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ABSTRACT

Let R be a commutative ring with identity. Let M be an R-module. Define Λ-rank (M) = m provided $\Lambda^m M \neq 0$ and $\Lambda^{m+1} M = 0$. If no such m exists, define Λ-rank (M) = ∞. Similarly define $\otimes$-rank and S-rank (symmetric rank).

Chapter I investigates the behavior of the ranks defined above relative to exact sequences.

Chapter II applies the results of Chapter I to prove, for V a DVR, that Λ-rank $(M \otimes N) \leq \Lambda$-rank $(M) \cdot \Lambda$-rank (N) and Λ-rank $(\Lambda^p M) \leq \binom{\Lambda\text{-rank}(M)}{p}$ for all V-modules M and N.

In Chapter III, weaker conclusions than those in Chapter II are obtained for R a Noetherian ring of finite Krull dimension.

In Chapter IV, we investigate what conditions on Λ-rank (M) follow from the condition "Λ-rank $(\Lambda^p M) = q$".

INTRODUCTION

Throughout, R is a commutative ring with identity and all R-modules are unitary. Unless otherwise indicated, module means R-module. M, N, A, B, and C always stand for modules; d, p, q, m, n, i, and k always stand for non-negative integers.

We will use F^p (or G^p) to stand for one of the functors $\otimes^p$, S^p or Λ^p: Category of R-modules and R-module homomorphisms $\longrightarrow$ Category of R-modules and R-module homomorphisms. If there is any question as to what ring we mean, we will use subscripts, e.g., F^p_R, $F^p_{R'}$.

<u>Definition</u>. F-rank(M) = m iff $F^m(M) \neq 0$ and $F^{m+1}(M) = 0$. Notice that m is unique, if it exists, because of the canonical R-module homomorphism [of <u>Proposition</u> <u>0.13</u>] of $F^{m+1}(M) \otimes F^k(M)$ onto $F^{(m+1)+k}(M)$. If such an <u>m</u> exists, we will say that M has finite F-rank or F-rank(M) $< \infty$. Note also that F-rank(M) = 0 if and only if M = 0.

We will make some straight-forward observations about F-rank (and, in particular, Λ-rank).

i) If $M \xrightarrow{h} N \longrightarrow 0$ is exact, then F-rank(M) $\geq$ F-rank(N) since $F^p(h): F^p(M) \longrightarrow F^p(N) \longrightarrow 0$ is also exact.

(F^p is neither left nor right exact nor does it preserve injections.)

Consider the following isomorphisms: [Prop 0.15]

$$\otimes^p(A \oplus C) \cong \bigoplus_{k=0}^{p} \binom{p}{k} (\otimes^{p-k} A \otimes \otimes^k C);$$

$$S^p(A \oplus C) \cong \bigoplus_{k=0}^{p} (S^{p-k}(A) \otimes S^k(C));$$

$$\Lambda^p(A \oplus C) \cong \bigoplus_{k=0}^{p} (\Lambda^{p-k} A \otimes \Lambda^k C).$$

Now suppose that F-rank(A) = a and F-rank(C) = c and consider, for $p = a + c + 1$, $F^{p-k}(A) \otimes F^k(C)$, where $0 \leq k \leq p$. Either $p-k \geq a + 1$ or $k \geq c + 1$; therefore either $F^{p-k}(A)$ or $F^k(C)$ is 0; hence, for $0 \leq k \leq p$, $F^{p-k}(A) \otimes F^k(C)$ is 0; hence the preceeding isomorphisms tell us that $F^p(A \oplus C) = 0$. Thus,

ii) F-rank$(A \oplus C) \leq$ F-rank(A) + F-rank(C).

Suppose M is an R-module and R' is an R algebra. The isomorphism $F^p_R(M) \otimes_R R' \underset{R'}{\cong} F^p_{R'}(M \otimes_R R')$ of Prop. 0.14 yields:

iii) F_R-rank(M) $\geq F_{R'}$-rank$(M \otimes_R R')$

A special case of Prop 0.14 is the following isomorphism, where S is any multiplication closed set of R ($0 \notin S$).

$$[F^p_R(M)]_S \underset{R_S}{\cong} F^p_{R_S}(M_S).$$

Since a module N is 0, if and only if, $N_Q = 0$ for every maximal ideal Q, we obtain from the above isomorphism:

iv) F_R-rank(M) = max$\{F_{R_Q}$-rank$(M_Q) \mid Q$ is a maximal ideal of $R\}$.

Throughout this paper, we will be concerned primarily with Λ-rank, dealing with $\otimes$-rank and S-rank only in Chapter I.

The main objective of the paper is to bound Λ-rank($\Lambda^p M$) by a number that is a function of p and Λ-rank(M) and to bound Λ-rank($M \otimes N$) by a number that is a function of Λ-rank(M) and Λ-rank(N). Unfortunately, I have not been able, in general, to do this, and, although we will find bounds when R is a Noetherian ring of finite Krull dimension, the bounds depend also on the Krull dimension and are almost certainly excessive. When R is a discrete valuation ring (DVR), we obtain the "best possible bounds" via theorems (in Chapter I) concerning rank and exact sequences.

We will also look at the problem of bounding Λ-rank(M) by a number that is a function of Λ-rank($\Lambda^p M$) and p.

Suppose now that M and N are free on _m_ and _n_ generators respectively. Then $\Lambda^p M$ is free on $\binom{m}{p}$ generators (by _Examples 0.12 c_), implying that Λ-rank(M) = m; therefore the Λ-rank of a finitely generated free module is the _free rank_ (the number of elements in a basis).

v) We have, for finitely generated free modules M and N, Λ-rank($M \otimes N$) = Λ-rank(M) $\cdot$ Λ-rank(N) (because $M \otimes N$ is free on mn generators) and Λ-rank($\Lambda^p M$) = $\binom{\Lambda\text{-rank}(M)}{p}$ (by the preceding paragraph).

vi) If R is quasi-local and M is minimally generated by q elements, then _Nakayama's Lemma_ and the fact that Λ commutes with ring homomorphisms enable us to reduce to the case of a field (over which all modules are free) to conclude that Λ-rank(M) = q. Since M finitely generated implies $\Lambda^p M$ finitely generated for every p > 1, Λ-rank($\Lambda^p M$) = $\binom{q}{p}$.

From vi) and the fact that Λ and tensor product localize well we have:

vii) If M and N are punctually finitely generated, then $\Lambda\text{-rank}(M \otimes N) \leq \Lambda\text{-rank}(M) \cdot \Lambda\text{-rank}(N)$ and $\Lambda\text{-rank}(\Lambda^p M) = \binom{\Lambda\text{-rank}(M)}{p}$.

Examples of non-finitely generated modules M for which $\Lambda\text{-rank}(\Lambda^p M) < \binom{\Lambda\text{-rank}(M)}{p}$ are easy to find.
[Consider the Z-module $M = Z \oplus Z \oplus Q/Z$. $\Lambda\text{-rank}(M) = 3$ but $\Lambda\text{-rank}(\Lambda^2 M) = 2 < \binom{3}{2} = 3$.]

It seems reasonable to ask whether or not the following always hold:

a) $\Lambda\text{-rank}(M \otimes N) \leq \Lambda\text{-rank}(M) \cdot \Lambda\text{-rank}(N)$

b) $\Lambda\text{-rank}(\Lambda^p M) \leq \binom{\Lambda\text{-rank}(M)}{p}$

I know of no examples of modules M and N for which a) and b) do not hold. On the other hand, I have been unable to show in general that the finiteness of $\Lambda\text{-rank}(M)$ implies the finiteness of $\Lambda\text{-rank}(\Lambda^p M)$ for $p > 1$.

(Note: For a ring R, the following statements are equivalent:

1) For all R-modules M, $\Lambda\text{-rank}(M) < \infty$ implies that $\Lambda\text{-rank}(\Lambda^p M) < \infty$ for all $p > 1$;

2) For all R-modules M, $\Lambda\text{-rank}(M) < \infty$ implies that $\Lambda\text{-rank}(\Lambda^2 M) < \infty$;

3) For all R-modules M and N, $\Lambda\text{-rank}(M)$, $\Lambda\text{-rank}(N) < \infty$ implies that $\Lambda\text{-rank}(M \otimes N) < \infty$.

<u>Proof</u>: 1) $\Rightarrow$ 2) obvious

2) $\Rightarrow$ 3) Suppose M and N have finite exterior rank. Then, by ii), $M \oplus N$ has finite exterior rank and by hypothesis $\Lambda^2(M \oplus N)$ has finite exterior rank. But $\Lambda^2(M \oplus N) \cong \Lambda^2 M \oplus (M \otimes N) \oplus \Lambda^2 N$. Hence $M \otimes N$ is a homomorphic image of $\Lambda^2(M \oplus N)$; therefore by i), Λ-rank$(M \otimes N) \leq \Lambda$-rank$(\Lambda^2(M \oplus N)) < \infty$.

3) $\Rightarrow$ 1) Suppose Λ-rank$(M) < \infty$. Then, by hypothesis and induction, Λ-rank$(\otimes^p M) < \infty$ and Λ-rank$(\Lambda^p M) < \infty$ by i) since $\Lambda^p M$ is a homomorphic image of $\otimes^p M$.)

CHAPTER 0. BACKGROUND MATERIAL

Herein are noted results necessary to the understanding of the body of the paper. All rings are commutative with identity and all modules are unitary.

Let R be a ring. Let A_i, A, and B be R-modules, n, i, p, t, and k stand for natural numbers, $a_t \in A_t$, and $r, r' \in R$.

Definition 0.1.

a) A function $\phi: A_1 X \ldots X A_n \longrightarrow B$ is called a multilinear map provided that $\phi(a_1, \ldots, ra_i + r'a_i', \ldots a_n) = r\phi(a_1, \ldots a_i, \ldots a_n) + r'\phi(a_1, \ldots a_i', \ldots, a_n)$.

b) A multilinear map $\psi: \underset{n}{X} A \longrightarrow B$ is called symmetric provided that $\psi(a_1, \ldots a_n) = \psi(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ for all permutations σ of $\{1, \ldots, n\}$.

c) A multilinear map $\theta: \underset{n}{X} A \longrightarrow B$ is called alternating provided that $\theta(a_1, \ldots, a_n) = 0$ whenever $a_i = a_j$ for some $i \neq j$.

d) A multilinear map $\gamma: \underset{n}{X} A \longrightarrow B$ is called skew-symmetric provided $\gamma(a_1, \ldots, a_n) = \text{sign}(\sigma)\gamma(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ for all permutations σ of $\{1, \ldots, n\}$.

Prop 0.2. An alternating multilinear map is skew-symmetric.

Proof: First suppose $\theta: A X A \longrightarrow B$ is an alternating multilinear map. To verify the above, we need only show that $\theta(a_1, a_2) = -\theta(a_2, a_1)$.

We have $\theta(a_1 + a_2, a_1 + a_2) = 0$ but $\theta(a_1 + a_2, a_1 + a_2)$

$= \theta(a_1,a_1) + \theta(a_1,a_2) + \theta(a_2,a_1) + \theta(a_2,a_2)$
$= \theta(a_1,a_2) + \theta(a_2,a_1)$.
Thus $\theta(a_1,a_2) + \theta(a_2,a_1) = 0$, or $\theta(a_1,a_2) = -\theta(a_2,a_1)$. By a similar argument, if $\theta: A \times \ldots \times A \longrightarrow B$ is an alternating multilinear map, then $\theta(a_1, \ldots a_{i-1}, a_i, a_{i+1}, \ldots a_{j-1}, a_j, a_{j+1}, \ldots, a_n) = -\theta(a_1, \ldots a_{i-1}, a_j, a_{i+1}, \ldots a_{j-1}, a_i, a_{j+1}, \ldots, a_n)$. Therefore, interchanging two coordinates of the tuple simply changes the sign of the image; but interchanging two coordinates is the same as operating on the indices with a transposition; and any permutation σ is a product of t transpositions, for some t, the sign of σ being $(-1)^t$. q.e.d.

<u>Prop</u>. <u>0.3</u>. Let R be a ring. Every skew-symmetric multilinear map is alternating $\Longleftrightarrow$ 2 is a unit in R.

<u>Proof</u>: $\Leftarrow$ Suppose 2 is a unit in R. Suppose $\gamma: \underbrace{A \times \ldots \times A}_{n\text{-factors}} \longrightarrow B$ is a skew-symmetric multilinear map.

$$\gamma(a_1, \ldots a_{i-1}, a_i, a_{i+1}, \ldots a_{j-1}, a_i, a_{j+1}, \ldots, a_n) =$$
$$-\gamma(a_1, \ldots a_{i-1}, a_i, a_{i+1}, \ldots a_{j-1}, a_i, a_{j+1}, \ldots, a_n)$$

or $2\gamma(a_1, \ldots a_{i-1}, a_i, a_{i+1}, \ldots a_{j-1}, a_i, a_{j+1}, \ldots, a_n) = 0$
But 2 is a unit in R; so $\gamma(a_1, \ldots a_{i-1}, a_i, a_{i+1}, \ldots, a_{j-1}, a_i, a_{j+1}, \ldots, a_n) = 0$ and γ is alternating.

$\Rightarrow$ Suppose that 2 is not a unit in R. Consider the map $\gamma: R \times R \longrightarrow R/2R$ given by $(r,r') \longrightarrow \overline{rr'}$. γ is bilinear and $\gamma(r,r') = -\gamma(r',r)$; hence γ is a skew-symmetric bilinear map but γ is not alternating because $\gamma(1,1) = \overline{1} \neq 0$.

The construction in ($\Rightarrow$) may make one think that, in order for γ not to be alternating, 2 must annihilate the image of γ. The following example shows that this is not the case.

Let R be a ring in which 2 is not a unit.
Let F be free on $\{e_1,e_2\}$.
Let $\gamma: F \times F \longrightarrow F\otimes F / \langle\{2e_1\otimes e_1,\ 2e_2\otimes e_2,\ e_1\otimes e_2 - e_2\otimes e_1\}\rangle$ by $(f_1,f_2) \longrightarrow \overline{f_1\otimes f_2}$ be the obvious bilinear map.
This map is also skew-symmetric because it is skew-symmetric on the generators. However $\gamma(e_1,e_1) \neq 0$ by the following argument: If $\gamma(e_1,e_1) = 0$, then $e_1\otimes e_1 = a(2e_1\otimes e_1) + b(2e_2\otimes e_2) + c(e_1\otimes e_2 - e_2\otimes e_1)$, $a, b, c \in R$.
The elements $e_1\otimes e_1$, $e_2\otimes e_2$, $e_1\otimes e_2 - e_2\otimes e_1$, are linearly independent over R because $\{e_1\otimes e_1,\ e_2\otimes e_2,\ e_1\otimes e_2 - e_2\otimes e_1,\ e_2\otimes e_1\}$ can be easily seen to form a basis for $F\otimes F$; thus we see that $2a = 1$, $2b = 0$, and $c = 0$. But $2a = 1 \Rightarrow$ 2 is a unit in R. $\rightarrow\leftarrow$

Tensor Products

Definition 0.4. The Tensor Product

Let $A_1, \ldots A_n$ be R-modules. Then $A_1\otimes \ldots \otimes A_n = F_{A_1 X \ldots X A_n}/K$, where $F_{A_1 X\ldots X A_n}$ is the free R-module on the elements of $A_1 X \ldots X A_n$ and K is the submodule of $F_{A_1 X \ldots X A_n}$ generated by elements of the form $(a_1, \ldots, ra_i + r'a_i', \ldots, a_n) - r(a_1, \ldots, a_i, \ldots, a_n) - r'(a_1, \ldots, a_i', \ldots, a_n)$, $r,r' \in R$; $a_t \in A_t$. Let $J: A_1 X \ldots X A_n \longrightarrow A_1\otimes \ldots \otimes A_n$ by $(a_1, \ldots, a_n) \longmapsto$

$a_1 \otimes \ldots \otimes a_n$. Notice that J is multilinear and that the image of J generates $A_1 \otimes \ldots \otimes A_n$.

The tensor product of $A_1, \ldots, A_n$ has the following universal property:

<u>Corollary</u> <u>0.4:1</u> Suppose that $\phi: A_1 X \ldots X A_n \longrightarrow B$ is a multilinear map. Then there exists a unique R-module homomorphism $\hat{\phi}: A_1 \otimes \ldots \otimes A_n \longrightarrow B$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & A_1 \otimes \ldots \otimes A_n & & \\ & J \nearrow & & \searrow \exists ! \hat{\phi} & \\ A_1 X \ldots X A_n & & \xrightarrow{\quad \phi \quad} & & B \end{array}$$

Proof: [Bbki.: Prop. 1, p. 244]

Some Facts About Tensor Product:

<u>Prop. 0.5</u>. Tensor product is associative. i.e. $(A \otimes B) \otimes C \cong A \otimes B \otimes C$ by $(a \otimes b) \otimes c \longmapsto a \otimes b \otimes c$.

<u>Proof</u>: Consider the function $\phi: A \ X \ B \ X \ C \longrightarrow (A \otimes B) \otimes C$ by $(a,b,c) \longmapsto (a \otimes b) \otimes c$. ϕ is R-multilinear; hence we get an R-module homomorphism $\hat{\phi}: A \otimes B \otimes C \longrightarrow (A \otimes B) \otimes C$ given by $a \otimes b \otimes c \longmapsto (a \otimes b) \otimes c$.

Now we need only construct an R-module homomorphism $\hat{\phi}' : (A \otimes B) \otimes C \longrightarrow A \otimes B \otimes C$ where $(a \otimes b) \otimes c \longmapsto a \otimes b \otimes c$. Then $\hat{\phi}$ and $\hat{\phi}'$ will be inverses of each other because their compositions are the identities. For every $c \in C$, let $\lambda_c: A \ X \ B \longrightarrow A \otimes B \otimes C$ by $(a,b) \longmapsto a \otimes b \otimes c$. λ_c is R-bilinear; hence we get an R-homomorphism $\hat{\lambda}_c: A \otimes B \longrightarrow$

$A \otimes B \longrightarrow A \otimes B \otimes C$. Now define $\phi':(\alpha,c) = \lambda_c(\alpha)$ where $\alpha \in A \otimes B$. ϕ' is R-bilinear; hence we get an R-module homomorphism $\hat{\phi}':(A \otimes B) \otimes C \longrightarrow A \otimes B \otimes C$ under which $(a \otimes b) \otimes c \longmapsto a \otimes b \otimes c$.

By a similar argument, we can get a generalized associative law for tensor product.

Prop. 0.6. Tensor product is commutative, i.e. $A \otimes B \cong B \otimes A$ by $a \otimes b \longmapsto b \otimes a$

Proof: Let $\phi:A \times B \longrightarrow B \otimes A$ by $(a,b) \longrightarrow b \otimes a$. ϕ is bilinear, hence induces an R-module homomorphism $\hat{\phi}$: $A \otimes B \longrightarrow B \otimes A$ given by $a \otimes b \longmapsto b \otimes a$. By a similar argument, we can obtain an R-module homomorphism $\hat{\phi}'$: $B \otimes A \longrightarrow A \otimes B$ such that $b \otimes a \longmapsto a \otimes b$. Thus the compositions are the identities on generating sets, hence are the identities.

Prop. 0.7. If $A \xrightarrow{f} B$ and $A' \xrightarrow{f'} B'$ where f and f' are R-linear (R-module homomorphisms), then the map $f \times f'$: $A \times A' \longrightarrow B \otimes B'$ by $f \times f'((a,a')) = f(a) \otimes f(a')$ is bilinear, hence induces uniquely an R-linear map $f \otimes f'$: $A \otimes A' \longrightarrow B \otimes B'$ by $a \otimes a' \longmapsto f(a) \otimes f(a')$. [Bbki; Remarks, p. 250] If $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an exact sequence of R-modules, and D is an R-module then $A \otimes D \xrightarrow{f \otimes id_D} B \otimes D \xrightarrow{g \otimes id_D} C \otimes D \longrightarrow 0$ is exact. [Bbki.; Prop. 5, p. 251]

Prop. 0.8. Tensor product distributes over direct sum. i.e., for 2 modules, $(A \oplus B) \otimes C \cong (A \otimes C) \oplus (B \otimes C)$ by $(a,b) \otimes c \longmapsto (a \otimes c,\ b \otimes c)$. [Bbki.; Prop. 7, p. 255]

Note: If there is any question as to what ring R over which we are taking the tensor product, we will use subscripts, e.g., $A \otimes_R B$.

Tensor, Symmetric, and Exterior Powers

Definition 0.9.

$\otimes^o A = S^o(A) = \Lambda^o A = R$; $\otimes^1 A = S^1(A) = \Lambda^1 A = A$;

If p is an integer > 1, then

a) $\otimes^p A = \overbrace{A \otimes \ldots \otimes A}^{\text{p-factors}}$

b) $S^p(A) = \otimes^p A/N_1$ where $N_1 = \langle\{a_1 \otimes \ldots \otimes a_p - a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(p)} \mid \sigma$ is a permutation of $\{1, 2, \ldots p\}\}\rangle$

c) $\Lambda^p(A) = \otimes^p A/N_2$ where $N_2 = \langle\{a_1 \otimes \ldots \otimes a_p \mid a_i = a_j$ for some $i \neq j\}\rangle$

For the image of $a_1 \otimes \ldots \otimes a_p$ in $S^p(A)$ (resp. $\Lambda^p A$) under the canonical homomorphism "mod N_1" (resp., "mod N_2") we write $a_1 * \ldots * a_p$ (resp., $a_1 \wedge \ldots \wedge a_p$).

Let $F^p(A)$ be $\otimes^p A$ (resp., $S^p(A)$; resp., $\Lambda^p A$)

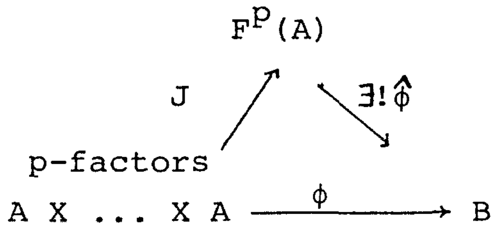
Let $J: \overbrace{A \times \ldots \times A}^{\text{p-factors}} \longrightarrow F^p(A)$ be the canonical map $(a_1, \ldots, a_p) \longmapsto a_1 * \ldots * a_p$. Then J is multilinear (resp., symmetric multilinear; resp., alternating multilinear).

Note: When we use $F^p(A)$ to stand for one of the three modules $\otimes^p A$, $S^p(A)$, or $\Lambda^p A$, then we will use $a_1 * \ldots * a_p$ to stand for the image of $(a_1, \ldots, a_p)$ under J.

Also if there is any doubt as to the ring involved we will write $F^p_R(A)$.

Prop. 0.10. The modules of Definition 0.9 have the following universal properties:

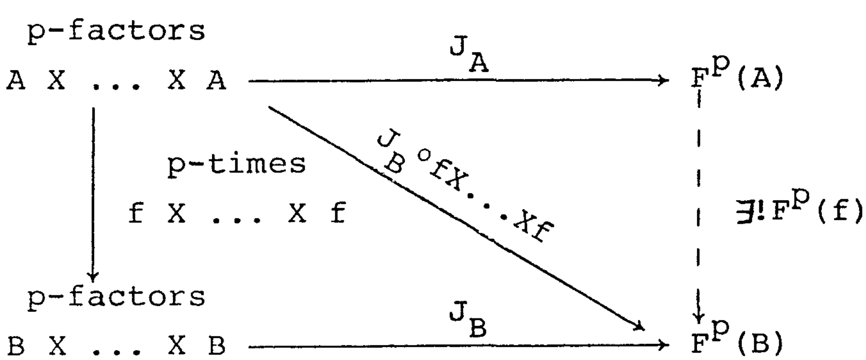
Suppose $\phi: \overbrace{A \times \ldots \times A}^{p\text{-factors}} \longrightarrow B$ is multilinear (resp., symmetric multilinear; resp., alternating multilinear). Then there exists a unique R-module homomorphism $\hat{\phi}: F^p(A) \longrightarrow B$ commuting the following diagram:



Proof: [Bbki: Prop 1, p. 485; Prop 6, p. 500; Prop 7, p. 511]

The universal properties yield the following functorial properties:

Prop. 0.11. Suppose $f: A \longrightarrow B$ is an R-module homomorphism. Consider the following diagram:



$J_B \circ (\overbrace{f \times \ldots \times f}^{p\text{-times}})$ is multilinear (resp., symmetric multilinear; resp., alternating multilinear); hence there exists a unique R-module homomorphism $F^p(f)$ that makes the diagram commutative. [Observe that $F^p(f)\ (a_1 *\dots* a_p) = f(a_1) *\dots* f(a_p)$.]

Note: If $A \xrightarrow{f} B \longrightarrow 0$ is exact, then $F^p(A) \xrightarrow{F^p(f)} F^p(B) \longrightarrow 0$ is also exact because $\{b_1 *\dots* b_p | b_i \in B\}$ generates $F^p(B)$ (because the set $\{b_1 \otimes \dots \otimes b_p\}$ generates $\otimes^p B$) and since, f is onto, each $b_1 *\dots* b_p$ is the image of an $a_1 *\dots* a_p \in F^p(A)$.

Examples 0.12

Let L be a free R-module with basis $\{e_1, \dots, e_n\}$.

a) For $p \geq 1$, $\otimes^p L$ is free on $\{e_{i_1} \otimes \dots \otimes e_{i_p}\}$ where $(i_1, i_2, \dots, i_p)$ ranges over all sequences of length p taking values from $\{1, \dots, n\}$

Proof: This follows by induction from the fact that tensor product distributes over direct sum and the fact that, for any ring R, $R \otimes R \otimes \dots \otimes R$ is free on $1 \otimes \dots \otimes 1$.

b) For $p \geq 1$, $S^p(L)$ is free on $\{e_{i_1} *\dots* e_{i_p}\}$ where $i_1 \leq i_2 \leq \dots \leq i_p$ and $i_j \in \{1, \dots, n\}$

Proof: Let $R[X_1, \dots, X_n]$ be the polynomial ring in n-variables over R. Consider the R-multilinear map $X^p L \xrightarrow{\phi} R[X_1, \dots, X_n]$ given by $(e_{i_1}, \dots, e_{i_p}) \longrightarrow X_{i_1} \cdot X_{i_2} \cdot \dots \cdot X_{i_p}$. Since the polynomial ring is commutative, ϕ is symmetric and induces an R-module homomorphism $\hat{\phi}: S^p(L) \longrightarrow R[X_1, \dots, X_n]$ given by $e_{i_1} *\dots* e_{i_p} \longmapsto X_{i_1} \cdot X_{i_2} \cdot \dots \cdot X_{i_p}$. Thus we have a subjective R-module homomorphism of $S^p(L)$ onto the homogeneous polynomials of degree p in $R[X_1, \dots X_n]$. $\{e_{i_1} *\dots* e_{i_p} | i_j \in \{1, \dots n\}$ and $i_1 \leq i_2 \leq \dots \leq i_p\}$ forms a generating set for $S^p(L)$ and its image under $\hat{\phi}$ is a basis for the homogeneous polynomials of degree p in $R[X_1, \dots, X_n]$. Thus $\{e_{i_1} * \dots * e_{i_p} \mid i_j \in \{1, \dots n\}$ and $i_1 \le i_2 \le \dots \le i_p\}$ is a basis for $S^p(L)$.

c) For $1 < p \le n$, $\Lambda^p L$ is free on $\beta = \{e_{i_1} \wedge \dots \wedge e_{i_p} \mid i_j \in \{1, \dots n\}$ and $i_1 < i_2 < \dots < i_p\}$; if $p > n$, $\Lambda^p L = 0$.

Proof: We first prove the statement for $p = n$. $\Lambda^n L$ is generated by $e_1 \wedge \dots \wedge e_n$ and we wish to see that $e_i \wedge \dots \wedge e_n$ is a free generator. It suffices to show that there exists a homomorphism of $\Lambda^n L$ to R that takes $e_i \wedge \dots \wedge e_n$ to 1. Consider the alternating multilinear map ϕ of $\overbrace{L \times \dots \times L}^{n\text{-factors}} \longrightarrow R$ given by $\left(\begin{pmatrix} r_{11} \\ \vdots \\ r_{n1} \end{pmatrix}, \begin{pmatrix} r_{12} \\ \vdots \\ r_{n2} \end{pmatrix}, \dots, \begin{pmatrix} r_{1n} \\ \vdots \\ r_{nn} \end{pmatrix}\right)$

$$\longrightarrow \det(r_{jk}). \quad \phi\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}\right] = 1. \quad \phi \text{ induces}$$

$$\hat{\phi}: \Lambda^n L \longrightarrow R \text{ such that } 1 = \hat{\phi}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \dots \wedge \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}\right) =$$

$\phi(e_1 \wedge e_2 \wedge \dots \wedge e_n)$. q.e.d.

Now suppose that $1 < p < n$. Let $\sum_{(i)} r_{(i)} e_{i_1} \wedge \dots \wedge e_{i_p}$ be a relation on the elements of β. We wish to show that $r_{(i)} = 0$ for every (i). Fixing (i'), let $i'_{p+1}, \dots, i'_n$ be the values of $\{1, \dots n\}$ that do not appear in $i'_1, \dots i'_p$.

Now consider the element $X = \sum_{(i)} r_{(i)} (e_{i_1} \wedge \dots \wedge e_{i_p} \wedge e_{i'_{p+1}} \wedge \dots \wedge e_{i'_n}) \in \Lambda^n L$.

$X = 0$ because X is the image under the canonical homomorphism $\gamma_{p,n-p}: \Lambda^p L \otimes \Lambda^{n-p} L \longrightarrow \Lambda^n L$ of the element $0 \otimes (e_{i'_{p+1}} \wedge \cdots \wedge e_{i'_n})$, which is 0.

All the terms in X have repeated components and are 0 except $r_{(i')}\, e_{i'_1} \wedge \cdots \wedge e_{i'_p} \wedge e_{i'_{p+1}} \wedge \cdots \wedge e_{i'_n}$. Therefore, $0 = r_{(i')}\, e_{i'_1} \wedge \cdots \wedge e_{i'_p} \wedge e_{i'_{p+1}} \wedge \cdots \wedge e_{i'_n} = \pm\, r_{(i')}\, e_1 \wedge \cdots \wedge e_n$ and since $e_1 \wedge \cdots \wedge e_n$ is a free generator for $\Lambda^n L$, $r_{(i')} = 0$, as desired.

If $p > n$ then $\Lambda^p L$ is generated by elements of the form $e_{i_1} \wedge \cdots \wedge e_{i_n} \wedge e_{i_{n+1}} \wedge \cdots \wedge e_{i_p}$ where $i_j \in \{1, \ldots n\}$. In each element of this form, we have at least one repeated component; hence each generator is 0 and $\Lambda^p L = 0$. q.e.d.

<u>Prop</u>. <u>0.13</u>. There exists a subjective R-module homomorphism ${}_F\gamma_{p,q}: F^p(A) \otimes F^q(A) \longrightarrow F^{p+q}(A)$ given by $(a_1 * \cdots * a_p) \otimes (a'_1 * \cdots * a'_q) \longmapsto a_1 * \cdots * a_p * a'_1 * \cdots * a'_q$.

<u>Proof</u>: If ${}_F\gamma_{p,q}$ exists, it is subjective because $F^{p+q}(A)$ is generated by elements of the form $a_1 * \cdots * a_p * a'_1 * \cdots a'_q$.

By the generalized associativity of tensor product, we get the homomorphism ${}_\otimes\gamma_{p,q}: \otimes^p A \otimes \otimes^q A \longrightarrow \otimes^{p+q} A$.

Suppose now that $F = S$ (resp., Λ). Consider the following diagram

$$\begin{array}{ccc} (\otimes^p A) \otimes (\otimes^q A) & \xrightarrow{\ {}_\otimes\gamma_{p,q}\ } & \otimes^{p+q} A \\ \Big\downarrow {\scriptstyle \Delta_F p \,\otimes\, \Delta_F q} & & \Big\downarrow {\scriptstyle \Delta_F p+q} \\ F^p(A) \otimes F^q(A) & \xrightarrow{\ {}_F\gamma_{p,q}\ } & F^{p+q}(A) \end{array}$$

where $\Delta_F p$, $\Delta_F q$, and $\Delta_F p+q$ are the canonical homomorphisms

$$a_1 \otimes \cdots \otimes a_k \overset{\Delta_F k}{\longmapsto} a_1 * \cdots * a_k$$

Since $\Delta_F p \otimes \Delta_F q$ is a surjection, we need only show that the kernel of $\Delta_F p \otimes \Delta_F q$ is contained in the kernel of $\Delta_F p{+}q \circ {}_{\otimes}\gamma_{p,q}$. But, by [Bbki: Prop. 6, p. 252], the kernel of $\Delta_F p \otimes \Delta_F q$ is the set of all elements of $(\otimes^p A) \otimes (\otimes^q A)$ that are images of $[\ker \Delta_F p \otimes \otimes^q A] + [\otimes^p A \otimes \ker \Delta F^q]$ under the map (inclusion $\otimes\ \mathrm{id}_{\otimes^q A}$) + ($\mathrm{id}_{\otimes^p A} \otimes$ inclusion).

If F = S (resp., Λ), the above image is contained in the kernel of $\Delta_F p{+}q \circ {}_{\otimes}\gamma_{p,q}$. Thus the homomorphism ${}_F\gamma_{p,q}$ always exists for $F = \otimes$, S, or Λ.

We have, by induction, a subjective homomorphism $\otimes^n F^p(A) \xrightarrow{\gamma} F^{np}(A)$ given by $(a_{11} * \cdots * a_{1p}) \otimes \cdots \otimes (a_{n1} * \cdots * a_{np}) \longmapsto a_{11} * \cdots * a_{np}$ (which is an isomorphism if $F = \otimes$).

<u>Corollary</u> <u>0.13.1</u>. Suppose p is odd or R has characteristic 2. Then there exists a unique subjective homomorphism $\hat{\gamma}$ that makes the following diagram commutative.

$$\begin{array}{ccc} \otimes^n(\Lambda^p A) & \xrightarrow{\gamma} & \Lambda^{np} A \\ {\scriptstyle \Lambda^p{}_A\Delta_n}\big\downarrow & \nearrow{\scriptstyle \hat{\gamma}} & \\ \Lambda^n(\Lambda^p A) & & \end{array}$$

<u>Proof</u>: If it exists, the homomorphism γ will be unique and a surjection because both ${}_\Lambda p_A \Delta_n$ and γ are surjections.

Since ${}_\Lambda p_A \Delta_n$ is surjective, $\hat{\gamma}$ exists, if and only if $\ker({}_\Lambda p_A \Delta_n) \subseteq \ker(\gamma)$. For simplicity, we will first deal with the case n = 2 and then generalize.

Suppose n = 2. The kernel of $\Lambda^p{}_A\Delta_2$ is generated by elements of the form $X \otimes X$ where $X \in \Lambda^p A$. Let $X = \sum_{i=1}^{k} a_{i1} \wedge \cdots \wedge a_{ip}$. Then $\gamma(X \otimes X) =$

$$\sum_{1 \le i \le i' \le k} a_{i1} \wedge \cdots \wedge a_{ip} \wedge a_{i'1} \wedge \cdots \wedge a_{i'p}.$$

Rewrite $\gamma(X \otimes X)$ as follows:

$$\sum_{i=1}^{k} a_{i1} \wedge \cdots \wedge a_{ip} \wedge a_{i1} \wedge \cdots \wedge a_{ip} + \sum_{1 \le i < i' \le k} [(a_{i1} \wedge \cdots \wedge a_{ip} \wedge a_{i'1} \quad \cdots \quad a_{i'p}) + (a_{i'1} \wedge \cdots \wedge a_{i'p} \wedge a_{i1} \wedge \cdots \wedge a_{ip})]$$

The elements $a_{i1} \wedge \cdots \wedge a_{ip} \wedge a_{i1} \wedge \cdots \wedge a_{ip}$ are 0 by definition of $\Lambda^{2p} A$.

If p is odd, then $a_{i'1} \wedge \cdots \wedge a_{i'p} \wedge a_{i1} \wedge \cdots \wedge a_{ip} =$

$-a_{i1} \wedge \cdots \wedge a_{ip} \wedge a_{i'1} \wedge \cdots \wedge a_{i'p}$ because the indices (i1, ... ip, i'1,..., i'p) can be obtained from the indices (i'1, ... i'p, i ,..., ip) by the permutation σ = (i'1,i1) (i'2,i2) ... (i'p,ip) which is an odd permutation because p is odd; thus $\gamma(X \otimes X) = 0$.

If p is even, but R has characteristic 2, then we have $a_{i'1} \wedge \cdots \wedge a_{i'p} \wedge a_{i1} \wedge \cdots \wedge a_{ip} = a_{i1} \wedge \cdots \wedge a_{ip} \wedge a_{i'1} \wedge a_{i'p}$ (here σ would be even) $= -a_{i1} \wedge \cdots \wedge a_{ip} \wedge a_{i'1} \wedge \cdots \wedge a_{i'}$ (since R has characteristic 2) and $\gamma(X \otimes X) = 0$. q.e.d.

For the general case, the kernel of $\Lambda^p{}_A\Delta_n$ is generated by elements of the form $Y_1 \otimes \cdots \otimes X \otimes \cdots \otimes Y_j \otimes \cdots \otimes X \otimes \cdots \otimes Y_{n-2}$ where X is as above and the Y_j's are "basic wedges" in $\Lambda^p A$ (i.e. of the form $a_1 \wedge \cdots \wedge a_p$). A similar argument shows that any element of the above form is in the kernel of γ.

Note a) Let n, p > 1. γ is almost never an isomorphism unless both $\Lambda^n(\Lambda^p A)$ and $\Lambda^{np}A$ are both 0. For example, if L is free on k generators, then $\Lambda^n(\Lambda^p L)$ is free on $\left(\begin{smallmatrix}\binom{k}{p}\\ n\end{smallmatrix}\right)$ generators and $\Lambda^{np}L$ is free on $\binom{k}{np}$ generators. Assume γ exists. Since γ is surjective, $\gamma:\Lambda^n(\Lambda^p L) \longrightarrow \Lambda^{np}L$ is an isomorphism, if and only if, $\binom{\binom{k}{p}}{n} = \binom{k}{np}$.

b) Let R be any ring of characteristic other than 2 and let L be the free R-module on $\{e_1, e_2, e_3, e_4\}$. Then $\Lambda^2 L$ is free on $\{e_1 \wedge e_2, e_1 \wedge e_3, e_1 \wedge e_4, e_2 \wedge e_3, e_2 \wedge e_4, e_3 \wedge e_4\}$ and $\Lambda^4 L$ is free on $e_1 \wedge e_2 \wedge e_3 \wedge e_4$.

Claim: The homomorphism $\otimes^2 \Lambda^2 L \longrightarrow \Lambda^4 L$ does not factor through $\Lambda^2 \Lambda^2 L$.

Proof of Claim: Under the homomorphism $\otimes^2 \Lambda^2 L \longrightarrow \Lambda^2\Lambda^2 L$, the element $(e_1 \wedge e_2 + e_3 \wedge e_4) \otimes (e_1 \wedge e_2 + e_3 \wedge e_4)$ goes to 0; however, the image of this element in $\Lambda^4 L$ is $(e_1 \wedge e_2 \wedge e_3 \wedge e_4) + (e_3 \wedge e_4 \wedge e_1 \wedge e_2) = 2e_1 \wedge e_2 \wedge e_3 \wedge e_4 \neq 0$. q.e.d.

Proposition 0.14. Chance of Rings. Let R' be an R-algebra. Let F^p stand for $\otimes^p$, S^p, or Λ^p. Then there is an R'-module isomorphism:

$$F^p_R(A) \otimes_R R' \xrightarrow{\cong} F^p_{R'}(A \otimes_R R') \text{ given by } (a_1 * \ldots * a_p) \otimes r' \longmapsto r' \cdot (a_1 \otimes 1) * (a_2 \otimes 1) * \ldots * (a_p \otimes 1).$$

Proof: [Bbki: Prop. 5, p. 489; Prop. 7, p. 502; Prop. 8, p. 514.]

Notice the following two special cases of Proposition 0.14:

1) If I is an ideal of R, then $F_R^p(A)/(I\ F_R^p(A)) \underset{R/I}{\cong} F_{R/I}^p(A/IA)$ by $\overline{a_1 * \ldots * a_p} \longmapsto \bar{a}_1 * \ldots * \bar{a}_p$

2) If S is a multiplicative system of R ($0 \notin S$), then $[F_R^p(A)]_S \underset{R_S}{\cong} F_{R_S}^p(A_S)$ by $\frac{a_1 * \ldots * a_p}{s} \longmapsto \frac{1}{s} \cdot \frac{a_1}{1} * \ldots * \frac{a_p}{1}$

Proposition 0.15. We have the following isomorphisms:

a) $\otimes^p(A_1 \oplus A_2) \xrightarrow{\cong} \underset{(\lambda)}{\oplus}(A_{\lambda_1} \otimes \ldots \otimes A_{\lambda_p})$

by $(a_{11}, a_{12}) \otimes \ldots \otimes (a_{p1}, a_{p2}) \longmapsto (a_{1\lambda_1} \otimes \ldots \otimes a_{p\lambda_p})_{(\lambda)}$ where $(\lambda) = (\lambda_1, \ldots \lambda_p)$ ranges over all sequences of length p whose range is $\{1,2\}$.

b) $S^p(A \oplus B) \xleftarrow{\cong} \bigoplus_{k=0}^{p} (S^{p-k}(A) \otimes S^k(B))$

by $(a_1, 0) * \ldots * (a_{p-k}, 0) * (0, b_1) * \ldots * (0, b_k) \longleftarrow\!\!\shortmid$ $(0, \ldots, 0, (a_1 * \ldots * a_{p-k}) \otimes (b_1 * \ldots * b_k), 0, \ldots, 0)$

c) $\Lambda^p(A \oplus B) \xleftarrow{\cong} \bigoplus_{k=0}^{p} (\Lambda^{p-k} A \otimes \Lambda^k B)$

by $(a_1, 0) \wedge \ldots \wedge (a_{p-k}, 0) \wedge (0, b_1) \wedge \ldots \wedge (0, b_k) \longleftarrow\!\!\shortmid$ $(0, 0, \ldots, 0, (a_1 \wedge \ldots \wedge a_{p-k}) \otimes (b_1 \wedge \ldots \wedge b_k), 0, \ldots, 0)$

Proof: a) This isomorphism is a direct consequence of the fact that tensor product distributes over direct sum. [Bbki: Prop. 7, p. 255.]

b) [Bbki: Corollary to Prop. 9, p. 505]

c) [Bbki: Corollary to Prop. 10, p. 517]

Purity

Definition 0.16. Suppose $E: 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact. Then we say that E is a pure exact sequence or f embeds A as a pure submodule of B provided that one of the following equivalent conditions holds:

a) For any R-module D the sequence $0 \longrightarrow A \otimes D \xrightarrow{f \otimes id_D} B \otimes D \xrightarrow{g \otimes id_D} C \otimes D \longrightarrow 0$ is exact.

b) Suppose $\begin{cases} r_{11}x_1 + \ldots + r_{1n}x_n = f(a_1) \\ \vdots \\ r_{m1}x_1 + \ldots + r_{mn}x_n = f(a_m) \end{cases}$ is any system of finitely many linear equations having a solution in $\underset{n}{X} B$. Then

$$\begin{cases} r_{11}x_1 + \ldots + r_{1n}x_n = a_1 \\ \vdots \\ r_{m1}x_1 + \ldots + r_{mn}x_n = a_m \end{cases}$$ has a solution in $\underset{n}{X} A$, $(r_{ij} \in R,\ a_i \in A)$.

c) Any homomorphism $\phi: C' \longrightarrow C$, where C' is finitely presented, factors through B; i.e., there exists a homomorphism $\hat{\phi}: C' \longrightarrow B$ such that $g\hat{\phi} = \phi$.

a $\Longleftrightarrow$ b [Cohn: Theorem 2.4, p. 384]

b $\Longleftrightarrow$ c [Fieldhouse: Prop. 7.2, p. 9]

We have, by c), that, if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is pure exact and C is finitely presented, then the sequence splits. Also notice that a split exact sequence is pure exact.

Proposition 0.17. Suppose that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is pure exact and $\phi: C' \longrightarrow C$. Then there exists a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0 \\ & & \Big\downarrow id_A & & \Big\downarrow \Phi & & \Big\downarrow \phi & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

where the top sequence is also pure exact, (B' being the pullback of $\left(\begin{array}{c} C' \\ \downarrow \phi \\ B \xrightarrow{g} C \end{array} \right)$.

Proof: The existence of the commutative diagram is well known [Mitchell: Prop. 13.1, p. 15.]. f' is a pure embedding because $\Phi \circ f' = f \circ id_A$, where id_A and f are pure embeddings, implying that $\Phi \circ f'$, hence f', is a pure embedding.

Direct Limits

Proposition 0.18. (Direct limits preserve exact sequences). Let $(A , \gamma^{\alpha}_{\beta})$, $(B , \Phi^{\alpha}_{\beta})$, $(C , \phi^{\alpha}_{\beta})$ be three directed systems of R-modules and (f_{α}) and (g_{α}) two directed systems of R-homomorphisms such that the sequences $A_{\alpha} \xrightarrow{f_{\alpha}} B_{\alpha} \xrightarrow{g_{\alpha}} C_{\alpha}$ are exact for all α. Then writing $f = \varinjlim f_{\alpha}$ and $g = \varinjlim g_{\alpha}$, the sequences $\varinjlim A_{\alpha} \xrightarrow{f} \varinjlim B_{\alpha} \xrightarrow{g} \varinjlim C_{\alpha}$ is exact.

Proof: [Bbki: Prop. 3, p. 287]

An immediate corollary to Prop. 0.18 is

Prop. 0.19. Let $(M_{\alpha}, \gamma^{\alpha}_{\beta})$ and $(N_{\alpha}, \phi^{\alpha}_{\beta})$ be directed systems of R-modules and let (h_{α}) be a directed system of R-homomorphisms. (i.e., For $\beta \geq \alpha$, the following diagram commutes:

$$\left.\begin{array}{ccc} M_\alpha & \xrightarrow{h_\alpha} & N_\alpha \\ \downarrow \gamma_\beta^\alpha & & \downarrow \phi_\beta^\alpha \\ M_\beta & \xrightarrow{h_\beta} & N_\beta \end{array}\right)$$

Let $h = \varinjlim h_\alpha$.

Then the exact sequence $0 \longrightarrow \ker h \xrightarrow{\text{inclusion}} \varinjlim M_\alpha \xrightarrow{h} \varinjlim N_\alpha \longrightarrow \operatorname{cok} h \longrightarrow 0$ is the limit of the sequences $0 \longrightarrow \ker h_\alpha \xrightarrow{\text{inclusion}} M_\alpha \xrightarrow{h_\alpha} N_\alpha \longrightarrow \operatorname{cok} h_\alpha \longrightarrow 0$ where, $\forall \beta > \alpha$, the map from $\ker h_\alpha$ to $\ker h_\beta$ is induced by restriction and the map from $\operatorname{cok} h_\alpha$ to $\operatorname{cok} h_\beta$ is the homomorphism induced on cokernels.

Proposition 0.20. (Tensor products commute with direct limits). Suppose $(M_\alpha, \gamma_\beta^\alpha)$ and $(N_\alpha, \phi_\beta^\alpha)$ are directed systems of R-modules. Then the directed system $(M_\alpha \otimes N_\alpha, \gamma_\beta^\alpha \otimes \phi_\beta^\alpha)$ has as its limit $\varinjlim M_\alpha \otimes \varinjlim N_\alpha$ where, for each α, the canonical map of $M_\alpha \otimes N_\alpha$ into $\varinjlim M_\alpha \otimes \varinjlim N_\alpha$ is $\gamma^\alpha \otimes \phi^\alpha$.

Proof: [Bbki: Prop. 7, p. 290]

Proposition 0.21. The direct limit of pure exact sequences is pure exact.

Proof: Suppose $A \xrightarrow{f} B \xrightarrow{g} C$ is the direct limit of the pure exact sequences $0 \longrightarrow A_\alpha \xrightarrow{f_\alpha} B_\alpha \xrightarrow{g_\alpha} C_\alpha \longrightarrow 0$. Let D be any R-module. If we can show that $0 \longrightarrow A \otimes D \xrightarrow{f \otimes id_D} B \otimes D \xrightarrow{g \otimes id_D} C \otimes D \longrightarrow 0$ is exact, we will be done by Definition 0.16. But since $0 \longrightarrow A_\alpha \xrightarrow{f_\alpha} B_\alpha \xrightarrow{g_\alpha} C_\alpha \longrightarrow 0$ is pure exact, $0 \longrightarrow A_\alpha \otimes D \xrightarrow{f_\alpha \otimes id_D} B_\alpha \otimes D \xrightarrow{g_\alpha \otimes id_D} C_\alpha \otimes D \longrightarrow 0$ is exact, again by Definition 0.16.

Since, by 0.20, tensor products commute with direct limits, $A \otimes D \xrightarrow{f \otimes id_D} B \otimes D \xrightarrow{g \otimes id_D} C \otimes D$ is the limit of the exact sequences $0 \longrightarrow A_\alpha \otimes D \xrightarrow{f_\alpha \otimes id_D} B_\alpha \otimes D \xrightarrow{g_\alpha \otimes id_D} C_\alpha \otimes D \longrightarrow 0$; hence $0 \longrightarrow A \otimes D \xrightarrow{f \otimes id_D} B \otimes D \xrightarrow{g \otimes id_D} C \otimes D \longrightarrow 0$ is exact since, by Prop. 0.18, the direct limit of exact sequences is exact. q.e.d.

Proposition 0.22. Let F^p be one of $\otimes^p$, S^p, or Λ^p. Then F^p commutes with direct limits in the following obvious way: If $M = \varinjlim (M_\alpha, \gamma^\alpha)$, then $F^p(M) = \varinjlim (F^p(M_\alpha), F^p(\gamma^\alpha))$.

Proof: For $\otimes^p$, we use induction and the fact that tensor product commutes with direct limits (Prop. 0.20); for S^p, [Bbki: Prop. 8, p. 503]; for Λ^p, [Bbki: Prop. 9, p. 515].

Proposition 0.23. Every module is the direct limit of a family of finitely presented modules.

Proof: [Lazard: Appendice, p. 125]

Proposition 0.24. Suppose $E: 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact. Suppose $\varinjlim (C_\alpha, \phi^\alpha) = C$. Let $E^\alpha: 0 \longrightarrow A_\alpha \xrightarrow{f_\alpha} B_\alpha \xrightarrow{g_\alpha} C_\alpha \longrightarrow 0$ be the α^{th} pullback sequence relative to $\phi^\alpha: C_\alpha \longrightarrow C$. Then $\{E_\alpha\}$ is a directed family of exact sequences and $E = \varinjlim \{E_\alpha\}$.

Proof: Suppose $\beta > \alpha$. Use the pullback property of B_β relative to g and ϕ^β to define uniquely $\Phi^\alpha_\beta: B_\alpha \longrightarrow B_\beta$ such that the following diagram is commutative with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{f_\alpha} & B_\alpha & \xrightarrow{g_\alpha} & C_\alpha & \longrightarrow & 0 \\
 & & \Vert & & \downarrow \Phi_\beta^\alpha & & \downarrow \phi_\beta^\alpha & & \\
0 & \longrightarrow & A & \xrightarrow{f_\beta} & B_\beta & \xrightarrow{g_\beta} & C_\beta & \longrightarrow & 0 \\
 & & \Vert & & \downarrow \Phi^\beta & & \downarrow \phi^\beta & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0
\end{array}$$

The above diagram induces a commutative diagram:

$$\begin{array}{ccccccccc}
 & & A & \xrightarrow{\lim f_\alpha} & \underrightarrow{\lim}\, B_\alpha & \xrightarrow{\lim g_\alpha} & \underrightarrow{\lim}\, C_\alpha = C & \longrightarrow & 0 \\
 & & \Vert & & \downarrow \Phi & & \Vert & & \\
0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0
\end{array}$$

(Since $\{\Phi^\alpha : B_\alpha \longrightarrow B\}$ has the property that for $\beta > \alpha$ $\Phi^\alpha = \Phi^\beta \circ \Phi_\beta^\alpha$ we have a homomorphism Φ induced from $\underrightarrow{\lim}\, B_\alpha$ to B.) Φ must be an isomorphism by the "5 Lemma".

CHAPTER I. RANK AND EXACT SEQUENCES

Let F be one of $\otimes$, S, or Λ; Let G be one of $\otimes$, S, or Λ. The aim of this chapter is to prove the following two theorems:

Theorem 1. Suppose, for $1 \le i \le n$, $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$ is an exact sequence. Then G-rank $(B_1 \otimes \ldots \otimes B_n) \le$ G-rank $\big((A_1 \oplus C_1) \otimes \ldots \otimes (A_n \oplus C_n)\big)$. If each f_i is a pure embedding, then G-rank $(B_1 \otimes \ldots \otimes B_n)$ = G-rank $\big((A_1 \oplus C_1) \otimes \ldots \otimes \ldots \otimes (A_n \oplus C_n)\big)$

Theorem 2. Suppose $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact. Then G-rank $\big(F^p(B)\big) \le$ G-rank $\big(F^p(A \oplus C)\big)$. If f is a pure embedding, then G-rank $\big(F^p(B)\big)$ = G-rank $\big(F^p(A \oplus C)\big)$.

We begin by considering the kernel of the homomorphism $F^p(g): F^p(B) \longrightarrow F^p(C)$.

Suppose $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact and $A \xrightarrow{f'} B \xrightarrow{g'} C \longrightarrow 0$ is exact; then, by [Bbki: Prop. 6, p. 252]

$$(A \otimes B') \oplus (B \otimes A') \xrightarrow{(f \otimes id_{B'}) + (id_B \otimes f')} B \otimes B' \xrightarrow{g \otimes g'} C \otimes C' \longrightarrow 0$$ is exact.

Applying this p-1 times, we have: If $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact, then $0 \longrightarrow \ker \otimes^p g \xrightarrow{\text{inclusion}} \otimes^p B \xrightarrow{\otimes^p g} \otimes^p C \longrightarrow 0$ is exact where $\ker \otimes^p g = \langle\{b_1 \otimes \ldots \otimes b_p \mid \text{all } b_i \in B, \text{ some } b_j \in F(A)\}\rangle$.

Now notice that F = S or Λ, we have the following commutative diagram of exact sequences, where T_B and T_C are the canonical homomorphisms:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \ker T_B & \xrightarrow{\otimes^p g \text{ restr.}} & \ker T_C & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \text{incl.}} & & \downarrow{\scriptstyle \text{incl.}} & & \\
0 & \longrightarrow & \ker \otimes^p g & \xrightarrow{\text{incl.}} & \otimes^p B & \xrightarrow{\otimes^p g} & \otimes^p C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle T_B \text{ restricted}} & & \downarrow{\scriptstyle T_B} & & \downarrow{\scriptstyle T_C} & & \\
0 & \longrightarrow & \ker F^p(g) & \xrightarrow{\text{incl.}} & F^p(B) & \xrightarrow{F^p(g)} & F^p(C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

From the fact that g is onto and by the way $F^p(X)$ is defined for X an R-module and F either S or Λ, we get that the map $\otimes^p g$ restricted: $\ker T_B \longrightarrow \ker T_C$ is onto. A diagram chase will now yield the fact that T_B restricted: $\ker \otimes^p g \longrightarrow \ker F^p(g)$ is onto.

Summarizing, we have:

<u>Lemma</u> <u>1.1</u>. Suppose $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact. Then $\ker F^p(g) = \langle \{b_1 * \ldots * b_p \mid b_i \in B, \text{ some } b_j \in f(A)\} \rangle$ If $F = S$ or Λ, $\ker F^p(g) = \langle \{f(a) * b_2 * \ldots * b_p \mid a \in A; b_2, \ldots, b_p \in B\} \rangle$

<u>Proposition</u> <u>1.2a</u>. Let F be S or Λ. Suppose $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact. Then there exist exact sequences:

$$\varepsilon_0: \quad F^p(A) \xrightarrow{\psi_0} F^p(B) \longrightarrow N_0 \longrightarrow 0$$

$$\varepsilon_1: \quad F^{p-1}(A) \otimes C \xrightarrow{\psi_1} N_0 \longrightarrow N_1 \longrightarrow 0$$

$$\vdots$$

$$\varepsilon_k: \quad F^{p-k}(A) \otimes F^k(C) \xrightarrow{\psi_k} N_{k-1} \longrightarrow N_k \longrightarrow 0$$

$$\vdots$$

$$\varepsilon_{p-1}: \quad A \otimes F^{p-1}(C) \xrightarrow{\psi_{p-1}} N_{p-2} \longrightarrow F^p(C) \longrightarrow 0.$$

If, in addition, f is a pure embedding, then for, $0 \leq k \leq p-1$, ψ_k is a pure embedding.

<u>Proof</u>: We first show the existence of the exact sequences ε_k for $0 \leq k \leq p-1$.

Let $\psi_0 = \theta_0 = F^p(f) : F^p(A) \longrightarrow F^p(B)$. For $1 \leq k \leq p-1$, let $\theta_k : F^{p-k}(A) \otimes F^k(B) \longrightarrow F^p(B)$ be the canonical homomorphism induced by f and id_B (i.e. $(a_1 * \ldots * a_{p-k}) \otimes (b_1 * \ldots * b_k) \longmapsto f(a_1) * \ldots * f(a_{p-k}) * b_1 * \ldots * b_k$, where $a_i \in A$, $b_j \in B$). For $0 \leq k \leq p-1$, let $N_k = \operatorname{cok} \theta_k$. Note that $N_{p-1} = \operatorname{cok} \theta_{p-1} = F^p(B)/\operatorname{im}(A \otimes F^{p-1}(B)) \cong F^p(C)$ by <u>Lemma 1.1</u>.

For $1 \leq k \leq p-1$, consider the following commutative diagram where the homomorphisms between the modules are the canonical homomorphisms induced by f and id_B:

$$D_{F,k} \qquad \begin{array}{ccc} \otimes^{p-k+1} A \otimes \otimes^{k-1} B & \xrightarrow{\Delta} & F^{p-k+1}(A) \otimes F^{k-1}(B) \\ \Big\downarrow \eta & & \Big\downarrow \theta_{k-1} \\ F^{p-k}(A) \otimes F^k(B) & \xrightarrow{\theta_k} & F^p(B) \end{array}$$

(We obtain η by composition of the following chain of canonical homomorphisms:

$$\bigotimes^{p-k+1} A \otimes \bigotimes^{k-1} B \longrightarrow \bigotimes^{p-k} A \otimes (A \otimes \bigotimes^{k-1} B)$$

$$\xrightarrow{\bigotimes^{p-k} id_A \otimes (f \otimes \bigotimes^{k-1} id_B)} \bigotimes^{p-k} A \otimes (B \otimes \bigotimes^{k-1} B)$$

$$\longrightarrow \bigotimes^{p-k} A \otimes \bigotimes^{k} B \longrightarrow F^{p-k}(A) \otimes F^{k}(B)\ )$$

The diagram $D_{F,k}$ induces a commutative diagram $\widetilde{D_{F,k}}$ with exact rows and columns obtained by tacking on cokernels as follows:

$$\widetilde{D_{F,k}} \qquad \begin{array}{ccccccc}
\bigotimes^{p-k+1} A \otimes \bigotimes^{k-1} B & \xrightarrow{\Delta} & F^{p-k+1}(A) \otimes F^{k-1}(B) & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow \eta & & \downarrow \theta_{k-1} & & \downarrow & & \\
F^{p-k}(A) \otimes F^{k}(B) & \xrightarrow{\theta_k} & F^{p}(B) & \longrightarrow & N_k & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\varepsilon_k : F^{p-k}(A) \otimes F^{k}(C) & \xrightarrow{\psi_k} & N_{k-1} & \longrightarrow & N_k & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Define ψ_k to be the homomorphism induced on cokernels as shown. (That N_{k-1} is cok θ_{k-1} and N_k is cok θ_k follows by definition; that $F^{p-k}(A) \otimes F^{k}(C)$ is canonically cok η follows from Lemma 1.1 and the fact that tensor product is right exact.)

Thus we have shown the existence of the exact sequences ε_k for $0 \leq k \leq p-1$.

Now we will show that, if f is a pure embedding, then, for $0 \leq k \leq p-1$, ψ_k is a pure embedding. We will begin with a lemma about split exact sequences.

Lemma 1.2a.1. If $E: 0 \longrightarrow A \underset{\pi}{\overset{f}{\rightleftarrows}} B \underset{\delta}{\overset{g}{\rightleftarrows}} C \longrightarrow 0$ splits, then, for $0 \leq k \leq p-1$, ψ_k is a split embedding.

Proof: $k = 0$. Let π be a left-hand splitting map For E; then $\psi_0 = F^p(f):F^p(A) \longrightarrow F^p(B)$ splits by $F^p(\pi)$.

Suppose k is fixed, $1 \leq k \leq p-1$. Let δ be a fixed right-hand splitting map for E. Write $B = f(A) \oplus \delta(C)$, the direct sum being internal. Identifying A with f(A) and C with $\delta(C)$, we notice that the projection homomorphism Γ_k of $F^p(B)$ onto $F^{p-k}(A) \otimes F^k(C)$ obtained from the isomorphism $F^p(B) \cong F^p(A) \oplus \ldots \oplus [(F^{p-k}(A) \otimes F^k(C)] \oplus \ldots \oplus F^p(C)$ (Proposition 0.15) has the following two properties:

i) $\Gamma_k[\text{im } \theta_{k-1}] = 0$

ii) $\Gamma_k(f(a_1) * \ldots * f(a_{p-k}) * \delta(c_1) * \ldots * \delta(c_k)) = (a_1 * \ldots * a_{p-k}) \otimes (c_1 * \ldots * c_k)$, where $a_i \in A$, $c_j \in C$.

i) tells us that Γ_k factors through cok $\theta_{k-1} = N_{k-1}$ uniquely; let $\hat{\Gamma}_k$ be the homomorphism induced by Γ_k from N_{k-1} to $F^{p-k}(A) \otimes F^k(C)$.

Consider, for $a_i \in A$, $c_j \in C$, $\hat{\Gamma}_k \circ \psi_k((a_1 * \ldots * a_{p-k}) \otimes (c_1 * \ldots * c_k)) = \Gamma_k (f(a_1) * \ldots * f(a_{p-k}) * b_1 * \ldots * b_k)$ where $b_j \in B$ and $g(b_1) = c_1$, $g(b_2) = c_2, \ldots, g(b_k) = c_k$.

Since $g(b_j) = c_j$ for $1 \leq j \leq k$ and δ is a splitting homomorphism for g, then $b_1 = f(a_1') + \delta(c_1)$, $b_2 = f(a_2') + \delta(c_2)$,..., $b_k = f(a_k') + \delta(c_k)$ where $a_j' \in A$. Therefore, $f(a_1) * \ldots * f(a_{p-k}) * b_1 * \ldots * b_k = [f(a_1) * \ldots * f(a_{p-k}) * \delta(c_1) * \ldots * \delta(c_k)] + X$ where $X \in \text{im}\, \theta_{k-1}$. By i), $\Gamma(X) = 0$, and $\Gamma_k\big(f(a_1) * \ldots * f(a_{p-k}) * b_1 * \ldots * b_k\big) = \Gamma_k\big(f(a_1) * \ldots * f(a_{p-k}) * \delta(c_1) * \ldots * \delta(c_k)\big) = (a_1 * \ldots * a_{p-k}) \otimes (c_1 * \ldots * c_k)$ by ii). We have just shown that $\hat{\Gamma}_k \circ \psi_k((a_1 * \ldots * a_{p-k}) \otimes (c_1 * \ldots * c_k)) = (a_1 * \ldots * a_{p-k}) \otimes (c_1 * \ldots * c_k)$. Since elements of the form $(a_1 * \ldots * a_{p-k}) \otimes (c_1 * \ldots * c_k)$ generate $F^{p-k}(A) \otimes F^k(C)$, $\hat{\Gamma}_k \circ \psi_k = id_{F^{p-k}(A) \otimes F^k(C)}$, as desired. q.e.d.

Now suppose that $E: 0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is pure exact. Let $(C_\alpha, \phi_\beta^\alpha)$ be a directed family of finitely presented modules whose direct limit is C. (Existence of such a family is given by Prop. 0.23.) For each α, let $E^\alpha: 0 \longrightarrow A_\alpha \xrightarrow{f_\alpha} B_\alpha \xrightarrow{g_\alpha} C_\alpha \longrightarrow 0$ be the (right) pullback sequence relative to $C_\alpha \xrightarrow{\phi_\alpha} C$. E^α splits since C_α is finitely presented and the pullback of a pure exact sequence is pure exact [Remark following Prop. 0.16 and Prop. 0.17].

$\psi_0 = F^p(f)$ is the direct limit of $(F^p(f_\alpha))$ since F^p commutes with direct limit (by Prop. 0.22). Each $F^p(f_\alpha)$ splits by Lemma 1.2a.1; hence ψ_0 is pure exact since it is the direct limit of split embeddings (Prop. 0.21).

For $1 \leq k \leq p-1$, consider the commutative diagram, k fixed:

$$D^{\alpha}_{F,k} \qquad \begin{array}{ccc} \otimes^{p-k+1}A \otimes \otimes^{k-1} B_{\alpha} & \xrightarrow{\Delta_{\alpha}} & F^{p-k+1}(A) \otimes F^{k-1}(B_{\alpha}) \\ \downarrow{\scriptstyle \eta_{\alpha}} & & \downarrow{\scriptstyle \theta_{\alpha_{k-1}}} \\ F^{p-k}(A) \otimes F^{k}(B_{\alpha}) & \xrightarrow{\theta_{\alpha_k}} & F^{p}(B_{\alpha}) \end{array}$$

where, as before, the homomorphisms between the modules are the canonical homomorphisms induced by f_{α} and $id_{B_{\alpha}}$. Since E is the direct limit of $\{E^{\alpha}\}$ [Prop. 0.24] and since $\otimes$ and F^n commute with direct limit, then $D_{F,k}$ is the limit of $\{D^{\alpha}_{F,k}\}$.

Let $\widetilde{D^{\alpha}_{F,k}}$ be, as before, the induced commutative diagram with exact rows and columns obtained by tacking on cokernels to $D^{\alpha}_{F,k}$:

$$\begin{array}{ccccccc} \otimes^{p-k+1}A \otimes \otimes^{k-1} B_{\alpha} & \xrightarrow{\Delta_{\alpha}} & F^{p-k+1}(A) \otimes F^{k-1}(B_{\alpha}) & \longrightarrow & 0 & \longrightarrow & 0 \\ \downarrow{\scriptstyle \eta_{\alpha}} & & \downarrow{\scriptstyle \theta_{\alpha_{k-1}}} & & \downarrow & & \\ F^{p-k}(A) \otimes F^{k}(B_{\alpha}) & \xrightarrow{\theta_{\alpha_k}} & F^{p}(B_{\alpha}) & \longrightarrow & N_{\alpha_k} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ F^{p-k}(A) \otimes F^{k}(C_{\alpha}) & \xrightarrow{\psi_{\alpha_k}} & N_{\alpha_{k-1}} & \longrightarrow & N_{\alpha_k} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & & 0 & & 0 & & \end{array}$$

Then, by Prop. 0.19, $\widetilde{D_{F,k}}$ is the direct limit of $\{\widetilde{D^{\alpha}_{F,k}}\}$;

in particular, $\varepsilon_k : F^{p-k}(A) \otimes F^k(C) \xrightarrow{\psi_k} N_{k-1} \longrightarrow N_k \longrightarrow 0$ is the direct limit of $\{\varepsilon_k^\alpha : F^{p-k}(A) \otimes F^k(C_\alpha) \longrightarrow N_{\alpha_{k-1}} \longrightarrow N_{\alpha_k} \longrightarrow 0\}$; but each ε_k^α splits since E^α splits (by Lemma 1.2a.1); hence ε_k is the direct limit of a family of split (hence pure) exact sequences; hence, by Prop. 0.21, ε_k is a pure exact (i.e. ψ_k is a pure embedding). q.e.d.

Let $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be exact. Looking at Proposition 1.2a, we notice that for $F = S$ or Λ, $F^p(A \oplus C)$ is isomorphic (by Prop. 0.15) to the direct sum of the left-hand terms of the sequences ε_k and the right hand term, $F^p(C)$, of ε_{p-1}. If $F^p(A \oplus C) = 0$, then all its direct summands are 0, yielding $F^p(B) \cong N_0 \cong N_1 \cong \ldots \cong N_{p-2} \cong F^p(C) = 0$.

We have just shown:

Corollary 1.3a. Let F be either S or Λ. Let A, B, and C be R-modules. If there exists an exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$, then F-rank (B) $\leq$ F-rank $(A \oplus C)$.

Looking again at Proposition 1.2a, suppose that $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is pure exact and that $F^p(B) = 0$. Then $0 = N_0 = N_1 = \ldots = N_{p-2} = F^p(C)$ because homomorphic images of the 0 module are the 0 module. For $0 \leq k \leq p-1$, the left-hand term of ε_k is also 0 since ψ_k is a (pure) embedding. But now $F^p(A \oplus C) = 0$ because it is the direct sum of terms all of which are 0. Along with Corollary 1.3a, this argument has proved:

Corollary 1.4a. Let F be either S or Λ. Let A, B, and C be R-modules. If there exists a pure exact sequence

$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$, then F-rank (B) = F-rank (A $\oplus$ C).

We now establish, directly, a result for $\otimes$-rank analogous to Corollary 1.3a.

Corollary 1.3b. Suppose A, B, and C are R-modules and there exists an exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$. Then $\otimes$-rank (B) $\leq$ $\otimes$-rank (A $\oplus$ C).

Proof: It suffices to show that, whenever $\otimes^p(A \oplus C) = 0$, then $\otimes^p B = 0$. Suppose $\otimes^p(A \oplus C) = 0$ and $b_1 \otimes \ldots \otimes b_p$ is a fundamental tensor in $\otimes^p B$. Since $\otimes^p C = 0$, $b_1 \otimes \ldots \otimes b_p \in \ker \otimes^p g$, and we can, by Lemma 1.1, write $b_1 \otimes \ldots \otimes b_p$ as a sum of fundamental tensors of the form $b'_1 \otimes \ldots \otimes b'_{i-1} \otimes f(a'_i) \otimes b'_{i+1} \otimes \ldots \otimes b'_p$, $b'_t \in B$, $a'_i \in A$. But $A \otimes \otimes^{p-1} C$ is also 0; thus we have, from Lemma 1.1 and the fact that tensor product is right exact, that $b'_1 \otimes \ldots \otimes b'_{i-1} \otimes f(a'_i) \otimes b'_{i+1} \otimes \ldots \otimes b'_p$ can be written as a sum of fundamental tensors of the form $b''_1 \otimes \ldots \otimes b''_{j-1} \otimes f(a''_j) \otimes b''_{j+1} \otimes \ldots \otimes b''_{k-1} \otimes f(a''_k) \otimes b''_{k+1} \otimes \ldots \otimes b''_p$, where either j or k = i and $b''_t \in B$, a''_j, $a''_k \in A$. Continuing in this manner, we obtain $b_1 \otimes \ldots \otimes b_p$ as a sum of fundamental tensors of the form $f(a_1) \otimes \ldots \otimes f(a_p)$ which are all 0 since $\otimes^p A = 0$; therefore $b_1 \otimes \ldots \otimes b_p = 0$, implying $\otimes^p B = 0$. q.e.d.

Next we prove a proposition somewhat similar to Proposition 1.2a:

Proposition 1.2b. Suppose $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is pure exact. Then, for $0 \leq k \leq p-1$, there exist pure embeddings ψ_k as follows:

$$\otimes^p A \xrightarrow{\psi_0} \otimes^p B$$

$$\otimes^{p-1} A \otimes C \xrightarrow{\psi_1} \otimes^p B/K_0$$

$$\vdots$$

$$\otimes^{p-k} A \otimes \otimes^k C \xrightarrow{\psi_k} \otimes^p B/K_{k-1}$$

$$\vdots$$

$$A \otimes \otimes^{p-1} C \xrightarrow{\psi_{p-1}} \otimes^p B/K_{p-2}$$

where $K_0 \subseteq K_1 \subseteq \ldots \subseteq K_{p-2}$ are certain submodules of $\otimes^p B$.

<u>Proof</u>: Define $\psi_0 = \theta_0 = \otimes^p f$. Define, for $1 \leq k \leq p-1$, $\theta_k : \otimes^{p-k} A \otimes \otimes^k B \longrightarrow \otimes^p B$ to be the canonical homomorphism induced by f and id_B [i.e. $\theta((a_1 \otimes \ldots \otimes a_{p-k}) \otimes (b_1 \otimes \ldots \otimes b_k)) = f(a_1) \otimes \ldots \otimes f(a_{p-k}) \otimes b_1 \otimes \ldots \otimes b_k$]. For $0 \leq t \leq p-2$, define K_t as the submodule of $\otimes^p B$ generated by fundamental tensors of the form $b_1 \otimes \ldots \otimes b_p$ where at least $p-t$ of the b_i's are in f(A).

For $1 \leq k \leq p-1$, consider the following commutative diagram where ψ_k is induced by θ_k:

$$\widetilde{D}_{\otimes,k} \qquad \begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
\otimes^{p-k}A \otimes (\ker \otimes^{k} g) & \xrightarrow{\theta_k \circ (\otimes^{p-k}\, id_A \otimes\, inc.)} & K_{k-1} \\
\downarrow {\scriptstyle \otimes^{p-k} id_A \,\otimes\, inclusion} & & \downarrow {\scriptstyle inclusion} \\
\otimes^{p-k}A \otimes \otimes^{k}B & \xrightarrow{\theta_k} & \otimes^{p}B \\
\downarrow {\scriptstyle \otimes^{p-k} id_A \otimes\ \otimes^{k} g} & & \downarrow \\
\otimes^{p-k}A \otimes \otimes^{k}C & \xrightarrow{\psi_k} & \otimes^{p}B/K_{k-1} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

(The left hand column is exact since tensor product is right exact; the right hand column is defined to be exact. Any element of $\otimes^{p-k}A \otimes (\ker \otimes^{k} g)$ can be written as a sum of fundamental tensors of the form $(a_1 \otimes \ldots \otimes a_{p-k}) \otimes (b_1 \otimes \ldots \otimes b_{i-1} \otimes f(a) \otimes b_{i+1} \otimes \ldots \otimes b_k)$. The image of a fundamental tensor of this form in $\otimes^p B$ under $\theta_k \circ (\otimes^{p-k}\, id_A \otimes \text{inclusion})$ is of the form $f(a_1) \otimes \ldots \otimes f(a_{p-k}) \otimes b_1 \otimes \ldots \otimes b_{i-1} \otimes f(a) \otimes b_{i+1} \otimes \ldots \otimes b_k$ which is in K_{k-1} since at least p-k+1 of the entries are in f(A); hence the top horizontal homomorphism exists. Let ψ_k be the homomorphism induced on the cokernels of the vertical homomorphisms.)

<u>Lemma</u> <u>1.2b.1</u>. If $E: 0 \longrightarrow A \underset{\pi}{\overset{f}{\rightleftarrows}} B \underset{\delta}{\overset{g}{\rightleftarrows}} C \longrightarrow 0$ splits then ψ_k is a split embedding for $0 \leq k \leq p-1$.

<u>Proof</u>: (The proof is analogous to the proof of

Lemma 1.2a.1) $k = 0$. Let π be a left hand splitting homomorphism for E. Then $\psi_0 = \otimes^p f$ splits by $\otimes^p \pi$.

Let k be fixed, $1 \leq k \leq p-1$. Let δ be a fixed right-hand splitting map for E. Write $B = f(A) \oplus \delta(C)$, the direct sum being internal. Identifying A with f(A) and C with $\delta(C)$, we note that the projection homomorphism Γ_k of $\otimes^p B$ onto $\otimes^{p-k} A \otimes \otimes^k C$ has the following two properties:

i) $\Gamma_k(K_{k-1}) = 0$

ii) $\Gamma_k\big(f(a_1) \otimes \ldots \otimes f(a_{p-k}) \otimes \delta(c_1) \otimes \ldots \otimes \delta(c_k)\big) = (a_1 \otimes \ldots \otimes a_{p-k}) \otimes (c_1 \otimes \ldots \otimes c_k)$ where $a_i \in A$, $c_j \in C$.

i) tells us that Γ_k factors through $\otimes^p B/K_{k-1}$; let $\hat{\Gamma}_k$ be the homomorphism induced by Γ_k from $\otimes^p B/K_{k-1}$ onto $\otimes^{p-k} A \otimes \otimes^k C$

If we could show, for $a_i \in A$, $c_j \in C$, $\hat{\Gamma}_k \circ \psi_k\big((a_1 \otimes \ldots \otimes a_{p-k}) \otimes (c_1 \otimes \ldots \otimes c_k)\big) = (a_1 \otimes \ldots \otimes a_{p-k}) \otimes (c_1 \otimes \ldots \otimes c_k)$, then $\hat{\Gamma}_k \circ \psi_k$ would be $id_{\otimes^{p-k} A \otimes \otimes^k C}$ since elements of the form $(a_1 \otimes \ldots \otimes a_{p-k}) \otimes (c_1 \otimes \ldots \otimes c_k)$ generate $\otimes^{p-k} A \otimes \otimes^k C$. $\hat{\Gamma}_k \circ \psi_k\big((a_1 \otimes \ldots \otimes a_{p-k}) \otimes (c_1 \otimes \ldots \otimes c_k)\big) = \Gamma_k\big(f(a_1) \otimes \ldots \otimes f(a_{p-k}) \otimes b_1 \otimes \ldots \otimes b_k\big)$ where, for $1 \leq j \leq k$, $b_j \in B$ and $g(b_j) = c_j$. Since δ is a splitting homomorphism for g, $b_1 = f(a_1') + \delta(c_1), \ldots, b_k = f(a_k') + \delta(c_k)$ where $a_j' \in A$. Therefore $f(a_1) \otimes \ldots \otimes f(a_{p-k}) \otimes b_1 \otimes \ldots \otimes b_k = [f(a_1) \otimes \ldots \otimes f(a_{p-k}) \otimes \delta(c_1) \otimes \ldots \otimes \delta(c_k)] + X$ where $X \in K_{k-1}$. By i) $\Gamma_k(X) = 0$; therefore, $\Gamma_k\big(f(a_1) \otimes \ldots \otimes f(a_{p-k}) \otimes b_1 \otimes \ldots \otimes b_k\big) = \Gamma_k\big(f(a_1) \otimes \ldots \otimes f(a_{p-k}) \otimes \delta(c_1) \otimes \ldots \otimes \delta(c_k)\big) = (a_1 \otimes \ldots \otimes a_{p-k}) \otimes (c_1 \otimes \ldots \otimes c_k)$ by ii). Therefore $\hat{\Gamma}_k \circ \psi_k\big((a_1 \otimes \ldots \otimes a_{p-k}) \otimes (c_1 \otimes \ldots \otimes c_k)\big) = (a_1 \otimes \ldots \otimes a_{p-k}) \otimes (c_1 \otimes \ldots \otimes c_k)$

and $\widehat{\Gamma}_k \circ \psi_k = id_{\otimes^{p-k}A \otimes \otimes^k C}$. q.e.d.

Let $(C_\alpha, \phi^\alpha_\beta)$ be a family of finitely-presented modules whose direct limit is C. For each α, let $E^\alpha: 0 \longrightarrow A \xrightarrow{f_\alpha} B_\alpha \xrightarrow{g_\alpha} C_\alpha \longrightarrow 0$ be the pullback sequence of $E: 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ relative to $C_\alpha \xrightarrow{\phi^\alpha} C$. Each E^α splits because f_α is a pure embedding and C_α is finitely presented.

For k = 0, $\psi_o = \otimes^p f : \otimes^p A \longrightarrow \otimes^p B$ is a pure embedding (by Prop. 0.21) since it is the direct limit of split embeddings $\otimes^p f_\alpha : \otimes^p A \longrightarrow \otimes^p B_\alpha$.

For $1 \le k \le p-1$, k fixed, consider the commutative diagram

$$\widetilde{D^\alpha_{\otimes,k}} \qquad \begin{array}{ccc} & & 0 \\ & & \downarrow \\ \otimes^{p-k}A \otimes \ker \otimes^k g_\alpha & \longrightarrow & K_{\alpha_{k-1}} \\ \downarrow & & \downarrow \text{inclusion} \\ \otimes^{p-k}A \otimes \otimes^k B_\alpha & \xrightarrow{\theta_{\alpha_k}} & \otimes^p B_\alpha \\ \downarrow \otimes^{p-k} id_A \otimes \otimes^k g_\alpha & & \downarrow \\ \otimes^{p-k}A \otimes \otimes^k C_\alpha & \xrightarrow{\psi_{\alpha_k}} & \otimes^p B_\alpha / K_{\alpha_{k-1}} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

where, θ_{α_k} is the canonical homomorphism induced by f_α and id_{B_α}, $K_{\alpha_{k-1}}$ is the submodule of $\otimes^p B_\alpha$ generated by fundamental tensors of the form $b_1 \otimes \ldots \otimes b_p$ where $b_i \in B_\alpha$ and at least p-k+1 of the b_i's are in

$f_\alpha(A)$, and, as before, ψ_{α_k} is the homomorphism induced on cokernels. (The same argument that showed the existence of $\widetilde{D_{\otimes,k}}$ shows the existence of $\widetilde{D^{\alpha}_{\otimes,k}}$.)

Since tensor product commutes with direct limits and all homomorphisms are canonical, $\widetilde{D_{\otimes,k}}$ is the direct limit of the family $\{\widetilde{D^{\alpha}_{\otimes,k}}\}$. In particular ψ_k is the direct limit of the family (ψ_{α_k}); hence ψ_k is a pure embedding (by Prop. 0.21) since each ψ_{α_k} is a split (hence pure) embedding (by Lemma 1.2b.1). q.e.d.

In the presence of Prop. 1.2b, suppose $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is pure exact and $\otimes^p B = 0$. Then $\otimes^p C = 0$ because $\otimes^p g$ is subjective. Since $\otimes^p B$ is 0, so are $\otimes^p B/K_0$, ..., $\otimes^p B/K_{p-2}$ all 0. Since for $0 \leq k \leq p-1$, ψ_k is a pure embedding, hence an embedding, $\otimes^p A$, $\otimes^{p-1} A \otimes C$, ..., $A \otimes \otimes^{p-1} C$ are all 0. But $\otimes^p (A \oplus C) = \bigoplus_{k=0}^{p} \binom{p}{k} [\otimes^{p-k} A \otimes \otimes^k C] = 0$. Hence, $\otimes$-rank $(A \oplus C) \leq \otimes$-rank (B). Recalling Corollary 1.3b., we have shown:

Corollary 1.4b. Suppose A, B, and C are R-modules; suppose there exists a pure exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$. Then $\otimes$-rank $(B) = \otimes$-rank $(A \oplus C)$.

Combining Corollaries 1.3a, 1.3b, 1.4a and 1.4b, we obtain:

Corollary 1.5. Let G be one of S, Λ, or $\otimes$; let A, B, and C be R-modules. If there exists an exact sequence $A \longrightarrow B \longrightarrow C \longrightarrow 0$, then G-rank $(B) \leq$ G-rank $(A \oplus C)$. If there exists a pure exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$, then G-rank $(B) =$ G-rank $(A \oplus C)$.

Immediately, we have:

Lemma 1.6. Direct Sum Replacement Lemma. Let G be one of $\otimes$, S or Λ. Suppose $M \xrightarrow{h} M' \xrightarrow{h'} M'' \longrightarrow 0$ is exact and N is any R-module. Then G-rank $(N \oplus M') \leq$ G-rank $(N \oplus M \oplus M'')$. If h is a pure embedding, then G-rank $(N \oplus M')$ = G-rank $(N \oplus M \oplus M'')$.

Proof: $N \oplus M \xrightarrow{id_N \oplus h} N \oplus M' \xrightarrow{0 \oplus h'} M'' \longrightarrow 0$ is exact. Hence, by Corollary 1.5, G-rank $(N \oplus M') \leq$ G-rank $((N \oplus M) \oplus M'')$. If h is a pure embedding, so is $id_N \oplus h$ and, again by Corollary 1.5, = s holds.

Lemma 1.7. Tensor Product Replacement Lemma. Let G be any one of $\otimes$, S, or Λ. Suppose $M \xrightarrow{h} M' \xrightarrow{h'} M'' \longrightarrow 0$ is exact and N is any R-module. Then G-rank $(N \otimes M') \leq$ G-rank $(N \otimes (M \oplus M''))$. If h is a pure embedding, then G-rank $(N \otimes M')$ = G-rank $(N \otimes (M \oplus M''))$.

Proof: $N \otimes M \xrightarrow{id_N \otimes h} N \otimes M' \xrightarrow{id_N \otimes h'} N \otimes M'' \longrightarrow 0$ is also exact; hence, by Corollary 1.5, G-rank $(N \otimes M') \leq$ G-rank $((N \otimes M) \oplus (N \otimes M''))$ but $(N \otimes M) \oplus (N \otimes M'') \cong N \otimes (M \oplus M'')$. If h is a pure embedding, then $id_N \otimes h$ is a pure embedding, and again by Corollary 1.5, = s holds.

Theorem 1 now follows from Lemma 1.7 by iteration.

Theorem 1. Let G be one of $\otimes$, S, or Λ. Suppose, for $1 \leq i \leq n$, $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$ is an exact sequence. Then G-rank $(B_1 \otimes \ldots \otimes B_n) \leq$ G-rank $((A_1 \oplus C_1) \otimes \ldots \otimes (A_n \oplus C_n))$. If each f_i is a pure embedding, then = s holds.

Proof: The exactness of $A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \longrightarrow 0$ for $1 \leq i \leq n$ and Lemma 1.7 imply:

$$\begin{aligned}\text{G-rank } (B_1 \otimes \cdots \otimes B_n) &\leq \text{G-rank } ((A_1 \oplus C_1) \otimes B_2 \otimes \cdots \otimes B_n) \\ &\leq \text{G-rank } ((A_1 \oplus C_1) \otimes (A_2 \oplus C_2) \otimes B_3 \otimes \cdots \otimes B_n)) \\ &\vdots \\ &\leq \text{G-rank } ((A_1 \oplus C_1) \otimes \cdots \otimes (A_n \oplus C_n))\end{aligned}$$

If each f_i is a pure embedding, then Lemma 1.7 says that we can replace "$\leq$" by "=" in each step above. q.e.d.

Theorem 2. Let F be one of $\otimes$, S, or Λ; let G be one of $\otimes$, S, or Λ. Suppose $A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is exact. Then G-rank $(F^p(B)) \leq$ G-rank $(F^p(A \oplus C))$. If f is a pure embedding, then = s holds.

Proof: $F = \otimes$. This case is simply a special case of Theorem 1.

$F = S$ or Λ. We write the exact sequences of Proposition 1.2a for $F^p(B)$:

$$\begin{aligned}&\varepsilon_0 : F^p(A) \xrightarrow{\psi_0} F^p(B) \longrightarrow N_0 \longrightarrow 0 \\ &\varepsilon_1 : F^{p-1}(A) \otimes C \xrightarrow{\psi_1} N_0 \longrightarrow N_1 \longrightarrow 0 \\ &\vdots \\ &\varepsilon_k : F^{p-k}(A) \otimes F^k(C) \xrightarrow{\psi_k} N_{k-1} \longrightarrow N_k \longrightarrow 0 \\ &\vdots \\ &\varepsilon_{p-1} : A \otimes F^{p-1}(C) \xrightarrow{\psi_{p-1}} N_{p-2} \longrightarrow F^p(C) \longrightarrow 0.\end{aligned}$$

Then G-rank $(F^p(B)) \leq$ G-rank $(F^p(A) \oplus N_0)$ by Cor. 1.5

$\leq$ G-rank $(F^p(A) \oplus [F^{p-1}(A) \otimes C] \oplus N_1)$ by Lemma 1.6

$\leq$

$\vdots$

$\leq$ G-rank $\left(F^p(A) \oplus \ldots \oplus [F^2(A) \otimes F^{p-2}(C)] \oplus N_{p-2}\right)$

by Lemma 1.6

$\leq$ G-rank $\left(F^p(A) \oplus \ldots \oplus [A \otimes F^{p-1}(C)] \oplus F^p(C)\right)$

by Lemma 1.6

But $F^p(A) \oplus \ldots \oplus [F^{p-k}(A) \otimes F^k(C)] \oplus \ldots \oplus F^p(C) \cong F^p(A \oplus C)$

If, in addition, f is a pure embedding, then, for $0 \leq k \leq p-1$, ψ_k is a pure embedding by Prop. 1.2a. Corollary 1.5 and Lemma 1.6 now enable us to replace "$\leq$" by "=" at each step. q.e.d.

CHAPTER II. APPLICATIONS OF THEOREMS 1 AND 2

Definition. Suppose that M is an R-module having exterior rank m. If there exists a family $\{C_i\}_{i=0}^{m}$ of submodules of M such that $0 = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_m = M$ and, for $1 \leq i \leq m$, Λ-rank $(C_i/C_{i-1}) = 1$, then we will call M an exteriorly solvable module or say that M admits an exterior composition series $\{C_i\}_{i=0}^{m}$.

Note: Theorem 2 gives us the fact that Λ-rank $(C_i) = i, \forall i \ni 0 \leq i \leq m$; hence each submodule C_i, $1 \leq i \leq m$, is exteriorly solvable.

[Verification of Note. Clearly if Λ-rank $C_i = i$, then the exterior composition series for M contains an exterior composition series for C_i; therefore, we need only see that Λ-rank $(C_i) = i$.

Claim. Λ-rank $(C_i) \leq i$.

Proof by induction. Clearly Λ-rank$(C_0) = 0$. For $i \geq 1$, suppose Λ-rank $(C_{i-1}) \leq i-1$. We have an exact sequence $0 \longrightarrow C_{i-1} \longrightarrow C_i \longrightarrow C_i/C_{i-1} \longrightarrow 0$ where C_i/C_{i-1} has Λ-rank 1. Thus by Theorem 2, Λ-rank $(C_i) \leq \Lambda$-rank (C_{i-1}) + Λ-rank $(C_i/C_{i-1}) \leq i-1+1 = i$.

Claim. Λ- rank $(C_i) \geq i$. Proof by reverse induction. By definition Λ-rank $(C_m) = m$. For $i \leq m-1$, suppose Λ-rank $C_{i+1} \geq i+1$. Consider $0 \longrightarrow C_i \longrightarrow C_{i+1} \longrightarrow C_{i+1}/C_i \longrightarrow 0$. Λ-rank $(C_{i+1}) \leq \Lambda$-rank (C_i) + Λ-rank (C_{i+1}/C_i) by Theorem 2. We have $i+1 \leq \Lambda$-rank $(C_i) + 1$ or Λ-rank $(C_i) \geq i$ as desired.]

Obviously, finitely-generated free modules are exteriorly solvable; therefore, by Nakayama's lemma, finitely generated modules over a quasi-local ring are exteriorly solvable. One might ask what conditions must be put on a ring R to guarantee that all modules having finite exterior rank are exteriorly solvable. The following example shows that even finitely-generated projectives over a non-local Noetherian ring need not be exteriorly solvable.

Example: Suppose R is a Noetherian domain having the following properties: [Samuel; Props. 9 and 10, p. 164-165.]

a) Finitely generated rank 1 projectives are free.

b) There exists a finitely generated rank 2 projective P that is not free. Then P is not exteriorly solvable.

Verification: By contradiction. Suppose $0 \subseteq C_1 \subseteq P$ is an exterior composition series for P. Consider the exact sequence of finitely generated modules $0 \longrightarrow C_1 \longrightarrow P \longrightarrow P/C_1 \longrightarrow 0$. Let Q be any maximal ideal of R. $0 \longrightarrow C_{1_Q} \longrightarrow R_Q \oplus R_Q \longrightarrow (P/C_1)_Q \longrightarrow 0$ is exact where C_{1_Q} and $(P/C_1)_Q$ have exterior rank 1; hence both C_{1_Q} and $(P/C_1)_Q$ are principal. C_1 then is a finitely generated torsion free module (over a domain) that is locally principal; hence C_1 is projective; hence C_1 is free by a). Tensoring with the quotient field of R_1 we see that P/C_1 cannot be a torsion module; but P/C_1 is locally principal, hence locally free; hence P/C_1 is projective. Again by a), P/C_1 is free. Thus we have $0 \longrightarrow R \longrightarrow P \longrightarrow R \longrightarrow 0$ is exact, implying $P \cong R \oplus R$, contradicting b).

We will prove that, if M is an exteriorly solvable R-module, then Λ-rank $(M \otimes N) \leq \Lambda\text{-rank}(M) \cdot \Lambda\text{-rank}(N)$ (where N is any R-module) and Λ-rank $(\Lambda^p M) \leq \binom{\Lambda\text{-rank}(M)}{p}$ $\forall\ p \geq 1$.

We pause for a lemma.

<u>Lemma 2.1</u>. If Λ-rank (A) = 1 and Λ-rank (B) $< \infty$ then Λ-rank $(A \otimes B) \leq \Lambda$-rank (B).

<u>Proof</u>. Let n = Λ-rank (B). Since $\Lambda^{n+1}B = 0$, elements of the form $(a_1 \otimes b_1) \otimes \ldots \otimes (a_i \otimes b_i) \otimes \ldots \otimes (a_j \otimes b_i) \otimes \ldots \otimes (a_{n+1} \otimes b_{n+1})$ generate $\otimes^{n+1} A \otimes B$. Since $\Lambda^2 A = 0$ we can replace this element by a linear combination of elements of the form $(a_1 \otimes b_1) \otimes \ldots \otimes (a'_i \otimes b_i) \otimes \ldots \otimes (a'_i \otimes b_i) \otimes \ldots \otimes (a_{n+1} \otimes b_{n+1})$. But any element of the last form is in the kernel of the map $\otimes^{n+1}(A \otimes B) \longrightarrow \Lambda^{n+1}(A \otimes B)$ given by $(a_1 \otimes b_1) \otimes \ldots \otimes (a_{n+1} \otimes b_{n+1}) \longmapsto (a_1 \otimes b_1) \wedge (a_2 \otimes b_2) \wedge \ldots \wedge (a_{n+1} \otimes b_{n+1})$; hence $\Lambda^{n+1} A \otimes B = 0$. q.e.d.

<u>Proposition 2.2</u>. If M is an exteriorly solvable R-module and N is an R-module, then Λ-rank $(M \otimes N) \leq$ Λ-rank $(M) \cdot \Lambda$-rank (N).

<u>Proof</u>: If Λ-rank N = ∞, then there is nothing to prove; therefore suppose Λ-rank (N) $< \infty$. We will induct on m = Λ-rank (M). If m = 0, then M = 0 and $M \otimes N = 0$. If m = 1, the proof is just <u>Lemma 2.1</u>. If m > 1 consider the exact sequence $0 \longrightarrow C_{m-1} \xrightarrow{\text{inclusion}} M \longrightarrow M/C_{m-1} \longrightarrow 0$ (obtained from an exterior composition series, $\{C_i\}$) where C_{m-1} is exteriorly solvable of exterior rank m-1 by the <u>Note</u>. We have $C_{m-1} \otimes N \longrightarrow M \otimes N \longrightarrow M/C_{m-1} \otimes N \longrightarrow 0$ is exact. Thus by Theorem 2, Λ-rank $(M \otimes N) \leq$ Λ-rank $[(C_{m-1} \otimes N) \oplus (M/C_{m-1} \otimes N)] \leq (m-1)\Lambda$-rank (N) + Λ-rank (N)

by Observation ii) of the Introduction and induction on m. But $(m-1)\Lambda\text{-rank}(N) + \Lambda\text{-rank}(N) = m\cdot\Lambda\text{-rank}(N)$

$$= \Lambda\text{-rank}(M)\cdot\Lambda\text{-rank}(N).$$ q.e.d.

Proposition 2.3. If M is an exteriorly solvable R-module, then, for every $p \geq 2$, $\Lambda\text{-rank}\ (\Lambda^p M) \leq \binom{\Lambda\text{-rank}\ (M)}{p}$.

Proof: Again we will induct on $m = \Lambda\text{-rank}\ (M)$. If $m = 0$, then $M = 0$ and $\Lambda^p M = 0$. If $m = 1$, then $\Lambda^p M = 0$ and we are done. Suppose now that $m > 1$. Consider the exact sequence (obtained from an exterior composition series, $\{C_i\}$ for M) $0 \longrightarrow C_{m-1} \xrightarrow{\text{inclusion}} M \longrightarrow M/C_{m-1} \longrightarrow 0$ where C_{m-1} has exterior rank m-1 and is also exteriorly solvable. We have,

$$\Lambda\text{-rank}(\Lambda^p M) \leq \Lambda\text{-rank}(\Lambda^p[(C_{m-1})\oplus(M/C_{m-1})])$$ by Theorem 2

$$= \Lambda\text{-rank}\big(\Lambda^p C_{m-1}\oplus(\Lambda^{p-1}C_{m-1}\otimes M/C_{m-1})\big)$$ by Prop. 0.15

$$\leq \Lambda\text{-rank}(\Lambda^p C_{m-1}) + \Lambda\text{-rank}(\Lambda^{p-1}C_{m-1}\otimes M/C_{m-1})$$
by Observation ii) of the Introduction

$$\leq \binom{m-1}{p} + \binom{m-1}{p-1}$$ by induction and Lemma 2.1.

But $\binom{m-1}{p} + \binom{m-1}{p-1} = \binom{m}{p}$ and we are done. q.e.d.

The following result is due to L. Fuchs:

Theorem (Fuchs: Theorem 32.3, p. 137). Suppose V is a discrete valuation ring (DVR) having quotient field Q and suppose that M is a V-module. Then there exists a pure exact sequence $0 \longrightarrow K \xrightarrow{\phi} M \longrightarrow D \longrightarrow 0$ where K is a direct sum of cyclic V-modules and D is a direct sum of copies of Q and Q/V. We say that ϕ embeds K as a basic submodule of M.

Theorem 2 tells us that Λ-rank $(M) = \Lambda$-rank $(K \oplus D)$ and Λ-rank $(\Lambda^p M) = \Lambda$-rank $(\Lambda^p(K \oplus D))$. Theorem 1 (or Lemma 1.7) tells us that Λ-rank $(M \otimes N) = \Lambda$-rank $((K \oplus D) \otimes N)$.

We are now ready to prove the following:

Proposition 2.4. Suppose V is a DVR and M and N are V-modules. Then Λ-rank $(M \otimes N) \leq \Lambda$-rank $(M) \cdot \Lambda$-rank (N) and Λ-rank $(\Lambda^p M) \leq \binom{\Lambda\text{-rank}(M)}{p}$ for all $p \geq 2$.

Proof: If Λ-rank $(M) = \infty$, there is nothing to prove, so suppose Λ-rank $(M) < \infty$.

Let $0 \longrightarrow K \xrightarrow{\phi} M \longrightarrow D \longrightarrow 0$ be exact where ϕ embeds K as a basic submodule of M and Λ-rank $(M) < \infty$. The comments following Fuch's theorem along with Propositions 2.2 and 2.3 will yield a proof providing that we can show $K \oplus D$ to be exteriorly solvable.

Verification that $K \oplus D$ is exteriorly solvable if Λ-rank $(M) < \infty$. Λ-rank $(M) < \infty$ and Λ-rank $(M) = \Lambda$-rank $(K \oplus D)$ together imply that Λ-rank $(K \oplus D) < \infty$. But we see that Λ-rank $(K \oplus D) < \infty$ if and only if the number of cyclic direct summands in K is finite and the number of direct summands of Q in D is finite. Write

$$K \oplus D = \left(\bigoplus_{a=1}^{f} V_a\right) \oplus \left(\bigoplus_{b=1}^{t} T_b\right) \oplus \left(\bigoplus_{c=1}^{q} Q_c\right) \oplus \left(\bigoplus_{\delta} Q/V\right)$$

where f, t, and q are integers ≥ 0, δ is some cardinal number, $V_a = V$ for $1 \leq a \leq f$, T_b is a cyclic torsion module for $1 \leq b \leq t$, and $Q_c = Q$ for $1 \leq c \leq q$.

We consider three cases to write an exterior composition series $\{C_i\}$ for $K \oplus D$:

Case 1. ($K\oplus D$ is free) $t = q = \delta = 0$. Then Λ-rank $(K\oplus D) = f$. Let $C_0 = 0$ and, for $1 \le i \le f$, let $C_i = \bigoplus_{a=1}^{i} V_a$.

Case 2. $t = q = 0$, $\delta > 0$. Then Λ-rank $(K\oplus D) = f + 1$. Let $C_0 = 0$ and, for $1 \le i \le f$, let $C_i = \bigoplus_{a=1}^{i} V_a$. Let $C_{f+1} = \bigoplus_{a=1}^{f} V_a \oplus (\bigoplus_{\delta} Q/V)$ $(= K\oplus D)$.

Case 3. Now suppose $\max\{t,q\} \ge 1$. Let $r = \max\{t,q\}$. Then Λ-rank $(K\oplus D) = f + r$. If $t > q$, define $Q_{q+1} = Q_{q+2} = \ldots = Q_r = 0$. If $t < q$, define $T_{t+1} = T_{t+2} = \ldots = T_r = 0$. Let $C_0 = 0$ and for $1 \le i \le f$ let $C_i = \bigoplus_{a=1}^{i} V_a$. For $f < i \le f+r$, let $C_i = (\bigoplus_{a=1}^{f} V_a) \oplus (\bigoplus_{\delta} Q/V) \oplus (\bigoplus_{b=1}^{i-f} T_b) \oplus (\bigoplus_{c=1}^{i-f} Q_c)$.

Thus we have shown that $K\oplus D$ is exteriorly solvable. q.e.d.

Since Λ localizes well, Prop. 2.4 holds with the hypothesis "V is a DVR" replaced by the hypothesis "for every maximal ideal Q of V, V_Q is either a field of a DVR". Further slight generalizations are immediate but they seem pointless.

CHAPTER III. NOETHERIAN RINGS OF FINITE KRULL DIMENSION

Let R be a Noetherian ring of finite Krull dimension d; let M and N be R-modules. We will:

a) bound Λ-rank $(M \otimes N)$ by a number that is a function of Λ-rank (M), Λ-rank (N), and d;

b) bound Λ-rank $(\Lambda^p M)$ by a number that is a function of Λ-rank (M), p, and d; and

c) bound Λ-rank (M) by a number that is a function of Λ-rank $(\Lambda^p M)$, p, and d.

Unfortunately, the bounds obtained in a) and b) are probably excessive; and, for p odd, Chapter IV gives, for an arbitrary ring R, a bound for Λ-rank (M), depending only on Λ-rank $(\Lambda^p M)$ and p, that is usually better than the bound obtained in c).

Chapter III is independent of the other chapters, the techniques being similar to techniques found in [Wiegand].

We begin with a trivial lemma:

Lemma. Suppose A and B are R-modules and $B = \text{Ann}(a)B$ for every $a \in A$. Then $A \otimes_R B = 0$.

Proof: Consider $a \otimes b \in A \otimes B$. $b = \sum_i r_i b_i$ where each $r_i \in \text{Ann}(a)$. $a \otimes b = a \otimes \sum_i r_i b_i = \sum_i (r_i a \otimes b_i) = \sum_i (0 \otimes b_i) = 0$. Since elements of the form $a \otimes b$ generate $A \otimes B$, $A \otimes B$ is 0. q.e.d.

Proposition 3.1. Suppose R is a Noetherian ring of finite Krull dimension $\leq$ d. Suppose M and N are R-modules with Λ-rank $(M) \leq m$ and Λ-rank $(N) \leq n$. Then Λ-rank $(M \otimes N) \leq (d+1)(mn+1) - 1$.

Proof: By induction on d. Suppose d = 0. We must show that $\Lambda_R^{mn+1}(M \otimes_R N) = 0$. We have, for any prime P, $[\Lambda_R^{mn+1}(M \otimes_R N)]_P \cong \Lambda_{R_P}^{mn+1}(M \otimes N)_P \cong \Lambda_{R_P}^{mn+1}(M_P \otimes_{R_P} N_P)$ where

Λ_{R_P}-rank$(M_P) \leq m$ and Λ_{R_P}-rank $(N_P) \leq n$. Since a module is 0 if, and only if, it localizes to 0 at every maximal ideal, we have reduced to the case in which R is local of Krull dimension 0.

Therefore, suppose that R, Q is local of Krull dimension 0 and Λ-rank $(M) \leq m$ and Λ_R-rank $(N) \leq n$. (Notice that Q is nilpotent.) Then $\Lambda_{R/Q}$-rank $(M/QM) \leq m$ and $\Lambda_{R/Q}$-rank $(N/QN) \leq n$. Since R/Q is a field, M/QM and N/QN are free and $\Lambda_{R/Q}$-rank $(M/QM \otimes_{R/Q} N/QN) \leq mn$. Therefore $\Lambda_{R/Q}^{mn+1}(M/QM \otimes_{R/Q} N/QN) = 0$. But $M/QM \otimes_{R/Q} N/QN \underset{R/Q}{\cong} M \otimes_R N/Q(M \otimes_R N)$ so that $\Lambda_{R/Q}^{mn+1}(M/QM \otimes_{R/Q} N/QN) \cong \Lambda_{R/Q}^{mn+1}[(M \otimes_R N)/Q(M \otimes_R N)] \underset{\text{(Prop. 0.14)}}{\cong}$ $\Lambda_R^{mn+1}(M \otimes_R N)/Q\Lambda_R^{mn+1}(M \otimes_R N)$; hence $\Lambda_R^{mn+1}(M \otimes_R N)/Q\Lambda_R^{mn+1}(M \otimes_R N) = 0$. But Q is nilpotent; hence $\Lambda_R^{mn+1}(M \otimes_R N) = 0$, as desired.

Suppose now that we have proved the proposition for rings of Krull dim. $\leq d-1$, $d \geq 1$. Let R be a Noetherian ring of Krull dimension $\leq d$. As in the case d = 0, we may assume that R is local with maximal ideal Q.

We first notice that $\Lambda_R^{mn+1}(M \otimes_R N) = Q(\Lambda_R^{mn+1}(M \otimes_R N))$ since

$$\Lambda_R^{mn+1}(M \otimes_R N) \big/ Q(\Lambda_R^{mn+1}(M \otimes_R N)) \underset{(\underline{Prop.}\ \underline{0.14})}{=} \Lambda_{R/Q}^{mn+1}(M/QM \otimes_{R/Q} N/QN)$$

$= 0$ (because R/Q is a field). Note also, by the inductive hypothesis, that for any non-maximal prime P, $[\Lambda_R^{d(mn+1)} M \otimes_R N]_P$ $= 0$ since $[\Lambda_R^{d(mn+1)} M \otimes_R N]_P \underset{(\underline{Prop.}\ \underline{0.14})}{\cong} \Lambda_{R_P}^{d(mn+1)}(M_P \otimes_{R_P} N_P)$, R_P has Krull dimension $\leq d-1$, Λ_{R_P}-rank $(M_P) \leq m$, and Λ_{R_P}-rank $(N_P) \leq n$.

Let $x \in \Lambda_R^{d(mn+1)} M \otimes_R N$. Since x localizes to 0 at every non-maximal prime P, $\mathrm{Ann}(x) \not\subseteq P$ for every non-maximal prime P. Hence $Q = \sqrt{\mathrm{Ann}(x)}$ and some power of Q, say Q^k, is contained in $\mathrm{Ann}(x)$.

But $\Lambda_R^{mn+1}(M \otimes_R N) = Q\, \Lambda_R^{mn+1}(M \otimes_R N)$ so that $\Lambda_R^{mn+1}(M \otimes_R N)$ $= Q^k\, \Lambda_R^{mn+1}(M \otimes_R N)$, implying: $\Lambda_R^{mn+1}(M \otimes_R N) = \mathrm{Ann}(x)(\Lambda_R^{mn+1} M \otimes_R N)$ since $Q^k \subseteq \mathrm{Ann}(x)$.

Since the above is true for every $x \in \Lambda_R^{d(mn+1)}(M \otimes_R N)$ we have $[\Lambda_R^{d(mn+1)}(M \otimes_R N)] \otimes_R [\Lambda_R^{mn+1}(M \otimes_R N)] = 0$ by the Lemma; but, by Prop. 0.13, $\Lambda_R^{(d+1)(mn+1)} M \otimes_R N$ is a homomorphic image of $[\Lambda_R^{d(mn+1)}(M \otimes_R N)] \otimes_R [\Lambda_R^{mn+1}(M \otimes_R N)]$ and is therefore 0. q.e.d.

Proposition 3.2. Suppose R is a Noetherian ring of finite Krull dimension $\leq$ d. Suppose M is an R-module having exterior rank $\leq$ m. Then Λ-rank $(\Lambda^p M) \leq (d+1)\left(\binom{m}{p}+1\right) - 1$.

Proof: The proof is analogous to the proof of Proposition 3.1.

Suppose d = 0. Since Λ localizes well (Prop. 0.14) we may assume R is local with maximal ideal Q (where Q is nilpotent). Then $\Lambda_{R/Q}^{\binom{m}{p}+1} M/QM = 0$ since $\Lambda_{R/Q}$-rank$(M/QM) \leq$ Λ_R-rank(M) (Observation iii) of Introduction) $\leq$ m and R/Q is a field.

But $\Lambda_{R/Q}^{\binom{m}{p}+1} M/QM \underset{\text{(Prop. 0.14)}}{\cong} \Lambda_R^{\binom{m}{p}+1} M / Q\Lambda_R^{\binom{m}{p}+1} M$; and, since Q is nilpotent, the fact that $\Lambda_R^{\binom{m}{p}+1} M / Q\Lambda_R^{\binom{m}{p}+1} M = 0$ implies that $\Lambda_R^{\binom{m}{p}+1} M = 0$.

Suppose now that the proposition has been shown for Krull dim. $\leq$ d-1. Suppose R is Noetherian of Krull dimension $\leq$ d and Λ_R-rank(M) $\leq$ m. Since Λ localizes well (Prop. 0.14), we may assume that R is local with maximal ideal Q. Since R/Q is a field and $\Lambda_{R/Q}$-rank(M/QM) $\leq$ m, we have $\Lambda_{R/Q}^{\binom{m}{p}+1} M/QM = 0$ or, by Prop. 0.14, $\Lambda_R^{\binom{m}{p}+1} M / Q\Lambda_R^{\binom{m}{p}+1} M = 0$, or $\Lambda_R^{\binom{m}{p}+1} M = Q\Lambda_R^{\binom{m}{p}+1} M$.

For every non-maximal prime P, we have, by induction, $\Lambda_{R_P}^{d(\binom{m}{p}+1)} M_P = 0$ since Λ_{R_P}-rank$(M_P) \leq \Lambda_R$-rank(M) $\leq$ m and Krull dimension $R_P \leq$ d-1. By Proposition 0.14, $[\Lambda_R^{d(\binom{m}{p}+1)} M]_P$ is also 0. Let $x \in \Lambda_R^{d(\binom{m}{p}+1)} M$. Since x localizes to zero at every non-maximal prime P, $\sqrt{\mathrm{Ann}(x)} = Q$. Thus, for some k, $Q^k \subseteq \mathrm{Ann}(x)$. Therefore $\Lambda_R^{\binom{m}{p}+1} M = \mathrm{Ann}(x)\Lambda_R^{\binom{m}{p}+1} M$ for every $x \in \Lambda_R^{d(\binom{m}{p}+1)} M$, and by the Lemma, $\Lambda_R^{d(\binom{m}{p}+1)} M \otimes_R \Lambda_R^{\binom{m}{p}+1} M = 0$

implying, via Prop. 0.13, that $\Lambda_R^{(d+1)(\binom{m}{p}+1)} M = 0$. q.e.d.

Proposition 3.3. Suppose R is a Noetherian ring of Krull dimension $\leq$ d. Suppose Λ-rank $(\Lambda^p M) \leq q$ and that $\binom{t}{p} \geq q$. Then Λ-rank $(M) \leq (d+1)(t+1) - 1$.

Proof: The proof is analogous to the proofs of Props. 3.1 and 3.2.

Suppose d = 0. Since Λ localizes well (Prop. 0.14), we may assume R is local with prime ideal Q (which is nilpotent).

Since R/Q is a field, $\Lambda_{R/Q}$-rank $(\Lambda_{R/Q}^p\ M/QM) \leq q$, and $\binom{t}{p} \geq q$, then $\Lambda_{R/Q}^{t+1}\ M/QM = 0$. (We have here the case of a free R/Q-module M/QM.)

But $\Lambda_{R/Q}^{t+1}\ M/QM \underset{(\text{Prop. } 0.14)}{\cong} \Lambda_R^{t+1} M/Q\Lambda_R^{t+1} M$; thus

$\Lambda_R^{t+1} M/Q\Lambda_R^{t+1} M = 0$, and, since Q is nilpotent, $\Lambda_R^{t+1} M = 0$.

Now suppose the proposition has been proved for all rings having Krull dimension $\leq$ d-1. Suppose R has Krull dimension $\leq$ d. Again, since Λ localizes well, we may assume R is local with maximal ideal Q.

Since R/Q is a field, Λ-rank $(\Lambda_{R/Q}^p\ M/QM) \leq q$, and $\binom{t}{p} \geq q$, we have $\Lambda_{R/Q}^{t+1}\ M/QM = 0$, or in the presence of

Prop. 0.14, $\Lambda_R^{t+1}M = Q\Lambda_R^{t+1}M$.

Let P be any non-maximal prime ideal of R. Then

$[\Lambda_R^{d(t+1)}M]_P \cong \Lambda_{R_P}^{d(t+1)}M_P$ by Proposition 0.14

$= 0$ by induction since Krull dimension $R_P \leq d-1$,

Λ-rank $(M_P) \leq q$ and $\binom{t}{p} \geq q$.

Let $x \in \Lambda_R^{d(t+1)}M$. Since x localizes to 0 at every non-maximal prime P, $\sqrt{\mathrm{Ann}(x)} = Q$ and, for some k, $Q^k \subseteq \underset{R}{\mathrm{Ann}}(x)$. Therefore, we have, for any $x \in \Lambda_R^{d(t+1)}M$, $\Lambda_R^{t+1}M = \mathrm{Ann}(x)\Lambda_R^{t+1}M$ implying, by the Lemma, that $\Lambda_R^{d(t+1)}M \otimes_R \Lambda_R^{t+1}M = 0$. Hence $\Lambda_R^{(d+1)(t+1)}M$ is 0 because it is a homomorphic image of $\Lambda_R^{d(t+1)}M \otimes_R \Lambda_R^{t+1}M$ (=0). q.e.d.

In Propositions 3.1, 3.2, and 3.3 we can replace the hypothesis "R is a Noetherian ring of Krull dimension $\leq$ d" with slightly weaker hypotheses (for example, "R has Krull dimension $\leq$ d and R_Q is Noetherian for every maximal ideal Q") and the same or similar proofs will go through. However, to generalize significantly the results obtained in these propositions would require a different approach.

CHAPTER IV. A LOOK AT A QUESTION OF H. FLANDERS

Suppose R is a ring, $p > 1$, and M is an R-module such that Λ-rank $(\Lambda^p M) = q$. What can we say about Λ-rank (M)?

Corollary 0.13.1 tells us that if p is odd or if R has characteristic 2, then the canonical homomorphism of $\otimes^{q+1} \Lambda^p M$ onto $\Lambda^{(q+1)p} M$ factors through $\Lambda^{q+1} \Lambda^p M$; thus, if $\Lambda^{q+1} \Lambda^p M = 0$, then $\Lambda^{(q+1)p} M = 0$. We have just shown:

Proposition 4.1. Suppose p is odd or R has characteristic 2. Suppose Λ-rank $(\Lambda^p M) = q$. Then Λ-rank $(M) \leq (q+1)p - 1$.

I do not know whether or not Proposition 4.1 holds for p even and R of characteristic $\neq 2$. In fact, I cannot in general prove that, if Λ-rank $(\Lambda^p M) < \infty$, then Λ-rank $(M) < \infty$. (The last statement is true for Noetherian rings of finite Krull dimension by Proposition 3.3.)

We will show, however, that the bound of Proposition 4.1 is as good as can be hoped for, showing that the answer to the following question of H. Flanders [Flanders: p. 359] is "No":

If Λ-rank $(\Lambda^p M) \leq q$, is Λ-rank $(M) \leq p + q - 1$?

Example: [Wiegand: Sections 2 and 3]. Let p, q be given $q \geq 1$, $p > 1$. We construct a ring R and R-modules $M_1, \ldots, M_{(q+1)p-1}$ having the following two properties

i) $M_i \otimes M_i = 0$ for every $1 \leq i \leq (q+1)p - 1$

ii) $M_1 \otimes M_2 \otimes \ldots \otimes M_{(q+1)p-1} \neq 0$.

Let $M = M_1 \oplus \ldots \oplus M_{(q+1)p-1}$.

Then Λ-rank $(\Lambda^p M) = q$ and Λ-rank $(M) = (q+1)p-1$.

<u>Proof</u>: Suppose that we have constructed R, M_i, and M as above.

<u>Proposition 0.15</u> and induction yield the following isomorphism:

$$* \qquad \Lambda^t(A_1 \oplus \ldots \oplus A_n) = \bigoplus_{(k_1, \ldots, k_n)} (\Lambda^{k_1}A_1 \otimes \ldots \otimes \Lambda^{k_n}A_n)$$

where $(k_1, \ldots, k_n)$ ranges over all sequences of non-negative integers of length n, the sum of whose terms is t. We make use of * to see:

$$\Lambda^{(q+1)p-1} M \cong M_1 \otimes \ldots \otimes M_{(q+1)p-1} \neq 0$$

by i) and ii), and $\Lambda^{(q+1)p}M = 0$ by i). Therefore Λ-rank $(M) = (q+1)p-1$.

What is Λ-rank $(\Lambda^p M)$? $\Lambda^q(\Lambda^p M) \neq 0$ because $M_1 \otimes \ldots \otimes M_p$, $M_{p+1} \otimes \ldots \otimes M_{2p}, \ldots, M_{(q-1)p+1} \otimes \ldots \otimes M_{qp}$ are all direct summands of $\Lambda^p M$ and, therefore,
$(M_1 \otimes \ldots \otimes M_p) \otimes (M_{p+1} \otimes \ldots \otimes M_{2p}) \otimes \ldots \otimes (M_{(q-1)p+1} \otimes \ldots \otimes M_{qp})$
is a direct summand of $\Lambda^q(\Lambda^p M)$. (And $M_1 \otimes \ldots \otimes M_{qp} \neq 0$ by ii).)
However, $\Lambda^{q+1}(\Lambda^p M) = 0$ because it is a homomorphic image of $\otimes^{q+1}(\otimes^p M) \cong \otimes^{(q+1)p} M$ which is 0 by i). Therefore Λ-rank$(\Lambda^p M) = q$.

To complete the proof, we need only construct, for any given $n > 1$, a ring R and R-modules $M_1, \ldots, M_n$ having the following two properties:

i) $M_i \otimes M_i = 0$ for $1 \leq i \leq n$

ii) $M_1 \otimes M_2 \otimes \ldots \otimes M_n \neq 0$.

<u>Construction</u>: Let $R = k[X_1, \ldots, X_n]$ where k is a field.

For $1 \leq i \leq n$ define $M_i = \varinjlim_{k \in N} (R/(X_i^k), \phi_{i_{k'}}^{k})$ where, for $k \leq k'$, $\phi_{i_{k'}}^{k} : R/(X_i^k) \longrightarrow R/(X_i^{k'})$ by $\bar{1} \longmapsto \overline{X_i^{k'-k}}$.

Verification of Property i). By Proposition 0.20, $M_i \otimes M_i \cong \varinjlim_{k \in N} (R/(X_i^k) \otimes R/(X_i^k), \phi_{i_{k'}}^{k} \otimes \phi_{i_{k'}}^{k})$. Identifying $R/(X_i^k) \otimes R/(X_i^k)$ with $R/(X_i^k)$ (via $\bar{1} \otimes \bar{1} \longmapsto \bar{1}$) the homomorphism $\phi_{i_{k'}}^{k} \otimes \phi_{i_{k'}}^{k} : R/(X_i^k) \otimes R/(X_i^k) \longrightarrow R/(X_i^{k'}) \otimes R/(X_i^{k'})$ is identified with $\psi_{i_{k'}}^{k} : R/(X_i^k) \longrightarrow R/(X_i^{k'})$ where $\psi_{i_{k'}}^{k}(\bar{1}) = \overline{X_i^{2(k'-k)}}$.

Let k be given and choose k' such that k' > k and $2(k'-k) \geq k'$. Then $\psi_{i_{k'}}^{k} = 0$; thus $0 = \varinjlim_{k\ N} (R/(X_i^k), \psi_{i_{k'}}^{k}) \cong M_i \otimes M_i$.

Verification of Property ii). By Proposition 0.20, $M_1 \otimes \ldots \otimes M_n \cong \varinjlim_{k \in N} (R/(X_1^k) \otimes \ldots \otimes R/(X_n^k), \phi_{1_{k'}}^{k} \otimes \ldots \otimes \phi_{n_{k'}}^{k})$. Identifying $R/(X_1^k) \otimes \ldots \otimes R/(X_n^k)$ with $R/(X_1^k, \ldots, X_n^k)$, $\phi_{1_{k'}}^{k} \otimes \ldots \otimes \phi_{n_{k'}}^{k}$ is identified with $\psi_{k'}^{k} : R/(X_1^k, \ldots X_n^k) \longrightarrow R/(X_1^{k'}, \ldots X_n^{k'})$ where $\psi_{k'}^{k}(\bar{1}) = \overline{X_1^{k'-k} \cdot X_2^{k'-k} \cdot \ldots \cdot X_n^{k'-k}}$, so that $M_1 \otimes \ldots \otimes M_n \cong \varinjlim_{k \in N} (R/(X_1^k, \ldots X_n^k), \psi_{k'}^{k})$.

Consider, for any $k' > 1$, $\psi_{k'}^{1}(\bar{1}) = \overline{X_1^{k'-1} \cdot X_2^{k'-1} \cdot \ldots \cdot X_n^{k'-1}} \in R/(X_1^{k'}, \ldots X_n^{k'})$. To say that $\overline{X_1^{k'-1} \cdot \ldots \cdot X_n^{k'-1}} = 0$ in $R/(X_1^{k'}, \ldots X_n^{k'})$ would be to say that $X_1^{k'-1} \cdot \ldots \cdot X_n^{k'-1} \in (X_1^{k'}, \ldots, X_n^{k'})R$

which cannot happen by a trivial degree argument. [Every monomial in the ideal $(X_1^{k'}, \ldots, X_n^{k'})R$ has degree at least k' in at least one of the variables.] Thus, in $\varinjlim_{k\in N}$ $(R/(X_1^k, \ldots X_n^k), \psi_{k'}^k)$, $\psi^1(\bar{1}) \neq 0$, implying that $0 \neq \varinjlim_{k\in N} R/(X_1^k, \ldots, X_n^k) \cong M_1 \otimes \ldots \otimes M_n$. q.e.d.

In <u>Observation vii)</u> of the <u>Introduction</u>, we saw that, if M is punctually finitely generated, then Λ-rank $(\Lambda^p M) = \binom{\Lambda\text{-rank}(M)}{p}$. In [Gardner], R. Gardner showed that the hypothesis "M is punctually finitely generated" can be replaced by the weaker hypothesis "$\Lambda^p M$ is punctually finitely generated":

<u>Proposition</u> (Gardner). Suppose that $\Lambda^p M$ is punctually finitely generated. Then Λ-rank $(\Lambda^p M) = \binom{\Lambda\text{-rank}(M)}{p}$.

<u>Proof</u>: First we need a lemma.

<u>Lemma</u> (Gardner). If $\Lambda^p M$ is finitely-generated, then, for $k \geq 1$, $\Lambda^{p+k} M$ is finitely generated.

<u>Proof</u>: It suffices to show that $\Lambda^{p+1} M$ is finitely generated.

Let $G_p = \{x_{11} \wedge \ldots \wedge x_{1p}, \ldots, x_{t1} \wedge \ldots \wedge x_{tp}\}$ be a finite generating set of "fundamental wedges" for $\Lambda^p M$.

Let $S = \{x_{11}, x_{12}, \ldots x_{t1}, \ldots, x_{tp}\}$.

<u>Claim</u>: $G_{p+1} = \{y_1 \wedge \ldots \wedge y_p | y_i \in S\}$ is a generating set for $\Lambda^{p+1} M$. (Clearly, since S is finite, G_{p+1} is finite.)

<u>Proof of Claim</u>: Let $z_1 \wedge \ldots \wedge z_{p+1}$ be a "fundamental wedge" in $\Lambda^{p+1} M$. Because of the canonical homomorphism of $\Lambda^p M \otimes M$ onto $\Lambda^{p+1} M$ we can write $z_1 \wedge \ldots \wedge z_p \wedge z_{p+1}$ as a linear

combination of fundamental wedges of the form $x_{i1} \wedge \dots \wedge x_{ip} \wedge z_{p+1}$ where $x_{i1} \wedge \dots \wedge x_{ip} \in G_p$. Each fundamental wedge of the form $x_{i1} \wedge \dots \wedge x_{ip} \wedge z_{p+1}$ can then be written as a linear combination of fundamental wedges of the form $x_{i1} \wedge x_{i'1} \wedge x_{i'2} \wedge \dots \wedge x_{i'p}$ where $x_{i'1} \wedge \dots \wedge x_{i'p} \in G_p$ (because of the canonical homomorphism of $M \otimes \Lambda^p M$ onto $\Lambda^{p+1} M$). But fundamental wedges of the form $x_{i1} \wedge x_{i'1} \wedge \dots \wedge x_{i'p}$ are in G_{p+1}. q.e.d.

For the sake of completeness, note that the Lemma holds (via the same proof) with Λ replaced with either $\otimes$ or S.

Suppose that we have shown the proposition for quasi-local rings. Recall that Λ localizes well, implying that

Λ_R-rank (M) = $\max\{\Lambda_{R_Q}\text{-rank } (M_Q) \mid Q \text{ is a maximal ideal of } R\}$

and Λ_R-rank $(\Lambda^p M)$ = $\max\{\Lambda_{R_Q}\text{-rank } (\Lambda^p_{R_Q} M_Q) \mid Q \text{ is a maximal ideal of } R\}$. Thus: Λ_R-rank $(\Lambda^p_R M)$ =

$$\max \{\Lambda_{R_Q}\text{-rank } (\Lambda^p_{R_Q} M_Q) \mid Q \text{ is a maximal ideal of } R\}$$

but, by the quasi-local case,

$$\Lambda_{R_Q}\text{-rank } (\Lambda^p_{R_Q} M_Q) = \binom{\Lambda_{R_Q}\text{-rank}(M_Q)}{p}; \text{ therefore}$$

$$\max\{\Lambda_{R_Q}\text{-rank } (\Lambda^p_{R_Q} M_Q) \mid Q \text{ is a maximal ideal of } R\} =$$

$$\max\{\binom{\Lambda_{R_Q}\text{-rank}(M_Q)}{p} \mid Q \text{ is a maximal ideal of } R\}.$$

But $x \longmapsto \binom{x}{p}$ is a non-decreasing function on the natural numbers so that

$$\max\{\binom{\Lambda_{R_Q}\text{-rank}(M_Q)}{p} \mid Q \text{ is a maximal ideal of } R\}$$

$= \binom{\max\{\Lambda_{R_Q}\text{-rank}(M_Q) \mid Q \text{ is a maximal ideal of } R\}}{p}$

$= \binom{\Lambda_R\text{-rank}(M)}{p}$ since $\Lambda_R\text{-rank}(M) = \max\{\Lambda_{R_Q}\text{-rank}(M_Q) \mid Q$ is a maximal ideal of $R\}$

Therefore, we may assume that R is quasi-local with maximal ideal Q and that $\Lambda_R^p M$ is finitely generated.

Case 1. If $\Lambda_R^p M = 0$, then Λ-rank $(M) < p$ and $\binom{\Lambda_R\text{-rank}(M)}{p} = 0 = \Lambda\text{-rank } (\Lambda_R^p M)$.

Case 2. Suppose $\Lambda^p M \neq 0$. Then, for $1 \leq k \leq p$, $\Lambda_{R/Q}^k\ M/QM \neq 0$ by the following argument:

For $1 \leq k \leq p$, $\Lambda_{R/Q}^k\ M/QM = 0$ implies that $\Lambda_{R/Q}^p\ M/QM = 0$ which implies by Prop. 0.14 that $\Lambda_R^p M/Q\Lambda_R^p M = 0$ which implies (by Nakayama's Lemma, since $\Lambda^p M$ is finitely generated) that $\Lambda_R^p M = 0$; $\rightarrow \leftarrow$.

Claim: $\Lambda_R^t M = 0 \Leftrightarrow \Lambda_{R/Q}^t\ M/QM = 0$.

Proof: $\Rightarrow$) Tensor $\Lambda_R^t M$ with R/Q and apply Prop. 0.14.

$\Leftarrow$) Suppose $\Lambda_{R/Q}^t\ M/QM = 0$. Then by the preceeding paragraph $t > p$. But then, by the Lemma, $\Lambda_R^t M$ is finitely generated and $0 = \Lambda_{R/Q}^t\ M/QM \underset{R/Q}{\cong} \Lambda_R^t M/Q\Lambda_R^t M$ and, by Nakayama's Lemma, $\Lambda_R^t M = 0$.

The claim shows: $\Lambda_R\text{-rank}(M) = \Lambda_{R/Q}\text{-rank } (M/QM)$.

Since $\Lambda_R^p M$ is finitely generated and R is quasi-local, $\Lambda_R\text{-rank}(\Lambda_R^p M)$ = (by observation vi of Introduction) the number of elements in a minimal generating set for $\Lambda_R^p M$ =

(by Nakayama's Lemma) the number of elements in a basis for $\Lambda_R^p M/Q\Lambda_R^p M$ as an R/Q vector space = $\Lambda_{R/Q}$-rank$(\Lambda_R^p M/Q\Lambda_R^p M)$

$\underset{(\text{Prop. } 0.14)}{=} \Lambda_{R/Q}$-rank $(\Lambda_{R/Q}^p(M/QM))$.

Since M/QM is a free R/Q-module, we have (by Observation v) of Introduction) $\Lambda_{R/Q}$-rank$(\Lambda_{R/Q}^p M/QM)$ =

$\binom{\Lambda_{R/Q}\text{-rank}(M/QM)}{p}$. Substituting equals for equals, we obtain

Λ_R-rank$(\Lambda_R^p M) = \binom{\Lambda_R\text{-rank}(M)}{p}$. q.e.d.

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