarrow T^i$, where $D \in \mathcal{D}$, corresponds to an obvious D^* -reversal $T^i \rightsquigarrow T^{i-1}$, and $D^* \in \mathcal{D}$. \square

If \mathcal{D} is a self-dual collection of asymmetric digraphs, the equivalence classes of n -tournaments relative to \mathcal{D} -reversals we be referred to simply as \mathcal{D} -classes. With this terminology, the problem which we shall investigate in

Chapters 1 and 2 may be formulated as follows.

THE REVERSAL PROBLEM. Given a collection \mathcal{D} (as above), determine the \mathcal{D} -classes. In particular, given an asymmetric digraph D , determine the $\{D, D^*\}$ -classes (these are just the D -classes when D is self-dual).

The history of this problem is fairly recent. In 1964, H.J. Ryser [27] proved a result concerning 0-1 matrices which, restated in the language of tournaments (see [19, Thm. 35]) is this: if T, U are n -tournaments with the same score sequence, then $T \sim U$ via 3-cycle-reversals. Since reversal of a 3-cycle (or of any length cycle) does not change any score, the converse of this result clearly holds; thus, Ryser's theorem completely characterizes the C_3 -classes. Following this, K.B. Reid in 1973 proved [22] that any two n -tournaments are equivalent via k -path-reversals, for any fixed integer k satisfying $1 \leq k \leq n-1$, and thus characterized the k -path-classes. More recently, the author [30] characterized the TT_3 -classes (TT_3 denotes the transitive triple, Figure 1) by proving that, if T, U are n -tournaments with the same number of vertices of even score (equivalently, odd score), then $T \sim U$ via TT_3 -reversals. (Again, the converse of this statement is trivial, because it is easily seen that a TT_3 -reversal always preserves score parity.)

In 1968, J.W. Moon specifically asked [19, p. 74] whether Ryser's theorem still holds if "3-cycle" is replaced

by "4-cycle." We give an affirmative answer to this question (Theorem 1.5.1 below), and show also (Theorem 1.6.3) that the same characterization obtains for the 5-cycle. An example is presented which shows that these results are best possible in the sense that the answer is negative for the k -cycle, where $k \geq 6$ is fixed (even when T, U are strong n -tournaments and $n \geq k$). We do, however, prove some general results regarding k -cycle-reversals in n -tournaments (subject to various restrictions on n, k and the tournaments involved), thereby giving a partial answer to [30, Prob. 1]. These results are Theorems 1.7.3 and 1.7.4.

A rough sketch of our development is as follows: in the next two sections of this chapter, we shall develop several useful tools for attacking the reversal problem. These tools (especially Proposition 1.4.4) are of a quite general nature, and yield an obvious proof technique which will then be utilized not only to derive the new results described above, but—in order to demonstrate the efficacy of our approach—to provide alternate proofs of all previously known results (cited above). Indeed, Ryser's theorem and the author's result on TT_3 -reversals are proved in §1.4, and Reid's k -path-reversal theorem is proved in Chapter 2. This latter proof (see §2.4) depends primarily on the 4-cycle theorem (i.e., 1.5.1), as do all of the subsequent results on cycle-reversals.

Finally, we mention some related work in the undirected case. In 1971, L.T. Ollmann and K.B. Reid [20] considered

the following question (in effect): given a graph G and a 2-edge-coloring of K_n , the complete (undirected) graph of order n , which other 2-edge-colorings of K_n are obtainable via a sequence of color-interchanges on copies of G in K_n ? That is to say, what are the "G-classes"? Now it will be clear by the end of the next section that the reversal problem is merely the directed version of this one, and, even though the authors of [20] gave a complete answer to their question for arbitrary G , the directional restrictions inherent in the reversal problem make things considerably more difficult. Still, some of our forthcoming results (notably Theorem 2.3.6) are just directed versions of those in [20].

1.3 The directed difference graph.

The first of the above-mentioned tools we develop is based on the following concept.

1.3.1 Definition. Let T, U be n -tournaments and $\varphi: V(T) \rightarrow V(U)$ a bijective vertex-map. Define a directed graph G as follows: $V(G) = V(T)$ and dominance in G is determined by $x \rightarrow y$ in G if and only if

$$x \rightarrow y \text{ in } T \text{ and } \varphi(y) \rightarrow \varphi(x) \text{ in } U.$$

The digraph G is called the directed difference graph (or DDG) of φ re T, U , and we write $G = \text{DDG}_{\varphi}(T, U)$.

Consequently, the arcs of G are those in T whose orientation is reversed by the map φ . Hence, G is discrete (i.e., arc-less) if and only if φ is an isomorphism,

and, at the other extreme, $G = T$ if and only if φ is a dual isomorphism (or anti-isomorphism). Thus, $|E(G)|$, the number of arcs of G , measures how "close" φ is to being an isomorphism: the smaller $|E(G)|$ is, the "closer" φ is.

The basic properties of the directed difference graph which we shall need are formulated in the next two propositions.

1.3.2 Proposition. Let φ, T, U , and G be as in Definition 1.3.1. Then for any vertex x of G :

- i) $O_G(x) = O_T(x) \cap \varphi^{-1}[I_U(\varphi(x))]$;
- ii) $od_G(x) - id_G(x) = od_T(x) - od_U(\varphi(x))$.

Proof: Fix $x \in V(G)$. Then, by definition of G ,

$$\begin{aligned}
 y \in O_G(x) &\Leftrightarrow x \rightarrow y \text{ in } G \\
 &\Leftrightarrow x \rightarrow y \text{ in } T \text{ and } \varphi(y) \rightarrow \varphi(x) \text{ in } U \\
 &\Leftrightarrow y \in O_T(x) \text{ and } \varphi(y) \in I_U(\varphi(x)) \\
 &\Leftrightarrow y \in O_T(x) \text{ and } y \in \varphi^{-1}[I_U(\varphi(x))] \\
 &\Leftrightarrow y \in O_T(x) \cap \varphi^{-1}[I_U(\varphi(x))]
 \end{aligned}$$

and part i) follows. By the principle of directional duality, the statement dual to i), namely

$$i^*) \quad I_G(x) = I_T(x) \cap \varphi^{-1}[O_U(\varphi(x))] ,$$

also holds.

In order to prove part ii), we shall first prove that

$$iii) \quad O_G(x) \cup \varphi^{-1}[O_U(\varphi(x))] = O_T(x) \cup I_G(x) .$$

Using i), i*), and the fact that φ is 1-1, we have that

$$\begin{aligned}
 O_G(x) \cup \varphi^{-1}[O_U(\varphi(x))] &= [O_T(x) \cap \varphi^{-1}[I_U(\varphi(x))]] \cup \varphi^{-1}[O_U(\varphi(x))] \\
 &= [O_T(x) \cup \varphi^{-1}[O_U(\varphi(x))]] \cap [\varphi^{-1}[I_U(\varphi(x))] \cup \varphi^{-1}[O_U(\varphi(x))]]
 \end{aligned}$$

$$\begin{aligned}
&= [O_T(x) \cup \varphi^{-1}O_U\varphi(x)] \cap \varphi^{-1}[I_U\varphi(x) \cup O_U\varphi(x)] \\
&= [O_T(x) \cup \varphi^{-1}O_U\varphi(x)] \cap \varphi^{-1}[V(U) - \{\varphi(x)\}] \\
&= [O_T(x) \cup \varphi^{-1}O_U\varphi(x)] \cap [V(T) - \{x\}] \\
&= [O_T(x) \cup \varphi^{-1}O_U\varphi(x)] \cap [O_T(x) \cup I_T(x)] \\
&= O_T(x) \cup [\varphi^{-1}O_U\varphi(x) \cap I_T(x)] = O_T(x) \cup I_G(x) ,
\end{aligned}$$

which establishes iii). Next, observe that $O_G(x) \cap \varphi^{-1}O_U\varphi(x) = \emptyset$, by i), and that $O_T(x) \cap I_G(x) = \emptyset$, by i*). Thus, upon taking cardinalities of both sides of iii), we obtain

$$|O_G(x)| + |\varphi^{-1}O_U\varphi(x)| = |O_T(x)| + |I_G(x)|$$

and hence (again using the fact that φ is 1-1)

$$od_G(x) + od_U(\varphi(x)) = od_T(x) + id_G(x) ,$$

from which part ii) follows immediately. This completes the proof of the proposition. \square

1.3.3 Proposition. Let φ, T, U , and G be as in Definition 1.3.1, and let π be a permutation of $V(T)$ so that $\varphi \circ \pi: V(T) \rightarrow V(U)$ is a bijection. Let G_π be the directed difference graph of $\varphi \circ \pi$ re T, U . Then

$$|E(G)| \equiv |E(G_\pi)| \pmod{2}$$

if and only if π is an even permutation.

Proof: Any permutation is a product of transpositions, and it is odd if and only if it is a product of an odd number of transpositions (e.g., see [4]). Therefore, in order to establish both implications in the conclusion of the proposition, it will be sufficient to show

(1) If π is a transposition, then $|E(G)| \not\equiv |E(G_\pi)| \pmod{2}$.

To do this, let $\pi = (xy)$, where $x, y \in V(T)$ are distinct, and let e be the arc of T incident with x, y . For $i = 0, 1, 2$, let $E_i(G) = \{a \in E(G) : a \text{ is incident in } G \text{ with exactly } i \text{ of } x, y\}$ and define $E_i(G_\pi)$ analogously. Then

$$|E(G)| = |E_0(G)| + |E_1(G)| + |E_2(G)|,$$

$$|E(G_\pi)| = |E_0(G_\pi)| + |E_1(G_\pi)| + |E_2(G_\pi)|.$$

Since π interchanges x, y , $E_2(G) = \{e\}$ or \emptyset accordingly as $E_2(G_\pi) = \emptyset$ or $\{e\}$. Hence $|E_2(G)| \not\equiv |E_2(G_\pi)| \pmod{2}$.

Since π fixes all other vertices, $E_0(G) = E_0(G_\pi)$. Thus, to prove (1), it is sufficient to show

$$(2) \quad |E_1(G)| \equiv |E_1(G_\pi)| \pmod{2}$$

and this is done as follows: applying Proposition 1.3.2 ii) to G, x and G, y in turn yields

$$\text{od}_G(x) - \text{id}_G(x) = \text{od}_T(x) - \text{od}_U(\varphi(x)),$$

$$\text{od}_G(y) - \text{id}_G(y) = \text{od}_T(y) - \text{od}_U(\varphi(y)).$$

Applying 1.3.2 ii) to G_π, x and G_π, y (and using $\pi(x)=y$, $\pi(y)=x$) we similarly obtain

$$\text{od}_{G_\pi}(x) - \text{id}_{G_\pi}(x) = \text{od}_T(x) - \text{od}_U(\varphi(y)),$$

$$\text{od}_{G_\pi}(y) - \text{id}_{G_\pi}(y) = \text{od}_T(y) - \text{od}_U(\varphi(x)).$$

From these four equalities, it is easy to see that

$$\begin{aligned} & \text{od}_G(x) - \text{id}_G(x) + \text{od}_G(y) - \text{id}_G(y) \\ &= \text{od}_{G_\pi}(x) - \text{id}_{G_\pi}(x) + \text{od}_{G_\pi}(y) - \text{id}_{G_\pi}(y) \end{aligned}$$

and this, together with the fact that $j \equiv -j \pmod{2}$, yields

$$\begin{aligned} (3) \quad & \text{od}_G(x) + \text{id}_G(x) + \text{od}_G(y) + \text{id}_G(y) \\ & \equiv \text{od}_{G_\pi}(x) + \text{id}_{G_\pi}(x) + \text{od}_{G_\pi}(y) + \text{id}_{G_\pi}(y) \pmod{2}. \end{aligned}$$

Now the LHS (left-hand side) of (3) counts the arcs of G incident with x or y once each except for e , which is counted twice if $e \in E(G)$ and not at all otherwise. Thus, the LHS of (3) equals $|E_1(G)| + 2|E_2(G)| \equiv |E_1(G)| \pmod{2}$. Similarly, the RHS of (3) is congruent to $|E_1(G_\pi)|$ modulo 2. Therefore, (3) implies that $|E_1(G)| \equiv |E_1(G_\pi)| \pmod{2}$, which proves (2). By our foregoing remarks, the proof of the proposition is now complete. \square

1.4 The refinement technique.

The following notation will prove helpful in this and subsequent sections (n is any positive integer).

\mathcal{U} : an arbitrary class of tournaments

\mathcal{T} : the class of all tournaments

\mathcal{T}_n : the class of n -tournaments

$\mathcal{T}_{\geq n}$: the class of tournaments of order at least n

\mathcal{S} : the class of strong tournaments

\mathcal{R} : the class of regular tournaments

\mathcal{NR} : the class of near-regular tournaments.

Other symbols (such as $\mathcal{T}_{\leq n}$, \mathcal{S}_n , $\mathcal{R}_{\geq n}$, etc.) are defined analogously.

1.4.1 Definition. Let \mathcal{D} be a collection of asymmetric digraphs and let \mathcal{U} be a class of tournaments. We say \mathcal{U} is closed under \mathcal{D} -reversals, or simply \mathcal{D} -closed, if, for any tournament $T \in \mathcal{U}$, any \mathcal{D} -reversal performed on T yields a tournament $T^1 \in \mathcal{U}$.

1.4.2 Examples. The class \mathcal{R} of regular tournaments

is closed under k -cycle-reversals ($k \geq 3$), i.e., C_k -closed (C_k will always denote a directed k -cycle). This follows immediately from the observation that a C_k -reversal does not change the (net) score of any vertex of the tournament on which the operation is performed, and (consequently) preserves the score sequence. Similar reasoning applies to the class \mathcal{NR} . Since the score sequence completely determines strong connectedness in tournaments, \mathcal{S} is also C_k -closed. Finally, we make the trivial observation that \mathcal{T}_n (also $\mathcal{T}_{\geq n}$ and $\mathcal{T}_{\leq n}$) is \mathcal{S} -closed, for any collection \mathcal{S} .

As we shall see, the following notion of "refinement," when used in conjunction with the directed difference graph, lays the foundation for an efficient proof technique for reversal theory.

1.4.3 Definition. Let $\mathcal{D}_1, \mathcal{D}_2$ be collections of asymmetric digraphs and \mathcal{U} a class of tournaments. We say \mathcal{D}_1 refines \mathcal{D}_2 in the class \mathcal{U} if, given any tournament $T \in \mathcal{U}$ and any copy $D_0 \subseteq T$ of any digraph $D \in \mathcal{D}_2$, there exists a (possibly null) sequence

$$T = T^0 \rightsquigarrow T^1 \rightsquigarrow \dots \rightsquigarrow T^m \quad (m \geq 0)$$

of \mathcal{D}_1 -reversals, beginning in T , which exactly reverses D_0 in the following sense: upon termination of the sequence, the resulting tournament T^m has $V(T^m) = V(T)$ and $E(T^m) = (E(T) - E(D_0)) \cup E(D_0^*)$.

When the class \mathcal{U} is understood, we sometimes say, for brevity, \mathcal{D}_1 refines \mathcal{D}_2 . Note that the definition requires that T^m be equal, not merely isomorphic, to the

tournament obtained from T by reversing the arcs of D_0 .

1.4.4 Proposition. Let $\mathcal{D}_1, \mathcal{D}_2$ be collections of asymmetric digraphs, \mathcal{U} a class of tournaments, and $T, U \in \mathcal{U}$ such that

- i) \mathcal{D}_1 refines \mathcal{D}_2 in the class \mathcal{U} ;
- ii) \mathcal{U} is \mathcal{D}_2 -closed;
- iii) $T \sim U$ via \mathcal{D}_2 -reversals.

Then also $T \sim U$ via \mathcal{D}_1 -reversals. Therefore, if $\mathcal{D}_1, \mathcal{D}_2$ are self-dual, then every \mathcal{D}_2 -class is contained in a \mathcal{D}_1 -class.

Proof: By assumption iii), there is a sequence

$$T = T^0 \rightsquigarrow T^1 \rightsquigarrow \dots \rightsquigarrow T^m \cong U$$

of \mathcal{D}_2 -reversals. Since \mathcal{D}_1 refines \mathcal{D}_2 in \mathcal{U} and $T \in \mathcal{U}$, the first \mathcal{D}_2 -reversal $T^0 \rightsquigarrow T^1$ may be replaced by a sequence

$$T^0 = T^{0,0} \rightsquigarrow T^{0,1} \rightsquigarrow \dots \rightsquigarrow T^{0,m_1} = T^1$$

of \mathcal{D}_1 -reversals. Since $T \in \mathcal{U}$ and \mathcal{U} is \mathcal{D}_2 -closed, $T^1 \in \mathcal{U}$. We may now repeat the process, replacing $T^1 \rightsquigarrow T^2$ by a sequence of \mathcal{D}_1 -reversals. By induction on m , it follows that $T \sim U$ via \mathcal{D}_1 -reversals.

The "Therefore,..." part of the conclusion follows immediately from the first part. \square

Utilization of Proposition 1.4.4 in obtaining subsequent results (and there will be a great many instances of this) will be called the refinement technique. The special case $\mathcal{D}_1 = \{D_1\}$, $\mathcal{D}_2 = \{D_2\}$ of the proposition will be of

particularly frequent use.

The next two preliminary results take up the issue of "cycle stacking"; they are needed for direct application of the refinement technique to the cycle-reversal theorems.

1.4.5 Lemma. Fix integers $i, j \geq 3$. Then in the class \mathcal{T} of all tournaments:

- i) $\{C_i, C_j\}$ refines C_{i+j-2} ;
- ii) C_i refines $C_{m(i-2)+2}$, for $m = 1, 2, \dots$.

Proof: i): Let T be any tournament and let

$$Z: z_1 \rightarrow z_2 \rightarrow \dots \rightarrow z_{i+j-2} \rightarrow z_1$$

be an arbitrary cycle in T of length $i+j-2$. Now in T , either $z_1 \rightarrow z_i$ (dominates) or $z_i \rightarrow z_1$. Suppose first that (a) $z_1 \rightarrow z_i$. Then successive reversals of the cycles

$$C_j: z_1 \rightarrow z_i \rightarrow z_{i+1} \rightarrow \dots \rightarrow z_{i+j-2} \rightarrow z_1,$$

$$C_i: z_1 \rightarrow z_2 \rightarrow \dots \rightarrow z_i \rightarrow z_1$$

exactly reverses Z (see Figure 2 (a) below; arcs of T irrelevant to the reversal procedure will not be shown in such diagrams) and C_j, C_i have the indicated lengths.

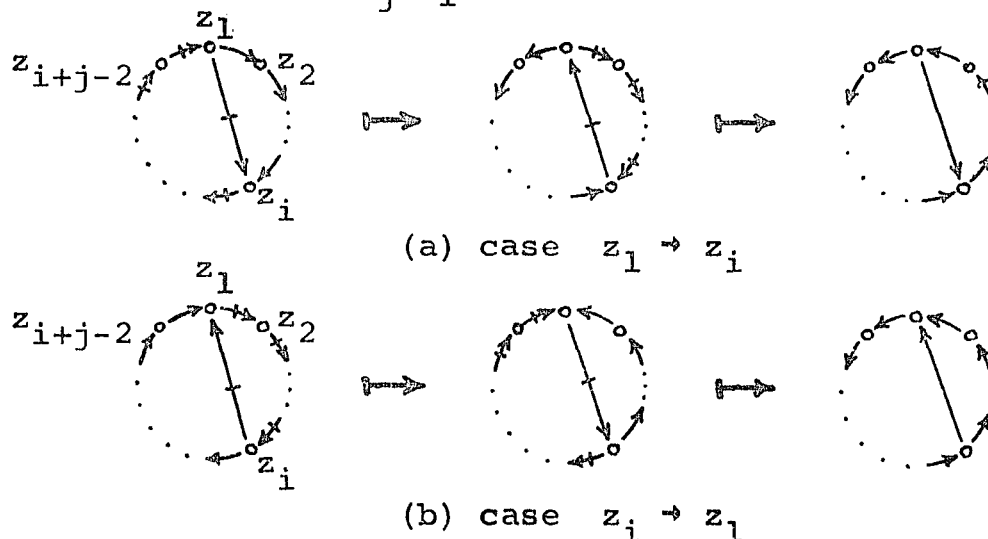


Figure 2. Cycle stacking

Now suppose that (b) $z_i \rightarrow z_1$. Then successive reversals of the cycles

$$C_i: z_1 \rightarrow z_2 \rightarrow \dots \rightarrow z_i \rightarrow z_1,$$

$$C_j: z_1 \rightarrow z_i \rightarrow z_{i+1} \rightarrow \dots \rightarrow z_{i+j-2} \rightarrow z_1$$

exactly reverses Z (Figure 2 (b)). In either case, Z can be exactly reversed by a sequence of $\{C_i, C_j\}$ -reversals, so $\{C_i, C_j\}$ refines C_{i+j-2} in \mathfrak{T} , proving assertion i).

ii): The class \mathfrak{T} will be understood throughout. We induct on m and use part i). If $m = 1$, ii) asserts that C_i refines C_i , which is certainly true. Suppose that C_i refines $C_{m(i-2)+2}$ for some $m \geq 1$. We shall show that C_i refines $C_{(m+1)(i-2)+2}$. It is clear that

$$(1) \quad C_i \text{ refines } \{C_i, C_{m(i-2)+2}\}$$

by the induction hypothesis. Since $i + (m(i-2)+2) - 2 = (m+1)(i-2) + 2$, we also have that

$$(2) \quad \{C_i, C_{m(i-2)+2}\} \text{ refines } C_{(m+1)(i-2)+2}$$

by part i). From (1) and (2) it is easy to see that C_i refines $C_{(m+1)(i-2)+2}$ (refinement is transitive, that is), which is what we wished to show. Part ii) now follows by induction. \square

An immediate consequence of Lemma 1.4.5 ii) is

1.4.6 Corollary. In the class \mathfrak{T} of all tournaments, the 3-cycle refines all cycles and the 4-cycle refines all cycles of even length. \square

We are now in a position to achieve some "concrete" results. First, we shall give a new proof of [27, Thm. 4.2],

which was also proved by Moon [19, pp. 73-74, 61-62] using the terminology of tournaments.

1.4.7 Theorem (H.J. Ryser, 1964). If T and U are n -tournaments with the same score sequence, then $T \sim U$ via 3-cycle-reversals.

Proof: Using the fact that T, U have the same score sequence, let $\varphi: V(T) \rightarrow V(U)$ be a score-preserving bijection, and let $G = \text{DDG}_{\varphi}(T, U)$, i.e., G is the directed difference graph of φ re T, U . By Proposition 1.3.2 ii), for every $x \in V(G)$,

$$\text{od}_G(x) - \text{id}_G(x) = \text{od}_T(x) - \text{od}_U(\varphi(x)) = 0,$$

the last equality following since φ is score-preserving. Therefore,

$$(3) \quad \text{od}_G(x) = \text{id}_G(x) \quad (\text{for all } x \in V(G)).$$

Now a digraph G satisfying property (3) (such digraphs are called regular) is easily seen to be an arc-disjoint union of (directed) cycles (this is [6, Exer. 10.3.2]).

We proceed by (backward) induction on $|E(G)| \geq 0$. If $|E(G)| = 0$, then φ is an isomorphism from T to U , and there is nothing to prove. So suppose $|E(G)| > 0$. Then, by the previous remarks, G contains a cycle Z of some length $k \geq 3$. By Corollary 1.4.6, Z may be exactly reversed by a sequence of 3-cycle-reversals beginning in T . This sequence yields a new tournament, say T^1 , and the same map $\varphi: V(T^1) \rightarrow V(U)$ is score-preserving relative to T^1, U (see Examples 1.4.2). Let $G^1 = \text{DDG}_{\varphi}(T^1, U)$. Then property (3) holds with G^1 in place of G (so that the

induction can be continued, if necessary), and we clearly have $|E(G^1)| = |E(G)| - k < |E(G)|$.

The theorem now follows, since repeating the above procedure eventually yields a tournament T^m for which $\varphi: V(T^m) \rightarrow V(U)$ is an isomorphism from T^m to U (cf. Definition 1.2.2). \square

Next, we shall give a new proof of the main theorem of [30]. The following preliminary result is [30, Lemma 4], in effect.

1.4.8 Lemma. In the class $\mathcal{T}_{\geq 4}$ of tournaments of order at least 4, TT_3 refines C_3 (and hence refines all cycles).

Proof: Let T be a tournament of order at least 4, and let $Z: z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow z_1$ be a 3-cycle in T . Let x be a vertex of T not on Z . Then either (a) x dominates at least two vertices of Z , or (b) x is dominated by at least two vertices of Z .

Suppose (a) holds. By relabelling Z , if necessary, we may assume that x dominates z_1, z_2 . Then the sequence

$$x \rightarrow z_1 \rightarrow z_2 \leftarrow x,$$

$$z_2 \rightarrow z_3 \rightarrow z_1 \leftarrow z_2,$$

$$z_1 \rightarrow z_2 \rightarrow x \leftarrow z_1$$

of TT_3 -reversals exactly reverses Z (Figure 3 below).

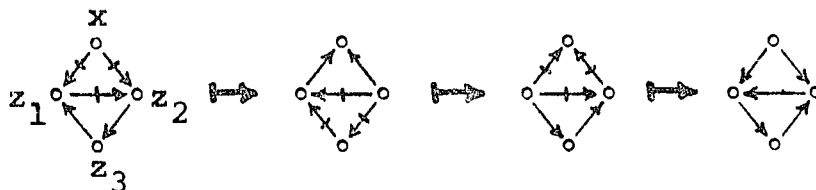


Figure 3. The transitive triple refines the cyclic triple

Case (b) follows from case (a) by directional duality, so the proof of the lemma is complete. \square

1.4.9 Theorem (C. Waldrop, 1976). If T and U are n -tournaments with the same number of vertices of even score (equivalently, odd score), then $T \sim U$ via TT_3 -reversals.

Proof: The conclusion clearly holds if $n < 4$, so assume that $n \geq 4$. By hypothesis, there exists a bijection $\varphi: V(T) \rightarrow V(U)$ which preserves score parity, i.e., $od_T(x) \equiv od_U(\varphi(x)) \pmod{2}$, for every $x \in V(T)$. Let $G = DDG_\varphi(T, U)$. Then, by Proposition 1.3.2 ii),

$$od_G(x) - id_G(x) = od_T(x) - od_U(\varphi(x)) \equiv 0 \pmod{2}$$

for all $x \in V(G)$, so

$$(4) \quad od_G(x) \equiv id_G(x) \pmod{2} \quad (\text{for all } x \in V(G)).$$

As before, we induct on $|E(G)| \geq 0$. If $|E(G)| = 0$, then φ is an isomorphism from T to U , and $T \sim U$ via the null sequence of TT_3 -reversals. Assume, therefore, that $|E(G)| > 0$. Let $\vec{xy} \in E(G)$. Then (4) implies that there exists $z \in V(G) - \{x, y\}$ such that either (a) $\vec{xz} \in E(G)$ or (b) $\vec{zx} \in E(G)$.

Suppose (a) $\vec{xz} \in E(G)$. Then $\langle x, y, z \rangle_T \cong TT_3$, and reversing this transitive triple yields a tournament T^1 and its associated $G^1 = DDG_\varphi(T^1, U)$. Clearly, $|E(G^1)| = |E(G)| - 3$ or $|E(G)| - 1$ accordingly as $e \in E(G)$ or $e \notin E(G)$, where e denotes the arc of T joining y, z . Thus, $|E(G^1)| < |E(G)|$.

Now suppose (b) $\vec{zx} \in E(G)$. If $\langle x, y, z \rangle_T \cong TT_3$, the previous argument for case (a) applies. On the other hand,

if $\langle x, y, z \rangle_T \cong C_3$, then (since $n \geq 4$) Lemma 1.4.8 insures that there is a sequence of TT_3 -reversals that exactly reverses $\langle x, y, z \rangle_T$, and again case (a) applies.

Therefore, T may always be transformed into a tournament T^1 via TT_3 -reversal(s) for which its associated directed difference graph G^1 has fewer arcs than does G . Since property (4) is always preserved by such reversals, we can repeat the above procedure until we obtain a copy of U . The theorem follows. \square

As remarked earlier, the converses of Theorems 1.4.7 and 1.4.9 are trivially true (a C_3 -reversal is a score-preserving transformation, and a TT_3 -reversal preserves score parity).

1.5 The 4-cycle theorem.

A simple parity argument using the directed difference graph shows that no digraph with an even number of arcs refines one with an odd number of arcs (in any tournament). Consequently, a proof of Theorem 1.5.1 below cannot be based solely on Ryser's theorem and the refinement technique. Something extra, namely Proposition 1.3.3, is needed.

1.5.1 Theorem. If T and U are n -tournaments with the same score sequence, then $T \sim U$ via 4-cycle-reversals.

Proof: Let $\varphi: V(T) \rightarrow V(U)$ be a score-preserving bijection, and let $G = DDG_\varphi(T, U)$.

We first reduce to the case where T, U are strong.

Let s^1, s^2, \dots, s^p be the strong components of T and let w^1, w^2, \dots, w^q be those of U . Because φ is score-preserving,

$$(1) \quad p = q \quad \text{and} \quad \varphi(V(s^i)) = V(w^i) \quad (i = 1, \dots, p)$$

and if $\varphi_i = \varphi|_{V(s^i)}$ (i.e., the restriction of φ), then

$$(2) \quad \varphi_i: V(s^i) \rightarrow V(w^i) \quad \text{is score-preserving from } s^i \text{ to } w^i \\ (i = 1, \dots, p)$$

(cf., comments in §1.1 regarding the score sequence). Since s^i dominates s^j in T and w^i dominates w^j in U if and only if $i < j$, no arc of G joins a vertex of s^i to one of s^j when $i \neq j$. Thus, letting $G_i = \text{DDG}_{\varphi_i}(s^i, w^i)$ ($i = 1, \dots, p$), we have that

$$(3) \quad E(G) = \bigcup_{i=1}^p E(G_i) .$$

By inducting on p and using (2) and (3), it is now easy to see that the theorem will follow if we can prove it in the special case in which T, U are strong.

Next, we reduce to the case where $|E(G)|$ is even. For suppose G has an odd number of arcs. Then choose distinct $x, y \in V(T)$ such that $\text{od}_T(x) = \text{od}_T(y)$ (if this cannot be done then T, U have score sequence $(0, 1, \dots, n-1)$, hence both are transitive and φ is an isomorphism from T to U , so $|E(G)| = 0$, but this contradicts the assumption that $|E(G)|$ is odd), let $\pi = (xy)$, note that the composition $\varphi \circ \pi: V(T) \rightarrow V(U)$ is also score-preserving, and that $|E(G_\pi)|$, where $G_\pi = \text{DDG}_{\varphi \circ \pi}(T, U)$, is even, by Proposition 1.3.3. Thus, by replacing φ with $\varphi \circ \pi$ and G with G_π , if necessary, we may assume $|E(G)|$ is even (and T, U are

strong) for the remainder of the proof.

As in the proof of Theorem 1.4.7, G is a regular digraph (i.e., G satisfies (3) of §1.4). We induct on $|E(G)| \geq 0$ as before. If $|E(G)| = 0$, then $T \cong U$ and there is nothing to prove, so assume $|E(G)| > 0$. As in the proof of 1.4.7, it will be sufficient to show that $|E(G)|$ can be "lowered" (i.e., to a new value $|E(G^1)|$) via a sequence of 4-cycle-reversals (and note that necessarily $|E(G^1)| \equiv |E(G)| \equiv 0 \pmod{2}$ for such reversals), for this will eventually convert φ into an isomorphism. By hypothesis, G contains a cycle of some length. In cases 1-3 below, we examine all of the possibilities.

Case 1: G contains a cycle Z of even length $2k$.

Then Z may be exactly reversed by a sequence of C_4 -reversals, by Corollary 1.4.6. This lowers $|E(G)|$ by $2k$. In addition, note that if T contains a $2k$ -cycle all but possibly one of whose arcs are in G , then reversal of this cycle lowers $|E(G)|$ by $2k$ or $2k - 2$, i.e., by at least 2.

Case 2: G contains a cycle Z of odd length $2k+1 \geq 5$.

Let $Z: z_1 \rightarrow z_2 \rightarrow \dots \rightarrow z_{2k+1} \rightarrow z_1$. Now if $z_1 \rightarrow z_3$ (dominance in T , unless otherwise stated), then the "in addition" part of case 1 applies, so we may assume that $z_3 \rightarrow z_1$ and, by similar considerations, that $z_5 \rightarrow z_3$ and $z_1 \rightarrow z_4$. But now successive reversals of the 4-cycles

$$z_1 \rightarrow z_4 \rightarrow z_5 \rightarrow z_3 \rightarrow z_1,$$

$$z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow z_4 \rightarrow z_1$$

lowers $|E(G)|$ by at least 2 (Figure 4 below).

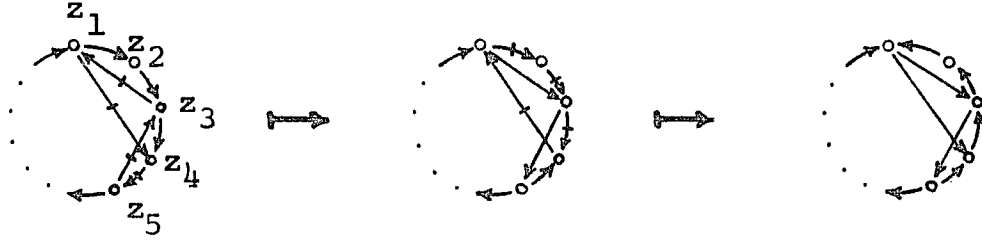


Figure 4. The 4-cycle proof, reduction

Therefore, we may assume that the only cycles in G are 3-cycles, and because $|E(G)|$ is even, we need only consider

Case 3: G contains two arc-disjoint 3-cycles, Z^1, Z^2 .

Let $Z^1: x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_1$ and $Z^2: y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_1$. If Z^1, Z^2 have a vertex in common, say $x_1 = y_1$, then by symmetry we may assume $x_2 \rightarrow y_2$, and successive reversals of

$$x_2 \rightarrow y_2 \rightarrow y_3 \rightarrow x_1 \rightarrow x_2,$$

$$y_2 \rightarrow x_2 \rightarrow x_3 \rightarrow x_1 \rightarrow y_2$$

reverses the arcs of both Z^1 and Z^2 (see Figure 5 (a), p. 27).

Assume, therefore, that Z^1 and Z^2 are vertex-disjoint. By symmetry, we may assume $x_1 \rightarrow y_1$. Now if $y_3 \rightarrow x_2$, then successive reversals of

$$y_3 \rightarrow x_2 \rightarrow x_3 \rightarrow x_1 \rightarrow y_1 \rightarrow y_2 \rightarrow y_3,$$

$$x_2 \rightarrow y_3 \rightarrow y_1 \rightarrow x_1 \rightarrow x_2$$

(Figure 5 (b)) reverses Z^1 and Z^2 (recall that the 4-cycle refines the 6-cycle), so we may assume $x_2 \rightarrow y_3$, and we may similarly assume $x_3 \rightarrow y_2$ (by repeating the procedure). Now if $y_3 \rightarrow x_1$, then successive reversals of

$$Y_3 \rightarrow x_1 \rightarrow Y_1 \rightarrow Y_2 \rightarrow Y_3 ,$$

$$Y_3 \rightarrow Y_1 \rightarrow x_1 \rightarrow x_2 \rightarrow Y_3 ,$$

$$Y_3 \rightarrow x_2 \rightarrow x_3 \rightarrow x_1 \rightarrow Y_3$$

reverses Z^1 and Z^2 (Figure 5 (c)), so we may assume

$x_1 \rightarrow y_3$. Repetition of the above reversal procedures now

allows us to assume that Z^1 dominates Z^2 (that is,

$x_i \rightarrow y_j$ for all $1 \leq i, j \leq 3$) without loss of generality.

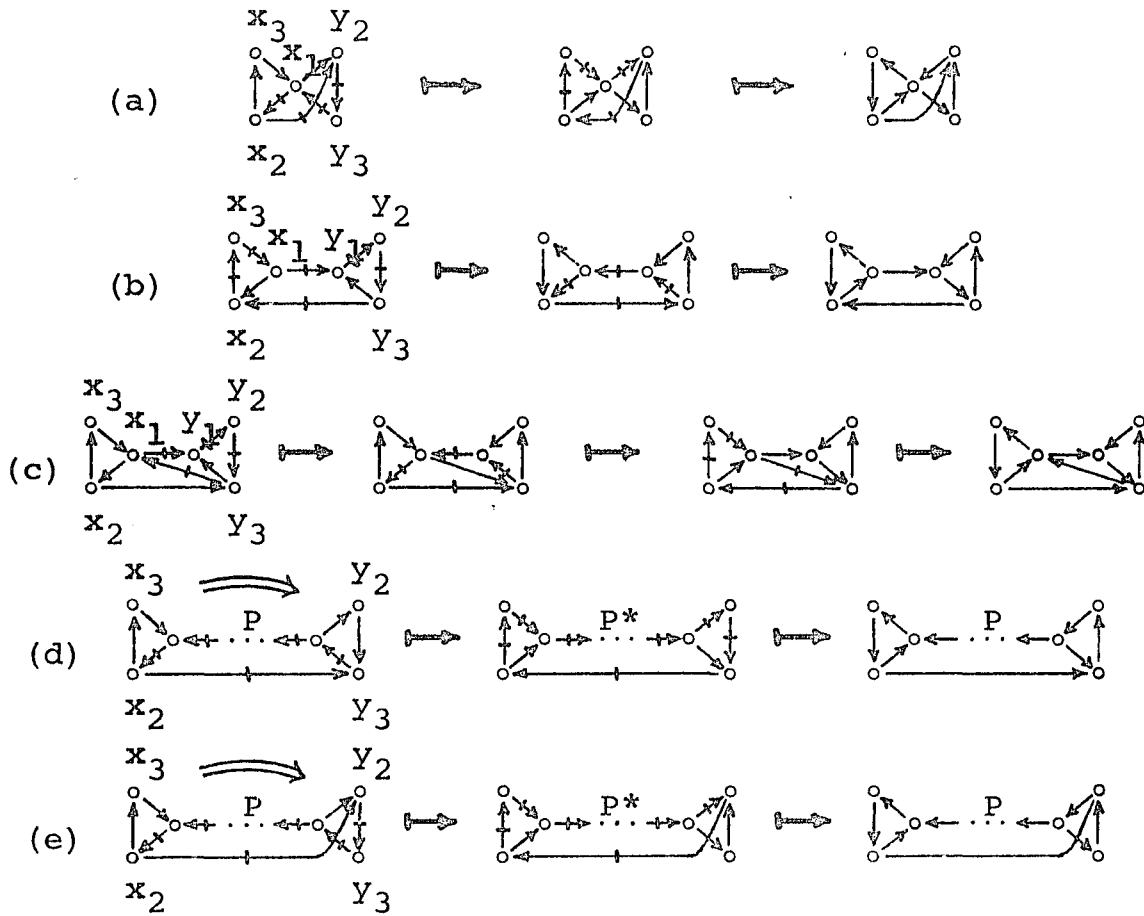


Figure 5. The 4-cycle proof, refinement

By assumption, T is strong, so there exists a path from some vertex of Z^2 to some vertex of Z^1 . Let P be a shortest such path in T , say from y_1 to x_1 , and note that (by minimality) the internal vertices of P do

not meet $V(Z^1) \cup V(Z^2)$. If P has odd length, then successive reversals of the cycles

$$x_2 \rightarrow y_3 \rightarrow P \rightarrow x_2 ,$$

$$x_2 \rightarrow x_3 \rightarrow P^* \rightarrow y_2 \rightarrow y_3 \rightarrow x_2$$

(both of which have even length) reverses Z^1 and Z^2 (see Figure 5 (d)). On the other hand, if P has even length, then successive reversals of

$$x_2 \rightarrow y_2 \rightarrow y_3 \rightarrow P \rightarrow x_2 ,$$

$$x_2 \rightarrow x_3 \rightarrow P^* \rightarrow y_2 \rightarrow x_2$$

reverses the arcs of Z^1 and Z^2 (Figure 5 (e)). In case 3, therefore, $|E(G)|$ may be lowered by 6 by exactly reversing $Z^1 \cup Z^2$.

The theorem follows by induction. \square

Before deriving the second of our new results, we pause to remark that, in view of the refinement technique and Corollary 1.4.6, Ryser's 3-cycle theorem (i.e., 1.4.7) follows at once from the 4-cycle theorem just proved. As we shall subsequently demonstrate, the 4-cycle theorem will be exceedingly useful for deriving additional results.

1.6 The 5-cycle theorem.

One possible approach to proving our second major result, the 5-cycle theorem, is to use the directed difference graph and the "cycle stacking" (i.e., Lemma 1.4.5 and Corollary 1.4.6) aspect of the refinement technique, but this appears to be considerably more difficult than the foregoing proof of the 4-cycle theorem (which such an approach would

mimic). Instead, the following lemma on "reverse cycle stacking" (the proof of which is straightforward, though tedious) and the 4-cycle theorem itself will be employed.

1.6.1 Lemma. In the class \mathcal{A}_6 of strong 6-tournaments, the 5-cycle refines the 4-cycle.

Proof: Let S be a strong 6-tournament, and let

$$Z: z_0 \rightarrow z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow z_0$$

be a 4-cycle in S . Let x, y be the vertices of S not on Z with (say) $x \rightarrow y$. We show that Z may be exactly reversed via a sequence of 5-cycle-reversals.

First, consider any two opposite vertices z_i, z_{i+2} of Z (where the subscripts are read modulo 4). If $z_i \rightarrow x$ and $y \rightarrow z_{i+2}$ (in which case the 3-path $z_i \rightarrow x \rightarrow y \rightarrow z_{i+2}$ is internally disjoint from $V(Z)$), then successive reversals of the 5-cycles

$$z_i \rightarrow x \rightarrow y \rightarrow z_{i+2} \rightarrow z_{i+3} \rightarrow z_i,$$

$$z_i \rightarrow z_{i+1} \rightarrow z_{i+2} \rightarrow y \rightarrow x \rightarrow z_i$$

exactly reverses Z (Figure 6 below).

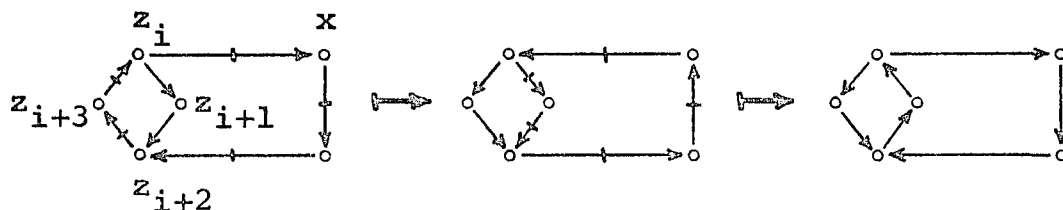


Figure 6. Reverse cycle stacking

Next, consider Figure 7, p. 30. There are eight arcs of S unaccounted for in this diagram (excluding those between opposite vertices of Z), giving a total of $2^8 = 256$ possible orientations. Any one of these eight arcs which

happens to be oriented from $\{z_0, z_2\}$ to $\{x, y\}$ or from $\{x, y\}$ to $\{z_1, z_3\}$ will be called an "up" arc.

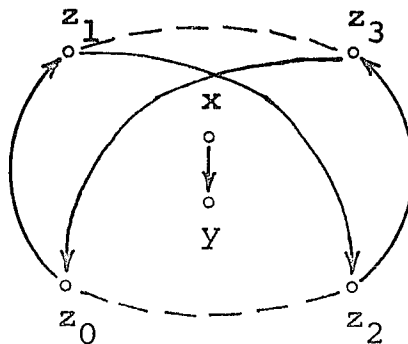


Figure 7.

By symmetry and other considerations, however, the list of 256 configurations may be reduced to 15, and this reduction procedure follows the simplifying assumptions A1)-A6) below.

A1) There are at most four "up" arcs.

Otherwise, relabelling Z via the permutation $(z_0 z_1 z_2 z_3)$ achieves the desired result.

A2) If either z_i or z_{i+2} dominates x , then a) the other dominates y , and b) if $i = 0$, then $z_0 \rightarrow x$.

We may assume a) via the reversal procedure of Figure 6, and b) by symmetry, i.e., relabelling Z via the permutation $(z_0 z_2)(z_1 z_3)$.

A3) x does not dominate all of the z_i 's, nor do all of the z_i 's dominate y .

Otherwise, S would not be strong.

A4) At most one of z_0, z_2 dominates x .

Otherwise, both also dominate y , by A2), for a total of four "up" arcs, but then, by A1), all the z_i 's dominate

y , contradicting A3).

A5) y dominates at most one of z_1, z_3 .

This is dual to A4).

A6) Other obvious symmetry considerations are assumed (e.g., the case in which $z_0 \vec{y}$ is the only "up" arc is similar to that in which $z_2 \vec{y}$ is the only one).

It is now straightforward to check that the 15 "up" arc configurations (a)-(o) of Figure 8 below are the only ones that need be considered (only the "up" arcs are depicted in the figure).

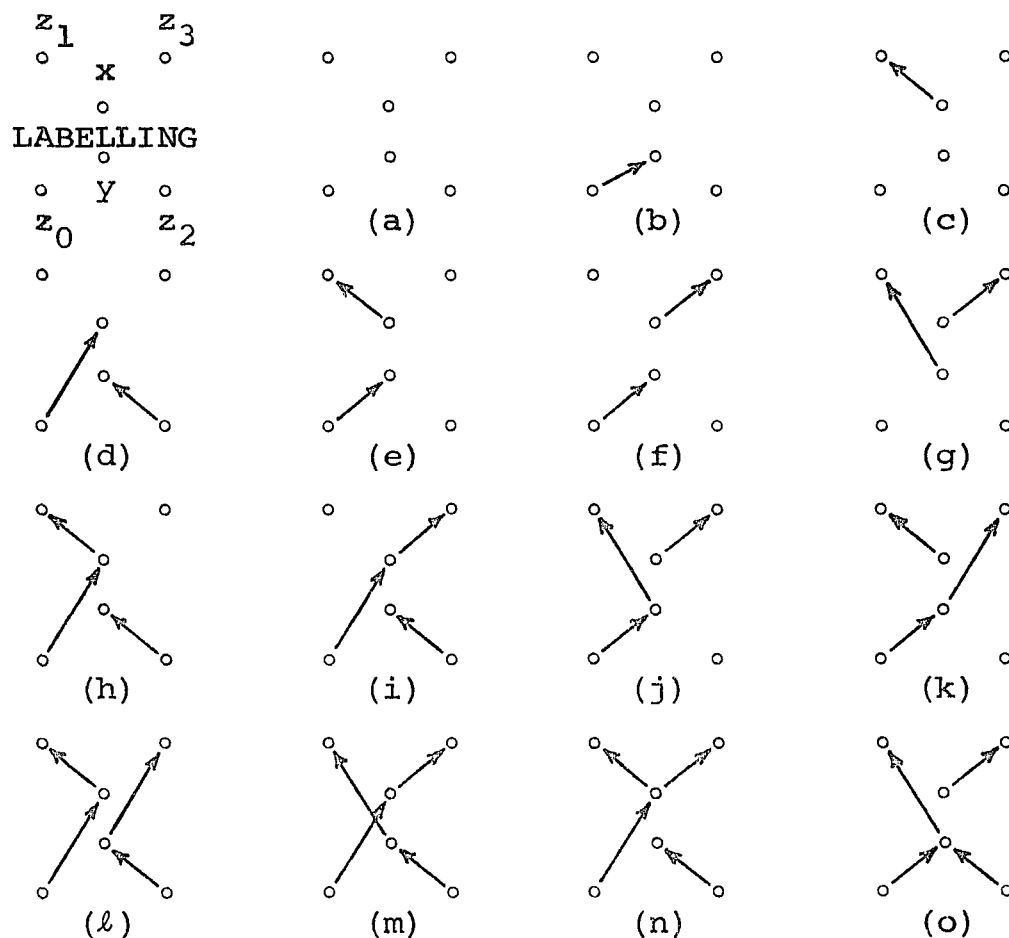


Figure 8. "Up" arc configurations

For each configuration (a)-(o), we list below a se-

quence of 5-cycle-reversals, beginning in S , that exactly reverses Z , using simplified notation; e.g., "x2301x , 032xy0 , ..." means "reverse $x \rightarrow z_2 \rightarrow z_3 \rightarrow z_0 \rightarrow z_1 \rightarrow x$, then reverse $z_0 \rightarrow z_3 \rightarrow z_2 \rightarrow x \rightarrow y \rightarrow z_0$, and so forth."

(a): x2301x , 032xy0 , 3x1y23 , y1x30y , 103yx1 , y3012y .

(b): 01x230 , 2x10y2 , 03x2y0 , 2x3012 .

(c): 3y0123 , 10y3x1 , 3y21x3 , 12y301 .

(d): if $z_3 \rightarrow z_1$: 0x23y0 , 01xy30 , 0312x0 , 0yx130 ;
 if $z_1 \rightarrow z_3$ and $z_2 \rightarrow z_0$: 0123y0 , 0x2130 , 3120y3 ,
 0213x0 , 1y0x31 , x0y12x ;
 if $z_1 \rightarrow z_3$ and $z_0 \rightarrow z_2$: 013xy0 , 02yx30 , 1xy231 ,
 0yx120 .

(e): 0y23x0 , 0x12y0 , 3y21x3 , 12y301 .

(f): 01x230 , 0y2x10 , 1x3y01 , y3x12y .

(g): if $z_0 \rightarrow z_2$: 1x23y1 , 012xy0 , 021y30 , 0yx120 ;
 if $z_2 \rightarrow z_0$: 01x3y0 , 0y23x0 , y120xy , 021y30 ,
 103yx1 , y3012y .

(h): 0x12y0 , 0y23x0 , 01x3y0 , 0x1y30 , 0321x0 , 1230y1 .

(i): 01x3y0 , 0x1230 , 3210y3 , 0123x0 .

(j): 01x230 , 0y12x0 , 0321y0 , 1230x1 .

(k): 0y23x0 , 012y30 , 3210x3 , 0123y0 .

(l): 0x2y30 , 0123x0 , 2103y2 , 3012x3 .

(m): 0123y0 , 0x3210 , 123xy1 , 01yx30 , 3210y3 , 0123x0 .

(n): if $z_1 \rightarrow z_3$: 0123y0 , 0x2y30 , x13y2x , y31x0y ;
 if $z_3 \rightarrow z_1$: 0123y0 , 0x2y30 , x31y2x , y13x0y .

(o): 01x230 , 0y12x0 , 0321y0 , 1230x1 .

The proof of the lemma is now complete. \square

A generalization of the fact, due to Harary and Moser [12, Thm. 7], that all strong tournaments are pancyclic (that is, have cycles of all possible lengths) is that they are vertex-pancyclic [19, Thm. 3]; in fact, Moon has obtained an even stronger result [17]: if T is a strong n -tournament and W is a strong r -subtournament of T , then for each value of s satisfying $r \leq s \leq n$ except possibly $s = r + 1$, there exists a strong s -subtournament S of T such that $W \subseteq S$. As a consequence of this result and Lemma 1.6.1, we have

1.6.2 Corollary. In the class $\mathcal{T}_{\geq 6}$ of strong tournaments of order at least 6, the 5-cycle refines the 4-cycle.

Proof: Let T be a strong n -tournament, $n \geq 6$, and let Z be a 4-cycle in T . Let $W = \langle Z \rangle_T$; then W is a strong 4-subtournament of T , and thus is contained in a strong 6-subtournament S of T . The cycle Z can be exactly reversed via 5-cycle-reversals in S and, a fortiori, in T . \square

1.6.3 Theorem. If T and U are n -tournaments with the same score sequence, then $T \sim U$ via 5-cycle-reversals.

Proof: Precisely as in the proof of the 4-cycle theorem, we may assume T, U are strong with no loss of generality. If $n \geq 6$, then Corollary 1.6.2, the refinement technique, and the 4-cycle theorem together imply that $T \sim U$ via 5-cycle-reversals. If $n \leq 4$, then since T, U have identical score sequences, $T \cong U$ and there is nothing

to prove. Finally, if $n = 5$, the two sequences of 5-cycle-reversals diagrammed in Figure 9 below show directly that $T \sim U$ via 5-cycle-reversals (cf. [19, Appendix, p. 92] for verification; the only strong 5-tournament not depicted in Figure 9 has score sequence $(2,2,2,2,2)$ and thus is the unique regular 5-tournament, denoted RT_5).

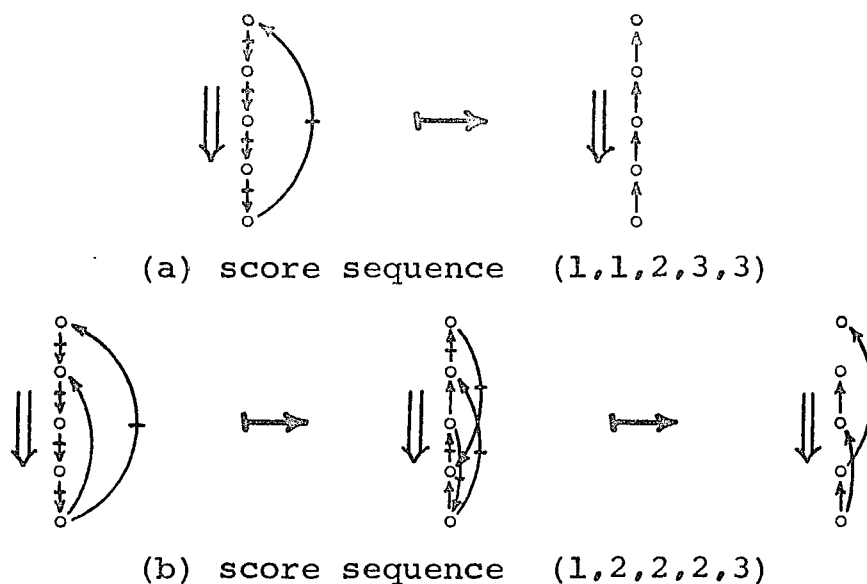


Figure 9. The 5-cycle proof, case $n = 5$

The theorem follows. \square

We summarize our results (Theorems 1.4.7, 1.5.1, and 1.6.3) on reversals of the "small" cycles in the more definitive Theorem 1.6.5 to follow, but we first require a preliminary lemma aimed at establishing its part iii).

1.6.4 Lemma. In the class $\mathcal{T}_{\geq 6}$, the 5-cycle refines the 3-cycle, and hence refines all cycles.

Proof: Let T be a strong n -tournament, $n \geq 6$, and let $Z: z_0 \rightarrow z_1 \rightarrow z_2 \rightarrow z_0$ be a 3-cycle in T . In view of

Corollary 1.6.2, we need only show that there is a sequence of $\{C_4, C_5\}$ -reversals that exactly reverses Z . To do this, observe first that if T contains a 3-path

$$P: z_i \rightarrow x \rightarrow y \rightarrow z_j$$

such that $z_i, z_j \in V(Z)$ and $x, y \notin V(Z)$, then Z can be exactly reversed via one of the reversal procedures (a) or (b) indicated in Figure 10 below, depending on whether P is of type I (i.e., $z_j = z_{i+1}$, subscripts modulo 3) or type II ($z_j = z_{i-1}$), respectively. We shall show that T possesses a type I or type II path by reductio ad absurdum.

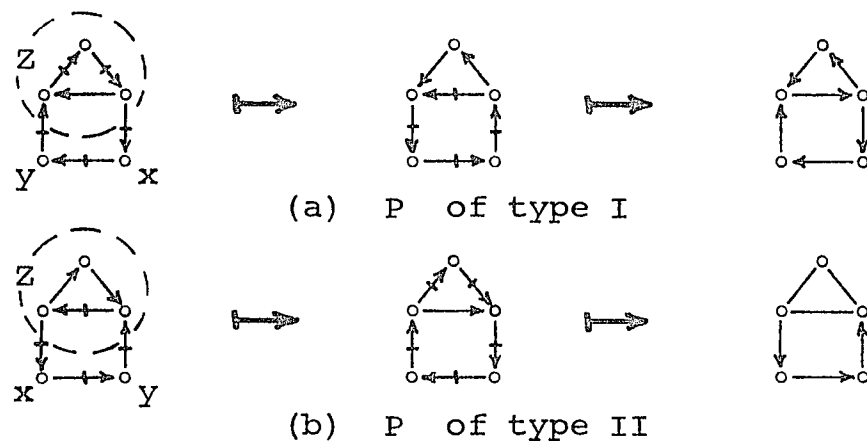


Figure 10.

Denote by X the set of vertices of T not on Z , and let $X_k = \{x \in X: x \text{ dominates exactly } k \text{ vertices of } Z\}$, for $k = 0, 1, 2, 3$. We claim first that $X_0 = \emptyset$, for suppose otherwise. Then some arc exits X_0 (by strongness of T), so there exist $x \in X_0$ and $y \in X_1 \cup X_2 \cup X_3$ such that $x \rightarrow y$. Now $y \rightarrow z_i \in V(Z)$ for some i , and then $z_{i-1} \rightarrow x \rightarrow y \rightarrow z_i$ is a path in T of type I, contradicting our assumption. Thus, $X_0 = \emptyset$. Dually, $X_3 = \emptyset$ as well. Hence at least one of X_1, X_2 has at least two vertices (as

$n \geq 6$), and we may suppose that X_1 does (by dualizing the tournament, if necessary). Let $x, y \in X_1$ with (say) $x \rightarrow y$. Now y dominates some vertex of Z , say z_0 for definiteness, and x , which dominates only one vertex of Z , must be dominated by z_1 or z_2 (or both). In the former case, however, $z_1 \rightarrow x \rightarrow y \rightarrow z_0$ is a path of type II, and in the latter case, $z_2 \rightarrow x \rightarrow y \rightarrow z_0$ is a path of type I. This final contradiction proves the lemma. \square

1.6.5 Theorem. Let T, U be strong n -tournaments, $\varphi: V(T) \rightarrow V(U)$ a bijective vertex-map, G the directed difference graph of φ relative to T, U , and for $k \geq 3$, let θ_k be the statement "There exists a sequence

$$T = T^0 \rightarrow T^1 \rightarrow \dots \rightarrow T^m \quad (m \geq 0)$$

of k -cycle-reversals such that $\varphi: V(T^m) \rightarrow V(U)$ is an isomorphism from T^m to U ." Then:

- i) θ_3 is true if and only if φ is score-preserving;
- ii) θ_4 is true if and only if φ is score-preserving and $|E(G)|$ is even;
- iii) in case $n \geq 6$, θ_5 is true if and only if φ is score-preserving.

Proof: Note that if θ_k is true for any $k \geq 3$, then φ must be score-preserving, since a k -cycle-reversal is a score-preserving operation.

i): θ_3 implies φ is score-preserving, as above. The converse is an easy corollary to the proof of Theorem 1.4.7.

ii): θ_4 implies φ is score-preserving (as above)

and $|E(G)|$ is even, the latter following from the observation that, when a 4-cycle (or any digraph with an even number of arcs) is reversed, the parity of the number of arcs in successive DDG's is the same (and the final DDG, G^m , has no arcs since φ is, by θ_4 , an isomorphism from T^m to U). The converse is an obvious corollary to the proof of Theorem 1.5.1.

iii): θ_5 implies φ is score-preserving, as above. The converse follows immediately from part i) and Lemma 1.6.4, by the refinement technique. \square

1.6.6 Remarks. Concerning Theorem 1.6.5:

a) The restriction that T, U be strong (in 1.6.5) is of no essential consequence, for the theorem applies to the strong components of the tournaments involved in any case. For example, if T, U are not assumed to be strong, conclusion ii) should be replaced by ii') θ_4 is true if and only if φ is score-preserving and $|E(G) \cap E(S)|$ is even for every strong component S of T .

b) By Proposition 1.3.2 ii), φ is score-preserving if and only if G is regular, i.e., a regular subdigraph of T .

c) By taking T, U to be strong tournaments on the same vertex-set and φ to be the identity map, the theorem tells one precisely when T may be identically transformed into U via k -cycle-reversals, $k = 3, 4, 5$ (e.g., in case $k = 4$, precisely when G is regular and has an even number of arcs), the only exception being $k = 5$ and $n \leq 5$ (which

we leave as an easy exercise for the reader).

1.7 Reversals of k-cycles.

We show first in this section that Ryser's theorem does not extend per se to the k-cycle, $k \geq 6$, that is, without imposing some restrictions on the tournaments and/or the score sequence involved (this will be done in Theorems 1.7.3 and 1.7.4 to follow). Indeed, the following example points out the severe limitations inherent in the case $k = n$ of Hamiltonian cycle-reversals.

1.7.1 Example. Fix an integer $n \geq 3$ and recall that TT_n denotes the transitive n-tournament. Let its vertices be labelled x_1, x_2, \dots, x_n so that x_i dominates x_j if $i < j$. For any path P in TT_n from x_1 to x_n , let $TT_n(P)$ denote the tournament obtained from TT_n by reversing the path P . We use special notation for two of these tournaments:

$$T^*T_n = TT_n(x_1 \rightarrow x_n) \quad \text{and}$$

$$T_*T_n = TT_n(x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow \dots \rightarrow x_n) .$$

(When $n = 5$, these are the tournaments of Figure 9 (a).)

For notational convenience, write the path P in the form

$$P: x_1 \rightarrow P_0 \rightarrow x_n$$

by allowing (possibly empty) subpaths P_0 of P . Let

$$Q: x_1 \rightarrow Q_0 \rightarrow x_n$$

be another such path in TT_n . We shall call $\{P, Q\}$ a

complementary path pair (or CPP) if $V(P_0) \cap V(Q_0) = \emptyset$ and $V(P_0) \cup V(Q_0) = \{x_2, x_3, \dots, x_{n-1}\}$.

For any paths P, Q in TT_n (from x_1 to x_n) for which $\{P, Q\}$ is a CPP, it is straightforward to verify the following comments:

a) $TT_n(P)$ has score sequence $(1, 1, 2, 3, \dots, n-3, n-2, n-2)$ and hence is strong (as $n \geq 3$); conversely (induct on n), any tournament with this score sequence has the form $TT_n(P')$ for some appropriate path P' .

b) $TT_n(P)$ has only one Hamiltonian cycle, namely

$$Z: x_1 \rightarrow Q_O \rightarrow x_n \rightarrow P_O^* \rightarrow x_1.$$

c) The directional dual of Z is

$$Z^*: x_1 \rightarrow P_O \rightarrow x_n \rightarrow Q_O^* \rightarrow x_1,$$

and this is (by b) above) the unique Hamiltonian cycle of $TT_n(Q)$.

d) Consequently, $TT_n(P) \rightsquigarrow TT_n(Q) \rightsquigarrow TT_n(P)$ via Hamiltonian cycle-reversals (i.e., $Z \rightsquigarrow Z^* \rightsquigarrow Z$), but no other tournaments are reachable from $TT_n(P)$ via such reversals. (The situation is depicted in Figure 11 below.)

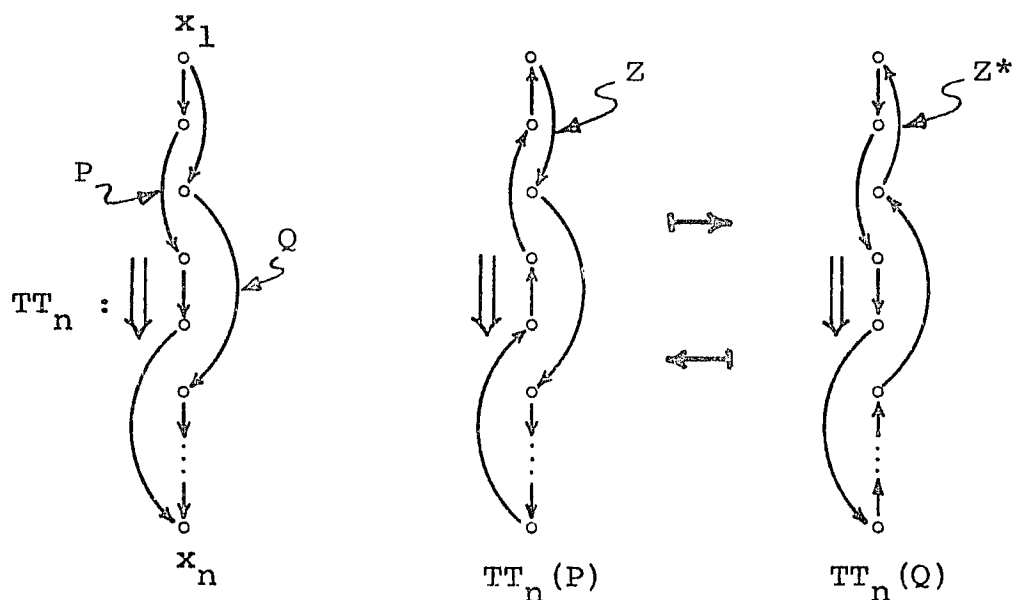


Figure 11. Limitations of Hamiltonian cycle-reversals

It follows that the C_n -classes (i.e., the equivalence classes of Hamiltonian cycle-reversals) of n -tournaments with score sequence $(1,1,2,3,\dots,n-3,n-2,n-2)$ are precisely $\{TT_n(P), TT_n(Q)\}$, where $\{P,Q\}$ ranges over all CPP's of TT_n . Evidently, $\{x_1 \rightarrow x_n, x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_n\}$ is a CPP, so $\{T^*T_n, T_{\#}^*T_n\}$ is one such C_n -class. Now $T^*T_5 \not\approx T_{\#}^*T_5$ and these are the only two (nonisomorphic) 5-tournaments with the score sequence in question. However, when $n \geq 6$, it is easily seen that there are more than two such n -tournaments (e.g., T^*T_n , $T_{\#}^*T_n$, and $TT_n(x_1 \rightarrow x_3 \rightarrow x_n)$ are nonisomorphic), their exact number being 2^{n-4} (valid for $n \geq 4$, note that this is the number of subsets of $\{x_3, x_4, \dots, x_{n-2}\}$).

In connection with the above, it bears mentioning that the tournaments $TT_n(P)$ form a proper subclass of those which admit a unique Hamiltonian cycle (for instance, the leftmost tournament of Figure 9 (b) has this property). This larger class has been enumerated by R.J. Douglas (see [23, p. 21]) and M.R. Garey, the latter having shown in [8] that the number of such n -tournaments, $n \geq 4$, is F_{2n-6} , the $(2n-6)^{\text{th}}$ term of the Fibonacci sequence $1,1,2,3,5,\dots$. An open problem is to determine the C_n -classes within this larger class (this appears to be quite difficult, as are most problems concerning Hamiltonian cycle-reversals; we expound on these and similar difficulties in subsection 1.7.5 below).

The first of our results (Theorem 1.7.3 below) on

k -cycle-reversals, where $k \geq 6$ is arbitrary, puts a sufficient condition on n in terms of k to insure that all strong n -tournaments which share any given score sequence fall into the same C_k -class. It is highly doubtful, however, that the condition $n \geq 8k - 27$ in the theorem is the best possible lower bound. Although the foregoing example shows that one cannot allow $n = k$, it is conceivable that $n \geq k + 1$ insures the desired conclusion. Before stating and proving the theorem, we introduce the following handy notation which will be used throughout this work.

1.7.2 Notation. Let T be a tournament and $X \subseteq V(T)$. Let $I_T(X) = \bigcap_{x \in X} I_T(x)$, the set of vertices of T that dominate every vertex of X , and (dually) $O_T(X) = \bigcap_{x \in X} O_T(x)$. We shall write $I_T(x, y, z)$ instead of the more cumbersome $I_T(\{x, y, z\})$, and similar notation will be used for other subsets of $V(T)$. For any subdigraph D of T , we let $I_T(D) = I_T(V(D))$, and dually for outsets. As usual, we shall drop the subscript " T " when the meaning is clear (but, to avoid ambiguity, this will be the only subscript omitted) and use analogous notation for more general directed graphs.

1.7.3 Theorem. Let T and U be strong n -tournaments with the same score sequence and suppose that $n \geq 8k - 27$, where $k \geq 6$ is an integer. Then $T \sim U$ via k -cycle-reversals.

Proof: As in the proof of the 5-cycle theorem via the 4-cycle theorem and the refinement technique, it will be

sufficient to show that any 4-cycle in T , say

$$Z: z_0 \rightarrow z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow z_0 ,$$

may be exactly reversed by a sequence of k -cycle-reversals.

For assume that Z cannot be so reversed. We shall derive a contradiction.

Observe first that T contains no path of the form $z_i \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_{k-3} \rightarrow z_{i+2}$ (subscripts of the z_i 's read modulo 4), where $\{x_1, x_2, \dots, x_{k-3}\} \cap V(Z) = \emptyset$, for otherwise Z is exactly reversed by the sequence

$$z_i \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_{k-3} \rightarrow z_{i+2} \rightarrow z_{i+3} \rightarrow z_i ,$$

$$z_{i+2} \rightarrow x_{k-3} \rightarrow \dots \rightarrow x_2 \rightarrow x_1 \rightarrow z_i \rightarrow z_{i+1} \rightarrow z_{i+2}$$

(Figure 6 shows the special case $k = 5$). Let us call such a path in T a "good" path. As T contains no such path, it follows easily that $|O(z_i) \cap I(z_{i+2})| \leq k-3$, for $i = 0, 1, 2, 3$, and so

$$(1) \quad \sum_{i=0}^3 |O(z_i) \cap I(z_{i+2})| \leq 4(k-3) .$$

We argue on similar grounds that $|O(Z)| \leq k-5$, for suppose not. Then (as T is strong) some vertex x of the terminal strong component of $\langle O(Z) \rangle$ dominates some vertex $y \in V(T) - O(Z)$. Now y dominates z_i for some $z_i \in V(Z)$, and x is the terminal vertex of some $(k-5)$ -path within $\langle O(Z) \rangle$, say $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_{k-4} = x$. But then $z_{i+2} \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_{k-4} \rightarrow y \rightarrow z_i$ is a good path in T , a contradiction. Therefore, $|O(Z)| \leq k-5$, as claimed. Dually, $|I(Z)| \leq k-5$; hence

$$(2) \quad |O(Z)| + |I(Z)| \leq 2(k-5) .$$

Next, we show that $|I(z_0, z_2) \cap O(z_1, z_3)| \leq k-3$, for

suppose otherwise. Let $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_{k-2}$ be a $(k-3)$ -path in $\langle I(z_0, z_2) \cap O(z_1, z_3) \rangle$. Then (we are assuming $k \geq 6$) the sequence

$$\begin{aligned} z_1 &\rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_{k-2} \rightarrow z_0 \rightarrow z_1, \\ x_{k-3} &\rightarrow x_{k-4} \rightarrow \dots \rightarrow x_1 \rightarrow z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow x_{k-3}, \\ x_{k-3} &\rightarrow z_2 \rightarrow z_1 \rightarrow x_1 \rightarrow \dots \rightarrow x_{k-5} \rightarrow z_0 \rightarrow x_{k-2} \rightarrow x_{k-3}, \\ z_0 &\rightarrow x_{k-5} \rightarrow \dots \rightarrow x_1 \rightarrow z_1 \rightarrow z_2 \rightarrow x_{k-3} \rightarrow z_3 \rightarrow z_0 \end{aligned}$$

of k -cycle-reversals exactly reverses Z , contradicting our assumption. Hence $|I(z_0, z_2) \cap O(z_1, z_3)| \leq k-3$, and dually. Therefore,

$$(3) \quad |I(z_0, z_2) \cap O(z_1, z_3)| + |O(z_0, z_2) \cap I(z_1, z_3)| \leq 2(k-3).$$

Now the left-hand sides of inequalities (1), (2), and (3) account for every vertex of T (some may be counted twice by the LHS of (1)). Thus,

$$n \leq 4(k-3) + 2(k-5) + 2(k-3) = 8k - 28,$$

but this contradicts the hypothesis that $n \geq 8k - 27$.

The theorem follows. \square

Our second result on k -cycle-reversals applies to many more cycle lengths (all save that of Hamiltonian cycles) than does the previous theorem. Its conclusion is achieved by imposing a "regularity" condition on the score sequence (s_1, s_2, \dots, s_n) sufficiently restrictive to insure high connectivity of all tournaments sharing this score sequence. Insofar as we make a study of tournament connectivity in Chapter 3, its proof (which depends in part on recent results of C. Thomassen) will be deferred to §3.3 to follow.

1.7.4 Theorem. Let T and U be n -tournaments with the same score sequence (s_1, s_2, \dots, s_n) and suppose that $n \geq 2(s_n - s_1) + 10$. Then $T \sim U$ via k -cycle-reversals, for each $k = 3, 4, \dots, n-1$.

1.7.5 Summary. In this subsection we offer some general comments concerning cycle-reversals and refinement, especially in regards to the above Theorems 1.7.3 and 1.7.4, and list below several problems and questions which are (to the author's knowledge) entirely unresolved.

a) As remarked previously, the bound on n in Theorem 1.7.3 is probably capable of improvement. The same may be said regarding the bound $n \geq 2(s_n - s_1) + 10$ in Theorem 1.7.4. Just what are the best possible bounds?

b) What are the equivalence classes of Hamiltonian cycle-reversals? Specifically, does the conclusion of Theorem 1.7.4 hold for $k = n$? While Example 1.7.1 answers the first question for a particular type of score sequence, it is not even known whether any two regular tournaments of a given (odd) order n are equivalent via C_n -reversals (note that 1.7.4 insures that they are equivalent via C_k -reversals, $3 \leq k < n$, provided $n \geq 11$).

c) Having established the 4-cycle theorem, our method of proof in deriving subsequent results on k -cycle-reversals, $k > 4$, was to show that (under appropriate conditions) the k -cycle refines the 4-cycle. Unfortunately, our adopted proof technique fails to handle the case $k = n$ (a glance at Figure 6 should supply the reason for this), which leads

us to ask: In what class of n -tournaments does the n -cycle refine the 4-cycle?

d) A general problem regarding cycle refinement may be stated thusly: Given integers $n \geq i, j \geq 3$, determine whether or not C_i refines C_j in the class \mathcal{S}_n of strong n -tournaments. We have already obtained partial results in this connection (for instance, Lemmas 1.4.5, 1.6.1, and 1.6.4, and the proof of Theorem 1.7.3), and it is certainly likely that further results would supply answers to some of the questions listed above.

Chapter 2

REVERSALS OF DIGRAPHS

2.1 Introduction.

In this chapter we continue our investigation of the reversal problem by determining the equivalence classes of n -tournaments relative to $\{D, D^*\}$ -reversals, for a variety of digraphs D other than directed cycles. The more important of the digraphs considered herein are the "generalized" 4-cycles (see Figure 12 and Theorem 2.3.6), directed k -paths (Theorem 2.4.5, due to K.B. Reid, is furnished with an alternative proof), and antidirected k -paths (Theorem 2.5.1). For the most part, our results completely characterize the equivalence classes, the only notable exception being those of antidirected Hamiltonian paths (see Remarks 2.5.4).

The proof techniques utilized are essentially those of Chapter 1 (i.e., properties of the directed difference graph, used in conjunction with the refinement technique), together with a new tool, the symmetric difference graph, developed in §2.2. Although our main results are contained in §2.3-§2.5, we have applied these techniques (particularly heavy use is made of refinement) to other types of digraphs, such as " k -claws" and the "3-cycle with sticker", in §2.6.

Lastly, in §2.7, we offer some closing comments on the current state of the reversal problem, including a number of open questions associated with it.

2.2 The symmetric difference graph.

The underlying graph of an asymmetric digraph D is the (undirected) graph Δ obtained from D by "ignoring" the orientations of its arcs, i.e., $V(\Delta) = V(D)$ and $E(\Delta) = \{xy: \overrightarrow{xy} \in E(D)\}$. Conversely, any digraph having Δ as its underlying graph is called an orientation of Δ ; for example, any n -tournament is an orientation of K_n , the complete n -graph. For many digraphs D , it will be seen that the equivalence classes re $\{D, D^*\}$ -reversals depend (in whole or in part) on its underlying graph Δ , which suggests the following notion, due (in essence) to the authors of [24].

2.2.1 Definition. Let T, U be n -tournaments and let $\varphi: V(T) \rightarrow V(U)$ be a bijective vertex-map. Let G be the directed difference graph associated with this map, i.e., $G = \text{DDG}_{\varphi}(T, U)$, and let H be the underlying graph of G . We call H the symmetric difference graph (or SDG) of φ re T, U , and write $H = \text{SDG}_{\varphi}(T, U)$.

Clearly, an edge xy is present in H if and only if φ reverses the orientation of the arc joining the vertices x, y of T , and the parameter $|E(H)|$ ($= |E(G)|$) measures how "close" φ is to being an isomorphism from T to U , just as in §1.3. Indeed, many properties of G carry over to H by simply ignoring orientations; for instance, Proposition 1.3.3 clearly holds with G, G_{π} , etc., replaced by, respectively, H, H_{π} , and so forth. To illustrate further, let us denote by $\text{td}_G(x)$ the total degree of any $x \in V(G)$,

that is, $td_G(x) = od_G(x) + id_G(x)$, and also let $N_H(x) = \{y \in V(H) : xy \in E(H)\}$, the neighborhood of $x \in V(H)$, and $d_H(x) = |N_H(x)|$, the degree of x (in H). (Naturally, we shall use similar notation for other digraphs and graphs.) Then we have $d_H(x) = td_G(x) = od_G(x) + id_G(x) \equiv od_G(x) - id_G(x) \pmod{2}$, and this holds for every $x \in V(T)$. Hence, by Proposition 1.3.2 ii), for every $x \in V(T)$,

$$d_H(x) \equiv od_T(x) - od_U(\varphi(x)) \pmod{2}.$$

Consequently, if φ preserves score parity (in particular, if φ is score-preserving), then H is even degreed in the sense of [20]: every vertex has even degree in H . In this case, H is the edge-disjoint union of (undirected) cycles, and conversely. Also in accordance with [20], we shall call any graph or digraph even if it has an even number of edges (or arcs, respectively), and odd otherwise.

2.3 Reversals of generalized 4-cycles.

If Δ is a graph, any orientation of Δ will be called a generalized Δ . There are four essentially different ways to orient the undirected 4-cycle, that is, four nonisomorphic generalized 4-cycles, and these are diagrammed and labelled in Figure 12 below.

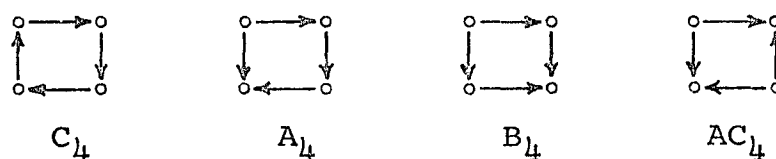


Figure 12. The generalized 4-cycles

All of the generalized 4-cycles are self-dual, clearly,

and we determine in this section the equivalence classes of each of them. Of course, this has already been done in §1.5 for C_4 , the directed 4-cycle. For the others, note that their classes are contained in the TT_3 -classes (since reversal of any one of them preserves score parity, this follows from Theorem 1.4.9); we shall show that this containment is not proper when n is sufficiently large.

2.3.1 Lemma. In the class $\mathcal{T}_{\geq 5}$ of tournaments of order at least 5, A_4 refines all of the generalized 4-cycles.

Proof: Let T be an n -tournament, $n \geq 5$. We show first that

$$(1) \quad A_4 \text{ refines } AC_4.$$

To do this, let

$$AC_4: z_1 \rightarrow z_2 \leftarrow z_3 \rightarrow z_4 \leftarrow z_1$$

be a copy of AC_4 in T (often called an antidirected 4-cycle), and fix a vertex $x \in V(T) - V(AC_4)$. Now if $x \rightarrow z_1$ and $z_3 \rightarrow x$ (dominance in T unless otherwise stated), then the sequence

$$z_3 \rightarrow x \rightarrow z_1 \rightarrow z_2 \leftarrow z_3,$$

$$z_1 \rightarrow x \rightarrow z_3 \rightarrow z_4 \leftarrow z_1$$

of A_4 -reversals exactly reverses AC_4 , and we are done.

We reach a similar conclusion if $x \rightarrow z_3$ and $z_1 \rightarrow x$, so we may assume that x "agrees" at z_1, z_3 in the sense that x either dominates both or is dominated by both z_1, z_3 .

Dually, we see that x agrees at z_2, z_4 (without loss of generality). By symmetry, we may assume $z_2 \rightarrow z_4$ as well.

Now if all of the z_i 's dominate x , then the sequence

$$\begin{aligned} z_3 &\rightarrow z_2 \rightarrow z_4 \rightarrow x \leftarrow z_3, \\ z_1 &\rightarrow x \rightarrow z_4 \rightarrow z_2 \leftarrow z_1, \\ z_2 &\rightarrow x \rightarrow z_1 \rightarrow z_4 \leftarrow z_2, \\ x &\rightarrow z_3 \rightarrow z_4 \rightarrow z_2 \leftarrow x \end{aligned}$$

exactly reverses AC_4 . The case in which x dominates all of the z_i 's is entirely dual, so there remain only two possibilities to consider. First, suppose that x dominates z_1, z_3 and is dominated by z_2, z_4 . Then the sequence

$$\begin{aligned} z_2 &\rightarrow x \rightarrow z_3 \rightarrow z_4 \leftarrow z_2, \\ x &\rightarrow z_1 \rightarrow z_4 \rightarrow z_2 \leftarrow x, \\ z_1 &\rightarrow z_2 \rightarrow z_4 \rightarrow x \leftarrow z_1, \\ z_3 &\rightarrow x \rightarrow z_4 \rightarrow z_2 \leftarrow z_3 \end{aligned}$$

reverses AC_4 . Finally, suppose that x dominates z_2, z_4 and is dominated by z_1, z_3 . Then the sequence

$$\begin{aligned} z_3 &\rightarrow x \rightarrow z_2 \rightarrow z_4 \leftarrow z_3, \\ z_1 &\rightarrow z_4 \rightarrow z_2 \rightarrow x \leftarrow z_1, \\ x &\rightarrow z_1 \rightarrow z_2 \rightarrow z_4 \leftarrow x, \\ z_4 &\rightarrow x \rightarrow z_3 \rightarrow z_2 \leftarrow z_4 \end{aligned}$$

exactly reverses AC_4 . This proves (1).

Next, we show that

$$(2) \quad \{A_4, AC_4\} \text{ refines } B_4.$$

Let

$$B_4: z_1 \rightarrow z_2 \rightarrow z_3 \leftarrow z_4 \leftarrow z_1$$

be a copy of B_4 in T . By symmetry, we may assume that

$z_2 \rightarrow z_4$. If $z_1 \rightarrow z_3$, then the sequence

$$z_1 \rightarrow z_3 \leftarrow z_2 \rightarrow z_4 \leftarrow z_1 ,$$

$$z_4 \rightarrow z_3 \rightarrow z_1 \rightarrow z_2 \leftarrow z_4$$

of $\{A_4, AC_4\}$ -reversals exactly reverses B_4 . On the other hand, if $z_3 \rightarrow z_1$, then the sequence

$$z_2 \rightarrow z_3 \rightarrow z_1 \rightarrow z_4 \leftarrow z_2 ,$$

$$z_1 \rightarrow z_3 \leftarrow z_4 \rightarrow z_2 \leftarrow z_1$$

of $\{A_4, AC_4\}$ -reversals reverses B_4 . This proves (2).

Finally, it is easy to show that

$$(3) \quad \{A_4, AC_4, B_4\} \text{ refines } C_4 .$$

We conclude from (1), (2), and (3) that A_4 refines $\{A_4, AC_4, B_4, C_4\}$ in $\mathfrak{T}_{\geq 5}$ (by transitivity of refinement and the obvious fact that A_4 refines itself), as was to be shown. \square

2.3.2 Remark. It is also easy to see that TT_3 , the transitive triple, refines all of the generalized 4-cycles, in any class of tournaments.

2.3.3 Lemma. In the class $\mathfrak{T}_{\geq 5}$, $\{B_4, AC_4\}$ refines $\{A_4, C_4\}$.

Proof: Let T be an n -tournament, $n \geq 5$. We show first that

$$(4) \quad \{B_4, AC_4\} \text{ refines } C_4 .$$

Let

$$C_4: z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow z_4 \rightarrow z_1$$

be a 4-cycle in T . By relabelling C_4 , if necessary, we may suppose that $z_1 \rightarrow z_3$ and $z_2 \rightarrow z_4$. Then the sequence

$$z_1 \rightarrow z_2 \rightarrow z_4 \leftarrow z_3 \leftarrow z_1 ,$$

$$z_2 \rightarrow z_3 \leftarrow z_4 \rightarrow z_1 \leftarrow z_2 ,$$

$$z_3 \rightarrow z_4 \rightarrow z_2 \leftarrow z_1 \leftarrow z_3$$

of $\{B_4, AC_4\}$ -reversals exactly reverses C_4 , which proves statement (4).

In order to complete the proof, we need only show

$$(5) \quad \{B_4, AC_4, C_4\} \text{ refines } A_4 .$$

To this end, let

$$A_4: z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow z_4 \leftarrow z_1$$

be a copy of A_4 in T , and choose $x \in V(T) - V(A_4)$. We distinguish four cases, depending on dominance between opposite vertices on A_4 .

Case 1: $z_1 \rightarrow z_3$ and $z_2 \rightarrow z_4$.

Then the sequence

$$z_1 \rightarrow z_3 \leftarrow z_2 \rightarrow z_4 \leftarrow z_1 ,$$

$$z_3 \rightarrow z_1 \rightarrow z_2 \leftarrow z_4 \leftarrow z_3$$

exactly reverses A_4 .

Case 2: $z_3 \rightarrow z_1$ and $z_4 \rightarrow z_2$.

Then the sequence

$$z_3 \rightarrow z_1 \rightarrow z_2 \leftarrow z_4 \leftarrow z_3 ,$$

$$z_1 \rightarrow z_3 \leftarrow z_2 \rightarrow z_4 \leftarrow z_1$$

exactly reverses A_4 (just as in case 1, the extra vertex x is not needed).

Case 3: $z_3 \rightarrow z_1$ and $z_2 \rightarrow z_4$.

We examine the subcases (a) $x \rightarrow z_1, z_3$; (b) $z_1, z_3 \rightarrow x$;
(c) $z_1 \rightarrow x$ and $x \rightarrow z_3$; and (d) $z_3 \rightarrow x$ and $x \rightarrow z_1$.

First, if subcase (a) holds, then the sequence

$$\begin{aligned}
x &\rightarrow z_1 \rightarrow z_4 \leftarrow z_3 \leftarrow x , \\
z_2 &\rightarrow z_3 \rightarrow z_1 \leftarrow z_4 \leftarrow z_2 , \\
z_1 &\rightarrow z_2 \leftarrow z_4 \rightarrow z_3 \leftarrow z_1 , \\
z_1 &\rightarrow x \leftarrow z_3 \rightarrow z_4 \leftarrow z_1
\end{aligned}$$

of $\{B_4, AC_4\}$ -reversals exactly reverses A_4 (see Figure 13 below, where we have indicated which type of digraph is being reversed at each stage).

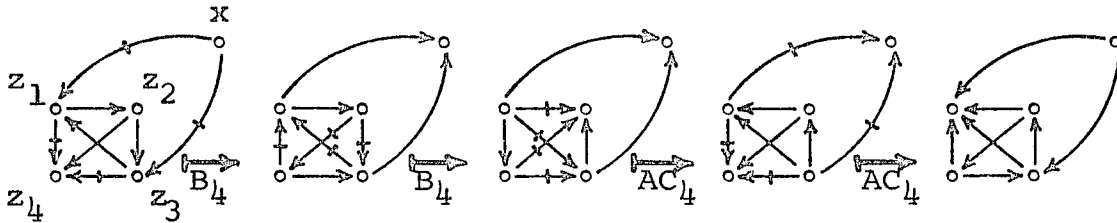


Figure 13.

In subcase (b), an entirely analogous sequence reverses A_4 , the succession being AC_4, B_4, AC_4, B_4 , where these digraphs have, respectively, the same underlying graphs as the sequence B_4, B_4, AC_4, AC_4 of Figure 13. Now suppose subcase (c) holds. Then the sequence

$$\begin{aligned}
z_1 &\rightarrow x \rightarrow z_3 \leftarrow z_2 \leftarrow z_1 , \\
z_2 &\rightarrow z_1 \leftarrow z_3 \rightarrow z_4 \leftarrow z_2 , \\
z_1 &\rightarrow z_4 \rightarrow z_2 \leftarrow z_3 \leftarrow z_1 , \\
x &\rightarrow z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow x
\end{aligned}$$

(which is a B_4, AC_4, B_4, C_4 sequence) exactly reverses A_4 . Finally, if subcase (d) holds, then an entirely analogous sequence (a C_4, AC_4, B_4, B_4 sequence, with the same respective underlying graphs as in (c)) exactly reverses A_4 . This finishes the argument for case 3, since one of the subcases (a)-(d) must hold.

Case 4: $z_1 \rightarrow z_3$ and $z_4 \rightarrow z_2$.

This reduces to case 3 by taking the directional dual, then relabelling A_4 via the permutation $(z_1 z_4)(z_2 z_3)$.

This proves statement (5) and completes the proof of the lemma. \square

2.3.4 Lemma. In the class $\mathcal{T}_{\geq 7}$, B_4 refines AC_4 , and therefore refines all of the generalized 4-cycles.

Proof: The second assertion in the lemma follows from the first part and Lemma 2.3.3. To prove that B_4 refines AC_4 in $\mathcal{T}_{\geq 7}$, let T be an n -tournament, $n \geq 7$, and let

$$AC_4: z_1 \rightarrow z_2 \leftarrow z_3 \rightarrow z_4 \leftarrow z_1$$

be an antidirected 4-cycle in T , labelled so that

$z_1 \rightarrow z_3$ and $z_2 \rightarrow z_4$ (in T) . Let $W = V(T) - V(AC_4)$.

Now if there exists $x \in W$ such that $z_1 \rightarrow x$ and $x \rightarrow z_3$, then the sequence

$$z_1 \rightarrow z_2 \rightarrow z_4 \leftarrow z_3 \leftarrow z_1 ,$$

$$z_1 \rightarrow x \rightarrow z_3 \leftarrow z_4 \leftarrow z_1 ,$$

$$z_3 \rightarrow x \rightarrow z_1 \leftarrow z_2 \leftarrow z_3 ,$$

$$z_3 \rightarrow z_1 \rightarrow z_2 \leftarrow z_4 \leftarrow z_3$$

of B_4 -reversals exactly reverses AC_4 . Also, if there exists $x \in W$ with $z_3 \rightarrow x$ and $x \rightarrow z_1$, then the sequence above with the second and third reversals interchanged exactly reverses AC_4 , so again we have accomplished our objective. We may therefore assume that $W = W_1 \cup W_2$, where $W_1 = W \cap I(z_1, z_3)$ and $W_2 = W \cap O(z_1, z_3)$. Since $|W| \geq 3$, one of the sets W_1, W_2 must contain at least two vertices. Suppose $|W_1| \geq 2$. Let $x, y \in W_1$ with $x \rightarrow y$. Then the

sequence

$$\begin{aligned}
 z_1 &\rightarrow z_2 \rightarrow z_4 \leftarrow z_3 \leftarrow z_1 , \\
 x &\rightarrow y \rightarrow z_1 \leftarrow z_3 \leftarrow x , \\
 z_3 &\rightarrow x \rightarrow z_1 \leftarrow z_2 \leftarrow z_3 , \\
 z_1 &\rightarrow x \rightarrow z_3 \leftarrow z_4 \leftarrow z_1 , \\
 z_1 &\rightarrow y \rightarrow x \leftarrow z_3 \leftarrow z_1 , \\
 z_3 &\rightarrow z_1 \rightarrow z_2 \leftarrow z_4 \leftarrow z_3
 \end{aligned}$$

of B_4 -reversals exactly reverses AC_4 . On the other hand, if $|W_2| \geq 2$, a very similar sequence suffices to reverse AC_4 . The lemma follows. \square

2.3.5 Lemma. In the class $\mathcal{T}_{\geq 9}$, AC_4 refines B_4 , and therefore refines all of the generalized 4-cycles.

Proof: Again, the second assertion in the lemma is just a consequence of the first. Let T be an n -tournament, $n \geq 9$, let

$$B_4: z_1 \rightarrow z_2 \rightarrow z_3 \leftarrow z_4 \leftarrow z_1$$

be a copy of B_4 in T , and let $W = V(T) - V(B_4)$. If there exists $x \in W$ such that x dominates both z_2, z_4 , then the sequence

$$\begin{aligned}
 x &\rightarrow z_2 \leftarrow z_1 \rightarrow z_4 \leftarrow x , \\
 z_2 &\rightarrow x \leftarrow z_4 \rightarrow z_3 \leftarrow z_2
 \end{aligned}$$

of AC_4 -reversals exactly reverses B_4 . If both z_2, z_4 dominate x , then a sequence dual to the above one reverses B_4 , so we may suppose no such vertex x exists, i.e., $W = W_1 \cup W_2$, where we set $W_1 = W \cap I(z_2) \cap O(z_4)$ and $W_2 = W \cap O(z_2) \cap I(z_4)$. Now $|W| \geq 5$, so one of W_1, W_2 must contain at least three vertices, and we may assume

that $|W_1| \geq 3$, by directional duality. Let $x_1, x_2, x_3 \in W_1$ be distinct, and choose $y \in W - \{x_1, x_2, x_3\}$. Now y either (a) dominates at least two of x_1, x_2, x_3 , or (b) is dominated by two of x_1, x_2, x_3 . Suppose (a) is the case; for definiteness, assume that y dominates x_1 and x_2 .

Then the sequence

$$\begin{aligned} y &\rightarrow x_1 \leftarrow z_4 \rightarrow x_2 \leftarrow y, \\ x_1 &\rightarrow z_2 \leftarrow z_1 \rightarrow z_4 \leftarrow x_1, \\ z_2 &\rightarrow z_3 \leftarrow z_4 \rightarrow x_1 \leftarrow z_2, \\ x_1 &\rightarrow y \leftarrow x_2 \rightarrow z_4 \leftarrow x_1 \end{aligned}$$

of AC_4 -reversals exactly reverses B_4 . Case (b) is entirely analogous, so the lemma follows. \square

We are now prepared to state and prove the main result of this section. It should be pointed out that this result bears the same relationship to Theorem 1.4.9 as the 4-cycle theorem (i.e., 1.5.1) does to the 3-cycle theorem (1.4.7). The proof given below should reinforce this analogy.

2.3.6 Theorem. In the class \mathcal{T}_n of all n -tournaments, the TT_3 -classes (characterized by Theorem 1.4.9) coincide precisely with:

- i) the A_4 -classes, for all $n \geq 5$;
- ii) the B_4 -classes, for all $n \geq 7$;
- iii) the AC_4 -classes, for all $n \geq 9$.

Proof: We prove all parts of the theorem simultaneously, and our proof will parallel that of the 4-cycle theorem, but with directions "ignored."

Let $X_4 \in \{A_4, B_4, AC_4\}$. Then X_4 refines the entire set $\{A_4, B_4, AC_4, C_4\}$ of generalized 4-cycles, for the appropriate range of n (depending on X_4), by the foregoing lemmas. We shall determine the X_4 -classes.

Let T, U be n -tournaments and $\varphi: V(T) \rightarrow V(U)$ a bijection which preserves score parity, i.e., for all $x \in V(T)$,

$$(6) \quad \text{od}_T(x) \equiv \text{od}_U(\varphi(x)) \pmod{2}.$$

Let $G = \text{DDG}_\varphi(T, U)$ and $H = \text{SDG}_\varphi(T, U)$. By interchanging the rôles of two vertices of T with the same score parity, if necessary, we may assume that G and H are even, by Proposition 1.3.3. By (6) and the comments in §2.2, H is even degreed, and therefore consists of an arc-disjoint union of (undirected) cycles. It follows that G , being an orientation of H , is an arc-disjoint union of generalized cycles. As in the proof of the 4-cycle theorem, we induct on $|E(G)| \geq 0$. If $|E(G)| = 0$, then φ is an isomorphism from T to U , so $T \sim U$ via (the null sequence of) X_4 -reversals. So assume $|E(G)| > 0$. Then G contains a generalized cycle Z of some length $\ell \geq 3$. As before, we distinguish three cases.

Case 1: $\ell = 2k$ is even.

Then Z may be exactly reversed via a sequence of $\{A_4, B_4, AC_4, C_4\}$ -reversals (for this is merely Corollary 1.4.6 with directions "ignored," i.e., its generalized counterpart), and hence via a sequence of X_4 -reversals, by the refinement technique. This "lowers" $|E(G)|$ by $2k$, retains property (6) because the underlying graph of X_4 is

even degreed, and leaves the parity of $|E(G)|$ unchanged (i.e., even) because X_4 is even.

Case 2: $\ell = 2k+1 \geq 5$.

Case 3: G contains two arc-disjoint generalized 3-cycles.

Just as in case 1, these two cases are the generalized counterparts of cases 2 and 3 in the proof of the 4-cycle theorem, and are obtained from the latter simply by ignoring directions. In all cases, $|E(G)|$ can be lowered via a sequence of X_4 -reversals, and this eventually converts φ into an isomorphism from some resulting tournament T^m to U as before. Thus, the X_4 -classes are characterized by Theorem 1.4.9, that is, they coincide with the TT_3 -classes, for the appropriate range of n . \square

In view of Remark 2.3.2, it should be clear that any one of the parts i), ii), or iii) of the generalized 4-cycle theorem implies Theorem 1.4.9, at least when n is large enough. But this is not all that Theorem 2.3.6 implies, for it is the directed analogue of [20, Corollary 12], and, as such, implies this latter result (when n is sufficiently large).

We also point out that the lower bound $n \geq 5$ in part i) of 2.3.6 is easily seen to be best possible, but that this is not necessarily the case in ii) and iii). It is conceivable that ii) holds for $n = 5$ or 6 , for instance, and that iii) holds for $n = 8$, though these would be the smallest possible such values. To see this in the case of

the latter, note that the regular 7-tournament QRT_7 (often called the quadratic residue tournament of order 7, as in [3; 23]) depicted in Figure 14 below contains no copy of AC_4 (making it unique among 7-tournaments in this respect), hence cannot be equivalent via AC_4 -reversals to any other tournament. Since, however, all three of the diagrammed 7-tournaments have the same number (zero) of vertices of even score, part iii) fails when $n = 7$ (and for smaller n as well).

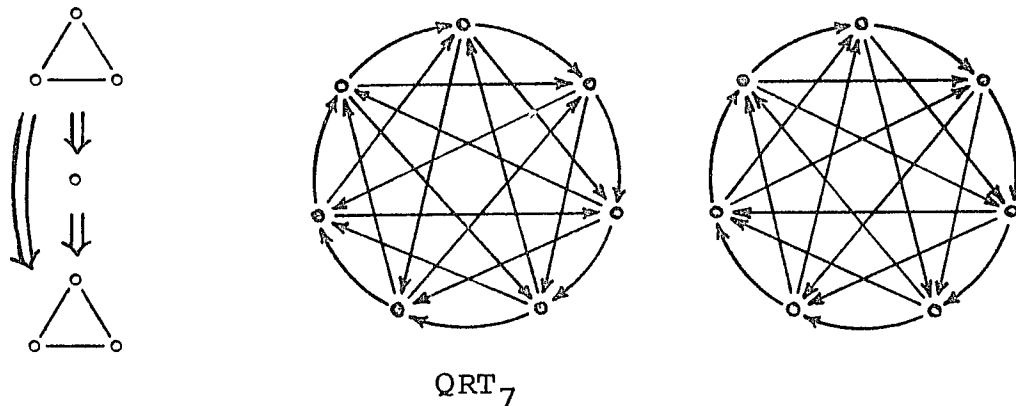


Figure 14.

We close this section with the comment that, with the exception of the directed 5-cycle, the equivalence classes of the generalized 5-cycles are unknown, and conjecture that they coincide eventually (i.e., for n sufficiently large) with the TT_3 -classes (certainly each such class is contained in a TT_3 -class, by refinement). Similar remarks apply to the generalized k -cycles, $k > 5$, although the situation here is further complicated by the fact (see, e.g., [14]) that not all of these digraphs are self-dual. Even so, perhaps some headway can be made using simple refinement ideas, such as

the (easily proven) observation that the generalized k -cycle of the form $z_1 \rightarrow z_2 \rightarrow \dots \rightarrow z_k \leftarrow z_1$ refines the directed $(2k-2)$ -cycle, in any tournament.

2.4 Reversals of paths: Reid's theorem.

We now show how the result in [22] of K.B. Reid cited earlier, which is stated as Theorem 2.4.5 below, can be obtained from the 4-cycle theorem, via the refinement technique and a few preliminary lemmas.

The following useful notion serves the purpose of measuring how "close" (in some restricted sense) a tournament comes to being regular.

2.4.1 Definition. For any tournament T , let $s_{\max}(T)$ ($s_{\min}(T)$) denote its maximum (resp., minimum) score, and set $q(T) = s_{\max}(T) - s_{\min}(T)$. The integer $q(T)$ is the quasiregularity of T , and we usually abbreviate the notation to s_{\max} , s_{\min} , q when T is understood.

Clearly (see §1.1), $q(T) = 0$ if and only if T is regular (in which case its order n must be odd), and $q(T) = 1$ if and only if n is even and T is near-regular.

2.4.2 Lemma. Let T be an n -tournament such that $q(T) \geq 2$, and let k be any integer satisfying $2 \leq k \leq n-1$. Then T contains a k -path from some vertex u of maximum score to some vertex v of minimum score.

Before beginning the proof, we remark that the lemma

can be strengthened somewhat (among other things, the condition that $q(T) \geq 2$ may be dropped), but it will be more appropriate to postpone this task to the next chapter.

Proof of lemma: Fix n , and we shall induct on k .

Let x and y be any vertices of T having maximum and minimum score, respectively. Now $O(x) \cap I(y) \neq \emptyset$, for if not, then $O(x) \subseteq O(y) \cup y$, and therefore

$$s_{\max} = od(x) \leq od(y) + 1 = s_{\min} + 1,$$

but this implies that $q(T) = s_{\max} - s_{\min} \leq 1$, a contradiction. Thus, letting $w \in O(x) \cap I(y)$, we see that $x \rightarrow w \rightarrow y$ is a 2-path in T , so the conclusion of the lemma holds for $k = 2$ with $u = x$, $v = y$. This anchors the induction.

Now assume $2 \leq k < n-1$, and suppose that there exists a k -path

$$P: x = x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_{k-1} \rightarrow x_k = y$$

in T from x to y . We shall show that there is a $(k+1)$ -path in T from u to v , where $od(u) = od(x)$ and $od(v) = od(y)$. Denote by W the set of vertices of T not on P , and let $I = I(x, y) \cap W$, $O = O(x, y) \cap W$, $B = O(x) \cap I(y) \cap W$, and $C = I(x) \cap O(y) \cap W$.

First, suppose there exists $w \in W$ such that $x_i \rightarrow w$ and $w \rightarrow x_j$, for some $0 \leq i < j \leq k$. Then, by fixing i and choosing the least j satisfying this property, we see that

$$x = x_0 \rightarrow \dots \rightarrow x_{j-1} \rightarrow w \rightarrow x_j \rightarrow \dots \rightarrow x_k = y$$

is a $(k+1)$ -path from x to y , and we are done (take

$u = x$, $v = y$) . Following [2] , we shall refer to this process as replacement , and we may certainly assume no such vertex w exists. In particular, we may assume $B = \emptyset$. Another obvious consequence of replacement is that, without loss of generality, I dominates $V(P)$ (i.e., each vertex of I dominates all the x_i 's) and $V(P)$ dominates \emptyset .

Next, we shall argue that $I = \emptyset$, for suppose otherwise. Let $w \in I$. Since $w \rightarrow x$ and $od(w) \leq od(x)$, there exists a vertex z of T such that $x \rightarrow z$ and $z \rightarrow w$. Now $z \notin V(P)$, because w dominates $V(P)$. Hence

$$x \rightarrow z \rightarrow w \rightarrow x_2 \rightarrow \dots \rightarrow y$$

is a suitable $(k+1)$ -path, and again we are finished (this is also a form of replacement, since we are just replacing the 2-path $x \rightarrow x_1 \rightarrow x_2$ with the 3-path $x \rightarrow z \rightarrow w \rightarrow x_2$) . We may therefore assume $I = \emptyset$, as claimed. Dually, we may assume $\emptyset = \emptyset$.

We now have that $C = W$, and this set is nonempty (as $k+1 < n$) , so pick a vertex $w \in C$. If $x_1 \rightarrow w$, then $x_i \rightarrow w$ for all $i > 1$ (otherwise, replacement applies), and hence $o(w) \subseteq (C \cup x) - w$, so $od(w) \leq |C| + 1 - 1 = |C|$. Also, $C \subseteq o(y)$, so $|C| \leq od(y)$. Combining these inequalities, we see that $od(w) \leq od(y)$, and hence (since y has minimum score) $od(w) = od(y)$. But then

$$x \rightarrow x_1 \rightarrow \dots \rightarrow y \rightarrow w$$

is a $(k+1)$ -path, and we are done (take $u = x$, $v = w$) .

Dually, if $w \rightarrow x_{k-1}$, either replacement applies or we obtain an appropriate $(k+1)$ -path with $u = w$, $v = y$. We may

consequently assume that $w \rightarrow x_1$ and $x_{k-1} \rightarrow w$.

Now suppose $x \rightarrow x_i$ and its predecessor $x_{i-1} \rightarrow y$, for some index i such that $2 \leq i \leq k-1$. Then

$$x \rightarrow x_i \rightarrow \dots \rightarrow x_{k-1} \rightarrow w \rightarrow x_1 \rightarrow \dots \rightarrow x_{i-1} \rightarrow y$$

is a suitable $(k+1)$ -path (see Figure 15 below), so we are done in this case. So suppose no such index i exists, and

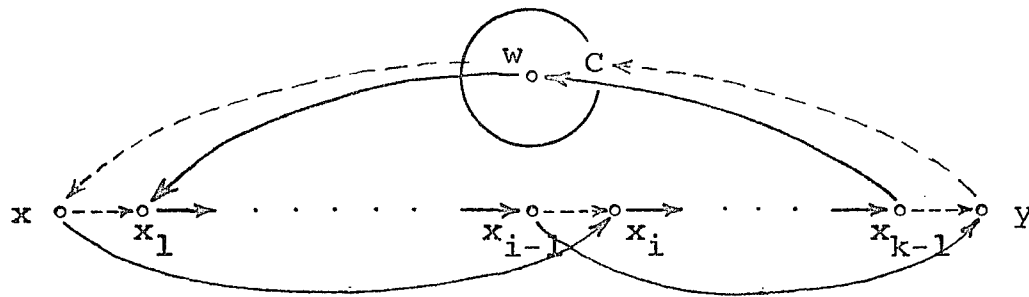


Figure 15.

let $X = \{x_i: x \rightarrow x_i, 2 \leq i \leq k-1\}$, $Y = \{x_{i-1}: x_i \in X\}$. Then x dominates X (by definition of X), so y dominates Y (since no index i exists, as above). Since x dominates no vertex of C , we have that $od(x) \leq |X| + 2$ (allowing for x_1 and possibly $y \in O(x)$); also, since y dominates both Y and C , and since (clearly) $|X| = |Y|$, we have that $od(y) \geq |Y| + |C| = |X| + |C| \geq |X| + 1$. It follows that

$$2 \leq q(T) = od(x) - od(y) \leq (|X| + 2) - (|X| + 1) = 1,$$

a contradiction. We must, therefore, conclude that an index i (as above) exists, so the path in Figure 15 suffices for the induction step. The lemma follows. \square

Now suppose we are given an n -tournament T , say with score sequence (s_1, s_2, \dots, s_n) , and an integer k satisfying $2 \leq k \leq n-1$. Then, assuming $q(T) (= s_n - s_1) \geq 2$,

Lemma 2.4.2 insures the existence of a k -path in T , say P , from some vertex u , $od_T(u) = s_n$, to some v , $od_T(v) = s_1$. If the path P is reversed, we obtain a new tournament, say T^1 , and clearly $od_{T^1}(u) = s_n - 1$, $od_{T^1}(v) = s_1 + 1$, but the scores of all other vertices remain unaltered. Thus, the scores in T^1 are more nearly equal than in T . If $q(T^1) \geq 2$, then the lemma can be used again, an appropriate k -path reversed, and a tournament T^2 obtained, in which the scores are more nearly equal than in T^1 . To be precise, if one uses (among other choices) the integer-valued measure

$$f(T^j) = \sum_{i=1}^n |s_i(T^j) - \frac{n-1}{2}|$$

(where $s_i(T^j)$ = the i^{th} score of T^j , and $T = T^0$) of the phrase "more nearly equal," one easily deduces that

$$f(T) > f(T^1) > f(T^2) > \dots,$$

so the process cannot be continued indefinitely. Eventually, therefore, a tournament T^m is obtained, for which $q(T^m) \leq 1$, i.e., T^m is regular or near-regular (depending on the parity of n). We have proved:

2.4.3 Corollary. For any n -tournament T and integer k , $2 \leq k \leq n-1$, there exists an n -tournament R such that

- i) $T \sim R$ via k -path-reversals;
- ii) $q(R) \leq 1$, i.e., R is regular or near-regular. \square

Of course, the corollary may be phrased differently, to wit: Every k -path-class includes at least one regular or near-regular tournament.

We acknowledge several suggestions of K. Wayland which were instrumental in proving the following result.

2.4.4 Lemma. Let R^1 and R^2 be regular or near-regular n -tournaments (depending on the parity of n), and assume $n \geq 11$. Then $R^1 \sim R^2$ via k -path-reversals, for each integer k , $2 \leq k \leq n-1$.

Proof: Since R^1, R^2 have the same score sequence (see §1.1), $R^1 \sim R^2$ via 4-cycle-reversals (Theorem 1.5.1). In view of this, and via the refinement technique, it will be sufficient to show that an arbitrary 4-cycle in R^1 , say

$$Z: z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow z_4 \rightarrow z_1,$$

can be exactly reversed via a sequence of k -path-reversals. Since this can obviously be done if $k = 2$, we shall assume $k \geq 3$ in the following, and we may also assume, by symmetry, that $z_1 \rightarrow z_3$ and $z_2 \rightarrow z_4$.

By using the facts that $n \geq 11$ and $q(R^1) \leq 1$, it is easy to see (and this is a direct application of Proposition 3.3.1 of the (independent) Chapter 3) that if $W \subseteq V(R^1)$ with $|W| \leq 3$, then $R^1 - W$, the (sub-)tournament obtained from R^1 by deleting the vertices of W , together with all incident arcs (thus, $R^1 - W = \langle V(R^1) - W \rangle$), is strong. In particular, $R^1 - \{z_2, z_3, z_4\}$ is a strong $(n-3)$ -tournament, and hence has a spanning path, i.e., an $(n-4)$ -path, starting at its vertex z_1 . As $0 \leq k-3 \leq n-4$, let

$$P: z_1 \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_{k-3}$$

be an initial subpath of this path (of length $k-3$, possibly zero). Similarly, $R^1 - \{z_1, z_2, z_4\}$ is strong, and by

dualizing the previous procedure, we can find within this tournament a $(k-3)$ -path

$$Q: y_1 \rightarrow y_2 \rightarrow \dots \rightarrow y_{k-3} \rightarrow z_3$$

ending at z_3 . Now, successive reversals of the k -paths

$$z_2 \rightarrow z_3 \rightarrow z_4 \rightarrow z_1 \rightarrow x_1 \rightarrow \dots \rightarrow x_{k-3},$$

$$x_{k-3} \rightarrow \dots \rightarrow x_1 \rightarrow z_1 \rightarrow z_3 \rightarrow z_2 \rightarrow z_4,$$

$$y_1 \rightarrow \dots \rightarrow y_{k-3} \rightarrow z_3 \rightarrow z_1 \rightarrow z_4 \rightarrow z_2,$$

$$z_4 \rightarrow z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow y_{k-3} \rightarrow \dots \rightarrow y_1$$

exactly reverses Z (see Figure 16; it should be noted that

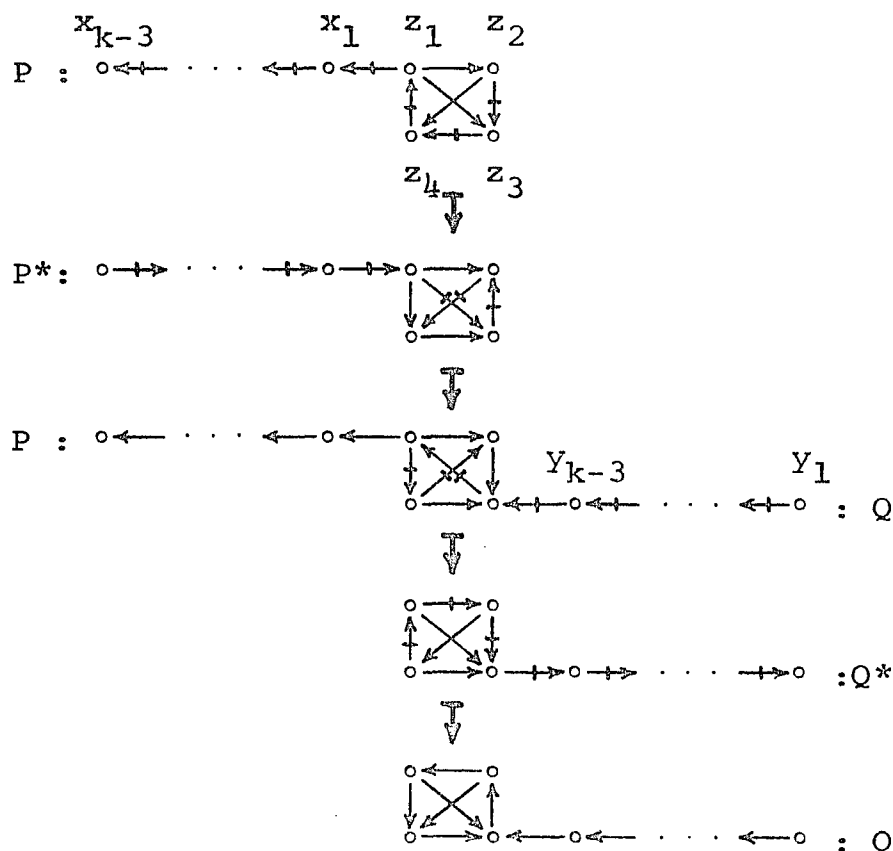


Figure 16. Refinement of a 4-cycle by a k -path whenever P, Q have arcs in common, the first reversal will destroy Q , but in this case the second will restore it; similarly, P is restored at the end of the sequence). As this accomplishes our objective, the proof is complete. \square

The proof given above does not illustrate one noteworthy point of which the reader may be unaware, namely that Reid's theorem (stated below) is quite easy to prove in the range $1 \leq k \leq n-3$, even without resort to the 4-cycle theorem. To see this, the case $k = 1$ is obvious, as then single arcs are being reversed, and the case $k = 2$ is easily handled using the DDG. Having accomplished this, assume $3 \leq k \leq n-3$, let T be an n -tournament, and let

$$P: x \rightarrow y \rightarrow z$$

be any 2-path in T . By the refinement technique, it will be sufficient to show that some sequence of k -path-reversals exactly reverses P . To do this, pick k distinct vertices y_1, \dots, y_k not on the path P , indexed such that $y_1 \rightarrow y_2 \rightarrow \dots \rightarrow y_k$ is a path in T . Now if either $y \rightarrow y_1$ or $y_k \rightarrow y$, then T contains a $(k-1)$ -path (in fact, a k -path) having y as one of its endvertices, and P is easily reversed in such a circumstance. So assume that $y_1 \rightarrow y$ and $y \rightarrow y_k$. If $y_k \rightarrow y_1$, T again has a $(k-1)$ -path with y as an endvertex, so assume that $y_1 \rightarrow y_k$. Choose i least such that $y \rightarrow y_i$. Then the sequence

$$\begin{aligned} & y_1 \rightarrow \dots \rightarrow y_{i-1} \rightarrow y \rightarrow y_i \rightarrow \dots \rightarrow y_k, \\ & x \rightarrow y \rightarrow y_{i-1} \rightarrow \dots \rightarrow y_1 \rightarrow y_k \rightarrow \dots \rightarrow y_{i+1}, \\ & y_{i+1} \rightarrow \dots \rightarrow y_k \rightarrow y_1 \rightarrow \dots \rightarrow y_{i-1} \rightarrow y \rightarrow z, \\ & y_k \rightarrow \dots \rightarrow y_i \rightarrow y \rightarrow y_{i-1} \rightarrow \dots \rightarrow y_1 \end{aligned}$$

of k -path-reversals exactly reverses P , which accomplishes our objective. The reader should note that the first reversal (in the above sequence) just brings about the desir-

able circumstance encountered previously, and does not disturb the arcs \vec{xy}, \vec{yz} , so that it can be "undone" at the end of the sequence. We have seen this trick many times before, of course, and the interested reader is invited to try his hand at proving Reid's theorem for $k = n-2$ and/or $k = n-1$ by using this same technique. The latter enterprise alone, if successful, would furnish (via [22, Thm. 4.6]) still another proof of the theorem, for the one given below is based instead on Corollary 2.4.3 and Lemma 2.4.4.

2.4.5 Theorem (K.B. Reid, 1973). Any two n -tournaments are equivalent via k -path-reversals, for each integer k , $1 \leq k \leq n-1$.

Proof: As noted above, the conclusion is obvious if $k = 1$, so assume throughout that $k \geq 2$. If $n \geq 11$, then Corollary 2.4.3 and Lemma 2.4.4 together imply the theorem, clearly (k -path-equivalence is an equivalence relation), so assume $1 \leq n \leq 10$. In view of the remarks above, we may also assume that $k = n-2$ or $n-1$. Depending on n , we distinguish three cases.

Case 1: $1 \leq n \leq 5$.

For each value of n in this range, there is a unique regular or near-regular n -tournament (they are $T_1, T_2, C_3, ST_4, RT_5$), and any two n -tournaments are equivalent to this one, by the corollary.

Case 2: $n = 6$ or 7 .

There are five near-regular 6-tournaments (see the appendix to [19]); we leave to the reader the task of showing

they are all 4- and 5-path-equivalent. There are three regular 7-tournaments (two of which are pictured in Figure 14), and it can easily be shown that they are all 5- and 6-path-equivalent. (In neither instance is a reversal procedure much more complicated than that shown in Figure 9 required.) Once this has been done, the corollary again applies.

Case 3: $n = 8, 9$, or 10 .

It is inadvisable to do these cases directly (since, for instance, B. Alspach (unpublished work) has determined that there are fifteen regular 9-tournaments), so we shall resort to a different argument. For definiteness, we shall suppose $n = 10$ in the sequel, and leave the cases $n = 8, 9$ to the reader, as these can be handled in a highly similar manner.

Let R be a near-regular 10-tournament, and let

$$Z: z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow z_4 \rightarrow z_1$$

be a 4-cycle in R . By symmetry, we may assume $z_1 \rightarrow z_3$ and $z_2 \rightarrow z_4$. As in the proof of Lemma 2.4.4, it is clearly sufficient to show that Z can be exactly reversed via k -path-reversals. Let the vertices of R not on Z be x_1, x_2, \dots, x_6 , arranged so that $x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_6$ is a path in R , and let $R^i = \langle z_i, x_1, x_2, \dots, x_6 \rangle$, for each of $i = 1, \dots, 4$. The sequence depicted in Figure 16 reverses Z if both R^1, R^3 are strong, or even if one or more fails to be strong, but z_1 lies in the initial component of R^1 and z_3 in the terminal component of R^3 . (As may easily be checked, a similar sequence exists which reverses Z in

case z_2 lies in the initial component of R^2 and z_4 in the terminal component of R^4). Let us first assume that z_1 is not in the initial component of R^1 . Then there is an index i such that $\{x_1, \dots, x_i\}$ dominates $\{z_1, x_{i+1}, \dots, x_6\}$. Using the fact that indegrees and outdegrees in R are either 4 or 5, it is easy to see that $i = 3$ or 4. Suppose that $i = 3$. Because x_1 dominates x_2, z_1, x_4, x_5, x_6 and $\text{od}(x_1) \leq 5$, it must be dominated by x_3, z_2, z_3, z_4 . Since now $x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_1$ forms a 3-cycle in R , similar reasoning applies to x_2 and x_3 as well, so $\{z_2, z_3, z_4\}$ dominates $\{x_1, x_2, x_3\}$. Hence

$$P: z_2 \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_6$$

is a spanning path of R^2 , so z_2 is in the initial component of R^2 . We shall show that z_4 is in the terminal component of R^4 , and this will finish the case $i = 3$.

If $x_6 \rightarrow z_4$, then

$$Q_1: x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_6 \rightarrow z_4$$

is a path in R^4 , ending at z_4 , and we are done (i.e., P, Q_1 provide a reversal procedure for Z), so assume that $z_4 \rightarrow x_6$. Then z_4 dominates z_1, x_1, x_2, x_3, x_6 , and hence must be dominated by the rest of the vertices; in particular, $x_5 \rightarrow z_4$. Also, x_6 is dominated by x_1, x_2, x_3, x_5, z_4 , so $x_6 \rightarrow x_4$. Thus,

$$Q_2: x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_6 \rightarrow x_4 \rightarrow x_5 \rightarrow z_4$$

is a path of the appropriate type. The case $i = 4$, being quite similar to the above, is omitted. Finally, the case in which z_3 is not in the terminal component of R^3 is

directionally dual to the one just considered, provided the sequence in Figure 16 is replaced by its reverse sequence (i.e., the reversals are performed in the opposite order) and Z is appropriately relabelled.

The theorem follows. \square

2.5 Reversals of antirected paths.

Recall from §2.3 that a generalized path P is a digraph whose underlying graph is a path (undirected), say $x_0, x_0x_1, x_1, x_1x_2, \dots, x_{k-1}x_k, x_k$. If the arcs of P satisfy the antirectedness condition

$x_i \rightarrow x_{i-1}$ if and only if $x_i \rightarrow x_{i+1}$ ($i = 1, \dots, k-1$), i.e., if consecutive arcs are oppositely oriented, then P is said to be antirected and is called an antirected path, or antirected k-path to emphasize that its length (i.e., number of arcs) is k . An antirected Hamiltonian (or spanning) path in a tournament (or digraph) T is one that includes all the vertices of T . An antirected cycle is defined in an analogous manner, and usage of such terms as antirected k-cycle, antirected spanning cycle, etc., should be obvious.

From the antirectedness condition, it should be clear that antirected cycles always have even length and are self-dual. Moreover, antirected $2k$ -paths come in two varieties (they are duals), viz.,

$$\text{Type 1: } x_0 \rightarrow x_1 \leftarrow x_2 \rightarrow \dots \leftarrow x_{2k},$$

$$\text{Type 2: } x_0 \leftarrow x_1 \rightarrow x_2 \leftarrow \dots \rightarrow x_{2k},$$

whereas those of odd length are unique (up to isomorphism) and self-dual. Since we consider in this work only reversals of self-dual collections of asymmetric digraphs (so as to obtain an equivalence relation), when we speak of "antidirected $2k$ -path-reversals," we shall mean that, at each stage of the reversal process, either of the Types 1 or 2 antidirected $2k$ -paths is (allowed to be) reversed.

The best-known of the existence results concerning antidirectedness in tournaments is a theorem of B. Grünbaum [10], which states that, with three exceptions, every (non-trivial) tournament has an antidirected Hamiltonian path. The exceptional tournaments are C_3 (Figure 1), RT_5 (the unique regular 5-tournament), and QRT_7 (Figure 14, center). Using an elegant proof technique, M. Rosenfeld in [26] gave a new, simpler proof of Grünbaum's theorem, and strengthened it to [26, Thm. 2 and the remark following proof]: Every vertex in a tournament T of order $n \geq 9$ is an endvertex of some antidirected Hamiltonian path in T . It is this latter result which, together with the special case $k = 2$ of Reid's theorem, yields an easy proof of:

2.5.1 Theorem. If $n \geq 11$, then any two n -tournaments are equivalent via antidirected k -path-reversals, for each integer k , $1 \leq k \leq n-2$.

Proof: Let T be an n -tournament, $n \geq 11$, and let

$$P: x \rightarrow y \rightarrow z$$

be any 2-path in T . Fix k as in the theorem. By the refinement technique and the case in Reid's theorem men-

tioned above, it is sufficient to show that P can be exactly reversed via antidirected k -path-reversals. The result is obvious if $k = 1$, so assume $k \geq 2$ (to avoid notational difficulties), and consider the $(n-2)$ -tournament $T - \{x, z\}$. Since $n-2 \geq 9$, the vertex y is an endvertex (the initial vertex, we may assume) of some antidirected spanning path Q in $T - \{x, z\}$, by Rosenfeld's theorem. Since Q has length $n-3$, it has one of the forms

$$\begin{aligned} Q^1: & y \rightarrow y_1 \leftarrow y_2 \rightarrow \dots y_{n-3} \\ \text{or} \quad Q^2: & y \leftarrow y_1 \rightarrow y_2 \leftarrow \dots y_{n-3}, \end{aligned}$$

where y_1, y_2, \dots, y_{n-3} are the vertices of T not on P .

Define antidirected $(k-1)$ -subpaths Q_{k-1}^1, Q_{k-1}^2 of these by

$$\begin{aligned} Q_{k-1}^1: & y \rightarrow y_1 \leftarrow y_2 \rightarrow \dots y_{k-1}, \\ Q_{k-1}^2: & y \leftarrow y_1 \rightarrow y_2 \leftarrow \dots y_{k-1}, \end{aligned}$$

and note that the notation makes sense since $1 \leq k-1 \leq n-3$.

If $Q = Q^1$, then $z \leftarrow Q$ is antidirected, and it is easy to see that successive reversals of the antidirected k -paths

$$\begin{aligned} z & \leftarrow Q_{k-1}^1, \\ x & \rightarrow Q_{k-1}^2 \end{aligned}$$

exactly reverses P ; if instead $Q = Q^2$, then the (reverse) sequence

$$\begin{aligned} x & \rightarrow Q_{k-1}^2, \\ z & \leftarrow Q_{k-1}^1 \end{aligned}$$

exactly reverses P . (Note that we could have invoked duality here.) This completes the proof. \square

The theorem just proved says nothing when $n < 11$, even when k is small. The next result will remedy that.

2.5.2 Theorem. Any two n -tournaments are equivalent via antidirected k -path-reversals, for each integer k , $1 \leq k \leq 2\lfloor \frac{n}{2} \rfloor - 3$, where $\lfloor . \rfloor$ denotes the greatest integer function.

Proof: We shall use the same technique as in the proof above, but with recourse only to Grünbaum's theorem. Let T be an n -tournament, and let

$$P: x \rightarrow y \rightarrow z$$

be a 2-path in T . We shall reverse P via antidirected k -path-reversals, and it is no loss of generality to assume $k = 2\lfloor \frac{n}{2} \rfloor - 3$, for the result for smaller k can be easily obtained by taking appropriate antidirected subpaths (as done in the previous proof).

Define a subtournament S of T as follows: if n is even, let $S = T - \{x, z\}$; otherwise, pick any $w \in V(T) - V(P)$ and let $S = T - \{x, z, w\}$. Note that S has order $2\lfloor \frac{n}{2} \rfloor - 2$, which is even, and therefore has, by Grünbaum's theorem, an antidirected spanning path (which is an antidirected k -path, in this case)

$$Q: y_0 \rightarrow y_1 \leftarrow y_2 \rightarrow \dots \rightarrow y_k.$$

Since $y \in V(Q)$, $y = y_i$ for some i . Now if $y_0 \rightarrow y_k$, then Q (and its vertex y) lies in the antidirected $(k+1)$ -cycle

$$Z: y_0 \rightarrow y_1 \leftarrow y_2 \rightarrow \dots \rightarrow y_k \leftarrow y_0,$$

whereupon P can be exactly reversed via the sequence in the previous proof, clearly. On the other hand, if $y_k \rightarrow y_0$, then we can create this desirable situation by first re-

versing Q , then utilizing the previous sequence to reverse P , and finally reversing Q^* (completing a double-reversal of Q). The particular case $k = 5$, $i = 3$ of this procedure is diagrammed in Figure 17 below.

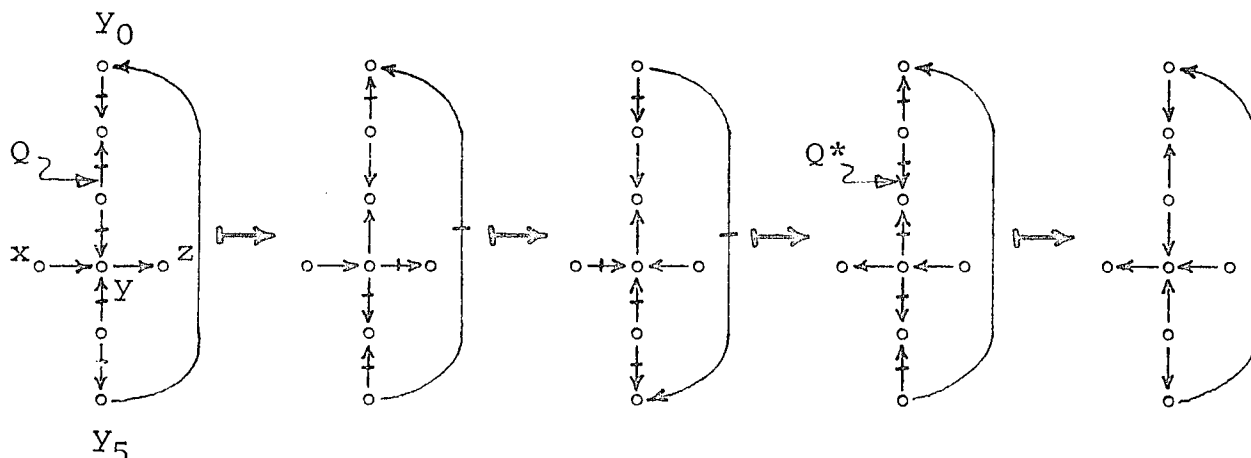


Figure 17. Refinement of a directed 2-path by an antidirected k -path

Since Q does not contain the arcs $\vec{x\bar{y}}, \vec{y\bar{z}}$, the procedure always suffices to reverse P . The proof is now complete. \square

Combining our results (Theorems 2.5.1 and 2.5.2) we obtain:

2.5.3 Corollary. Any two n -tournaments are equivalent via antidirected k -path-reversals, provided that:

- i) $k = 2$ and $n \geq 4$;
- ii) $k = 3$ and $n \geq 6$;
- iii) $k = 4$ or 5 and $n \geq 8$;
- iv) $k = 6$ or 7 and $n \geq 10$;
- v) $8 \leq k \leq n-2$ and $n \geq 11$. \square

(The case $k = 1$ is uninteresting, hence is omitted.)

2.5.4 Remarks. The bounds (on n in terms of k) in Corollary 2.5.3 are not necessarily best possible, except in part i) (since $C_3 \not\sim TT_3$). In fact, we have ascertained (jointly with K. Wayland) that when $k = 3$, $n \geq 4$ is sufficient, which leads us to believe that the bounds in parts iii)-v) are not best possible either. This problem has not been investigated, even for the smaller values of k .

Also open is the determination of the antidirected Hamiltonian path-classes (about which our results say nothing). We conjecture that, for $n \neq 3, 5, 7$, any two n -tournaments are equivalent via antidirected Hamiltonian path-reversals, although there may be a few more exceptional orders n (the given values are definitely ruled out by Grünbaum's theorem [10]).

Needless to say, nothing is known concerning reversals of antidirected k -cycles either, when $k \geq 6$ (and even), for Theorem 2.3.6 iii) only handles the case $k = 4$. The situation here is even more complex than in the case of directed cycles, perhaps because there are no known "stacking" results comparable with the directed case (in fact, there are tournaments, such as QRT_7 , that contain antidirected 6-cycles but no antidirected 4-cycle).

Some existence results regarding antidirected cycles might be of aid in connection with any of the problems just mentioned. Grünbaum in [10] conjectured that every tournament of even order $n \geq 10$ possesses an antidirected Hamiltonian cycle, and C. Thomassen [28] verified this for even

$n \geq 50$. Shortly thereafter, Rosenfeld [25] extended this result to even $n \geq 28$. (It should be mentioned that the remaining cases in a problem of this sort are virtually immune from attack by computers, unless an extremely efficient algorithm is employed, because of sheer numerousness, e.g., there are over nine million 10-tournaments and over 1.5×10^{11} of order 12 .)

A possible generalization of both Theorems 2.4.5 and 2.5.1 to include generalized paths would certainly be of interest. We conjecture that, for all n sufficiently large, any two n -tournaments are equivalent via $\{G, G^*\}$ -reversals, for each generalized path G of length $k \leq n-1$ (i.e., all possible lengths). We have verified this, via the refinement technique, for $k \leq 3$, and for most G 's with length 4 , but it seems very likely that general progress in this direction will have to await further existence results, for present-day theory is rather limited in this area. Of possible use is a theorem, due to R. Forcade [7, Cor. 2.2], which insures that every tournament of order $n = 2^m$ ($m \in \mathbb{Z}^+$) contains generalized paths of all possible types (Forcade actually proved a stronger result, viz. [7, Thm. 2.1]; for an excellent discussion of these and related results, conjectures, etc., see [23]). Unfortunately, almost all other orders n remain to be settled, although it is conjectured by Rosenfeld [25] (and others) that the paths in question do, in fact, exist (with finitely many exceptions, as above).

2.6 Reversals of other digraphs.

All digraphs that we have considered up to this point, i.e., in connection with the reversal problem, had one common characteristic: their underlying graphs were either cycles or paths. The techniques we have developed, however, are general enough to apply to many other "small" digraphs.

In this section we list, without proof, the $\{D, D^*\}$ -classes, for various digraphs D , that have not been hitherto considered. Only the refinement technique, i.e., in conjunction with our previous results, need be used to compute these classes in all save (possibly) one case, that of the digraph σ (below).

Let LL_k , Λ_k ($k = 1, 2, \dots$), σ , and GP_3 be the digraphs depicted in Figure 18 below.

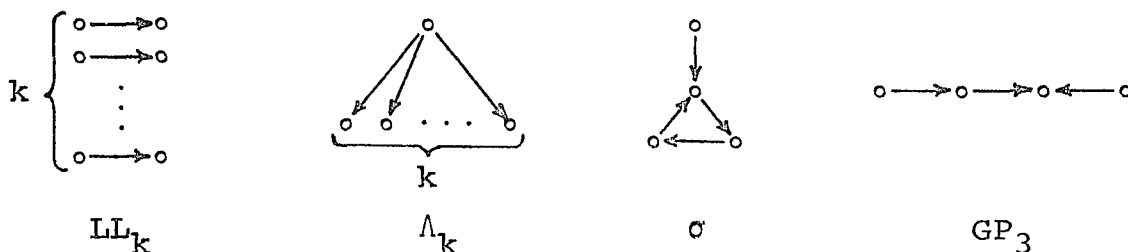


Figure 18.

The following paragraphs describe the equivalence classes of each of these.

LL_k -classes: Universal for all $n \geq 2k$ (i.e., all n -tournaments fall into the same class); trivial for all $n < 2k$ (i.e., each tournament forms a class by itself).

$\{\Lambda_k, \Lambda_k^*\}$ -classes: Universal for all $n \geq 2k$, and this is best possible (by considering a regular $(2k-1)$ -tourna-

ment); the classes when $n < 2k$ have not been computed.

$\{\sigma, \sigma^*\}$ -classes: All n -tournaments except the transitive one fall into the same class (for all n); TT_n is in a class by itself, clearly.

$\{GP_3, GP_3^*\}$ -classes: Universal for all $n \geq 4$, and otherwise trivial, clearly.

Not all "small" digraphs are handled as easily as these. For example, the TT_k -classes, $k \geq 4$, are unknown.

2.7 Open problems.

We have listed below three problems subsidiary to the reversal problem, the answers (or partial answers) to any of which might prove interesting. We have included cross-references by section number (§) and subsection (§§) where there is duplication with previous commentary.

Problem 1. Determine the equivalence classes of the following:

Hamiltonian cycles (§§1.7.5)

k -cycles, $k \geq 6$, not otherwise covered by our previous results (§§1.7.3-1.7.5)

antidirected k -cycles, even $k \geq 6$ (§§2.5.4)

generalized k -cycles, $k \geq 5$ (§§2.3.6)

antidirected Hamiltonian paths (§§2.5.4)

generalized k -paths, $k \geq 4$ (§§2.5.4)

Information concerning any of these digraphs is quite likely to yield information concerning the others, by (of course) the refinement technique.

Problem 2. Given a digraph D , what properties are preserved under D -reversals? An answer for a particular D whose classes are known (or partially known) would obviously yield the information that all tournaments in a given class share the preserved properties (or lack of them), and thus allow one to apply reversal theory to other aspects of tournament theory. For example, k -strongness (see §3.2) is clearly preserved by cycle-reversals (of any length), so this property is completely determined by score sequence, according to any of Theorems 1.4.7, 1.5.1, or 1.6.3; of course, this is easily seen to be true by other considerations, too. Are there any nontrivial examples?

Problem 3. Given two digraphs D_1, D_2 , in what class of tournaments does D_1 refine D_2 ? This has been completely determined in [20] for the undirected case. Until an answer to this question is forthcoming (and we have seen how it depends, at least in part, on "existence" theory, as in §2.5), no general answer to the reversal problem seems possible.

Chapter 3

PANCONNECTIVITY IN TOURNAMENTS

3.1 Introduction.

This chapter is independent of the rest of the work. Our interest here is with tournaments possessing rather restrictive path connectivity properties, especially the second property in the following definition, taken from [29].

3.1.1 Definition. An n -tournament (or n -digraph) T is weakly panconnected if, given any distinct vertices x, y of T and any integer k , $3 \leq k \leq n-1$, there is a k -path in T from x to y or from y to x (i.e., connecting x, y); T is strongly panconnected if, given any distinct vertices x, y of T and any integer k , $3 \leq k \leq n-1$, there are k -paths in T from x to y and from y to x .

No (nontrivial) necessary and sufficient conditions are known for a tournament to be strongly panconnected. In recent work, however, C. Thomassen [29] has completely characterized weakly panconnected tournaments, as well as deriving a number of other interesting results in this same paper. We outline some of these, including his characterization, in §3.2, for we shall rely heavily upon them in subsequent sections.

In §3.3, we present a proof of Theorem 1.7.4 which is based on Thomassen's characterization, and generalize Lemma 2.4.2 in §3.4. Our main result is Theorem 3.5.3 of the fi-

nal section, which gives a sufficient condition for a tournament to be strongly panconnected. Specifically, it will be shown that an n -tournament T is strongly panconnected if $n \geq \max \{ 5q(T) + 4, 2q(T) + 13 \}$. This condition, being less restrictive than most that have been imposed in the past (see, e.g., [2; 15; 29]), broadens considerably the class of tournaments known to be strongly panconnected, and the bound is also best possible when the quasiregularity $q(T) \geq 3$.

3.2 Weak panconnectivity: Thomassen's characterization.

Standard in tournament literature are two natural ways of measuring "how strong" a tournament is, in terms of a positive integer k . A tournament T is k -strong if, for every nonempty, proper subset U of its vertices, there are at least k arcs exiting U (equivalently, entering U); T is k -connected if, for every subset U of fewer than k of its vertices, the subtournament $T - U$ is strong.

It is clear that the following concepts all coincide in the case of tournaments: strong, 1-strong, 1-connected. It is also not difficult to show that k -connected implies k -strong, but not conversely when $k > 1$. In [9], Goldberg and Moon have listed some of the basic properties enjoyed by k -strong tournaments. It is easy to see that a tournament with score sequence (s_1, s_2, \dots, s_n) is k -strong if and only if $\sum_{i=1}^j s_i \geq \binom{j}{2} + k$, for $j = 1, 2, \dots, n-1$, for one thing. Consequently, k -strongness is purely a function of the score

sequence. In contrast, k -connectedness is not determined completely by the score sequence parameter, as can be seen by considering appropriate examples.

Two path connectivity notions closely allied to those of Definition 3.1.1 are these: a tournament T is weakly Hamiltonian connected if, given any distinct vertices x, y of T , there is a Hamiltonian path in T connecting x, y , and is strongly Hamiltonian connected if (analogously) the direction of the path can be specified.

In [29], C. Thomassen has made a study of all four of these path connectivity properties. Some of his results (mentioned in the introduction) that we shall need follow.

Theorem [29, Thms. 2.3 and 3.2, combined]. For any tournament T with at least three vertices, these are equivalent:

- i) T is weakly Hamiltonian connected;
- ii) T is weakly panconnected;
- iii) T satisfies a), b), and c) below:
 - a) T is strong
 - b) For each $x \in V(T)$, $T - x$ has at most two components
 - c) $T \not\cong XT_6, XT_6^*$ (see Figure 19 for XT_6).

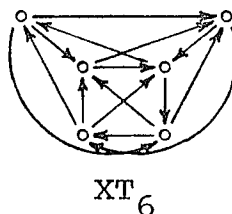


Figure 19.

We remark that the result that i) implies ii) in this theorem is achieved by showing that if x, y are vertices of an n -tournament, $n \geq 5$, connected by a Hamiltonian path, then x, y are also connected by an $(n-2)$ -path. An immediate consequence is

Corollary [29, Cor. 3.3]. A 2-connected tournament is weakly panconnected unless it is isomorphic to XT_6 or XT_6^* .

One more result of Thomassen's that we shall find useful, particularly in the proof of our main result, is

Theorem [29, Cor. 3.4]. If x, y, z are distinct vertices of a strong n -tournament and k is any integer satisfying $1 \leq k \leq n-1$, then there is a k -path connecting two of x, y, z .

3.3 Connectivity.

By definition, the connectivity (also called the strong connectivity) of a tournament T is the largest integer k such that T is k -connected. Recall from Definition 2.4.1 that $q(T)$, the quasiregularity of T , is $s_{\max} - s_{\min}$, the maximum difference between its scores. Nearly all of our results will be stated in terms of this last parameter, and we begin by determining the minimum connectivity of a tournament, given its quasiregularity.

3.3.1 Proposition. If T is an n -tournament such that $n \geq 2q(T) + 3k - 2$ ($k \in \mathbb{Z}^+$), then T is k -connected.

Proof: It is well-known (e.g., [12, Cor. 12a]) that a tournament S is strong provided that the maximum difference between its scores is less than $\frac{1}{2}|V(S)|$, i.e., if $q(S) \leq \frac{1}{2}(|V(S)| - 1)$. Let $U \subseteq V(T)$ with $|U| \leq k-1$. To establish that $T-U$ is strong, it will therefore be sufficient to show that $q(T-U) \leq \frac{1}{2}(n - |U| - 1)$, and since, clearly, $q(T-U) \leq q(T) + |U|$, it will be sufficient to show that

$$q(T) + |U| \leq \frac{1}{2}(n - |U| - 1) ,$$

which is equivalent to

$$(1) \quad n \geq 2q(T) + 3|U| + 1 .$$

Now $n \geq 2q(T) + 3k - 2 = 2q(T) + 3(k-1) + 1 \geq 2q(T) + 3|U| + 1$, by hypothesis, so (1) holds and the proposition follows. \square

We now demonstrate that the second of the k -cycle-reversal theorems follows easily from Thomassen's [29, Cor. 3.3].

Proof of Theorem 1.7.4: To avoid trivial difficulties, fix k , $6 \leq k \leq n-1$. Let

$$Z: z_0 \rightarrow z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow z_0$$

be a 4-cycle in T . As in the proof of Theorem 1.7.3 (and many others), it is sufficient to show that Z can be exactly reversed via a sequence of k -cycle-reversals. Since, by hypothesis, $n \geq 2q(T) + 10$, T is 4-connected, by the foregoing proposition. It follows at once from the definition of k -connected that $T - \{z_0, z_2\}$ is 2-connected. Since this subtournament is obviously not XT_6 or XT_6^* (it has at

least eight vertices), it is weakly panconnected, by [29, Cor. 3.3]. Consequently (and since $4 \leq k-2 \leq n-3$), there is a $(k-2)$ -path, say P , in $T - \{z_0, z_2\}$ connecting its vertices z_1, z_3 . It is clear that, regardless of the direction of the path P , it may be used to exactly reverse Z via k -cycle-reversals (the obvious generalization of Figure 6). Upon noticing that cycle-reversals will not affect quasiregularity, and hence will preserve the connectivity (i.e., at least 4-connectedness) of all resulting tournaments, we see that the proof is complete. \square

3.4 Strong panconnectivity.

The first known condition sufficient to insure strong panconnectivity in tournaments was given by Alspach, Reid, and Roselle in [2]. Their statement also included an earlier result of Alspach [1].

Theorem [2, Thm. 6]. Every regular n -tournament, $n \geq 7$, is strongly panconnected. (RT_5 is not.)

They also pointed out [2, Thm. 7] that no condition on the degree sequence of an asymmetric digraph (and, in particular, on the score sequence of a tournament) will insure 2-paths from each vertex to every other vertex. Thus, the range $3 \leq k \leq n-1$ given in Definition 3.1.1 is, in some sense, the "natural" one for what we have in mind.

O.S. Jakobsen [15] proved that each arc of a near-regular n -tournament, $n \geq 8$, is contained in cycles of all

lengths ℓ , $4 \leq \ell \leq n$, and Thomassen extended this to

Theorem [15; 29, Cor. 4.9]. Every near-regular n -tournament, $n \geq 10$, is strongly panconnected. (There are two exceptions when $n = 8$.)

These two theorems take care of the cases $q(T) = 0, 1$, so we shall concentrate our attention on tournaments T with $q(T) \geq 2$. Our first result concerning these tournaments emphasizes "local" strong panconnectivity, extends Lemma 2.4.2, and is a simple generalization of its proof. We credit K. Wayland with some useful observations in the ensuing argument.

3.4.1 Theorem. Let T be an n -tournament with quasi-regularity $q(T) \geq 2$, and let x and y be vertices of T having maximum and minimum score, respectively. Then for each integer k , $2 \leq k \leq n-2$, there is a k -path in T from x to y . Moreover, there is a Hamiltonian path in T from x to y unless $n \geq 5$ and there exists a vertex w of T such that T has one of the forms depicted in Figure 20 (a) or (b) (they are duals with x, y interchanged) and y (resp., x) has the stated property, in which case T has no such path.

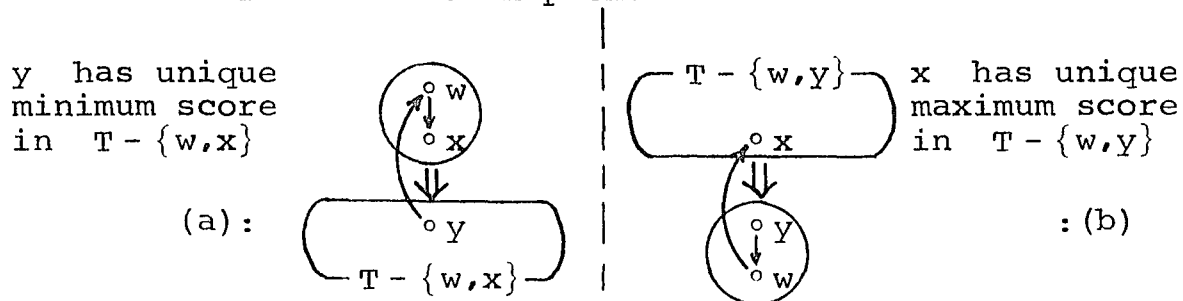


Figure 20.

Proof: We refer the reader to the proof of Lemma 2.4.2, in which x, y were chosen to satisfy $od(x) = s_{\max}$, $od(y) = s_{\min}$, but otherwise arbitrary. We were able to find k -paths from x to y for $k = 2$, and in all cases in which one of the sets B , I , or \emptyset was nonempty, we were able to replace a k -path with a $(k+1)$ -path from x to y , thereby completing the induction step. We may therefore assume that $C = W$, as before, that we are given a k -path, $2 \leq k \leq n-2$, from x to y , say

$$P: x = x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_{k-1} \rightarrow x_k = y,$$

and we shall assume that k is maximum relative to there existing paths in T from x to y of all lengths k' , $2 \leq k' \leq k$ (since it is clear that neither of the tournaments of Figure 20 have a Hamiltonian path from x to y).

Let $w \in C$ ($C \neq \emptyset$, since $k+1 < n$). First, we show that $k = n-2$. By the maximality of k and the proof of the lemma (the last part of same), either $x_1 \rightarrow w$ or $w \rightarrow x_{k-1}$. By dualizing, if necessary, we may assume that $x_1 \rightarrow w$. Then $x_i \rightarrow w$, for all $i > 1$ (otherwise, we can replace P by a longer path from x to y), so it follows that $|C| \leq od_T(y) \leq od_T(w) = 1 + od_{\langle C \rangle}(w) \leq 1 + |C| - 1 = |C|$, and hence we have equality throughout. Thus,

$$od_{\langle C \rangle}(w) = |C| - 1,$$

so w is a transmitter in $\langle C \rangle$, i.e., it has zero indegree and therefore dominates all other vertices of C . Now suppose $k < n-2$. Then $C \neq \{w\}$, so let $w' \in C$, $w' \neq w$. Clearly, $\langle C \rangle$ cannot have two transmitters, so $w' \rightarrow x_1$,

by the above argument with w' in place of w , and also $w' \rightarrow x_{k-1}$, again as above. The latter implies that $w' \rightarrow x_i$ for all $i < k-1$ (by maximality of k), and we also have that $w \rightarrow w'$, since w is a transmitter in $\langle C \rangle$. Now note that $\text{od}_T(x) \leq k$ and $\text{od}_T(y) \geq |C| \geq 2$, which together imply that $2 \leq \text{od}_T(y) \leq \text{od}_T(x) - 2 \leq k-2$ (using $q(T) \geq 2$), whence $k \geq 4$. Putting all these facts together, we see that

$$x \rightarrow x_1 \rightarrow w \rightarrow w' \rightarrow x_3 \rightarrow \dots \rightarrow y$$

is a $(k+1)$ -path in T from x to y , but this contradicts the assumed maximality of k . Therefore $k = n-2$, as we wished to show.

We now have that $\text{od}_T(w) = 1$ (w dominates x only), so $\text{od}_T(y) = 1$, by minimality (we already know y dominates w), and hence $T - \{w, y\}$ dominates y . All arcs are now oriented precisely as shown in Figure 20 (b). Because x has maximum score in T , it must have strictly larger score in $T - \{w, y\}$ than any other vertex in this subtournament. Finally, T is strong (the tournament in Figure 20 (b) is strong), and no strong tournament with $q(T) \geq 2$ can have fewer than five vertices, so $n \geq 5$.

Naturally, if we had assumed that $w \rightarrow x_{k-1}$, we would have obtained the dual tournament of Figure 20 (b). The proof is complete. \square

The tournaments depicted in Figure 20 are strong, but not 2-strong. Also, x dominates y , in both cases. The following result is therefore immediate.

3.4.2 Corollary. Let T be an n -tournament and x and y vertices of T with maximum and minimum score, respectively. Then there are paths in T from x to y of all lengths k , $2 \leq k \leq n-1$, if any one of the following holds:

- i) T is not strong;
- ii) T is 2-strong and $q(T) \geq 2$;
- iii) y dominates x .

Proof: By the previous theorem, i) or ii) immediately give the result. That condition iii) is sufficient follows from [1], [15], and an easy examination of the "small" cases $n = 4, 6$ which are left to the reader. \square

3.5 Strongly panconnected tournaments.

Thomassen in [29] has introduced the following notion as a measure of the "regularity" of a tournament. The irregularity of an n -tournament T , denoted $i(T)$, is defined by $i(T) = \max_{x \in V(T)} |\text{od}(x) - \text{id}(x)|$. This is equivalent to letting

$$i(T) = 2 \cdot \max \left\{ s_{\max} - \frac{n-1}{2}, \frac{n-1}{2} - s_{\min} \right\}.$$

Just as in the case of quasiregularity, $i(T) = 0$ if and only if T is regular (note that $\frac{n-1}{2}$ is the average score of any n -tournament) and $i(T) = 1$ if and only if T is near-regular. The two notions $i(T)$, $q(T)$ coincide for a large class of tournaments, in fact (for example, those with self-dual score sequences). Their exact relationship is given by the following proposition.

3.5.1 Proposition. For any n -tournament T , we have:

- i) $i(T) = q(T) + |(s_{\max} + s_{\min}) - (n-1)|$;
- ii) $q(T) \leq i(T) \leq 2q(T)$, and the second inequality is strict unless $i(T) = q(T) = 0$.

Proof: For any real numbers $a, b \geq 0$, $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$. It follows that

$$\begin{aligned}
 i(T) &= 2 \max\left\{s_{\max} - \frac{n-1}{2}, \frac{n-1}{2} - s_{\min}\right\} \\
 &= s_{\max} - \frac{n-1}{2} + \frac{n-1}{2} - s_{\min} + \left|s_{\max} - \frac{n-1}{2} - \frac{n-1}{2} + s_{\min}\right| \\
 &= s_{\max} - s_{\min} + |(s_{\max} + s_{\min}) - (n-1)| \\
 &= q(T) + |(s_{\max} + s_{\min}) - (n-1)| ,
 \end{aligned}$$

which proves assertion i).

For ii), note that $q(T) \leq i(T)$ by part i). For the second inequality in ii), we have that

$$\begin{aligned}
 i(T) &= q(T) + |(s_{\max} + s_{\min}) - (n-1)| \\
 &= q(T) + \left|(s_{\max} - \frac{n-1}{2}) + (s_{\min} - \frac{n-1}{2})\right| \\
 &\leq q(T) + \left|s_{\max} - \frac{n-1}{2}\right| + \left|s_{\min} - \frac{n-1}{2}\right| \\
 &= q(T) + s_{\max} - \frac{n-1}{2} + \frac{n-1}{2} - s_{\min} \\
 &= q(T) + s_{\max} - s_{\min} \\
 &= 2q(T) .
 \end{aligned}$$

Note also that the only inequality in this chain (which comes from the triangle inequality) is strict unless $s_{\min} \geq \frac{n-1}{2}$, whence $s_{\min} = \frac{n-1}{2}$ (the minimum score cannot exceed the average score), T is regular, and therefore $i(T) = q(T) = 0$. This proves the proposition. \square

Using the irregularity parameter, Thomassen in [29] has greatly extended the results cited in the previous section (that almost all regular and near-regular tournaments

are strongly panconnected) by proving:

Theorem [29, Thm. 4.5]. Let T be an n -tournament and x, y vertices of T . Then:

- i) If \vec{yx} is an arc of T and $n \geq 5i(T) + 3$, then \vec{yx} is contained in cycles of all lengths ℓ , $4 \leq \ell \leq n$;
- ii) If \vec{xy} is an arc of T and $n \geq 5i(T) + 9$, then \vec{xy} has bypasses (i.e., there are x -to- y paths) of all lengths ℓ , $3 \leq \ell \leq n-1$.

In particular, an n -tournament T is strongly panconnected if $n \geq 5i(T) + 9$.

As suspected by Thomassen, the bound $n \geq 5i(T) + 9$ for strong panconnectivity is not quite best possible, since he knew of counterexamples when $n = 5i(T) + 3$ [29, Lemma 4.4]. As we shall show later, $n \geq 5i(T) + 4$ is sufficient when $q(T) \geq 3$.

We shall now work toward a similar result, and, as remarked earlier, it will be stated in terms of the "weaker" (in view of Proposition 3.5.1) notion of quasiregularity.

For any arc \vec{uv} of a tournament T , let

$$BY_T(\vec{uv}) = \{x \in V(T) : u \rightarrow x \text{ and } x \rightarrow v\},$$

$$CY_T(\vec{uv}) = \{x \in V(T) : v \rightarrow x \text{ and } x \rightarrow u\}.$$

These are called, respectively, the bypass and cycle sets of \vec{uv} (in T). Their respective cardinalities will be denoted by $by_T(\vec{uv})$, $cy_T(\vec{uv})$, and we shall omit the subscript " T " when T is understood in all of these expres-

sions, as usual. The following simple counting lemma involving these sets will be used repeatedly in the proof of the main result.

3.5.2 Lemma. For any arc \vec{uv} of a tournament T ,
 $(by(\vec{uv})+1) - cy(\vec{uv}) = od(u) - od(v)$ and therefore
 $|(by(\vec{uv})+1) - cy(\vec{uv})| \leq q(T)$.

Proof: It is clear that $od(u) = by(\vec{uv}) + 1 + |O(u,v)|$ and that $od(v) = cy(\vec{uv}) + |O(u,v)|$, from which the first conclusion follows at once. The second is an obvious consequence of this. \square

We shall also be using repeatedly the fact that in any tournament S , we can find a vertex x with outdegree at least $\frac{1}{2}(|V(S)| - 1)$ (i.e., in S) and a vertex y with outdegree at most this much, and we can do better unless S is regular (in particular, if $|V(S)|$ is even). Before entering the proof, it will be helpful to introduce the following notation: for each pair of integers $i, j \geq 0$, let $D_{i,j}^2$ and $D_{i,j}^3$ denote the digraphs depicted in Figure 21 below (the vertices shown for each of these are assumed distinct, and note that we allow the "degenerate" paths corresponding to $i = 0$ and/or $j = 0$).

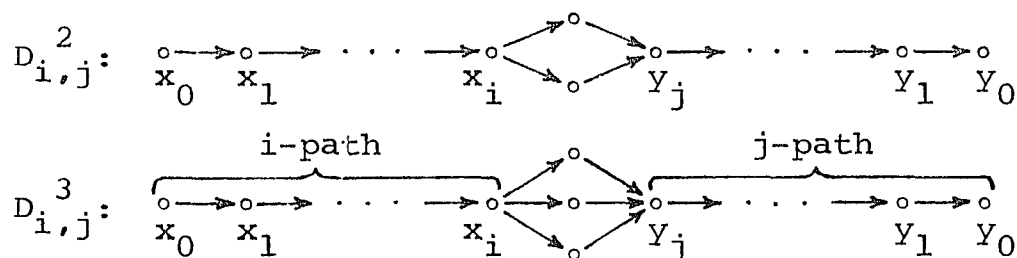


Figure 21.

3.5.3 Theorem. An n -tournament T is strongly pan-connected provided that $n \geq \max \{ 5q(T) + 4, 2q(T) + 13 \}$.

Proof: Let $q = q(T)$. If $q = 0$, the conclusion of the theorem follows from [2, Thm. 6] (since $n \geq 13$, by hypothesis), and if $q = 1$, from [15; 29, Cor. 4.9], results cited in the previous section. Assume, therefore, that $q \geq 2$. Let x, y be an ordered pair of distinct vertices of T . In steps 1-5 below, we show that there is a k -path in T from x to y , for all integers k satisfying $3 \leq k \leq n-1$.

Step 1: There is a 3-path from x to y .

Let $B = O(x) \cap I(y)$, $C = I(x) \cap O(y)$, $I = I(x, y)$, and $O = O(x, y)$, as shown in Figure 22 below.

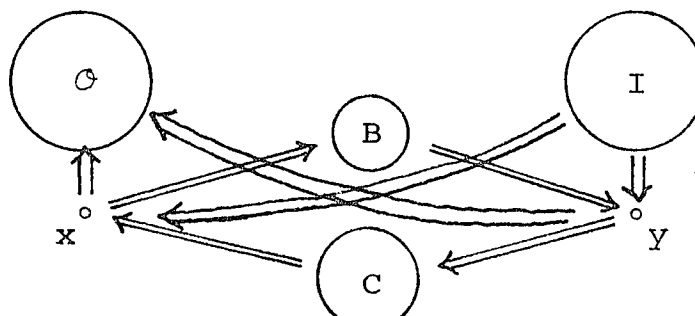


Figure 22.

Now if there exist $b_1, b_2 \in B$ with $b_1 \rightarrow b_2$, then $x \rightarrow b_1 \rightarrow b_2 \rightarrow y$ is a 3-path from x to y , so we may assume that $|B| \leq 1$. If $x \rightarrow y$, then Lemma 3.5.2 implies that $|C| \leq q+2$. If, on the other hand, $y \rightarrow x$, the lemma implies that $|C| \leq q$, so in any case $|C| \leq q+2$. Hence

$$|O| + |I| = n - (|B| + |C| + 2) \geq (5q+4) - (q+5) = 4q-1,$$

so one of the sets O or I must have at least $2q$ vertices. Suppose first that $|O| \geq 2q$. Then there exists

$x_1 \in O$ with $\text{id}_{\langle O \rangle}(x_1) \geq q$. These q (or more) vertices provide bypasses (i.e., 2-bypasses) of the arc \vec{yx}_1 . Hence $\text{cy}(\vec{yx}_1) \geq 1$, by the lemma. Letting $w \in \text{Cy}(\vec{yx}_1)$, we see that $x \rightarrow x_1 \rightarrow w \rightarrow y$ is an appropriate 3-path in T . A similar conclusion in case $|I| \geq 2q$ is reached by selecting $y_1 \in I$ such that $\text{od}_{\langle I \rangle}(y_1) \geq q$, and considering the arc $\vec{y_1x}$ (clearly, the two cases are directional duals).

Step 2: There is a k -path from x to y , for $k = 4, 5$.

First note that either the minimum outdegree or the minimum indegree in T is at least $2q+2$, for suppose otherwise. Then $s_{\min} \leq 2q+1$ and

$$s_{\max} \geq (n-1) - (2q+1) \geq (5q+3) - (2q+1) = 3q+2;$$

hence $s_{\max} - s_{\min} \geq q+1$, but this contradicts the quasi-regularity assumption. Therefore, the claim holds, and we may assume $s_{\min} \geq 2q+2$, by working in T^* , if necessary.

We shall prove a slightly stronger result than required in this step, by induction on k , namely: The existence of a $(k-1)$ -path from x to y implies the existence of a k -path from x to y , for $4 \leq k \leq 2q+2$.

To this end, let

$$P: x = x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_{k-1} \rightarrow x_k = y$$

be a $(k-1)$ -path in T . Denote by W the set of vertices not on P . Since $\text{od}(x) \geq 2q+2$ (outdegree in T understood if not written) and $\text{od}_{\langle P \rangle}(x) \leq k-1 \leq 2q+1$, it follows that $x \rightarrow u$ for some $u \in W$. Now if $u \rightarrow x_i$ for some (least index) i , $2 \leq i \leq k$, then

$$x \rightarrow x_2 \rightarrow \dots \rightarrow x_{i-1} \rightarrow u \rightarrow x_i \rightarrow \dots \rightarrow y$$

is an appropriate k -path, and we are done. Hence we may assume every x_i dominates u . Then $O(u) \subseteq W$, and if some vertex $v \in O(u)$ dominates some x_i for which $3 \leq i \leq k$, then

$$x \rightarrow x_2 \rightarrow \dots \rightarrow x_{i-2} \rightarrow u \rightarrow v \rightarrow x_i \rightarrow \dots \rightarrow y$$

is a k -path. We may therefore assume that $\{x_3, x_4, \dots, y\}$ dominates $O(u)$.

Now $|O(u)| \geq 2q+2$, so choose a vertex $v \in O(u)$ such that $\text{id}_{\langle O(u) \rangle}(v) \geq q+1$. Then the arc $\vec{y}v$ is bypassed by at least $q+2$ vertices (counting u), so $\text{cy}(\vec{y}v) \geq 3$ according to the lemma. Conceivably two of these three (or more) vertices, i.e., in the cycle set of $\vec{y}v$, could lie on the path P , but no more than two (as x, x_2 are the only possibilities), so there exists a vertex $w \in \text{Cy}(\vec{y}v) \cap W$. Then

$$x \rightarrow \dots \rightarrow x_{k-3} \rightarrow u \rightarrow v \rightarrow w \rightarrow y$$

is a k -path from x to y .

This completes the induction step, and since $q \geq 2$, step 2 is complete.

Step 3: T is 5-connected.

Since $n \geq 2q+13 = 2q + 3 \cdot 5 - 2$, T is 5-connected by Proposition 3.3.1.

Step 4: If T contains a copy of any one of the digraphs $D_{0,0}^2$, $D_{0,1}^2$, $D_{1,0}^2$, or $D_{1,1}^3$ with $x = x_0$, $y = y_0$, then there are k -paths in T from x to y , for all integers k , $6 \leq k \leq n-1$.

Fix k , $6 \leq k \leq n-1$, and suppose first that T con-

tains a copy of $D_{0,1}^2$ with $x = x_0$, $y = y_0$, and other vertices labelled as shown in Figure 23 below.

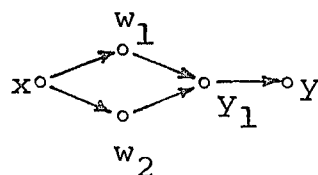


Figure 23.

Let $S = T - \{x, y_1, y\}$. Since T is 5-connected, the subtournament S is 2-connected (at least). Hence, by [29, Cor. 3.3] (which applies since S has at least ten vertices), there is a $(k-3)$ -path P in S connecting w_1, w_2 (because $3 \leq k-3 \leq n-4 = |V(S)| - 1$). Then the augmented path $x \rightarrow P \rightarrow y_1 \rightarrow y$ is the desired k -path in T .

In case T contains $D_{1,0}^2$ or $D_{0,0}^2$, the argument is analogous to that above (and in the case of the latter, only 4-connectedness of T is needed).

Suppose, finally, that T contains the digraph $D_{1,1}^3$ depicted in Figure 24 below.

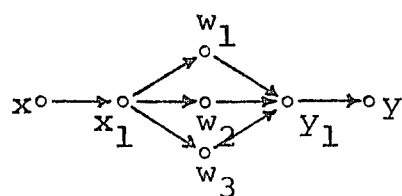


Figure 24.

Let $S' = T - \{x, x_1, y_1, y\}$. Then S' is strong (as T is 5-connected), so by [29, Cor. 3.4] there is a $(k-4)$ -path P' in S' connecting two of w_1, w_2, w_3 . Then, regardless of the endvertices of P' , the augmented path

$$x \rightarrow x_1 \rightarrow P' \rightarrow y_1 \rightarrow y$$

is a k -path in T from x to y . This completes the proof of step 4.

Step 5: T contains at least one of the digraphs described in step 4.

Assume that T contains none of the digraphs $D_{0,0}^2$, $D_{0,1}^2$, $D_{1,0}^2$ with $x = x_0, y = y_0$. We shall establish the existence of an appropriate copy of $D_{1,1}^3$ in T .

Denote by B, C, \mathcal{O} , and I the sets of vertices in step 1 of the proof. Since we are assuming T does not contain $D_{0,0}^2$ with $x = x_0, y = y_0$, clearly $|B| \leq 1$. We shall examine only the case $|B| = 1$, leaving the case in which B is empty to the reader with the remark that a (somewhat easier) counting argument similar to the sequel exists.

Let $B = \{b\}$. As proved in step 1, $|C| \leq q+2$ and $|\mathcal{O}| + |I| \geq 4q-1$, so one of the sets \mathcal{O} or I has at least $2q$ vertices. By duality, we may assume that $|\mathcal{O}| \geq 2q$ without loss of generality. Now if $|\mathcal{O}| \geq 2q+2$, then there exists $x_1 \in \mathcal{O}$ such that $\text{id}_{\langle \mathcal{O} \rangle}(x_1) \geq q+1$, which means that $\text{by}(\vec{yx_1}) \geq q+1$, and hence $\text{cy}(\vec{yx_1}) \geq 2$ by the lemma. Choosing distinct $w_1, w_2 \in \text{Cy}(\vec{yx_1})$, it is then easily seen that the vertices x, x_1, w_1, w_2 , and y (together with the appropriate arcs) form a copy of $D_{1,0}^2$ in T of the excluded type, a contradiction. Therefore, $|\mathcal{O}| \leq 2q+1$, and it follows from this and the lower bound on $|\mathcal{O}| + |I|$ that $|I| \geq 2q-2$.

Since $|\mathcal{O}| \geq 2q$ and $|I| \geq 2q-2$, we can select (and

fix) vertices $x_1 \in \mathcal{O}$ and $y_1 \in \mathcal{I}$ for which $|X| \geq q$ and $|Y| \geq q-1$, where we set

$$X = I_T(x_1) \cap \mathcal{O} \quad \text{and} \quad Y = O_T(y_1) \cap \mathcal{I}.$$

Now either x_1 dominates y_1 or conversely, and we examine each possibility in turn.

Suppose that $x_1 \rightarrow y_1$. Then y_1 dominates X (to avoid a copy of $D_{0,1}^2$ in T) and Y dominates x_1 (to avoid $D_{1,0}^2$). Also, $y_1 \rightarrow b$ and $b \rightarrow x_1$ for the same reasons, so it follows that

$$cy(x_1 \vec{y}_1) \geq |X| + |Y| + 3 \geq 2q+2.$$

This in turn yields $by(x_1 \vec{y}_1) \geq q+1 \geq 3$, again by Lemma 3.5.2. Choose distinct $w_1, w_2, w_3 \in By(x_1 \vec{y}_1)$. Then T contains the copy of $D_{1,1}^3$ depicted in Figure 24 above, precisely as labelled in this figure.

Finally, suppose that $y_1 \rightarrow x_1$. Then at most one vertex of X dominates y_1 , and x_1 dominates at most one vertex of Y (to avoid the excluded digraphs). Since also $x, y \in By(y_1 \vec{x}_1)$, we have that

$$by(y_1 \vec{x}_1) \geq (|X|-1) + (|Y|-1) + 2 = |X| + |Y|,$$

and note that this lower bound on $by(y_1 \vec{x}_1)$ can be improved to $|X| + |Y| + 1$ unless both of the following occur:

- a) Exactly one $u \in X$ dominates y_1 ;
- b) x_1 dominates exactly one $v \in Y$.

If both a) and b) occur, however, then $y_1 \rightarrow b$ (to avoid $D_{0,1}^2$) and $b \rightarrow x_1$ (dually), that is, $b \in By(y_1 \vec{x}_1)$, and we obtain one more vertex. In any case, therefore, we have that $by(y_1 \vec{x}_1) \geq |X| + |Y| + 1 \geq 2q$, and therefore, again

using the counting lemma, $cy(y_1 \vec{x}_1) \geq q+1 \geq 3$. Choosing distinct $w_1, w_2, w_3 \in Cy(y_1 \vec{x}_1)$, we again obtain the digraph of Figure 24.

This proves step 5 and completes the proof of the theorem. \square

3.5.4 Example. The tournament depicted in Figure 25 below (one can use almost any tournament for S , but R^1, R^2 must be regular; the orders of each are indicated) fails to have a 3-path from x to y . Its order $n = 5q + 3$ and it is easy to see that $2q+1 \leq od(v) \leq 3q+1$ for every vertex $v \in V(T)$, and the extreme values here are attained, i.e., $q = q(T)$.

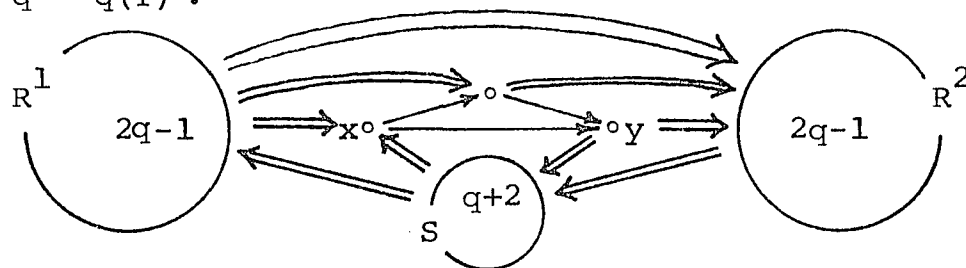


Figure 25. An example of a non-strongly panconnected n -tournament T with $n = 5q(T) + 3$.

Thomassen in [29] has constructed the same example (it has $i(T) = q(T)$).

Since the bound in Theorem 3.5.3 just reduces to $n \geq 5q(T) + 4$ when $q(T) \geq 3$, the above example shows that it is best possible when $q(T) \geq 3$, and so is the bound $n \geq 5i(T) + 4$, by combining Theorem 3.5.3 with Proposition 3.5.1.

The following combines results from [1; 2; 15; and 29] as well as Theorem 3.5.3.

3.5.5 Corollary [1; 2; 15; 29; Thm. 3.5.3] Any one of the following conditions implies that an n -tournament T is strongly panconnected:

- i) $q(T) = 0$ and $n \geq 7$;
- ii) $q(T) = 1$ and $n \geq 10$;
- iii) $q(T) = 2$ and $n \geq 17$;
- iv) $q(T) \geq 3$ and $n \geq 5q(T) + 4$.

Moreover, these bounds on n are best possible except possibly in case iii) $q(T) = 2$. \square

3.5.6 Example. This example will show that Theorem 3.5.3 has an advantage over [29, Thm. 4.5] when $i(T)$ is nearly twice $q(T)$. Let T be a tournament of order 42 with score sequence

$$(17, \underbrace{20, \dots, 20}_{17}, \underbrace{21, \dots, 21}_{24}) .$$

(Such a tournament exists.) T has $i(T) = 7$ and $q(T) = 4$ and it is straightforward to check that, by Theorem 3.5.3, T is not only strongly panconnected, but for any subset of three or fewer of its vertices, say S , the subtournament $T - S$ is also strongly panconnected, since $T - S$ has quasi-regularity at most 7 and has at least 39 vertices. However, [29, Thm. 4.5] does not imply that T itself is strongly panconnected.

BIBLIOGRAPHY

- [1] B. Alspach, Cycles of each length in regular tournaments, Canad. Math. Bull., 10(1967), 283-286.
- [2] B. Alspach, K.B. Reid, and D.P. Roselle, Bypasses in asymmetric digraphs, J. Comb. Theory, Ser. B 17(1974), 11-18.
- [3] L.W. Beineke and R.J. Wilson, A survey of recent results on tournaments, preprint, 1975.
- [4] C. Berge, "Principles of Combinatorics," Academic Press, New York, 1971.
- [5] J.C. Bermond, Graphes orientes fortement k-connexes et graphes k-arc Hamiltoniens, C. R. Acad. Sci. Paris, Ser. A 271(1970), 141-144.
- [6] J.A. Bondy and U.S.R. Murty, "Graph Theory with Applications," American Elsevier, New York, 1976.
- [7] R. Forcade, Parity of paths and circuits in tournaments, Discrete Math., 6(1973), 115-118.
- [8] M.R. Garey, On enumerating tournaments that admit exactly one Hamiltonian circuit, J. Comb. Theory, Ser. B 13(1972), 266-269.
- [9] M. Goldberg and J.W. Moon, Cycles in k-strong tournaments, Pacific J. Math., 40 No. 1(1972), 89-96.
- [10] B. Grünbaum, Antidirected Hamiltonian paths in tournaments, J. Comb. Theory, 11(1971), 249-257.
- [11] F. Harary, "Graph Theory," Addison-Wesley, Reading, Mass., 1969.
- [12] F. Harary and L. Moser, The theory of round robin tournaments, Amer. Math. Monthly, 73(1966), 231-246.
- [13] F. Harary, R.Z. Norman, and D. Cartwright, "Structural Models: An Introduction to the Theory of Directed Graphs," John Wiley and Sons, New York, 1965.

- [14] F. Harary, E. Palmer, and C. Smith, Which graphs have only self-converse orientations?, Canad. Math. Bull., 10 No. 3(1967), 425-429.
- [15] O.S. Jakobsen, Cycles and paths in tournaments, Ph.D. thesis, Univ. of Aarhus, Denmark, 1972.
- [16] H.G. Landau, On dominance relations and the structure of animal societies: III. The condition for a score structure, Bull. Math. Biophys., 15(1953), 143-148.
- [17] J.W. Moon, On cycles in tournaments, Mat. Časopis, 19 No. 2(1969), 121-125.
- [18] J.W. Moon, On subtournaments of a tournament, Canad. Math. Bull., 9(1966), 297-301.
- [19] J.W. Moon, "Topics on Tournaments," Holt, Rinehart and Winston, New York, 1968.
- [20] L.T. Ollmann and K.B. Reid, Edge sets generated by symmetric difference, Proc. Second Louisiana Conf. on Combinatorics, Graph Theory, and Computing (R.C. Mullin, et al., Eds.), Congressus Numerantium III, Univ. of Manitoba, Winnipeg, 1971, pp. 411-456.
- [21] L. Rédei, Ein kombinatorischer Satz, Acta Litt. Szeged, 7(1934), 39-43.
- [22] K.B. Reid, Equivalence of n-tournaments via k-path reversals, Discrete Math., 6(1973), 263-280.
- [23] K.B. Reid, Paths and cycles in tournaments, preprint, 1973.
- [24] K.B. Reid and C. Thomassen, Strongly self-complementary and hereditarily isomorphic tournaments, preprint, 1975.
- [25] M. Rosenfeld, Antidirected Hamiltonian circuits in tournaments, preprint, 1974.
- [26] M. Rosenfeld, Antidirected Hamiltonian paths in tournaments, J. Comb. Theory, 12(1972), 93-99.
- [27] H.J. Ryser, Matrices of zeros and ones in combinatorial mathematics, in: Recent Advances in Matrix Theory (H. Schneider, Ed.), Univ. of Wisconsin Press, Madison, Wisc., 1964, pp. 103-124.

- [28] C. Thomassen, Antidirected Hamiltonian circuits and paths in tournaments, Math. Ann., 201(1973), 231-238.
- [29] C. Thomassen, Hamiltonian connected tournaments, preprint, 1977.
- [30] C. Waldrop, An arc-reversal theorem for tournaments, Proc. Seventh Southeastern Conf. on Combinatorics, Graph Theory, and Computing (F. Hoffman, et al., Eds.), Congressus Numerantium XVII, Univ. of Manitoba, Winnipeg, 1976, pp. 501-507.

VITA

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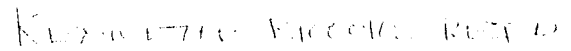
EXAMINATION AND THESIS REPORT

Candidate: Claybourne Waldrop, Jr.

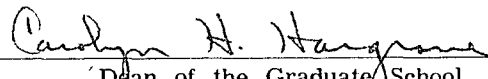
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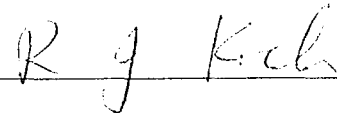


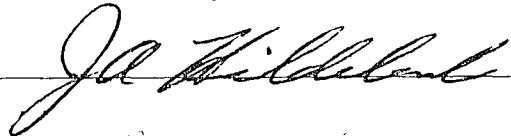
Major Professor and Chairman

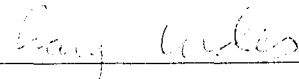


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