6-1-2010

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Super-Resolution at the Shot-Noise Limit with Coherent States and Photon-Number-Resolving Detectors

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(Dated: October 23, 2018)

There has been much recent interest in quantum optical interferometry for applications to metrology, sub-wavelength imaging, and remote sensing, such as in quantum laser radar (LADAR). For quantum LADAR, atmospheric absorption rapidly degrades any quantum state of light, so that for high-photon loss the optimal strategy is to transmit coherent states of light, which suffer no worse loss than the Beer law for classical optical attenuation, and which provides sensitivity at the shot-noise limit. This approach leaves open the question — what is the optimal detection scheme for such states in order to provide the best possible resolution? We show that coherent light coupled with photon number resolving detectors can provide a super-resolution much below the Rayleigh diffraction limit, with sensitivity no worse than shot-noise in terms of the detected photon power.

PACS: 42.50.St, 03.67.-a, 42.50.Dv, 95.75.Kk

Ever since the work of Boto *et al.* in 2000, it has been realized that by exploiting quantum states of light, such as N00N states, it is possible to beat the Rayleigh diffraction limit in imaging and lithography (super-resolution) while also beating the shot-noise limit in phase estimation (super-sensitivity) [1, 2, 3]. However such quantum states of light are very susceptible to photon loss [4, 5, 6]. Recent work has shown that in the presence of high loss, the optimal phase sensitivity achieved is always $\Delta\varphi = 1/\sqrt{\bar{n}}$, where $\bar{n}$ is the average number of photons to arrive at the detector [7, 8]. These results suggest that, given the difficulty in making quantum states of light, as well as their susceptibility to loss, that the most reasonable strategy for quantum LADAR is to transmit coherent states of light, $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{k=0}^{\infty}(\alpha^k/\sqrt{k!})|k\rangle$ where $|k\rangle$ is a k-photon Fock state to mitigate a super-Beer's law in loss and maximize sensitivity [3]. It is well known that such an approach can only ever achieve at best shot-noise limited sensitivity [9]. However, such a conclusion leaves open the question as to what is the best detection strategy to optimize resolution. Recent experimental results have indicated that such a coherent-state strategy can still be super-resolving, provided a quantum detection scheme is employed [10]. In this paper we derive an optimal quantum detection scheme, that is super-resolving, and which can be implemented with photon number resolving detectors. Our proposed scheme exploits coherent states of light, is shot-noise limited in sensitivity, and can resolve features by an arbitrary amount below the Rayleigh diffraction limit. Our scheme would have applications to quantum optical remote sensing, metrology, and imaging. In Fig. 1 we illustrate schematically a two-mode interferometric quantum LADAR scheme. The source at the left is assumed to contain a laser producing a coherent state $|\alpha\rangle_A$ in upper mode A with vacuum in lower mode B, which is illustrated at line I in Fig. 1. The state is incident on a 50-50 beam splitter (BS), which mixes this coherent state with the vacuum state $|0\rangle_B$ in lower mode B. The output of such a mixing is computed by the well known beam splitter transformation, and is the state,

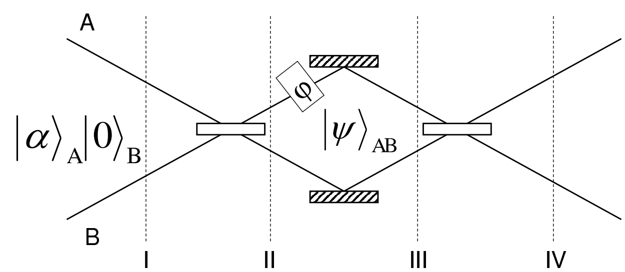

FIG. 1: Here we indicate the Mach-Zehnder interferometer used in the calculations. The coherent state is incident in mode A and vacuum in mode B at the left at Line I. After the first beam splitter transformation we have the two-mode coherent state of Eq. (1), as indicated at Line II. After the phase shifter φ this state becomes the two-mode coherent state of Eq. (2). At Line III we also implement the detection schemes corresponding to the operators $\widehat{N}_{AB}$, $\hat{\nu}_{AB}$, and $\hat{\mu}_{AB}$. Finally after the final beam splitter, we implement the parity operator $\widehat{\Pi}_A$ detection of Eq. (13) in the upper mode.

$$|\alpha/\sqrt{2}, \alpha/\sqrt{2}\rangle = e^{-\bar{n}/2}\sum_{n,m=0}^{\infty}\frac{\left(\alpha/\sqrt{2}\right)^{n+m}}{\sqrt{n!m!}}|n, m\rangle, \quad (1)$$

where the notation $|p, q\rangle \equiv |p\rangle_A|q\rangle_B$, $\bar{n} = |\alpha|^2$, and without loss of generality, a BS phase factor of $i = e^{i\pi/2}$ has been dropped for clarity. (This physically corresponds to a symmetric 50-50 BS with a quarter-wave plate in one of the output ports.) This is the state at line II in Fig. 1. As shown in Fig. 1 the upper mode A imparts a phase shift φ on this state, yielding,

$$
\begin{aligned}
|\psi\rangle_{AB} &= |\alpha e^{i\varphi}/\sqrt{2}, \alpha/\sqrt{2}\rangle \\
&= e^{-\bar{n}/2} \sum_{n,m=0}^{\infty} \frac{(e^{i\varphi}\sqrt{\bar{n}/2})^n(\sqrt{\bar{n}/2})^m}{\sqrt{n!m!}} |n, m\rangle,
\end{aligned} \tag{2}
$$

where, without loss of generality, we take $\alpha = \sqrt{\bar{n}}$, which amounts to setting an irrelevant overall phase to zero. This is the state at the line III in Fig. 1. In the coherent state basis, this state is obviously separable. However it is less obviously separable in the number basis, and it is easy to see that the double sum contains the N00N states $|N::0\rangle^{\varphi} = e^{iN\varphi}|N, 0\rangle + |0, N\rangle$ as well as MM' states $|M::M'\rangle^{\varphi} = e^{iM\varphi}|M, M'\rangle + e^{iM'\varphi}|M', M\rangle$, both of which are known to be $N = M - M'$ fold super-resolving. Indeed this observation suggests a strategy, similar to that employed by Resch *et al.*, of projecting the state $|\psi\rangle_{AB}$ of Eq. (2) onto the N00N basis through the implementation of the detection operator $\widehat{N}_{AB} = |N, 0\rangle\langle 0, N| + |0, N\rangle\langle N, 0|$ [10]. This results in an expectation value of $_{AB}\langle\psi|\widehat{N}_{AB}|\psi\rangle_{AB} = (\bar{n}/2)^N e^{-\bar{n}} 2\cos N\varphi/N!$ that is clearly N-fold super-resolving. However the factor of $(\bar{n}/2)^N e^{-\bar{n}}/N!$ indicates that the visibility of this expectation value, as a function of φ, is much less than unity. Let us optimize this visibility by tuning the return power $\bar{n}$. The maximal visibility occurs when $\bar{n} = N$ or when the return power is equal to the desired super-resolution factor. For example, with $\bar{n} = N = 10$ we achieve 10-fold super-resolution in $_{AB}\langle\psi|\widehat{N}_{AB}|\psi\rangle_{AB}$ with a visibility of about 0.024%. In a similar fashion we may now estimate the minimal phase sensitivity variance via the usual Gaussian error propagation formula [3],

$$
\Delta\varphi_N^2 = \frac{\Delta\widehat{N}_{AB}^2}{|\frac{\partial\langle\widehat{N}_{AB}\rangle}{\partial\varphi}|^2} = \frac{2^N e^{\bar{n}} N! - 2\bar{n}^N \cos^2 N\varphi}{2\bar{n}^N \sin^2 N\varphi} \frac{1}{N^2}, \tag{3}
$$

where the factor of $1/N^2$ would provide Heisenberg limited sensitivity, as it would be in the case of a pure N00N state with this detection scheme [3], if it were not for the Poisson weight factors inherited from the coherent state. This expression, Eq.(3), can be minimized by inspection, again by taking $N = \bar{n}$ and simultaneously choosing φ (which can be done by putting a phase shifter in arm B) such that $N\varphi = \pi/2$, yielding,

$$
\Delta\varphi_N^2 = \frac{2^{\bar{n}} e^{\bar{n}} \bar{n}!}{2\bar{n}^{\bar{n}}} \frac{1}{\bar{n}^2} \cong \sqrt{\frac{\pi}{2}} \frac{2^{\bar{n}}}{\bar{n}^{3/2}}, \tag{4}
$$

where we have used Stirling's approximation, and where again the Heisenberg limit behavior of $1/\bar{n}^2$ appears in tantalizing fashion. However, the exact expression has a minimum at $\bar{n} = 2$ of $\Delta\varphi_N^2|_{\bar{n}=2} \cong 1.85$, and hence for all values of $\bar{n}$ always does worse than the shot-noise and Heisenberg limits of $\Delta\varphi_{\text{SNL}}^2 = 1/\bar{n} > \Delta\varphi_{\text{HL}}^2 = 1/\bar{n}^2$, in agreement with the conclusion of [10].

It is easy to see that the reason this above strategy does worse than the shot-noise limit (SNL), which can be achieved with simple photon intensity difference counting [3], is that the projective measurement on the operator $\widehat{N}_{AB}$, defined above, throws away very many photon number amplitudes in the coherent state, keeping only the two terms in Eq. (2) with $n = N$ and $m = 0$ or $n = 0$ and $m = N$. Such an observation suggests that we introduce a new operator, which projects onto all of the maximally super-resolving terms, that is, $\hat{\nu}_{AB} = \sum_{N=0}^{\infty} \widehat{N}_{AB}$, which corresponds to the phase-bearing anti-diagonal terms in the two-mode density matrix describing the interferometer [6]. We may now carry out the expectation with respect to the dual-mode coherent state of Eq. (2) to get,

$$
\begin{aligned}
\langle\hat{\nu}_{AB}\rangle &= 2e^{-\bar{n}+\bar{n}\cos\varphi/2}\cos(\bar{n}\sin\varphi/2) \\
&\cong 2e^{-\bar{n}/2}\cos(\bar{n}\varphi/2),
\end{aligned} \tag{5}
$$

where we have approximated the expression near $\varphi = 0$, where the function is sharply peaked. We can see from this expression that by carefully choosing the return power $\bar{n} = N$, we recover super resolution. However the exponential pre-factor produces a loss in visibility. Choosing $\bar{n} = N = 20$, to obtain again 10-fold super-resolution, we obtain a visibility of only 0.009%. This low visibility does not bode well for the sensitivity, which we may compute as per Eq. (3) to get,

$$
\begin{aligned}
\Delta\varphi_\nu^2 &= \left\{e^{\bar{n}} + e^{3\bar{n}/2} - e^{\bar{n}\cos\varphi}[1 + \cos(\bar{n}\sin\varphi)]\right\} \\
&\quad \times e^{-\bar{n}\cos\varphi} \csc^2\left(\varphi + \frac{\bar{n}}{2}\sin\varphi\right) \frac{2}{\bar{n}^2},
\end{aligned} \tag{6}
$$

which again displays the tempting Heisenberg-limiting pre-factor of $1/\bar{n}^2$. This expression is singular at the phase origin, but has a minimum nearby $\varphi = \pi/2\bar{n}$, which for large photon number is approximately at $\bar{n} = N$, which by again choosing this to be a large integer, simplifies Eq. (6) to,

$$
\Delta\varphi_\nu^2|_{\varphi=\pi/2\bar{n}} \cong \frac{16\pi e^{\bar{n}/2}}{(2\bar{n} + \pi)^2}, \tag{7}
$$

which only approaches the SNL and HL near $\bar{n} = 1$ and then rapidly diverges to be much worse than the SNL for large $\bar{n}$. Hence, from these examples, we see there is a high price to pay for N00N-like super-resolution with coherent states — so many photon amplitudes are discarded that we always do far worse than shot noise. This analysis then suggests our final protocol — what if we choose a measurement scheme that includes all of the phase-carrying off-diagonal terms in the two-mode density matrix? Such a scheme is to consider the operator constructed from all the MM' projectors [6], that is,

$$
\hat{\mu}_{AB} = \sum_{M,M'=0}^{\infty} |M', M\rangle\langle M, M'|, \tag{8}
$$

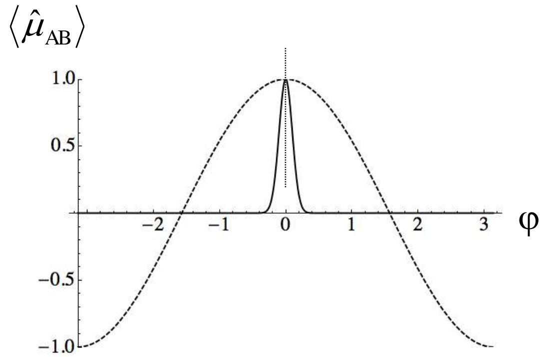

FIG. 2: This plot shows the expectation value $\langle\hat{\mu}_{AB}\rangle$ of Eq. (9) plotted as a function of the phase shift φ (solid curve) for a return power of $\bar{n}=100$. For reference we plot the normalized "classical" two-port difference signal (dashed curve). We see that the plot of the $\langle\hat{\mu}_{AB}\rangle$ is super-resolving and is narrower than the classical curve by a factor of $\delta\varphi = 1/\sqrt{\bar{n}} = 1/10$, as given in Eq. (11).

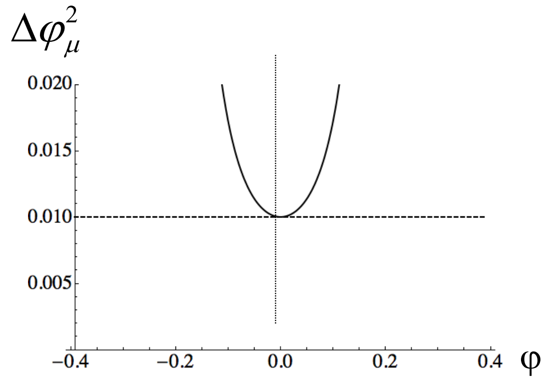

FIG. 3: In this plot we depict the sensitivity expression $\Delta\varphi_\mu^2$ of Eq. (10), again for the return power of $\bar{n}=100$ (solid curve). The horizontal dashed line indicates the shot-noise limit of $\Delta\varphi^2_{\rm SNL} = 1/\bar{n} = 1/100$. We see that the sensitivity of the super-resolving $\hat{\mu}_{AB}$ detection scheme hits the SNL at $\varphi=0$, as indicated by expanding Eq. (10) in a power series.

where we note this is evidently not a resolution of the identity operator. It is easy to show that this operator of Eq. (8) is both Hermitian and idempotent, that is $\hat{\mu}^\dagger_{AB} = \hat{\mu}_{AB}$ and $\hat{\mu}^2_{AB} = \hat{I}_{AB}$, respectively, where $\hat{I}_{AB}$ is the two-mode identity operator. Using these properties, with a bit of algebra, we establish, with respect to the two-mode coherent state of Eq. (2), the following results,

$$\langle\hat{\mu}_{AB}\rangle = e^{-2\bar{n}\sin^2(\varphi/2)}, \tag{9}$$

$$\Delta\varphi_\mu^2 = \frac{e^{4\bar{n}\sin^2(\varphi/2)}-1}{\bar{n}^2\sin^2\varphi}, \tag{10}$$

where once again the Heisenberg limit term of $1/\bar{n}^2$ appears accompanied by an exponential factor in the sensitivity estimate. We plot as a solid curve the expectation value of Eq. (9), for a return power of $\bar{n}=100$, in Fig. 2, along with the standard (classical) photon difference detection interferogram (dashed curve) [3]. Clearly $\langle\hat{\mu}_{AB}\rangle$ has a visibility of 100% now, and is periodic in φ with period 2π, and highly peaked at the phase origin where $\varphi=0$. This curve is not super-resolving in the usual sense of the word, as there are no multiple narrow peaks as would be the case in a N00N-state scheme, but it is super-resolving in the sense that there is a well defined narrow feature that is clearly sub-Rayleigh limited in resolution. Such a feature would be useful, for example, in LADAR ranging or laser Doppler velocimetry, where one would lock onto the side of such a feature and then monitor how it changes in time with a feedback loop in the interferometer.

To estimate the width of this central peak we note that in the small phase angle limit, Eq. (9) may be approximated as,

$$\langle\hat{\mu}_{AB}\rangle|_{\varphi\cong 0} \cong e^{-\bar{n}\varphi^2/2}, \tag{11}$$

which is clearly a Gaussian of width $\delta\varphi = 1/\sqrt{\bar{n}}$. Hence by choosing a return power of $\bar{n}=100$, we are 10-fold super resolving in terms of this central feature of the expectation value.

Now we check the sensitivity of this scheme. In Fig. 3 we plot as a solid curve the variance of Eq. (10), again for $\bar{n}=100$, near the phase origin over $-\pi/8 < \varphi < \pi/8$. We include as a dashed curve the shot-noise limit. We see that the sensitivity is shot-noise limited about the phase origin. This may be established analytically by first noting that the expression of Eq. (10) for the sensitivity has a removable singularity at the origin, and then by expanding it in a power series around $\varphi=0$ to get $\delta\varphi_\mu|_{\varphi\cong 0} \cong 1/\sqrt{\bar{n}}$, which is precisely the shot-noise limit. Hence by counting all the photons in an off-diagonal fashion, the detection scheme embodied in the operator $\hat{\mu}_{AB}$ of Eq. (8) produces a new kind of super resolution, and performs at the shot-noise limit in sensitivity. Since all of this information is extracted at the detector, and only coherent states are used at the source and in the interferometer, this scheme will be no worse in sensitivity in the presence of absorption or loss than an equivalent classical LADAR scheme, but it will in addition have super-resolving capabilities.

It remains to be understood how the observable of Eq. (8) might be detected in the laboratory. We note that at this stage of the analysis the operator $\hat{\mu}_{AB}$ is to be carried out with respect to the two-mode coherent state at the line III in the Fig. 1. That is we have not yet applied the second beam splitter. To the right of the second beam splitter, at the line IV in Fig. 1, we wish to now carry out the well-known parity operator [11] on output mode A in the upper arm,

$$\hat{\Pi}_A = (-1)^{\hat{n}_A} = e^{i\pi\hat{a}^\dagger\hat{a}}, \tag{12}$$

which simply indicates whether an even or odd number of photons exits that port. Here $\hat{n}_A = \hat{a}^\dagger\hat{a}$ is the number operator for that mode. Such a detection scheme is easily implemented by placing a highly efficient photon-number-resolving detector at this port, and such detectors with 95% efficiency and number resolving capabilities in the tens of photons have been demonstrated [12]. The connection between these two operators may be established via the easily proved identity,

$$_{AB}\langle\psi|\hat{\mu}_{AB}|\psi\rangle_{AB} \equiv \langle\alpha, 0|U^{\dagger}(\widehat{\Pi}_A \otimes \hat{I}_B)U|\alpha, 0\rangle, \quad (13)$$

where $\hat{I}_B$ is the B-mode identity operator, and $U = e^{-i\hat{J}_y\varphi}$ with $\hat{J}_y = i(\hat{a}\hat{b}^{\dagger} - \hat{a}^{\dagger}\hat{b})/2$ denotes the transformation of the usual Mach-Zehnder interferometer. That is, the effect of measuring the two-mode coherent state with respect to the operator $\hat{\mu}_{AB}$ to the left of the second BS at line III is equivalent to measuring it with respect to the parity operator $\widehat{\Pi}_A$ to the right of the second BS at line IV, all indicated in Fig. 1. Hence Eq. (14) establishes that the two schemes have the same super-resolving and shot-noise limiting properties, with the important point that the parity operator is perhaps far easier to implement in the laboratory, and it has been show to be a universal detection scheme in quantum interferometry [13, 14].

In summary we have provided a super-resolving interferometric metrology strategy, which achieves the shot-noise limit. The protocol has the appealing feature that it requires only the production and transmission of ordinary laser beams in the form of coherent states of light. Hence, unlike the issues concerning the propagation of non-classical states of light, such as squeezed light or entangled Fock states, this scheme clearly suffers no worse degradation in the presence of absorption and loss than a classical coherent LADAR system. All of the quantum trickery, which provides the super resolution, is carried out in the detection, which can be carried out with current photon number-resolving technology. However, a scheme by which we count the number of photons and then decide if that number is even or odd is overkill. The parity operation only requires that we know the sign — even or odd — independently of the actual number. Hence counting photon number is sufficient but perhaps not necessary, if a general scheme to determine the parity of a photon state could be found that did not require photon number counting. We conjecture such a scheme exists, perhaps through the exploitation of optical nonlinearities [15], or projective measurements, and this is an area of ongoing research. Our protocol for super-resolving phase measurements at the shot-noise limit has applications to quantum imaging, metrology, and remote sensing. In particular, for applications such as Doppler velocimetry of rapidly moving objects, one does not have the luxury to integrate the data for long periods of time in order to push down the signal to noise. In such scenarios the signal resolution in the form of the Rayleigh criteria is the usual limit to the system performance. It is in such situation that we anticipate this protocol to be most useful.

We would like to acknowledge interesting and useful discussions with Alan Migdall, Seth Lloyd, Sae-Woo Nam, and Jeffrey Shapiro. We would also like to acknowledge support from the Army Research Office, the Boeing Corporation, the Defense Advanced Research Projects Agency, the Foundational Questions Institute, the Intelligence Advanced Research Projects Activity, and the Northrop-Grumman Corporation.

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