Stabilizations of Periodic Maps on Manifolds.

David Calvin Royster

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STABILIZATIONS OF PERIODIC MAPS ON MANIFOLDS

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ON MANIFOLDS

A Dissertation

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in

The Department of Mathematics

by
David Calvin Royster
B.A., University of the South, 1973
August, 1978
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ABSTRACT

In this paper we study the unrestricted cobordism ring $I_\ast(\mathbb{Z}_2)$ and an endomorphism, $\Gamma$, on this ring. We then extend these results, as far as possible, to the oriented cobordism ring $\Omega^\ast_{SO}(\mathbb{Z}_p)$, $p$ an odd prime, and an endomorphism, $\Gamma_0$.

We first introduce some ideals, $\mathcal{A}_n$, in $I_\ast(\mathbb{Z}_2)$ by the use of $\Gamma$. For $n \geq 0$ define

$$\mathcal{A}_n = \{ x + \Gamma^n(x) \mid x \in I_\ast(\mathbb{Z}_2) \text{ and } \epsilon(\Gamma^j(x)) = 0 \text{ for } 0 \leq j < n \}.$$  

We first show that $I_\ast(\mathbb{Z}_2)/\mathcal{A}_1$ is a polynomial ring over $\mathbb{Z}_2$ and over $\mathbb{Z}_2$. The same cannot be said for $I_\ast(\mathbb{Z}_2)/\mathcal{A}_n$ for $n > 1$. Using this result we then prove the Five-Halves Theorem of J.M. Boardman. The proof of this result is especially nice because we do not need to use an inverse limit argument, as was necessary previously. Next, we see that some elements of the form $x^r + \Gamma^n(x^r)$ within the ideals $\mathcal{A}_n$ behave like the polynomials $1+t^n$ in $\mathbb{Z}_2[t]$. We first present the general theory for factoring $1+t^n$ in $\mathbb{Z}_2[t]$, and we then apply this to these elements in $I_\ast(\mathbb{Z}_2)$. What we...
see is that $\Gamma_*(\mathbb{Z}_2)$ behaves much like a polynomial ring.

We next extend our definition of $\Gamma$ to oriented cobordism. We make a constructive definition of the map $\Gamma_\mathcal{O}$ and see wherein the pitfalls lie. We have to consider a quotient ring of $\mathcal{O}^\mathcal{SO}_*(\mathbb{Z}_p)$ in order to have $\Gamma_\mathcal{O}$ well-defined. One of these ideals which must be factored out is easily determined by the Pontrjagin numbers of the fixed point data. We devote a chapter to showing that this ideal is preserved by $\Gamma_\mathcal{O}$. We then prove the analogous results for $\Gamma_\mathcal{O}$ that are found in [1] for $\Gamma$. We then look at polynomials of the form $1+t^n$ in $\mathbb{Z}_p[t]$, and notice that we have a mod $p$ analogue of Chapter V.
INTRODUCTION

In this paper we will study the graded unrestricted unoriented cobordism ring, $I_* (\mathbb{Z}_2)$, and an endomorphism, $\Gamma$, of $I_* (\mathbb{Z}_2)$ of degree $+1$. In the last four chapters we will study the oriented analogues of these objects.

In Chapter I we will give some introductory material, define the rings and modules with which we will be working in the unoriented case, and define the endomorphism $\Gamma$.

In Chapter II we use this map $\Gamma$ to define ideals, $C_n$, in $I_* (\mathbb{Z}_2)$ for $n \geq 0$ by

$$C_n = \{ x + \Gamma^n(x) \mid x \in I_* (\mathbb{Z}_2) \text{ and } \epsilon(\Gamma^j(x)) = 0 \text{ for } 0 \leq j < n \}. $$

The ideal $C_1$ shall play a more important part in our theory than the remainder of these ideals. We prove that $I_* (\mathbb{Z}_2)/C_1$ is a polynomial ring over $MO_*$ and over $\mathbb{Z}_2$, using the split exact sequence of unoriented cobordism modules

$$0 \to I_* (\mathbb{Z}_2) \to \mathcal{M}_* (\mathbb{Z}_2) \to MO_* (\mathbb{Z}_2) \to 0.$$

The quotient rings, $I_* (\mathbb{Z}_2)/C_n$, however, are not seen to be polynomial rings via this proof if $n > 1$. We do show, however, that they do inject into a polynomial ring over $MO_*$ and
we compute the cokernel of this injection as an $M_{O^*}$-module.

In Chapter III we apply our result about $I_*(\mathbb{Z}_2)/\mathcal{A}_1 = \Lambda(\mathbb{Z}_2)$ to find another proof of the Five-Halves Theorem of J.M. Boardman. We induce a map $\overline{\varphi}: \Lambda(\mathbb{Z}_2) \to M_{O^*}[t]$, the subring of "homogeneous" formal power series of the ring of formal power series over $M_{O^*}$. We show that $\Lambda(\mathbb{Z}_2) \approx M_{O^*}(BO)$, which is a polynomial ring over $\mathbb{Z}_2$. Using several results from [2], we construct some explicit polynomial generators for $\Lambda(\mathbb{Z}_2)$ whose augmentations are polynomial generators for $M_{O^*}$. We then introduce two filtrations on $\Lambda(\mathbb{Z}_2)$. An increasing filtration is given by considering the dimensions of the various components of fixed point sets of differentiable involutions on closed smooth manifolds. A decreasing filtration on $\Lambda(\mathbb{Z}_2)$ is induced by $\overline{\varphi}$ from a decreasing filtration on $M_{O^*}[t]$. After seeing how these filtrations act on the polynomial generators and on polynomials in these generators, the Five-Halves Theorem and its converse follow.

In Chapter IV we look at an application of the Five-Halves Theorem to a conjecture about flat manifolds. Though we do not prove or disprove this conjecture, we can see applications should it be true, and a possible method of attack should the conjecture be false.

In Chapter V we see that certain elements in the ideals $\mathcal{A}_n$ behave very much like the polynomial $1+t^n$ in $\mathbb{Z}_2[t]$. Knowing how to factor the cyclotomic polynomials and hence $1+t^n$ in $\mathbb{Z}_2[t]$, we then factor the elements $x^r + r^n(x^r)$ in $\mathcal{A}_n$ in the same way.

In Chapter VI we extend our map $\Gamma$ to a map $\Gamma_\mathcal{G}$, an
endomorphism of oriented cobordism rings. We define the map $\Gamma_\mathcal{O}$ of degree $+2$ and see what conditions we must satisfy in order for $\Gamma_\mathcal{O}$ to be a well-defined endomorphism on a quotient of the ring $\mathcal{O}^*_p\mathbb{Z}$, $p$ an odd prime.

In Chapter VII we let $(T, M^n)$ be an orientation preserving differentiable periodic map of period $p$ on an oriented smooth closed weakly complex manifold $M^n$ which preserves the weakly complex structure; and we let $F = \bigcup_{m=0}^{n} F^m$ be the fixed point set of $T$ on $M^n$. We then prove that if all of the tangential and normal Chern numbers as well as those Chern numbers arising from the products of the tangential and normal Chern classes are divisible by $p$ for each component of $F$, then all of the Chern numbers of $M^n$ are divisible by $p$. This then gives us an ideal $\mathcal{O}_{\mathcal{O}}$ of $\mathcal{O}^*_p\mathbb{Z}$ for which

$$\Gamma_\mathcal{O}: \mathcal{O}^*_p\mathbb{Z}/\mathcal{O}_{\mathcal{O}} \longrightarrow \mathcal{O}^*_p\mathbb{Z}/\mathcal{O}_{\mathcal{O}}$$

is a well-defined endomorphism of degree $+2$.

In Chapter VIII we note that the results of Chapter VII are true for Pontrjagin numbers. Thus, we get an ideal $\mathcal{O}_{\mathcal{O}}$ of $\mathcal{O}^*\mathbb{Z}$, and we have that $\Gamma_\mathcal{O}$ is a well-defined endomorphism on $\mathcal{O}^*\mathbb{Z}/\mathcal{O}_{\mathcal{O}}$. We then prove the results for $\Gamma_\mathcal{O}$ analogous to those results for $\Gamma$ found in [1; §2].

Chapter IX is the mod $p$ analogue of Chapter V.
TABLE OF NOTATIONS

<table>
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<th>Symbol</th>
<th>Description</th>
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<tr>
<td>( \mathbb{C} )</td>
<td>the field of complex numbers</td>
</tr>
<tr>
<td>( \mathbb{C}_p )</td>
<td>the cyclic group of integers mod ( p )</td>
</tr>
<tr>
<td>( \mathbb{F}_p )</td>
<td>the finite field with ( p ) elements</td>
</tr>
<tr>
<td>( \mathbb{Q} )</td>
<td>the field of rational numbers</td>
</tr>
<tr>
<td>( \mathbb{R} )</td>
<td>the field of real numbers</td>
</tr>
<tr>
<td>( \mathbb{Z} )</td>
<td>the ring of integers</td>
</tr>
<tr>
<td>( \mathbb{Z}_n )</td>
<td>the group of integers modulo ( n )</td>
</tr>
<tr>
<td>([\ ])</td>
<td>the oriented cobordism class of a manifold in ( \text{MSO}_\ast )</td>
</tr>
<tr>
<td>([\ ]_2)</td>
<td>the unoriented cobordism class of a manifold in ( \text{MO}_\ast )</td>
</tr>
<tr>
<td>{})</td>
<td>the oriented cobordism class of a manifold in ( \text{OSO}_\ast (\mathbb{Z}_p) )</td>
</tr>
<tr>
<td>{}(_2)</td>
<td>the unoriented cobordism class of a manifold in ( \text{I}_\ast (\mathbb{Z}_2) )</td>
</tr>
<tr>
<td>( \theta^n )</td>
<td>a trivial real ( n )-plane bundle</td>
</tr>
<tr>
<td>( \xi, \eta, \nu, \zeta, \tau )</td>
<td>vector bundles</td>
</tr>
<tr>
<td>( w_M )</td>
<td>the orientation class of a manifold ( M )</td>
</tr>
</tbody>
</table>

We will use \([\ ; \ ]\) to denote the references; e.g., \([3;28.1]\) means reference 3, Theorem 28.1. The symbol // will denote the end of a proof or the need for no further proof.
CHAPTER I: PRELIMINARY MATERIAL

We will use $I_\ast(\mathbb{Z}_2)$ to denote the graded unrestricted unoriented cobordism ring of smooth manifolds with involution; $MO_\ast(\mathbb{Z}_2)$, the graded unoriented cobordism ring of smooth manifolds with fixed point free involutions; $MO_\ast$, the graded unoriented Thom cobordism algebra; and $\mathcal{M}_\ast(\mathbb{Z}_2)$, the graded unoriented cobordism ring of principal $O(k)$ bundles with

$$\mathcal{M}_n(\mathbb{Z}_2) = \sum_{j=0}^{n} MO_j(BO(n-j))$$

and $MO_n(BO(0)) = MO_n$, by definition.

Conner and Floyd completely determined the additive structure of $I_\ast(\mathbb{Z}_2)$ in [3;28.1].

I.1. **THEOREM**: The sequence of $MO_\ast$-modules

$$0 \rightarrow I_n(\mathbb{Z}_2) \xrightarrow{\iota_\ast} \mathcal{M}_n(\mathbb{Z}_2) \xrightarrow{J} MO_{n-1}(\mathbb{Z}_2) \rightarrow 0$$

is split exact.///

The maps, $\iota_\ast$ and $J$, and the splitting maps are defined as follows.

I.3. $\iota_\ast$. Let $\{T,M^l\} \subset I_n(\mathbb{Z}_2)$ and let $F^m$ denote the $m$-dimensional component of the fixed point set of $T$ on $M^n$. 

1
Let $\xi_m: E_m \to F^m$ denote the normal bundle to $F^m$. Define $\iota_*$ by

$$\iota_*([T_* M^n])_2 = \sum_{m=0}^{n-1} [\xi_m]_2 \in \mathcal{M}_n(\mathbb{Z}_2).$$

I.4. $\rho: \mathcal{M}_n(\mathbb{Z}_2) \to I_n(\mathbb{Z}_2)$, the splitting map for $\iota_*$. Let $\xi: E \to V^m$ be a differentiable linear $O(n-m)$ bundle, and let $\xi' = \xi \otimes R: E' \to V^m$ with fiber $R^m \times R$. Define $T'(v,t) = (-v,-t)$ and $S(v,t) = (-v,t)$ on $E'$. Restrict to the $(n-m)$ sphere bundle, $B'$, in $E'$. Then $S$ induces an involution $S$ on the $\mathbb{R}P(n-m)$ bundle $B'/T'$. Define

$$\rho([\xi])_2 = [\hat{A}, B'/T']_2 \in I_n(\mathbb{Z}_2).$$

I.5. $J: \mathcal{M}_n(\mathbb{Z}_2) \to \mathcal{M}_{n-1}(\mathbb{Z}_2)$. Let $\xi: E \to V^n$ be a differentiable $k$-dimensional vector space bundle. Consider the associated $(k-1)$-dimensional sphere bundle with fiber $S^{k-1}$ and structure group $O(k)$, say $q: S(\xi) \to V^n$. $S(\xi)$ is a closed $(n+k-1)$-dimensional manifold. The antipodal map lies in the center of $O(k)$, so there is a fiber preserving differentiable fixed point free involution $(T, S(\xi))$ which on each fiber reduces to the antipodal map. Then define $J: \mathcal{M}_n(BO(k)) \to \mathcal{M}_{n+k-1}(\mathbb{Z}_2)$ by

$$J([\xi])_2 = [T, S(\xi)]_2.$$  

We then define $J: \mathcal{M}_n(\mathbb{Z}_2) = \sum_{j=0}^{n} \mathcal{M}_j(BO(n-j)) \to \mathcal{M}_{n-1}(\mathbb{Z}_2)$ to be the sum of the above homomorphisms. Note that $J(\mathcal{M}_n) = 0$.

I.6. $K: \mathcal{M}_{n-1}(\mathbb{Z}_2) \to \mathcal{M}_n(\mathbb{Z}_2)$, the splitting map to $J$. Since the antipodal maps on the spheres form a homogeneous basis
for \( \text{MO}_\ast(\mathbb{Z}_2) \), we have for any \( [T, V^{n-1}]_2 \in \text{MO}_{n-1}(\mathbb{Z}_2) \),

\[
[T, V^{n-1}]_2 = \sum_{m=0}^{n-1} [W^{n-m-1}]_2 [A, S^m]_2
\]

with \( [W^{n-m-1}]_2 \in \text{MO}_{n-m-1} \) for all \( 0 \leq m \leq n-1 \). Let \( \xi_m : E_m \to W^{n-m-1} \) be the trivial \((m+1)\)-dimensional real vector space bundle over \( W^{n-m-1} \). Define

\[
K([T, V^{n-1}]_2) = \sum_{m=0}^{n-1} [\xi_m]_2.
\]

I.7. Lemma: For \( n \geq 0 \), \( \rho \tau_\ast = \text{identity} \).

**Proof:** This is [3;28.2]. Consider a differentiable involution \((T, M^n)\) on a closed smooth manifold. Consider the manifold \( M^n \times S^1 \) with the following three involutions.

\[
\begin{align*}
T_1 (m, z) &= (T(m), -z), \\
T_2 (m, z) &= (m, \overline{z}), \quad \text{and} \\
T_3 (m, z) &= (T(m), z);
\end{align*}
\]

where we think of \( S^1 \) as \( \{ z \in \mathbb{C} \mid |z|=1 \} \) and where \( \overline{z} \) denotes the complex conjugate of \( z \). Notice that these three involutions commute. Thus \( T_2 \) and \( T_3 \) induce involutions \((T', M^n \times S^1 / T_1)\) and \((S, M^n \times S^1 / T_1)\) on the manifold \( M^n \times S^1 / T_1 \).

\( T_1 \) is fixed point free. The fixed point set of \( T_2 \) on \( M^n \times S^1 \) is \( M^n \times \{-1, 1\} \). The coincidence of \( T_1 \) and \( T_2 \) is \( F \times \{-i, i\} \), where \( F \) is the fixed point set of \( T \) on \( M^n \). Thus the fixed point set of \( T' \) on \( M^n \times S^1 / T_1 \) is \( F \cup M_n \), the disjoint union of \( F \) and \( M^n \). The normal bundle to this fixed point set is \( (\nu \oplus \theta) \cup \theta \), where \( \nu \) is the normal bundle to \( F \) in \( M^n \) and \( \theta \) is the trivial line bundle.

Note that \( T' \) restricted to the normal sphere bundle to its fixed point set in \( M^n \times S^1 / T_1 \) reduces to the bundle involution.
Moreover, $S$ and $T'$ commute. Let $V^{n+1} \subset (M^n \times S^1)/T_1$ be the compact submanifold with boundary, invariant under $S$ and $T'$, obtained by removing the interior of a tubular neighborhood about the fixed point set of $T'$. Since $T'$ acts freely on $V^{n+1}$, we get an involution $(\hat{\alpha}, \hat{V}^{n+1}/T')$ on a compact manifold with boundary. Examination of $(\hat{\alpha}, \hat{V}^{n+1}/T')$ shows that one component of the boundary is $(T, M^n)$ and the other is $p_{\nu *}(\{T, M^n\}_2)$.///

We now want to define the endomorphism $\Gamma$. We want to consider the circle as $S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$. Define the map $\Gamma: I_n(\mathbb{Z}_2) \to I_{n+1}(\mathbb{Z}_2)$ as follows. Take $\{T, M^n\}_2 \in I_n(\mathbb{Z}_2)$ and consider the manifold $S^1 \times M^n$ with the two involutions

$$T_1(z, m) = (-z, T(m)) \quad \text{and} \quad T_2(z, m) = (\overline{z}, m)$$

for $z \in S^1$ and $m \in M^n$. Note that $T_1$ is fixed point free, so the quotient of $S^1 \times M^n$ by $T_1$ is a manifold; call it $M'$. $T_1$ and $T_2$ commute, so $T_2$ induces an involution, say $T'$, on the quotient manifold $S^1 \times M^n/T_1$. Put

$$\Gamma(\{T, M^n\}_2) = \{T', S^1 \times M^n/T_1\}_2 = \{T', M'\}_2 \in I_{n+1}(\mathbb{Z}_2).$$

Note that this definition of $\Gamma$ differs from that given in [1]. Since we want to use the product formula [1;1.3], we must resolve this difference. Consider a differentiable involution $(T, M^n)$ and put $\Gamma(\{T, M^n\}_2) = \{T', M'\}_2$. Recall that $T_1$ is fixed point free on $S^1 \times M^n$. The fixed point set of $T_2$ on $S^1 \times M^n$ is $\{-1, 1\} \times M^n$. The coincidence of $T_1$ and $T_2$ is then $\{-i, i\} \times F(T)$, where $F(T)$ is the fixed point set of $T$ on $M^n$. Thus, the fixed point set of $T'$ on $M'$ is the
disjoint union of \( F(T) \) and \( \mathbb{M}^n \). The normal bundle to this fixed set is \( (\nu \oplus \theta^1) \cup \theta^1 \), where \( \theta^1 \) is the trivial real line bundle and \( \nu \) is the normal bundle to \( F(T) \) in \( \mathbb{M}^n \). Note that the second factor of \( \theta^1 \) appears as the normal bundle to \( \mathbb{M}^n \) in \( S^1 \times \mathbb{M}^n / T_1 \). This is exactly the same fixed point data as the map \( \Gamma \) in [1]. So, these two maps agree on the bordism level.

Now, we can use the following results about \( \Gamma \) from [1]. The map \( \Gamma \) is well-defined, additive, and an \( MO_\ast \)-module map. Further, \( \Gamma(MO_\ast) = 0 \). We also have the following product formula for \( \Gamma \): for any \( x, y \in I_n(\mathbb{Z}/2) \)

\[
(1.8) \quad \Gamma(xy) = x \cdot \Gamma(y) + \Gamma(x) \cdot \epsilon(y)
\]

\[
= \Gamma(x) \cdot y + \epsilon(x) \cdot \Gamma(y),
\]

where \( \epsilon \) is the augmentation map, \( \epsilon : I_\ast(\mathbb{Z}/2) \to MO_\ast \), given by \( \epsilon([T, \mathbb{M}^n]_2) = [\mathbb{M}^n]_2 \).
Consider the set
\[ \mathcal{A}_1 = \{ x + \Gamma(x) \mid x \in I_\star(\mathbb{Z}_2) \text{ and } \epsilon(x) = 0 \} \]
in \( I_\star(\mathbb{Z}_2) \). Let \( y \in I_\star(\mathbb{Z}_2) \), then
\[ (x + \Gamma(x)) y = xy + \Gamma(x)y = xy + \Gamma(xy) + \epsilon(x)\Gamma(y) = xy + \Gamma(xy) \in \mathcal{A}_1, \]
since
\[ \epsilon(xy) = \epsilon(x)\epsilon(y) = 0. \]
Also, note that \( y(x + \Gamma(x)) = yx + \Gamma(yx) \) is in \( \mathcal{A}_1 \). Clearly, if \( y + \Gamma(y) \) is also in \( \mathcal{A}_1 \), then \( (x + y) + \Gamma(x + y) = x + \Gamma(x) + y + \Gamma(y) \) is in \( \mathcal{A}_1 \), since \( \epsilon \) and \( \Gamma \) are both additive. Hence, \( \mathcal{A}_1 \) is a two-sided ideal of \( I_\star(\mathbb{Z}_2) \).

Define a set, \( \mathcal{A}_n \), for each positive integer \( n \), as follows:
\[ \mathcal{A}_n = \{ x + \Gamma^n(x) \mid x \in I_\star(\mathbb{Z}_2) \text{ and } \epsilon(\Gamma^j(x)) = 0 \text{ for all } 0 \leq j < n \}. \]
The sets \( \mathcal{A}_n \) are also two-sided ideals for each \( n \). The only thing to see in this case is that \( \Gamma^n(xy) = \Gamma^n(x)y \) if \( x + \Gamma^n(x) \in \mathcal{A}_n \). We know that this is the case if \( n = 1 \) and \( \epsilon(x) = 0 \). Assume that this is true for \( n \) and \( \epsilon(\Gamma^j(x)) = 0 \) for \( 0 \leq j < n \). Now assume that \( \epsilon(\Gamma^j(x)) = 0 \) for \( 0 \leq j < n+1 \).
We wish to show that $\Gamma^{n+1}(xy) = \Gamma^{n+1}(x)y$.

$\Gamma^{n+1}(xy) = \Gamma(\Gamma^n(xy))$

$= \Gamma(\Gamma^n(x)y)$, by the inductive step

$= \Gamma^{n+1}(x)y + \varepsilon(\Gamma^n(x))\Gamma(y)$

$= \Gamma^{n+1}(x)y$, since $\varepsilon(\Gamma^n(x)) = 0$.

Since $\mathcal{A}_n$ is a two-sided ideal of $I_*(\mathbb{Z}_2)$ for each $n > 0$, we may form the quotient ring $I_*(\mathbb{Z}_2)/\mathcal{A}_n$. We would like to know the structure of this quotient ring. For $n > 0$, let $\mathcal{O}_n$ be the principal ideal in $M_*(\mathbb{Z}_2)$ generated by the element $1 + [\theta^n \to \text{pt}]._2$, where $\theta^n$ is the trivial real $n$-plane bundle. For $n = 1$ we have the following result.

**II.1. Theorem:** $I_*(\mathbb{Z}_2)/\mathcal{A}_1 \approx M_*(\mathbb{Z}_2)/\mathcal{O}_1$, as rings.

**Proof:** We have the following diagram of exact sequences of $\text{MO}_*\text{-modules}$.

![Diagram](image)

We will show that for all $n > 0$, $\text{Ker}(\pi \circ \iota_*) = \mathcal{A}_n$. However, only for $n = 1$ is the map $\pi \circ \iota_*$ an epimorphism. Before going any further, we must note that $\iota_*$ and $\pi$ are ring homomorphisms.
as well as being $\text{MO}_*$-module homomorphisms. Further still, $\iota_*$ is a ring monomorphism. The maps $\iota$ and $J$ are only $\text{MO}_*$-module maps.

(i) For $n = 1$, $\pi \circ \iota_*$ is onto.

The only elements in $\mathcal{M}(\mathbb{Z}_2)$ that are not in the image of $\iota_*$ are the elements of the form $M^{n-r} \times D^r$; i.e., the trivial $r$-plane bundles, $\Theta^r$. Now, $M^{n-r} \times D^0 \in \text{Im}(\iota_*)$ as $\iota_*\{(\text{id},M^{n-r})\}_2$. In the quotient ring $\mathcal{M}(\mathbb{Z}_2)/\Theta_1$, we are identifying $\Theta$ and $1$; in other words, we are identifying $M^{n-r}$ with $D^1$. Hence, we identify $M^{n-r}$ with $D^r$. Thus, $\{(\text{id},M^{n-r})\}_2$ is a preimage of $M^{n-r} \times D^r$ under $\pi \circ \iota_*$. 

(ii) $\ker(\pi \circ \iota_*) = \mathcal{A}_n$, for $n \geq 1$.

To show that $\mathcal{A}_n \subseteq \ker(\pi \circ \iota_*)$, let $x + \Gamma^n(x) \in \mathcal{A}_n$. Let $F$ denote the fixed point set from $x$ and let $\nu$ denote its normal bundle. Now, $\Gamma^n(x)$ has fixed point set $F \cup \bigcup_{m=0}^{n-1} \Gamma^m(x)$ with normal bundle $(\nu \oplus \Theta^n) \cup \bigcup_{m=1}^n \Theta^m$; where $\Theta^m$ sits over $\Gamma^{n-m}(x)$.

Since $x + \Gamma^n(x) \in \mathcal{A}_n$, we know that $\epsilon(\Gamma^m(x)) = 0$ for all $0 \leq m < n$. Thus, the terms $\Gamma^m(x)$ contribute nothing to the bordism class of the normal bundle over $\Gamma^n(x)$, for $0 \leq m < n$. Thus,

$$\iota_*(x + \Gamma^n(x)) = [\nu]_2 + [\nu \oplus \Theta^n]_2 = [\nu]_2(1 + [\Theta^n \rightarrow \text{pt}])_2.$$ 

Thus, $\iota_*(x + \Gamma^n(x)) \in \Theta_n$, and thus $\mathcal{A}_n \subseteq \ker(\pi \circ \iota_*)$.

For the opposite inclusion, let $\alpha \in \ker(\pi \circ \iota_*)$ be non-zero. Then, of course, $\pi(\iota_*(\alpha)) = 0$. Since $\iota_*$ is a monomorphism, $\iota_*(\alpha) \neq 0$. Thus, $\iota_*(\alpha) \in \Theta_n$. Let

$$\iota_*(\alpha) = [\xi + \xi \oplus \Theta^n]_2.$$

Now, $J([\xi + \xi \oplus \Theta^n]_2) = 0$ since the sequence (1.2) is exact. Therefore,
There is no reason to believe that $[\xi]_2$ is a homogeneous element in $M_*(\mathbb{Z}_2)$. So, let

$$[\xi]_2 = (0, \ldots, 0, [\xi]_2, \ldots, [\xi]_{k+m}, 0, \ldots).$$

For dimensional reasons

\[(\text{II.2}) \quad J([\xi_j \otimes \theta^n]_2) = 0\]

for $k+m-n < j \leq k+m$. From [3;26.4] we have the following commutative diagram

\[
\begin{array}{ccc}
M_{n+m}(\mathbb{Z}_2) & \rightarrow & MO_{n+m-1}(\mathbb{Z}_2) \\
\theta^m \uparrow & & \downarrow \Delta^m \\
M_n(\mathbb{Z}_2) & \rightarrow & MO_{n-1}(\mathbb{Z}_2)
\end{array}
\]

where $\Delta$ is the Smith homomorphism. From this and (II.2) we get that

$$J([\xi]_2) = 0 \quad \text{for} \quad k+m-n < j \leq k+m.$$

But then we have that

$$J([\xi_j \otimes \theta^n]_2) = 0 \quad \text{for} \quad k+m-2n < j \leq k+m-n$$

and hence

$$J([\xi]_2) = 0 \quad \text{for} \quad k+m-2n < j \leq k+m-n.$$

Continuing this reasoning on blocks of $n$ in $[\xi]_2$, we get that $J([\xi_j]_2) = 0$ for $k \leq j \leq k+m$, or $J([\xi]_2) = 0$. Since the sequence (I.2) is exact, we have that $[\xi]_2 \in \text{Im}(\iota_*)$. Let $x \in I_*(\mathbb{Z}_2)$ be such that $\iota_*(x) = [\xi]_2$. We need to show that $\iota_*(\Gamma^n(x)) = [\xi \otimes \theta^n]_2$ and that $e(\Gamma^m(x)) = 0$ for $0 \leq m < n$.

To compute $e([T, M^n]_2)$ we can look at the normal bundle to the fixed point set, add a trivial line bundle, pass to the associated sphere bundle, and factor out the involution to get the real projective space bundle. Now,
\[ \tau_{\ast}(\Gamma^n(x)) = [\xi \oplus \theta^n_\ast]_2 + [\theta^n_\ast \to x]_2 + [\theta^{n-1}_\ast \to \Gamma(x)]_2 + \ldots + [\theta^1_\ast \to \Gamma^{n-1}(x)]_2. \]

We have that \( J(\tau_{\ast}(\Gamma^n(x))) = 0 \) and \( J([\xi \oplus \theta^n_\ast]_2) = 0. \) So,

\[ \sum_{m=1}^{n} J([\theta^m_\ast \to \Gamma^{n-m}(x)]_2) = 0. \]

But these appear in different dimensions, so we must have that \( J([\theta^m_\ast \to \Gamma^{n-m}(x)]_2) = 0 \) for \( m = 1, \ldots, n. \) But this is passing to the sphere bundle of the normal bundle to the fixed point set. At this stage, we already have 0. Therefore, we have that \( e(\Gamma^m(x)) = 0 \) for \( 0 \leq m < n. \) Thus,

\[ \tau_{\ast}(\Gamma^n(x)) = [\xi \oplus \theta^n_\ast]_2, \]

or

\[ \rho([\xi \oplus \theta^n_\ast]_2) = \Gamma^n(x). \]

Therefore,

\[ a = \rho_{\ast}(a) = \rho([\xi]_2 + [\xi \oplus \theta^n_\ast]_2) = \rho([\xi]_2) + \rho([\xi \oplus \theta^n_\ast]_2) = x + \Gamma^n(x) \]

and \( e(\Gamma^m(x)) = 0 \) for \( 0 \leq m < n. \) Thus, \( a \in \mathcal{A}_n \) and \( \mathcal{A}_n = \text{Ker}(\pi \circ \tau_{\ast}). \)

We then have shown that the sequence of rings and ideals

\[ 0 \longrightarrow \mathcal{A}_1 \longrightarrow I_{\ast}(\mathbb{Z}_2) \longrightarrow \mathcal{M}_\ast(\mathbb{Z}_2)/\mathcal{I}_1 \longrightarrow 0 \]

is exact.\///

**II.3. Corollary:** \( I_{\ast}(\mathbb{Z}_2)/\mathcal{A}_1 \) is a polynomial ring over \( \mathbb{M}_0 \ast \)

and over \( \mathbb{Z}_2. \)

**Proof:** Recall that \( \mathcal{M}_\ast(\mathbb{Z}_2) \) is a polynomial ring over \( \mathbb{M}_0 \ast \) in a countable number of indeterminants. Recall further that \( \theta \) is one of these indeterminants. Thus, \( \mathcal{M}_\ast(\mathbb{Z}_2)/\mathcal{I}_1 \) is
a polynomial ring over \( \mathbb{Z}_2 \). Noting that \( \mathbb{Z}_2 \) completes the proof.

\[ 
\text{II.4. Corollary: For } n > 1, \text{ the quotient } I_*^n(\mathbb{Z}_2)/\mathcal{A}_n \text{ is a subring of } \mathcal{M}_n(\mathbb{Z}_2)/\mathcal{A}_n, \text{ which as an } \mathcal{M}_n \text{-submodule of } \\
\mathcal{M}_n(\mathbb{Z}_2)/\mathcal{A}_n \text{ has cokernel the } \mathcal{M}_n \text{-submodule generated by } \\
\overline{\mathcal{A}}, \ldots, \overline{\mathcal{A}}, \text{ where } \overline{\mathcal{A}} \text{ denotes the image in } \mathcal{M}_n(\mathbb{Z}_2)/\mathcal{A}_n \text{ of } \\
[\mathcal{A} \mapsto \text{pt}]_2 \text{ in } \mathcal{M}_n(\mathbb{Z}_2). \\
\]

\textbf{Proof:} We have shown in the proof of Theorem (II.2) that \( \mathcal{A}_n = \ker(\pi \circ \iota_*) \) where \( \pi: \mathcal{M}_n(\mathbb{Z}_2) \rightarrow \mathcal{M}_n(\mathbb{Z}_2)/\mathcal{A}_n \) is the canonical quotient map. Thus, we have a ring monomorphism from \( I_*^n(\mathbb{Z}_2)/\mathcal{A}_n \) into \( \mathcal{M}_n(\mathbb{Z}_2)/\mathcal{A}_n \). We can describe those elements in \( \mathcal{M}_n(\mathbb{Z}_2) \) which are not in the image of \( \iota_* \). They are the elements of the form \( M^{n-k} \times D^k \), or \( [\mathcal{A}^{k}]_2 \). By passing to the quotient ring, \( \mathcal{M}_n(\mathbb{Z}_2)/\mathcal{A}_n \), we have identified \( [\mathcal{A}^{n}]_2 \) with 1; i.e., \( [\mathcal{A}^{n}]_2 = \mathcal{A} = 1 \). Of course, 1 \( \in \text{Im}(\iota_*) \) as the image of the trivial \( \mathbb{Z}_2 \)-action on a point. Consider, though, \( [\mathcal{A}^{k}]_2 \) in \( \mathcal{M}_n(\mathbb{Z}_2)/\mathcal{A}_n \), where \( k < n \). We may just as well consider the element \( M^{n-k} \times D^k \). Now, the only way that \( M^{n-k} \times D^k \) could be in the image of \( \iota_* \) is for it to have a preimage. However,
\[ 
\iota_*([\mathcal{A}^{k}(\text{id},M^{n-k})_2]) = [\mathcal{A}^{k}]_2 + \sum_{m=0}^{k-1} [\mathcal{A}^{m} \rightarrow [\mathcal{A}^{k-m}(\mathcal{M}^{n-k})]_2. \\
\]
In order that this be the same as \( [\mathcal{A}^{k}]_2 \), we must then have that \( \in(\mathcal{M}^m(M^{n-k})) = 0 \) for all \( 0 \leq m \leq k-1 \). Thus, we have that \( [M^{n-k}]_2 = 0 \) in \( \mathcal{M}_{n-k} \). So, for a manifold \( M^m \) which does not bound, \( [\mathcal{A}^{k}]_2 \not\in \text{Im}(\iota_*) \), for \( 1 \leq k < n \). This gives us the
There are some very interesting properties of the elements of the form \( x + \Gamma^n(x) \) which we will consider in Chapter V. First, we want to look at an application of Theorem (II.2).
CHAPTER III: BOARDMAN'S FIVE-HALVES THEOREM REVISITED

Let \( \Lambda(\mathbb{Z}_2) = I_*(\mathbb{Z}_2)/\varphi_1 \approx m_*(\mathbb{Z}_2)/\varphi_1 \), which is a polynomial ring over \( MO_* \) and over \( \mathbb{Z}_2 \). Let

\[
MO_* [[t]] = \{ \sum_{k=0}^{\infty} [\nu^k]_2 t^k | [\nu^k]_2 \in MO_k \}
\]

be the subring of "homogeneous" power series of the ring of formal power series over \( MO_* \). Note that we require that the dimension of the coefficient from \( MO_* \) be equal to the exponent of \( t \).

Define a map \( \varphi: I_*(\mathbb{Z}_2) \rightarrow MO_* [[t]] \) by

\[
\varphi(\{T,M^n\}_2) = \sum_{j=0}^{\infty} \epsilon(\Gamma^j(\{T,M^n\}_2)) t^{n+j}.
\]

Clearly, \( \varphi \) is additive since \( \Gamma \) and \( \epsilon \) are additive. Furthermore, we have that \( \varphi \) is multiplicative. To see this, let \( x,y \in I_*(\mathbb{Z}_2) \).

\[
\varphi(xy) = \sum_{j=0}^{\infty} \epsilon(\Gamma^j(xy)) t^{k+j}, k = \text{dim}(x) + \text{dim}(y)
\]

\[
= \epsilon(xy) t^k + \sum_{j=1}^{\infty} \epsilon(\Gamma^j(xy)) t^{k+j}
\]

\[
= \epsilon(xy) t^k + \epsilon(x\Gamma(y) + \epsilon(y)\Gamma(x)) t^{k+1} + \sum_{j=2}^{\infty} \epsilon(\Gamma^j(xy)) t^{k+j}
\]
\[ e(x)e(y)tk + \sum_{j=2}^{\infty} e(I^j(xy))t^{k+j} + (e(x)e(F(y)) + e(y)e(r(x)))tk+1 \]

since \( e(e(y)) = e(y) \), i.e., the augmentation map is the identity on \( MO_* \). Now,

\[ e(\Gamma^2(xy)) = e(\Gamma(x\Gamma(y) + e(y)\Gamma(x))) = e(e(x)\Gamma^2(y) + \Gamma(x)\Gamma(y) + e(y)\Gamma^2(x)) = e(x)e(\Gamma^2(y)) + e(\Gamma(x))e(\Gamma(y)) + e(y)e(\Gamma^2(x)) \]

By using the formula for \( \Gamma \) on products, we see that

\[ e(\Gamma^n(xy)) = \sum_{j=0}^{n} e(I^j(x))e(\Gamma^{n-j}(y)) \]

Thus, \( \phi(xy) = \phi(x)\phi(y) \), and \( \phi \) is multiplicative.

**III.1. Lemma:** \( \phi(\mathcal{A}_1) = 0 \), or \( \mathcal{A}_1 \subseteq \ker(\phi) \).

**Proof:** Let \( x + \Gamma(x) \in \mathcal{A}_1 \); i.e., \( e(x) = 0 \). Then

\[ \phi(x + \Gamma(x)) = \phi(x) + \phi(\Gamma(x)) = e(x) + \sum_{j=1}^{\infty} e(I^j(x))t^{n+j} + \sum_{j=1}^{\infty} e(I^j(x))t^{n+j} = 0 + 0 = 0./// \]

Thus, \( \phi \) induces a homomorphism

\[ \overline{\phi} : \Lambda(\mathbb{Z}_2) \rightarrow MO_* [t] \]

Note that this \( \overline{\phi} \) is well-defined, additive, and multiplicative.

Before continuing any further, note that each \( I_k(\mathbb{Z}_2) \) injects into \( \Lambda(\mathbb{Z}_2) \) for all \( k \geq 0 \). To see this, note that
if \( x \) is a homogeneous element of \( \text{I}_*(\mathbb{Z}_2) \), for example \( x \in \text{I}_n(\mathbb{Z}_2) \), then \( x \) and \( \Gamma(x) \) cannot have the same dimension. In other words, \( x + \Gamma(x) \) is not contained in any \( \text{I}_k(\mathbb{Z}_2) \) for \( k \geq 0 \). Thus, we do not introduce any relations on the homogeneous \( \text{I}_n(\mathbb{Z}_2) \) in the quotient \( \Lambda(\mathbb{Z}_2) \) when we identify \( x \) with \( \Gamma(x) \).

As in [2], \( \text{MO}_*(\text{BO}) \) may be interpreted by stabilizing \( \text{m}_*(\mathbb{Z}_2) \) by ignoring the addition of trivial line bundles. In other words, we impose the relation

\[
[\xi]_2 = [\xi \oplus \theta]_2 = [\xi]_2 [\theta]_2
\]

for any vector bundle \( \xi \). This is clearly equivalent to requiring that \( 1 + [\theta - \text{pt}]_2 = 0 \). Thus, we have that \( \text{MO}_*(\text{BO}) \cong \Lambda(\mathbb{Z}_2) \).

We will follow [2] closely for the next few definitions, examples, and theorems. To make the computations simpler, let us define \( \sigma_n(\xi) \), \( e_n(\xi) \), and \( b_n(\xi) \) as in [2]. We will put off the definition of \( b_n(\xi) \) until after Theorem (III.4).

Let \( \xi \) be a real vector bundle with base \( B \). In terms of the Stiefel-Whitney classes, \( w_i(\xi) \), define \( \sigma_n(\xi) \in H^n(B; \mathbb{Z}_2) \) to be that polynomial which expresses the sums of powers \( \sum_i s_i^n \) in terms of the elementary symmetric polynomials of the indeterminants \( s_i \). These characteristic classes satisfy

(i) naturality: \( \sigma_n(f^*\xi) = f^*\sigma_n(\xi) \)

(ii) additivity: \( \sigma_n(\xi \oplus \eta) = \sigma_n(\xi) + \sigma_n(\eta) \)

(iii) \( \sigma_n(\xi) = w_1(\xi)^n \) when \( \xi \) is a line bundle.

See [2; p. 132]. If the base space \( B \) is an \( n \)-dimensional manifold with fundamental class \( w_B \in H_n(B; \mathbb{Z}_2) \), define

\[
e_n(\xi) = \langle \sigma_n(\xi), w_B \rangle \in \mathbb{Z}_2.
\]
As usual, if \( T \) denotes the tangent bundle to \( B \), we define \( \sigma_n(B) = \sigma_n(T) \) and \( e_n(B) = e_n(T) \).

Let us consider some examples at this point. We will actually need these calculations later, so it will be time well spent.

### III.2. Real Projective Space

Let \( \text{RP}(n) \) denote the real projective \( n \)-space and let \( \alpha \in H^1(\text{RP}(n); \mathbb{Z}_2) \) be the generator of the cohomology ring. By adding a trivial line bundle to the tangent bundle, we get the Whitney sum of \( (n+1) \) copies of the canonical line bundle over \( \text{RP}(n) \). Thus,

\[ \sigma_n(\text{RP}(n)) = (n+1)\alpha^n, \quad \text{and} \]
\[ e_n(\text{RP}(n)) = (n+1) \mod 2. \]

### III.3. The Milnor Hypersurface \( H(m,n) \)

\( H(m,n) \) is the surface in \( \text{RP}(m) \times \text{RP}(n) \) defined by the equation

\[ u_0v_0 + u_1v_1 + \ldots + u_mv_m = 0 \]

where \( u_0, \ldots, u_m \) are homogeneous coordinates in \( \text{RP}(m) \) and \( v_0, \ldots, v_n \) are homogeneous coordinates in \( \text{RP}(n) \), and we assume that \( m \leq n \). Let \( \alpha \) and \( \beta \) be the generators of \( H^1(\text{RP}(m) \times \text{RP}(n); \mathbb{Z}_2) \) inherited from \( \text{RP}(m) \) and \( \text{RP}(n) \), respectively. If \( v \) is the normal line bundle to \( H(m,n) \) in \( \text{RP}(m) \times \text{RP}(n) \), then \( w_1(v) = \alpha + \beta \). Thus,

\[ \sigma_{m+n-1}(H(m,n)) = \sigma_{m+n-1}(\text{RP}(m)) + \sigma_{m+n-1}(\text{RP}(n)) + (\alpha + \beta)^{m+n-1} \]
\[ \sigma_{m+n-1}(H(m,n)) = (m+1)\alpha^{m+n-1} + (n+1)\beta^{m+n-1} + (\alpha + \beta)^{m+n-1}. \]

The first two terms vanish if \( 2 \leq m \leq n \) for dimensional
reasons. Note that for any class $\gamma \in H^*(\mathbb{RP}(m) \times \mathbb{RP}(n); \mathbb{Z}_2)$ we have

$$<j^*\gamma, w_{\mathbb{H}(m,n)}> = <\gamma \cup (\alpha + \beta), w_{\mathbb{RP}(m) \times \mathbb{RP}(n)}>$$

where $j: \mathbb{H}(m,n) \to \mathbb{RP}(m) \times \mathbb{RP}(n)$ is the inclusion. To see this, note that the one-dimensional cohomology class in $H^*(\mathbb{RP}(m) \times \mathbb{RP}(n); \mathbb{Z}_2)$ dual to the submanifold $\mathbb{H}(m,n)$ must be $\alpha + \beta$, cf. [5; p.69]. Thus,

$$j_* w_{\mathbb{H}(m,n)} = w_{\mathbb{RP}(m) \times \mathbb{RP}(n)} \cap (\alpha + \beta).$$

Then

$$<j^*\gamma, w_{\mathbb{H}(m,n)}> = <\gamma, j_* w_{\mathbb{H}(m,n)}> = <\gamma, w_{\mathbb{RP}(m) \times \mathbb{RP}(n)} \cap (\alpha + \beta)> = <\gamma \cup (\alpha + \beta), w_{\mathbb{RP}(m) \times \mathbb{RP}(n)}>.$$

Thus, to calculate $e_{m+n-1}(\mathbb{H}(m,n))$, we need the coefficient of $\alpha^m \beta^m$ in $(\alpha + \beta)^{m+n}$, which is by definition $\binom{m+n}{m}$ mod 2. Thus

$$e_{m+n-1}(\mathbb{H}(m,n)) = \binom{m+n}{m} \mod 2.$$ 

If $m = 1$, we need to modify the computation, slightly. Since $\mathbb{H}(1,1)$ is just $S^1$, let us assume that $n \geq 2$. Then, $e_n(\mathbb{H}(1,n))$ is the coefficient of $\alpha \beta^n$ in

$$(n+1)\beta^n (\alpha + \beta) + (\alpha + \beta)^{n+1}.$$ 

This is easily seen to be 0, so

$$e_n(\mathbb{H}(1,n)) = 0.$$ 

III.4. THEOREM: [2; Theorem 1] The ring $MO_*$ is a graded polynomial ring over $\mathbb{Z}_2$ with one generator in each dimension not of the form $2^q - 1$. The class $[V^k]_2$ in $MO_*$ serves as a polynomial generator in dimension $k$ if and only if $e_k(V^k) = 1.$
Not only will we want to find the generators of $MO_*(BO)$, but we shall also want to detect generators of the subring $MO_*$ of $MO_*(BO)$. To do this we define

$$b_n(\xi) = e_n(\nu^n)$$

whenever $\xi$ is a vector bundle over a manifold $\nu^n$.

III.5. THEOREM: [2;Lemma 5] Suppose that we are given for each positive $n$, elements $y_n'$ and $y_n''$ of $MO_*(BO)$, with $y_n'$ absent if $n$ has the form $2^q-1$. These elements form a system of polynomial generators of the ring $MO_*(BO)$ over $\mathbb{Z}_2$ if and only if for each $n$ the pairs of numbers $(b_n(y_n'),e_n(y_n'))$, $(b_n(y_n''),e_n(y_n''))$ and $(0,0)$ are distinct.

**Proof:** See[2;p.137].//

Note that since $\Lambda(\mathbb{Z}_2) \approx MO_*(BO)$, we may replace $MO_*(BO)$ in Theorem (III.5) by $\Lambda(\mathbb{Z}_2)$.

III.6. THEOREM: There exist elements $\{\tau, M^n\}_2$ in $\Lambda(\mathbb{Z}_2)$ for each $n \geq 0$ such that:

(i) $\{[M^n]_2\}_{n=0}^{\infty}$ generates $MO_*$ as a polynomial ring over $\mathbb{Z}_2$, for $n$ not of the form $2^q-1$;

(ii) $\{[\tau, M^n]_2\}_{n=0}^{\infty}$ generates $\Lambda(\mathbb{Z}_2)$ as a polynomial ring over $\mathbb{Z}_2$.

**Proof:** We again appeal to [2] for these elements and
the necessary calculations. This theorem is a part of [2; Lemma 16].

Define the involution \( T_i \) on \( \mathbb{RP}(2i) \) by
\[
T_i([x_0, x_1, \ldots, x_i, x'_i]) = [x_0, x_1, \ldots, x_i, -x'_i, \ldots, -x'_i]
\]
in terms of the homogeneous coordinates for \( \mathbb{RP}(2i) \). The fixed point set of \( T_i \) is the disjoint union of \( \mathbb{RP}(i) \) and \( \mathbb{RP}(i-1) \). Letting \( \alpha \in H^1(\mathbb{RP}(2i); \mathbb{Z}_2) \) be the generator of this cohomology ring, (III.2) gives us that
\[
\sigma_{2i}(\mathbb{RP}(2i)) = (2i + 1)\alpha^{2i} = \alpha^{2i}.
\]
For the normal bundle \( \nu_1 \) to \( \mathbb{RP}(i) \) in \( \mathbb{RP}(2i) \), we have
\[
\sigma_i(\nu_1) = i\alpha^i.
\]
For the normal bundle \( \nu_2 \) to \( \mathbb{RP}(i-1) \) in \( \mathbb{RP}(2i) \), we have
\[
\sigma_{i-1}(\nu_2) = (i + 1)\alpha^{i-1}.
\]
We also want to consider an involution on the Milnor hypersurface \( H(2i, 2j) \) in \( \mathbb{RP}(2i) \times \mathbb{RP}(2j) \) defined by the equation
\[
x_0u_0 + \ldots + x_iu_i + x'_1u'_1 + \ldots + x'_i u'_i = 0
\]
where \( u_0, u'_1, \ldots, u_j, u'_1, \ldots, u_j \) are homogeneous coordinates in \( \mathbb{RP}(2j) \) and \( i \leq j \). The product involution \( T_i \times T_j \) on \( \mathbb{RP}(2i) \times \mathbb{RP}(2j) \) maps \( H(2i, 2j) \) into itself. The fixed point set of \( T_i \times T_j \) restricted to \( H(2i, 2j) \) is the various intersections of \( H(2i, 2j) \) with the four fixed point sets of \( T_i \times T_j \) on \( \mathbb{RP}(2i) \times \mathbb{RP}(2j) \). Now, \( \mathbb{RP}(i) \times \mathbb{RP}(j) \) intersects \( H(2i, 2j) \) in a copy of \( H(i, j) \). The normal bundle, \( \nu_1 \), of \( H(i, j) \) in \( H(2i, 2j) \) is the restriction to \( H(i, j) \) of the normal bundle of \( \mathbb{RP}(i) \times \mathbb{RP}(j) \) in \( \mathbb{RP}(2i) \times \mathbb{RP}(2j) \). So,
\[
\sigma_{i+j-1}(\nu_1) = i\alpha^{i+j-1} + j\beta^{i+j-1}.
\]
The two sets $\text{RP}(i) \times \text{RP}(j-1)$ and $\text{RP}(i-1) \times \text{RP}(j)$ are both contained in $H(2i,2j)$. The normal bundle, $v_2$, of $\text{RP}(i) \times \text{RP}(j-1)$ becomes, if we add the normal bundle of $H(2i,2j)$ in $\text{RP}(2i) \times \text{RP}(2j)$, the normal bundle of $\text{RP}(i) \times \text{RP}(j-1)$ in $\text{RP}(2i) \times \text{RP}(2j)$. Thus,

$$\sigma_{i+j-1}(v_2) = ia^{i+j-1} + (j+1)b^{i+j-1} + (a+b)^{i+j-1}.$$ 

The similar fact is true for the normal bundle, $v_3$, of $\text{RP}(i-1) \times \text{RP}(j)$ in $H(2i,2j)$. So,

$$\sigma_{i+j-1}(v_3) = (i+1)a^{i+j-1} + jb^{i+j-1} + (a+b)^{i+j-1}.$$ 

$\text{RP}(i-1) \times \text{RP}(j-1)$ intersects $H(2i,2j)$ in a copy of $H(i-1,j-1)$ which has dimension $i+j-3$. Because of the dimension, we may safely ignore this factor.

Since we are working in $\text{MO}_*(BO) \approx \Lambda(\mathbb{Z}_2)$, we need only consider $e_j(\{\tau, M^n\}_2)$ and $b_j(\{\tau, M^n\}_2)$ where $j$ is the maximum of the dimensions of the various components of the fixed point set of $\tau$ on $M^n$. We want $j$ to be the highest dimension of the components of the fixed point set which are non-zero on the bordism level; i.e., if $F^m$ is a component of the fixed point set and $v_m$ denotes its normal bundle with $m > j$, then we require that $[v_m \sim F^m]_2$ be zero. In computing the numbers, $e_j$ and $b_j$, we may ignore components of the fixed point set with lower dimensions, cf. [2]. We now apply Theorem (III.5) and we then need only consider the following four cases.

$n = 4k - 2$: Take $\{\tau, M^{4k-2}\}_2 = \{T_{2k-1}, \text{RP}(4k-2)\}_2$. We only have to consider the fixed point set $\text{RP}(2k-1)$, on which we must evaluate $\sigma_{2k-1}(\text{RP}(2k-1)) = (2k-1)a^{2k-1} = \alpha^{2k-1}$. Now,

$$e_{2k-1}(\{\tau, M^{4k-2}\}_2) = 1,$$ 

and
\[ b_{2k-1}(\{\tau, M^{\frac{1}{4k-2}}\})_2 = e_{2k-1}(\text{RP}(2k-1)) \]
\[ = 2k \alpha^{2k} = 0. \]

\( n = \frac{1}{4k-1} \); \( n \) not of the form \( 2^q-1 \): We can choose positive integers \( i \) and \( j \) such that

(i) \( i + j = k \)
(ii) \( \binom{k}{i} = 1 \text{ (mod 2).} \)

Take \( \{\tau, M^{\frac{1}{4k-1}}\}_2 = \{T_{2i} \times T_{2j} | H(4i, 4j), H(4i, 4j)\}_2 \). The fixed point dimension is \( 2i + 2j - 1 = 2k - 1 \). To find \( e_{2k-1}(\{\tau, M^{\frac{1}{4k-1}}\})_2 \) we add up the evaluations of \( \sigma_{2k-1}(v_i) \) on the fundamental classes of the various components. For \( H(2i, 2j) \),

\[ \sigma_{2k-1}(v_1) = 2ia^{2k-1} + 2j\beta^{2k-1} = 0. \]

For evaluating \( \sigma_{2k-1}(v_2) \) on \( \text{RP}(2i) \times \text{RP}(2j-1) \) we need the coefficient of \( \alpha^{2i} \beta^{2j-1} \) in \( (\alpha + \beta)^{2k-1} \), which is \( \binom{2k-1}{2i} \).

Also, \( \text{RP}(2i-1) \times \text{RP}(2j) \) contributes \( \binom{2k-1}{2j} \). Thus,

\[ e_{2k-1}(\{\tau, M^{\frac{1}{4k-1}}\}_2) = \binom{2k-1}{2i} + \binom{2k-1}{2i-1} = \binom{2k}{2i} = \binom{k}{i} = 1. \]

Furthermore, \( b_{2k-1}(\{\tau, M^{\frac{1}{4k-1}}\})_2 = e_{2k-1}(H(2i, 2j)) = 1. \)

\( n = \frac{1}{4k} \): Take \( \{\tau, M^{\frac{1}{4k}}\}_2 = \{T_{2k}, \text{RP}(\frac{1}{4k})\}_2 \). For the normal bundle to the fixed point set \( \text{RP}(2k) \) we have

\[ \sigma_{2k}(v) = 2k \alpha^{2k} = 0. \]

Thus, \( e_{2k}(\{\tau, M^{\frac{1}{4k}}\}_2) = 0. \) Also, \( b_{2k}(\{\tau, M^{\frac{1}{4k}}\}_2) = e_{2k}(\text{RP}(2k)) = 1. \)

\( n = \frac{1}{4k + 1} \): Take \( \{\tau, M^{\frac{1}{4k+1}}\}_2 = \{T_1 \times T_{2k} | H(2, 4k), H(2, 4k)\}_2 \).

For \( H(1, 2k) \), we have \( \sigma_{2k}(v_1) = \alpha^{2k} + 2k\beta^{2k} = \alpha^{2k} \). Thus, evaluating this on \( H(1, 2k) \) we get 0 from (III.3). For \( \text{RP}(1) \times \text{RP}(2k-1) \),

\[ \sigma_{2k}(v_2) = \alpha^{2k} + (2k+1)\beta^{2k} + (\alpha + \beta)^{2k}. \]
in which the coefficient of $\alpha \beta^{2k-1}$ is $\begin{pmatrix} 2k \\ 1 \end{pmatrix} = 0 \pmod{2}$. For $RP(0) \times RP(2k)$, we have

$$\sigma_{2k}(v_3) = 2\alpha^{2k} + 2k\beta^{2k} + (\alpha + \beta)^{2k}$$

which contains the term $\beta^{2k}$. Thus,

$$e_{2k}(\{r, M^{4k+1}\}) = 1$$

and

$$b_{2k}(\{r, M^{4k+1}\}) = e_{2k}(H(1,2k)) = 0.$$ 

This completes (ii).

For (i) all we need to do is compute the following numbers and apply Theorem (III.4).

$$e_{4k-2}(M^{4k-2}) = e_{4k-2}(RP(4k-2))$$

$$= \langle (4k-1)\alpha^{-1}, \omega_{RP}(4k-2) \rangle$$

$$= 1.$$ 

$$e_{4k-1}(M^{4k-1}) = e_{4k-1}(H(4i,4j))$$

$$= \begin{pmatrix} \frac{4k}{i} \\ \frac{k}{j} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= 1 \pmod{2}, \text{ by the choice of } i.$$ 

$$e_{4k}(M^{4k}) = e_{4k}(RP(4k)) = 1.$$ 

$$e_{4k+1}(M^{4k+1}) = e_{4k+1}(H(2,4k))$$

$$= \begin{pmatrix} 4k+2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2k+1 \\ 1 \end{pmatrix}$$

$$= 1 \pmod{2}. \text{///}$$

III.7. **Corollary:** $\overline{\varphi}: \Lambda(\mathbb{Z}_2) \to \mathcal{M}_* \llbracket t \rrbracket$ is a monomorphism.

**Proof:** By checking $\overline{\varphi}$ on the generators, this is clear. \text{///}

Let us introduce two filtrations on $\Lambda(\mathbb{Z}_2)$. The first
filtration, \( \text{fil}_{FP} \), is an increasing filtration. We shall say that \( \text{fil}_{FP}(\{T,V^n\}_2) = k \) for \( \{T,V^n\}_2 \in I_n(\mathbb{Z}_2) \subseteq \Lambda(\mathbb{Z}_2) \) if \( k \) is the maximum of the dimensions of the various components of the fixed point set.

There is a decreasing filtration of \( \mathcal{MO}_* \) given by \( \text{fil}(x) = n \) if the first non-zero coefficient in the power series for \( x \) is the coefficient of \( t^n \). Thus, \( \overline{\varphi} \) induces a decreasing filtration on \( \Lambda(\mathbb{Z}_2) \) denoted by \( \text{fil}_{\overline{\varphi}} \). For an element \( \{T,V^n\}_2 \in I_n(\mathbb{Z}_2) \) we have that 
\[
\text{fil}_{\overline{\varphi}}(\{T,V^n\}_2) = n + j
\]
if \( j(\{T,V^n\}_2) \neq 0 \) and all of the preceding powers of \( \Gamma \) on \( \{T,V^n\}_2 \) do augment to 0.

III.8. Lemma: For the genera \( \{\tau,M^n\}_2 \) given in Theorem (III.6), we have
\[
\begin{align*}
\text{(i) } \text{fil}_{FP}(\{\tau,M^n\}_2) &= \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even,} \\
\frac{n-1}{2} & \text{if } n \text{ is odd;}
\end{cases} \\
\text{(ii) } \text{fil}_{\overline{\varphi}}(\{\tau,M^n\}_2) &= n.
\end{align*}
\]

Proof: This is clear from the construction of these generators in Theorem(III.6).////

We would like to know what happens to these two filtrations on products and sums. The following facts are clear from the definitions of the filtrations.

\[
\begin{align*}
\text{(i) } \text{fil}_{FP}(xy) &= \text{fil}_{FP}(x) + \text{fil}_{FP}(y), \text{ for any } x \neq 0 \text{ and } y \neq 0.
\end{align*}
\]
(ii) \( \text{fil}_\varphi(xy) = \text{fil}_\varphi(x) + \text{fil}_\varphi(y) \), for any \( x \neq 0 \) and \( y \neq 0 \).

(iii) \( \text{fil}_{FP}(x + y) = \max\{\text{fil}_{FP}(x), \text{fil}_{FP}(y)\} \) if \( x \) and \( y \) have no monomials in common.

(iv) \( \text{fil}_\varphi(x + y) = \min\{\text{fil}_\varphi(x), \text{fil}_\varphi(y)\} \) if \( x \) and \( y \) have no monomials in common.

It then follows that the fixed point filtration, \( \text{fil}_{FP} \), of a polynomial in the generators \( \{\tau, M^n\} \) is the maximum of the fixed point filtrations of its terms. For the \( \varphi \)-filtration, \( \text{fil}_\varphi \), of a polynomial in the generators, we have the minimum of the \( \varphi \)-filtrations of its terms.

Note that for any generator \( \{\tau, M^n\} \), \( n \) not of the form \( 2^q-1 \), we have that

\[
\text{fil}_\varphi(\{\tau, M^n\}) \leq \frac{5}{2} \text{fil}_{FP}(\{\tau, M^n\}).
\]

This is clear if \( n \) is even and bigger than 5. For \( n = 2 \), we have \( 2 \leq \frac{5}{2}(1) \); and for \( n = 4 \), \( 4 \leq \frac{5}{2}(2) = 5 \). If \( n \) is odd and \( n \geq 5 \), we have

\[
n \leq \frac{5}{2}(\frac{n-1}{2}) = \frac{5n-5}{4}, \text{ or } 4n \leq 5n - 5.
\]

We do not have to consider \( n = 1 \) and \( n = 3 \), since both are of the form \( 2^q-1 \). We cannot improve on this inequality, for we have equality when \( n = 5 \); i.e.,

\[
\text{fil}_\varphi(\{\tau, M^5\}) = \frac{5}{2} \text{fil}_{FP}(\{\tau, M^5\}).
\]

III.9. THEOREM: (Boardman's Five-Halves Theorem) Let \( T \) be a smooth involution on a closed manifold \( V^n \) of dimension \( n \) and let \( k \) be the fixed point dimension; i.e., the maximum of
the dimensions of the various components of the fixed point set which are non-zero on the bordism level. If \( V^n \) does not bound, \( [V^n]_2 \neq 0 \), then \( n \leq \frac{5}{2} k \).

\[ \text{Proof:} \] Note first that \( \{T, V^n\}_2 \in I_n(\mathbb{Z}_2) \subset \wedge(\mathbb{Z}_2) \). Since \( [V^n]_2 \neq 0 \) in \( MO_n \), we have that
\[ \text{fil}_\phi([T, V^n]_2) = n. \]
Write \( \{T, V^n\}_2 \) as a polynomial in the generators \( \{t, M^n\}_2 \) and call this polynomial \( p_\phi(V^n) \). Then we have,
\[ n = \text{fil}_\phi([T, V^n]_2) \leq \text{fil}_\phi(p_\phi(V^n)) \leq \frac{5}{2} \text{fil}_{FP}(p_\phi(V^n)) \leq \frac{5}{2} \text{fil}_{FP}([T, V^n]_2) = \frac{5}{2} k. \]

This last equality is due to our hypothesis that \( \text{fil}_{FP}([T, V^n]_2) = k \).

Since \( n \) must be an integer, we actually have that \( n \leq \lceil \frac{5k}{2} \rceil \), the greatest integer in \( \frac{5k}{2} \). We have the following corollary.

\[ \text{III.10. Corollary:} \] Under the same assumptions as Theorem (III.9), we have
\[ n \leq \begin{cases} \frac{5k}{2} & \text{if } k \text{ is even,} \\ \frac{5k-1}{2} & \text{if } k \text{ is odd.} \end{cases} \]

\[ \text{III.11. Corollary:} \] Let \( T \) be a smooth involution on a smooth closed manifold \( V^n \), and let \( \text{fil}_{FP}([T, V^n]_2) = k \). If \( [V^n]_2 \) is indecomposable in \( MO_* \) then
\[
\begin{cases}
n \leq 2k + 1 	ext{ if } n \text{ is odd, or} \\
n \leq 2k & \text{if } n \text{ is even.}
\end{cases}
\]

Proof: In this case, we need only to appeal to the fixed point filtration. If \([V^n]_2\) is indecomposable in \(M_0\), then the generator \(\{T,M^n\}_2\) appears by itself in the polynomial for \(\{T,V^n\}_2; \text{i.e.,}\)

\[\{T,V^n\}_2 = \{T,M^n\}_2 + \text{decomposables.}\]

Then, for \(n\) odd,

\[k = \text{fil}_{fp}(\{T,V^n\}_2) \geq \text{fil}_{fp}(\{T,M^n\}_2) = \frac{n-1}{2};\]

or \(n \leq 2k + 1\). For \(n\) even, we have,

\[k = \text{fil}_{fp}(\{T,V^n\}_2) \geq \text{fil}_{fp}(\{T,M^n\}_2) = \frac{n}{2};\]

or \(n \leq 2k.///\)

For the \(\bar{\varphi}\)-filtration, the Five-Halves Theorem tells us that for at least one \(m\) with \(0 \leq m \leq \frac{5k}{2}\), \(\epsilon(\Gamma^m(\{T,V^n\}_2) \neq 0\) in \(M_{0+n+m}\), with \(k = \text{fil}_{fp}(\{T,V^n\}_2)\). Actually, since \(n \geq k\), we have that \(0 \leq m \leq \frac{3k}{2}\).

III.12. Corollary: Let \(T\) be a differentiable involution on a smooth closed manifold \(V^n\) and let \(\text{fil}_{fp}(\{T,V^n\}_2) = k\). If \(n > \frac{5k}{2}\), then \(\{T,V^n\}_2\) bounds in \(I_*(\mathbb{Z}_2)\).

Proof: Recall that \(\bar{\varphi}(\{T,V^n\}_2) = \sum_{j=0}^{\infty} \epsilon(\Gamma^j(\{T,V^n\}_2)) t^{n+j}\). Since \(n > \frac{5k}{2}\), by Theorem (III.9) we know that \([V^n]_2 = \epsilon(\{T,V^n\}_2) = 0\). If there is a \(j\) such that \(\epsilon(\Gamma^j(\{T,V^n\}_2)) \neq 0\), by our previous discussion we know that \(j > \frac{3k}{2}\). Otherwise, Theorem (III.9) would imply that \(n \leq \frac{5k}{2}\). The fixed point set
of $\Gamma^j_j(T, V^n_2)$ is $F(T) \cup \bigcup_{i=0}^{j} \Gamma^i_i(V^n)$, where $F(T)$ denotes the fixed point set of $T$ on $V^n$ and this is a disjoint union.

Let $j$ be the first integer for which $\epsilon(\Gamma^j_j(T, V^n_2)) \neq 0$. Then, $\text{fil}_\varphi(\Gamma^j_j(T, V^n_2)) = n + j$ and $\text{fil}_\text{FP}(\Gamma^j_j(T, V^n_2)) = k$, since $\epsilon(\Gamma^i_i(T, V^n_2)) = 0$ for all $i < j$. Thus,

$$\text{fil}_\varphi(\Gamma^j_j(T, V^n_2)) > \frac{5}{2} \text{fil}_\text{FP}(\Gamma^j_j(T, V^n_2)).$$

But then by Theorem (III.9), $\epsilon(\Gamma^j_j(T, V^n_2)) = 0$. Thus, we must have that $\varphi(T, V^n_2) = 0$. Since $\varphi$ is a monomorphism, we have that $\{T, V^n_2\} = 0$ in $\Lambda(\mathbb{Z}_2)$. Again, $I_n(\mathbb{Z}_2)$ injects into $\Lambda(\mathbb{Z}_2)$. Therefore, we have that $\{T, V^n_2\}$ bounds in $I_n(\mathbb{Z}_2)$.///
chapter iv: an application to flat manifolds

in this section we apply both the map $\Gamma$ and the five-
halves theorem to a conjecture about flat manifolds. we
do not make any claims about the validity of this conjecture.
if the conjecture is true, we have an interesting corollary;
if it is false, we have a way in which a counterexample might
be constructed. the reader interested in the background of
this conjecture should see [9], [10], and [11].

an abstract group, $B$, is a bieberbach group if $B$ has a
normal free abelian subgroup, $B^*$, of finite index. $M^n$ is a
flat manifold if $\pi_1(M^n)$ is a bieberbach group and the rank
of the normal free abelian subgroup is $n$.

if $M^n$ is a flat manifold, then so is $\Gamma(M^n)$. to see this,
consider the fibration

\[
M^n \longrightarrow \Gamma(M^n) \longrightarrow \mathbb{R}P(1).
\]

from the homotopy exact sequence for a fibration, we have
that

\[
0 \longrightarrow \pi_1(M^n) \longrightarrow \pi_1(\Gamma(M^n)) \longrightarrow \mathbb{Z} \longrightarrow 1
\]

is exact. this sequence is actually split exact, so we have
that $\pi_1(\Gamma(M^n)) \approx \pi_1(M^n) \cdot \mathbb{Z}$, the semi-direct product. then,
the normal abelian subgroup of $\pi_1(\Gamma(M^n))$ will be the direct
sum of that of $\pi_1(M^n)$ and the subgroup of index 2 in $\mathbb{Z}$. The rank of this subgroup is $n+1$. Thus, $\pi_1(\Gamma(M^n))$ is a Bieberbach group and $\Gamma(M^n)$ is a flat manifold.

There is a standing conjecture that all flat manifolds bound mod 2, [9] and [10]. Let $T$ be a smooth involution on a flat manifold $M^n$ with $\text{fil}_{FP}(\{T,M^n\}_2) \geq 0$; i.e., the fixed point set is non-empty. Since $M^n$ is flat, $\Gamma^j(M^n)$ is a flat manifold for all $j$. If this conjecture is true, we have that

$$\bar{\varphi}(\{T,M^n\}_2) = \sum_{j=0}^{\infty} \epsilon(\Gamma^j(\{T,M^n\}_2)) t^{n+j}$$

$$= \sum_{j=0}^{\infty} [\Gamma^j(M^n)]_2 t^{n+j}$$

$$= 0.$$  

Since $\bar{\varphi}$ is a monomorphism, $\{T,M^n\}_2 = 0$ in $I_n(\mathbb{Z}_2)$, using the injection of $I_n(\mathbb{Z}_2)$ into $\Lambda(\mathbb{Z}_2)$. If $T$ is a fixed point free involution, we already know that $\{T,M^n\}_2 = 0$ in $I_n(\mathbb{Z}_2)$. Thus, if the conjecture is true and if $M^n$ is a flat manifold, then any smooth involution on $M^n$ bounds in $I_n(\mathbb{Z}_2)$. If the conjecture is false, then $\epsilon(\Gamma^j(\{T,M^n\}_2)) \neq 0$ for some $j$. This may be a way of constructing a counterexample.
CHAPTER V: FACTORIZATION IN $I_*(\mathbb{Z}_2)$

In this section we shall return to the elements of the form $x + \Gamma^n(x)$ in $I_*(\mathbb{Z}_2)$ with $\epsilon(\Gamma^j(x)) = 0$ for all $0 \leq j < n$. Since $I_*(\mathbb{Z}_2)/\mathfrak{a}_1 \cong \mathfrak{m}_*(\mathbb{Z}_2)/\mathfrak{a}_1$, Theorem(II.1), and $\mathfrak{m}_*(\mathbb{Z}_2)/\mathfrak{a}_1$ is a polynomial ring over $\mathbb{Z}_2$, we might expect elements in $I_*(\mathbb{Z}_2)/\mathfrak{a}_1$ to behave like polynomials. In fact they do, and so do some elements in $I_*(\mathbb{Z}_2)$.

In $\mathfrak{m}_*(\mathbb{Z}_2)$ we have that
\[
1 + \theta^n = (1 + \theta)(\sum_{j=0}^{n-1} \theta^j).
\]
Thus, $1 + \theta^n = 0$ in $\mathfrak{m}_*(\mathbb{Z}_2)/\mathfrak{a}_1$. We should expect a similar occurrence in $I_*(\mathbb{Z}_2)$.

V.1. Proposition: $(x + \Gamma(x))(\sum_{j=0}^{n-1} \Gamma^j(x)) = x^2 + \Gamma^n(x^2)$ if $\epsilon(\Gamma^j(x)) = 0$ for $0 \leq j < n$.

Proof: We prove this by induction on $n$.

$n = 1$. This is clear since $\mathfrak{a}_1$ is an ideal.

$n = 2$. $(x + \Gamma(x))(x + \Gamma(x)) = x^2 + x\Gamma(x) + x\Gamma(x) + (\Gamma(x))^2 = x^2 + (\Gamma(x))^2$.
Now, $\Gamma^2(x^2) = \Gamma(\Gamma(xx)) = \Gamma(x\Gamma(x) + \epsilon(x)\Gamma(x))$

$= \Gamma(x\Gamma(x))$, since $\epsilon(x) = 0$,

$= \Gamma(x)\Gamma(x) + \epsilon(x)\Gamma^2(x)$

$= (\Gamma(x))^2$.

So, $(x + \Gamma(x))(x + \Gamma(x)) = x^2 + \Gamma^2(x^2)$.

Now assume that the result is true for $n; i.e.,$

$(x + \Gamma(x))(\sum_{j=0}^{n-1} \Gamma^j(x)) = x^2 + \Gamma^n(x^2)$

with $\epsilon(\Gamma^j(x)) = 0$ for $0 \leq j < n$. Assuming that $\epsilon(\Gamma^j(x))$ is

zero for $0 \leq j < n+1$, we have that

$(x + \Gamma(x))(\sum_{j=0}^{n} \Gamma^j(x)) = (x + \Gamma(x))(\sum_{j=0}^{n-1} \Gamma^j(x) + \Gamma^n(x))$

$= x^2 + \Gamma^n(x^2) + x\Gamma^n(x) + \Gamma(x)\Gamma^n(x)$.

Now,

$\Gamma^n(x^2) = \Gamma(x\Gamma^{n-1}(x)) = x\Gamma^n(x) + \epsilon(\Gamma^{n-1}(x))\Gamma(x) = x\Gamma^n(x)$,

and

$\Gamma^{n+1}(x^2) = \Gamma(x\Gamma^n(x)) = \Gamma(x)\Gamma^n(x) + \epsilon(x)\Gamma^{n+1}(x)$

$= \Gamma(x)\Gamma^n(x)$.

These two facts are just the repeated use of the product

formula for $\Gamma$ and the fact that $\epsilon(\Gamma^j(x)) = 0$ for $0 \leq j < n+1$.

Thus,

$(x + \Gamma(x))(\sum_{j=0}^{n} \Gamma^j(x)) = x^2 + \Gamma^n(x^2)$.

Now, $\mathbb{M}_x(\mathbb{Z}_2)$ is a polynomial ring over $\mathbb{Z}_2$ in a countable

number of indeterminants, one of which is $\theta$. Using results

from commutative algebra, we have that $\mathbb{M}_x(\mathbb{Z}_2) = (\mathbb{Z}_2[\theta])[Y_\alpha]$;

i.e., $\mathbb{M}_x(\mathbb{Z}_2)$ is a polynomial ring over $\mathbb{Z}_2[\theta]$ in a countable

number of indeterminants. We will be interested in the ring

$\mathbb{Z}_2[\theta]$ for our results. We are interested in how the poly-
nominal \( 1 + \theta^n \) factors in \( \mathbb{Z}_2[\theta] \). This factorization then gives us relations in \( \mathcal{I}_n(\mathbb{Z}_2) \) with remarkable similarities.

We want to know how \( \theta^n + 1 \) factors in \( \mathbb{Z}_2[\theta] \), or the mod 2 factorization of \( t^n - 1 \) over \( \mathbb{Z}[t] \). This, of course, has to do with cyclotomic field extensions. We shall factor \( t^n - 1 \) over \( \mathbb{Z}[t] \mod p \), \( p \) any prime. We, at present, only need the factorization for \( p = 2 \), but we will need the general result in Chapter IX. The reference for the following is [12; Chapter 7].

Consider the polynomial \( t^n - 1 \) over \( \mathbb{Q} \) and let \( \zeta_n \) denote the primitive \( n^{th} \) root of unity. Let \( E = \mathbb{Q}(\zeta_n) \). The minimal polynomial for \( \zeta_n \) is the \( n^{th} \) cyclotomic polynomial, denoted by \( \phi_n(t) \). The following are consequences of the definitions.

(i) \( \phi_n(t) = \prod_{(n,m)=1} (t - \zeta_n^m) \).

(ii) \( t^n - 1 = \prod_{d|n} \phi_d(t) \), where \( d|n \) means that \( d \) divides \( n \).

(iii) \( \phi_n(t) = (t^n - 1)/ \prod_{d|n, d\not|n} \phi_d(t) \).

Let \( p \) be a positive prime integer.

v.2. THEOREM: If \( p \not| n \), then \( p \) factors in \( \mathbb{Q}(\zeta_n) \) into the product of \( r \) distinct prime ideals of degree \( f \), where \( rf = \varphi(n) \) and \( f \) is the smallest positive integer such that \( p^f \equiv 1 \pmod{n} \). (NOTE: \( \varphi \) is the Euler \( \varphi \)-function.)
\textbf{Proof:} See [12; 7-2-4].

\section*{V.3. Corollary}

If \( p \mid n \), then \( p \) splits completely in \( \mathbb{Q}(\zeta_n) \)
if and only if \( p = 1(\text{mod } n) \).

\textbf{Proof:} See [12; 7-2-5].

\section*{V.4. Theorem}

If \( p \mid n \), write \( n = p^{s}n' \) with \( (p,n') = 1 \).
Then \( p \) factors in \( \mathbb{Q}(\zeta_n) \) in the form
\[
p^{0}_{\mathbb{Q}(\zeta_n)} = (\mathfrak{b}_1 \ldots \mathfrak{b}_r)^{\phi(p^s)}
\]
where \( \mathfrak{b}_1, \ldots, \mathfrak{b}_r \) are distinct prime ideals of \( \mathbb{Q}(\zeta_n) \) of degree \( f \) with \( rf = \phi(n') \) and \( f \) being the smallest positive integer such that \( p^f = 1(\text{mod } n') \).

\textbf{Proof:} See [12; 7-4-3].

\section*{V.5. Theorem}

\( \{1, \zeta_n, \ldots, \zeta_n^{\phi(n')-1}\} \) is an integral basis for \( \mathbb{Q}(\zeta_n) \) over \( \mathbb{Q} \); i.e., \( \mathbb{Q}(\zeta_n) = \mathbb{Z}[\zeta_n] \).

\textbf{Proof:} See [12; 7-5-4].

Let \( n = p^{s}n' \) with \( s \geq 0 \) and \( (p,n') = 1 \).

\section*{V.6. Theorem}

If \( p \) factors in \( \mathbb{Z}[\zeta_n] \) as
\[
p^{0}_{\mathbb{Q}(\zeta_n)} = (\prod_{i=1}^{r} \mathfrak{b}_i)^{\phi(p^s)}
\]
with the degree of \( \mathfrak{b}_i \) being \( f \), \( rf = \phi(n') \), \( f \) being the smallest integer such that \( p^f = 1(\text{mod } n') \), and the \( \mathfrak{b}_i \) being distinct prime ideals in \( \mathbb{Q}(\zeta_n) = \mathbb{Z}[\zeta_n] \) for \( i = 1, \ldots, r \);
then
$$\phi_n(t) \equiv \left( \prod_{i=1}^{r} p_i(t) \right)^g(p^s) \mod p,$$
with the $p_i(t)$ being distinct polynomials for $i = 1, \ldots, r$; \deg(p_i) = f for $i = 1, \ldots, r$; and the same conditions on $r$ and $f$.

**Proof:** We have the following diagram of rings and fields:

\[
\begin{array}{ccc}
\mathbb{Z}[t]/(\phi_n(t)) & \cong & \mathbb{Z}[\zeta_n] \\
\mathbb{Z} & \longrightarrow & \mathbb{Z}/p \mathbb{Z} \\
\mathbb{Z}/p \mathbb{Z} & \longrightarrow & \mathbb{Z}[\zeta_n]/\mathfrak{p}_i = \kappa_i
\end{array}
\]

By Theorem (V.5) $Q(\zeta_n) = \mathbb{Z}[\zeta_n]$ and this is a Dedekind domain. Thus, $\mathfrak{p}_i$, being prime, is a maximal ideal in $\mathbb{Z}[\zeta_n]$.

Thus $\kappa_i = \mathbb{Z}[\zeta_n]/\mathfrak{p}_i$ is a field; in fact, an extension field of $F_p = \mathbb{Z}/p \mathbb{Z}$. Now, $[\kappa_i:F_p] = f$, by assumption. Take $\zeta_n \in \kappa_i$ and let $p_i(s)$ be its minimal polynomial. Now, $p_i(t)^g(p^s)$ divides $\phi_n(t)$ modulo $p$. Doing this for all $i = 1, \ldots, r$, we have our result.///

We now restrict our attention to $p = 2$. This factorization of $\phi_n(t)$ modulo 2 tells us how to factor $1 + \varphi^n$ in $\mathbb{F}_2(\varphi) = \mathbb{F}_2[\varphi][\mathbb{Y}_n]$. See Appendix I.

We need some computational tools for the endomorphism $\Gamma$. Let us assume that $n+m < N$ and $\epsilon(\Gamma^j(x)) = 0$ for all $0 \leq j \leq N$.

**V.7. Lemma:** If $n+m < N$ and $\epsilon(\Gamma^j(x)) = 0$ for $0 \leq j < N$, then
$$\Gamma^n(x)\Gamma^m(x) = \Gamma^{n+m}(x^2).$$
Proof: We shall use induction.

\[ n = 1, m = 1; \Gamma(x) \Gamma(x) = \Gamma^2(x^2). \]

\[ \Gamma^2(x^2) = \Gamma(\Gamma(x)) \]
\[ = \Gamma(x \Gamma(x) + \epsilon(x) \Gamma(x)) \]
\[ = \Gamma(x \Gamma(x)) \]
\[ = \Gamma(x) \Gamma(x) + \epsilon(x) \Gamma^2(x) \]
\[ = (\Gamma(x))^2. \]

Assume that \( \Gamma(x) \Gamma^k(x) = \Gamma^{k+1}(x^2) \) for all \( k \leq m \). We need to show that \( \Gamma(x) \Gamma^{m+1}(x) = \Gamma^{m+2}(x^2) \).

\[ \Gamma^{m+2}(x^2) = \Gamma(\Gamma^{m+1}(x^2)) \]
\[ = \Gamma(\Gamma(x) \Gamma^m(x)), \text{ by the inductive hypothesis,} \]
\[ = \Gamma(x) \Gamma^{m+1}(x) + \Gamma^2(x) \epsilon(\Gamma^m(x)) \]
\[ = \Gamma(x) \Gamma^{m+1}(x), \text{ since } \epsilon(\Gamma^m(x)) = 0. \]

Thus, we have that \( \Gamma(x) \Gamma^m(x) = \Gamma^{m+1}(x^2) \) for all \( m \). Now, we induct on \( n \). Assume that \( \Gamma^k(x) \Gamma^m(x) = \Gamma^{k+m}(x^2) \) for all \( k \leq n \). We want to show that \( \Gamma^{n+1}(x) \Gamma^m(x) = \Gamma^{n+m+1}(x^2) \) for an arbitrary \( m \).

\[ \Gamma^{n+m+1}(x^2) = \Gamma(\Gamma^{n+m}(x^2)) \]
\[ = \Gamma(\Gamma^n(x) \Gamma^m(x)) \]
\[ = \Gamma^{n+1}(x) \Gamma^m(x) + \epsilon(\Gamma^n(x)) \Gamma^{m+1}(x) \]
\[ = \Gamma^{n+1}(x) \Gamma^m(x), \text{ since } \epsilon(\Gamma^n(x)) = 0. /// \]

V.8. Lemma: If \( \epsilon(\Gamma^j(x)) = 0 \) for \( 0 \leq j < N \) and if \( n+1 < N \), then

\[ \Gamma(x) \Gamma^n(x^r) = \Gamma^{n+1}(x^{r+1}). \]

Proof: \( \Gamma^{n+1}(x^{r+1}) = \Gamma^n(\Gamma(x^{r+1})) \)
\[ = \Gamma^n(x^r \Gamma(x) + \epsilon(x) \Gamma(x^r)) \]
\[ = \Gamma^n(x^r \Gamma(x)) \]
\[ = \Gamma^{n-1}(\Gamma(x^r) \Gamma(x) + \epsilon(x^r) \Gamma^2(x)) \]
\[ = \Gamma^{n-1}(\Gamma(x^r) \Gamma(x)) \]
\[ = \Gamma^{n-2}(\Gamma^2(x^r) \Gamma(x) + \epsilon(\Gamma(x^r)) \Gamma^2(x)) \]
\[ = \Gamma^{n-2}(\Gamma^2(x^r) \Gamma(x)) \]

since \( \epsilon(\Gamma(x^r)) = \epsilon(x \Gamma(x^{r-1})) = 0 \). Continuing this process, we get
\[ \Gamma^{n+1}(x^{r+1}) = \Gamma(x) \Gamma^n(x^r)./// \]

This can be proved more easily using an induction argument as in the proof of Lemma (V.7).

By combining Lemmas (V.7) and (V.8), we get the following proposition.

V.9. Proposition: If \( n+m < N \) and \( \epsilon(\Gamma^j(x)) = 0 \) for \( 0 \leq j < N \), then
\[ \Gamma^n(x^r) \Gamma^m(x^s) = \Gamma^{n+m}(x^{r+s})./// \]

Let \( p(t) \) be a polynomial over \( \mathbb{Z}_2 \) defined by
\[ p(t) = \sum_{j=0}^{n} a_j t^j, \quad a_j \in \mathbb{Z}_2. \]
Define the polynomial operator \( p(\Gamma) \) by
\[ p(\Gamma)(x) = (\sum_{j=0}^{n} a_j \Gamma^j)(x) \]
\[ = \sum_{j=0}^{n} a_j \Gamma^j(x), \quad a_j \in \mathbb{Z}_2. \]

If \( q(\Gamma) \) is another polynomial operator
\[ q(\Gamma)(x) = \sum_{j=0}^{m} b_j \Gamma^j(x), \quad b_j \in \mathbb{Z}_2; \]
define the product of these two polynomial operators to be
\[ p(\Gamma)q(\Gamma) = pq(\Gamma) \]
\[ = \sum_{k=0}^{n+m} c_k r^k \]
where \( c_k = \sum_{i=0}^{k} a_i b_{k-i} \in \mathbb{Z}_2 \). Note that the above lemmas give us that
\[ [p(\Gamma)(x)][q(\Gamma)(x)] = (pq(\Gamma))(x^2). \]
From this, from the above lemmas, and from Theorem (V.6) we have proven the following result.

V.10. THEOREM: If \( \theta^n + 1 = \prod_{d|n} (\prod_{j_d=1}^{k_d} p_{j_d})^{s_d} \) with the \( p_{j_d} \) not necessarily distinct for different values of \( d \) and \( d = 2^sd' \) with \( (d',2) = 1 \), then
\[ \prod_{d|n} [(p_{j_d}(\Gamma)(x))]^{s_d} \in (2^sd') = x^{k+1} + r^n(x^{k+1}) \]
where \( k+1 = \sum_{d|n} k_d s_d \).

This theorem is not as bad as it looks, nor is it hard to use. For some explicit calculations using this theorem see Appendix II.

This theorem tells us some very interesting things about \( I_* (\mathbb{Z}_2) \). For each \( n \geq 1 \) there are elements in \( \mathcal{G}_n \) which factor in exactly the same way that \( 1 + \theta^n \) factors in \( M_* (\mathbb{Z}_2) \). This introduces some interesting relations into \( I_* (\mathbb{Z}_2) \). Also, it tells us that \( I_* (\mathbb{Z}_2) \) acts much like a polynomial ring. This should not really be unexpected since the sequence (I.2) is split exact.
CHAPTER VI: AN ORIENTED ANALOGUE OF THE MAP $\Gamma$

Let $p$ be a fixed odd prime integer and let $\mathbb{Z}_p$ and $C_p$ denote the cyclic group $\mathbb{Z}/p\mathbb{Z}$. $\mathbb{Z}_p$ is the standard notation when talking about differentiable maps of period $p$. In the following $\text{MSO}_*$ will denote the graded oriented Thom cobordism algebra; $\text{MSO}_*(\mathbb{Z}_p)$, the graded oriented cobordism ring of smooth oriented manifolds with a fixed point free orientation preserving differentiable periodic map of period $p$; $\mathcal{O}_*^{SO}(\mathbb{Z}_p)$, the graded unrestricted oriented cobordism ring of smooth oriented manifolds with orientation preserving differentiable periodic maps of period $p$; and $\mathcal{M}_*^{SO}(\mathbb{Z}_p)$, the graded oriented cobordism ring of principal $U(k)$ bundles where

$$\mathcal{M}_n^{SO}(\mathbb{Z}_p) = \sum_{j=0}^{[\frac{n}{2}]} \text{MSO}_{n-2j}(BU(j))$$

and $\text{MSO}_n(BU(0)) = \text{MSO}_n$.

Throughout the rest of this paper we will work with a fixed representation of $\mathbb{Z}_p$, and $\mathbb{Z}_p$ will act by this one action in each coordinate.

We want to define a map similar to $\Gamma$ but in the oriented case. We would like to have a map $\Gamma_0: \mathcal{O}_n^{SO}(\mathbb{Z}_p) \to \mathcal{O}_{n+2}^{SO}(\mathbb{Z}_p)$. 
We shall attempt to define such a map and see wherein the troubles lie.

Let \( \rho = \exp(2\pi i/p) \) be the primitive \( p \)th root of unity. Let \( T \) be an orientation preserving diffeomorphism of period \( p \) on the closed oriented manifold, \( M^n \). Let \( F \) denote the fixed point set of \( T \) on \( M^n \), and let \( v \) denote its normal bundle. Consider the manifold \( M^n \times S^1 \) with the map

\[
\Delta(x, z) = (T(x), \rho z),
\]

where we think of \( S^1 \) as the set \( \{ z \in \mathbb{C} \mid |z| = 1 \} \). By [3;35.1] this diagonal action is the same as the map \( T_M^* : M^n \times S^1 \to M^n \times S^1 \) given by

\[
T_M(x, z) = (x, \rho z)
\]
on the bordism level. Add a trivial complex line bundle, \( C \), to the normal bundle \( v \). The sphere bundle, \( S(v \oplus C) \), is endowed with a differentiable fixed point free map, \( T' \), of period \( p \). This \( T' \) is the restriction to the unit sphere in each fiber of the diagonal map \( T \times T_1 \) on the total space of \( v \oplus C \) with \( T_1(z) = \rho \cdot z \). By [3;35.2] we have that \( (T_M^*, M^n \times S^1) \) is freely cobordant to \( (T', S(v \oplus C)) \). That is, there is a manifold \( W^{n+2} \) with a free action \( \tau \) and with boundary such that one component of this boundary is \( (T_M^*, M^n \times D^2) \) and the other is \( -(T', B(v \oplus C)) \).

We would like to define \( \Gamma_\mathcal{O} : \mathcal{O}^{SO}_n(\mathbb{Z}_p) \to \mathcal{O}^{SO}_{n+2}(\mathbb{Z}_p) \) by \( \Gamma_\mathcal{O}(\{T, M^n\}) = \{\tau, W^{n+2}\} \). Unfortunately, such a definition of \( \Gamma_\mathcal{O} \) is not well-defined. The problem lies in \( \{\tau, W^{n+2}\} \). The manifold with the free action of \( \mathbb{Z}_p \) connection \( (T_M^*, M^n \times S^1) \) and \( (T', S(v \oplus C)) \) gives us trouble. According to [3;19.4], a free action on a closed oriented manifold is divisible by \( p \) in \( \mathcal{O}^{SO}_*(\mathbb{Z}_p) \).
In order to keep the above definition of $\Gamma_\mathcal{O}$, we must pass to a quotient of $O^\text{SO}_* (\mathbb{Z}_p)$. Clearly, the ideal which must be factored out of this ring must contain $\{(c_p, c_p)\}$, which is the set of all bordism classes divisible by $p$. Unfortunately, just factoring out the ideal $\{(c_p, c_p)\}$ is not enough, because $\Gamma_\mathcal{O}$ does not preserve this ideal. If $F$ is the fixed point set of $T$ on $M^n$ and $\nu$ is its normal bundle, the fixed point data for $\Gamma_\mathcal{O}(\{T, M^n\})$ with the above definition is

$$\nu \otimes \mathcal{C} \cup -\mathcal{C}$$

$$\downarrow$$

$$F \cup -M^n.$$

Thus, from this fixed point data we can easily see that $\Gamma_\mathcal{O}$ will not preserve $\{(c_p, c_p)\}$.

Let $\mathcal{P}$ be an ideal of $O^\text{SO}_* (\mathbb{Z}_p)$ such that

$$\Gamma_\mathcal{O} : O^\text{SO}_* (\mathbb{Z}_p) / \mathcal{P} \rightarrow O^\text{SO}_* (\mathbb{Z}_p) / \mathcal{P}$$

is a well-defined endomorphism. Let $\mathcal{U}$ be the set of all ideals in $O^\text{SO}_* (\mathbb{Z}_p)$ which satisfy this condition. Clearly, $O^\text{SO}_* (\mathbb{Z}_p) \in \mathcal{U}$, so $\mathcal{U} \neq \emptyset$. Furthermore, from our previous discussion, for every $\mathcal{P} \in \mathcal{U}$, we must have that $\{(c_p, c_p)\} \subset \mathcal{P}$.

We need to exhibit a proper ideal of $O^\text{SO}_* (\mathbb{Z}_p)$ that is contained in $\mathcal{U}$. Let $B = \{ \{T, M\} \in O^\text{SO}_* (\mathbb{Z}_p) \mid \text{ some non-empty component of } F(T) \text{ is divisible by } p \text{ on the bordism level} \}$. $B$ is an ideal of $O^\text{SO}_* (\mathbb{Z}_p)$. Let $B' = (B \cup \{c_p, c_p\})$. Then $B' \in \mathcal{U}$, but $B' \neq O^\text{SO}_* (\mathbb{Z}_p)$. Partially order $\mathcal{U}$ by $\mathcal{P} \preceq \mathcal{P}'$ if

(i) $\mathcal{P} \subset \mathcal{P}'$, and

(ii) the following diagram commutes.
Clearly, the intersection of ideals serves as a lower bound for any totally ordered chain in \( \mathcal{U} \). So, by Zorn's Lemma there is a minimal ideal, \( P_1 \), such that

\[
\Gamma_G : \mathcal{O}_*^{SO}(\mathbb{Z}_p)/P_1 \to \mathcal{O}_*^{SO}(\mathbb{Z}_p)/P_1
\]

is a well-defined endomorphism. This raises two questions immediately.

1. What, precisely, is this ideal \( P_1 \)?
2. Can we specifically exhibit an ideal in \( \mathcal{U} \) with which we can easily work?

We have an affirmative answer to this second question. Consider the set of all \( \{T,M^n\} \) in \( \mathcal{O}_*^{SO}(\mathbb{Z}_p)/\{C_p,C_p\} \) such that all of the Pontrjagin numbers of the fixed point data are divisible by \( p \). This is clearly an ideal of \( \mathcal{O}_*^{SO}(\mathbb{Z}_p)/\{C_p,C_p\} \). Call it \( \mathcal{I} \). In order for \( \Gamma_G \) to preserve this ideal we must have an affirmative answer to the following question.

3. If all of the Pontrjagin numbers of the fixed point data of \( (T,M^n) \) are divisible by \( p \), then are all of the Pontrjagin numbers of the ambient manifold, \( M^n \), divisible by \( p \)?

We devote the next chapter to answering this question.
CHAPTER VII: DIVISIBILITY OF CHERN AND PONTRYAGIN NUMBERS

We will be considering the two manifolds $M^n \times S^1$ and $S(\nu \oplus \mathbb{C})$ as in Chapter VI. We are trying to find the Pontrjagin numbers of $M^n$ subject to a condition which will be stated shortly. We are interested in the characteristic numbers of the free $\mathbb{Z}_p$-actions on $S(\nu \oplus \mathbb{C})$ and on $M^n \times S^1$. We will use [3; 35.2] and we have that

$$\sum_{m=0}^{n} [T', S(\nu_m \oplus \mathbb{C})] = [T_1, S^1][M^n]$$

in $\text{MSO}_{n+1}(\mathbb{Z}_p)$. Thus, the Chern or Pontrjagin numbers of $M^n$ will be given by the characteristic numbers of $(T', S(\nu_m \oplus \mathbb{C})).$

In order to make these calculations less horrendous, we want to put some restrictions on our manifolds. Since Chern classes and Chern numbers are much easier with which to work than are the Pontrjagin classes and Pontrjagin numbers, we will restrict our attention to manifolds which are smooth, closed, and weakly complex.

A manifold is said to be weakly complex if there is an "almost complex" structure on its tangent bundle; i.e.,
if for some \( r, n \in \mathbb{Z}^+ \), \( \tau \oplus \theta^r \approx \epsilon^n \). In this case, the oriented cobordism class of a weakly complex manifold is completely determined by its Chern numbers. We must also restrict our maps to being differentiable periodic maps of period \( p \) which preserve the weakly complex structure on the manifold. In this situation we do not have to exclude the prime 2 from our considerations, since we are only concerned with Chern classes which are integral classes. When we do pass back to \( \mathcal{C}_*^{SO}(\mathbb{Z}_p) \), however, we will again require that \( p \) must be an odd prime.

We will use the notations \( \mathcal{C}_*^{SO}(\mathbb{Z}_p) \), \( \mathcal{M}_*^{SO}(\mathbb{Z}_p) \), \( \mathcal{G}_*^U(\mathbb{Z}_p) \), and \( \mathcal{M}_*^U(\mathbb{Z}_p) \). However, these will denote the original rings with the fixed point free involutions already having been identified to zero. This means that our fixed point sets will be non-empty. Unless otherwise stated we will be working in \( \mathcal{G}_*^U(\mathbb{Z}_p) \) and \( \mathcal{M}_*^U(\mathbb{Z}_p) \), which are the same as \( \mathcal{C}_*^{SO}(\mathbb{Z}_p) \) and \( \mathcal{M}_*^{SO}(\mathbb{Z}_p) \) except that the manifolds are required to be weakly complex and the maps must preserve this weakly complex structure.

Let \( (T, M^n) \) be an orientation preserving differentiable periodic map of period \( p \) on the smooth, closed, weakly complex manifold \( M^n \). Let \( F \) denote the fixed point set of \( T \) on \( M^n \). Note that \( F \neq \emptyset \), from the previous paragraph. \( F = \bigcup_{m=0}^{n} F^m \) where \( F^m \) is a submanifold of \( M^n \) of dimension \( m \) for each \( m \). Also, note that each component of the fixed point set \( F \) is a weakly complex manifold. Let \( \nu_m \) denote the normal bundle to \( F^m \) in \( M^n \). Note that \( \nu_m \) is a complex bundle for each \( m \), [3].
We shall say that \((T, M^n)\) satisfies condition \((D)\) if:

\[(D) - \text{All of the tangential and normal Chern numbers as well as all of the Chern numbers associated to the products of the tangential and normal Chern classes of each component of } F \text{ are divisible by } p.\]

Note that this puts \(\{T, M^n\}\) into the ideal of \(O_x^U(\mathbb{Z}_p)\) which we want to factor out in order to have \(\Gamma^e\) well-defined.

Assume that \((T, M^n)\) satisfies condition \((D)\). Add a trivial complex line bundle to \(v_m\) and pass to the associated sphere bundle, \(S(v_m \oplus \mathbb{C})\), over \(F^m\). There is a free \(\mathbb{Z}_p\) action on this bundle, so we may pass to the Lens space bundle, \(L_p(v_m \oplus \mathbb{C}) = S(v_m \oplus \mathbb{C})/\mathbb{Z}_p\). Our assumption that \((T, M^n)\) satisfies condition \((D)\) tells us that the Lens space bundle, \(L_p(v_m) = S(v_m)/\mathbb{Z}_p\), bounds as a manifold. We have two methods of computing the Chern classes of \(L_p(v \oplus \mathbb{C})\), which will be the sum of the Chern classes of the \(L_p(v_m \oplus \mathbb{C})\).

First, we have \(L_p(v_m)\) contained in \(L_p(v_m \oplus \mathbb{C})\) as a codimension 2 submanifold. Thus, there is a dual cocycle to \(L_p(v_m)\) in \(H^2(L_p(v_m \oplus \mathbb{C}); \mathbb{Z})\). If \(i: L_p(v_m) \rightarrow L_p(v_m \oplus \mathbb{C})\) denotes the embedding, then this cocycle, \(\lambda\), is uniquely determined by \([5]\)

\[
(VII.1) \quad \lambda \cap w_{L_p(v_m \oplus \mathbb{C})} = i_*(w_{L_p(v_m)}),
\]

where \(w(M)\) denotes the orientation class of the oriented manifold \(M\). This cohomology class \(\lambda\) also has the property that if \(\eta\) denotes the normal bundle to the embedding of \(L_p(v_m)\) in \(L_p(v_m \oplus \mathbb{C})\), then,
(VII.2) \[ c(\eta) = 1 + i^*(\lambda), \]
where \( c(\eta) \) is the total Chern class of \( \eta \), cf. [5; 4.8.1].

Now, if \( \alpha \) is any cohomology class in \( H^*(L_p^*(\nu_m \otimes \mathcal{C}); \mathbb{Z}) \), we have that

(VII.3) \[ i_*(i^*(\alpha) \cap w_{L_p^*(\nu_m)}) = \alpha \cap i_*(w_{L_p^*(\nu_m)}) \]
\[ = \alpha \lambda \cap w_{L_p^*(\nu_m \otimes \mathcal{C})}. \]

We can find the Chern classes of \( L_p^*(\nu_m \otimes \mathcal{C}) \) by considering

\[ i_*((1+\lambda)^{-1}c(L_p^*(\nu_m \otimes \mathcal{C})) \cap w_{L_p^*(\nu_m)}) , \]
where we think of \( c(L_p^*(\nu_m \otimes \mathcal{C})) \) as being a linear polynomial
in the indeterminants \( c_0, \ldots, c_t \) with \( c_i \in H^{2i}(L_p^*(\nu_m \otimes \mathcal{C}); \mathbb{Z}) \).

This term is then set equal to \( c(L_p^*(\nu_m)) \), the total Chern class of \( L_p^*(\nu_m) \). We then bring these classes back up to \( L_p^*(\nu_m \otimes \mathcal{C}) \) via (VII.3).

In the other method the tangent bundle to \( L_p^*(\nu_m \otimes \mathcal{C}), \)
\( \tau(L_p^*(\nu_m \otimes \mathcal{C})) \), can be identified with the Whitney sum of
the tangent bundle along the fibers, \( \tau_f(L_p^*(\nu_m \otimes \mathcal{C})) \), and
the pullback of the tangent bundle to \( F_m, \tau(F_m); \) i.e.,

\[ \tau(L_p^*(\nu_m \otimes \mathcal{C})) \cong \tau_f(L_p^*(\nu_m \otimes \mathcal{C})) \oplus \pi^*(\tau(F_m)). \]

VII.4. THEOREM: Let \( \xi \) be a U(n) vector bundle over a manifold \( M \), then there exists a bundle \( \mathcal{A}_2 \) such that

\[ \tau_f(L_p^*(\xi)) \oplus \theta = \mathcal{A}_2 \otimes \pi^*(\xi). \]

Proof: This is a \( \mathbb{Z}_p \) analogue of results in [8].

Let \( \xi \) be a U(n) vector bundle over \( M \) and let
\( \eta = (S^{2n-1}, \nu, L_p^{2n-1}) \) be the standard principal \( \mathbb{Z}_p \) bundle.
Let us make the following definitions.

(i) \( S(\xi) = (E(\xi) \times_{U(n)} S^{2n-1}, \pi_S, M) \) will denote the sphere bundle associated to \( \xi \) with fiber \( S^{2n-1} \).

(ii) \( L_p(\xi) = (E(\xi) \times_{U(n)} L_p^{2n-1}, \pi_{L_p}, M) \) will denote the Lens space bundle associated with \( \xi \) and \( S(\xi) \) with fiber \( L_p^{2n-1} \).

(iii) \( \zeta_1 = (E(\xi) \times S^{2n-1}, \pi_{\zeta_1}, E(L_p(\xi))) \) is a principal \( \mathbb{Z}_p \times U(n) \) bundle.

(iv) \( \zeta_2 = (E(S(\xi)), \pi_{\zeta_2}, E(L_p(\xi))) \) is a principal \( \mathbb{Z}_p \) bundle.

Let \( \psi: S^{2n-1} \to \mathbb{C}^n \) be the usual embedding and let \( \gamma: \mathbb{Z}_p \times U(n) \to U(n) \) be the multiplication map. We can define this map \( \gamma \) since \( \mathbb{Z}_p \) is in the center of \( U(n) \), or at least can be identified with a subgroup of this center. Now, \( \psi \) is equivariant relative to this representation, \( \gamma \).

We can now apply [8; 1.3], and we have

\[(VII.5) \ T^f(L_p(\xi)) \oplus (T^f(\zeta_2)/\mathbb{Z}_p) \oplus (\gamma/\mathbb{Z}_p) = \gamma(\zeta_1).\]

Here, \( \gamma \) denotes the normal bundle to the embedding \( \phi: E(S(\xi)) \to E(\xi) \times_{U(n)} \mathbb{C}^n \) induced by \( \psi \). \( \gamma(\zeta_1) \) is defined as follows.

\[E(\gamma(\zeta_1)) = E(\zeta_1) \times \mathbb{Z}_p \times U(n) \ U(n),\]

where \( \mathbb{Z}_p \times U(n) \) acts on \( U(n) \) by \( gg' = \gamma(g)g' \).

Clearly, \( T^f(\zeta_2)/\mathbb{Z}_p \) is the zero dimensional vector bundle over \( M \), so (VII.5) now yields

\[(VII.6) \ T^f(L_p(\xi)) \oplus (\gamma/\mathbb{Z}_p) = \gamma(\zeta_1).\]

Now let \( \gamma_1: \mathbb{Z}_p \times U(n) \to \mathbb{Z}_p \) and \( \gamma_2: \mathbb{Z}_p \times U(n) \to U(n) \) be the canonical projection maps. We then have that

\[(VII.7) \ \gamma(\zeta_1) = \gamma_1(\zeta_1) \otimes \gamma_2(\zeta_1)\]
where \( \gamma_1(\zeta_1) \) and \( \gamma_2(\zeta_1) \) are defined in the same manner as \( \gamma(\zeta_1) \). Furthermore, we see that \( \gamma_1(\zeta_1) = \zeta_2 \), the line bundle associated to \( \zeta_2 \); and \( \gamma_2(\zeta_1) = \pi_{L_p}(\xi) \), the bundle over \( E(L_p(\xi)) \) induced by \( \pi_{L_p} \) from \( E(\xi) \). We exhibit the following diagram in an attempt to keep things straight.

\[
\begin{array}{c}
E(\xi) \times S^{2n-1} \xrightarrow{1 \times \psi} E(\xi) \times \mathbb{C}^n \\
P_1 \downarrow & \downarrow P_2 \\
E(S(\xi)) = E(\xi) \times U(n) \xrightarrow{2n-1} E(\xi) \times U(n) \mathbb{C}^n \\
\pi_2 \downarrow & \downarrow \pi_{L_p} \\
E(L_p(\xi)) = E(\xi) \times U(n) \xrightarrow{2n-1} M
\end{array}
\]

Now, the embedding \( \psi: S^{2n-1} \to \mathbb{C}^n \) has an \( \mathbb{Z}_p \times U(n) \) equivariant normal field, so \( \frac{\nu}{\varphi} = \theta \). Therefore, (VII.6), (VII.7), and this give us that

\[
\tau_p(E(L_p(\xi))) \otimes \theta = \bigotimes_{i,j \in I} \pi_{L_p}^*(\xi).
\]

Applying (VII.4) to \( \tau_p(E(L_p(\nu \otimes \xi))) \), we have

\[
\tau_p(E(L_p(\nu \otimes \xi))) \otimes \theta = \eta \otimes \pi_{L_p}^*(\nu \otimes \xi).
\]

Now, we can compute the Chern classes of \( L_p(\nu \otimes \xi) \). Assume that \( c(\nu_m) = \prod_{i=0}^{k_m} (1 + x_{i,m}) \) where \( k_m = \text{dim}_c(\nu_m) \) and the cohomological dimension of \( x_{i,m} \) is 2 for each \( i \). Then from (VII.2), (VII.8), and [5; 4.4.3], we have

(VII.9) \[
c(\tau_p(E(L_p(\nu \otimes \xi)))) = \prod_{i=0}^{k_m} (1 + x_{i,m} - \lambda),
\]

since \( c(\nu \otimes \xi) = c(\nu_m) \). Thus, \( c_j(\tau_p(E(L_p(\nu \otimes \xi)))) \) is the \( j^{th} \) symmetric polynomial of the set \( \{x_{1,m} - \lambda, \ldots, x_{k,m} - \lambda\} \).

VII.10. Lemma: If \( \sigma_j \) denotes the \( j^{th} \) symmetric polynomial, then

\[
\sigma_j(\{x_{i,m} - \lambda\}_{i=1}^{k_m}) = \sum_{r=0}^{k-r} \binom{k-r}{j-r} (-\lambda)^{j-r} \sigma_r(\{x_{i,m}\}_{i=1}^{k_m}).
\]
Proof: This is a straightforward calculation using symmetric polynomials.

Thus, (VII.9) and (VII.10) give us

\[ c_j(\tau_f(L_p(\nu_m \oplus \mathcal{C}))) = \sum_{r=0}^{j} \binom{k-r}{j-r}(-\lambda)^{j-r}c_r(\nu_m). \]

Since \( \tau(L_p(\nu_m \oplus \mathcal{C})) = \tau_f(L_p(\nu_m \oplus \mathcal{C})) \oplus \tau^*(\tau(F^n)), \) Whitney duality gives us that

\[ c(L_p(\nu_m \oplus \mathcal{C})) = c(\tau_f(L_p(\nu_m \oplus \mathcal{C}))) \cdot c(F^n). \]

Thus,

\[ c_s(L_p(\nu_m \oplus \mathcal{C})) = \sum_{t=0}^{s} c_t(\tau_f(L_p(\nu_m \oplus \mathcal{C})))c_{s-t}(F^n) \]

\[ = \sum_{t=0}^{s} [c_{s-t}(F^n) \cdot \sum_{r=0}^{t} \binom{k-r}{t-r}(-\lambda)^{t-r}c_r(\nu_m)]. \]

(VII.11) \[ c_s(L_p(\nu_m \oplus \mathcal{C})) = \sum_{t=0}^{s} \sum_{r=0}^{t} \binom{k-r}{t-r}(-\lambda)^{t-r}c_r(\nu_m)c_{s-t}(F^n). \]

Now, if \( \dim_{\mathbb{C}}(\nu_m) = k_m, \) then \( \dim_{\mathbb{R}}(L_p(\nu_m \oplus \mathcal{C})) = 2k_m + 1. \)

Before we can compute Chern numbers, though, we must know the cohomology ring of \( L_p(\nu_m \oplus \mathcal{C}). \) \( H^*(L_p^{2n+1};\mathbb{F}_p) \) is generated by the set \( \{\delta, \lambda, \delta \lambda, \lambda^2, \delta \lambda^2, \ldots, \lambda^n, \delta \lambda^n\} \) where \( \delta \in H^1(L_p^{2n+1};\mathbb{F}_p), \lambda \in H^2(L_p^{2n+1};\mathbb{F}_p), \) and \( \beta(\delta) = \lambda \) with \( \beta \) the Bockstein homomorphism, cf. [6].

Consider the class

(VII.12) \[ c = \delta \lambda \prod_{i=2}^{r} c_{i,j}(L_p(\nu_m \oplus \mathcal{C})) \]

where \( l + \sum_{j=1}^{r} i_j = 2k_m + 1. \) Our Chern numbers will then be given by evaluating these classes against the fundamental class, \( w_{L_p}(\nu_m \oplus \mathcal{C}). \) From (VII.11), (VII.12), and our assumption that \( (T,M^n) \) satisfies condition (D), we see that
the only number which, \textit{a priori}, might not be zero mod $p$
would be:
\[ (+\delta \lambda^m) \cap u_{L_p}(\nu_m \oplus \mathbb{C}) \cdot \]

By (VII.3),
\[ i^*(+\delta \lambda^m) \cap u_{L_p}(\nu_m) = +\delta \lambda^m \cap u_{L_p}(\nu_m \oplus \mathbb{C}) = 0, \]

since $2k_m + 3 > \dim \mathcal{L}(\nu_m \oplus \mathbb{C})$. So, by taking the class
$\pm \delta \lambda^m$ down to $L_p(\nu_m)$ and evaluating it against $u_{L_p}(\nu_m)$,
we get zero. Therefore, this number must have been 0.

Therefore, all of the Chern numbers of $L_p(\nu_m \oplus \mathbb{C})$
are divisible by $p$ for every $m$; and, thus, all of the Chern
numbers of $L_p(\nu \oplus \mathbb{C})$ are divisible by $p$. Therefore, all of
those of $(T', S(\nu \oplus \mathbb{C}))$ are divisible by $p$. We have proved
the following theorem, making use of [3; 35.2].

VII.13. \textsc{Theorem}: Let $(T, M^n)$ be a differentiable periodic map
of period $p$ on a smooth closed weakly complex manifold, $M^n$. Let $F \neq \emptyset$ denote the fixed point set of $T$ on $M^n$. If $(T, M^n)$
satisfies condition (D), then all of the Chern numbers of
$M^n$ are divisible by $p$.}
CHAPTER VIII: THE MAP $\Gamma_{\mathcal{O}}$

Let $\mathcal{O}_{\mathcal{O}}$ be the ideal of $\mathcal{O}_{\mathcal{O}}^U(\mathbb{Z}_p)/(\{c_p, c_p\})$ consisting of all elements $\{T, M^n\}$ which satisfy condition (D). Note that the set of elements $\{T', M'\}$, where the normal bundle to the fixed point set is bordant to a trivial bundle, is contained in $\mathcal{O}_{\mathcal{O}}$, cf. [3; 42.2].

Recall that $\Gamma_{\mathcal{O}}: \mathcal{O}_{\mathcal{O}}^U(\mathbb{Z}_p) \to \mathcal{O}_{\mathcal{O}}^U(\mathbb{Z}_p)$ has normal data $\otimes \mathbb{C} \cup \mathcal{E}$ over $F \cup M^n$. By (VII.13) and the fact that $\{T, M^n\}$ is in $\mathcal{O}_{\mathcal{O}}$, all of the Chern numbers of $M^n$ are divisible by $p$. Furthermore, since the normal bundle to $M^n$ in $\Gamma_{\mathcal{O}}(\{T, M^n\})$ is trivial, there will be no Chern numbers emanating from it. Thus, we see that $\Gamma_{\mathcal{O}}(\{T, M^n\}) \in \mathcal{O}_{\mathcal{O}}$ if $\{T, M^n\} \in \mathcal{O}_{\mathcal{O}}$. So, the mapping

$$\Gamma_{\mathcal{O}}: (\mathcal{O}_{\mathcal{O}}^U(\mathbb{Z}_p)/(\{c_p, c_p\}))/\mathcal{O}_{\mathcal{O}} \to (\mathcal{O}_{\mathcal{O}}^U(\mathbb{Z}_p)/(\{c_p, c_p\}))/\mathcal{O}_{\mathcal{O}}$$

is well-defined.

We want to consider smooth orientable closed manifolds rather than restrict ourselves to closed weakly complex manifolds. The analogous argument will give us the same results for Pontrjagin numbers. We define the ideal $\mathcal{O}_{\mathcal{O}}$ in $\mathcal{O}_{\mathcal{O}}^S(\mathbb{Z}_p)/(\{c_p, c_p\})$ as above, and we have

50
(VII.1) $\Gamma_\mathcal{O} : (\mathcal{O}_*^\text{SO}(\mathbb{Z}_p)/\{(c_p,c_p)\})/\mathcal{O} \to (\mathcal{O}_*^\text{SO}(\mathbb{Z}_p)/\{(c_p,c_p)\})/\mathcal{O}$ is a well-defined endomorphism of degree +2.

VIII.2. THEOREM: Let $\overline{\mathcal{O}}_*^\text{SO}(\mathbb{Z}_p)/\mathcal{O}$ denote the ring $(\mathcal{O}_*^\text{SO}(\mathbb{Z}_p)/\{(c_p,c_p)\})/\mathcal{O}$. The map $\Gamma_\mathcal{O}$ as has been defined has the following properties.

(i) $\Gamma_\mathcal{O}$ is well-defined.

(ii) $\Gamma_\mathcal{O}$ is additive.

(iii) $\Gamma_\mathcal{O}$ is an MSO$_*$-module map.

(iv) $\Gamma_\mathcal{O}(\text{MSO}_*) = 0$.

(v) If $a,b \in \overline{\mathcal{O}}_*^\text{SO}(\mathbb{Z}_p)/\mathcal{O}$ and $\overline{c} : \overline{\mathcal{O}}_*^\text{SO}(\mathbb{Z}_p)/\mathcal{O} \to \text{MSO}_*/(\text{MSO}_* \cap \mathcal{O})$ is the augmentation map, then

$$\Gamma_\mathcal{O}(ab) = \Gamma_\mathcal{O}(a) \cdot b + \overline{c}(a) \cdot \Gamma_\mathcal{O}(b) = a \cdot \Gamma_\mathcal{O}(b) + \Gamma_\mathcal{O}(a) \cdot \overline{c}(b).$$

Proof:

(i) We have just shown this.

(ii) We want to show that

$$\Gamma_\mathcal{O}([T_1,M_1] + [T_2,M_2]) = \Gamma_\mathcal{O}([T_1,M_1]) + \Gamma_\mathcal{O}([T_2,M_2]).$$

Recall that $[T_1,M_1] + [T_2,M_2] = [T_1 \sqcup T_2,M_1 \sqcup M_2]$. Consider

$$(M_1 \sqcup M_2) \times S^1 = (M_1 \times S^1) \sqcup (M_2 \times S^1)$$

with the diagonal action $\Delta(x,z) = (T_i(x),\rho z). x \in M_i, i = 1,2,$ and

$$\rho = \exp(2\pi i/p).$$

Again, this action is the same as

$$T_M : (M_1 \sqcup M_2) \times S^1 \to (M_1 \sqcup M_2) \times S^1$$

given by $T_M(x,z) = (x,\rho z)$. By adding a trivial complex line bundle to the normal bundle to the fixed point set of $T_1 \sqcup T_2$ on $M_1 \sqcup M_2$, we are adding a trivial complex line bundle to the normal bundle to the fixed point set of $T_i$ on $M_i$ for $i = 1,2$. Thus, we have that
(T'_M \times M'_1 \times S^1) \sqcup (T'_M \times M'_2 \times S^1) = (T'_M, (M'_1 \sqcup M'_2) \times S^1) is freely
cobordant to (T', S(v'_1 \oplus C) \sqcup S(v'_2 \oplus C)) =
(T', S(v'_1 \oplus C)) \sqcup (T', S(v'_2 \oplus C)), where T' is defined as when
we first defined \( \Gamma_G \). This then gives us that \( \Gamma_G \) is additive.

(iii) Recall that the MSO*-module structure of \( \mathcal{OS}_* (\mathbb{Z}_p) \)
is given by \( [V^k] \{T, M^n\} = \{T \times \text{id}, M^n \times V^k\} \). We want to see
that \( \Gamma_G([V^k]\{T, M^n\}) = [V^k] \cdot \Gamma_G({\{T, M^n\}}) \). It is sufficient to
see that the normal data on each side of the equation agrees.

The fixed point set of \( T \times \text{id} \) on \( M^n \times V^k \) is \( F \times V^k \) with
normal bundle, \( v \times 0 \), where 0 denotes the 0-plane bundle
over \( V^k \). Thus, the normal data of \( \Gamma_G([V^k]\{T, M^n\}) \) is
\[
((v \times 0) \oplus C) \cup -C \quad \text{over} \quad (F \times V^k) \cup -(M^n \times V^k).
\]
That of \( [V^k] \cdot \Gamma_G({\{T, M^n\}}) \) is
\[
((v \oplus C) \cup -C) \times 0 \quad \text{over} \quad (F \cup -M^n) \times V^k.
\]
Passing to the bordism level gives us the desired result.

(iv) Recall that MSO* is embedded in \( \mathcal{OS}_* (\mathbb{Z}_p) \) by
sending \( [V^k] \) to \( \{\text{id}, V^k\} = [V^k] \cdot \{\text{id}, \text{pt}\} \). So,
\[
\Gamma_G({\{\text{id}, V^k\}}) = \Gamma_G([V^k]\{\text{id}, \text{pt}\}) = [V^k] \cdot \Gamma_G({\{\text{id}, \text{pt}\}}) = [V^k] \cdot 0 = 0.
\]

(v) \( \Gamma_G({\{T_1, M_1\} \cdot \{T_2, M_2\}}) = \{T_1, M_1\} \cdot \Gamma_G({\{T_2, M_2\}}) +
\Gamma_G({\{T_1, M_1\} \cdot \text{c}({\{T_2, M_2\}}}) \). Let \( F_i \) with normal bundle \( v_i \)
be the fixed point data for \( \{T_i, M_i\} \), for \( i = 1, 2 \).

The fixed point data for \( \Gamma_G({\{T_1, M_1\} \cdot \{T_2, M_2\}}) \) is
\[
((v_1 \times v_2) \oplus C) \cup -C \quad \text{over} \quad (F_1 \times F_2) \cup -(M_1 \times M_2).
\]
The fixed point data for \( \{T_1, M_1\} \cdot \Gamma_G({\{T_2, M_2\}}) \) is
(VIII.3) $(v_1 \times v_2) \oplus \mathcal{C}) \cup -(v_1 \oplus \mathcal{C}) \text{ over } (F_1 \times F_2) \sqcup -(F_1 \times M_2)$. 

The fixed point data for $\tau_G\left(\{T_1,M_1\}\right) \cdot \bar{\varepsilon}\left(\{T_2,M_2\}\right)$ is

(VIII.4) $(v_1 \oplus \mathcal{C}) \cup -\mathcal{C} \text{ over } (F_1 \times M_2) \sqcup -(M_1 \times M_2)$. 

The result follows when we pass to the bordism level, noting that the $v_1 \oplus \mathcal{C}$ over $F_1 \times M_2$ have opposite orientations in (VIII.3) and (VIII.4) and hence add out.///
Consider the sets
\[ \mathfrak{a}_n = \{-x + \Gamma^n_\varnothing(x) \mid x \in \mathcal{O}_{\ast}^\text{SO}(\mathbb{Z}_p)/\varnothing_{\varnothing} \text{ and } \overline{e}(\Gamma^j_\varnothing(x)) = 0 \text{ for } 0 \leq j < n.\} \]

As in the unoriented case each \( \mathfrak{a}_n \) is an ideal of \( \mathcal{O}_{\ast}^\text{SO}(\mathbb{Z}_p)/\varnothing_{\varnothing} \).

We do have an unfortunate problem which arises as follows.

From [4] we have that the following long exact sequence of graded oriented bordism rings as modules over \( \text{MSO}_{\ast} \):
\[
\begin{array}{cccccc}
\mathcal{O}_{\ast}^\text{SO}(\mathbb{Z}_p) & \overset{\beta}{\longrightarrow} & \mathcal{M}_{\ast}^\text{SO}(\mathbb{Z}_p) & \overset{\gamma}{\longrightarrow} & \text{MSO}_{\ast}(\mathbb{Z}_p) & \longrightarrow & 0
\end{array}
\]

Fortunately, we know that \( \text{Ker}(\beta) = \langle \{ c_p, c_p \} \rangle \), so this yields the following short exact sequence:
\[
(\text{IX.1}) \quad 0 \to \mathcal{O}_{\ast}^\text{SO}(\mathbb{Z}_p)/\langle \{ c_p, c_p \} \rangle \overset{\bar{\beta}}{\longrightarrow} \mathcal{M}_{\ast}^\text{SO}(\mathbb{Z}_p) \overset{\gamma}{\longrightarrow} \text{MSO}_{\ast}(\mathbb{Z}_p) \to 0.
\]

We actually would like to have \( \mathcal{O}_{\ast}^\text{SO}(\mathbb{Z}_p)/\langle \{ c_p, c_p \} \rangle/\varnothing_{\varnothing} = \mathcal{O}_{\ast}^\text{SO}(\mathbb{Z}_p)/\langle \{ c_p, c_p \} \rangle/\varnothing_{\varnothing} \) as our left hand term in the above sequence. In order to factor \( \varnothing_{\varnothing} \) out of the present term, we must consider \( \mathcal{M}_{\ast}^\text{SO}(\mathbb{Z}_p) \) and \( \text{MSO}_{\ast}(\mathbb{Z}_p) \) as modules over \( \mathcal{O}_{\ast}^\text{SO}(\mathbb{Z}_p) \) which is \( \mathcal{O}_{\ast}^\text{SO}(\mathbb{Z}_p)/\langle \{ c_p, c_p \} \rangle \) via \( \bar{\beta} \) and \( \gamma \circ \bar{\beta} \), respectively. Note that this makes \( \text{MSO}_{\ast}(\mathbb{Z}_p) \) into a trivial \( \mathcal{O}_{\ast}^\text{SO}(\mathbb{Z}_p)/\varnothing_{\varnothing} \)-module. We then tensor the sequence (IX.1) with \( \mathcal{O}_{\ast}^\text{SO}(\mathbb{Z}_p)/\varnothing_{\varnothing} \).
We get
\[ \text{Tor}_1 \left( \mathcal{O}_\ast^*(Z_p) / \mathcal{O}_\ast^*(Z_p), \mathcal{MSO}_\ast^*(Z_p) \right) \to \mathcal{O}_\ast^*(Z_p) / \mathcal{O}_\ast^*(Z_p) \to \mathcal{MSO}_\ast^*(Z_p) \to 0. \]

It is not known if Tor_1 \left( \mathcal{O}_\ast^*(Z_p) / \mathcal{O}_\ast^*(Z_p), \mathcal{MSO}_\ast^*(Z_p) \right) is always 0. Even if we knew that to be true, we still have a problem identifying \[ \mathcal{M}_\ast^*(Z_p) / \mathcal{O}_\ast^*(Z_p) \] as a ring.

Though we do not see an analogous result to (II.1) for the oriented case, we can still factor elements in \[ \mathcal{O}_n \], as before, like the cyclotomic polynomial mod \( p \). The method we used in the unoriented case (V) was based only on the mod 2 factorization of \( t^n + 1 \) and the product formula for \( \Gamma \). Since we have the same product formula for \( \Gamma_G \) and the mod \( p \) factorization of \( t^n - 1 \), we will proceed in the same way.

Let us note that \( p \in \mathcal{O}_G \). Actually, we should say that
\[ p \cdot \mathcal{O}_\ast^*(Z_p) = p \cdot \{\text{id, pt}\} \in \mathcal{O}_G. \]
This is clear because the Euler class of the trivial \( Z_p \)-action on a point is +1. \( p\{\text{id, pt}\} \) is the \( Z_p \)-action on \( p \) points and clearly has Euler class divisible by \( p \).

Let us assume that \( n+m \leq N \) and \( \bar{e}(\Gamma_G^j(x)) = 0 \) for \( 0 \leq j < N \) and \( x \in \mathcal{O}_\ast^*(Z_p) / \mathcal{O}_G \).

**IX.2. Lemma:** Under these assumptions,
\[ \Gamma_G^n(x) \cdot \Gamma_G^m(x) = \Gamma_G^{n+m}(x^2). \]

**Proof:** This is exactly the same proof as for (V.7). Just replace \( \Gamma \) by \( \Gamma_G \) and \( e \) by \( \bar{e} \).} ///
IX.3. Lemma: Under the above assumptions,
\[ \Gamma(x) \cdot \Gamma^nx^r = \Gamma^{n+1}(x^{r+1}). \]

Proof: This is the same proof as for (V.7), again replacing \( \Gamma \) by \( \Gamma_0 \) and \( e \) by \( \bar{e} \).

IX.4. Lemma: Under the above assumptions,
\[ \Gamma_0^n(\Gamma_0^nx^r) \cdot \Gamma_0^m(x^s) = \Gamma_0^{n+m}(x^{r+s}). \]

Proof: This follows from the two preceding lemmas.

IX.5. Corollary: Under the above assumptions,
\[ (\Gamma_0(x))^n = \Gamma_0^nx^n. \]

Now, with the preliminaries over, we want to divide the positive integers into two disjoint collections. We will say that \( n \in C_o \) if \( n = p^s \) for some \( s \geq 1 \); otherwise, \( n \in C_1 \).

IX.6. Theorem: If \( n \in C_o \), then
\[ (\Gamma_0(x) - x)^n = \Gamma_0^nx^n - x^n \]
if \( \bar{e}(\Gamma_0^j(x)) = 0 \) for \( 0 \leq j < n \).

Proof: Expanding the left hand side, we have
\[ (\Gamma_0(x) - x)^n = \sum_{j=0}^{n} \binom{n}{j} (\Gamma_0(x))^j x^{n-j}. \]

Since \( n \) is a power of \( p \), \( p \) divides each of the coefficients.
\( \binom{n}{j} \) for \( 0 < j < n \). Since \( p \in \mathcal{O} \), each \( \binom{n}{j}(\Gamma_{\mathcal{O}}(x))^j x^{n-j} = 0 \) in \( \mathbb{Q}_x^\mathcal{O}(\mathbb{Z}_p)/\mathcal{O} \) for \( 0 < j < n \). Thus,

\[
(\Gamma_{\mathcal{O}}(x) - x)^n = (\Gamma_{\mathcal{O}}(x))^n - x^n = \Gamma_{\mathcal{O}}^n(x^n) - x^n
\]

by (IX.5).///

Now, let \( n \in \mathbb{C}_1 \). We must factor \( t^n - 1 \) into irreducible factors mod \( p \). However, this was done in (V.1) and (V.5).

We have

\[
t^n - 1 = \prod_{d \mid n} \phi_d(t).
\]

Let \( d = p^s d' \) with \( s \geq 0 \) and \( (p, d') = 1 \). We then have that

\[
\phi_d(t) \equiv (\prod_{p \mid d} p_i(t))\phi(p^s) \pmod{p}
\]

with degree \( (p_i(t)) = f_d, r_d f_d = \phi(d') \), and \( f_d \) being the smallest positive integer such that \( p^{f_d} = 1 \pmod{d'} \). Then

\[
t^n - 1 \equiv \prod_{d \mid n} (\prod_{p \mid d} p_i(t))\phi(p^s)
\]

where we know the number of factors and their degrees.

Let \( p(y) \) be a polynomial over \( \mathbb{Z}_p \). As before, define the polynomial operator \( p(\Gamma_{\mathcal{O}}) \) and the product of two such operators \( p(\Gamma_{\mathcal{O}})q(\Gamma_{\mathcal{O}}) = pq(\Gamma_{\mathcal{O}}) \) in the exact same manner. We then have that

\[
(p(\Gamma_{\mathcal{O}})(x))(q(\Gamma_{\mathcal{O}})(x)) = pq(\Gamma_{\mathcal{O}})(x^2).
\]

Now, (V.1), (V.5), and the preceding lemmas give us the following theorem.

**IX.7. Theorem:** If \( n \in \mathbb{C}_1 \) and \( t^n - 1 = \prod_{d \mid n} (\prod_{p \mid d} p_i(t))\phi(p^s) \),

with $f_d$, $r_d$, $d'$, and $d$ as above, then

$$\Gamma G^n(x^k) - x^k = \prod_{d|n} (\prod_{i=1}^{r_d} (p_i (\Gamma \partial))(x)) \phi(p^s)$$

with $k = \sum_{d} r_d \phi(p^s)$. ///

Notice that our factorizations of these elements do not depend on our ideal $\Theta \partial$. All we need is the fact that $\Gamma \partial$ is well-defined, the product formula for $\Gamma \partial$ is as given, and $p$ is contained in the ideal. Thus, any ideal of the collection $\mathcal{U}$ of Chapter VI which contains $p$ will suffice. The reason for considering the ideal $\Theta \partial$ is that it is easily defined in geometric terms.
BIBLIOGRAPHY


### APPENDIX I

$t^n - 1, \hat{n}(t)$, and the mod 2 factorization of $t^n - 1$

<table>
<thead>
<tr>
<th>$t^n - 1$</th>
<th>$\hat{n}(t)$</th>
<th>Mod 2 Factorization of $t^n - 1$</th>
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<td>$t^1 - 1$</td>
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<td>$t + 1$</td>
</tr>
<tr>
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<td>$t + 1$</td>
<td>$(t + 1)^2$</td>
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<td>$(t + 1) (t^2 + t + 1)$</td>
</tr>
<tr>
<td>$t^4 - 1$</td>
<td>$t^2 + 1$</td>
<td>$(t + 1)^4$</td>
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<td>$(t+1)^2 (t^2+t+1)^2$</td>
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<td>$t^{17} - 1$</td>
<td>$\sum_{i=0}^{16} t^i$</td>
<td>$(t+1) (t^8+t^5+1+t^3+1) (t^8+t^7+6+t^2+t+1)$</td>
</tr>
<tr>
<td>$t^{18} - 1$</td>
<td>$t^6 - t^3 + 1$</td>
<td>$[(t+1)(t^2+t+1)(t^6+t^3+1)]^2$</td>
</tr>
<tr>
<td>-------------</td>
<td>----------------</td>
<td>----------------------------------</td>
</tr>
<tr>
<td>$t^{19} - 1$</td>
<td>$\sum_{i=0}^{18} t^i$</td>
<td>$(t+1)(\sum_{i=0}^{18} t^i)$</td>
</tr>
<tr>
<td>$t^{20} - 1$</td>
<td>$t^8 - t^6 + t^{\frac{3}{4}} - t^2 + 1$</td>
<td>$[(t+1)(t^\frac{3}{4}+t^2+t+1)]^4$</td>
</tr>
<tr>
<td>$t^{21} - 1$</td>
<td>$t^{12} - t^{11} + t^9 - t^8 + t^6 - t^{\frac{3}{4}} + t^{\frac{5}{2}} - t + 1$</td>
<td>$(t+1)(t^2+t+1)(t^3+t^2+1) \cdot (t^3+t+1)(t^6+t^{\frac{3}{4}}+t^2+t+1) \cdot (t^6+t^3+t\frac{1}{2}+t^2+1)$</td>
</tr>
<tr>
<td>$t^{22} - 1$</td>
<td>$\sum_{i=0}^{10} (-1)^i t^i$</td>
<td>$[(t+1)(\sum_{i=0}^{10} t^i)]^2$</td>
</tr>
<tr>
<td>$t^{23} - 1$</td>
<td>$\sum_{i=0}^{22} t^i$</td>
<td>$(t+1) \cdot (t^{11}+t^{10}+t^6+t^{\frac{5}{4}}+t^2+1) \cdot (t^{11}+t^9+t^7+t^6+t^5+t+1)$</td>
</tr>
<tr>
<td>$t^{24} - 1$</td>
<td>$t^8 - t^\frac{1}{4} + 1$</td>
<td>$[(t+1)(t^2+t+1)]^8$</td>
</tr>
<tr>
<td>$t^{25} - 1$</td>
<td>$t^{20} + t^{15} + t^{10} + t^5 + 1$</td>
<td>$(t+1) (t^\frac{1}{4}+t^3+t^2+t+1) \cdot (t^{20}+t^{15}+t^{10}+t^5+1)$</td>
</tr>
<tr>
<td>$t^{26} - 1$</td>
<td>$\sum_{i=0}^{12} (-1)^i t^i$</td>
<td>$[(t+1)(\sum_{i=0}^{12} t^i)]^2$</td>
</tr>
<tr>
<td>$t^{27} - 1$</td>
<td>$t^{18} + t^9 + 1$</td>
<td>$(t+1)(t^2+t+1)(t^6+t^3+1) \cdot (t^{18}+t^9+1)$</td>
</tr>
<tr>
<td>$t^{28} - 1$</td>
<td>$t^{12} - t^{10} + t^8 - t^6 + t^{\frac{1}{4}} - t^2 + 1$</td>
<td>$[(t+1)(t^3+t^2+1)(t^3+t+1)]^4$</td>
</tr>
<tr>
<td>$t^{29} - 1$</td>
<td>$\sum_{i=0}^{28} t^i$</td>
<td>$((t+1)(\sum_{i=0}^{28} t^i)$</td>
</tr>
<tr>
<td>$t^{30} - 1$</td>
<td>$t^8 + t^7 - t^5 - t^{\frac{3}{4}} - t^3 + t + 1$</td>
<td>$[(t+1)(t^2+t+1) \cdot (t^{\frac{4}{3}}+t^3+t^2+t+1) \cdot (t^{\frac{4}{3}}+t^3+t^2+1) \cdot (t^{\frac{4}{3}}+t^3+t^2+t+1) \cdot (t^{\frac{4}{3}}+t^3+t^2+1)$</td>
</tr>
<tr>
<td>$t^{31} - 1$</td>
<td>$\sum_{i=0}^{30} t^i$</td>
<td>$(t+1)(t^5+t^2+1)(t^5+t^3+1) \cdot (t^5+t^4+t^3+t^2+1) \cdot (t^5+t^4+t^3+t^2+1) \cdot (t^5+t^4+t^3+t^2+t+1) \cdot (t^5+t^4+t^3+t^2+1)$</td>
</tr>
<tr>
<td>$t^{32} - 1$</td>
<td>$t^{16} + 1$</td>
<td>$(t+1)^{32}$</td>
</tr>
</tbody>
</table>
APPENDIX II

Calculations from Theorem (V.10)

(Use Appendix I as a comparison, especially the first and last columns.) Note that $\epsilon(T_j(x)) = 0$ for $0 \leq j < n$.

\[
x + \Gamma(x) = x + \Gamma(x)
\]

\[
x^2 + \Gamma^2(x^2) = (x + \Gamma(x))^2
\]

\[
x^2 + \Gamma^3(x^2) = (x + \Gamma(x))(x + \Gamma(x) + \Gamma^2(x))
\]

\[
x^4 + \Gamma^4(x^4) = (x + \Gamma(x))^4
\]

\[
x^2 + \Gamma^5(x^2) = (x + \Gamma(x))(x + \Gamma(x) + \Gamma^2(x) + \Gamma^3(x) + \Gamma^4(x))
\]

\[
x^4 + \Gamma^6(x^4) = [(x + \Gamma(x))(x + \Gamma(x) + \Gamma^2(x))]^2
\]

\[
x^3 + \Gamma^7(x^3) = (x + \Gamma(x))(x + \Gamma^2(x) + \Gamma^3(x))(x + \Gamma(x) + \Gamma^3(x))
\]

\[
x^8 + \Gamma^8(x^8) = (x + \Gamma(x))^8
\]

\[
x^3 + \Gamma^9(x^3) = (x + \Gamma(x))(x + \Gamma(x) + \Gamma^2(x))(x + \Gamma^3(x) + \Gamma^6(x))
\]

\[
x^4 + \Gamma^{10}(x^4) = [(x + \Gamma(x))(x + \Gamma(x) + \Gamma^2(x) + \Gamma^3(x) + \Gamma^4(x))]^2
\]

\[
x^2 + \Gamma^{11}(x^2) = (x + \Gamma(x))(\sum_{i=0}^{10} \Gamma^i(x))
\]

\[
x^8 + \Gamma^{12}(x^8) = [(x + \Gamma(x))(x + \Gamma(x) + \Gamma^2(x))]^4
\]

\[
x^2 + \Gamma^{13}(x^2) = (x + \Gamma(x))(\sum_{i=0}^{12} \Gamma^i(x))
\]

\[
x^6 + \Gamma^{14}(x^6) = [(x + \Gamma(x))(x + \Gamma^2(x) + \Gamma^3(x))(x + \Gamma(x) + \Gamma^3(x))]^2
\]

\[
x^5 + \Gamma^{15}(x^5) = (x + \Gamma(x))(x + \Gamma(x) + \Gamma^2(x))(x + \Gamma(x) + \Gamma^4(x))
\]

\[
x^6 + \Gamma^{16}(x^6) = (x + \Gamma(x))^6
\]

\[
x^3 + \Gamma^{17}(x^3) = (x + \Gamma(x))(x + \Gamma^3(x) + \Gamma^4(x) + \Gamma^5(x) + \Gamma^8(x))
\]

\[
x^6 + \Gamma^{18}(x^6) = [(x + \Gamma(x))(x + \Gamma(x) + \Gamma^2(x))(x + \Gamma^3(x) + \Gamma^6(x))]^2
\]
\[ x^2 + \Gamma^{19}(x^2) = (x + \Gamma(x)) \left( \sum_{i=0}^{18} \Gamma^i(x) \right) \]
\[ x^8 + \Gamma^{20}(x^8) = [(x+\Gamma(x)) (x+\Gamma(x)+\Gamma^2(x)+\Gamma^3(x)+\Gamma^4(x))]^4 \]
\[ x^6 + \Gamma^{21}(x^6) = (x+\Gamma(x)) (x+\Gamma(x)+\Gamma^2(x)) (x+\Gamma(x)+\Gamma^3(x)) \cdot (x+\Gamma(x)+\Gamma^3(x)) (x+\Gamma(x)+\Gamma^2(x)+\Gamma^4(x)+\Gamma^6(x)) \cdot (x+\Gamma^2(x)+\Gamma^4(x)+\Gamma^6(x)) \]
\[ x^4 + \Gamma^{22}(x^4) = [(x+\Gamma(x)) (x+\Gamma(x)+\Gamma^2(x)+\Gamma^3(x)+\Gamma^4(x))]^2 \]
\[ x^3 + \Gamma^{23}(x^3) = (x+\Gamma(x)) (x+\Gamma(x)+\Gamma^2(x)+\Gamma^3(x)+\Gamma^4(x)+\Gamma^5(x)+\Gamma^6(x)+\Gamma^7(x)+\Gamma^8(x)+\Gamma^9(x)+\Gamma^{10}(x)+\Gamma^{11}(x)) \cdot (x+\Gamma(x)+\Gamma^5(x)+\Gamma^6(x)+\Gamma^7(x)+\Gamma^8(x)+\Gamma^9(x)+\Gamma^{10}(x)+\Gamma^{11}(x)) \]
\[ x^{16} + \Gamma^{24}(x^{16}) = [(x+\Gamma(x)) (x+\Gamma(x)+\Gamma^2(x))]^8 \]
\[ x^3 + \Gamma^{25}(x^3) = (x+\Gamma(x)) (x+\Gamma(x)+\Gamma^2(x)+\Gamma^3(x)+\Gamma^4(x)) \cdot (x+\Gamma^5(x)+\Gamma^6(x)+\Gamma^7(x)+\Gamma^8(x)+\Gamma^9(x)+\Gamma^{10}(x)+\Gamma^{11}(x)+\Gamma^{12}(x)) \]
\[ x^4 + \Gamma^{26}(x^4) = [(x+\Gamma(x)) (x+\Gamma(x)+\Gamma^2(x)+\Gamma^3(x)+\Gamma^4(x)) (x+\Gamma(x)+\Gamma^3(x)+\Gamma^4(x))]^4 \]
\[ x^2 + \Gamma^{27}(x^2) = (x + \Gamma(x)) \left( \sum_{i=0}^{28} \Gamma^i(x) \right) \]
\[ x^{10} + \Gamma^{30}(x^{10}) = [(x+\Gamma(x)) (x+\Gamma(x)+\Gamma^2(x)) (x+\Gamma(x)+\Gamma^2(x)+\Gamma^3(x)+\Gamma^4(x)) \cdot (x+\Gamma(x)+\Gamma^6(x)) (x+\Gamma^3(x)+\Gamma^4(x))]^2 \]
\[ x^7 + \Gamma^{31}(x^7) = (x+\Gamma(x)) (x+\Gamma^2(x)+\Gamma^3(x)) (x+\Gamma^3(x)+\Gamma^4(x)) \cdot (x+\Gamma^2(x)+\Gamma^3(x)+\Gamma^4(x)+\Gamma^5(x)) \cdot (x+\Gamma(x)+\Gamma^3(x)+\Gamma^4(x)+\Gamma^5(x)) \cdot (x+\Gamma(x)+\Gamma^2(x)+\Gamma^3(x)+\Gamma^5(x)) \cdot (x+\Gamma(x)+\Gamma^2(x)+\Gamma^3(x)+\Gamma^5(x)) \cdot (x+\Gamma(x)+\Gamma^2(x)+\Gamma^3(x)+\Gamma^5(x)) \]
\[ x^{32} + \Gamma^{32}(x^{32}) = (x + \Gamma(x))^{32} \]
VITA

The author was born in Lexington, Kentucky, on February 16, 1952. He attended Sayre School in Lexington, Kentucky, from which he graduated in June, 1970. During the last two years of this high school education, he was a part-time student at the University of Kentucky in Lexington, taking one course in Mathematics each of the four semesters. He enrolled in the University of the South at Sewanee, Tennessee, in September, 1970. He graduated from there with a B.A. in Mathematics in June, 1973.

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Approved:

Major Professor and Chairman

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