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Yang Gao
Louisiana State University

Hwang Lee
Louisiana State University

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Sub-Shot-Noise Quantum Optical Interferometry: A Comparison of Entangled State Performance within a Unified Measurement Scheme

Yang Gao* and Hwang Lee†
Hearne Institute for Theoretical Physics, Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803
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Phase measurement using a lossless Mach-Zehnder interferometer with certain entangled N-photon states can lead to a phase sensitivity of the order of $1/N$, the Heisenberg limit. However, previously considered output measurement schemes are different for different input states to achieve this limit. We show that it is possible to achieve this limit just by the parity measurement for all the commonly proposed entangled states. Based on the parity measurement scheme, the reductions of the phase sensitivity in the presence of photon loss are examined for the various input states.

The notion of quantum entanglement holds great promise for certain computational and communication tasks. It is also at the heart of metrology and precision measurements in extending their capabilities beyond the so-called standard quantum limit [1, 2, 3]. For example, the phase sensitivity of a usual two-port interferometer has a shot-noise limit (SL) that scales as $1/\sqrt{N}$, where N is the number of the photons entering the input port. However, a properly correlated Fock-state input for the Mach-Zehnder interferometer can lead to an improved phase sensitivity that scales as $1/N$, i.e., the Heisenberg limit (HL) [4, 5, 6, 7]. In the subsequent development, the dual Fock-state [8] and the so-called intelligent state [9, 10] were proposed to reach a sub-shot-noise sensitivity as well. Recently, much attention has been paid to the so-called NOON state to reach the exact HL in interferometry as well as super-resolution imaging [11, 12, 13, 14].

The utilization of those quantum correlated input states are accompanied by various output measurement schemes. In some cases the conventional measurement scheme of photon-number difference is used, whereas a certain probability distribution [15, 16, 17, 18], a specific adaptive measurement [19, 20, 21], and the parity measurement are used for other cases.

Gerry and Campos first showed the use of the parity measurement for the "maximally entangled state"–the NOON state–of light to reach the exact HL [22], following the earlier suggestion of the HL spectroscopy with N two-level atoms [23]. Campos, Gerry, and Benmoussa later suggested that the parity measurement scheme can also be used for the dual Fock state inputs by comparing the quantum state *inside* the interferometer with the NOON state [24]. In this paper we show that the parity measurement can be used as a detection scheme for sub-shot-noise interferometry with the correlated Fock states first proposed by Yurke, McCall, and Klauder [4], as well as with the intelligent states first suggested by Hillery and Mlodinow [9]. Extension of its use for all these input states then promote the parity measurement to a kind of universal detection scheme for quantum interferometry. Then, based on such a universal detection scheme comparisons of performance of various quantum states can be made in a common ground. As an example, we present a comparison of the phase sensitivity reduction for various quantum states of light in the presence of photon loss.

In order to describe the notations, we briefly review the group theoretical formalism of Mach-Zehnder interferometer. The key point of such a formalism is that any passive lossless four-port optical system can be described by the SU(2) group [4]. First, we use the mode annihilation operators $a_{in(out)}$ and $b_{in(out)}$, which satisfy boson commutation relations, to represent the two light beams entering (leaving) the beam splitter (BS), respectively. Then the action of BS takes the form

$$\begin{pmatrix} a_{out} \\ b_{out} \end{pmatrix} = \begin{pmatrix} e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ -e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix} \begin{pmatrix} a_{in} \\ b_{in} \end{pmatrix}. \quad (1)$$

Here α, β, and γ denote the Euler angles parameterizing SU(2), and they are related to the complex transmission and reflection coefficients. Through the Schwinger representation of angular momentum we can construct the operators for the angular momentum and for the occupation number from the mode operators a and b,

$$\mathbf{J} = \begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} ab^\dagger + ba^\dagger \\ i(ab^\dagger - ba^\dagger) \\ aa^\dagger - bb^\dagger \end{pmatrix}, \quad (2)$$

and $N = a^\dagger a + b^\dagger b$. The commutation relations $[a, b] = [a, b^\dagger] = 0$ and $[a, a^\dagger] = [b, b^\dagger] = 1$ lead to the relation $\mathbf{J} \times \mathbf{J} = i\mathbf{J}$. The Casimir invariant has the form $J^2 = J_x^2 + J_y^2 + J_z^2 = (N/2)(N/2 + 1)$ that commutes with J_i and N. Next, it was shown that the operation of the BS is equivalent to [4]

$$\mathbf{J}_{out} = e^{i\alpha J_z} e^{i\beta J_y} e^{i\gamma J_z} \mathbf{J}_{in} e^{-i\gamma J_z} e^{-i\beta J_y} e^{-i\alpha J_z}, \quad (3)$$

in the Heisenberg picture, and to

$$|\text{out}\rangle = e^{-i\alpha J_z} e^{-i\beta J_y} e^{-i\gamma J_z} |\text{in}\rangle, \quad (4)$$

in the Schrödinger picture. If we use the symbols j and m to indicate the eigenvalues of $N/2$ and J_z, then the theory of angular momentum tells that the representation Hilbert space is spanned by the complete

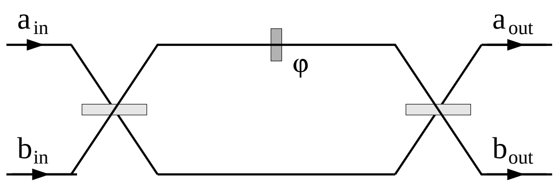

FIG. 1: Schematic of the Mach-Zehnder interferometer. The angle φ denotes the relative phase difference between the arms. Note that the Yurke, dual-Fock, and intelligent states are inserted to the left of the first beam splitter, and NOON to the right.

orthonormal basis $|j,m\rangle$ with $m \in [-j,j]$, which can also be labeled by the Fock states of the two modes, $|j,m\rangle = |j+m\rangle_a |j-m\rangle_b$. In terms of this language, we may make the geometrical interpretation of the elements of a Mach-Zehnder interferometer. For example, the effect of a 50/50 BS, which leads a $\pm\pi/2$ rotation around the x axis (given by the unitary transformation $e^{\pm i(\pi/2)J_x}$), is equivalent to the transformation

$$\begin{pmatrix} a_{out} \\ b_{out} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & \mp i \\ \mp i & 1 \end{pmatrix} \begin{pmatrix} a_{in} \\ b_{in} \end{pmatrix}. \tag{5}$$

Similarly, the relative phase shift φ acquired between the two arms of the Mach-Zehnder interferometer can be expressed by $a_{out} = a_{in}$, $b_{out} = e^{i\varphi} b_{in}$, or by the unitary transformation $e^{-i\varphi J_z}$ equivalently. The Mach-Zehnder interferometer can be illustrated schematically in FIG. 1, where the two light beams a and b first enter the BS_+, and then acquires a relative phase shift φ, and finally pass through the BS_-. The photons leaving the BS_- are counted by detectors $\mathrm{D_a}$ and $\mathrm{D_b}$. Therefore, in the language of the group theory, the input states of BS_+ and the output states of BS_- is connected by a simple unitary transformation $U = e^{i(\pi/2)J_x} e^{-i\varphi J_z} e^{-i(\pi/2)J_x} = e^{-i\varphi J_y}$ [25].

The information on the phase shift φ is inferred from the photon statistics of the output beams. There are many statistical methods to extract such information. The most common one is to use the difference between the number of photons in the two output modes, $N_d = a_{out}^\dagger a_{out} - b_{out}^\dagger b_{out}$, or equivalently, $J_{z,out} = N_d/2$. The minimum detectable phase shift then can be estimated by [26]

$$\delta\varphi = \frac{\Delta J_{z,out}}{|\partial\langle J_{z,out}\rangle/\partial\varphi|}, \tag{6}$$

where $\Delta J_{z,out} = \sqrt{\langle J_{z,out}^2\rangle - \langle J_{z,out}\rangle^2}$. The expectation value of $J_{z,out}$ and $J_{z,out}^2$ are calculated by $\langle J_{z,out}\rangle = \langle \mathrm{in}|J_{z,out}|\mathrm{in}\rangle = \langle \mathrm{in}|U^\dagger J_{z,in} U|\mathrm{in}\rangle$, $\langle J_{z,out}^2\rangle = \langle \mathrm{in}|J_{z,out}^2|\mathrm{in}\rangle = \langle \mathrm{in}|U^\dagger J_{z,in}^2 U|\mathrm{in}\rangle$, and $U^\dagger J_{z,in}^n U = (-\sin\varphi J_{x,in} + \cos\varphi J_{z,in})^n$.

Now the application of the group formalism to analyze the phase sensitivity of the ideal Mach-Zehnder interferometer is straightforward. Let us first consider the correlated photon-number states [4, 5, 7]. In particular, the so-called Yurke state has the form $|\mathrm{in}\rangle = [|j,0\rangle + |j,1\rangle]/\sqrt{2}$, which is one of the earliest proposals of utilizing the correlated photon-number states [4]). A simple calculation for the Yurke-state input gives

$$\delta\varphi = \frac{\left\{[j(j+1)-1]\sin^2\varphi + \cos^2\varphi\right\}^{1/2}}{|\sqrt{j(j+1)}\cos\varphi + \sin\varphi|}, \tag{7}$$

which has its minimum value $\delta\varphi_{min} \approx 1/\sqrt{j(j+1)}$ when $\sin\varphi \approx 0$. Hence, when the Yurke state is fed into the input ports of an interferometer, the minimum of $\delta\varphi$ has the order of $2/N$ limit since $j = N/2$. We should bear in mind that the minimum phase sensitivity is achieved only at particular values of $\varphi \approx 0$. For other values of φ the phase sensitivity is decreased. However, one can always control the phase shift by a feed-back loop which keeps φ at any particular value.

On the other hand, the parity measurement, represented by the observable $P = (-1)^{b^\dagger b} = e^{i\pi(j-J_z)}$ has an advantage when the simple photon number counting method ceases to be appropriate to infer the phase shift and provides a wider applicability than J_z. The parity measurement scheme was first introduced by Bollinger, Itano, Wineland, and Heinzen for spectroscopy with trapped ions of maximally entangled form [23]. Gerry and Campos adopted such a measurement scheme to the optical interferometry with the NOON state [22]. The NOON state can be formally written as $|\mathrm{NOON}\rangle = [|j,j\rangle + |j,-j\rangle]/\sqrt{2}$. Note that the NOON state is *not* the input state of MZI, but the state *after* the first beam splitter BS_+. Hence the output state is described as $|\mathrm{out}\rangle = e^{i(\pi/2)J_x} e^{-i\varphi J_z}|\mathrm{NOON}\rangle$.

The expectation value for the parity operator is then given by $\langle P\rangle == i^N \langle \mathrm{NOON}|e^{i\varphi J_z} e^{i\pi J_y} e^{-i\varphi J_z}|\mathrm{NOON}\rangle = i^N \left[e^{iN\varphi} + (-1)^N e^{-iN\varphi}\right]/2$, so that we have

$$\langle P\rangle \;=\; \begin{cases} i^{N+1}\sin N\varphi, & N \ \text{odd}, \\ i^N \cos N\varphi, & N \ \text{even}, \end{cases} \tag{8}$$

where the identity $e^{-i(\pi/2)J_x} e^{-i\pi J_z} e^{i(\pi/2)J_x} = e^{i\pi J_y}$ is applied. Since $P^2 = 1$, the equation (8) then immediately leads to the result $\delta\varphi = 1/N$, exactly.

Now, let us consider the dual Fock-state as the input state, $|j,0\rangle = |j\rangle_a |j\rangle_b$. Here, if we still use $J_{z,out}$ as our observable, we have $\langle J_{z,out}\rangle = \langle j,0| - \sin\varphi J_x + \cos\varphi J_z |j,0\rangle = 0$. The expectation value of the difference of the output photon number is now independent of the phase shift. Therefore, in this case the measurement of $J_{z,out}$ contains no information about the phase shift. A method of reconstruction of the probability distribution has been proposed to avoid this phase independence and to reach the Heisenberg limit [8, 17, 18]. More recently, Campos, Gerry, and Benmoussa suggested the use of the parity measurement for the dual Fock-state inputs [24].

The expectation value of P can be derived from $\langle P_{out}\rangle = \langle \text{in}|e^{i\varphi J_y} P_{in} e^{-i\varphi J_y}|\text{in}\rangle$ and $\langle P_{out}^2\rangle = \langle \text{in}|\text{in}\rangle = 1$. For the dual Fock-state, we have $\langle P_{out}\rangle^{\text{d-Fock}} = \langle j,0|e^{i\varphi J_y}(-1)^{j-J_z}e^{-i\varphi J_y}|j,0\rangle = (-1)^j \ d^j_{0,0}(2\varphi)$, where $d^j_{m,n}$ denotes the rotation matrix element: $e^{-i\varphi J_y}|j,n\rangle = \sum_{m=-j}^{j} d^j_{m,n}(\varphi)|j,m\rangle$, and

$$d^j_{m,n}(\varphi) = (-1)^{m-n}2^{-m}\sqrt{\frac{(j-m)!(j+m)!}{(j-n)!(j+n)!}}$$
$$\times \ P_{j-m}^{(m-n,m+n)}(\cos\varphi) \ (1-\cos\varphi)^{\frac{m-n}{2}}(1+\cos\varphi)^{\frac{m+n}{2}},$$

where $P_n^{(\alpha,\beta)}(x)$ represents the Jacobi polynomial. Thus the phase sensitivity is obtained as $\delta\varphi^{\text{d-Fock}} = \{1-[d^j_{0,0}(2\varphi)]^2\}^{1/2}/|\partial d^j_{0,0}(2\varphi)/\partial\varphi|$ for the dual Fock-state, and in the limit of $\varphi \to 0$, we have $\delta\varphi^{\text{d-Fock}} \to 1/\sqrt{2j(j+1)} \sim \sqrt{2}/N$.

If we use the parity measurement scheme for the Yurke-state input, we obtain

$$\begin{aligned}\langle P_{out}\rangle^{\text{Yurke}} &= \langle \text{in}|e^{i\varphi J_y}(-1)^{j-J_z}e^{-i\varphi J_y}|\text{in}\rangle \\ &= \sum_{m=-j}^{j}\frac{(-1)^{j-m}}{2}\left(d^{j*}_{m,0}+d^{j*}_{m,1}\right)\left(d^j_{m,0}+d^j_{m,1}\right) \\ &= \frac{(-1)^j}{2}\left[d^j_{0,0}+d^j_{0,1}-d^j_{1,0}-d^j_{1,1}\right](2\varphi),\end{aligned} \quad (9)$$

where have used the following properties of the matrix element [27] in the last line of (9):

$$\begin{aligned} d^{j*}_{m,n} = d^j_{m,n} = (-1)^{m-n}d^j_{n,m} = d^j_{-n,-m} \\ \sum_{m=-j}^{j} d^j_{k,m}(\varphi_1) \ d^j_{m,n}(\varphi_2) = d^j_{k,n}(\varphi_1+\varphi_2).\end{aligned} \quad (10)$$

Again, using $\delta\varphi = \sqrt{1-[\langle P_{out}\rangle^{\text{Yurke}}]^2}/\left|\partial\langle P_{out}\rangle^{\text{Yurke}}/\partial\varphi\right|$, we have $\delta\varphi^{\text{Yurke}} \to 1/\sqrt{j(j+1)} \sim 2/N$, in the limit of $\varphi \to 0$. This shows that, for the Yurke state, the parity measurement scheme leads to the same phase sensitivity as the $J_{z,out}$ measurement scheme. The dual-Fock state then performs better than the Yurke-state by a factor of $\sqrt{2}$ within the parity measurement scheme.

We can also use parity observable for the intelligent state entering the first beam splitter BS_+ in FIG. 1. The intelligent state is defined as the solution of the equation $(J_y + i\eta J_z)\,|j,m_0,\eta\rangle = \beta|j,m_0,\eta\rangle$, where $\eta^2 = (\Delta J_y)^2/(\Delta J_z)^2$ and m_0 is an integer belonging to $[-j,j]$ [9]. The eigenvalue corresponding to $|j,m_0,\eta\rangle$ is $\beta = i\ m_0\sqrt{\eta^2-1}$ and the eigenvector $|j,m_0,\eta\rangle = \sum_{k=-j}^{j} C_k|j,k\rangle$, where an explicit form of the expansion coefficient C_k is given in Ref. [10]. The expectation value of the parity operator is then obtained as $\langle P_{out}\rangle^{\text{Int}} = (-1)^j\sum_{k,n=-j}^{j} C_k^* C_n (-1)^k d^j_{k,n}(2\varphi)$. It follows that from the explicit form of C_k's the phase sensitivity scales better with a larger η and a smaller $|m_0|$ As $\eta \to \infty$, the phase sensitivity becomes

$$\delta\varphi^{\text{Int}} \to \frac{1}{\sqrt{2(j^2-m_0^2+j)}} \sim \frac{\sqrt{2}}{N}. \quad (11)$$

On the other hand, as $\eta \to 1$, we have $\delta\varphi^{\text{Int}} \to 1/\sqrt{2j} \sim 1/\sqrt{N}$, which is the standard shot-noise limit. So the minimum value of $\delta\varphi$ is only accessible for $m_0 = 0$. This limiting behavior is the same as the phase sensitivity with J_z measurement at $\varphi = 0$ [10]. We note that, within the parity measurement scheme, of all states considered here only the NOON state reaches exactly the HL [28].

Now that we have seen we can adopt the parity measurement as a universal detection scheme for all the commonly used entangled states, we will use it as a common ground to compare the effect of photon loss on phase sensitivity, thus we can put each input state on the same footing.

The effect of photon loss has been recently studied for the NOON states. Gilbert and coworkers applied a model for loss as a series of beam splitters in the propagation paths [29]. Rubin and Kaushik applied a single beam-splitter model for loss on the detection operator [30]. Whereas the two approaches are equivalent, we adopt the one given in Ref. [29] by putting the the effect of photon loss in the following form [31]: $a_{out} = e^{(-i\eta_a\omega/c-K_a/2)L_a}a_{in} + i\sqrt{K_a}\int_0^{L_a} dz \ e^{(-i\eta_a\omega/c-K_a/2)(L_a-z)}d(z)$, where η_i is the index of refraction for arm i of the interferometer, K_i is the absorption coefficient, and L_i is the path length. The annihilation operator $d(z)$ is the modes into which photons are scattered. A similar expression for the mode b is obtained by replacing a with b.

The observable used for the output detection schemes in both Refs. [29, 30], namely, $A = |N,0\rangle\langle 0,N| + |0,N\rangle\langle N,0|$, is equivalent to the parity measurement for the NOON state [11]. In addition, if we now only consider the measurement performed in the N*-photon subspace* of the output state, we can ignore the scattering term of the above transformation.

Following Ref. [29], we assume that the losses are present only in the one of the two arms of the interferometer and set $e^{-K_aL_a} = 1$ and $e^{-K_bL_b} \equiv \lambda$. The associated operation of the lossy Mach-Zehnder interferometer then can be expression as

$$\begin{pmatrix} a_{out} \\ b_{out}\end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\lambda e^{i\varphi} & -i(1-\lambda e^{i\varphi}) \\ i(1-\lambda e^{i\varphi}) & 1+\lambda e^{i\varphi}\end{pmatrix}\begin{pmatrix} a_{in} \\ b_{in}\end{pmatrix}, \quad (12)$$

which is non-unitary unless $\lambda = 1$. In the angular momentum representation, this transformation can be rephrased as $L(\varphi) = e^{iJ_x\frac{\pi}{2}}\Lambda e^{-iJ_x\frac{\pi}{2}}e^{iJ_x\frac{\pi}{2}}e^{-iJ_z\varphi}e^{-iJ_x\frac{\pi}{2}} = e^{iJ_x\frac{\pi}{2}}\Lambda e^{-iJ_x\frac{\pi}{2}}e^{-iJ_y\varphi}$, where Λ is a matrix representing the effect of path absorption. Then we get

$$\begin{aligned} L^\dagger P_N L &= e^{iJ_y\varphi}e^{iJ_x\frac{\pi}{2}}\Lambda e^{-iJ_x\frac{\pi}{2}}P_N e^{iJ_x\frac{\pi}{2}}\Lambda e^{-iJ_x\frac{\pi}{2}}e^{-iJ_y\varphi} \\ &= \lambda^N e^{iJ_y\varphi}P_N e^{-iJ_y\varphi} \equiv \mathcal{Y}_1.\end{aligned} \quad (13)$$

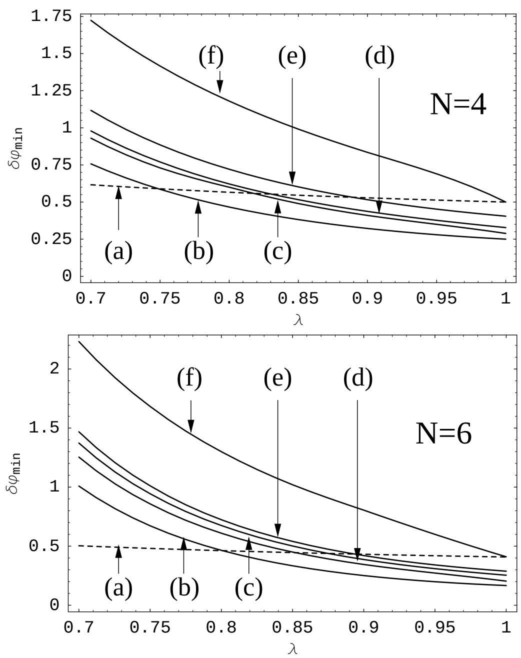

FIG. 2: The minimum phase sensitivity, $\delta\varphi_{\min}$, for the various entangled states as a function of λ, the transmission coefficient. The upper and lower figures are for $N = 4$ and $N = 6$, respectively. The dotted line (a) represents that of the uncorrelated input state [29]. The solid lines represent (b) the NOON state, (c) the dual Fock state, (d) the intelligent ($\eta = 10$) state, (e) the Yurke, and (f) the intelligent ($\eta = 1$) state, respectively.

with $P_N = P \otimes \sum_{m=-j}^{m=j} |j,m\rangle\langle j,m|$ denoting the N-photon projected parity operator. That is to say, one needs to detect all the N photons, even though that probability decreases exponentially with N.

Similarly, we find

$$L^\dagger P_N^2 L = L^\dagger L = e^{iJ_y\varphi} e^{iJ_x\frac{\pi}{2}} \Lambda^2 e^{-iJ_x\frac{\pi}{2}} e^{-iJ_y\varphi} = e^{iJ_x\frac{\pi}{2}} \Lambda^2 e^{-iJ_x\frac{\pi}{2}} \equiv \mathcal{Y}_2, \quad (14)$$

where the commutability of $\mathcal{Y}_2$ and $e^{-iJ_y\varphi}$ is applied, which can be simply proved in the spinor representation. Now, for a general input state, $|\text{in}\rangle = \sum_{m=-j}^{j} c_m|j,m\rangle$, we obtain $\langle P_N\rangle_{\text{out}} = \langle\mathcal{Y}_1\rangle_{\text{in}} = (-1)^j\lambda^{2j}\sum_{m,n} c_m^* c_n (-1)^m d_{mn}^j(2\varphi)$, and $\langle P_N^2\rangle_{\text{out}} = \langle\mathcal{Y}_2\rangle_{\text{in}} = (1/2)\sum_{m,n} c_m^* c_n [Q_{mn} + Q_{nm}](\lambda)$. Here, the polynomial $Q_{mn}(\lambda)$ is defined as the matrix element $\langle j,m|\mathcal{Y}_2|j,n\rangle$ such that

$$Q_{mn}(\lambda) = \frac{i^{-m-n}}{(j-n+1)_{j+n}} \frac{(2j)!\sqrt{(j+n)!}}{\sqrt{(j-m)!(j+m)!(j-n)!}} \times \left(\frac{x^2-1}{4}\right)^j \left(\frac{1+x}{1-x}\right)^{j+n-m} \times P_{j+n}^{(-2j-1,m-n)}\left(1-\frac{8x}{(x+1)^2}\right), \quad (15)$$

where $x \equiv \lambda^2$ and $p_q \equiv \Gamma(p+q)/\Gamma(p)$.

We now compare the phase sensitivity for different entangled states in the presence of photon loss. The plots depicted in FIG. 2 show the reduced phase sensitivity due to the photon loss, in this case as a function of λ (the transmission coefficient). All the commonly proposed entangled states are compared to the lossy-environment shot-noise limit. Among the entangled states, the best possible phase sensitivity can be achieved by the NOON state, and it gets worse in the following order: the dual Fock state, the $\eta = 10$ intelligent state, the Yurke state, and then the $\eta = 1$ intelligent state. Within the restricted parity measurement scheme the NOON states show the best performance for phase detection and can still beat the shot-noise limit if the transmittance of interferometer is not too small and the photon number is not too large. We see that beating the shot-noise limit (dotted line, represented by the uncorrelated input state) requires less attenuation as the number of photons increases. For example, the lowest solid line (representing the NOON states) requires 75% transmission for $N = 4$ and 80% for $N = 6$.

To summarize, we showed that the utilization of the parity measurement in sub-shot-noise interferometry is applicable to a wide range of quantum entangled input states, so far known entangled states of light. Comparison of the performance of the various quantum states then can be made within such a unified output measurement scheme. Furthermore, it may lead to a great reduction of the efforts in precise quantum state preparation as well as in various optimization strategies involving quantum state engineering for the sub-shot-noise interferometry [32].

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* Electronic address: ygao1@lsu.edu
† Electronic address: hwlee@phys.lsu.edu

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