The Radon-Gauss transform

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THE RADON-GAUSS TRANSFORM

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Abstract

Gaussian measure is constructed for any given hyperplane in an infinite dimensional Hilbert space, and this is used to define a generalization of the Radon transform to the infinite dimensional setting, using Gauss measure instead of Lebesgue measure. An inversion formula is obtained and a support theorem proved.
Introduction

The purpose of this paper is to extend the theory of Radon transforms to infinite dimensions. Since there is no useful version of Lebesgue measure in infinite dimensions, and Gauss measure is the most useful standard measure in this setting, we use Gauss measure as the background measure for the transform. We obtain an inversion formula and a support theorem (of course, there may be many different support theorems as there are in the finite dimensional case).

The Radon transform (invented by Radon in 1917, see reproduction in [1]) of a function \( f \) on \( \mathbb{R}^n \) is a function which associates to each hyperplane \( \xi \subset \mathbb{R}^n \) the value \( \int_{\xi} f \, dm \), where \( m \) is Lebesgue measure on \( \xi \). Our transform takes place in the setting of a real, infinite-dimensional, separable Hilbert space \( V \). As is known, and we describe in Chapter 1 (Gaussian Measures on Infinite Dimensional Spaces), there is a probability space \((\Omega, \mathcal{F}, \mu)\) and a linear map \( x \mapsto \hat{x} \), associating to each \( x \) in a dense linear subspace \( V_0 \subset V \) a measurable function \( \hat{x} \) on \( \Omega \), such that each \( \hat{x} \), viewed as a random variable, is Gaussian with mean 0 and variance \( |x|^2 \), and the random variables \( \hat{v} \) generate the \( \sigma \)-algebra \( \mathcal{F} \); this leads to a linear map \( V \to L^2(\mu) : x \mapsto \hat{x} \), with each \( \hat{x} \) Gaussian of mean 0 and variance \( |x|^2 \). This is generally taken as the standard Gaussian measure “on” the Hilbert space \( V \), though \( \Omega \) is not equal to \( V \) in any natural sense.

Let us now summarize some of the results and constructions of this paper, referring to the notation set up above. A hyperplane in \( V \) is a subset of the form \( \xi = ru + u^\perp \), where \( u \) is a unit vector in \( V \) and \( r \) a non-negative real number; when \( r \), which is the distance \( d(0, \xi) \) of \( \xi \) from the origin, is positive, the hyperplane \( \xi \) determines \( r \) as \( d(0, \xi) \) and \( u \) as the unit normal vector from the origin onto \( \xi \).
In Chapter 3 (Gaussian Measure on Hyperplanes. Finite Dimensional Case) we describe the Gauss measure on a hyperplane \( \xi = \omega^\perp + p\omega \) in \( \mathbb{R}^n \) by:

\[
d\mu_\xi(x) = \frac{1}{(\sqrt{2\pi})^{n-1}} e^{\frac{x^2}{2}} e^{-\frac{|y_1|^2}{2}} dm(x).
\]

The Gaussian measure on the hyperplane in \( \mathbb{R}^n \) is essentially the same, with appropriate transformations, as the standard Gauss measure on \( \mathbb{R}^{n-1} \). Also in Lemma 3.3 find its characteristic function \( \varphi \) given by:

\[
\varphi(y) = \int_\xi e^{i(x,y)} d\mu_\xi(x) = e^{i\langle p, \omega \rangle - \frac{|y_1|^2}{2}}
\]

where \( y_1 \) is the orthogonal projection of \( y \) on \( \omega^\perp \).

In Chapter 5 (Gaussian measure on Hyperplanes. The Infinite Dimensional Case), we show that on the \((\Omega, \mathcal{F})\) there is a probability measure \( \mu_\xi \), and then each function \( \hat{v} \) is a Gaussian random variable with respect to \( \mu_\xi \) with mean \( r\langle v, u \rangle \) and variance \( \langle v, P_{u^\perp} v \rangle \). At a coarse level (and we prove this), one can view \( \mu_\xi \) as the Gauss measure \( \mu \) conditioned to satisfy \( \hat{u} = r \); however, the general construction of such conditional measures provide existence for almost every \( r \) whereas we construct \( \mu_\xi \) as a probability measure for each given value of \( (r, u) \). One other issue to observe here is that, as we prove in Theorem 5.3, the measures \( \mu_\xi \) lives on a set of \( \mu \)-measure 0, and so, a priori, the random variable \( \hat{v} \), for \( v \in V \), when viewed as elements in \( L^2(\mu) \), do not have meaning as elements of \( L^2(\mu_\xi) \).

In Chapter 4 (Radon-Gauss Transform for Finite Dimensional Spaces), we introduce the Radon-Gauss transform for the finite dimensional case. If \( f \in C^\infty(\mathbb{R}^n) \) is a function such that the function \( x \to f(x)e^{-\frac{|y|^2}{2}} \) is in \( \mathcal{S}(\mathbb{R}^n) \), then its Radon-Gauss transform \( Gf \) associated to each hyperplane \( \xi \) in \( \mathbb{R}^n \), the value

\[
Gf(\xi) := \int_\xi f(x)d\mu_\xi(x),
\]
and work out an inversion formula given by:

\[
    f(x) = \frac{1}{(2\pi)^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} e^{i x \cdot y} \left( \int_{\mathbb{R}} Gf \left( \frac{y}{|y|}, p \right) e^{-ip|y|^2 - \frac{p^2}{2}} dp \right) dy,
\]

and few examples.

In Chapter 6 (Radon-Gauss Transform for Infinite Dimensional Spaces. The Inversion Formula) we formally introduce the Radon-Gauss transform. In brief, if \( f \) is a suitable measurable function on \((\Omega, \mathcal{F})\), then its Radon-Gauss transform \( Gf \) associates to each hyperplane \( \xi \) in the Hilbert space \( V \), the value

\[
    Gf(\xi) = \int_{\Omega} f d\mu_{\xi}
\]

and work out a few examples. You can look at the Radon-Gauss transform of \( f \) as the conditional expectation of \( f \),

\[
    E_{\mu} [f \mid \hat{u} = r] = \int_{\Omega} f \ d\mu_{ur+u^\perp}.
\]

In the finite-dimensional case, it is known (see Chapter 2, Theorem 2.2 and Theorem 2.4) that there is an inversion formula using powers of the Laplacian and another formula using the Fourier transform. We have not been able to give an appropriate meaning to an infinity-power of the Laplacian in our context (though the possibility of a meaningful definition remains), so we proceeded using the Segal-Bargmann transform \( S \), and in Theorem 6.3 we establish a relation which allows inversion of \( G \) by inverting \( S \).
1. Gaussian Measure on Infinite Dimensional Spaces

In this chapter we introduce some basic facts from probability theory, and construct the Gaussian measure on infinite dimensional spaces.

1.1 Kolmogorov’s Existence Theorem

The fundamental result we shall use for measures in infinite dimensions is Kolmogorov’s theorem:

**Theorem 1.1. Kolmogorov’s Existence Theorem**

Suppose that for each \( n \in \{1, 2, 3, \ldots\} \) we have a Borel probability measure \( \mu_n \) on \( \mathbb{R}^n \) satisfying the Kolmogorov consistency condition

\[
\mu_{n+1}(E \times \mathbb{R}) = \mu_n(E)
\]

for all Borel sets \( E \subseteq \mathbb{R}^n \). Then there is a unique probability measure \( \mu \) on the product \( \sigma \)-algebra of \( \mathbb{R}^\infty = \mathbb{R} \{1, 2, 3, \ldots\} \) such that

\[
\mu(E \times \mathbb{R} \times \mathbb{R} \times \ldots) = \mu_n(E)
\]

for each Borel set \( E \subseteq \mathbb{R}^n \).

1.2 Gaussian Measure for Finite Dimensional Spaces

1.2.1 The Borel measure \( \mu \) on \( \mathbb{R} \) given by

\[
\mu(A) = \int_A \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(x-m)^2}{2\sigma^2}} \, dx \tag{1.1}
\]

is called the Gaussian measure with mean \( m \) and variance \( \sigma^2 \) on \( \mathbb{R} \), where \( m, \sigma \in \mathbb{R} \) and \( \sigma \neq 0 \).
Recall that if \( X \) is a random variable on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) then the expectation of \( X \) is

\[
E(X) = \int X \, d\mathbb{P}
\]

and the variance of \( X \) is

\[
Var(X) = E[(X - E(X))^2].
\]

Moreover, the distribution of \( X \) is the probability measure \( \mu_X \) on \( \mathbb{R} \) given by

\[
\mu_X(A) = P[X \in A].
\]

Observations:

(a) If \( X \) is a random variable with Gaussian distribution with mean \( m \) and variance \( \sigma^2 \), then \( E(X) = m \) and \( Var(X) = E(X - m)^2 = \sigma^2 \).

(b) If \( X \) is a random variable with Gaussian distribution and \( m = 0 \) and \( \sigma^2 = 1 \) then \( X \) is called a standard Gaussian random variable.

1.2.2 The characteristic function \( \hat{\mu} \) of a probability measure \( \mu \) on \( \mathbb{R} \) is given by

\[
\hat{\mu}(y) = \int_{\mathbb{R}} e^{ixy} \, d\mu(x) \tag{1.2}
\]

for all \( y \in \mathbb{R} \).

It is known that the characteristic function of any Borel probability measure on \( \mathbb{R} \) uniquely determines the measure.

1.2.3 Let \( X \) be a random variable. The characteristic function of \( X \) is

\[
\hat{\mu}_X(y) = \int_{\mathbb{R}} e^{ixy} \, d\mu_X(x) = E(e^{ixy}) \tag{1.3}
\]

1.2.4 If \( X \) is a Gaussian random variable with mean \( m \) and variance \( \sigma^2 \) then it is
well known that the characteristic function of $X$ is given by

$$
\tilde{\mu}_X(y) = E(e^{iyX}) = e^{imy - \frac{y^2}{2\sigma^2}}.
$$

(1.4)

We now work with a finite dimensional Hilbert space $V$, $\dim V = n$. Then the standard Gaussian measure on $V$ is given by:

$$
d\mu_V(x) = \frac{1}{(\sqrt{2\pi})^n} e^{-\frac{|x|^2}{2}} \, dx,
$$

with $dx$ Lebesgue measure on $V$, and $|x|$ denotes the norm of $x$. (For infinite dimensions, this expression does not make sense, because $\frac{1}{(\sqrt{2\pi})^n}$ would be zero, and Lebesgue measure does not exist in infinite dimensions.)

For any $x \in V$, we have a linear map

$$
\hat{x} : V \to \mathbb{R} : y \mapsto \hat{x}(y) = \langle x, y \rangle
$$

(1.5)

where $\langle , \rangle$ is the inner product on $V$.

On the finite dimensional space $V$ the Borel $\sigma$-algebra is generated by the functions $\hat{x}$ with $x$ running over $V$.

The characteristic function of the random variable $\hat{x}$ is given by

$$
\int_V e^{i\lambda \hat{x}} \, d\mu_V = e^{-\lambda^2 |x|^2/2}.
$$

(1.6)

We see that from 1.4, for all $\lambda \in \mathbb{R}$, $\hat{x}$ is a Gaussian random variable with mean zero and variance $|x|^2$. As noted earlier, the condition (1.6) uniquely specifies the measure $\mu_V$.

### 1.3 Gaussian Measure for Infinite Dimensional Spaces

Let $V$ be a real, separable, infinite dimensional Hilbert space.
In the infinite dimensional case we want to have random variables like \( \hat{x} \) on a probability space which will give us the Gaussian measure on \( V \). So we need to find a probability space \((\Omega, \mathcal{F}, P)\), and for each \( x \) a linear map
\[
V \to L^2(\Omega, \mathcal{F}, P) : x \mapsto \hat{x},
\]
such that each \( \hat{x} \) is a Gaussian random variable with mean zero and variance \(|x|^2\), i.e. \( \hat{x} \) satisfies
\[
\int_{\Omega} e^{i\lambda \hat{x}} \, dP = e^{-\lambda^2 |x|^2 / 2}.
\]
Let \( \{e_n\}_{n \in I} \) be an orthonormal basis of \( V \). Each element \( x \in V \) can be written as a convergent sum \( x = \sum_{n \in I} x_n e_n \), where \( x_n = \langle x, e_n \rangle \).

Define :
\[
J_0 : V \to \mathbb{R}^I : x \mapsto (x_n)_{n \in I}
\]
with \( J_0(x) \) being the string of coordinates of \( x \) with respect to the basis \( \{e_n\} \). The mapping \( J_0 \) is injective and its image is
\[
\text{Im}(J_0) = J_0(V) = \left\{ (y_n)_{n \in I} ; \sum_{n \in I} y_n^2 < \infty \text{ i.e. } \sum_{n \in I} y_n e_n \text{ convergent} \right\}.
\]
This is a proper subset of \( \mathbb{R}^I \).

We have the standard Gaussian measure \( \mu_{\mathbb{R}} \) on \( \mathbb{R} \), given by
\[
d\mu_{\mathbb{R}}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2 / 2} \, dx.
\]
Then by Kolmogorov’s Theorem we have the product measure \( \mu_{\mathbb{R}^I} \) on \( \mathbb{R}^I \). The \( \sigma \)-algebra is the product \( \sigma \)-algebra \( \mathcal{F} \) generated by all the coordinate projections
\[
\pi_j : \mathbb{R}^I \to \mathbb{R} : (y_n)_{n \in I} \mapsto y_j.
\]
Let \( \Omega = \mathbb{R}^I \), equipped with the product Gaussian measure \( \mu_{\mathbb{R}^I} \) on the product \( \sigma \)-algebra \( \mathcal{F} \).
Define $\tilde{e}_j$ to be the random variable on $\Omega$ given by the coordinate projection $\pi_j$. Then this is a Gaussian random variable with mean 0 and variance 1.

For $x \in V$ define $\tilde{x}$ to be the sum of the random variables $x_n \tilde{e}_n$,

$$\tilde{x} = \sum_{n \in I} x_n \tilde{e}_n$$

where $x_n = \langle x, e_n \rangle$.

We have

$$\sum_{n \in I} |x_n \tilde{e}_n|^2_{L^2(\Omega, \mu_{\mathbb{R}})} = \sum_{n \in I} x_n^2 = \|x\|^2 < \infty$$

in $L^2(\Omega, \mu_{\mathbb{R}})$, so the series for $\tilde{x}$ defined as $L^2$ limit of the partial sums, specifies $\tilde{x}$ as a random variable on $\Omega$. The random variable $\tilde{x}$ is Gaussian because an $L^2$-limit of Gaussians is Gaussian as we now show:

**Lemma 1.2.** Suppose $X_1, X_2, X_3, \ldots$ are Gaussian random variable on a probability space $(\Omega, \mathcal{F}, P)$ and suppose the sequence $(X_n)_n$ converges in $L^2(P)$ to a random variable $X$. Then $X$ is also Gaussian.

**Proof.** The characteristic function of $X$ is given by

$$\varphi_X(t) = \int e^{itX} \, dP = \langle e^{itX}, 1 \rangle_{L^2(P)}$$

Since $X_n \to X$ in $L^2$, it follows that

$$e^{itX_n} \to e^{itX}$$

in $L^2(P)$, because

$$\int |e^{itX_n} - e^{itX}|^2 \, dP \leq \int |X_n - X|^2 \, dP \to 0.$$ 

Then

$$\langle e^{itX_n}, 1 \rangle_{L^2(P)} \to \langle e^{itX}, 1 \rangle_{L^2(P)}.$$
i.e.

\[ \varphi_{X_n}(t) \to \varphi_X(t) \]

for all \( t \in \mathbb{R} \), as \( n \to \infty \).

Because \( X_n \) is Gaussian, we have

\[ \varphi_{X_n}(t) = e^{itE(X_n) - \frac{t^2}{2}Var(X_n)}. \]

Since \( X_n \to X \) in \( L^2 \), we have

\[ |E(X_n) - E(X)| \leq \sqrt{E|X_n - X|^2} \to 0 \]

and

\[ Var(X_n - X) = E(X_n - X)^2 - [E(X_n - X)]^2 = \|X_n - X\|_{L^2}^2 - [EX_n - EX]^2 \to 0. \]

So, as \( n \to \infty \), \( E(X_n) \to E(X) \) and \( Var(X_n) \to Var(X) \).

Consequently,

\[ \varphi_X(t) = \lim_{n \to \infty} \varphi_{X_n}(t) = e^{itE(X) - \frac{t^2}{2}Var(X)} \]

So \( X \) is a Gaussian random variable.

Note that the map

\[ J_0 : V \to L^2(\Omega, \mathcal{F}, \mu_{\mathbb{R}^I}) : x \mapsto \tilde{x} \quad (1.11) \]

is linear.

So we have our probability space \((\Omega, \mathcal{F}, \mu_{\mathbb{R}^I})\), and for each \( x \in V \) we have the Gaussian random variable \( \tilde{x} \) with mean 0 and variance \( |x|^2 \), depending linearly on \( x \).

**Proposition 1.3.** The proper subset \( J_0(V) \) of \( \mathbb{R}^I \) on the probability space \((\Omega, \mathcal{F}, \mu_{\mathbb{R}^I})\) has measure zero.
Proof. First we observe that
\[
J_0(V) = \left\{ (x_n)_{n \in I} \in \mathbb{R}^I : \sum_{n \in I} x_n^2 < \infty \right\} \\
= \left\{ \omega \in \Omega : \sum_{n=1}^{\infty} \tilde{e}_n(\omega)^2 < \infty \right\}.
\]

Now \(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \ldots\) are independent standard Gaussian random variables which are not identically zero. So for any \(\epsilon > 0\), and any \(n \geq 1\),
\[
\mu_{\mathbb{R}^I} \left( \{ \omega : |\tilde{e}_n(\omega)| \leq \epsilon, |\tilde{e}_{n+1}(\omega)| \leq \epsilon, \ldots \} \right) = \mu_{\mathbb{R}^I} \left( \bigcap_{k=n}^{\infty} \{ \omega : |\tilde{e}_n(\omega)| \leq \epsilon, \ldots |\tilde{e}_k(\omega)| \leq \epsilon \} \right) \\
= \lim_{k \to \infty} \mu_{\mathbb{R}^I} \{ \omega : |\tilde{e}_n(\omega)| \leq \epsilon, \ldots |\tilde{e}_k(\omega)| \leq \epsilon \} \\
= \lim_{k \to \infty} \mu_{\mathbb{R}^I} \left( [-\epsilon, \epsilon] \right)^{k-n+1} = 0
\]
because \(\mu_{\mathbb{R}}([-\epsilon, \epsilon]) < 1\). So
\[
\mu_{\mathbb{R}^I} \left( \bigcup_{n \geq 1} \{ \omega : |\tilde{e}_n(\omega)| \leq \epsilon, |\tilde{e}_{n+1}(\omega)| \leq \epsilon, \ldots \} \right) = 0.
\]

Now
\[
\left\{ \omega : \lim_{n \to \infty} \tilde{e}_n(\omega) = 0 \right\} \subset \bigcup_{n \geq 1} \{ \omega : |\tilde{e}_n(\omega)| \leq \epsilon, |\tilde{e}_{n+1}(\omega)| \leq \epsilon, \ldots \}.
\]
So the probability of \(\{ \omega : \lim_{n \to \infty} \tilde{e}_n(\omega) = 0 \}\) is zero.

Consequently, the subset \(J_0(V)\) also has \(\mu_{\mathbb{R}^I}\)-measure zero.

The following result gives us a proper subspace of \(\mathbb{R}^I\) on which \(\mu_{\mathbb{R}^I}\) lives:

**Proposition 1.4.** Let \(c_1, c_2, \ldots > 0\) be such that \(\sum c_n^2 < \infty\). Let
\[
V_1 = \left\{ (x_n)_{n \in I} : \sum c_n^2 x_n^2 < \infty \right\}.
\]

Then \(\mu_{\mathbb{R}^I}(V_1) = 1\).\[\boxed{10}\]
Proof. Observe that
\[ \int \sum_{n \in I} c_n^2 \tilde{e}_n^2 \, d\mu_{\mathbb{R}^I} = \sum_{n \in I} c_n^2 < \infty, \]
since \( \tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \ldots \) are independent standard Gaussian random variables on \( \Omega \).
Therefore,
\[ \sum_{n \in I} c_n^2 \tilde{e}_n^2 < \infty \quad a.e. \]
and so
\[ \mu_{\mathbb{R}^I}(V_1) = 1. \]

Observe that \( V_1 \) is a Hilbert space with the inner product
\[ \langle (x_n)_{n \in I}, (y_n)_{n \in I} \rangle = \sum_{n \in I} c_n^2 x_n y_n. \]
Since the set \( V_1 \) is of full measure, we can take \( \Omega \) to be \( V_1 \) instead of the set \( \mathbb{R}^I \).
In conclusion, the Gaussian measure for a separable Hilbert space \( V \) lies on the Hilbert space \( V_1 \) defined as in the previous proposition. The numbers \( c_1, c_2, \ldots > 0 \) can be chosen in any way subject to \( \sum_n c_n^2 < \infty \).
2. Radon Transform on Finite Dimensional Spaces

This chapter describes the Radon Transform on $\mathbb{R}^n$ and a corresponding inversion formula. This material is from existing literature.

2.4 The Radon Transform on $\mathbb{R}^n$

2.1.1 Let $S(\mathbb{R}^n)$ be the set of all complex-valued rapidly decreasing functions on $\mathbb{R}^n$, i.e.

$$f : \mathbb{R}^n \rightarrow \mathbb{C}$$

is in $S(\mathbb{R}^n)$ if for each polynomial $P$, and each integer $m \geq 0$,

$$\sup_x |x|^m P(\partial_1, \ldots, \partial_n)f(x) < \infty,$$

where $|x|=$ norm of $x$.

2.1.2 Let $S(S^{n-1} \times \mathbb{R})$ be the space of $C^\infty$ functions on $S^{n-1} \times \mathbb{R}$ which for any integers $k, l \geq 0$ and any differential operator $D$ on $S^{n-1}$ satisfy

$$\sup_{\omega \in S^{n-1}, r \in \mathbb{R}} \left| (1 + |r|^k)^{\frac{d^l}{d^l}} (D\phi)(\omega, r) \right| < \infty. \quad (2.12)$$

2.1.3 Then we define the space

$$S(P^n) = \{ \phi \in S(S^{n-1} \times \mathbb{R}) : \phi(\omega, p) = \phi(-\omega, -p) \}. \quad (2.13)$$

Observation:

It will be useful to bring in “polar coordinates” by means of the map

$$S^{n-1} \times \mathbb{R} \rightarrow \mathbb{R}^n : (\omega, p) \mapsto p \omega,$$

where $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$.

2.1.4 Let $P^n$ to be the set of all hyperplanes in $\mathbb{R}^n$.

Observations:
(a) A hyperplane $\xi \in P^n$ is specified by a unit normal vector $\omega$ and by the “distance” $p$ of $\xi$ from the origin in the direction of $\omega$. Thus we can associate $(\omega, p)$ to $\xi$ as follows:

$$\xi := (\omega, p) := \{ x \in \mathbb{R}^n : \langle x, \omega \rangle = p \}$$

(2.14)

where $\langle , \rangle$ denotes the inner product in $\mathbb{R}^n$.

(b) The pairs $(\omega, p)$ and $(-\omega, -p)$ give the same hyperplane $\xi$.

We also have $S^{n-1} \times \mathbb{R} \cong P^n$ by means of the map $(\omega, p) \rightarrow \xi$.

2.1.5 The Radon Transform of a function $f \in S(\mathbb{R}^n)$ is the function $R(f)$ defined on $P^n$ by :

$$R(f)(\xi) = R(f)(\omega, p) := \int_{\xi} f(x) \ d\mu(x)$$

(2.15)

where $\xi$ is any hyperplane in $\mathbb{R}^n$, and $d\mu(x)$ is the Lebesgue measure on $\xi$.

2.1.6 Let $\phi$ be a continuous complex-valued function on $P^n$. For the transformation $f \rightarrow Rf$, we have the dual transform $\phi \rightarrow \hat{\phi}$, where the function $\hat{\phi}$ on $\mathbb{R}^n$ is defined by

$$\hat{\phi}(x) := \int_{x \in \xi} \phi(\xi) \ d\mu(\xi),$$

(2.16)

where $d\mu$ is the rotation-invariant unit mass measure on the compact space

$$\{ \xi \in P^n : x \in \xi \},$$

of all the hyperplanes passing through a fix point $x \in \mathbb{R}^n$.

2.1.7 Let $S_H(P^n)$ be the set of all $F \in S(P^n)$ such that for every $k$ a positive integer,

$$\int_{\mathbb{R}} F(\omega, p) \ p^k \ dp$$

is a homogeneous polynomial in $\omega_1, \omega_2, \ldots, \omega_n$ of degree $k$.

2.1.8 The Fourier Transform of a function $f \in S(\mathbb{R}^n)$ is the function $\hat{f}$ defined
by
\[ \Im f(y) := \int_{\mathbb{R}^n} f(x) e^{-i(x,y)} \, dx \]  
where \( y \in \mathbb{R}^n \) and \( dx \) is the Lebesgue measure on \( \mathbb{R}^n \).

It is known that:

a) The Fourier transform of a function \( f \in S(\mathbb{R}^n) \) is again in \( S(\mathbb{R}^n) \).

b) The Radon Transform of a function \( f \in S(\mathbb{R}^n) \) is a one-to-one map of \( S(\mathbb{R}^n) \) onto \( S_H(P^n) \), see the Schwartz theorem from [2].

2.5 Inversion Formulas

First we shall describe the relationship between the Radon transform and the Fourier transform.

**Proposition 2.5.** Let \( \Im f \) to be the Fourier transform of \( f \in S(\mathbb{R}^n) \). Then for the usual Radon transform we have

\[ \Im f(s\omega) = \int_{\mathbb{R}} Rf(\omega, p)e^{-isp} dp, \]

where \( s \in \mathbb{R} \) and \( \omega \) is any unit vector in \( \mathbb{R}^n \).

**Proof.** From the definition of \( \Im f(s\omega) \) we have:

\[ \Im f(s\omega) = \int_{\mathbb{R}^n} e^{-i(x,s\omega)} f(x) dx, \]

Decomposing \( \mathbb{R}^n \) as \( \mathbb{R}^n \cong \mathbb{R} \times \xi \) by \( (p, x) \rightarrow p\omega + x \), we have

\[
\begin{align*}
\Im f(s\omega) &= \int_{\mathbb{R}} \int_{\xi = \{x \in \mathbb{R}^n : \langle x, \omega \rangle = p\}} e^{-i\langle x, s\omega \rangle} f(x) dm(x) \, dp \\
\Im f(s\omega) &= \int_{\mathbb{R}} \int_{\xi = \{x \in \mathbb{R}^n : \langle x, \omega \rangle = p\}} e^{-isp} f(x) dm(x) \, dp \\
\Im f(s\omega) &= \int_{\mathbb{R}} e^{-isp} \left( \int_{\xi = \{x : \langle x, \omega \rangle = p\}} f(x) dm(x) \right) dp \\
\Im f(s\omega) &= \int_{\mathbb{R}} e^{-isp} Rf(\omega, p) dp.
\end{align*}
\]
Part of the conclusion from Fubini’s theorem insure that the integral make sense.

Inversion of the Fourier transform yields an inversion formula for the Radon transform.

**Theorem 2.6. The Inversion Formula for the Radon Transform**

If $f \in S(\mathbb{R}^n)$, then we have the following inversion formula for the Radon transform of $f$ given by:

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x, y \rangle} \left( \int_{\mathbb{R}} e^{-i|y|p} Rf \left( \frac{y}{|y|}, p \right) dp \right) dy.$$  

**Proof.** The Inverse Fourier Transform gives us

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x, y \rangle} (\text{ℑ} f)(y) dy,$$

where $y \in \mathbb{R}^n$.

By Proposition 2.5, we have

$$\text{ℑ} f(y) = \int_{\mathbb{R}} e^{-i|y|p} Rf \left( \frac{y}{|y|}, p \right) dp,$$

(note that the integral is undefined only at one point $y = 0$).

So:

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\langle x, y \rangle} \left( \int_{\mathbb{R}} e^{-i|y|p} Rf \left( \frac{y}{|y|}, p \right) dp \right) dy.$$

We now describe another standard approach to the inversion formula. We have not been able to generalize this approach to the infinite-dimensional case, but we state the final case since the possibility of extension to infinite dimensions remains.
2.2.1 Let \( S^r(x) = \{ y : |y - x| = r \} \subset \mathbb{R}^n \) be the sphere with radius \( r \) in \( \mathbb{R}^n \), where \( |x| = \sqrt{(x, x)} \) denote the norm of \( x \). Let us denote the area of the sphere \( S^r(x) \) by \( A(r) \), and the area of the unit sphere in \( \mathbb{R}^n \) by \( \Omega_n := \frac{2\pi^{n/2}}{\Gamma(n/2)} \) (2.18)

Let \( B^r(x) = \{ y : |y - x| < r \} \subset \mathbb{R}^n \) be the open ball in \( \mathbb{R}^n \).

2.2.2 The mean value of a continuous function \( f \) on \( S^r(x) \) is defined by
\[
(M^r f)(x) := \frac{1}{A(r)} \int_{S^r(x)} f(\omega) \, d\omega
\]
(2.19)

where \( d\omega \) is the Euclidean measure on \( S^r(x) \).

Observe that if \( y \in \mathbb{R}^n \) and \( r = |y| \) then:
\[
(M^r f)(x) = \int_K f(x + k \cdot y) \, dk,
\]
(2.20)

where \( K := O(n) \) is the orthogonal group (the group of invertible linear transformations that preserve the inner product on \( \mathbb{R}^n \) \( (T \in K \iff T^{-1} = T^T) \)), and \( dk \) is the Haar measure, normalized by \( \int_K dk = 1 \).

Observe also that for equation 2.15 we have:
\[
\tilde{\phi}(x) = \int_{\xi \in \xi} \phi(\xi) \, d\mu(\xi) = \int_K \phi(x + k \cdot \xi_0) \, dk
\]
(2.21)

with \( \xi_0 \) some fixed hyperplane through origin.

2.2.3 Let \( f \in C^2(\mathbb{R}^n) \), with \( f \) a radial function i.e. \( f(x) = F(r), \ r = |x| \). Then
\[
(Lf)(x) = \frac{d^2F}{dr^2} + \frac{n - 1}{r} \frac{dF}{dr}
\]
(2.22)

where \( L \) is the Laplacian on \( \mathbb{R}^n \).

**Lemma 2.7.** For each \( r > 0 \) we have
\[
LM^r = M^r L,
\]
(2.23)
where \( L \) is the Laplacian on \( \mathbb{R}^n \) and \( M^r \) on \( C^2(S^r(x)) \) is as in 2.19. Also for \( f \in C^2(S^r(x)) \) and the mean value of \( f \), \( (M^r f)(x) \), we have

\[
L_x(M^r L)(x) = \left[ \frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} \right] (M^r f)(x).
\]  

(2.24)

This is the Darboux Equation and can also be written as

\[
L_x F(x, y) = L_y F(x, y),
\]

(2.25)

where \( F(x, y) = M^{|y|} f(x) \), with \( y \in \mathbb{R}^n \).

Proof. Let

\[
T_z : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto x + z
\]

and

\[
R_k : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto kx
\]

where \( k \) is an integer, \( z \in \mathbb{R}^n \), and let \( r = |y| \), with \( y \in \mathbb{R}^n \).

Then the Laplacian \( L \) is invariant under \( R_k \) and \( T_z \). We use equation 2.20, so

\[
(L_x M^r f)(x) = L_x \int_K f(x + ky) \, dk
\]

\[
= \int_K (L_x f)(x + ky) \, dk
\]

\[
= \int_K (Lf)(x + ky) \, dk
\]

\[
= (M^r Lf)(x)
\]

\[
= \int_K [(Lf) \circ T_x \circ R_k](y) \, dk
\]

\[
= \int_K L(f \circ T_x \circ R_k)(y) \, dk
\]

\[
= L_y \int_K f(x + ky) \, dk
\]

\[
= (L_y M^r f)(x).
\]
Then $LM^r = M^rL$ and $L_x F(x, y) = L_y F(x, y)$ holds for $y \in \mathbb{R}^n$; $|y| = r$. If in equation 2.22 we take instead of $f$ the mean value of $f$, $M^rf$ the equation 2.24 will hold, and then the Lemma is proved.

Let $f \in S(\mathbb{R}^n)$ and $\phi$ be a continuous complex-valued function on $P^n$. Fix a hyperplane $\xi_0$ passing through origin and $g \in M(n)$ an isometry. As $k$ runs through $K$, $gk \cdot \xi_0$ runs through the set of hyperplanes through $g \cdot 0$. So we have by equation 2.21

$$\check{\phi}(g \cdot 0) = \int_K \phi(g \cdot 0 + k\xi_0) \, dk = \int_K \phi(gk \cdot \xi_0) \, dk. \tag{2.26}$$

Let take $Rf$ instead of $g$, in equation 2.26, then

$$(\check{R}f)(g \cdot 0) = \int_K Rf(gk \cdot \xi_0) \, dk$$

$$= \int_K \int_{gk \cdot \xi_0} f(x) \, dm(x) \, dk$$

$$= \int_K \int_{\xi_0} f(gk \cdot y) \, dm(y) \, dk$$

$$= \int_{\xi_0} \left( \int_K f(g \cdot 0 + k\xi_0) \, dk \right) \, dm(y)$$

$$= \int_{\xi_0} (M^{|y|}f)(g \cdot 0) \, dm(y)$$

so

$$(\check{R}f)(g \cdot 0) = \int_{\xi_0} (M^{|y|}f)(g \cdot 0) \, dm(y). \tag{2.27}$$

**Theorem 2.8. Inversion Formula for the Radon Transform**

The function $f$ can be recovered from its Radon transform by means of the following inversion formula:

$$cf = L^{-n-1} [\check{R}f], \tag{2.28}$$
where \( f \in \mathcal{S}(\mathbb{R}^n) \), \( c \) is a constant with
\[
c = (-4\pi)^{n-1} \frac{\Gamma(n/2)}{\Gamma(1/2)}.
\]

**Proof.** From the equation (2.24) we have:
\[
(\mathcal{R}f)(x) = \int_{\xi_0} (M^{[y]}f)(x) \, dm(y)
= \int_{\mathbb{R}} \int_{S^{n-1}(r,x)} (M^r f)(x) \, d\omega \, dr
= \int_{\mathbb{R}} (M^r f)(x) \int_{S^{n-1}(r,x)} d\omega \, dr
= \int_{\mathbb{R}} (M^r f)(x) \Omega_{n-1} r^{n-2} \, dr
= \Omega_{n-1} \int_{\mathbb{R}^+} r^{n-2} (M^r f)(x) \, dr.
\]
So,
\[
(\mathcal{R}f)(x) = \Omega_{n-1} \int_{\mathbb{R}^+} r^{n-2} (M^r f)(x) \, dr \quad (2.29)
\]
where \( y \in \mathbb{R}^n \), with \( |y| = r \). We applied the Laplacian to the equation 2.29, so:
\[
L_x(\mathcal{R}f)(x) = L_x(\Omega_{n-1} \int_0^\infty r^{n-2} (M^r f)(x) \, dr)
= \Omega_{n-1} \int_0^\infty r^{n-2} L_x(M^r f)(x) \, dr
\]
and using Lemma 2.7 we have
\[
L(\mathcal{R}f)(x) = \Omega_{n-1} \int_0^\infty r^{n-2} \left( \frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} \right) (F(x, r)) \, dr \quad (2.30)
\]
where
\[
F(x, r) = F(r) = (M^r f)(x) = \int_{S(r)} \frac{1}{|x-y|} f(x) \, d\omega \quad \text{and} \quad F(0) = f(x) \quad \text{and} \quad \lim_{r \to \infty} r^k F(r) = 0.
\]
So integrating by parts in equation 2.29

\[
(L\tilde{R}f)(x) = \Omega_{n-1} \int_0^\infty \left[ r^{n-2} \frac{d^2 F(r)}{dr^2} + r^{n-3} (n-1) \frac{dF(r)}{dr} \right] dr
\]

\[
= \Omega_{n-1} \left[ \int_0^\infty r^{n-2} \frac{d^2 F(r)}{dr^2} dr + \int_0^\infty r^{n-3} (n-1) \frac{dF(r)}{dr} dr \right]
\]

\[
= \Omega_{n-1} \left[ \left. \frac{dF(r)}{dr} r^{n-2} \right|_0^\infty - \int_0^\infty (n-2) r^{n-3} \frac{dF(r)}{dr} dr \right] + \Omega_{n-1} \left[ (n-1) r^{n-3} F(r) \biggr|_0^\infty - \int_0^\infty (n-1)(n-3) r^{n-4} F(r) dr \right]
\]

\[
= \Omega_{n-1} \left[ 0 - (n-2) \int_0^\infty r^{n-3} \frac{dF(r)}{dr} dr \right] - \Omega_{n-1} (n-1)(n-3) \int_0^\infty r^{n-4} F(r) dr
\]

\[
= \Omega_{n-1} \left[ -(n-2) F(r) r^{n-3} \biggr|_0^\infty + (n-2)(n-3) \int_0^\infty r^{n-4} F(r) dr \right] - \Omega_{n-1} (n-1)(n-3) \int_0^\infty r^{n-4} F(r) dr
\]

\[
= \Omega_{n-1} \left[ 0 - (n-3) \int_0^\infty r^{n-4} F(r) dr \right] = -\Omega_{n-1} (n-3) \int_0^\infty r^{n-4} F(r) dr
\]

So, we have

\[
L(\tilde{R}f)(x) = \begin{cases} 
-\Omega_{n-1} f(x) & \text{if } n = 3 \\
-\Omega_{n-1} (n-3) \int_0^\infty r^{n-4} F(r) dr & \text{if } n > 3
\end{cases} \tag{2.31}
\]

Let us again apply the Laplacian in equation 2.31, then

\[
L_x \int_0^\infty (M^r f)(x) r^k \, dr = \int_0^\infty r^k (L_x M^r f)(x) \, dr
\]

\[
= \int_0^\infty r^k \left( \frac{d^2}{dr^2} + \frac{n-1}{r} \frac{d}{dr} \right) (F(x, r)) \, dr
\]

\[
= (k-1)(k-n+1) \int_0^\infty r^{k-2} (M^r f)(x) \, dk
\]
So if \( A_k = \int_0^\infty (M^r f)(x)r^k \, dr \), then

\[
LA_k = (k - 1)(k + n - 1)A_{k-2}.
\] (2.32)

If \( n \) is odd the inversion formula for the Radon Transform 2.25 follows by iteration. If \( n \) is even we use the definition of the fraction power \( L^{\frac{n-1}{2}} \) in terms of Riesz potentials \( I^\gamma \) to obtain the inversion formula 2.25 for the Radon transform. Let us use the formula 2.19, then

\[
(\tilde{R}f)(x) = \int_K Rf(x + k \cdot \xi_0) \, dk
\]

\[
= \int_K \left( \int_{\xi_0} f(x + ky) \, dm(y) \right) \, dk
\]

\[
= \int_{\xi_0} (M|y| f)(x) \, dm(y)
\]

\[
= \Omega_{n-1} \int_{\mathbb{R}^n} r^{n-2} \left( \frac{1}{\Omega_n} \int_{S^{n-1}} f(x + r\omega) \, d\omega \right) \, dr
\]

\[
= \frac{\Omega_{n-1}}{\Omega_n} \int_{\mathbb{R}^n} |x - y|^{-1} f(y) \, dy
\]

\[
= 2^{n-1} \pi^{\frac{n}{2} - 1} \Gamma^{n-1} f
\] (2.33)

where \( I^\gamma \) is the Riesz potential

\[
(I^\gamma)(x) = \frac{1}{H_n(\gamma)} \int_{\mathbb{R}^n} f(y) |x - y|^{-n} \, dy
\] (2.34)

with

\[
H_n(\gamma) = 2^n \pi^{\frac{n}{2}} \frac{\Gamma(\frac{n}{2})}{\Gamma(\frac{n-\gamma}{2})},
\]

and the fraction power of \( L \):

\[
(-L)^p f = I^{-2p}(f).
\]
In this chapter we describe Gaussian measure on hyperplanes.

Let $\xi$ be a hyperplane in $\mathbb{R}^n$. We have

$$\xi = (\omega, p) = \{x \in \mathbb{R}^n : \langle x, \omega \rangle = p\};$$

where $p$ is the distance from the hyperplane $\xi$ to the origin, and $\omega$ a unit vector in $\mathbb{R}^n$, normal to $\xi$.

Observe that:

$$\xi = p\omega + \omega^\perp.$$

**Definition 3.9.** The Gaussian measure $\mu_\xi$ on the hyperplane $\xi$ is given by:

$$d\mu_\xi(x) = c_p e^{-\frac{|x|^2}{2}} dm(x) \quad (3.35)$$

where $dm(x)$ is the Lebesgue measure on $\xi$ and the value of the constant $c_p$ is chosen to insure that $\mu_\xi$ is a probability measure on $\xi$. The value of $c_p$ is computed in:

**Proposition 3.10.** The value of $c_p$ is

$$c_p = \frac{1}{(\sqrt{2\pi})^{n-1}} e^{\frac{p^2}{2}} \quad (3.36)$$

**Proof.** The requirement that $\mu_\xi$ is a probability measure on $\xi$ means that $\mu_\xi(\xi) = 1$, i.e.

$$\int_\xi c_p e^{-\frac{|x|^2}{2}} dm(x) = \int_\xi d\mu_\xi = 1.$$

As we have noted,

$$\xi = \omega^\perp + p\omega,$$
with every element $x$ in $\xi$ of the form

$$x = z + p\omega,$$

where $z \in \omega^\perp$. So we have:

$$c_p \int_{\omega^\perp} e^{-\frac{|p\omega|^2}{2} - \frac{|z|^2}{2}} \, dz = 1$$

and the term $e^{-\frac{|p\omega|^2}{2}}$ can be taken of the integral:

$$c_p e^{-\frac{|p\omega|^2}{2}} \int_{\omega^\perp} e^{-\frac{|z|^2}{2}} \, dz = 1.$$

Since $\omega^\perp$ has dimension $n - 1$, we have

$$\int_{\omega^\perp} e^{-\frac{|z|^2}{2}} \, dz = \left(\frac{\sqrt{2\pi}}{\sqrt{n-1}}\right)^n$$

moreover, $|p\omega| = p$. So

$$c_p e^{-\frac{p^2}{2}} \left(\frac{\sqrt{2\pi}}{\sqrt{n-1}}\right)^n = 1,$$

which gives the value

$$c_p = \frac{1}{\left(\frac{\sqrt{2\pi}}{\sqrt{n-1}}\right)^n e^{\frac{p^2}{2}}}.$$

\[\square\]

**Lemma 3.11.** The characteristic function $\varphi$, defined on $\mathbb{R}^n$, of the Gaussian measure $\mu_\xi$ on $\xi = (\omega, p) = \omega^\perp + p\omega$ is given by:

$$\varphi(y) = \int_\xi e^{i(x,y)} \, d\mu_\xi(x) = e^{ip(\omega, y) - \frac{|y_1|^2}{2}} \quad (3.37)$$

where $y_1$ is the orthogonal projection of $y$ on $\omega^\perp$.

**Proof.** Since $\xi = \omega^\perp + p\omega$, each point $x \in \xi$ can be written as

$$x = x_1 + p\omega$$
with \( x_1 \in \omega^\perp \). Using this, we have

\[
\varphi(y) = \int_\xi e^{i(y,x)} \, d\mu_\xi(x)
\]

\[
= \int_\xi e^{i(y,x)} \frac{1}{(\sqrt{2\pi})^{n-1}} e^{\frac{x_1^2}{2}} e^{-\frac{|x|^2}{2}} \, dm(x)
\]

\[
= \left(\frac{2\pi}{\sqrt{2}}\right)^{1-n} \int_{\omega^\perp} e^{i(y,x_1)} e^{ip(y,\omega)} e^{\frac{x_1^2}{2}} e^{-\frac{|x|^2}{2}} \, dx_1
\]

\[
= \left(\frac{2\pi}{\sqrt{2}}\right)^{1-n} \int_{\omega^\perp} e^{i(y,x_1)} e^{ip(y,\omega)} e^{-\frac{|x|^2}{2}} \, dx_1
\]

\[
= \left(\frac{2\pi}{\sqrt{2}}\right)^{n-1} \int_{\omega^\perp} e^{i(y,x_1)} e^{ip(y,\omega)} e^{-\frac{|x|^2}{2}} \, dx_1
\]

\[
= e^{ip(y,\omega)} \int_{\mathbb{R}^{n-1}} e^{i(y,x_1)} \, d\mu_{\omega^\perp}(x_1)
\]

\[
= e^{ip(y,\omega)} e^{-\frac{|x|^2}{2}} = e^{ip(y,\omega) - \frac{|x|^2}{2}}
\]

where \( d\mu_{\omega^\perp} \) is the Gaussian measure on the \((n-1)\)-dimensional space \( \omega^\perp \) and \( y_1 \) is the orthogonal projection of \( y \) on \( \omega^\perp \).

Recall that for each \( x \in \xi \), we have the map \( \hat{x} \) defined by

\[
\hat{x} : \mathbb{R}^n \rightarrow \mathbb{R} : \ y \mapsto \hat{x}(y) = \langle x, y \rangle
\]

(3.38)

where \( \langle \ , \ \rangle \) is the inner product on \( \mathbb{R}^n \).

**Proposition 3.12.** Let \( \xi \) be a hyperplane in \( \mathbb{R}^n \), and \( \mu_\xi \) the Gaussian measure on \( \xi \) given by 3.1. For any \( x \in \mathbb{R}^n \), the characteristic function of the random variable \( \hat{x} \) with respect to \( \mu_\xi \) is given by

\[
\int_{\mathbb{R}^n} e^{i\lambda \hat{x}} \, d\mu_\xi = e^{i\lambda p(x,\omega) - \lambda^2 \frac{|x_1|^2}{2}}
\]

(3.39)

for any \( \lambda \in \mathbb{R} \), and \( x_1 \) is the orthogonal projection of \( x \) on \( \omega^\perp \).
Proof. For any $x \in \mathbb{R}^n$, and $\lambda \in \mathbb{R}$ we have

$$
\int_{\mathbb{R}^n} e^{i\lambda \hat{x}} \, d\mu_\xi = \int_{\mathbb{R}^n} e^{i\lambda \hat{y}(y)} \, d\mu_\xi (y)
$$

$$
= \int_{\mathbb{R}^n} e^{i\lambda \langle x, y \rangle} \, d\mu_\xi (y)
$$

$$
= \int_{\mathbb{R}^n} e^{i\lambda \langle x, y \rangle} \frac{1}{(\sqrt{2\pi})^{n-1}} e^{\frac{y^2}{2}} e^{-\frac{|y|^2}{2}} \, dm(y)
$$

$$
= (\sqrt{2\pi})^{1-n} \int_{\omega^\perp} e^{i\lambda \langle x, y_1 \rangle} e^{i\lambda p \langle x, \omega \rangle} e^{\frac{y_1^2}{2}} e^{-\frac{|y_1|^2}{2}} \, dy_1
$$

$$
= \frac{e^{i\lambda p \langle x, \omega \rangle}}{(\sqrt{2\pi})^{n-1}} \int_{\mathbb{R}^{n-1}} e^{i\lambda \langle x, y_1 \rangle} e^{-\frac{|y_1|^2}{2}} \, dy_1
$$

$$
= e^{i\lambda p \langle x, \omega \rangle} e^{-\frac{\lambda^2 |x|^2}{4}}
$$

(3.40)

using $y = y_1 + p\omega$, $y_1 \in \omega^\perp$, where $x_1$ is the orthogonal projection of $x$ on $\omega^\perp$.  \qed

So by the above Proposition, we have that $\hat{x}$ is a Gaussian random variable with mean $p \langle x, \omega \rangle$ and variance $|x_1|^2$. The Gaussian measure $\mu_\xi$ on $\xi$ is uniquely determined by the relation 3.39.
4. Radon-Gauss Transform for Finite Dimensional Spaces

In this chapter we shall introduce the Radon-Gauss transform on $\mathbb{R}^n$ and obtain an inversion formula.

Recall that the usual Radon transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ is

$$Rf(\xi) = \int_\xi f(x) dm(x),$$

$\xi$ is a hyperplane in $\mathbb{R}^n$, $dm$ is the Lebesgue measure on $\xi$.

Notation:

Every hyperplane $\xi$ in $\mathbb{R}^n$ can be viewed as a pair, $\xi := (\omega, p)$, where $\omega \in \mathbb{R}^n$ is a unit vector and $p \in \mathbb{R}$; specifically,

$$\xi = \{ x \in \mathbb{R}^n | <x, \omega> = p \}.$$

Observation:

Let $C^\infty_b(\mathbb{R}^n)$ be the set of all functions $f \in C^\infty(\mathbb{R}^n)$ such that $f$, along with all its derivatives, is exponentially bounded.

Consequently, the function

$$x \rightarrow f(x) e^{-\frac{|x|^2}{2}}$$

is in $\mathcal{S}(\mathbb{R}^n)$.

**Definition 4.13.** Let $f \in C^\infty_b(\mathbb{R}^n)$. The **Radon-Gauss Transform** of $f$ is the function $Gf$, defined on the space $P^n$ of hyperplanes in $\mathbb{R}^n$, by

$$Gf(\xi) := \int_\xi f(x) d\mu_\xi(x) = \int_\xi c_p f(x) e^{-\frac{|x|^2}{2}} dm(x)$$

where the value of $c_p$ is as in (3.36), and $\xi = (\omega, p)$, and $c_p e^{-\frac{|x|^2}{2}} dm(x) = d\mu_\xi(x)$ is Gaussian measure on $\xi$. 

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The inversion formula for the Radon transform allows us to invert the Radon-Gauss transform.


Let \( f \in C^\infty(\mathbb{R}^n) \) be such that \( f(x)e^{-\frac{|x|^2}{2}} \) is in \( \mathcal{S}(\mathbb{R}^n) \). For instance, \( f \) may be \( C^\infty(\mathbb{R}^n) \) exponentially bounded. Then:

\[
f(x) = \frac{e^{\frac{|x|^2}{2}}}{(2\pi)^\frac{n+1}{2}} \int_{\mathbb{R}^n} e^{i(x,y)} \left( \int_{\mathbb{R}} Gf \left( \frac{y}{|y|}, p \right) e^{-\frac{|y|^2|p|^2}{2}} dp \right) dy
\]

(4.41)

**Proof.** Let \( p \in \mathbb{R} \), \( \omega \) a unit vector, normal to \( \xi = (\omega, p) \).

We have

\[ Gf(\omega, p) = c_p \int_{(\omega, p)} f(x)e^{-\frac{|x|^2}{2}} dx \]

the integration being over the hyperplane \((\omega, p) = p\omega + \omega^\perp\), with \( y = \omega p \) and \( \omega = \frac{y}{|y|} \). In terms of Radon transform \( R \), we then have

\[ Gf(\omega, p) = c_p R \left( f(x)e^{-\frac{|x|^2}{2}} \right), \]

and so

\[ R \left( f(x)e^{-\frac{|x|^2}{2}} \right) = \frac{1}{c_p} Gf(\omega, p). \]

Using the inversion formula for Radon transform \( R \) given in Proposition 2.5, we then have

\[
f(x)e^{-\frac{|x|^2}{2}} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x,y)} \left( \int_{\mathbb{R}} R \left( f(x)e^{-\frac{|x|^2}{2}} \right) \left( \frac{y}{|y|}, p \right) e^{-\frac{|y|^2|p|^2}{2}} dp \right) dy
\]

\[
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x,y)} \left( \int_{\mathbb{R}} c_p Gf \left( \frac{y}{|y|}, p \right) e^{-\frac{|y|^2|p|^2}{2}} dp \right) dy
\]

\[
= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i(x,y)} \left( \int_{\mathbb{R}} (2\pi)^\frac{n-1}{2} e^{-\frac{|p|^2}{2}} Gf \left( \frac{y}{|y|}, p \right) e^{-\frac{|y|^2|p|^2}{2}} dp \right) dy
\]

\[
= \frac{1}{(2\pi)^\frac{n+1}{2}} \int_{\mathbb{R}^n} e^{i(x,y)} \int_{\mathbb{R}} Gf \left( \frac{y}{|y|}, p \right) e^{-\frac{|y|^2|p|^2}{2}} dp dy.
\]

So

\[
f(x) = \frac{e^{\frac{|x|^2}{2}}}{(2\pi)^\frac{n+1}{2}} \int_{\mathbb{R}^n} e^{i(x,y)} \left( \int_{\mathbb{R}} Gf \left( \frac{y}{|y|}, p \right) e^{-\frac{|y|^2|p|^2}{2}} dp \right) dy.
\]
We shall now work out the Radon-Gauss transform of some simple functions.

**Example 4.15.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be the projection on the first coordinate:

$$f(x_1, x_2, x_3, \ldots, x_n) = x_1$$

Note: this is not in $\mathcal{S}^{(\mathbb{R}^n)}$. Let us find the Radon-Gauss transform of this $f$.

**Solution**

Let $\xi = (\omega, p)$, the hyperplane $\omega^\perp + p\omega$, when $\omega \in \mathbb{R}^n$ is a unit vector and $p \in \mathbb{R}$. Then

$$(Gf)(\xi) = \int_{\omega^\perp} e^{\frac{p^2}{2}} \frac{1}{(\sqrt{2\pi})^{n-1}} e^{-\frac{|y|^2}{2}} x_1 \, dm(x)$$

$$= e^{\frac{p^2}{2}} (\sqrt{2\pi})^{1-n} \int_{\omega^\perp} x_1 e^{-\frac{|y|^2}{2}} \, dm(x)$$

Since each element of $\xi$ is uniquely of the form $x = y + p\omega$, with $y \in \omega^\perp$. Let us write $p_1 = p\omega_1$, the first component of $p\omega$. We have

$$\begin{align*}
(Gf)(\xi) &= e^{\frac{p^2}{2}} (\sqrt{2\pi})^{1-n} \int_{\omega^\perp} (y_1 + p_1) e^{-\frac{|y+p\omega|^2}{2}} \, dy \\
&= e^{\frac{p^2}{2}} (\sqrt{2\pi})^{1-n} \int_{\mathbb{R}^{n-1}} (y_1 e^{-\frac{|y|^2}{2}} e^{-\frac{p_1^2}{2}} + p_1 e^{-\frac{|y|^2}{2}} e^{-\frac{p_1^2}{2}}) \, dy \\
&= e^{\frac{p^2}{2}} (\sqrt{2\pi})^{1-n} \left( \int_{\mathbb{R}^{n-1}} y_1 e^{-\frac{|y|^2}{2}} \, dy + \int_{\mathbb{R}^{n-1}} p_1 e^{-\frac{|y|^2}{2}} \, dy \right) \\
&= (\sqrt{2\pi})^{1-n} p_1 \int_{\mathbb{R}^{n-1}} e^{-\frac{|y|^2}{2}} \, dy \\
&= p_1 (\sqrt{2\pi})^{1-n} (\sqrt{2\pi})^{n-1} \\
&= p_1.
\end{align*}$$

**Example 4.16.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by

$$f(x_1, x_2, x_3, \ldots, x_n) = x_1^2.$$ 

Note that this is not in $\mathcal{S}(\mathbb{R}^n)$. We find the Radon-Gauss transform of this $f$
Solution

Let \( \xi = (\omega, p) \), as usual, with \( x = y + p\omega \), with \( y \in \omega^\perp \) and \( p_1 = p\omega_1 \). Then

\[
Gf(\xi) = \int_{\xi} e^{-\frac{|x|^2}{2}} x_1 e^{\frac{p_1^2}{2}} (\sqrt{2\pi})^{1-n} \, dm(x)
\]

\[
= e^{\frac{p_1^2}{2}} (\sqrt{2\pi})^{1-n} \int_{\omega^\perp} (y_1 + p_1)^2 e^{-\frac{|y + p\omega|^2}{2}} \, dy
\]

\[
= e^{\frac{p_1^2}{2}} (\sqrt{2\pi})^{1-n} \int_{\mathbb{R}^{n-1}} (y_1^2 + 2y_1p_1 + p_1^2) e^{-\frac{|y|^2}{2}} \, dy
\]

\[
= (\sqrt{2\pi})^{1-n} \left( \int_{\mathbb{R}^{n-1}} y_1^2 e^{-\frac{|y|^2}{2}} \, dy \right) +
\]

\[
+ (\sqrt{2\pi})^{1-n} \left( \int_{\mathbb{R}^{n-1}} 2y_1p_1 e^{-\frac{|y|^2}{2}} \, dy + \int_{\mathbb{R}^{n-1}} p_1^2 e^{-\frac{|y|^2}{2}} \, dy \right)
\]

\[
= (\sqrt{2\pi})^{1-n} \int_{\mathbb{R}^{n-1}} y_1^2 e^{-\frac{|y|^2}{2}} \, dy + 0 + p_1^2
\]

\[
= (\sqrt{2\pi})^{1-n} (\sqrt{2\pi})^{n-1} + p_1^2 = 1 + p_1^2
\]

Because

\[
\int_{\mathbb{R}} y^2 e^{-\frac{y^2}{2}} \, dy = -ye^{-\frac{y^2}{2}} \bigg|_{-\infty}^{+\infty} + \int_{\mathbb{R}} e^{-\frac{y^2}{2}} \, dy = 0 + \sqrt{2\pi} = \sqrt{2\pi}
\]

and

\[
\int_{\mathbb{R}^{n-1}} y_1^2 e^{-\frac{|y|^2}{2}} \, dy = \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} y_1^2 e^{-\frac{y_1^2}{2}} e^{-\frac{y_2^2}{2}} \ldots e^{-\frac{y_{n-1}^2}{2}} \, dy_1 \ldots dy_{n-1}
\]

\[
= \int_{\mathbb{R}} \ldots \left[ \int_{\mathbb{R}} y_1^2 e^{-\frac{y_1^2}{2}} \, dy_1 \right] e^{-\frac{y_2^2}{2}} dy_2 \ldots e^{-\frac{y_{n-1}^2}{2}} \, dy_{n-1}
\]

\[
= (\sqrt{2\pi})^{n-1}.
\]

Example 4.17. Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be given by

\[
f(x_1, x_2, x_3, \ldots, x_n) = x_1^\alpha,
\]

\( \alpha \geq 0 \) an integer. We find the Radon-Gauss transform of \( f \).

Solution
Let $\xi = (\omega, p)$, as usual and $p_1 = p\omega_1$. Then

$$G_f(\xi) = \int e^{x^2/2}(\sqrt{2\pi})^{1-n}x^\alpha e^{-|x|^2/2} \, dm(x)$$

$$= \int e^{y_\perp^2/2}(\sqrt{2\pi})^{1-n}(y_1 + p_1)^\alpha e^{-|y|^2/2} \, dy$$

$$= e^{y_\perp^2/2}(\sqrt{2\pi})^{1-n} \sum_{k=0}^\infty C^k \alpha 1^{\alpha-k} (p_1)^k e^{-|y|^2/2} \, dy$$

$$= (\sqrt{2\pi})^{1-n} \sum_{k=0}^\infty C^k \alpha 1^{\alpha-k} \int_{\mathbb{R}^{n-1}} y_1^{\alpha-k} e^{-|y|^2/2} \, dy$$  \hspace{1cm} (4.43)

Let us denote by

$$I_{\alpha-k} := \int_{\mathbb{R}} y_1^{\alpha-k} e^{-y_1^2/2} \, dy_1,$$

and $\alpha - k = s$. Then

$$I_s := \int_{\mathbb{R}} y_1^{s-1} e^{-y_1^2/2} \, dy_1,$$

$$I_s = y_1^{s-1} \left( e^{-y_1^2/2} \right) \bigg|_{-\infty}^{+\infty} + \int_{\mathbb{R}} (s-1) y_1^{s-2} e^{-y_1^2/2} \, dy_1$$

$$= 0 + (s-1) I_{s-2}.$$

So

$$I_s = (s-1) I_{s-2} \hspace{1cm} (4.44)$$

We have by equation 4.44:

$$I_1 = \int_{\mathbb{R}} y_1 e^{-y_1^2/2} \, dy_1 = e^{-y_1^2/2} \bigg|_{-\infty}^{+\infty} = 0$$

$$I_2 = \sqrt{2\pi}$$

$$I_3 = 2I_1 = 0,$$

$$I_4 = 3I_2 = 3 \cdot \sqrt{2\pi}$$

$$I_5 = 0$$

$$I_6 = 5I_4 = 3 \cdot 5\sqrt{2\pi}$$
So, inductively,

\[ I_s = \begin{cases} 
0 & \text{if } s \text{ is odd} \\
1 \cdot 3 \cdots (s - 1) \sqrt{2\pi} & \text{if } s \text{ is even}
\end{cases} \tag{4.45} \]

Then

\[ I_{\alpha-k} = \begin{cases} 
0 & \text{if } \alpha - k \text{ is odd} \\
1 \cdot 3 \cdots (\alpha - k - 1) \sqrt{2\pi} & \text{if } \alpha - k \text{ is even}
\end{cases} \tag{4.46} \]

Now

\[ (\sqrt{2\pi})^{1-n} \int_{\mathbb{R}^{n-1}} y_1^{\alpha-k} e^{-\frac{|y|^2}{2}} \, dy = \begin{cases} 
0 & \text{if } \alpha - k \text{ is odd} \\
1 \cdot 3 \cdot 5 \cdot 7 \cdots (\alpha - k - 1) & \text{if } \alpha - k \text{ is even}
\end{cases} \tag{4.47} \]

because using equation 4.46, for \( \alpha - k \) even, we have

\[ (\sqrt{2\pi})^{1-n} \int_{\mathbb{R}^{n-1}} y_1^{\alpha-k} e^{-\frac{|y|^2}{2}} \, dy = (\sqrt{2\pi})^{1-n} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} [1 \cdot 3 \cdot 5 \cdot 7 \cdots (s - 1)(\sqrt{2\pi})] e^{-\frac{y_1^2}{2}} \cdots e^{-\frac{y_{n-1}^2}{2}} \, dy_2 \cdots dy_{n-1} = 1 \cdot 3 \cdot 5 \cdot 7 \cdots (\alpha - k - 1) \]

Then the expression for the Radon-Gauss transform \( Gf(\xi) \), using equation 4.47 is

\[ Gf(\xi) = \begin{cases} 
0 & \text{if } \alpha - k \text{ is odd} \\
\sum_{k=0}^{\alpha} \binom{\alpha}{k} (\frac{k}{p})^{k} [1 \cdot 3 \cdot 5 \cdots (\alpha - k - 1)] & \text{if } \alpha - k \text{ is even}
\end{cases} \tag{4.48} \]
Example 4.18. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be given by

\[
f(x_1, x_2, x_3, \ldots, x_n) = x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}
\]

where \( i_1, \ldots, i_k \) are positive integers. Let us find the Gauss-Radon transform of this \( f \).

Solution

Let \( \xi = (\omega, p) \), and \( p_1 = p \omega_1 \) as usual.

\[
Gf(\xi) = \int_{\mathbb{R}^n} e^{\frac{p^2}{2}} \left( \sqrt{2\pi} \right)^{1-n} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} e^{-\frac{|x|^2}{2}} \mathrm{d}m(x)
\]

\[
= \int_{\mathbb{R}^n} e^{\frac{p^2}{2}} \left( \sqrt{2\pi} \right)^{1-n} (y_1 + p_1)^{i_1} (y_2 + p_2)^{i_2} \cdots (y_k + p_k)^{i_k} e^{-\frac{|y|^2}{2}} e^{-\frac{2}{2}} \mathrm{d}y
\]

\[
= e^{-\frac{p^2}{2}} \left( \sqrt{2\pi} \right)^{1-n} \int_{\mathbb{R}^n} \sum_{i=1}^{n-1} C_{i_j}^{i_1} y_1^{i_1-j_1} (p_1)^{j_1} \times \cdots \sum_{j_k=0}^{i_k} C_{j_k}^{i_k} y_k^{i_k-j_k} (p_k)^{j_k} \mathrm{d}y
\]

\[
= (\sqrt{2\pi})^{1-n} \sum_{i=1}^{n-1} \sum_{j_1=0}^{i_1} C_{i_1}^{i_1} (p_1)^{j_1} \cdots \sum_{j_k=0}^{i_k} C_{j_k}^{i_k} (p_k)^{j_k} \times \int_{\mathbb{R}^n} y_1^{i_1-j_1} \cdots y_k^{i_k-j_k} e^{-\frac{|y|^2}{2}} \mathrm{d}y
\]

\[
= (\sqrt{2\pi})^{1-n} \sum_{i=1}^{n-1} \sum_{j_1=0}^{i_1} C_{i_1}^{i_1} (p_1)^{j_1} \cdots \sum_{j_k=0}^{i_k} C_{j_k}^{i_k} (p_k)^{j_k} \times \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}} y_1^{i_1-j_1} \cdots y_k^{i_k-j_k} e^{-\frac{y_1^2}{2}} \cdots e^{-\frac{y_k^2}{2}} \mathrm{d}y_1 \cdots \mathrm{d}y_k
\]

\[
= (\sqrt{2\pi})^{1-n} \sum_{i=1}^{n-1} \sum_{j_1=0}^{i_1} C_{i_1}^{i_1} (p_1)^{j_1} \cdots \sum_{j_k=0}^{i_k} C_{j_k}^{i_k} (p_k)^{j_k} \times \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \left( \int_{\mathbb{R}} y_1^{i_1-j_1} e^{-\frac{y_1^2}{2}} \mathrm{d}y_1 \right) (y_2^{i_2-j_2} e^{-\frac{y_2^2}{2}}) \cdots (y_k^{i_k-j_k} e^{-\frac{y_k^2}{2}}) \mathrm{d}y_2 \cdots \mathrm{d}y_k
\]

Since we have already discussed the Gauss integrals

\[
\int_{\mathbb{R}} y^s e^{-\frac{y^2}{2}} \frac{\mathrm{d}y}{\sqrt{2\pi}}
\]

in equation 4.45, a complete expression for \( Gf(\xi) \) may be obtained.
Example 4.19. Let

\[ f(x) = e^{i(k,x)} = e^{ik}, \]

where \( k \in \mathbb{R}^n \). Let us find \( Gf \) for this \( f \).

Solution

\[ Gf(\xi) = Gf(\omega^\perp + p_\omega) = \int_\xi e^{i(k,y)}d\mu_\xi(y) = e^{k(p_\omega) - \frac{|k_1|^2}{2}} \]

by Proposition 3.38, where \( k_1 \) is the orthogonal projection of \( k \) onto \( \omega^\perp \).

Example 4.20. Consider

\[ f(x) = e^{(a,x) + i(b,x)}, \]

with \( a, b \in \mathbb{C}^n \).

Write \( z = a + ib \in \mathbb{C}^n \), and \( c \cdot b := c_1d_1 + \cdots + c_nd_n \) for \( c, d \in \mathbb{C}^n \).

Then \( f(x) = e^{zx} \). Then

\[ \int_{\mathbb{R}^n} f d\mu_{\omega^\perp + p_\omega} = e^{z(p_\omega + \frac{z_1z_1}{2})}, \]

where \( z_1 = a_1 + ib_1 \), with \( a_1, b_1 \) being the orthogonal projections of \( a, b \) onto \( \omega^\perp \).
Proof.

\[
\int_{\mathbb{R}^n} e^{\hat{a} + i \hat{b}} d\mu_{\omega^+ + p\omega} (y) = \int_{\mathbb{R}^n} e^{(a,y) + i(b,y)} d\mu_{\omega^+ + p\omega} (y)
\]

\[
= \int_{\mathbb{R}^n} e^{(a,y) + i(b,y)} e^{\frac{\|y\|^2}{2} e^{-\frac{\|y\|^2}{2}}} (\sqrt{2\pi})^{1-n} dy
\]

\[
= (\sqrt{2\pi})^{1-n} \int_{\mathbb{R}^n} e^{(a,y_1 + p\omega) + i(b,y_1 + p\omega)} e^{\frac{\|y_1\|^2}{2} e^{-\frac{\|y_1\|^2}{2}}} dy_1
\]

\[
= (\sqrt{2\pi})^{1-n} \int_{\mathbb{R}^n} e^{(a_1 + ib_1) y_1} e^{(a,p\omega) + i(b,p\omega)} e^{-\frac{\|y_1\|^2}{2}} dy_1
\]

\[
= (\sqrt{2\pi})^{1-n} e^{(a + ib) p\omega} \int_{\omega^\perp} e^{(a_1 + ib_1) y_1} e^{-\frac{\|y_1\|^2}{2}} dy_1
\]

\[
= e^{(a + ib) p\omega} e^{\frac{z_1, z_1}{2}}
\]

\[
\square
\]
5. Gaussian Measure on Hyperplanes
The Infinite Dimensional Case

In this chapter we define Gaussian measure on hyperplanes of infinite dimensional spaces.

Let $V$ be a separable infinite dimensional real Hilbert space. Let $\xi$ be a hyperplane in $V$, defined by

$$\xi = u^\perp + ru,$$

where $u$ is a normal, unit vector in $V$, and $r \in \mathbb{R}$ is the distance from origin to the hyperplane $\xi$. For our Hilbert space $V$ take an orthonormal basis $(e_n)_{n \geq 1}$, then every element $v$ in $V$ can be written as

$$v = \sum_{n=1}^{\infty} v_n e_n,$$

where $v_n = \langle v, e_n \rangle$ for $n = 1, 2, \ldots$, and $\langle \cdot, \cdot \rangle$ is the inner product in $V$.

We have from the finite dimension case (which we studied in Chapter 3) a Gaussian measure $\mu_\xi$ on a hyperplane $\xi$ in $\mathbb{R}^n$ given by

$$d\mu_\xi(x) = \frac{1}{(\sqrt{2\pi})^{n-1}} e^{\frac{1}{2} r^2} e^{-\frac{|x|^2}{2}} dm(x),$$

(5.49)

where $n = \dim V$, $r = d(\xi, 0)$ and $x \in \mathbb{R}^n$ and $|\cdot|$ is the norm on $\mathbb{R}^n$. In Lemma 3.11 we proved that the characteristic function $\varphi$, defined on $\mathbb{R}^n$, of the Gaussian measure $\mu_\xi$ on $\xi = (\omega, r) = \omega^\perp + r\omega$ is given by:

$$\varphi(y) = \int_{\xi} e^{i(x,y)} d\mu_\xi(x) = e^{ip(\omega \omega^\perp) - \frac{|y|^2}{2}}$$

(5.50)

where $x \in \xi$ is of the form $x = x_1 + r\omega$ with $x_1 \in \omega^\perp$ and $y_1$ is the orthogonal projection of $y$ on $\omega^\perp$. So using this characteristic function and also following the same procedure as in Chapter 1, we will construct the Gaussian measure on the hyperplane $\xi$ from $V$, where the dimension of $V$ is infinite.
Theorem 5.21. Let $V$ be a real, separable, infinite-dimensional Hilbert space. Let $\xi$ be a hyperplane in $V$, $\xi = u^\perp + ru$, where $u$ is a unit vector in $V$, and $r$ is the distance from the origin to the hyperplane. Then there is a probability measure $\mu_\xi$ on the product space $(\mathbb{R}^\infty, \mathcal{F})$, where $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots$ and $\mathcal{F}$ is the product $\sigma$-algebra on $\mathbb{R}^\infty$, and, for each $v \in V$ a Gaussian random variable $\hat{v}$ on $(\mathbb{R}^\infty, \mathcal{F}, \mu_\xi)$, depending linearly on $v$, such that:

$$
\int_{\mathbb{R}^\infty} e^{i\hat{v}} d\mu_\xi = e^{ir\langle v, u \rangle - \frac{1}{2}|P_{u^\perp}v|^2},
$$

for all $v \in V$, and $P_{u^\perp}$ is the orthogonal projection onto the subspace $u^\perp$ orthogonal to $u$.

Proof. Let $(e_n)_{n \geq 1}$ be an orthonormal basis for $V$. We have $\xi$ defined by

$$
\xi = u^\perp + ru,
$$

with $u \in V$ a unit vector. Then we have

$$
u = \sum_{n \geq 1} u_n e_n
$$

where $u_n = \langle u, e_n \rangle$, $n = 1, 2, 3, \cdots$. Take the first $n$ coordinates of $u$ and denote by $u_{1n}$ the vector $(u_1, u_2, \cdots, u_n) \in \mathbb{R}^n$. Because the vector $u_{1n}$ is not a unit vector in $\mathbb{R}^n$ let us normalize it:

$$
u_{1n}' = \frac{u_{1n}}{|u_{1n}|}
$$

First we construct a Gaussian measure $\mu_n$ on $\mathbb{R}^n$ whose characteristic function is

$$
\varphi_n(v) = e^{ir\langle v, u_{1n} \rangle - \frac{1}{2}(|v|^2 - \langle v, u_{1n} \rangle^2)},
$$

$n$ chosen large enough so that $u_{1n}$ is not 0. Let

$$
A_n = (1 - |u_{1n}|^2) I + |u_{1n}|^2 P_{u_{1n}^\perp}(v),
$$
where $I$ is the identity operator, and $P_{u_{1n}^\perp}$ is the orthogonal projection on $u_{1n}^\perp$.

Relative to an orthonormal basis of $\mathbb{R}^n$, with first vector being $u_1'$, $A_n$ has matrix

$$
\begin{pmatrix}
1 - |u_{1n}|^2 & & \\
 & 1 & \\
 & & \ddots \\
 & & & 1
\end{pmatrix}
$$

and so

$$
\det A_n = 1 - |u_{1n}|^2,
$$

which is not 0 if $|u_{1n}| < 1$. Now if $v \in V$ then

$$
P_{u_{1n}^\perp} (v) = v - \left\langle v, \frac{u_{1n}}{|u_{1n}|} \right\rangle \frac{u_{1n}}{|u_{1n}|}
= v - \frac{1}{|u_{1n}|^2} \left\langle v, u_{1n} \right\rangle u_{1n}
$$

So

$$
A_n (v) = \left(1 - |u_{1n}|^2\right) v + |u_{1n}|^2 P_{u_{1n}^\perp} (v)
= v - |u_{1n}|^2 v + |u_{1n}|^2 \left( v - \frac{1}{|u_{1n}|^2} \left\langle v, u_{1n} \right\rangle u_{1n} \right)
= v - |u_{1n}|^2 v + |u_{1n}|^2 v - \left\langle v, u_{1n} \right\rangle u_{1n}
= v - \left\langle v, u_{1n} \right\rangle u_{1n}
$$

and

$$
\left\langle v, A_n (v) \right\rangle = \left\langle v, v - \left\langle v, u_{1n} \right\rangle u_{1n} \right\rangle
= \left\langle v, v \right\rangle - \left\langle v, \left\langle v, u_{1n} \right\rangle u_{1n} \right\rangle
= |v|^2 - \left\langle v, u_{1n} \right\rangle \left\langle v, u_{1n} \right\rangle
= |v|^2 - \left\langle v, u_{1n} \right\rangle^2
$$

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Then the function \( \varphi_n \) can be written as:

\[
\varphi_n(v) = e^{i r (v, u_{1n}) - \frac{1}{2} (|v|^2 - \langle v, u_{1n} \rangle^2)}
\]

\[
= e^{i r (v, u_{1n}) - \frac{1}{2} (v, A_n(v))}
\] (5.52)

Assume for the moment that \(|u_{1n}| < 1\). Then \( A_n \neq 0 \). We will now show that the probability measure \( \mu_n \)

\[
d\mu_n(x) = \frac{1}{(\sqrt{2\pi})^n (\det A_n)^{\frac{1}{2}}} e^{-\frac{1}{2} \langle x - ru_{1n}, A_n^{-1} (x - ru_{1n}) \rangle} dx
\] (5.53)

has characteristic function given by \( \varphi_n(v) \). Now for \( x = ru_{1n} + A_n^{\frac{1}{2}} (y) \) we have \( dx = (\det A_n)^{\frac{1}{2}} dy \), and so:

\[
\int_{\mathbb{R}^n} e^{i (v, x)} d\mu_n(x) = \int_{\mathbb{R}^n} e^{i (v, x)} \frac{1}{(\sqrt{2\pi})^n \sqrt{\det A_n}} e^{-\frac{1}{2} \langle x - ru_{1n}, A_n^{-1} (x - ru_{1n}) \rangle} dx
\]

\[
= \int_{\mathbb{R}^n} e^{i (v, ru_{1n} + A_n^{\frac{1}{2}} y)} (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2} \langle A_n^{\frac{1}{2}} y, A_n^{-1} A_n^{\frac{1}{2}} y \rangle} (\det A_n)^{-\frac{1}{2} - \frac{1}{2}} dy
\]

\[
= \int_{\mathbb{R}^n} e^{i (v, ru_{1n})} e^{i \langle A_n^{\frac{1}{2}} v, y \rangle} (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2} \langle A_n^{-\frac{1}{2}} y, A_n^{\frac{1}{2}} y \rangle} dy
\]

\[
= e^{i (v, ru_{1n})} \int_{\mathbb{R}^n} e^{i \langle A_n^{\frac{1}{2}} v, y \rangle} (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2} \langle A_n^{-\frac{1}{2}} y, A_n^{\frac{1}{2}} y \rangle} dy
\]

\[
= e^{i (v, ru_{1n})} \int_{\mathbb{R}^n} e^{i \langle A_n^{\frac{1}{2}} v, y \rangle} (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2} \langle y, y \rangle} dy
\]

\[
= e^{i (v, ru_{1n})} \int_{\mathbb{R}^n} e^{i \langle A_n^{\frac{1}{2}} v, y \rangle} d\mu_{\mathbb{R}^n}(y)
\]

\[
= e^{i (v, ru_{1n})} e^{-\frac{|A_n^{\frac{1}{2}} v|^2}{2}}
\]

\[
= e^{i (v, ru_{1n})} e^{-\frac{1}{2} \langle A_n^{\frac{1}{2}} v, A_n^{\frac{1}{2}} v \rangle}
\]

\[
= e^{i (v, ru_{1n})} e^{-\frac{1}{2} \langle v, A_n^{\frac{1}{2}} A_n^{\frac{1}{2}} v \rangle}
\]

\[
= e^{i (v, ru_{1n})} e^{-\frac{1}{2} (v, A_n v)},
\]

which is exactly the value \( \varphi_n(v) \), by (5.52).
So for $u_{1n} \neq u$, which means $\|u_{1n}\| < 1$, equation (5.52) represents the characteristic function for the Gaussian measure $\mu_n$ defined in (5.53).

If $u_{1n} = u$ then $A_n = P_u$, because $\|u_{1n}\| = 1$.

Comparing $\varphi_n$, as given in (5.52), with the characteristic function $\varphi$, given in (5.50), we see that $\mu_n$ is the Gaussian measure $\mu_\xi$ on the hyperplane $\xi$ from equation (5.49).

Now from Chapter 1 we know that a measure is uniquely determined by its characteristic function. So for $n = 1, 2, 3, 4, \cdots$ we have a family of probability measures $\{\mu_n\}_{n \geq 1}$.

Now we will check that if

$$x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$$

and

$$x' = (x_1, x_2, \cdots, x_n, 0) \in \mathbb{R}^{n+1}$$

then $\varphi_{n+1}(x') = \varphi_n(x)$. We have:

$$\langle ru_{1n}, x' \rangle = r \sum_{j=1}^{n+1} u_j x'_j$$

$$= r \sum_{j=1}^{n} u_j x'_j + u_{n+1} x'_{n+1}$$

$$= r \langle u_{1n}, x \rangle$$

So

$$\varphi_{n+1}(x') = e^{-ir\langle u_{1n}, x' \rangle - \frac{1}{2}(\|x'\|^2 - \langle x', u_{1n} \rangle^2)}$$

$$= e^{-ir\langle u_{1n}, x \rangle - \frac{1}{2}(\|x\|^2 - \langle x, u_{1n} \rangle)}$$

$$= \varphi_n(x)$$

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It follows, by uniqueness of the measure corresponding to a given characteristic function, that

$$\mu_{n+1}(A \times \mathbb{R}) = \mu_n(A),$$

for any Borel set $A \in \mathbb{R}^n$. Now we apply Kolmogorov’s Existence Theorem from Chapter 1 to this family of probability measure $\{\mu_n\}_{n \geq 1}$, for $n = 1, 2, 3, \cdots$ on $\mathbb{R}^n$.

It follows that, there is a unique probability measure $\mu_\xi$ on the product $\sigma$-algebra of $\mathbb{R}^\infty$, such that

$$\mu_\xi(E \times \mathbb{R} \times \mathbb{R} \times \cdots) = \mu_n(E) = \mu_\xi(pr_n^{-1}(E))$$

where $E$ is any Borel subset of $\mathbb{R}^n$, and $pr_n$ is the projection map:

$$pr_n : \mathbb{R}^\infty \to \mathbb{R}^n$$

$$(x_1, x_2, x_3, \ldots) \mapsto (x_1, x_2, \cdots x_n).$$

So, on the space $(\mathbb{R}^\infty, \mathcal{F})$ we have the probability measure $\mu_\xi$. Now, let $V_0$ be the subspace of $V$ given by the linear span of $e_1, e_2, e_3, \cdots$.

For $v \in V_0$ and $v_i = \langle v, e_i \rangle$, $i = 1, 2, 3, \cdots$, take $v_i = 0$ for $i > n$.

Then we have a random variable $\hat{v}$ on $(\mathbb{R}^\infty, \mathcal{F}, \mu_\xi)$, defined by:

$$\hat{v}(x) = \sum_{j=1}^{\infty} v_j x_j = \sum_{j=1}^{n} v_j x_j,$$

where $x_j = \langle x, e_j \rangle$. 

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We have

\[ \int_{\mathbb{R}^\infty} e^{i\hat{v}} d\mu_\xi = \int_{\mathbb{R}^\infty} e^{i\hat{v}(x)} d\mu_\xi (x) \]

= \int_{\mathbb{R}^n} e^{\sum_{j=1}^n v_j x_j} d\mu_n (x_1, x_2, \cdots x_n)

= \varphi_n (v_1, v_2, \cdots v_n)

= e^{ir\langle v, u_1 \rangle - \frac{1}{2} (|v|^2 - \langle v, u_1 \rangle^2)}

= e^{ir\langle v, u \rangle - \frac{1}{2} (|v|^2 - \langle v, u \rangle^2)} \quad (5.54) \]

because

\[ v = \sum_{i=1}^n v_i e_i, \]

is a element of \( V \), and

\[ \langle v, u \rangle = \sum_{j=1}^n v_j u_j = \langle v, u_1 \rangle. \]

Observe that

\[ \int_{\mathbb{R}^\infty} e^{i\hat{v}} d\mu_\xi = e^{ir\langle v, u \rangle - \frac{1}{2} (|v|^2 - \langle v, u \rangle^2)} \]

= \[ e^{ir\langle v, u \rangle - \frac{1}{2} \langle v, A_n v \rangle} \]

= \[ e^{ir\langle v, u \rangle - \frac{1}{2} \langle v, P u \perp v \rangle} \]

Also we have

\[ \int_{\mathbb{R}^\infty} e^{it\hat{v}} d\mu_\xi = e^{irt\langle v, u \rangle - t^2 \frac{1}{2} \langle v, P u \perp v \rangle}. \quad (5.55) \]

Then the random variable \( \hat{v} \) is a Gaussian random variable defined everywhere on \( \mathbb{R}^\infty \) with mean \( r \langle v, u \rangle \) and variance \( \langle v, P u \perp v \rangle = |P u \perp v|^2 \), because

\[ |P u \perp v|^2 = \langle P u \perp v, P u \perp v \rangle \]

= \[ \langle v, P^2 u \perp v \rangle \]

= \[ \langle v, P u \perp v \rangle. \]
Let $V_0$ be the linear span of the basis vectors $e_n$, in $V$. Now we verify that the linear map

$$V_0 \rightarrow L^2(\mathbb{R}^\infty, \mu_\xi) : v \mapsto \hat{v}$$

(5.56)
can be extended to a continuous linear mapping:

$$V \rightarrow L^2(\mathbb{R}^\infty, \mu_\xi) : v \mapsto \hat{v}.$$  

(5.57)

We will use the standard relation:

$$\text{Var}X = E(X^2) - (EX)^2,$$

and then the Cauchy-Schwarz inequality:

$$\langle x, y \rangle \leq |x| |y|$$

for all $x, y \in V$. To show the extension is possible, we check:

$$|\hat{v}|^2_{L^2(\mathbb{R}^\infty, \mu_\xi)} = \int_{\mathbb{R}^\infty} |\hat{v}|^2 \, d\mu_\xi$$

$$= E(\hat{v}^2)$$

$$= \text{Var}(\hat{v}) + E(\hat{v})^2$$

$$= \langle v, P_{u^\perp} v \rangle + r^2 \langle v, u \rangle^2$$

$$\leq |v| |P_{u^\perp} v| + r^2 |v|^2 |u|^2$$

$$\leq |v| |v| + r^2 |v|^2$$

$$= (1 + r^2) |v|^2$$

(5.58)

The inequality (5.58) tells us that the linear mapping (5.56) is bounded, and so it can be extended to a continuous linear mapping (5.57), which is also bounded:

$$|\hat{v}|^2_{L^2(\mathbb{R}^\infty, \mu_\xi)} \leq (1 + r^2) |v|^2,$$
for all \( v \in V \). Also we can choose a sequence \((v'_n)_n \in V_0\) such that \( v'_n \) converges to \( v \in V \), and because \( |v'_n|_{L^2(\mu_\xi)} \leq r^2 |v'_n|^2 \) we have that \( \hat{v}_n \) converges to \( \hat{v} \) in \( L^2(\mu_\xi) \). Also we know from equation 5.55 that \( \hat{v}'_n \) are Gaussian random variable with mean \( r \langle v'_n, u \rangle \) and variance \( \langle v'_n, P_{u^\perp} v'_n \rangle \). Then by Lemma 1.2 we have that for each \( v \in V \) the random variable \( \hat{v} \) is Gaussian with mean \( r \langle v, u \rangle \) and variance \( \langle v, P_{u^\perp} v \rangle \).

In conclusion, for a real, separable, infinite-dimensional Hilbert space \( V \), with orthonormal basis \((e_n)_{n \geq 1}\), and for the product space \( \mathbb{R}^\infty \) with the product \( \sigma \)-algebra \( F \), and for a hyperplane \( \xi \) in \( V \), given by

\[
\xi = u^\perp + ru
\]

with \( r \in \mathbb{R} \) and \( u \in V \) a unit vector, we have constructed a probability measure \( \mu_\xi \) on the product space \((\mathbb{R}^\infty, F)\). Furthermore, for each \( v \in V \) we constructed a Gaussian random variable \( \hat{v} \) on \((\mathbb{R}^\infty, F, \mu_\xi)\), which depends linearly on \( v \in V \), defined by

\[
\hat{v}(x) = \sum_{n \geq 1} v_n x_n
\]

where \( v_n = \langle v, e_n \rangle \) for \( n = 1, 2, 3, \ldots \) and \( x = (x_1, x_2, x_3, \ldots) \in \mathbb{R}^\infty \), such that

\[
\int_{\mathbb{R}^\infty} e^{i\hat{v} \cdot x} \, d\mu_\xi = e^{ir \langle v, u \rangle} - \frac{1}{2} |P_{u^\perp} v|^2.
\]

We shall now show that \( \mu_\xi \) assigns mass 1 to a certain “hyperplane” in \( \mathbb{R}^\infty \).

**Theorem 5.22.** Suppose \( V \) is a real, separable Hilbert space, and \( \xi \) a hyperplane given by

\[
\xi = u^\perp + ru,
\]

where \( r \in \mathbb{R} \) and \( u \) is a unit vector in \( V \). Let \( \mu_\xi \) be the probability measure on \((\mathbb{R}^\infty, F)\) constructed in Theorem 5.21, and for each \( v \in V \), let \( \hat{v} \) be the Gaussian random variable on \((\mathbb{R}^\infty, F, \mu_\xi)\), from Theorem 5.21. Then

\[
\hat{u}(x) = r \text{ for } \mu_\xi\text{-almost-every } x \in \Omega = \mathbb{R}^\infty,
\]

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so

\[ \mu_\xi(\hat{u}^{-1}(r)) = 1 \text{ and } \mu(\hat{u}^{-1}(r)) = 0, \]

where \( \mu \) is the product Gaussian measure on \((\Omega, \mathcal{F})\).

Proof. From equation (5.59) we have

\[
\int_{\mathbb{R}^\infty} e^{it\hat{u}} d\mu_\xi = e^{ir\langle u, u \rangle - \frac{1}{2} \| P_{u\perp} u \|^2}
\]

\[ = e^{irt|u| - 0} \]

\[ = e^{irt} \]

\[ = \int_{\mathbb{R}} e^{its} d\delta_r(s), \]

where \( \delta_r \) is the delta measure i.e.

\[
\delta_r(A) = \begin{cases} 
1 & \text{if } r \in A \\
0 & \text{if } r \not\in A 
\end{cases} \quad (5.60)
\]

for every Borel set \( A \in \mathbb{R} \). Since the characteristic function of a random variable uniquely specifies the distribution, it follows that the random variable \( \hat{u} \) has the distribution \( \delta_r \). So,

\[
\delta_r(\{r\}) = \mu_\xi(\hat{u}^{-1}(r))
\]

\[ = \mu_\xi(\{x \in \mathbb{R}^\infty : \hat{u}(x) = r\}) = 1 \]

Then \( \hat{u} \) has the constant value \( r \), \( \mu_\xi \)-almost everywhere.

Consider any unit vector \( u \in V_0 \). Then, relative to the probability measure \( \mu \), the function \( \hat{u} \) is a Gaussian random variable for variable \( |u|^2 > 0 \). On the other hand, the same function \( \hat{u} \), is almost everywhere constant with respect to \( \mu_\xi \). Thus the set \( \hat{u}^{-1}(p) \) has \( \mu \)-measure 0 but \( \mu_\xi \)-measure 1.

\( \square \)
In this chapter we introduce the Radon transform for infinite dimensional spaces. The usual Radon transform uses integration with respect to Lebesgue measure which does not exist in infinite dimensions. So we use Gauss measure as the background measure, and call the resulting transform the Radon-Gauss transform.

6.6 The Radon-Gauss Transform

Let $V$ be a separable, infinite dimensional, real Hilbert space with orthonormal basis. Let $(e_n)_{n \geq 1}$ be an orthonormal basis of $V$. Let $\xi$ be a hyperplane in $V$ as usual, given by

$$\xi = u^\perp + ru,$$

where $u$ is a unit vector in $V$, and $r \in \mathbb{R}$ is the distance from origin to the hyperplane.

We work with the product space $(\Omega, \mathcal{F}, \mu)$, where $\Omega = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots = \mathbb{R}^\infty$ is the product space, $\mathcal{F}$ is the product $\sigma$-algebra, and $\mu$ is the product Gaussian measure on $(\Omega, \mathcal{F})$. Let $V_0$ be the linear subspace of $V$ spanned by the vectors $e_1, e_2, e_3, \cdots$.

Now, we know that for each $v \in V_0$ we have a Gaussian random variable $\hat{v}$ on $(\Omega, \mathcal{F}, \mu)$ defined by

$$\hat{v}(x) = \sum_{n \geq 1} v_n x_n,$$

where $v_n = \langle v, e_n \rangle$ and $x_n = \langle x, e_n \rangle$ for $n = 1, 2, 3, \cdots$.

In Chapter 5 we have shown that the linear map $v \mapsto \hat{v}$ extends to a linear isometry $V \to L^2(\mu) : v \mapsto \hat{v}$, with $\hat{v}$ also a Gaussian random variable with mean 0 and variance $|v|^2$. 

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Recall from Theorem 5.21 the measure $\mu_\xi$ on any hyperplane $\xi$ in $V$.

Let $\mu_\xi$ be the probability measure from Theorem 5.21, and $\mu$ the product probability measure, both of them on $(\Omega, F)$. Let us fix $u$ a unit vector, and let $r$ run over $\mathbb{R}$. Then we have the following property:

**Proposition 6.23.** Let $g$ be any bounded measurable function on $\mathbb{R}$. For $f \in W$, where $W$ is the linear span of all functions $e^{i\hat{v}}$ with $v$ running over $V_0$, we have:

$$
\int_{\Omega} g(\hat{u}) f \, d\mu = \int_{\mathbb{R}} \left[ \int_{\Omega} f \, d\mu_{u^+ + ur} \right] g(r) \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}} \, dr,
$$

(6.61)

where $u$ is any unit vector in $V$. Thus, the conditional expectation $E_{\mu} [f \mid \sigma(\hat{u})]$ is given by:

$$
E_{\mu} [f \mid \hat{u} = r] = \int_{\Omega} f \, d\mu_{ur + u^+},
$$

(6.62)

where $\sigma(\hat{u})$ is the $\sigma$-algebra generated by $\hat{u}$.

Note that, in general, conditional expectations are well-defined only almost-everywhere, not pointwise, and so cannot be used to define the measure $\mu_\xi$ for a given $\xi$.

**Proof.** Let $f = e^{i\hat{v}}$, for some $v \in V_0$ (this tells us that $\hat{v}$ is well defined everywhere on $\Omega$), and take $g(r) = e^{i sr}$ for some $s \in \mathbb{R}$.
Then we have

\[
\int_{\Omega} g(\hat{u}) f \, d\mu = \int_{\Omega} e^{is\hat{u}} e^{i\hat{v}} \, d\mu
\]

\[
= \int_{\Omega} e^{is\hat{u} + i\hat{v}} \, d\mu
\]

\[
= \int_{\Omega} e^{is\hat{u} + v} \, d\mu
\]

\[
= e^{-\frac{1}{2} ||su+v||^2}
\]

\[
= e^{-\frac{1}{2} ||P_{u\perp}(su+v)+(su+v, u)||^2}
\]

\[
= e^{-\frac{1}{2} ||P_{u\perp}(u)+(v, u)u||^2}
\]

\[
= e^{-\frac{1}{2} ||P_{u\perp}(v)||^2} e^{-\frac{1}{2} ||u||^2(s+(v, u))^2}
\]

\[
= e^{-\frac{1}{2}(s+(v, u))^2 - \frac{1}{2} ||P_{u\perp}(v)||^2}
\]

(6.63)

Now take the right side of equation 6.62 for the choices of \( f \) and \( g \) made:

\[
\int_{\mathbb{R}} \left[ \int_{\Omega} f \, d\mu_{u\perp + ur} \right] g(\hat{u}) e^{-\frac{r^2}{2}} \frac{1}{\sqrt{2\pi}} \, dr = \int_{\mathbb{R}} \left[ \int_{\Omega} e^{i\hat{v}} d\mu_{u\perp + ur} \right] e^{isr} e^{-\frac{r^2}{2}} \frac{1}{\sqrt{2\pi}} \, dr
\]

\[
= \int_{\mathbb{R}} \left[ e^{ir(v, u) - \frac{r^2}{2} ||P_{u\perp}v||^2} \right] e^{isr} e^{-\frac{r^2}{2}} \frac{1}{\sqrt{2\pi}} \, dr
\]

\[
= e^{-\frac{1}{2} ||P_{u\perp}v||^2} \int_{\mathbb{R}} e^{ir(v, u)+s} e^{-\frac{r^2}{2}} \frac{1}{\sqrt{2\pi}} \, dr
\]

\[
= e^{-\frac{1}{2} ||P_{u\perp}v||^2} e^{-\frac{1}{2}(s+(v, u))^2}
\]

\[
= e^{-\frac{1}{2}(s+(v, u))^2 - \frac{1}{2} ||P_{u\perp}v||^2}
\]

(6.64)

We see that this is exactly (6.63). So equation (6.62) is true for the choices of \( f \) and \( g \) made. Because \( W \) is the linear span of all functions \( e^{i\hat{v}} \), then by linearity, equation (6.62) holds for every \( f \in W \).

The linear span \( W_{\mathbb{R}} \) of the function \( r \mapsto e^{isr} \) for every \( r \in \mathbb{R} \), is dense in \( L^2(\mu_{\mathbb{R}}) \), with \( d\mu_{\mathbb{R}}(r) = \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}} \, dr \). Then for a given \( g \in L^2(\mu_{\mathbb{R}}) \), we can choose a sequence of functions \( g_n \in W_{\mathbb{R}} \) such that \( g_n \) converges to \( g \) in \( L^2(\mu_{\mathbb{R}}) \), and \( g_n(\hat{u}) \) converges to \( g(\hat{u}) \) in \( L^2(\mu) \).
In particular equation (6.62) holds for every \( f \in W \) and every \( L^2(\mu) \) measurable, bounded \( g \) on \( \mathbb{R} \) (in fact for every \( g \in L^2(\mu) \)). \( \square \)

**Definition 6.24.** Let \( f \) be any measurable function on \((\Omega, F)\) for which the integral \( \int f d\mu_\xi \) exists for all hyperplanes \( \xi \) in \( V \). The Radon-Gauss transform of \( f \) is the function \( Gf \) defined on the space of hyperplanes of \( V \) given by:

\[
(Gf)(\xi) = \int f d\mu_\xi
\]

where \( \mu_\xi \) is the probability measure from Theorem 5.21

**Proposition 6.25.** Let \( f \in L^2(\mu) \), and \( u \) a unit vector in the Hilbert space \( V \). Then \( f \in L^2(\mu_r u + u^\perp) \) for almost every \( r \in \mathbb{R} \) and

\[
\|f\|^2_{L^2(\mu)} = \int_{\mathbb{R}} \|f\|^2_{L^2(\mu_r u + u^\perp)} \frac{e^{-r^2/2}}{\sqrt{2\pi}} \, dp
\]

In particular, if \( f \) equals 0 \( \mu \)-almost-everywhere, then \( Gf(ru + u^\perp) \) equals 0 for almost every \( r \in \mathbb{R} \).

*Proof.* This follows from the disintegration formula (6.62). \( \square \)

### 6.7 Inversion Formula for the Radon-Gauss Transform

Let \( \mu_n \) be the standard Gaussian measure on \( \mathbb{R}^n \). The behavior of a measure under translation is of interest.

If we have a translation:

\[
\tau_v : \mathbb{R}^n \to \mathbb{R}^n : y \mapsto y + v,
\]

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then we have a new measure \((\tau_v)_* \mu\), given by:

\[
(\tau_v)_* \mu_n (A) = \mu_n (A - v)
\]

\[
= \int_{A-v} \mathrm{d}\mu_n
\]

\[
= \int_{A-v} e^{-\frac{|x|^2}{2}} \frac{1}{(\sqrt{2\pi})^n} \, dx
\]

\[
= \int_{A} \frac{1}{\sqrt{2\pi}} e^{-\frac{|x-v|^2}{2}} \, dx
\]

\[
= \int_{A} e^{(x,v) - \frac{|v|^2}{2}} \frac{1}{\sqrt{2\pi}} \, dx
\]

\[
= \int_{A} e^{\hat{v} - \frac{|v|^2}{2}} \, dx
\]

\[
= \int_{A} e^{\hat{v} - \frac{|v|^2}{2}} \, d\mu_n,
\]

where, as usual,

\[
\hat{v} : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto (x, v).
\]

So

\[
\frac{d (\tau_v)_* \mu_n}{d\mu_n} = e^{-\frac{|v|^2}{2}}.
\]

Consider now a real, separable, Hilbert space \(V\), and the corresponding product Gaussian \(\mu\) on \((\Omega = \mathbb{R}^\infty, \mathcal{F})\), constructed using an orthonormal basis \(\{e_n\}_{n \geq 1}\) of \(V\).

For \(v \in V_0\), the linear span of \(\{e_n\}_{n \geq 1}\), we have the translation \(\tau_v\) on \(\mathbb{R}^n\) given by

\[
\tau_v (x_n)_{n \geq 1} = (x_n + \langle v, e_n \rangle)_{n \geq 1}.
\]

Then

\[
\frac{d (\tau_v)_* \mu}{d\mu} = e^{\hat{v} - \frac{|v|^2}{2}}.
\]

Let \(f\) be a bounded, measurable function on \(\mathbb{R}\).

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**The Segal-Bargmann** transform (or S-transform) of \( f \) is the function \( S_f \) defined by:

\[
S_f(t) = \int_{\mathbb{R}} f(t + v) e^{-\frac{v^2}{2\sqrt{2\pi}}} dv = \int_{\mathbb{R}} f(v) e^{tv} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2\sqrt{2\pi}}} dv.
\]

(6.67)

Let \( f \in L^2(\mathbb{R}^n, \mu_{\mathbb{R}^n}) \).

The **S-transform** of \( f \) is defined by:

\[
S_f(x) = \int_{\mathbb{R}^n} f(y + x) e^{-\frac{|y|^2}{2}} \frac{1}{(\sqrt{2\pi})^{\frac{n}{2}}} dy = \int_{\mathbb{R}^n} f(y) e^{\langle x, y \rangle - \frac{|y|^2}{2}} \frac{e^{-\frac{x^2}{2}}}{(\sqrt{2\pi})^{\frac{n}{2}}} dy.
\]

(6.68)

This map has a holomorphic extension to a function, also denoted \( S_f \), on \( \mathbb{C}^n \). The Segal Bargmann Theorem says that the map:

\[
S : L^2 \left( \mathbb{R}^n; \frac{1}{(\sqrt{2\pi})^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}} dx \right) \to H \left( \mathbb{C}^n; \frac{1}{\pi^{\frac{n}{2}}} e^{-|z|^2} |dz| \right)
\]

\[
f \mapsto Sf
\]

is a unitary isomorphism, where \( H \left( \mathbb{C}^n; \frac{1}{\pi^{\frac{n}{2}}} e^{-|z|^2} |dz| \right) \) is the Hilbert space of holomorphic functions on \( \mathbb{C}^n \), square-integrable with respect to the Gaussian measure \( \frac{1}{\pi^{\frac{n}{2}}} e^{-|z|^2} |dz| \), and \( dz \) is the Lebesgue measure on \( \mathbb{C}^n \).

Also, this unitary isomorphism is meaningful when \( \mathbb{R}^n \) is replaced by an infinite dimensional linear space equipped with an appropriate Gaussian measure.

Let \( f \) be a bounded, measurable function on \( (\mathbb{R}^\infty, \mathcal{F}, \mu) \), with \( \mathcal{F} \) the product \( \sigma \)-algebra, and \( \mu \) the product Gaussian measure. Then the S-transform of \( f \) is the function \( Sf \) defined by:

\[
Sf(k) = \int_{\mathbb{R}^\infty} fe^{\hat{k} \cdot \frac{|y|^2}{2}} d\mu = \int_{\mathbb{R}^\infty} f(x + \hat{k}) d\mu(x),
\]

(6.69)

where \( \hat{k} \in \mathbb{R}^\infty \) is the coordinates vector

\[
\hat{k} = (\langle k, e_j \rangle)_{j \geq 1},
\]

and \( \hat{k}(x) = \langle k, x \rangle \) with \( k \in V \).
The $S$-transform, and its inversion, has been well-studied in the literature, see [2] for infinite dimensional case, and [7] for finite dimensional case.

**Theorem 6.26.** Let $f$ be a bounded, measurable function on $\Omega$. Then $f$ can be recovered from $Gf$ by inverting the $S$-transform in the following formula:

$$(SF_u)(t) = (Sf)(tu)$$

where, for each unit vector $u \in V$, $F_u$ is a function on $\mathbb{R}$ given by:

$$F_u(r) = (Gf)(ru + u^\perp)$$

**Proof.** Let $u \in V$, with $|u| = 1$, and

$$F_u : \mathbb{R} \rightarrow \mathbb{R}$$

$$F_u(r) = (Gf)(u^\perp + ur), \ r \in \mathbb{R}$$

Then we have:

$$(SF_u)(t) = \int_{\mathbb{R}} F_u(t + r) \ d\mu_{\mathbb{R}}(r)$$

$$= \int_{\mathbb{R}} F_u(t + r) e^{-\frac{r^2}{2}} \frac{dr}{\sqrt{2\pi}}$$

$$= \int_{\mathbb{R}} F_u(r) e^{-\frac{(r - t)^2}{2}} \frac{dr}{\sqrt{2\pi}}$$

$$= \int_{\mathbb{R}} F_u(r) e^{-\frac{r^2}{2} + r^2} e^{-\frac{t^2}{2}} e^{-\frac{rt}{2}} \frac{dr}{\sqrt{2\pi}}$$

$$= \int_{\mathbb{R}} (Gf)(u^\perp + ur) e^{-\frac{r^2}{2} + r^2} e^{-\frac{t^2}{2}} e^{-\frac{rt}{2}} \frac{dr}{\sqrt{2\pi}}$$

$$= \int_{\mathbb{R}} \left[ \int_{\Omega} \left| \int_{\mu_{u^\perp + ur}} f \right| \ d\mu \right] e^{-\frac{r^2}{2} + r^2} e^{-\frac{t^2}{2}} e^{-\frac{rt}{2}} \frac{dr}{\sqrt{2\pi}}$$

$$= \int_{\Omega} f e^{i\hat{u}_t - \frac{\|u\|^2}{2}} d\mu$$

$$= \int_{\Omega} f e^{i\hat{u}_t - \frac{\|u\|^2}{2}} d\mu$$

$$= (Sf)(tu).$$

(6.72)
So this theorem allows to find $f$ from equation (6.67), by inverting the $S$-transform.

Note that, in particular, $G$ is injective, if $Gf = Gh$ then $f = h$, $\mu$-almost-everywhere. We observe also that the preceding result allows the possibility of defining the conditional expectations $Gf(\xi)$ of a distribution $f$, by inverting the $S_R$-transform.
References


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