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PERVERSE POISSON SHEAVES ON THE NILPOTENT CONE

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
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Abstract

For a reductive complex algebraic group, the associated nilpotent cone is the variety of nilpotent elements in the corresponding Lie algebra. Understanding the nilpotent cone is of central importance in representation theory. For example, the nilpotent cone plays a prominent role in classifying the representations of finite groups of Lie type. More recently, the nilpotent cone has been shown to have a close connection with the affine flag variety and this has been exploited in the Geometric Langlands Program.

We make use of the following important fact. The nilpotent cone is invariant under the coadjoint action of G on the dual Lie algebra and admits a canonical Poisson structure which is compatible in a strong way with the action of G. We exploit this connection to develop a theory of perverse sheaves on the nilpotent cone that is suitable for the G-equivariant Poisson setting. Building on the work of Beilinson–Bernstein–Deligne and Deligne–Bezrukavnikov, we define a new category, the equivariant Poisson derived category and endow it with a new semiorthogonal filtration, the perverse Poisson t-structure. In order to construct the perverse Poisson t-structure, we also prove an axiomatized gluing theorem for semiorthogonal filtrations in the general setting of triangulated categories which generalizes the construction of the perverse coherent sheaves of Deligne–Bezrukavnikov.

Chapter 1
Introduction

If G is a reductive complex algebraic group with Lie algebra $\mathfrak{g}$, we can identify the variety $\mathcal{N}$ of nilpotent elements of $\mathfrak{g}$ with a subvariety of the dual Lie algebra $\mathfrak{g}^*$ via the Killing form. Understanding the nilpotent cone is of central importance in representation theory. For example, the nilpotent cone plays a primary role in the Springer correspondence (see [BM]) and in classifying the representations of finite groups of Lie type (see, for example, [Lus1]). The nilpotent cone has a close connection with the affine flag variety and therefore many results concerning the nilpotent cone have implications on the Geometric Langlands Program also (see [Bez4]).

The nilpotent cone is attractive both for its importance in representation theory and because it is a variety with particularly favorable properties. Most importantly for us, $\mathcal{N}$ is invariant under the coadjoint action of G on $\mathfrak{g}^*$ and admits a canonical Poisson structure which is compatible in a strong way with the action of G. Since the symplectic leaves of the Poisson structure coincide with the coadjoint orbits of $\mathcal{N}$, we are able to consider the equivariant and Poisson structures simultaneously in order to develop a theory of perverse Poisson sheaves. Building on the work of Beilinson–Bernstein–Deligne ([BBD]) and Deligne–Bezrukavnikov ([Bez1]), we define a new category, the equivariant Poisson derived category, and endow it with a new semiorthogonal filtration, the perverse Poisson t-structure. The heart of this t-structure is what we will call the category of perverse Poisson sheaves.

In order to construct the perverse Poisson t-structure, we prove an axiomatized gluing theorem for semiorthogonal filtrations in the general setting of triangulated categories (Theorem 3.4) which generalizes the construction of the perverse coherent sheaves of Deligne–Bezrukavnikov (Theorem 3.12). This novel construction of perverse coherent sheaves has the advantage that it is analogous to the construction of classical perverse sheaves found in [BBD]. Because of the unified approach to perverse sheaves (classical, coherent and Poisson), we are able to find a connection between irreducible local systems, certain vector bundlesand simple perverse Poisson sheaves in Theorem 4.49.

We now outline the organization of the thesis. We begin in Chapter 2 by reviewing the ideas that will serve as the foundation of this work. We attempt to strike a balance between providing a context for the current work and focusing on aspects of the general theory that apply to our particular setting. This results in including material which some would consider trivial and excluding material which is tangentially related but would be relevant to the discussion. Section 1 contains introductory material on triangulated categories, which were first described in the thesis of Verdier ([SGA $4\frac{1}{2}$, Catégories dérivées, Etat 0]) and are in many ways the most natural setting to discuss homological algebra. The definitions of the derived category of an abelian category and that of a t-structure are particularly important, as these are central to the definition of perverse Poisson sheaves. In Sections 2 and 3, we then review the construction and basic properties of perverse sheaves both in the classical (constructible) setting found in [BBD] and in the setting of coherent sheaves. The latter type

were described in a preprint by Bezrukavnikov ([Bez1]), though these results were known for some time by Deligne.

Since the presence of a G-action will be extremely important for us, we recall the definitions of equivariant sheaves in Section 4 and show that for an affine scheme, the corresponding equivariant module categories have enough injectives. This technical fact is important since it is one of the main ingredients in constructing the Poisson derived sheaf functors in Chapter 4. Sections 5 and 6 contain the background material on Poisson schemes and modules. Our approach to Poisson structures is entirely algebraic in nature. We review the generalizations of Poisson algebras to the setting of schemes developed by Kaledin ([Kal2], [Kal1]), Ginzburg–Kaledin ([GK]), and Polishchuck ([Pol]). We then discuss the appropriate notion of a compatible G-action on a Poisson scheme, that of a Hamiltonian action. This is followed by a review of the construction of the Poisson enveloping algebra, which was introduced by Oh ([Oh]). In the case of an affine scheme, this will allow us to view Poisson equivariant sheaves as ordinary equivariant modules over the Poisson enveloping algebra in much the same way that one can view representations of a Lie algebra as modules over its universal enveloping algebra. Finally, we introduce formally our primary object of study, the nilpotent cone in the dual of the Lie algebra associated to a reductive algebraic group G. In particular, we show how the concepts discussed above apply in this case. We describe the natural Poisson structure (due to Kostant–Kirillov), discuss the structure of the coordinate ring of $\mathcal{N}$, and show that the coadjoint action of G is Hamiltonian.

We then proceed to the main chapters of the thesis which contain original work. In Chapter 3, we develop a method of gluing semiorthogonal filtrations in triangulated categories. This will serve as a first step in the direction of constructing the perverse Poisson t-structure. Section 1 involves developing a sufficient condition for transferring a semiorthogonal filtration across a triangulated functor. This will provide one ingredient for the main theorem in this chapter—Theorem 3.4—which axiomatizes sufficient conditions that allow gluing of semiorthogonal filtrations in a triangulated category. As a first application of the theorem, in Section 3 we give a new construction of the perverse coherent t-structure by iteratively gluing shifts of the standard t-structure on the orbits. This is analogous to the construction of classical perverse sheaves in [BBD], but it was previously unknown that a similar construction would work in the coherent setting. The main obstacle is that many of the functors used in the classical case take values in larger categories and so are not available for use in the bounded derived category of coherent sheaves. The gluing theorem allows us to overcome this problem by separating the geometry from the homological algebra. By transferring a certain amount of the complexity to the homological algebra, we are then able to use non-functorial geometric methods to construct perverse coherent sheaves.

In the final chapter, we develop a theory of Poisson equivariant sheaves and Poisson sheaf functors. In Section 1, we define the equivariant Poisson derived category. We prove fundamental lemmas in Section 2 on the existence of derived Poisson sheaf functors and determine a sufficient condition for the existence of an equivariant Poisson dualizing complex. In Section 3, we then pass to the specific case of the nilpotent cone where we are able to develop the theory further. For technical reasons, we also pass to a particular quotient category of the equivariant Poisson derived category. It is in this setting that we are then able to give a construction of the perverse Poisson t-structure and its heart, the category

of perverse Poisson sheaves. Finally, in Section 4 we build on the work of Polishchuk ([Pol]) which shows how to realize certain vector bundles as Poisson sheaves. Specifically, on a G-orbit, every equivariant vector bundle with a flat connection arises from an equivariant local system, and we are then able to use the framework of perverse Poisson sheaves to gain information about the Poisson analogues of Green functions for simple perverse Poisson sheaves.

Chapter 2
Preliminaries

In order to provide both a context for this work and a foundation upon which later chapters will be based, it will be advantageous for us to review some preliminary concepts. We will assume the basics of category theory, the representation theory of reductive complex algebraic groups (and reductive complex Lie algebras), algebraic geometry, and homological algebra. We refer the reader unfamiliar with these topics to [Lan], [Hum], [Har2], and [GM1], respectively. In addition, we will provide references throughout the text to related works in the literature.

We begin by discussing triangulated categories and semiorthogonal filtrations. Of particular importance is the notion of a t-structure on a triangulated category. In one sense, this entire thesis is written to explain the construction of a single t-structure on a certain triangulated category. We then proceed to a discussion of perverse sheaves, where we review the construction of both classical (topological, constructible) perverse sheaves and perverse coherent sheaves. As the title of the thesis suggests, our goal is to extend these ideas in a new direction. Specifically, we would like to consider the case when we have both an equivariant structure and a Poisson structure at our disposal. Hence we review the basic theory of equivariant sheaves and Poisson schemes. Finally, we provide background information on the nilpotent cone, which will serve as our primary example of a Poisson variety.

2.1 Triangulated Categories

A triangulated category is an additive category that retains enough of the structure of an abelian category to allow us to use the basics of homological algebra. Originally defined in the thesis of Verdier (partially reprinted as *Catégories dérivées, Etat* 0 in [SGA $4\frac{1}{2}$]), triangulated categories have provided a fertile context for research over the last half century. A more modern treatment is [Nee], where one can find a much more general approach than the classical literature. We will not have use for this generality, however, since the triangulated categories that we will be concerned with will arise in the classical way (as derived categories of abelian categories). We begin by defining a triangulated category.

Definition 2.1. A *triangulated category* is an additive category $\mathcal{D}$ with an autoequivalence $[1] : \mathcal{D} \to \mathcal{D}$ and a collection of diagrams (called distinguished triangles)

$$\{A \to B \to C \to A[1]\}$$

such that the following axioms hold

(TR1) (a) $A \xrightarrow{id} A \to 0 \to A[1]$ is a distinguished triangle.

(b) Any triangle $A \to B \to C \to A[1]$ which is isomorphic to a distinguished triangle is distinguished.

(c) Any morphism $A \to B$ can be completed to a distinguished triangle

$$A \to B \to C \to A[1].$$

(TR2) The triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$ is a distinguished triangle if and only if $B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$ is a distinguished triangle.

(TR3) Given a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ A' & \longrightarrow & B' \end{array}$$

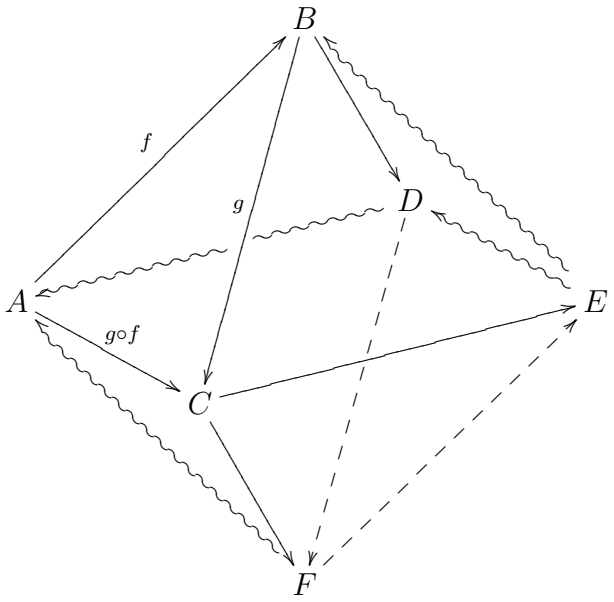
there exists a morphism h such that the square can be completed to a morhism of distinguished triangles

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & A'[1] \end{array}$$

(TR4) Given a composition $(g \circ f) : A \xrightarrow{f} B \xrightarrow{g} C$, consider the following three distinguished triangles (guaranteed by (TR1)(c))

$$A \xrightarrow{f} B \to D \to A[1]$$
$$B \xrightarrow{g} C \to E \to B[1]$$
$$A \xrightarrow{g \circ f} C \to F \to A[1].$$

Then there are morphisms such that $D \to F \to E \to D[1]$ is a distinguished triangle and these four distinguished triangles fit into an octahedral diagram



where the dashed arrows represent new morphisms, a morphism $X \rightsquigarrow Y$ means $X \to Y[1]$, all triangles containing exactly one such a morphism are distinguished, and all other triangles commute.

The last three axioms have common descriptions. Namely, (TR2) is the rotation axiom, (TR3) is called square completion, and (TR4) is often referred to as the octahedral axiom. It turns out that these axioms are not quite independent as (TR3) is a consequence of the others. We include it both for historical reasons and because it is a useful property. Many arguments involving distinguished triangles amount to writing down a diagram of distinguished triangles and then using these axioms to manipulate it into a diagram of distinguished triangles having a desired property.

Definition 2.2. If $\mathcal{D}'$ is another triangulated category, a *triangulated functor* $F : \mathcal{D} \to \mathcal{D}'$ is a functor which commutes with the respective shifts on $\mathcal{D}$ and $\mathcal{D}'$ and takes distinguished triangles to distinguished triangles.

These are sometimes also called *exact functors* in the literature since an exact functor (in the sense that it takes exact sequences to exact sequences) between abelian categories gives rise in a canonical way to a triangulated functor on their derived categories (see Definition 2.15 below). We reserve the term exact functor for a functor of abelian categories.

Definition 2.3. If $A, C \in \mathcal{D}$, then B is an *extension of C by A* if there is a distinguished triangle $A \to B \to C \to A[1]$. For any subcategory $\mathcal{A}$ of $\mathcal{D}$, the *extension stable subcategory* of $\mathcal{A}$ is the smallest strictly full subcategory of $\mathcal{D}$ which is stable under extensions and will be denoted by $E_{\mathcal{D}}(\mathcal{A})$.

Definition 2.4. Let $\mathcal{D}$ be a triangulated category. A strictly full triangulated subcategory $\mathcal{A}$ is called *thick* if it is also closed under direct summands.

The following notation is a convenient way to compactly write statements about diagrams of distinguished triangles.

Definition 2.5. Let $\mathcal{A}, \mathcal{B}$ be subcategories of $\mathcal{D}$, and $\mathcal{C}$ a triangulated category of $\mathcal{D}$. Also let $[A], [B]$ denote the isomorphism classes of the objects $A, B \in \mathcal{C}$. Then we define an operation on isomorphism classes

$$[A] *_{\mathcal{C}} [B] = [C]$$

when there is a distinguished triangle $A \to C \to B \to$ in $\mathcal{C}$. It is easy to check that this is a well defined operation on isomorphism classes. Similarly on subcategories, we define

$$\mathcal{A} *_{\mathcal{C}} \mathcal{B} = \{C \in \mathcal{C} \mid A \to C \to B \to \text{ is distinguished for some } A \in \mathcal{A}, B \in \mathcal{B}\}.$$

Lemma 2.6. *The operation $*_{\mathcal{C}}$ is associative.*

Proof. It suffices to show associativity in the case of isomorphism classes and here the condition follows directly from the octahedral axiom. □

Note that the subscript is important when one is working with several triangulated categories simultaneously. For example, if $\mathcal{A} \subset \mathcal{B} \subset \mathcal{D}$ is a chain of proper subcategories of a triangulated category $\mathcal{D}$, then in general $\mathcal{A} *_{\mathcal{B}} \mathcal{A} \neq \mathcal{A} *_{\mathcal{D}} \mathcal{A}$.

Definition 2.7. Let $\mathcal{A}$ be a collection of objects $\mathcal{A} \subset \mathrm{Obj}(\mathcal{D})$. For each $n \geq 1$ define strictly full subcategories

$$\mathcal{A}^n_{\mathcal{D}} = \underbrace{\mathcal{A} *_{\mathcal{D}} \cdots *_{\mathcal{D}} \mathcal{A}}_{n \text{ terms}}$$

and then note that

$$E_{\mathcal{D}}(\mathcal{A}) = \bigcup_{n \geq 1} \mathcal{A}^n_{\mathcal{D}}.$$

When the category $\mathcal{D}$ is clear, we will drop it from the notation.

Definition 2.8. For any collection of objects $\mathcal{A} \subset \mathrm{Obj}(\mathcal{D})$, the *triangulated subcategory of* $\mathcal{D}$ *generated by* $\mathcal{A}$ is the smallest strictly full triangulated subcategory of $\mathcal{D}$ containing $\mathcal{A}$. If $\mathcal{A}$ is closed under shifts in both directions, then $E_{\mathcal{D}}(\mathcal{A})$ is the triangulated category generated by $\mathcal{A}$.

We now come to one of the primary objects that we will study.

Definition 2.9. A t-structure on a triangulated category $\mathcal{D}$ is a pair of strictly full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ such that

(i) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 1} \subset \mathcal{D}^{\geq 0}$

(ii) $\mathrm{Hom}(A, B) = 0$ for $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$

(iii) For any $A \in \mathcal{D}$, there exists a distinguished triangle $B \to A \to C \to B[1]$ with $B \in \mathcal{D}^{\leq 0}$ and $C \in \mathcal{D}^{\geq 1}$

where $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$.

We call the intersection $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ the *heart* (fr. *coeur*) of the t-structure. Because of its similarity in both form and meaning, the English word *core* is sometimes also found in the literature.

A fundamental result of Beilinson–Bernstein–Deligne shows why t-structures play such an important role in the structure of a triangulated category.

Theorem 2.10. *The heart $\mathcal{C}$ of any t-structure on a triangulated category $\mathcal{D}$ is an abelian category. Moreover, there is a cohomological δ-functor ${}^tH^0 : \mathcal{D} \to \mathcal{C}$.*

The following two definitions are of additional structures that one can place on a triangulated category. These will not play an important role in our study, but several statements that we will need concerning t-structures can be stated and proved easily also for these structures. The basic theory of baric structures can be found in [AT].

Definition 2.11. A *baric structure* on a triangulated category $\mathcal{C}$ is a pair of collections of thick subcategories $\left(\left\{\mathcal{C}^{\leq w}\right\}, \left\{\mathcal{C}^{\leq w}\right\}\right)_{w \in \mathbb{Z}}$ satisfying

(i) $\mathcal{C}^{\leq w} \subset \mathcal{C}^{\leq w+1}$ and $\mathcal{C}^{\geq w+1} \subset \mathcal{C}^{\geq w}$ for all $w \in \mathbb{Z}$.

(ii) $\mathrm{Hom}(A, B) = 0$ for all $A \in \mathcal{C}^{\leq w}$ and $B \in \mathcal{C}^{\geq w+1}$.

(iii) For any object $X \in \mathcal{C}$, there is a distinguished triangle

$$A \to X \to B \to A[1]$$

with $A \in \mathcal{C}^{\leq w}$ and $B \in \mathcal{C}^{\geq w+1}$.

Definition 2.12. A *co-t-structure* on a triangulated category $\mathcal{C}$ (see [Pau]) is a pair of strictly full subcategories $\left(\mathcal{C}^{\leq 0}, \mathcal{C}^{\geq 0}\right)$ such that

(i) $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$ are closed under direct summands

(ii) $\mathcal{C}^{\leq 0} \subset \mathcal{C}^{\leq 1}$ and $\mathcal{C}^{\geq 1} \subset \mathcal{C}^{\geq 0}$

(iii) $\mathrm{Hom}(A, B) = 0$ for all $A \in \mathcal{C}^{\leq 0}$ and $B \in \mathcal{C}^{\geq 1}$

(iv) For any object $X \in \mathcal{C}$, there is a distinguished triangle

$$A \to X \to B \to A[1]$$

with $A \in \mathcal{C}^{\leq 0}$ and $B \in \mathcal{C}^{\geq 1}$

where $\mathcal{C}^{\leq n} = \mathcal{C}^{\leq 0}[n]$ and $\mathcal{C}^{\geq n} = \mathcal{C}^{\geq 0}[n]$.

The notation for co-t-structures here is different from that in [Pau] and elsewhere ([Bon], for example). We use the current notation (nearly that of the weight structures in [Wil]) to shorten many statements involving both t-structures and co-t-structures. This makes the exposition cleaner, but also obscures the differences between t-structures and co-t-structures. Most notably, though the definitions appear almost identical, the fact that $\mathcal{C}^{\leq n}$ and $\mathcal{C}^{\geq n}$ are defined differently is significant.

Definition 2.13. A *semiorthogonal filtration* on a triangulated category $\mathcal{D}$ is a family of pairs of strictly full subcategories $\left(\left\{\mathcal{D}^{\leq n}\right\}, \left\{\mathcal{D}^{\leq n}\right\}\right)_{n \in \mathbb{Z}}$ satisfying

(i) $\mathcal{D}^{\leq n} \subset \mathcal{D}^{\leq n+1}$ and $\mathcal{D}^{\geq n+1} \subset \mathcal{D}^{\geq n}$.

(ii) $\mathrm{Hom}(A, B) = 0$ for all $A \in \mathcal{D}^{\leq n}$ and $B \in \mathcal{D}^{\geq n+1}$.

Notice in particular that t-structures, co-t-structures and baric structures are semiorthogonal filtrations. This is a weaker, but clearly related idea to that of semiorthogonal decompositions, as found in [Orl]. By semiorthogonal filtration, we will always mean one of the three structures defined above.

We now recall a theorem that allows us to take the quotient of triangulated category by a thick subcategory. This was first proved in the thesis of Verdier, which was published as the last chapter in SGA $4\frac{1}{2}$, [SGA $4\frac{1}{2}$]. For a more recent and expansive treatment of the subject, we refer the reader to [Kra] or [Nee], where the authors refer to what follows as Verdier localization. In fact, the theory of localization in triangulated categories is a substantial area of research that has many recent developments. We state here only a simple version of localization that will suffice for our purposes. The following proposition, whose proof can be found in [Nee, §2.1], shows the naturality of the definition of a thick subcategory and provides a method for ensuring that functors descend to a quotient.

Proposition 2.14. *Let $\mathcal{D}$ be a triangulated category and $\mathcal{A}$ a thick subcategory. Then there is a triangulated category $\mathcal{D}/\mathcal{A}$ and a universal functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{A}$ with the property that any triangulated functor $F : \mathcal{D} \to \mathcal{D}'$ such that $F(A) \simeq 0$ for all $A \in \mathcal{A}$ factors through $\mathcal{D}/\mathcal{A}$ as $F = F' \circ Q$, where $F' : \mathcal{D}/\mathcal{A} \to \mathcal{D}'$ is triangulated.*

One of the primary uses for triangulated categories (and the historical reason for their invention) is that of the derived category of an abelian category. Since the focus of our study will be on certain derived categories, we give the necessary background information here.

Definition 2.15. Let $\mathcal{A}$ be an abelian category and $\mathrm{Kom}(\mathcal{A})$ the category of cochain complexes of objects of $\mathcal{A}$ with morphisms the cochain maps. Define the *derived category of* $\mathcal{A}$, denoted $\mathcal{D}(A)$ to be the category obtained from $\mathrm{Kom}(\mathcal{A})$ by localizing at the quasi-isomorphisms (this is equivalent to taking the quotient of $\mathrm{Kom}(\mathcal{A})$ by the subcategory of complexes quasi-isomorphic to the 0 complex). Then $\mathcal{D}(A)$ is a triangulated category where [1] is given by shifting the degrees of the cochain complex and the distinguished triangles come from taking mapping cones of morphisms. For details on this construction see [GM1].

It is difficult to overstate the importance of derived categories and the impact that they have had on representation theory over the last half century.

Definition 2.16. Let $\mathcal{A}$ be an abelian category and $\mathcal{D}(\mathcal{A})$ its derived category. Given any complex $A^\bullet$

$$\cdots \xrightarrow{\delta^{i-3}} A^{i-2} \xrightarrow{\delta^{i-2}} A^{i-1} \xrightarrow{\delta^{i-1}} A^i \xrightarrow{\delta^i} A^{i+1} \xrightarrow{\delta^{i+1}} A^{i+2} \xrightarrow{\delta^{i+2}} \cdots$$

of objects of $\mathcal{A}$, we can form the i^{th} cohomology object

$$H^i(A^\bullet) = \ker \delta^i / \mathrm{im}\ \delta^{i-1}.$$

Now define two strictly full subcategories of $\mathcal{D}(\mathcal{A})$ by

$$\begin{aligned} {}^{\mathrm{std}}\mathcal{D}^{\leq 0} &= \{A^\bullet \in \mathcal{D} \mid H^k(A^\bullet) = 0 \text{ for all } k > 0\} \\ {}^{\mathrm{std}}\mathcal{D}^{\geq 0} &= \{A^\bullet \in \mathcal{D} \mid H^k(A^\bullet) = 0 \text{ for all } k < 0\}. \end{aligned}$$

This defines a t-structure on $\mathcal{D}(\mathcal{A})$, which we will call the *standard t-structure*.

The following is a nontrivial theorem is due to Verdier and will be used extensively in constructing the categories of perverse sheaves below.

Theorem 2.17. *The heart of the standard t-structure is isomorphic to $\mathcal{A}$.*

Remark 2.18. Let $\mathcal{A}$ be an abelian category and $\mathcal{D}(\mathcal{A})$ its derived category. We can think of giving a t-structure on $\mathcal{D}(\mathcal{A})$ as a mechanism for constructing an abelian category by taking the heart of the t-structure. By Theorem 2.17, we see that we can recover the original abelian category $\mathcal{A}$ in this manner. As we will see in the case of perverse sheaves below, the heart of a non-standard t-structure can be highly nontrivial and worthy of investigation in its own right.

2.2 Classical Perverse Sheaves

Perverse sheaves have had a profound impact on mathematics since their discovery in the early 1980's. The description that we will give is almost purely algebraic, but this idea has far reaching implications in many fields. For instance, a recent result of Mirković and Vilonen ([MV]) is the geometric Satake equivalence, which says that the category of perverse sheaves on the affine Grassmannian associated to a reductive algebraic group is equivalent to the category of representations of the Langlands dual group. Perverse sheaves also appear as one ingredient in a construction by Borho and MacPherson of the Springer correspondence (see [BM]). This correspondence assigns irreducible local systems on nilpotent orbits to representations of the Weyl group and is central to understanding the representation theory of finite groups of Lie type. In a similar vein, Lusztig and others have used perverse sheaves to study quiver varieties, for example, to get canonical bases in the quantized enveloping algebras associated to a quiver (see [Lus2]).

In this section, we briefly review the construction of perverse sheaves found in [BBD] and state their basic properties. We will work in the setting of sheaves of complex vector spaces on a complex variety for clarity, though the construction is valid and important in positive characteristic. The category of perverse sheaves on a variety X is related to the category of sheaves (of vector spaces) on X, and in many ways is easier to work with. For example, the category of perverse sheaves is artinian (every object has finite length). For basic definitions about sheaves and sheaf functors, see [KS1] or [Dim]. An English alternative to [BBD] for much of this material is [GM1].

Let X be a complex algebraic variety and suppose we are given a finite stratification of X by locally closed strata

$$X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n,$$

such that the closure of each stratum is a union of strata and each stratum is a complex manifold. For example, we could take the Whitney stratification (see [Whi]). For technical reasons, let us assume for the remainder of this section that the stratification is a Whitney stratification.

Definition 2.19. A sheaf $\mathcal{F}$ on X is called *constructible* (relative to the given stratification) if the restriction $\mathcal{F}|_{X_i}$ is a locally constant sheaf on X_i for $1 \leq i \leq n$.

Theorem 2.20. *Let $i : Z \hookrightarrow X$ be a closed subvariety with open complement $j : U \hookrightarrow X$. Let $\mathcal{D}^b_c(X), \mathcal{D}^b_c(U), \mathcal{D}^b_c(Z)$ denote the bounded derived categories of sheaves of complex vector spaces with constructible cohomology sheaves on X, U and Z, respectively. Then we have the following diagram of functors*

$$\mathcal{D}^b_c(Z) \underset{i^{-1}}{\overset{i^!}{\rightleftarrows}} \mathcal{D}^b_c(X) \underset{i_*}{\overset{j_!}{\rightleftarrows}} \mathcal{D}^b_c(U)$$

where

- $(i^{-1}, i_*), (i_*, i^!), (j^{-1}, j_*)$, *and* $(j_!, j^{-1})$ *are adjoint pairs such that the natural morphisms*

$$i^{-1}i_*\mathcal{F} \to \mathcal{F} \to i^!i_*\mathcal{F}$$
$$j^{-1}j_*\mathcal{F} \to \mathcal{F} \to j^{-1}j_!\mathcal{F}$$

  are all isomorphisms.

- $j^*i_* = 0$.

- *there exist canonical morphisms* $i_*i^{-1}\mathcal{F} \to j_!j^{-1}\mathcal{F}[1]$ *and* $j_*j^{-1}\mathcal{F} \to i_*i^!\mathcal{F}[1]$ *such that*

$$j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^{-1}\mathcal{F} \to j_!j^*\mathcal{F}[1]$$
$$i_*i^!\mathcal{F} \to \mathcal{F} \to j_*j^{-1}\mathcal{F} \to i_*i^!\mathcal{F}[1]$$

  are distinguished triangles in $\mathcal{D}^b_c(X)$.

Theorem 2.21. *Let* $\left(\mathcal{D}^{\leq 0}_Z, \mathcal{D}^{\geq 0}_Z\right)$ *and* $\left(\mathcal{D}^{\leq 0}_U, \mathcal{D}^{\geq 0}_U\right)$ *be t-structures on* $\mathcal{D}^b_c(Z)$ *and* $\mathcal{D}^b_c(U)$*, respectively. Then there is a unique t-structure* $\left(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0}\right)$ *on* $\mathcal{D}^b_c(X)$ *such that*

$$\mathcal{D}^{\leq 0} = \left\{\mathcal{F} \in \mathcal{D}^b_c(X) \mid j^{-1}\mathcal{F} \in \mathcal{D}^{\leq 0}_U \text{ and } i^{-1}\mathcal{F} \in \mathcal{D}^{\leq 0}_Z\right\}$$
$$\mathcal{D}^{\geq 0} = \left\{\mathcal{F} \in \mathcal{D}^b_c(X) \mid j^{-1}\mathcal{F} \in \mathcal{D}^{\geq 0}_U \text{ and } i^!\mathcal{F} \in \mathcal{D}^{\geq 0}_Z\right\}.$$

In fact, the theorem proved in [BBD] is much more general than this. Any three triangulated categories and six functors satisfying the conditions of Theorem 2.20 comprise a set of *gluing data*, allowing us to glue t-structures as in Theorem 2.21.

Inductively, Theorem 2.21 enables us to choose arbitrary t-structures on (the derived category of sheaves on) the strata of a stratified complex variety and obtain a unique t-structure on the entire variety. We now want to choose a particular t-structure for each stratum that is relatively simple to work with and apply the theorem to get a highly nontrivial t-structure on the variety.

Definition 2.22. Let $\mathcal{S}$ be the set of strata of X. A *perversity* in its most basic form is just a function $p : S \to \mathbb{Z}$. We say that a perversity is *monotone* if whenever a stratum S is contained in the closure of another stratum T, we have $p(T) \leq p(S)$.

The *dual perversity* is the function $\overline{p} : \mathcal{S} \to \mathbb{Z}$ given by $\overline{p}(S) = -\dim_{\mathbb{R}} S - p(S)$. A perversity is *comonotone* if the dual perversity is monotone. The notions of *strictly monotone* and *strictly comonotone* are defined similarly, replacing the inequality with a strict inequality.

The most common perversity by far is the following. It was originally considered by Goresky and MacPherson in [GM2] and [GM3].

Definition 2.23. Define a perversity called the *middle perversity* by setting

$$p(S) = -\frac{1}{2}\dim_{\mathbb{R}} S.$$

From the definitions, we see that this perversity is strictly monotone and comonotone; in fact, $\overline{p} = p$.

We will assume for the remainder of this section that we are working with the middle perversity. For the general setting, simply replace any references to the dimension of a stratum with the value of the perversity on the stratum.

Definition 2.24. For each stratum $i_S : S \hookrightarrow X$ in $\mathcal{S}$, define the perverse t-structure on $\mathcal{D}_c^b(S)$ to be

$$
\begin{aligned}
{}^p\mathcal{D}_S^{\leq 0} &= {}^{\text{std}}\mathcal{D}_S^{\leq 0}[p(S)] \\
{}^p\mathcal{D}_S^{\geq 0} &= {}^{\text{std}}\mathcal{D}_S^{\geq 0}[p(S)].
\end{aligned}
$$

Using Theorem 2.21, we then obtain a t-structure on $\mathcal{D}_c^b(X)$ given by

$$
\begin{aligned}
{}^p\mathcal{D}^{\leq 0} &= \left\{ \mathcal{F} \in \mathcal{D}_c^b(X) \mid i_S^{-1}\mathcal{F} \in {}^p\mathcal{D}_S^{\leq 0} \text{ for all } S \in \mathcal{S} \right\} \\
{}^p\mathcal{D}^{\geq 0} &= \left\{ \mathcal{F} \in \mathcal{D}_c^b(X) \mid i_S^{!}\mathcal{F} \in {}^p\mathcal{D}_S^{\geq 0} \text{ for all } S \in \mathcal{S} \right\}.
\end{aligned}
$$

We call this the *perverse t-structure* and objects in its heart $\mathcal{M}(X) = {}^p\mathcal{D}^{\leq 0} \cap {}^p\mathcal{D}^{\geq 0}$ are called *perverse sheaves.*

Notice that perverse sheaves on a single stratum are particularly easy to describe. Since the perverse t-structure on a stratum S is a shift of the standard t-structure and objects in the heart of the standard t-structure are just constructible sheaves, we see that a perverse sheaf on S must be a local system $\mathcal{L}$ concentrated in degree $-\dim_{\mathbb{C}}(S)$.

Definition 2.25. Let $\mathcal{H}om$ be the sheaf-Hom (internal Hom) functor. A *dualizing complex* for a scheme X is an object $\omega_X \in \mathcal{D}_c^b(X)$ such that there is a natural isomorphism

$$\mathcal{F} \to \mathcal{H}om(\mathcal{H}om(\mathcal{F}, \omega_X), \omega_X)$$

for any $\mathcal{F} \in \mathcal{D}_c^b(X)$. We call the autoequivalence $\mathbb{D} : \mathcal{F} \mapsto \mathcal{H}om(\mathcal{F}, \omega_X)$ a *dualizing* functor.

Definition 2.26. Let $f : X \to \{pt\}$ be the map sending X to a point. For any complex variety, the object $\omega_X = f^!\mathbb{C}$ is a dualizing complex for X. We call the associated functor $\mathbb{D}$ the *Verdier dual.*

Lemma 2.27. *We have $\mathbb{D}\left({}^p\mathcal{D}^{\leq 0}\right) = {}^p\mathcal{D}^{\geq 0}$ and so Verdier duality takes perverse sheaves to perverse sheaves.*

Theorem 2.28. *For any stratum S, there is a fully faithful functor*

$$IC(S, -) : \mathcal{M}(S) \to \mathcal{M}(X).$$

Moreover, every simple object of $\mathcal{M}(X)$ is of the form $IC(S, \mathcal{L}[p(S)])$ for some stratum S and some irreducible local system $\mathcal{L}$.

Corollary 2.29. *The cohomology sheaves of a simple perverse sheaf are supported on the closure of a single stratum.*

Although in general $\mathcal{M}(X)$ is not a semisimple category (every object is not a direct sum of simple objects), it is always artinian. Therefore, if we understand the simple perverse sheaves we can use finite composition series to inductively study the category as a whole. Hence it is

an extremely important problem to understand the structure of the simple perverse sheaves. One way of studying a simple perverse sheaf $\mathcal{F} = IC(S, \mathcal{L}[p(S)]) \in \mathcal{M}(X)$ is by studying the restrictions of the cohomology sheaves of $\mathcal{F}$ to strata contained in the closure of S.

When the variety in question is the nilpotent cone of a reductive algebraic group stratified by G-orbits (in which case the category of perverse sheaves is actually semisimple), Shoji and Lusztig developed an algorithm to compute these restrictions (see [Lus1]). Thus, for a particular irreducible local system on an orbit C and a particular orbit C' in the closure of C, we have a complete description of the restrictions of the cohomology sheaves of $IC(C, \mathcal{L}[p(C)])$. There have been few results, however, that give general information about the restrictions of simple perverse sheaves to orbits. This is often considered to be a very difficult problem.

2.3 Perverse Coherent Sheaves

In this section we review the theory of perverse coherent sheaves first worked out by Deligne and communicated by Bezrukavnikov ([Bez1]). We include the results that are needed either to restate the construction of the perverse coherent t-structure and (for convenience of the reader) those that we will need to adapt to the Poisson setting later. This work is the starting point for many of the ideas that are included in this thesis. In particular, the results in Section 3.2 stemmed from an attempt to unravel the argument of the construction of the perverse coherent t-structure. Although relatively recent, perverse coherent sheaves have already proved useful. For example, they were used in [Bez2] to prove a conjecture of Lusztig and Vogan involving a bijection between dominant weights and pairs consisting of a nilpotent orbit and an irreducible representation of the centralizer of an element of that orbit. Bezrukavnikov again made use of perverse coherent sheaves in [Bez3] studying the cohomology of tilting modules over quantum groups.

We will use the following notation which, while different from that used in [Bez1], is consistent with the notation used in Chapters 2 and 3. Let X be a Noetherian scheme over a base scheme S and G an affine group scheme of finite type over S which is flat and Gorenstein. We denote by $\mathcal{C}_G(X)$ (resp. $\mathcal{Q}_G(X)$) the category of coherent (resp. quasicoherent) sheaves of $\mathcal{O}_X$-modules on X. We denote the bounded derived category of equivariant coherent sheaves on X by $\mathcal{D}^b_G(X)$. We will use $\underline{Y}$ for the underlying topological space of a scheme Y.

The primary difficulty in working with coherent sheaves rather than sheaves of vector spaces is that one does not have the six functors which comprise the gluing data. Indeed, the coherent functors i^* and $i^!$ do not take values in the subcategory of the derived category consisting of bounded complexes, the functor j_* does not in general preserve coherence, and the functor $j_!$ takes values in the category of pro-coherent sheaves (see [Del1]). To work around the problem of boundedness, we can work in the category of unbounded (on one side) quasicoherent sheaves and then use the following proposition when needed.

Lemma 2.30. *The categories $\mathcal{D}^b_G(X)$ and $\mathcal{D}^-_G(X)$ are equivalent to the respective subcategories of $\mathcal{D}^b(\mathcal{Q}_G(X))$ and $\mathcal{D}^-(\mathcal{Q}_G(X))$ consisting of complexes with coherent cohomology.*

Unfortunately, for an open inclusion $j : U \to X$, the functor $j^* = j^{-1}$ is not full, but nearly so. This will be enough for our purposes.

Lemma 2.31. *Let $j : U \to X$ be an open G-invariant subscheme.*

(a) For any morphism $f : \mathcal{F} \to \mathcal{G}$ in $\mathcal{D}^b_G(U)$, there exist objects $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{G}}$ in $\mathcal{D}^b_G(X)$ and a morphism $\tilde{f} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$ such that $\tilde{f}|_U \simeq f$.

(b) If $\mathcal{F}, \mathcal{G} \in \mathcal{D}^b_G(X)$ such that $f : \mathcal{F}|_U \simeq \mathcal{G}|_U$, then there exists $\mathcal{H} \in \mathcal{D}^b_G(X)$ and morphisms $g : \mathcal{G} \to \mathcal{H}$ and $h : \mathcal{F} \to \mathcal{H}$ so that restricting to U gives a commuting diagram of isomorphisms

$$\begin{array}{ccc} \mathcal{F}|_U & \xrightarrow{f|_U} & \mathcal{G}|_U \\ & {\scriptstyle h|_U}\searrow \quad \swarrow{\scriptstyle g|_U} & \\ & \mathcal{H}|_U & \end{array}.$$

Lemma 2.32. *Let $i : Z \hookrightarrow X$ be a closed subscheme*

(a) For any $\mathcal{F} \in \mathcal{D}^-_G(X)$ and $\mathcal{G} \in \mathcal{D}^+_G(X)$ we have an isomorphism

$$Hom(\mathcal{F}, i_* i^! \mathcal{G}) \simeq \varinjlim_{Z'} Hom(\mathcal{F}, i_{Z'*} i^!_{Z'} \mathcal{G})$$

where $i^!$ is the topological functor, $i^!_{Z'}$ is the coherent inverse image with supports and the limit is over all closed subschemes with underlying topological space $\underline{Z}$.

(b) If the cohomology sheaves of $\mathcal{F} \in \mathcal{D}^b_G(X)$ are supported topologically on $\underline{Z}$, then there exists a closed subscheme $i' : Z' \hookrightarrow X$ with underlying topological space $\underline{Z}$ and a sheaf $\mathcal{F}' \in \mathcal{D}^b_G(Z')$ such that $\mathcal{F} \simeq i'_* \mathcal{F}'$.

Let p be a perversity that is monotone and comonotone (see Definition 2.22) with respect to the stratification of X by G-orbits. For x a point of X (possibly nonclosed), define $p(x) = p(C)$ where C is the orbit containing x. We can now define the perverse coherent t-structure.

Theorem 2.33. *The subcategories*

$$\mathcal{D}^{p,\leq 0} = \left\{ \mathcal{F} \in \mathcal{D}^b_G(X) \mid i^*_x \in \mathcal{D}^{\leq p(x)}(\mathcal{O}_x\text{-}mod) \text{ for all } x \in X \right\}$$
$$\mathcal{D}^{p,\geq 0} = \left\{ \mathcal{F} \in \mathcal{D}^b_G(X) \mid i^!_x \in \mathcal{D}^{\geq p(x)}(\mathcal{O}_x\text{-}mod) \text{ for all } x \in X \right\}$$

define a t-structure on $\mathcal{D}^b(\mathcal{C}_G(X))$.

Definition 2.34. We call objects in the heart of this t-structure *perverse coherent sheaves* and denote the category of perverse coherent sheaves by $\mathcal{M}_{\mathrm{coh}^G}(X)$.

Theorem 2.35. *If p is strictly monotone and comonotone, then for each locally closed orbit C there is a fully faithful functor $\mathcal{IC}(C, -) : \mathcal{M}_{\mathrm{coh}^G}(C) \to \mathcal{M}_{\mathrm{coh}^G}(X)$. Every simple perverse coherent sheaf on X is of the form $\mathcal{IC}(C, \mathcal{E}[p(C)])$ for some orbit C and some irreducible equivariant vector bundle $\mathcal{E}$ on C.*

We can use the $\mathcal{IC}$ functors then to show the following useful corollary. This shows one sense in which the category of perverse coherent sheaves is better behaved than the corresponding category of coherent sheaves.

Corollary 2.36. *The category $\mathcal{M}_{\mathrm{coh}^G}(X)$ is artinian.*

One of the differences between perverse sheaves and perverse coherent sheaves is that one defines the middle perversity in the coherent setting using the complex (algebraic) dimension. A consequence of this is that the middle perversity is only strictly monotone and comonotone if orbits have even dimension and dimensions of adjacent orbits differ by at least two. Thankfully, as we will see below, these conditions are satisfied for the nilpotent cone.

Similarly to the classical case, it is of primary importance to understand the simple perverse coherent sheaves. The situation is not nearly as clear in this case however and very little is know in general. The main goal of the thesis is to make progress toward understanding certain simple perverse coherent sheaves on the nilpotent cone of a reductive complex algebraic group by utilizing a Poisson structure that is compatible with the equivariant structure.

2.4 Equivariant Module Categories

In Chapter 4 we will construct the equivariant Poisson derived category. In order to extend functors defined on the abelian category of Poisson sheaves, we will need to know that we have enough acyclic objects for those functors. Since injective objects are always acyclic for any left exact functor, it will be advantageous for us to see that we indeed have enough injectives. We will need the following classical result of Grothendieck.

Theorem 2.37. *If $\mathcal{C}$ is an abelian category satisfying*

(i) $\mathcal{C}$ is cocomplete, i.e. $\mathcal{C}$ is closed under arbitrary coproducts (direct sums)

(ii) given an object X and a totally ordered family of subobjects $X_i \to X$, for any other subobject $Y \to X$ we have

$$\left(\sum_i X_i\right) \cap Y = \sum_i (X_i \cap Y)$$

(iii) $\mathcal{C}$ has a generator, i.e. an object U such that for any monomorphism that is not an epimorphism $f : X \to Y$, there is a morphism $U \to Y$ which does not factor through X,

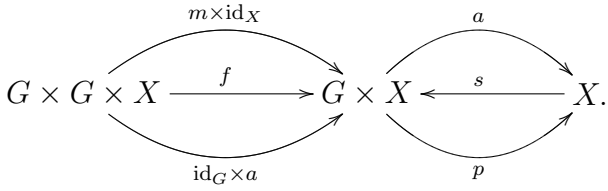
then $\mathcal{C}$ has enough injectives.

Proof. See [Gro, Theorem 1.10.1]. □

Corollary 2.38. *Let A be an associative $\mathbb{C}$-algebra with 1. Then A-mod (the category of left A-modules) has enough injectives.*

Note that (*iii*) in Theorem 2.37 could be replaced by the *a priori* weaker condition requiring only a set of generators $\{U_i \mid i \in I\}$ since in a cocomplete category these are equivalent notions (the coproduct of such a set is itself a generator). An abelian category satisfying (*i*) and (*ii*) is said to satisfy AB5. An abelian category satisfying all three conditions is sometimes called a *Grothendieck (abelian) category*.

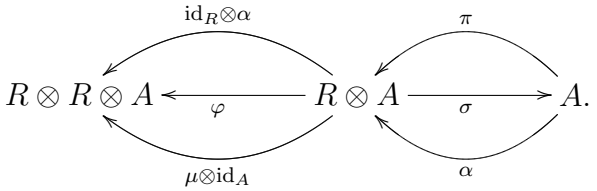
Definition 2.39. Let X be a locally ringed space and G a group acting on X. Then if $a : G \times X \to X$ denotes the action map, $p : G \times X \to X$ is the projection onto the second factor, $f : G \times G \times X \to G \times X$ forgets the first term, and s is the map sending $x \mapsto (e, x)$, we have a diagram



A *G-equivariant sheaf of modules on X* is a sheaf of modules $\mathcal{F}$ on X along with an isomorphism $\Psi : a^*\mathcal{F} \xrightarrow{\sim} p^*\mathcal{F}$ such that

$$f^*\Psi \circ (\mathrm{id}_G \times a)^*\Psi = (m \times \mathrm{id}_X)^*\Psi$$
$$s^*\Psi = \mathrm{id}_{\mathcal{F}}.$$

Remark 2.40. If $X = \operatorname{Spec} A$ is an affine scheme over a $\mathbb{C}$ and $G = \operatorname{Spec} R$ is a complex algebraic group acting linearly on X, a G-equivariant quasicoherent sheaf $\mathcal{F}$ on X is the same as an A-module M with a compatible G-action. Diagrammatically, on coordinate rings we maps



The maps α and π give $R \otimes A$ two different (right) A-module structures. A G-equivariant A-module is an A-module M along with an isomorphism

$$\Psi : (R \otimes_{\mathbb{C}} A)_\pi \otimes_A M \xrightarrow{\sim} (R \otimes_{\mathbb{C}} A)_\alpha \otimes_A M$$

such that conditions analogous to the sheaf version hold. One can characterize an equivariant structure on M by saying that there is an action such that the module structure morphism $A \otimes M \to M$ is a G-equivariant map in the sense that $g.(b.m) = (g.b).(g.m)$ for any $g \in G$, $b \in A$, and $m \in M$.

We extend the notion of an equivariant module to the setting of a noncommutative ring A by taking the characterization in this remark as a definition.

Lemma 2.41. *Let A be an associative $\mathbb{C}$-algebra with 1. Then A-mod^G (the category of G-equivariant left A-modules) has enough injectives.*

Proof. We adapt an argument from [Bez1]. Let M be an A-module and consider the A-module

$$Av(M) = ((R \otimes_{\mathbb{C}} A)_\pi \otimes_A M)_\alpha,$$

where $(-)_\star$ indicates the pullback of modules along the map $\star$. Then $Av(M)$ is a G-equivariant A-module and $Av : A\text{-mod} \to A\text{-mod}^G$ is right adjoint to the forgetful functor

$F : A\text{-mod}^G \to A\text{-mod}$. Since Av is an exact functor and the map $M \to Av(M)$ is injective, the lemma follows from Corollary 2.38 above. □

Notice that we did not require that the algebra A in the previous proposition to be commutative. If we only cared about the commutative situation, then we would have stated the argument of Bezrukavnikov verbatim and retained the scheme language since this is much easier to write down. In fact, as we will see below, one reasonable approach to perverse Poisson sheaves would be to rework all of [Bez1] using noncommutative geometry. The author does not know if this is possible, but it might provide a more sophisticated theory.

2.5 Poisson Schemes and Modules

We begin by laying out the general definitions of Poisson rings, algebras, schemes, modules, and related concepts that will be used extensively in what follows. We will not have much use for the detail that is provided in the definitions and rather rely on the properties outlined below when working with Poisson schemes and sheaves. Let X be a scheme of finite type over $\mathbb{C}$. Some of what follows can be done over more general fields, but restricting to the complex case is sufficient for us. The material in this section comes from [Kal2], [Kal1], and [Pol].

Definition 2.42. A ring A is a *Poisson ring* if it is equipped with a bilinear product $\{\ ,\ \}$ (the *Poisson bracket*) satisfying

(i) $\{a, a\} = 0$ for all $a \in A$.

(ii) $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$ for all $a, b, c \in A$.

(ii) $\{a, bc\} = \{a, b\}c + b\{a, c\}$ for all $a, b, c \in A$.

That is, $\{\ ,\ \}$ is a Lie bracket which satisfies the Leibnitz identity in each variable. In particular, an associative algebra over a field k is a *Poisson algebra* if it is also endowed with a Poisson bracket.

An ideal $I \subset A$ is a *Poisson ideal* if in addition $\{i, a\} \in I$ for all $i \in I$ and $a \in A$.

It is straightforward to pass from the previous definition to a scheme version. The level of detail provided here is certainly more than we will use, but it is instructive to see.

Definition 2.43. The scheme X is a *Poisson scheme* if the structure sheaf $\mathcal{O}_X$ is equipped with a Poisson bracket. Explicitly, this means that in the category of $\mathcal{O}_X$-modules there is a morphism

$$\{\ ,\ \} : \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{O}_X$$

satisfying

i. (Skew-symmetry) If $\iota : \mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X \otimes \mathcal{O}_X$ is the morphism which exchanges factors, then in $\mathrm{Hom}(\mathcal{O}_X \otimes \mathcal{O}_X, \mathcal{O}_X)$

$$\{\ ,\ \} = -\{\ ,\ \} \circ \iota$$

ii. (Jacobi identity) If $\sigma : \mathcal{O}_X \otimes \mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X \otimes \mathcal{O}_X \otimes \mathcal{O}_X$ is the morphism which cyclically permutes the factors, then in $\mathrm{Hom}(\mathcal{O}_X \otimes \mathcal{O}_X \otimes \mathcal{O}_X, \mathcal{O}_X)$

$$\{\ ,\ \} \circ \mathrm{id} \otimes \{\ ,\ \} + \{\ ,\ \} \circ \mathrm{id} \otimes \{\ ,\ \} \circ \sigma + \{\ ,\ \} \circ \mathrm{id} \otimes \{\ ,\ \} \circ \sigma \circ \sigma = 0$$

iii. (Leibniz identity) If $\mu : \mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X$ is the multiplication morphism and ι is as in (i), then in $\mathrm{Hom}(\mathcal{O}_X \otimes \mathcal{O}_X \otimes \mathcal{O}_X, \mathcal{O}_X)$

$$\{\ ,\ \} \circ \mathrm{id} \otimes \mu = \mu \circ \{\ ,\ \} \otimes \mathrm{id} + \mu \circ \{\ ,\ \} \otimes \mathrm{id} \circ \mathrm{id} \otimes \iota$$

where the tensor product is over $\mathbb{C}$ in all cases. Let $f : X \to Y$ be a morphism between two Poisson schemes. Then $f_*\mathcal{O}_X$ is naturally a sheaf of Poisson rings on Y. We say that f is a *morphism of Poisson schemes* if the corresponding morphism of structures sheaves $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ respects the Poisson structures.

We point out a nearly trivial, but vital fact. Let $j : U \hookrightarrow X$ be an open subscheme of a Poisson scheme X. Then since $\mathcal{O}_U$ is just the restriction of $\mathcal{O}_X$ to U, we see that $\mathcal{O}_U$ inherits a Poisson structure also. Thus any open subscheme of a Poisson scheme is Poisson.

Definition 2.44. If there exists a nonzero local section f of $\mathcal{O}_X$ such that the morphism $\{f, -\} : \mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is identically zero, we say that the Poisson structure is degenerate. A Poisson structure which is nondegenerate is called *symplectic*.

Since a Poisson bracket gives a derivation when one variable is fixed, there is an intimate link between the Poisson structure and both the cotangent and tangent bundles of the scheme X. In order to state this, we recall the definitions of the cotangent bundle and the tangent bundle.

Definition 2.45. We define the *cotangent space* of X at a point x to be

$$\Omega^1_x = \mathfrak{m}_x/\mathfrak{m}_x^2,$$

where $\mathfrak{m}_x$ is the is the maximal ideal of the local ring $\mathcal{O}_x$. Thus Ω^1_x is a vector space over the residue field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$.

Definition 2.46. If μ is the multiplication morphism from Definition 2.43, then we call the $\mathcal{O}_X$-module $\Omega^1_X = \ker\mu/(\ker\mu)^2$ the sheaf of *Kähler differentials*. We denote by d the morphism $\mathcal{O}_X \to \Omega^1_X$ given locally by the formula $f \longmapsto (1 \otimes f - f \otimes 1) + (\ker\mu)^2$. This is the *cotangent bundle*.

Lemma 2.47. *For any point x of X, the stalk of Ω^1_X at x is Ω^1_x.*

Definition 2.48. Define the *tangent space* of X at a point x to be the vector space dual to the cotangent space, *i.e.*

$$T_xX = \mathrm{Hom}_{k(x)}(\Omega^1_x, k(x)).$$

From Definition 2.48, it is straightforward to see that we have the following description of the tangent bundle.

Lemma 2.49. *We have*

$$T_X = Der(\mathcal{O}_X, \mathcal{O}_X).$$

Lemma 2.50. *Let $H : \Omega^1_X \to T_X$ be an $\mathcal{O}_X$-linear homomorphism. Then $\{\ ,\ \} : \mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X$ defined locally by $\{f, g\} = H(df)(g)$ gives a Poisson structure on X.*

We call such an H the *Hamiltonian* of the Poisson bracket.

Remark 2.51. If X is a Poisson scheme, then any local section f of $\mathcal{O}_X$ defines a local derivation $\{f, -\}$ on the structure sheaf. Thus we get a morphism of sheaves $\mathcal{O}_X \to T_X$. Vector fields (local sections of the tangent bundle) in the image of this morphism are called *Hamiltonian vector fields.*

We now recall the basic definitions and facts that we will need below concerning the interaction between a group action and a Poisson structure. We assume that G is an reductive complex algebraic group acting on X with finitely many orbits. We denote the Lie algebra of G by $\mathfrak{g}$. We refer the reader to [CG, Chapter 1], [Pol, Section 1], [Vai, Section 7.3] and [Kal2, Section 1] for more details on Hamiltonian actions. There is certainly much more that one could say here, but the facts that we will need rely only on the definition of a Hamiltonian action.

Definition 2.52. The action of G on X is a *Poisson action* if G acts by Poisson automorphisms. That is, for a fixed $g \in G$ the map sending $x \mapsto g.x$ induces an isomorphism $\mathcal{O}_X \to \mathcal{O}_X$ which preserves the Poisson bracket.

Definition 2.53. A Poisson action of G on X induces an action of $\mathfrak{g}$ on X by derivations in $\mathcal{O}_X$. We say that the action of G is *Hamiltonian* if the associated morphism $\mathfrak{g} \to T_X$ factors through the morphism $\mathcal{O}_X \to T_X$ described in Remark 2.51.

A primary example of a Hamiltonian action (and the one which is the central example in this thesis) is the coadjoint action on the nilpotent cone.

Definition 2.54. We define a Poisson module over X to be an quasicoherent $\mathcal{O}_X$-module $\mathcal{F}$ which is also equipped with Poisson bracket. Again, very explicitly, this means that in the category of quasicoherent $\mathcal{O}_X$-modules, there is a morphism $\{\ ,\ \} : \mathcal{O}_X \otimes \mathcal{F} \to \mathcal{F}$ such that

i. (Jacobi-like identity) If $\iota : \mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X \otimes \mathcal{O}_X$ is the morphism which exchanges factors, then in $\mathrm{Hom}(\mathcal{O}_X \otimes \mathcal{O}_X \otimes \mathcal{F}, \mathcal{F})$

$$\{\ ,\ \} \circ \mathrm{id} \otimes \{\ ,\ \} = \{\ ,\ \} \circ \mathrm{id} \otimes \{\ ,\ \} \circ \iota + \{\ ,\ \} \circ \{\ ,\ \} \otimes \mathrm{id}.$$

ii. (Compatibility with multiplication) If $\mu : \mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X$ is the multiplication morphism and $\phi : \mathcal{O}_X \otimes \mathcal{F} \to \mathcal{F}$ is the module action morphism, then in $\mathrm{Hom}(\mathcal{O}_X \otimes \mathcal{O}_X \otimes \mathcal{F}, \mathcal{F})$

$$\{\ ,\ \} \circ \mathrm{id} \otimes \phi = \phi \circ \{\ ,\ \} \otimes \mathrm{id} + \phi \circ \mathrm{id} \otimes \{\ ,\ \} \circ \iota$$
$$\{\ ,\ \} \circ \mu \otimes \mathrm{id} = \phi \circ \mathrm{id} \otimes \{\ ,\ \} + \phi \circ \mathrm{id} \otimes \{\ ,\ \} \circ \iota.$$

Define $\mathcal{Q}_{\mathrm{Poi}}(X)$ to be the category whose objects are quasicoherent $\mathcal{O}_X$-modules with a Poisson bracket and whose morphisms are those $\mathcal{O}_X$-module morphisms which respect the Poisson structure. I.e., if $\mathcal{F}, \mathcal{G} \in \mathcal{C}_{\mathrm{Poi}}(X)$, then $f : \mathcal{F} \to \mathcal{G}$ is a Poisson morphism if

$$\{\ ,\ \} \circ \mathrm{id} \otimes f = f \circ \{\ ,\ \}. \tag{$*$}$$

Remark 2.55. Unlike [Kal1], we do not require that Poisson modules (also called Poisson sheaves in the sequel) be coherent sheaves on X, but rather quasicoherent, since this is the proper setting for this work. We will restrict to complexes with coherent cohomology when we pass to the derived category below. The well known benefit of considering quasicoherent sheaves as opposed to all $\mathcal{O}_X$-modules is that for an affine scheme $X = \mathrm{Spec}\ A$, we have an equivalence $\mathcal{Q}(X) \simeq A$-mod (for example, see [Har2, Proposition 5.4]).

Definition 2.56. A closed subscheme $k : Y \hookrightarrow X$ of X is a *Poisson subscheme* if the corresponding ideal sheaf $\mathcal{I}_Y$ is a Poisson submodule of $\mathcal{O}_X$ and Y is equipped with the induced Poisson structure. A locally closed subscheme is then Poisson if it is the intersection of a Poisson closed subscheme and an open subscheme.

2.6 Poisson Enveloping Algebras

Analogously to the case of Lie algebras, Oh ([Oh]) has shown that it is possible to construct a universal associative algebra over any Poisson algebra which encodes the Poisson structure in the multiplication of the algebra. This allows us to identify modules over the Poisson algebra with ordinary modules over the enveloping algebra. Therefore, as long as we are willing to sacrifice finite generation, we can work in the Poisson enveloping algebra and translate the results back to the Poisson algebra.

Definition 2.57. Let A be a Poisson algebra over a base field k and $\mathcal{U}$ an associative k-algebra. We consider $\mathcal{U}$ as a Lie algebra with the usual commutator bracket. Then $\mathcal{U}$ is a Poisson enveloping algebra if it satisfies the following universal property. There is a k-algebra homomorphism $f : A \to \mathcal{U}$ and a k-Lie algebra homomorphism $g : A \to \mathcal{U}$ such that

$$\begin{aligned} f(\{a, b\}) &= g(a)f(b) - f(b)g(a) \\ g(ab) &= f(a)g(b) + f(b)g(a), \end{aligned} \tag{$*$}$$

and for any other associative k-algebra $\mathcal{W}$ and homomorphisms $h, \ell : A \to \mathcal{W}$ satisfying $(*)$ there exists a unique k-algebra homomorphism $\gamma : \mathcal{U} \to \mathcal{W}$ such that the diagrams

commute.

Theorem 2.58. *For any Poisson algebra A, there exists a unique universal Poisson enveloping algebra $\mathcal{U}_{\mathrm{Poi}}(A)$.*

Proof. For convenience and later reference, we reproduce the construction from [Oh] here. Fix a basis $\{x_i \mid i \in I\}$ for A and let V be an isomorphic vector space with basis $\{y_i \mid i \in I\}$ and isomorphism $\varphi : A \to V$ sending $x_i \mapsto y_i$. Consider the k-tensor algebra $\mathcal{T}(A \oplus V)$ which has as a subalgebra the k-tensor algebra $\mathcal{T}(A)$. Let $\psi : \mathcal{T}(A) \to A$ be the algebra homomorphism defined by $x_i \mapsto x_i$. Now let J be the ideal in $\mathcal{T}(A \oplus V)$ generated by $\ker \psi$ and elements in $\mathcal{T}(A \oplus V)$ of the form

- $y_i \otimes y_j - y_j \otimes y_i - \varphi(\{x_i, x_j\})$
- $y_i \otimes x_j - x_j \otimes y_i - \{x_i, x_j\}$
- $x_i \otimes y_j + x_j \otimes y_i - \varphi(x_i x_j)$.

Then define $\mathcal{U}_{\mathrm{Poi}}(A) = \mathcal{T}(A \oplus V)/J$. The homomorphisms f and g that we need are just the maps defined by $f(x_i) = x_i + J$ and $g(x_i) = y_i + J$. The construction of J ensures that these maps satisfy the required conditions. The proof that $\mathcal{U}_{\mathrm{Poi}}(A)$ does in fact satisfy the universal property is straightforward and we refer the reader to [Oh] for details. As usual, uniqueness follows. □

Theorem 2.59. *For any Poisson algebra A, there is an isomorphism of categories A-mod$_{Poi} \simeq \mathcal{U}_{\mathrm{Poi}}(A)$-mod.*

Proof. For a Poisson A-module M, there are maps $\alpha, \beta : A \to \mathrm{End}_k(M)$ satisfying $(*)$ above. By the universal property for $\mathcal{U}_{\mathrm{Poi}}(A)$, we then get a k-algebra homomorphism $\mu : \mathcal{U}_{\mathrm{Poi}}(A) \to \mathrm{End}_k(M)$ which gives M a (left) $\mathcal{U}_{\mathrm{Poi}}(A)$-module structure.

On the other hand, it is easy to check that $a.m = f(a).m$ and $\{b, m\} = g(b).m$ give any $\mathcal{U}_{\mathrm{Poi}}(A)$-module M a Poisson A-module structure. □

Corollary 2.60. *Let G be a reductive complex algebraic group which acts on A by Poisson automorphisms. Then there is a corresponding G-action on $\mathcal{U}_{\mathrm{Poi}}(A)$ and an equivalence of categories A-mod$_{Poi}^{G} \simeq \mathcal{U}_{\mathrm{Poi}}(A)$-mod^G.*

Proof. Using the notation from the construction of $\mathcal{U}_{\mathrm{Poi}}(A)$, define a G-action on $\mathcal{U}_{\mathrm{Poi}}(A)$ by letting $g.(x \otimes y + J) = g.x \otimes g.y + J$, where the G-action on V is induced via φ by the G-action on A. So by definition, φ is equivariant. This is well defined (G takes J to J) since G acts by Poisson automorphisms. The equivalence of categories then follows from the theorem. □

Next, we want to be able to realize $\mathcal{U}_{\mathrm{Poi}}(A)$ as a free module over the Poisson algebra A. We can find a theorem of this type in the literature already ([OPS, Theorem 3.8]), but checking that this theorem applies (as stated) to the Poisson algebras that we will be concerned with (e.g. the coordinate ring of the nilpotent cone described in Section 2.7) involves introducing additional concepts which we would not otherwise need. In fact, the theorem in *loc. cit.* applies only to Poisson algebras which can be realized as a quotient of a polynomial ring with Poisson structure induced from one on the polynomial ring where the ideal defining the

quotient satisfies a condition relating to Gröbner–Shirshov bases. In some circumstances (for example, if one needs a particular basis for computation), it might be necessary to restrict to this setting. For our purposes, however, it suffices to see that a $\mathbb{C}$-basis exists for any Poisson enveloping algebra which makes it obvious that $\mathcal{U}_{\text{Poi}}(A)$ is a free Poisson A-module. Fortunately, this is a straightforward corollary of the PBW theorem for Lie algebras.

Theorem 2.61. *(Poicaré–Birkhoff–Witt) Let L be a complex Lie algebra with ordered basis $\{x_i \mid i \in I\}$. Then $\mathcal{U}(L) = \mathcal{T}(L)/J$ has a $\mathbb{C}$-basis consisting of monomials $x_{i_1} \cdots x_{i_n}$, where J is a certain ideal and $n, i_j \in \mathbb{N}$ for all j and $i_1 \leq \cdots \leq i_n$ and $x_{i_1} \cdots x_{i_n}$ is the image of $x_{i_1} \otimes \cdots \otimes x_{i_n}$ in the quotient.*

Proof. See, for instance, [Hum, §17]. □

Corollary 2.62. *Let A be a Poisson algebra. Then the morphism $f : A \to \mathcal{U}_{\text{Poi}}(A)$ is injective.*

Proof. Let $f, g : A \to \mathcal{U}_{\text{Poi}}(A)$ be the universal maps from Definition 2.57. We follow [OPS, Proposition 2.2] in showing that f is injective. Define $\mu, \delta : A \to \text{End}_{\mathbb{C}}(A)$ by

$$\begin{aligned} \mu(a)(b) &= ab \\ \delta(a)(b) &= \{a, b\}. \end{aligned}$$

Then

$$\begin{aligned} \mu(\{a,b\})(c) = \{a,b\}c = \{a,bc\} - b\{a,c\} &= \delta(a) \circ \mu(b)(c) - \mu(b) \circ \delta(a)(c) \\ \delta(ab)(c) = \{ab,c\} = a\{b,c\} + b\{a,c\} &= \mu(a) \circ \delta(b)(c) + \mu(b) \circ \delta(a)(c). \end{aligned}$$

Thus the universal property of $\mathcal{U}_{\text{Poi}}(A)$ gives an algebra homomorphism

$$\gamma : \mathcal{U}_{\text{Poi}}(A) \to \text{End}_{\mathbb{C}}(A)$$

such that $\gamma \circ f = \mu$. Then if $a \in A$ and $f(a) = 0$, we must also have $\mu(a) : A \to A$ the zero map. Clearly this only happens when $a = 0$. □

Corollary 2.63. *The Poisson enveloping algebra $\mathcal{U}_{\text{Poi}}(A)$ is free as an A-module.*

Proof. Recall that $\mathcal{U}_{\text{Poi}}(A) = \mathcal{T}(A \oplus V)/J$ and that a basis for the tensor algebra is the set of all finite length words in $X = \{x_i \mid i \in I\} \cup \{y_j \mid j \in I\}$. We know from basic linear algebra then, that we can find a subset of the set of words in X that form a basis for $\mathcal{U}_{\text{Poi}}(A)$. Moreover, the fact that elements of the form

$$y_i \otimes x_j - x_j \otimes y_i - \{x_i, x_j\}$$

and $\ker \psi$ are in J ensure that we can choose this basis to consist of elements of the form $\{x_i y_{j_1} \cdots y_{j_m}\}$. Now since A acts on the left by multiplication, we see that $\mathcal{U}_{\text{Poi}}(A)$ is indeed free. □

2.7 The Nilpotent Cone

The origin of this work is in studying the geometry of the nilpotent cone associated to a reductive complex algebraic group. The nilpotent variety is an object of central importance in representation theory. The classification of representations of finite groups of Lie type reduces to the calculation of stalks of certain simple perverse sheaves on the nilpotent cone (references for the work of Lusztig on character sheaves can be found in [Lus1]). The nilpotent cone is also related to the geometric Langlands program, through work of Bezrukavnikov *et. al.* (see [Bez4]). In this section we give the definition of the nilpotent cone and state the basic properties which will form the foundation of the work later.

We begin by recalling a few of the basic definitions from the theory of complex algebraic groups and Lie algebras. Let G be a reductive complex algebraic group. Recall that we can identify the Lie algebra $\mathfrak{g}$ of G with the tangent space to G at the identity. We denote the (vector space) dual of the Lie algebra by $\mathfrak{g}^*$.

The action of G on itself by conjugation differentiates to an action of G on $\mathfrak{g}$, the *adjoint* action. We can then take the dual (contragredient) of this representation to get an action of G on $\mathfrak{g}^*$, the *coadjoint* action. If we differentiate the map $G \to \mathrm{Aut}(\mathfrak{g})$ (resp. $G \to \mathrm{Aut}(\mathfrak{g}^*)$), we also get an action of $\mathfrak{g}$ on itself (resp. on $\mathfrak{g}^*$), which we will also call the adjoint (resp. coadjoint) action. The adjoint action of $\mathfrak{g}$ has a simple description; it is given by the Lie bracket

$$\mathrm{ad}_x(y) = [x, y],$$

where $x, y \in \mathfrak{g}$. The coadjoint action of $\mathfrak{g}$ then is just the dual Lie representation. Thus we have

$$\mathrm{ad}^*_x(f)(y) = -f([x, y]),$$

where $x, y \in \mathfrak{g}$ and $f \in \mathfrak{g}^*$. We define a symmetric bilinear form on $\mathfrak{g}$ called the *Killing form* by

$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_x \mathrm{ad}_y),$$

the traceform of the adjoint representation. A Lie algebra is semisimple if and only if the Killing form is nondegenerate. In this case, we can identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Killing form in the usual way. Recall that a reductive Lie algebra $\mathfrak{g}$ can be written $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z}$, where $\mathfrak{s}$ is a semisimple Lie algebra and $\mathfrak{z}$ is the center of $\mathfrak{g}$.

Definition 2.64. We say that an element of $\mathfrak{g}$ is *nilpotent* if it corresponds to a nilpotent endomorphism in every representation of $\mathfrak{g}$. The *nilpotent cone* $\mathcal{N} \subset \mathfrak{g}$ is the set of all nilpotent elements in $\mathfrak{g}$.

Remark 2.65. Restricting the Killing form to the semisimple part of $\mathfrak{g}$ allows us to realize $\mathcal{N}$ as a subalgebra of $\mathfrak{g}^*$. Since $\mathcal{N}$ is G-invariant under the adjoint action of G on $\mathfrak{g}$, we see that $\mathcal{N}$ is G-invariant under the coadjoint action of G when we consider $\mathcal{N} \subset \mathfrak{g}^*$. Moreover, G acts with finitely many orbits (see, for example, [CG, Proposition 3.2.9]).

Definition 2.66. We recall the Kostant–Kirillov Poisson structure on $\mathfrak{g}^*$ (for additional details, see [BBT, §14.2]). Since $\mathfrak{g}^*$ is a vector space, the differential $d\alpha$ of any regular

function α on $\mathfrak{g}^*$ is linear. That is, $d\alpha \in (\mathfrak{g}^*)^* \simeq \mathfrak{g}$. Given two functions $\alpha, \beta \in \mathbb{C}[\mathfrak{g}^*]$, define

$$\{\alpha, \beta\}(f) = f([d\alpha, d\beta]).$$

It follows from the properties of differentials that this is a Poisson bracket.

One consequence of the PBW theorem (Theorem 2.61 above) is that $\mathbb{C}[\mathfrak{g}^*] \simeq S(\mathfrak{g})$, where $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$. The following is a fundamental theorem which tells us a great deal about the structure of the nilpotent cone. The result is due to Kostant and two proofs can be found in [CG, Section 3.2].

Theorem 2.67. *An element $x \in \mathfrak{g}^*$ is in $\mathcal{N}$ if and only if every G-invariant polynomial in $S(\mathfrak{g})$ of strictly positive degree vanishes at x.*

This theorem implies that $\mathcal{N}$ is a closed affine subvariety of $\mathfrak{g}^*$, with coordinate ring

$$\mathbb{C}[\mathcal{N}] = S(\mathfrak{g})/I,$$

where I is the ideal generated by G-invariant polynomials of strictly positive degree. Combining this with the definition of the Kostant–Kirilov bracket gives us the following proposition, which will be essential to our study of the nilpotent cone.

Proposition 2.68. *For any reductive complex algebraic group G with Lie algebra $\mathfrak{g}$, we have $\mathfrak{g} \subset \mathcal{O}_{\mathcal{N}}$ such that the Poisson bracket on $\mathcal{O}_{\mathcal{N}}$ is induced by the Lie bracket on $\mathfrak{g}$.*

Proof. From [Dix, Lemma 8.1.1], we see that the ideal I corresponding to $\mathcal{N}$ in $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$ can be generated by G-invariant polynomials of degree at least 2. Thus the natural embedding $\mathfrak{g} \hookrightarrow S(\mathfrak{g})$ gives $\mathfrak{g} \subset \mathcal{O}_{\mathcal{N}}$. The definition of the Kostant–Kirillov Poisson bracket guarantees that it is induced by the Lie bracket on $\mathfrak{g}$ (extending by bilinearity and the Liebnitz rule). □

Lemma 2.69. *The algebra $\mathbb{C}[\mathcal{N}]$ is generated as an algebra by a basis for the Lie algebra $\mathfrak{g}$.*

Proof. Any basis for $\mathfrak{g}$ generates $S(\mathfrak{g})$ as an algebra, of which $\mathbb{C}[\mathcal{N}]$ is a quotient. □

Recall that a symplectic structure is a nondegenerate Poisson structure. It is well known that given a Poisson variety X, there is a stratification of X by maximally symplectic subvarieties which are called *symplectic leaves*.

Proposition 2.70. *The stratification of $\mathcal{N}$ by coadjoint orbits coincides with the stratification of $\mathcal{N}$ by the symplectic leaves of the Poisson structure.*

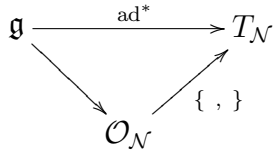
Proof. This follows from the fact that the kernel of the Kostant–Kirillov bracket is precisely the set of functions which are constant on the coadjoint orbits. See the discussion in [BBT, Section 14.2]. □

In particular, this tells us that each G-orbit is a smooth complex variety with even (complex) dimension since any symplectic variety must have even dimension. Therefore, as we mentioned in Section 2.3, the middle perversity is strictly monotone and comonotone on the nilpotent cone.

Lemma 2.71. *The coadjoint action of G on $\mathcal{N}$ is a Hamiltonian action.*

Proof. The coadjoint action of G differentiates to the coadjoint action of $\mathfrak{g}$, which can be given in terms of the Lie bracket on $\mathfrak{g}$. The lemma follows then from Proposition 2.68. □

Remark 2.72. In fact, Lemma 2.69 shows that more is true in this case. Not only do we get a factorization



we can say something more about the map $\mathfrak{g} \to \mathcal{O}_{\mathcal{N}}$. Namely, that it is an injective Lie homomorphism whose image generates $\mathcal{O}_{\mathcal{N}}$ as an algebra. This property will be key in Chapter 4 below in relating the equivariant and Poisson structures on the nilpotent cone.

Chapter 3
Gluing Semiorthogonal Filtrations

The work contained in this chapter grew out of an attempt to extract sufficient conditions from the proof of the construction of the perverse coherent t-structure ([Bez1, Theorem 1] that would allow us to glue together t-structures defined on G-orbits. We have accomplished that goal in Theorem 3.4 and are then able to recover the construction of the perverse coherent t-structure by gluing (Theorem 3.12). There are several advantages to the approach that we have taken. As mentioned above, this allows for a unified approach to perverse coherent sheaves and classical perverse sheaves. We also have been able to show that given a t-structure defined on the derived category of coherent sheaves on a closed subscheme, we can extend this t-structure to each of the formal neighborhoods of the closed subscheme and to the limit of these formal neighborhoods in the derived category of coherent sheaves on the entire scheme. In this chapter, a semiorthogonal filtration will always mean either a t-structure, a co-t-structure, or a baric structure.

3.1 Semiorthogonal Filtrations and Triangulated Functors

In this section, we prove a proposition giving sufficient conditions for transferring a semiorthogonal filtration across a functor. The situation that we have in mind is that where the functor is the pushforward of coherent sheaves along the inclusion of a closed subscheme. It is convenient to fix a triangulated category $\mathcal{D}$.

Definition 3.1. Let $\mathcal{A}$ be a triangulated category and suppose that we have a triangulated functor $F : \mathcal{A} \to \mathcal{D}$. Let $F(\mathcal{A})$ be the essential image of F and $\widetilde{\mathcal{A}}$ the triangulated subcategory of $\mathcal{D}$ generated by $F(\mathcal{A})$.

If $(\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0})$ is a (co-)t-structure on $\mathcal{A}$ we can define strictly full subcategories of $\widetilde{\mathcal{A}}$

$$\widetilde{\mathcal{A}}^{\leq 0} = \bigcup_{n>0} \widetilde{\mathcal{A}}^{n,\leq 0}$$
$$\widetilde{\mathcal{A}}^{\geq 0} = \bigcup_{n>0} \widetilde{\mathcal{A}}^{n,\geq 0}$$

where

$$\widetilde{\mathcal{A}}^{n,\leq 0} = \underbrace{F(\mathcal{A}^{\leq 0}) *_{\mathcal{D}} \cdots *_{\mathcal{D}} F(\mathcal{A}^{\leq 0})}_{n \text{ terms}}$$
$$\widetilde{\mathcal{A}}^{n,\geq 0} = \underbrace{F(\mathcal{A}^{\geq 0}) *_{\mathcal{D}} \cdots *_{\mathcal{D}} F(\mathcal{A}^{\geq 0})}_{n \text{ terms}}.$$

We say that F *respects* a t-structure $(\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0})$ on $\mathcal{A}$ if for all $A \in \mathcal{A}^{\leq 0}$ and $B \in \mathcal{A}^{\geq 0}$ the natural map

$$\mathrm{Hom}_{\mathcal{A}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$$

induced by F is surjective.
Slightly less restrictively, the functor F *respects* a baric structure $\left(\left\{\mathcal{A}^{\leq w}\right\}, \left\{\mathcal{A}^{\leq w}\right\}\right)$ on $\mathcal{A}$ if for each $w \in \mathbb{Z}$ we have

$$\mathrm{Hom}_{\mathcal{D}}(F(A), F(B)) = 0 \text{ if } A \in \mathcal{A}^{\leq w} \text{ and } B \in \mathcal{A}^{\geq w+1}.$$

Similarly, the functor F *respects* a co-t-structure $(\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0})$ on $\mathcal{A}$ if

$$\mathrm{Hom}_{\mathcal{D}}(F(A), F(B)) = 0 \text{ if } A \in \mathcal{A}^{\leq 0} \text{ and } B \in \mathcal{A}^{\geq 1},$$

as well as the additional requirement that $\widetilde{\mathcal{A}}^{\leq 0}$ and $\widetilde{\mathcal{A}}^{\geq 0}$ are closed under direct summands.

We are now able to give the main proposition in this section.

Proposition 3.2. *Let $F : \mathcal{A} \to \mathcal{D}$ be a triangulated functor which respects the (co-)t-structure $(\mathcal{A}^{\leq 0}, \mathcal{A}^{\geq 0})$ on $\mathcal{A}$. Then the subcategories $(\widetilde{\mathcal{A}}^{\leq 0}, \widetilde{\mathcal{A}}^{\geq 0})$ define a (co-)t-structure on $\widetilde{\mathcal{A}}$. Moreover, $(\widetilde{\mathcal{A}}^{\leq 0}, \widetilde{\mathcal{A}}^{\geq 0})$ is the unique (co-)t-structure with*

(i) $F(\mathcal{A}^{\leq 0}) = \widetilde{\mathcal{A}}^{\leq 0} \cap F(\mathcal{A})$

(ii) $F(\mathcal{A}^{\geq 0}) = \widetilde{\mathcal{A}}^{\geq 0} \cap F(\mathcal{A})$.

Proof. Most of the arguments here are identical for t-structures and co-t-structures; the small variations are identified as they occur. The proof is by induction on n.

If $A \in \widetilde{\mathcal{A}}^{2,\leq 0}$ and $B \in \widetilde{\mathcal{A}}^{2,\geq 1}$, then we have two distinguished triangles

$$A' \to A \to A'' \to$$
$$B' \to B \to B'' \to$$

with $A', A'' \in F(\mathcal{A}^{\leq 0}), B', B'' \in F(\mathcal{A}^{\geq 1})$. Then applying $\mathrm{Hom}_{\mathcal{D}}$, we get the following diagram which has exact rows and columns:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(A'', B') & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(A'', B) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(A'', B'') \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(A, B') & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(A, B) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(A, B'') \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(A', B') & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(A', B) & \longrightarrow & \mathrm{Hom}_{\mathcal{D}}(A', B'') \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
\end{array}$$

Since F respects the (co-)t-structure, we see that

$$\mathrm{Hom}_{\mathcal{D}}(A'', B') = \mathrm{Hom}_{\mathcal{D}}(A'', B'') = \mathrm{Hom}_{\mathcal{D}}(A', B') = \mathrm{Hom}_{\mathcal{D}}(A', B'') = 0.$$

Hence the groups $\mathrm{Hom}_{\mathcal{D}}(A, B')$ and $\mathrm{Hom}_{\mathcal{D}}(A, B'')$ (among others) are also zero, which in turn implies the desired $\mathrm{Hom}_{\mathcal{D}}(A, B) = 0$.

For the inclusions, the argument is basically the same in each case. As an example, we show that for a t-structure, $\widetilde{\mathcal{A}}^{2,\leq 0} \subset \widetilde{\mathcal{A}}^{2,\leq 1}$. Suppose $B \in \widetilde{\mathcal{A}}^{2,\leq 0} = F(\mathcal{A}^{\leq 0}) *_{\mathcal{D}} F(\mathcal{A}^{\leq 0})$. Then there is a distinguished triangle

$$A \to B \to C \to$$

with $A, C \in F(\mathcal{A}^{\leq 0})$. Thus $A[1], C[1] \in F(\mathcal{A}^{\leq -1}) \subset F(\mathcal{A}^{\leq 0})$, and so we get a distinguished triangle

$$A[1] \to B[1] \to C[1] \to,$$

which shows that $B \in \widetilde{\mathcal{A}}^{2,\leq 1}$.

For general $n > 2$, the inclusion argument can be repeated *mutatis mutandis* using the induction hypothesis. We get an analogous Hom diagram and the Hom vanishing result this time follows from the induction hypothesis and the fact that $(F(\mathcal{A}))^{n-1}$ is a full subcategory.

To prove the existence of the necessary distinguished triangles, it is convenient to use the $*$ notation from definition 2.5 (the symbol $*$ in this proof will always mean $*_{\mathcal{D}}$). In this notation, we need to see that $(F(\mathcal{A}))^n \subset \widetilde{\mathcal{A}}^{n,\leq 0} * \widetilde{\mathcal{A}}^{n,\geq 1}$.

Toward this end, we need two facts:

(i) $F(\mathcal{A}) \subset F(\mathcal{A}^{\leq 0}) * F(\mathcal{A}^{\geq 1})$

(ii) $F(\mathcal{A}^{\geq 1}) * F(\mathcal{A}^{\leq 0}) \subset F(\mathcal{A}^{\leq 0}) * F(\mathcal{A}^{\geq 1})$.

We can prove (i) in a straightforward and simple way. If $B \in F(\mathcal{A})$, then $B \simeq F(B')$, for some $B' \in \mathcal{A}$. Then we can find a distinuished triangle $A' \to B' \to C' \to A'[1]$ with $A' \in \mathcal{A}^{\leq 0}$ and $C' \in \mathcal{A}^{\geq 1}$. The image of this distinguished triangle under F shows that $B \in F(\mathcal{A}^{\leq 0}) * F(\mathcal{A}^{\geq 1})$.

For (ii), we need a slightly more involved argument. Let $B \in F(\mathcal{A}^{\geq 1}) * F(\mathcal{A}^{\leq 0})$ with distinguished triangle

$$A \longrightarrow B \longrightarrow C \xrightarrow{u} A[1].$$

We can assume that $A = F(A')$ for some $A' \in \mathcal{A}^{\geq 1}$ and $C = F(C')$ for some $C' \in \mathcal{A}^{\leq 0}$. Then we have $A'[1] \in \mathcal{A}^{\geq 0}$ and, in the case of a t-structure, $u = F(v)$ for some $v \in \mathrm{Hom}(C', A'[1])$ since F respects the t-structure. Thus $F(\mathrm{cone}(v)) \simeq \mathrm{cone}(u)$ and so $\mathrm{cone}(u) \in F(\mathcal{A})$. Then the diagram

$$\begin{array}{ccccccc} C & \xrightarrow{u} & A[1] & \longrightarrow & \mathrm{cone}(u) & \longrightarrow & C[1] \\ \| & & \| & & & & \| \\ C & \xrightarrow{u} & A[1] & \longrightarrow & B[1] & \longrightarrow & C[1], \end{array}$$

shows that $B[1]$ is isomorphic to $\mathrm{cone}(u)$ and so

$$B \simeq \mathrm{cone}(u)[-1] \simeq F(\mathrm{cone}(v))[-1] \simeq F(\mathrm{cone}(v)[-1]) \in F(\mathcal{A}).$$

Thus using (i), we see that $B \in F(\mathcal{A}^{\leq 0}) * F(\mathcal{A}^{\geq 1})$. In the case of a co-$t$-structure, we can be more precise. Since $A'[1] \in \mathcal{A}^{\geq 1}$, we see that u is just the 0 morphism and so B is actually isomorphic to the direct sum $A \oplus C$. Thus the triangle showing that $B \in F(\mathcal{A}^{\leq 0}) * F(\mathcal{A}^{\geq 1})$ is just

$$C \to B \to A \to C[1].$$

Now that we have (i) and (ii), we can see that

$$\begin{aligned}(F(\mathcal{A}))^n &= \underbrace{F(\mathcal{A}) * \cdots * F(\mathcal{A})}_{n \text{ terms}} \\ &\subset \underbrace{(F(\mathcal{A}^{\leq 0}) * F(\mathcal{A}^{\geq 1})) * \cdots * (F(\mathcal{A}^{\leq 0}) * F(\mathcal{A}^{\geq 1}))}_{n \text{ pairs}} \\ &\subset \underbrace{F(\mathcal{A}^{\leq 0}) * \cdots * F(\mathcal{A}^{\leq 0})}_{n \text{ terms}} * \underbrace{F(\mathcal{A}^{\geq 1}) * \cdots * F(\mathcal{A}^{\geq 1})}_{n \text{ terms}} \\ &= \widetilde{\mathcal{A}}^{n,\leq 0} * \widetilde{\mathcal{A}}^{n,\geq 1},\end{aligned}$$

as desired.

The properties that must be verified to see that the unions

$$\begin{aligned}\widetilde{\mathcal{A}}^{\leq 0} &= \bigcup_{n>0} \widetilde{\mathcal{A}}^{n,\leq 0} \\ \widetilde{\mathcal{A}}^{\geq 0} &= \bigcup_{n>0} \widetilde{\mathcal{A}}^{n,\geq 0}\end{aligned}$$

define a (co-)t-structure can now be checked by working in $\widetilde{\mathcal{A}}^n$ for appropriate choices of n.

In order to prove uniqueness, suppose that $(\widehat{\mathcal{A}}^{\leq 0}, \widehat{\mathcal{A}}^{\geq 0})$ is another (co-)t-structure on $\widetilde{\mathcal{A}}$ with

$$\begin{aligned}F(\mathcal{A}^{\leq 0}) &= \widehat{\mathcal{A}}^{\leq 0} \cap F(\mathcal{A}) \\ F(\mathcal{A}^{\geq 0}) &= \widehat{\mathcal{A}}^{\geq 0} \cap F(\mathcal{A}).\end{aligned}$$

We show by induction that for any $n > 0$, we have the equalities

$$\begin{aligned}\widetilde{\mathcal{A}}^{n,\leq 0} &= \widehat{\mathcal{A}}^{\leq 0} \cap \widetilde{\mathcal{A}}^n \\ \widetilde{\mathcal{A}}^{n,\geq 0} &= \widehat{\mathcal{A}}^{\geq 0} \cap \widetilde{\mathcal{A}}^n.\end{aligned}$$

This is just a restatement of the above for $n = 1$. Suppose $n > 1$. If $A \in \widetilde{\mathcal{A}}^{n,\leq 0}$, then there is a distinguished triangle

$$A' \to A \to A'' \to A'[1]$$

with $A' \in \widetilde{\mathcal{A}}^{1,\leq 0} = F(\mathcal{A}^{\leq 0})$ and $A'' \in \widetilde{\mathcal{A}}^{n-1,\leq 0}$. Using the induction hypothesis, we have $A', A'' \in \widehat{\mathcal{A}}^{\leq 0}$. So for any $B \in \widehat{\mathcal{A}}^{\geq 1}$, we get an exact sequence

$$\cdots \to \mathrm{Hom}(A'', B) \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A', B) \to \cdots,$$

which shows that $\mathrm{Hom}(A, B) = 0$. Hence $A \in \widehat{\mathcal{A}}^{\leq 0}$. On the other hand, if $A \in \widehat{\mathcal{A}}^{\leq 0} \cap \widetilde{\mathcal{A}}^n$, then for any $B \in \widetilde{\mathcal{A}}^{n,\geq 1}$ write

$$B' \to B \to B'' \to B'[1]$$

with $B' \in \widetilde{\mathcal{A}}^{1,\geq 1} = F(\mathcal{A}^{\geq 1})$ and $B'' \in \widetilde{\mathcal{A}}^{n-1,\geq 1}$. By induction, $B', B'' \in \widehat{\mathcal{A}}^{\geq 1}$ and so the exact sequence

$$\cdots \to \mathrm{Hom}(A, B') \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B'') \to \cdots,$$

shows that $\mathrm{Hom}(A, B) = 0$. Now $(\widetilde{\mathcal{A}}^{n,\leq 0}, \widetilde{\mathcal{A}}^{n,\geq 0})$ does not necessarily form a (co-)t-structure on $\widetilde{\mathcal{A}}^n$ (in fact, $\widetilde{\mathcal{A}}^n$ may not even be triangulated), but nonetheless we still have the fact that $\mathrm{Hom}(A, B) = 0$ for all $B \in \widetilde{\mathcal{A}}^{n,\geq 1}$ implies that $A \in \widetilde{\mathcal{A}}^{n,\leq 0}$. This follows from the truncation distinguished triangle in the case of t-structures and from the fact that $\widetilde{\mathcal{A}}^{n,\leq 0}$ is closed under direct summands in the case of co-t-structures. Now taking the union over all $n > 0$ gives the desired equality between the (co-)t-structures. □

Proposition 3.3. *Let $F : \mathcal{A} \to \mathcal{D}$ be a triangulated functor which respects the baric structure $\left(\{\mathcal{A}^{\leq w}\}, \{\mathcal{A}^{\geq w}\}\right)_{w\in\mathbb{Z}}$ on $\mathcal{A}$. Then the pairs of strictly full subcategories $\left(\{\widetilde{\mathcal{A}}^{\leq w}\}, \{\widetilde{\mathcal{A}}^{\geq w}\}\right)_{w\in\mathbb{Z}}$ define a baric structure on $\widetilde{\mathcal{A}}$, where*

$$\widetilde{\mathcal{A}}^{\leq w} = \bigcup_{n>1} \widetilde{\mathcal{A}}^{n,\leq w} = \bigcup_{n>1} \underbrace{F(\mathcal{A}^{\leq w}) *_{\mathcal{D}} \cdots *_{\mathcal{D}} F(\mathcal{A}^{\leq w})}_{(n \text{ terms})}$$

$$\widetilde{\mathcal{A}}^{\geq w} = \bigcup_{n>1} \widetilde{\mathcal{A}}^{n,\geq w} = \bigcup_{n>1} \underbrace{F(\mathcal{A}^{\geq w}) *_{\mathcal{D}} \cdots *_{\mathcal{D}} F(\mathcal{A}^{\geq w})}_{n \text{ terms}}.$$

Moreover, this is the unique baric structure having the property that for all w we have

(i) $F(\mathcal{A}^{\leq w}) = \widetilde{\mathcal{A}}^{\leq w} \cap F(\mathcal{A})$

(ii) $F(\mathcal{A}^{\geq w}) = \widetilde{\mathcal{A}}^{\geq w} \cap F(\mathcal{A})$.

Proof. The proofs of the axioms for a baric structure are completely analogous to the previous proposition. All that remains is to show that each of these subcategories is thick. That is, we need to see that for each $w \in \mathbb{Z}$, the subcategories $\widetilde{\mathcal{A}}^{\leq w}$ and $\widetilde{\mathcal{A}}^{\geq w}$ are closed under shifts (in both directions), cones, and direct summands. Closure under shifts follows directly from rotation. Let $A \xrightarrow{f} B$ be a morphism between two objects of $\widetilde{\mathcal{A}}^{n,\leq w}$ (resp. $\geq w$). Then by associativity of $*$ and rotation, $cone(f)$ is in $\widetilde{\mathcal{A}}^{2n,\leq w}$ (resp. $\geq w$) and hence in $\widetilde{\mathcal{A}}^{\leq w}$ (resp. $\geq w$). Closure under direct summands follows from the orthogonality of $\widetilde{\mathcal{A}}^{\leq w}$ and $\widetilde{\mathcal{A}}^{\geq w+1}$ in the same way that it does for t-structures. Uniqueness of the baric structure is also proved nearly identically to the statement for t-structures. □

Note that Propositions 3.2 and 3.3 justify the terminology of definition 3.1. Using this terminology, the propositions show that if a triangulated functor respects some semiorthogonal filtration, then we get a corresponding filtration on the triangulated category generated by the essential image of the functor.

3.2 Gluing Semiorthogonal Filtrations

In this section, we abstract the proof of the main theorem in [Bez1] to the more general setting of triangulated categories. The basic idea of the proof is based on the proof of [Bez1, Theorem 1], but the abstraction allows for applications to other settings and also elucidates previously unrecognized facets. The primary impact of our work is that we have separated the geometry from the homological algebra. On a technical note, even though truncation is not functorial when we have a co-t-structure, we will use the notation of truncation functors to simplify the exposition. The (non)functoriality of truncation will not play a role in the proof of this theorem.

Theorem 3.4. *Let $\mathcal{D}$, $\mathcal{D}_Z$ and $\mathcal{D}_U$ be triangulated categories and suppose that we have (co-)t-structures $(\mathcal{D}_Z^{\leq 0}, \mathcal{D}_Z^{\geq 0})$ on $\mathcal{D}_Z$ and $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ on $\mathcal{D}_U$. Furthermore, suppose that*

(G1) There is a triangulated functor $i_* : \mathcal{D}_Z \to \mathcal{D}$ and an essentially surjective triangulated functor $j^* : \mathcal{D} \to \mathcal{D}_U$.

(G2) There is a (co-)t-structure $(\widetilde{\mathcal{D}_Z}^{\leq 0}, \widetilde{\mathcal{D}_Z}^{\geq 0})$ on $\widetilde{\mathcal{D}_Z}$ (the triangulated category generated by the essential image of i_*) such that

$$i_*\left(\mathcal{D}_Z^{\leq 0}\right) \subset \widetilde{\mathcal{D}_Z}^{\leq 0}$$
$$i_*\left(\mathcal{D}_Z^{\geq 0}\right) \subset \widetilde{\mathcal{D}_Z}^{\geq 0}.$$

(G3) (i) For every $A \in \mathcal{D}$ (and any choice of $\tau_{\leq 0}^U j^* A$) there is an object $A^- \in \mathcal{D}$ and a morphism $A^- \to A$ such that

(a) $j^* A^- \simeq \tau_{\leq 0}^U j^* A$.

(b) $Hom_{\mathcal{D}}(A^-, B) = 0$ for all $B \in \widetilde{\mathcal{D}_Z}^{\geq 1}$.

(ii) For every $A \in \mathcal{D}$ (and any choice of $\tau_{\geq 1}^U j^* A$) there is an object $A^+ \in \mathcal{D}$ and a morphism $A \xrightarrow{g} A^+$ such that

(a) $\tau_{\geq 1}^U j^* A \simeq j^* A^+$.

(b) $Hom_{\mathcal{D}}(B, A^+) = 0$ for all $B \in \widetilde{\mathcal{D}_Z}^{\leq 0}$.

(G4) If $A \xrightarrow{f} B$ in $\mathcal{D}$ with $j^* f = 0$, then f factors through an object of $\widetilde{\mathcal{D}_Z}$.

(G5) $\widetilde{\mathcal{D}_Z} = \{A \in \mathcal{D} \mid j^* A = 0\}$.

Then the strictly full subcategories of $\mathcal{D}$

$$\mathcal{D}^{\leq 0} = \{A \in \mathcal{D} \mid j^* A \in \mathcal{D}_U^{\leq 0}, Hom_{\mathcal{D}}(A, B) = 0 \text{ for all } B \in \widetilde{\mathcal{D}_Z}^{\geq 1}\}$$
$$\mathcal{D}^{\geq 0} = \{B \in \mathcal{D} \mid j^* B \in \mathcal{D}_U^{\geq 0}, Hom_{\mathcal{D}}(A, B) = 0 \text{ for all } A \in \widetilde{\mathcal{D}_Z}^{\leq -1}\}$$

give a (co-)t-structure on $\mathcal{D}$. Moreover, this is the unique (co-)t-structure with

(i) $i_*(\mathcal{D}_Z^{\leq 0}) = \mathcal{D}^{\leq 0} \cap i_*(\mathcal{D}_Z)$ *and* $i_*(\mathcal{D}_Z^{\geq 0}) = \mathcal{D}^{\geq 0} \cap i_*(\mathcal{D}_Z)$

(ii) $j^*(\mathcal{D}^{\leq 0}) \subset \mathcal{D}_U^{\leq 0}$ *and* $j^*(\mathcal{D}^{\geq 0}) \subset \mathcal{D}_U^{\geq 0}$.

Proof. Denote the truncation associated to the (co-)t-structure on $\widetilde{\mathcal{D}_Z}$ by $\tau_\bullet^{\widetilde{\mathcal{D}_Z}}$, where $\bullet$ is $\leq n$ or $\geq n$ for some integer n. The inclusion axiom for a (co-)t-structure follows from the fact that the translation functor commutes with i_* and j^*.

Let $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$. If $f \in \mathrm{Hom}(A, B)$, then from the definitions we see that $j^* f = 0$, and so by (G4) we get a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & E & \end{array}$$

with $E \in \widetilde{\mathcal{D}_Z}$. Using [BBD, Proposition 1.1.9], we see that the morphisms g and h extend to morphisms of distinguished triangles

$$\begin{array}{ccccccc} A & \xrightarrow{\mathrm{id}} & A & \longrightarrow & 0 & \longrightarrow & A[1] \\ \vdots & & \downarrow{\scriptstyle g} & & \vdots & & \vdots \\ \tau_{\leq 0}^{\widetilde{\mathcal{D}_Z}} E & \longrightarrow & E & \longrightarrow & \tau_{\geq 1}^{\widetilde{\mathcal{D}_Z}} E & \longrightarrow & \tau_{\leq 0}^{\widetilde{\mathcal{D}_Z}} E[1] \\ \vdots & & \downarrow{\scriptstyle h} & & \vdots & & \vdots \\ 0 & \longrightarrow & B & \xrightarrow{\mathrm{id}} & B & \longrightarrow & 0 \end{array}$$

where each square commutes. From this it is clear that $f = 0$.

Now we show that $\mathcal{D} = \mathcal{D}^{\leq 0} * \mathcal{D}^{\geq 1}$. Let $A \in \mathcal{D}$. Then (G3) gives an object $A^- \in \mathcal{D}$ and a morphism $f : A^- \to A$. Since $j^* A^- \in \mathcal{D}_U^{\leq 0}$, the statement of (G3)(i)(b) says that $A^- \in \mathcal{D}^{\leq 0}$. Moreover, we can complete the morphism $A^- \to A$ to get a distinguished triangle in $\mathcal{D}$

$$A^- \longrightarrow A \longrightarrow A' \longrightarrow A^-[1]$$

which restricts to the truncation distinguished triangle

$$j^* A^- \longrightarrow j^* A \longrightarrow \tau_{\geq 1}^U j^* A \longrightarrow j^* A^-[1]$$

in $\mathcal{D}_U$. Hence $j^* A' \in \mathcal{D}_U^{\geq 1}$.

On the other hand, (G3) gives an object $A'^+ \in \mathcal{D}$ and an isomorphism $\tau_{\geq 1}^U j^* A' \simeq j^* A'^+$. Together with (G3)(ii)(b), this implies that $A'^+ \in \mathcal{D}^{\geq 1}$. The axiom (G3) also gives a morphism $A' \to A'^+$, and so we can define $A'' = \mathrm{cone}(A' \to A'^+)[-1]$ and get a distinguished triangle in $\mathcal{D}$

$$A'' \longrightarrow A' \longrightarrow A'^+ \longrightarrow A''[1]$$

which restricts to the truncation distinguished triangle

$$j^*A'' \longrightarrow j^*A' \longrightarrow \tau^U_{\geq 1} j^*A' \longrightarrow j^*A''[1].$$

Since $j^*A' \in \mathcal{D}_U^{\geq 1}$, we see that $j^*A'' \simeq 0$ and then (G5) tells us that $A'' \in \widetilde{\mathcal{D}_Z}$. Now define

$$A''^- = \left(\tau^{\widetilde{\mathcal{D}_Z}}_{\leq 0} A''\right)$$
and
$$A''^+ = \left(\tau^{\widetilde{\mathcal{D}_Z}}_{>0} A''\right).$$

Then the facts that $A''^- \in \mathcal{D}^{\leq 0}$ and $A''^+ \in \mathcal{D}^{\geq 1}$ follow directly from (G5) and the Hom vanishing property of a (co-)t-structure. Thus, as desired, we get

$$\begin{aligned} A \in [A^-] * [A''^-] * [A''^+] * [A'^+] &\subset \mathcal{D}^{\leq 0} * \mathcal{D}^{\leq 0} * \mathcal{D}^{\geq 1} * \mathcal{D}^{\geq 1} \\ &\subset \mathcal{D}^{\leq 0} * \mathcal{D}^{\geq 1}. \end{aligned}$$

Now it remains to see that for a co-t-structure, the subcategories $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}$ are closed under direct summands. This is equivalent to seeing that the Hom vanishing characterizes these subcategories. Suppose $A \in \mathcal{D}$ with $\mathrm{Hom}(A, B) = 0$ for all $B \in \mathcal{D}^{\geq 1}$. Then $\widetilde{\mathcal{D}_Z}^{\geq 1} \subset \mathcal{D}^{\geq 1}$, so $\mathrm{Hom}(A, \widetilde{\mathcal{D}_Z}^{\geq 1}) = 0$. Now we need to see that $j^*A \in \mathcal{D}_U^{\leq 0}$. Since j^* is essentially surjective, it suffices to show that $\mathrm{Hom}(j^*A, j^*C) = 0$ for all $C \in \mathcal{D}$ with $j^*C \in \mathcal{D}_U^{\geq 1}$. Take such a C and find C^+ as in (G3) with $j^*C^+ \simeq j^*C$. Thus $C^+ \in \mathcal{D}^{\geq 1}$, so $\mathrm{Hom}(A, C^+) = 0$ and since j^* is full, we see that $\mathrm{Hom}(j^*A, j^*C) = \mathrm{Hom}(j^*A, j^*C^+) = 0$. The argument for $\mathcal{D}^{\geq 0}$ is completely analogous.

The uniqueness statement follows directly from the definitions and the uniqueness of the induced (co-)t-structure on $\widetilde{\mathcal{D}_Z}$. □

Note that the assumption that j^* is essentially surjective is only used to prove that the two halves of the co-t-structure are closed under direct summands. In order to construct a minimal extension functor in the next section, however, we will need to assume that j^* is essentially surjective. Hence in the case of t-structures this can be omitted. Also, the statement in (G3) necessitating the existence of A^- and A^+ for each choice of truncation may also be omitted in the case of t-structures since all such choices are isomorphic here. Without much additional work, we can see that a similar gluing theorem also holds for baric structures. We follow the notation in [AT] and use $\beta_{\leq w}, \beta_{\geq w}$ for the baric truncation functors.

Theorem 3.5. *Let $\mathcal{D}$, $\mathcal{D}_Z$ and $\mathcal{D}_U$ be triangulated categories and suppose that we have baric structures $(\{\mathcal{D}_Z^{\leq w}\}, \{\mathcal{D}_Z^{\geq w}\})_{w\in\mathbb{Z}}$ on $\mathcal{D}_Z$ and $(\{\mathcal{D}_U^{\leq w}\}, \{\mathcal{D}_U^{\geq w}\})_{w\in\mathbb{Z}}$ on $\mathcal{D}_U$. Furthermore, suppose that*

(G1) There is a triangulated functor $i_* : \mathcal{D}_Z \to \mathcal{D}$ and a full triangulated functor $j^* : \mathcal{D} \to \mathcal{D}_U$.

(G2) There is a baric structure $\left(\{\widetilde{\mathcal{D}_Z}^{\leq w}\}, \{\widetilde{\mathcal{D}_Z}^{\geq w}\}\right)$ *on* $\widetilde{\mathcal{D}_Z}$ *such that for all* $w \in \mathbb{Z}$

$$i_*\left(\mathcal{D}_Z^{\leq w}\right) \subset \widetilde{\mathcal{D}_Z}^{\leq w}$$
$$i_*\left(\mathcal{D}_Z^{\geq w}\right) \subset \widetilde{\mathcal{D}_Z}^{\geq w}.$$

(G3) *(i) For every* $A \in \mathcal{D}$ *and every* $w \in \mathbb{Z}$ *there is an object* $A_w^- \in \mathcal{D}$ *and a morphism* $A_w^- \to A$ *such that*

(a) $j^*A_w^- \simeq \beta^U_{\leq w} j^*A$.

(b) $Hom_{\mathcal{D}}(A_w^-, B) = 0$ *for all* $B \in \widetilde{\mathcal{D}_Z}^{\geq w+1}$.

(ii) For every $A \in \mathcal{D}$ *there is an object* $A_w^+ \in \mathcal{D}$ *and a morphism* $A \to A_w^+$

(a) $j^*A_w^+ \simeq \beta^U_{\geq w+1} j^*A$.

(b) $Hom_{\mathcal{D}}(B, A_w^+) = 0$ *for all* $B \in \widetilde{\mathcal{D}_Z}^{\leq w}$.

(G4) If $A \xrightarrow{f} B$ *in* $\mathcal{D}$ *with* $j^*f = 0$*, then* f *factors through an object of* $\widetilde{\mathcal{D}_Z}$.

(G5) $\widetilde{\mathcal{D}_Z} = \{A \in \mathcal{D} \mid j^*A = 0\}$.

Then the strictly full subcategories of $\mathcal{D}$

$$\mathcal{D}^{\leq w} = \{A \in \mathcal{D} \mid j^*A \in \mathcal{D}_U^{\leq w}, Hom_{\mathcal{D}}(A, B) = 0 \text{ for all } B \in \widetilde{\mathcal{D}_Z}^{\geq w+1}\}$$
$$\mathcal{D}^{\geq w} = \{B \in \mathcal{D} \mid j^*B \in \mathcal{D}_U^{\geq w}, Hom_{\mathcal{D}}(A, B) = 0 \text{ for all } A \in \widetilde{\mathcal{D}_Z}^{\leq w-1}\}$$

give a baric structure on $\mathcal{D}$*. Moreover, this is the unique baric structure such that for every* $w \in \mathbb{Z}$ *we have*

(i) $i_*(\mathcal{D}_Z^{\leq w}) = \mathcal{D}^{\leq w} \cap i_*(\mathcal{D}_Z)$ *and* $i_*(\mathcal{D}_Z^{\geq w}) = \mathcal{D}^{\geq w} \cap i_*(\mathcal{D}_Z)$

(ii) $j^*(\mathcal{D}^{\leq w}) \subset \mathcal{D}_U^{\leq w}$ *and* $j^*(\mathcal{D}^{\geq w}) \subset \mathcal{D}_U^{\geq w}$.

Proof. As in Proposition 3.3 above, the axioms for a baric structure are satisfied by nearly identical arguments to those in the theorem. What remains to be checked is thickness. Closure under shifts and direct summands follow in the obvious way from the axioms, and so we only need to see that $\mathcal{D}^{\leq w}$ and $\mathcal{D}^{\geq w}$ are closed under cones. If $A, B \in \mathcal{D}^{\leq w}$ and $f \in \mathrm{Hom}(A, B)$, then the distinguished triangle

$$A \xrightarrow{f} B \to \mathrm{cone}(f) \to A[1]$$

yields the exact sequence

$$\cdots \to \mathrm{Hom}(A[1], C) \to \mathrm{Hom}(\mathrm{cone}(f), C) \to \mathrm{Hom}(B, C) \cdots \to$$

for any $C \in \widetilde{\mathcal{D}_Z}^{\geq w+1}$. Since $\mathcal{D}^{\leq w}$ is closed under shifts, we see the first and last terms here are 0. Thus $\mathrm{cone}(f) \in \mathcal{D}^{\leq w}$ also. The proof that $\mathcal{D}^{\geq w}$ is closed under cones is similar. $\square$

3.3 A Minimal Extension Functor

In this section, we show that with a slightly stronger hypothesis, we are able to find a fully faithful functor from the heart of the t-structure on $\mathcal{D}_U$ to the heart of the t-structure on $\mathcal{D}$ obtained by gluing. This generalizes the construction of the $\mathcal{IC}$ (or minimal extension) functor discussed in [Bez1]. Again in this case, the abstraction is the key that removes any mystery from the construction of the $\mathcal{IC}$ functor. Suppose we are in the setup of the previous section in the case of t-structures but that we have replaced (G3) with the stronger condition

(G3′) (i) For every $A \in \mathcal{D}$ there is an object $A^- \in \mathcal{D}$ and a morphism $A^- \to A$ such that
  (a) $j^*A^- \simeq \tau^U_{\leq 0} j^*A$.
  (b) $\mathrm{Hom}_{\mathcal{D}}(A^-, B) = 0$ for all $B \in \widetilde{\mathcal{D}_Z}^{\geq 0}$.

(ii) For every $A \in \mathcal{D}$ there is an object $A^+ \in \mathcal{D}$ and a morphism $A \to A^+$
  (a) $j^*A^+ \simeq \tau^U_{\geq 1} j^*A$.
  (b) $\mathrm{Hom}_{\mathcal{D}}(B, A^+) = 0$ for all $B \in \widetilde{\mathcal{D}_Z}^{\leq 1}$.

Then we can use $(\widetilde{\mathcal{D}_Z}^{\leq -1}, \widetilde{\mathcal{D}_Z}^{\geq -1})$ or $(\widetilde{\mathcal{D}_Z}^{\leq 1}, \widetilde{\mathcal{D}_Z}^{\geq 1})$ in addition to $(\widetilde{\mathcal{D}_Z}^{\leq 0}, \widetilde{\mathcal{D}_Z}^{\geq 0})$ in the gluing theorem to get two new t-structures $({}^-\mathcal{D}^{\leq 0}, {}^-\mathcal{D}^{\geq 0})$ and $({}^+\mathcal{D}^{\leq 0}, {}^+\mathcal{D}^{\geq 0})$ in addition to $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. Denote the associated truncation functors by $\tau^-_\bullet$ and $\tau^+_\bullet$, respectively. Let $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$, while $\widetilde{\mathcal{C}_Z} = \widetilde{\mathcal{D}_Z}^{\leq 0} \cap \widetilde{\mathcal{D}_Z}^{\geq 0}$ and $\mathcal{C}_U = \mathcal{D}_U^{\leq 0} \cap \mathcal{D}_U^{\geq 0}$.

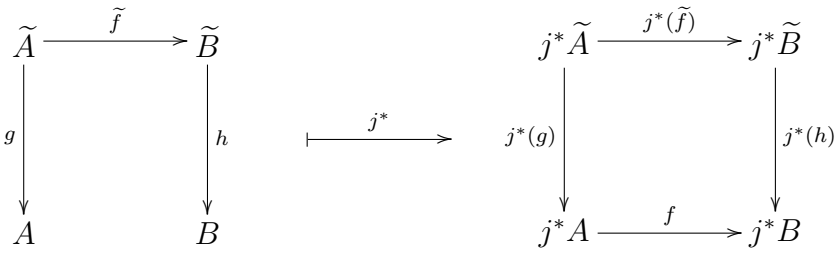
Definition 3.6. Define a full subcategory of $\mathcal{D}$

$$\widetilde{\mathcal{C}_U} = {}^-\mathcal{D}^{\leq 0} \cap {}^+\mathcal{D}^{\geq 0}.$$

Notice that $\widetilde{\mathcal{C}_U} \subset \mathcal{C}$ and in particular, $j^*\widetilde{\mathcal{C}_U} \subset \mathcal{C}_U$.

In order to get a fully faithful functor from the heart of the t-structure on $\mathcal{D}_U$ to the heart of the t-structure on $\mathcal{D}_Z$, we also need to make an additional assumption about the functor j^* which is immediately implied by Lemma 2.31 in the case of coherent sheaves. The additional assumption can be summarized by replacing (G1) with the stronger condition

(G1′) There is a triangulated functor $i_* : \mathcal{D}_Z \to \mathcal{D}$ and a triangulated functor $j^* : \mathcal{D} \to \mathcal{D}_U$ such that for any two objects $A, B \in \widetilde{\mathcal{C}_U}$ with a morphism $f : j^*A \to j^*B$, there exist objects $\widetilde{A}$ and $\widetilde{B}$ in $\widetilde{\mathcal{C}_U}$ and diagrams of morphisms



where $j^*(g)$ and $j^*(h)$ are isomorphisms.

Proposition 3.7. *The functor $j^*|_{\widetilde{\mathcal{C}_U}} : \widetilde{\mathcal{C}_U} \to \mathcal{C}_U$ is an equivalence of categories.*

Proof. First, let us see that $j^*|_{\widetilde{\mathcal{C}_U}}$ is essentially surjective. Let $A \in \mathcal{C}_U$ and $\widetilde{A}$ an object of $\mathcal{D}$ such that $j^*\widetilde{A} \simeq A$. Define

$$\overline{A} = \tau^-_{\leq 0}\tau^+_{\geq 0}\widetilde{A}.$$

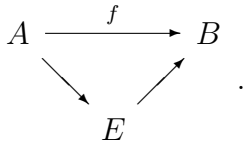
Clearly $\overline{A} \in {}^-\mathcal{D}^{\leq 0}$. From the distinguished triangle

$$(\tau^-_{\geq 1}\tau^+_{\geq 0}\widetilde{A})[-1] \to \overline{A} \to \tau^+_{\geq 0}\widetilde{A} \to,$$

we see that $\overline{A} \in \widetilde{\mathcal{C}_U}$ if ${}^-\mathcal{D}^{\geq 2} \subset {}^+\mathcal{D}^{\geq 0}$ since for any t-structure, each half is closed under extensions. This is obvious, however, when one writes down the definitions. The fact that $j^*\overline{A} \simeq A$ follows by applying j^* to all of the appropriate diagrams in the gluing theorem and following the isomorphisms.

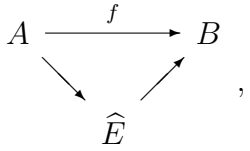
Now we need to see that $j^*|_{\widetilde{\mathcal{C}_U}}$ is full. First, suppose that $A, B \in \widetilde{\mathcal{C}_U}$ with a morphism $f : j^*A \to j^*B$. Let $\widetilde{A}$, $\widetilde{B}$, g and h be as in (G1′). Then since $j^*(g)$ and $j^*(h)$ are isomorphisms, the kernel an cokernel of g and h must be in $\widetilde{\mathcal{D}_Z}$. But $\widetilde{\mathcal{D}_Z} \cap \widetilde{\mathcal{C}_U} = \{0\}$ and so g and h are isomorphisms. Thus the composition $h \circ \widetilde{f} \circ g^{-1} : A \to B$ is sent to f by j^*.

It only remains to see that $j^*|_{\widetilde{\mathcal{C}_U}}$ is faithful. So let $A, B \in \widetilde{\mathcal{C}_U}$ and $f : A \to B$ with $j^*f = 0$. Then f factors through an object of $\widetilde{\mathcal{D}_Z}$, and we get a diagram



.

Applying $\tau^-_{\leq 0}\tau^+_{\geq 0}$ to this diagram gives



,

where $\widehat{E}$ is also in $\widetilde{\mathcal{D}_Z}$. But $\widetilde{\mathcal{D}_Z} \cap \widetilde{\mathcal{C}_U} = \{0\}$, so we see that $f = 0$. □

Definition 3.8. We define $\mathcal{IC} : \mathcal{C}_U \to \widetilde{\mathcal{C}_U} \subset \mathcal{C}$ to be the inverse equivalence of $j^*|_{\widetilde{\mathcal{C}_U}}$.

Proposition 3.9. *Every simple object of $\mathcal{C}$ is either isomorphic to a simple object of $\widetilde{\mathcal{C}_Z}$ or to a simple object of $\widetilde{\mathcal{C}_U}$.*

Proof. Let $A \in \mathcal{C}$ be simple with A not in the essential image of $\mathcal{IC}$. Then either there is a nonzero morphism in $\mathrm{Hom}(A, B)$ for some $B \in \widetilde{\mathcal{D}_Z}^{\geq 0}$, or there is a nonzero morphsim in $\mathrm{Hom}(B, A)$ for some $B \in \widetilde{\mathcal{D}_Z}^{\leq 0}$. In the first case, let $f' : A \to B$ be such a morphism. Since $\tau_{\leq 0}$ is adjoint to the inclusion, f' corresponds to some nonzero morphism $f : A \to \tau_{\leq 0}B$ in the abelian category $\mathcal{C}$. Taking the cone of f gives a distinguished triangle

$$A \xrightarrow{f} \tau_{\leq 0}B \to \mathrm{cone}(f) \to A[1], \qquad (*)$$

which yields the exact sequence

$$0 \to H^{-1}\text{cone}(f) \to A \to \tau_{\leq 0}B \to H^0\text{cone}(f) \to 0$$

after taking t-cohomology. Since A is simple, we must have $H^{-1}\text{cone}(f) = 0$, and so we see that $H^0\text{cone}(f) \simeq \text{cone}(f)$. In particular, $\text{cone}(f) \in \mathcal{C}$. From $(*)$ we see that $j^*\text{cone}(f) \simeq j^*A[1]$, but $j^*\text{cone}(f) \in \mathcal{C}_U$ while $j^*A[1] \in \mathcal{C}_U[1]$. Hence $\text{cone}(f) \in \widetilde{\mathcal{C}_Z}$ and so also $A \in \widetilde{\mathcal{C}_Z}$. The other case is completely analogous. □

3.4 Equivariant Coherent Sheaves on a Noetherian Scheme

The motivating case for the previous two sections (as the notation suggests) is when $\mathcal{D}$ is the bounded derived category of equivariant coherent sheaves on a reasonable scheme X. Specifically, let X be a scheme of finite type over a noetherian base scheme S and G an affine group scheme of finite type over S acting on X. We restrict to the case where the morphism $f_G : G \to S$ is flat of finite type and Gorenstein and G acts on X with finitely many orbits. We will write the structure sheaf of X as $\mathcal{O}_X$. In what follows, a subscheme will always mean a G-invariant subscheme unless otherwise stated. Denote the G-equivariant coherent sheaves on X by $\mathcal{C}_G(X)$ and the bounded derived category of equivariant coherent sheaves on X by $\mathcal{D}^b_G(X)$ or usually just $\mathcal{D}_X$ or $\mathcal{D}$ when X is understood. If $k : Y \hookrightarrow X$ is a subscheme of X, then $\mathcal{D}_Y$ is shorthand for $\mathcal{D}^b_G(Y)$. The bounded derived category of G-equivariant quasicoherent sheaves will be denoted $\mathcal{D}^b_G(\mathcal{Q}_X)$ or just $\mathcal{D}(\mathcal{Q})$ if the other information is obvious. If $i : Z \hookrightarrow X$ is a subscheme of X, we will denote the underlying topological space of Z by $\underline{Z}$. We say that $\mathcal{F} \in \mathcal{D}_X$ is set-theoretically supported on the topological space $\underline{Y}$ of a subscheme $k : Y \to X$ if its cohomology sheaves are supported on $\underline{Y}$. We will use $\tau^{\text{std}}_\bullet$ for the truncation functor associated to the standard t-structure.

We wish to show now that we can in fact recover the perverse t-structure of Deligne given in Theorem 2.33 by applying the gluing theorem repeatedly. Let X/G be the set of G-orbits in X (assumed to be finite) and Z^{gen} the set of generic points for any subscheme Z. Let $p : X/G \to \mathbb{Z}$ be a monotone perversity. That is, for every G-orbit C contained in the closure of another G-orbit C', we have

$$p(C') \leq p(C).$$

Moreover, suppose that p is comonotone: the dual perversity defined by $\overline{p}(C) = -\dim C - p(C)$ is also monotone.

Lemma 3.10. *If $i : Z \hookrightarrow X$ is a reduced closed subscheme and $i' : Z' \hookrightarrow X$ any closed subscheme with underlying topological space $\underline{Z}$, then $i'_*\mathcal{D}_{Z'} \subset (i_*\mathcal{D}_Z)^n_{\mathcal{D}}$ for large enough n.*

Proof. Let $\mathcal{I}$ be the ideal sheaf corresponding to the closed subscheme Z, and let Z^n be the closed subscheme corresponding to the ideal sheaf $\mathcal{I}^n$. We will start by showing that $\mathcal{D}_{Z'} \subset \mathcal{D}_{Z^n}$. To see this, let $\mathcal{F} \in \mathcal{D}_{Z'}$ be represented by the bounded complex $\mathcal{G}^\bullet$ of objects in $\mathcal{C}_G(Z')$ and let $\mathcal{J}$ be the ideal sheaf corresponding to Z'. Then we can translate the desired

statement into a statement about modules. It is enough to show that any $\mathcal{G}^i \in \mathcal{C}_G(Z')$ is also in $\mathcal{C}_G(Z^n)$ for some n. This is the same as saying that there exists an n such that $\mathcal{I}^n \subset \mathcal{J}$, which follows from looking at the primary decompositions of $\mathcal{I}$ and $\mathcal{J}$ over any open subscheme. Thus if $i^n : Z^n \hookrightarrow X$ is the closed inclusion, then $i'_*\mathcal{D}_{Z'} \subset i^n_*\mathcal{D}_{Z^n}$.

Now, to see that $i^n_*\mathcal{D}_{Z^n} \subset (i_*\mathcal{D}_Z)^n_{\mathcal{D}}$, take $\mathcal{F} \in i^n_*\mathcal{D}_{Z^n}$ represented by the bounded complex $\mathcal{H}^\bullet$ of objects in $\mathcal{C}_G(Z^n)$. Then each $\mathcal{H}^i$ corresponds to an $\mathcal{O}_X/\mathcal{I}^n$-module, and we have the exact sequence

$$0 \to \mathcal{I}^{n-1}\mathcal{H}^i \to \mathcal{H}^i \to \mathcal{H}^i/\mathcal{I}^{n-1}\mathcal{H}^i \to 0.$$

Then since $i^n_* : \mathcal{C}_G(Z^n) \to \mathcal{C}_G(X)$ is fully faithful and i factors through i^n, when we pass back to the derived category, we see that $\mathcal{F} \in (i_*\mathcal{D}_Z)^n_{\mathcal{D}}$ and so $i'_*\mathcal{D}_{Z'} \subset (i_*\mathcal{D}_Z)^n_{\mathcal{D}}$ as desired.

In fact, $Obj((i_*\mathcal{D}_Z)^n_{\mathcal{D}}) = Obj(i^n_*\mathcal{D}_{Z^n})$: if $\mathcal{F} \in (i_*\mathcal{D}_Z)^n_{\mathcal{D}}$, then we have a distinguished triangle

$$\mathcal{G} \to \mathcal{F} \to \mathcal{H} \to$$

with $\mathcal{G} \in (i_*\mathcal{D}_Z)^{n-1}_{\mathcal{D}}$ and $\mathcal{H} \in i_*\mathcal{D}_Z$. If $\mathcal{G}'^\bullet, \mathcal{F}'^\bullet, \mathcal{H}'^\bullet$ are bounded complexes representing these objects, then $\mathcal{H}'^i$ is an $\mathcal{O}_X/\mathcal{I}$-module and by induction $\mathcal{G}'^i$ is an $\mathcal{O}_X/\mathcal{I}^{n-1}$-module. This implies then that $\mathcal{F}'^i$ is an $\mathcal{O}_X/\mathcal{I}^n$-module, as desired. It should be noted, however, that the subcategory $i^n_*\mathcal{D}_{Z^n}$ is not necessarily full, whereas the subcategory $(i_*\mathcal{D}_Z)^n_{\mathcal{D}}$ was defined as a full subcategory of $\mathcal{D}_X$. □

Lemma 3.11. *With the assumptions in the introduction, the scheme X admits a dualizing complex.*

Proof. See [Bez1, Proposition 1]. □

Theorem 3.12. *For each G-orbit $i_C : C \hookrightarrow X$, suppose $\mathcal{D}^b_G(\mathcal{C}_C)$ is given the t-structure*

$$\mathcal{D}^{\leq 0}_C = {}^{std}\mathcal{D}^{\leq p(C)}$$
$$\mathcal{D}^{\geq 0}_C = {}^{std}\mathcal{D}^{\geq p(C)}.$$

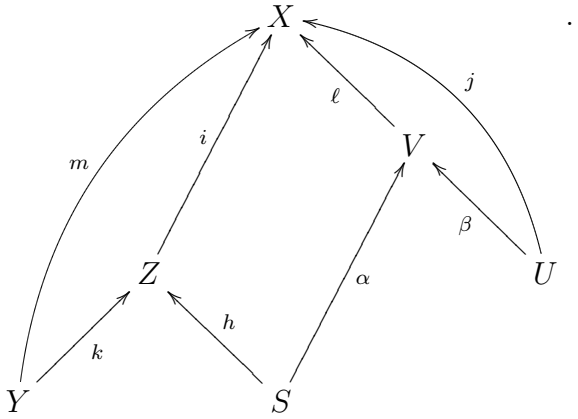
Then the t-structure $({}^p\mathcal{D}^{\leq 0}, {}^p\mathcal{D}^{\geq 0})$ on $\mathcal{D}^b_G(\mathcal{C}_X)$ obtained by iteratively gluing the respective t-structures on the G-orbits according to Theorem 3.4 above coincides with the perverse t-structure $(\mathcal{D}^{p,\leq 0}, \mathcal{D}^{p,\geq 0})$ of Deligne.

Proof. We use induction on the number of orbits. If there is only one orbit, then we can use [Bez1, Lemma 2(a)] to see that the t-structures trivially coincide.

In the general situation, choose an open orbit $j : U \hookrightarrow X$ (possible since we are assuming that G acts with finitely many orbits) and let $i : Z \hookrightarrow X$ be its closed complement in X with the reduced subscheme structure. We assume that the t-structure on $\mathcal{D}_Z$ has been built up by iteratively gluing the t-structures on the orbits contained in Z. We want to apply the gluing theorem once more, so we need to see that the hypotheses are satisfied and then verify that the t-structures indeed coincide. For the latter, it will be enough to check that ${}^p\mathcal{D}^{\leq 0} = \mathcal{D}^{p,\leq 0}$ since the other halves can be characterized as the right orthogonal complement to these.

The exact functors $i_* : \mathcal{C}_Z \to \mathcal{C}_X$ (direct image) and $j^* : \mathcal{C}_X \to \mathcal{C}_U$ (coherent inverse image) extend to triangulated functors of the respective derived categories in the usual way and so satisfy axiom (G1).

We show the existence of an induced t-structure on $\widetilde{\mathcal{D}_Z}$ required in (G2) by applying the gluing theorem within the induction argument. If Z is a single orbit, this follows from Proposition 3.2 since there i_* is fully faithful. Otherwise, let $h : S \hookrightarrow Z$ be an orbit which is open in Z with closed complement $k : Y \hookrightarrow Z$. Also define $\ell : V \to X$ be the complement of Y in X. For clarity, we summarize this information in the following diagram (with additional maps identified)

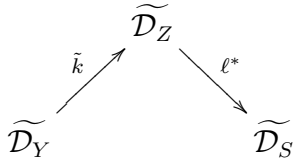


We may assume by induction that any closed subscheme $i_{Z'} : Z' \hookrightarrow X$ with fewer orbits than Z satisfies (G2) with respect to its inclusion in X. That is, if Z' has fewer orbits than Z, there is a t-structure on $\widetilde{\mathcal{D}_{Z'}} \subset \mathcal{D}_X$ containing $i_{Z'*}(\mathcal{D}_{Z'}^{\leq 0})$ and $i_{Z'*}(\mathcal{D}_{Z'}^{\geq 0})$. The base case is clear: if there is only one orbit in a given closed subscheme, this statement follows from Proposition 3.2 since the pushforward from any orbit is fully faithful on the heart of the standard t-structure.

Therefore, since Y is a closed subscheme with fewer orbits, we get a t-structure on $\widetilde{\mathcal{D}_Y} \subset \mathcal{D}_X$. Similarly, S is a closed subscheme of V and has fewer orbits than Z, so we get a t-structure on $\widetilde{\mathcal{D}_S} \subset \mathcal{D}_V$. Now m factors through i and so we get an inclusion

$$\tilde{k} : \widetilde{\mathcal{D}_Y} \hookrightarrow \widetilde{\mathcal{D}_Z} \subset \mathcal{D}_X.$$

Also, $\ell^*|_{\widetilde{\mathcal{D}_Z}}$ has essential image in $\widetilde{\mathcal{D}_S}$ since any sheaf supported topologically on the intersection of Z and V must be supported topologically on S. So we have



satisfying (G1). Now (G2) is easy in this case since $\tilde{k}$ is just the inclusion of a full subcategory. For (G3), fix $\mathcal{F} \in \widetilde{\mathcal{D}_Z}$. We will use $\tau_\bullet^S$ for the truncation functor associated to the t-structure on $\widetilde{\mathcal{D}_S}$. It is easy to see by following the proof of Proposition 3.2 that this is just a shift of the standard t-structure on $\widetilde{\mathcal{D}_S}$ by $p(S)$. To define $\mathcal{F}^+$ and $\mathcal{F}^-$, consider $\overline{\ell} : \overline{V} \to X$, the closure of V in X. Let

$$\mathcal{F}^- = \tau^{\mathrm{std}}_{\leq p(S)} \overline{\ell}_* \overline{\ell}^! \mathcal{F}$$
$$\mathcal{F}^+ = \mathbb{D} \tau^{\mathrm{std}}_{< \overline{p}(S)} \overline{\ell}_* \overline{\ell}^! \mathbb{D} \mathcal{F}.$$

Then since $\ell^* = \ell^*\overline{\ell}_*\overline{\ell}^!$ and $\tau^{\mathrm{std}}_{\leq p(S)}$ is the same as $\tau^S_{\leq 0}$ on $\widetilde{\mathcal{D}_S}$, we have

$$\ell^*\mathcal{F}^- \simeq \tau^{\mathrm{std}}_{\leq p(S)}\ell^*\mathcal{F} \simeq \tau^S_{\leq 0}\ell^*\mathcal{F}.$$

Now suppose $\mathcal{G} \in \widetilde{\mathcal{D}_Y}^{\geq 1}$. Then $\mathcal{G}$ is supported on some closed subscheme $i' : Y' \hookrightarrow X$ with the same topological space as Y, say $\mathcal{G} \simeq i'_*\mathcal{G}'$ and so

$$\mathrm{Hom}(\mathcal{F}^-, \mathcal{G}) \simeq \mathrm{Hom}(Li'^*\mathcal{F}^-, \mathcal{G}').$$

Since Li'^* is right t-exact with respect to the standard t-structure, we see that any bounded truncation of $Li'^*\mathcal{F}^-$ is in ${}^{\mathrm{std}}\mathcal{D}_{Y'}{}^{\leq p(S)}$.The monotonicity of the perversity then guarantees that ${}^{\mathrm{std}}\mathcal{D}_{Y'}{}^{\leq p(S)} \subset \widetilde{\mathcal{D}_Y}^{\leq 0}$. The morphism $\mathcal{F}^- \to \mathcal{F}$ is just the one induced from the counit of the adjunction $(\overline{\ell}_*, \overline{\ell}^!)$. For $\mathcal{F}^+$ we have the dual statements. Since $\ell^* = \ell^*\overline{\ell}_*\overline{\ell}^!$ and $\mathbb{D}\tau^{\mathrm{std}}_{<\overline{p}(S)} = \tau^{\mathrm{std}}_{\geq p(S)}\mathbb{D}$ on $\widetilde{\mathcal{D}_S}$,

$$\ell^*\mathcal{F}^+ \simeq \tau^S_{\geq 1}\ell^*\mathcal{F}.$$

The Hom vanishing statement for $\mathcal{F}^+$ follows by duality and the morphism $\mathcal{F} \to \mathcal{F}^+$ is the one induced by the dual of the counit of the adjunction $(\overline{\ell}_*, \overline{\ell}^!)$.

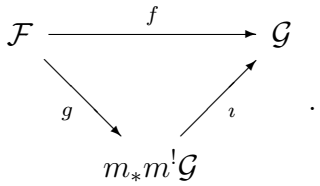
To show that the next axiom is satisfied, suppose we have a morphism $\mathcal{F} \xrightarrow{f} \mathcal{G}$ in $\widetilde{\mathcal{D}_Z}$ with $\ell^* f = 0$. Consider the distinguished triangle in $\mathcal{D}^+_G(\mathcal{Q}_X)$

$$m_*m^!\mathcal{G} \xrightarrow{\imath} \mathcal{G} \to \ell_*\ell^*\mathcal{G} \to m_*m^!\mathcal{G}[1],$$

which gives an exact sequence

$$\cdots \to \mathrm{Hom}(\mathcal{F}, m_*m^!\mathcal{G}) \xrightarrow{(\alpha \mapsto \alpha \circ \imath)} \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\ell^*\mathcal{F}, \ell^*\mathcal{G}) \to \cdots.$$

Since $\ell^* f = 0$, we see that f factors through $m_*m^!\mathcal{G}$ and we get the diagram



Now if $\widetilde{\mathcal{F}}^\bullet$ and $\widetilde{\mathcal{G}}^\bullet$ are chain complexes representing $\mathcal{F}$ and $m_*m^!\mathcal{G}$, respectively and $\widetilde{g}$ is a chain map representing the morphism g, then the image of $\widetilde{\mathcal{F}}^\bullet$ under $\widetilde{g}$ is a bounded subcomplex of $\widetilde{\mathcal{G}}^\bullet$ consisting of coherent sheaves. Viewing this image in the derived category gives an object of $\widetilde{\mathcal{D}_Y}$ which satisfies (G4).

The axiom (G5) is satisfied since Lemmas 2.32 and 3.10 show that the objects of $\widetilde{\mathcal{D}_Y}$ are precisely the objects in $\mathcal{D}_X$ which are set-theoretically supported on $\underline{Y}$.

Thus we can apply the gluing theorem and get a t-structure on $\widetilde{\mathcal{D}_Z}$ satisfying (G2).

For (G3)–(G5), we use an identical argument as above for these axioms with U in the place of V and Z in the place of Y. Fix $\mathcal{F} \in \mathcal{D}_X$. We will use $\tau^U_\bullet$ for the truncation functor

associated to the t-structure given in the statement of the theorem on U. To define $\mathcal{F}^+$ and $\mathcal{F}^-$, consider $\bar{\jmath} : \overline{U} \to X$, the closure of U in X. Let

$$\mathcal{F}^- = \tau^{\mathrm{std}}_{\leq p(U)} \bar{\jmath}_* \bar{\jmath}^! \mathcal{F}$$
$$\mathcal{F}^+ = \mathbb{D}\tau^{\mathrm{std}}_{<\overline{p}(U)} \bar{\jmath}_* \bar{\jmath}^! \mathbb{D}\mathcal{F}.$$

Then since $j^* = j^* \bar{\jmath}_* \bar{\jmath}^!$ and $\tau^{\mathrm{std}}_{\leq p(U)}$ is the same as $\tau^U_{\leq 0}$ on U, we have

$$j^* \mathcal{F}^- \simeq \tau^{\mathrm{std}}_{\leq p(U)} j^* \mathcal{F} \simeq \tau^U_{\leq 0} j^* \mathcal{F}.$$

Now suppose $\mathcal{G} \in \widetilde{\mathcal{D}_Z}^{\geq 1}$. Then $\mathcal{G}$ is supported on some closed subscheme $i' : Z' \hookrightarrow X$ with the same topological space as Z, say $\mathcal{G} \simeq i'_* \mathcal{G}'$ and so

$$\mathrm{Hom}(\mathcal{F}^-, \mathcal{G}) \simeq \mathrm{Hom}(Li'^* \mathcal{F}^-, \mathcal{G}').$$

Since Li'^* is right t-exact with respect to the standard t-structure, we see that any bounded truncation of $Li'^* \mathcal{F}^-$ is in ${}^{\mathrm{std}}\mathcal{D}_{Z'}{}^{\leq p(U)}$.The monotonicity of the perversity then guarantees that ${}^{\mathrm{std}}\mathcal{D}_{Z'}{}^{\leq p(U)} \subset \widetilde{\mathcal{D}_Z}^{\leq 0}$. The morphism $\mathcal{F}^- \to \mathcal{F}$ is just the one induced from the counit of the adjunction $(\bar{\jmath}_*, \bar{\jmath}^!)$. For $\mathcal{F}^+$ we have the dual statements. Since $j^* = j^* \bar{\jmath}_* \bar{\jmath}^!$ and $\mathbb{D}\tau^{\mathrm{std}}_{<\overline{p}(U)} = \tau^{\mathrm{std}}_{\geq p(U)} \mathbb{D}$ on U,

$$j^* \mathcal{F}^+ \simeq \tau^U_{\geq 1} j^* \mathcal{F}.$$

The Hom vanishing statement for $\mathcal{F}^+$ follows by duality and the morphism $\mathcal{F} \to \mathcal{F}^+$ is the one induced by the dual of the counit of the adjunction $(\bar{\jmath}_*, \bar{\jmath}^!)$.

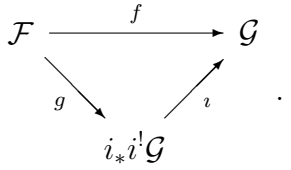
To show that the next axiom is satisfied, suppose we have a morphism $\mathcal{F} \xrightarrow{f} \mathcal{G}$ in $\mathcal{D}_X$ with $j^* f = 0$. Consider the distinguished triangle in $\mathcal{D}^+_G(\mathcal{Q}_X)$

$$i_* i^! \mathcal{G} \xrightarrow{\imath} \mathcal{G} \to j_* j^* \mathcal{G} \to i_* i^! \mathcal{G}[1],$$

which gives an exact sequence

$$\cdots \to \mathrm{Hom}(\mathcal{F}, i_* i^! \mathcal{G}) \xrightarrow{(\alpha \mapsto \alpha \circ \imath)} \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(j^* \mathcal{F}, j^* \mathcal{G}) \to \cdots .$$

Since $j^* f = 0$, we see that f factors through $i_* i^! \mathcal{G}$ and we get the diagram



Now if $\widetilde{\mathcal{F}}^\bullet$ and $\widetilde{\mathcal{G}}^\bullet$ are chain complexes representing $\mathcal{F}$ and $i_* i^! \mathcal{G}$, respectively and $\widetilde{g}$ is a chain map representing the morphism g, then the image of $\widetilde{\mathcal{F}}^\bullet$ under $\widetilde{g}$ is a bounded subcomplex of $\widetilde{\mathcal{G}}^\bullet$ consisting of coherent sheaves. Viewing this image in the derived category gives an object of $\widetilde{\mathcal{D}_Z}$ which satisfies (G4).

The axiom (G5) is satisfied since the two lemmas above show that the objects of $\widetilde{\mathcal{D}_Z}$ are precisely the objects in $\mathcal{D}_X$ which are set-theoretically supported on $\underline{Z}$.

So we can apply the gluing theorem to see that $({}^p\mathcal{D}^{\leq 0}, {}^p\mathcal{D}^{\geq 0})$ is in fact a t-structure $\mathcal{D}^b_G(\mathcal{C}_X)$. As mentioned at the beginning of this proof, to see that the t-structures coincide it suffices to check that ${}^p\mathcal{D}^{\leq 0} = \mathcal{D}^{p,\leq 0}$. Now the proposition follows from the fact that for any subscheme $i : Z \hookrightarrow X$ and any generic point $x \in Z^{gen}$, we have that the top cohomology of $i_x^* i^* \mathcal{F}$ and $i_x^* \mathcal{F}$ are equal and occur in the same degree, e.g. see [Bez1, Lemma 2(a)].

□

The t-structure obtained by gluing the respective t-structures on the orbits does not depend on the choice of open oribt in the proof of the proposition. This fact follows not only from the uniqueness of the perverse t-structure as defined by Deligne (see Theorem 2.33, where it is defined by conditions of sheaves which are defined on all of X), but also from the uniqueness in the gluing theorem.

Chapter 4
Perverse Poisson Sheaves

We begin this chapter by defining the Poisson derived category. We will not be interested in studying the Poisson derived category itself, however, but rather the analogous derived category when we also have a G-equivariant structure. Thus, the first section will be devoted to defining the equivariant Poisson derived category and developing the necessary theory that will allow us to construct the accompanying derived functors in Section 2. We will then leave the general setting and restrict to the case of the nilpotent cone associated to a reductive complex algebraic group. In this setting we will be able to use the machinery developed in the previous sections and the gluing theorem for triangulated categories proved above to develop a theory of perverse Poisson sheaves. Finally, using the framework of perverse Poisson sheaves, we are able to state a theorem on the Poisson analogues to Green functions.

Let us fix the following terminology. Let X be a noetherian Poisson scheme over $\mathbb{C}$. We will use G for a reductive complex algebraic group which has a Hamiltonian action on X and we assume that X has finitely many G-orbits. The fact that the G-action is Hamiltonian guarantees that G-orbits (and hence any G-invariant subscheme) are Poisson subschemes. Unless otherwise noted, we will only consider G-invariant Poisson subschemes. We refer the reader to Section 2.5 for the basic theory of Poisson sheaves (Poisson $\mathcal{O}_X$-modules) and Hamiltonian group actions.

4.1 The Equivariant Poisson Derived Category

We will denote the category of coherent Poisson sheaves on X by $\mathcal{C}_{\mathrm{Poi}}(X)$ which is a full subcategory of the cateogry of quasicoherent Poisson sheaves $\mathcal{Q}_{\mathrm{Poi}}(X)$. Each of these is an abelian category since there are naturally induced Poisson structures on kernels, cokernels, etc. and the composition of Poisson morphisms is again Poisson.

Definition 4.1. Let $\mathcal{D}(\mathcal{Q}_{\mathrm{Poi}}(X))$ be the derived category of quasicoherent Poisson $\mathcal{O}_X$-modules. We will call $\mathcal{D}_{\mathrm{Poi}}(X) = \mathcal{D}_{\mathrm{coh}}(\mathcal{Q}_{\mathrm{Poi}}(X))$ the *Poisson derived category* of X, i.e. the full subcategory of $\mathcal{D}(\mathcal{Q}_{\mathrm{Poi}}(X))$ whose objects are complexes with coherent cohomology. We will denote the bounded above, bounded below, and bounded Poisson derived categories by $\mathcal{D}^-_{\mathrm{Poi}}(X), \mathcal{D}^+_{\mathrm{Poi}}(X)$, and $\mathcal{D}^b_{\mathrm{Poi}}(X)$, respectively.

Lemma 4.2. *The categories $\mathcal{D}_{\mathrm{Poi}}(X), \mathcal{D}^-_{\mathrm{Poi}}(X), \mathcal{D}^+_{\mathrm{Poi}}(X)$, and $\mathcal{D}^b_{\mathrm{Poi}}(X)$ are triangulated.*

Proof. Since $\mathcal{D}_{\mathrm{Poi}}(X)$ is obviously closed under shifts, we only need to use the fact that the cone of a morphism between objects with coherent cohomology has coherent cohomology itself. This is not difficult to show directly, and can also be seen by forgetting the Poisson structure and then using the same statement which is already known in $\mathcal{D}(\mathcal{Q}(X))$. From the definition of cone (see, for instance, [GM1, §III.3.2]), we see that the cone of a bounded (below, above, or both) complex is again bounded. □

Remark 4.3. Let $\mathcal{P}_X$ be the abelian category of Poisson sheaves on X, i.e. the category of coherent Poisson $\mathcal{O}_X$-modules. Then $\mathcal{D}(\mathcal{P}_X) \subset \mathcal{D}_{\mathrm{Poi}}(X)$ and so we can embed $\mathcal{P}_X \hookrightarrow \mathcal{D}_{\mathrm{Poi}}(X)$.

We will now add in the G-equivariant structure. Note that the category $\mathcal{C}_{\mathrm{Poi}^G}(X)$ (resp. $\mathcal{Q}_{\mathrm{Poi}^G}(X)$) of G-equivariant coherent (resp. quasicoherent) Poisson sheaves is an abelian category since the equivariant structure is compatible with the Poisson structure (the action of G is Hamiltonian). The following proposition will be vital for defining derived functors below.

Proposition 4.4. *For any Poisson scheme X, the category $\mathcal{Q}_{\mathrm{Poi}^G}(X)$ has enough injectives.*

Proof. We make use of *Godement sheaves* (see [GK, Appendix A.6]). Let $\overline{X}$ be the disjoint union of all (scheme-theoretic) points of X and $\gamma : \overline{X} \to X$ the natural map. Then $God = \gamma_*\gamma^* : \mathcal{Q}_{\mathrm{Poi}^G}(X) \to \mathcal{Q}_{\mathrm{Poi}^G}(X)$ can be given by

$$God(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x.$$

Then $God(\mathcal{F})$ has a natural Poisson structure given coordinate-wise in the product and an equivariant structure induced from the equivariant structure on $\mathcal{F}$ by restricting to stalks. Hence the functor God takes values in the category $\mathcal{C}_{\mathrm{Poi}^G}(X)$ as claimed above. Moreover, for any $\mathcal{F} \in \mathcal{Q}_{\mathrm{Poi}^G}(X)$, there is a canonical embedding $\mathcal{F} \to God(\mathcal{F})$ which locally sends a section over U to its images in the stalks of $\mathcal{F}$.

Now let $\mathcal{I}_x$ be an injective object in the module category over the local ring $\mathcal{O}_x$ such that there is an embedding $\mathcal{F}_x \hookrightarrow \mathcal{I}_x$ for all x. We can choose $\mathcal{I}_x$ to be an equivariant Poisson module following Corollary 2.60 and Lemma 2.41. Then if $\mathcal{I} = \prod_x \mathcal{I}_x$, we have an embedding $\gamma^*\mathcal{F} \hookrightarrow \mathcal{I}$ in $\mathcal{Q}_{\mathrm{Poi}^G}(\overline{X})$. Since γ_* is left exact and takes injectives to injectives, we then get an embedding $God(\mathcal{F}) \hookrightarrow \gamma_*\mathcal{I}$. Combining this with the canonical embedding $\mathcal{F} \to God(\mathcal{F})$ gives $\mathcal{F}$ as a subobject of an injective. □

Proposition 4.5. *Let X be an affine Poisson scheme. For any locally closed Poisson subscheme $Y \hookrightarrow X$, the category $\mathcal{Q}_{\mathrm{Poi}^G}(Y)$ has enough locally free objects.*

Proof. First we show that there are enough free objects in $\mathcal{Q}_{\mathrm{Poi}^G}(X)$. Suppose that $X = \operatorname{Spec} A$ and consider an equivariant Poisson A-module M as an equivariant $\mathcal{U}_{\mathrm{Poi}}(A)$-module. Let $\{m_i \mid i \in I\}$ be a set of generators of M as an A-module, and E a G-stable complex vector space containing the m_i. Consider $\mathcal{U}_{\mathrm{Poi}}(A) \otimes E$, which we make into a free $\mathcal{U}_{\mathrm{Poi}}(A)$-module by letting $\mathcal{U}_{\mathrm{Poi}}(A)$ act on the first factor. The module $\mathcal{U}_{\mathrm{Poi}}(A) \otimes E$ can also be given the usual G-equivariant structure (G acts on both terms in the tensor). Moreover, the map $\mathcal{U}_{\mathrm{Poi}}(A) \otimes E \to M$ sending $a \otimes x \mapsto a.x$ is surjective and equivariant. Viewing $\mathcal{U}_{\mathrm{Poi}}(A) \otimes E$ as a Poisson A-module now, the fact that $\mathcal{U}_{\mathrm{Poi}}(A)$ is free as an A-module (Corollary 2.63) shows that $\mathcal{U}_{\mathrm{Poi}}(A) \otimes E$ is still free over A.

Now if $i : Z \hookrightarrow X$ is a closed subscheme, then we could repeat the above with A/I for some Poisson ideal I in A. For an open subscheme $j : U \hookrightarrow X$, the restriction j^* takes locally free objects to locally free objects and is exact. Since j^* is essentially surjective, we then have enough locally free objects in $\mathcal{Q}_{\mathrm{Poi}^G}(U)$. Combining these (any locally closed subscheme of

a noetherian scheme can be realized as an open subscheme of a closed subscheme), we see that for any locally closed subscheme Y, the category $\mathcal{Q}_{\mathrm{Poi}^G}(Y)$ has enough locally free objects. □

Definition 4.6. Denote the *G-equivariant Poisson derived category* by

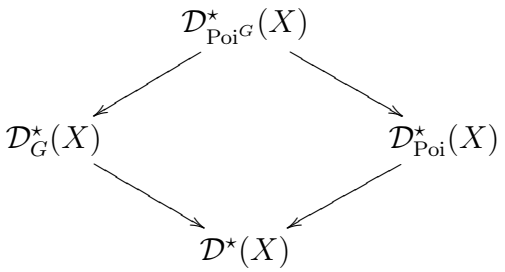
$$\mathcal{D}_{\mathrm{Poi}^G}(X) = \mathcal{D}_{\mathrm{coh}}(\mathcal{Q}_{\mathrm{Poi}^G}(X)).$$

That is, $\mathcal{D}_{\mathrm{Poi}^G}(X)$ is the full subcategory of the derived category of the category of G-equivariant quasicoherent Poisson sheaves whose objects have coherent cohomology. We also have the full subcategories $\mathcal{D}^\star_{\mathrm{Poi}^G}$ where $\star$ stands for $-, +, b$, i.e. bounded above, bounded below, and bounded, as above.

Lemma 4.7. *The categories $\mathcal{D}_{\mathrm{Poi}^G}(X), \mathcal{D}^-_{\mathrm{Poi}^G}(X), \mathcal{D}^+_{\mathrm{Poi}^G}(X)$, and $\mathcal{D}^b_{\mathrm{Poi}^G}(X)$ are triangulated.*

Proof. As above, the only thing that requires justification is that the subcategory $\mathcal{D}_{\mathrm{Poi}^G}(X)$ is closed under taking cones. Thus forgetting both the Poisson and equivariant structures allows us to use the analogous statement in $\mathcal{D}(\mathcal{Q}(X))$ to see that the cone of a morphism of objects with coherent cohomology must have coherent cohomology. As in Lemma 4.2, the bounded versions are closed under shifts and cones as well. □

Remark 4.8. We have the following commutative diagram of forgetful functors



Here $\mathcal{D}_G(X) = \mathcal{D}(\mathcal{C}_G(X))$ which is studied, for instance, in [Bez1]. The category $\mathcal{D}(X) = \mathcal{D}(\mathcal{C}(X))$ is the derived category of coherent sheaves, and $\star$ stands for b or $-$ (in these cases it is known that the full subcategory of the derived category of quasicoherent sheaves consisting of complexes with coherent cohomology is equivalent to the derived category of coherent sheaves, see Lemma 2.30).

4.2 Poisson Sheaf Functors

The Poisson equivariant derived category described in the previous section is morally the category that we would like to work in. This category is more subtle, however, than if we forget the Poisson structure. In the non-Poisson setting, we were able to construct the perverse coherent t-structure by shifting the standard t-strucutre on orbits. This was possible because we could induce a t-structure on the subcategory of the derived category consisting of those complexes supported topologically on a closed subscheme from the shifted standard t-structure on the reduced subscheme. As the following example shows, this is not the case for Poisson sheaves.

Example 4.9. Let $\mathcal{N}$ be the nilpotent cone in $\mathfrak{sl}_2$. It is straightforward to show that $\mathcal{N}$ can be realized as

$$\mathcal{N} = \text{ Spec } A,$$

where $A = \mathbb{C}\left[\frac{1}{2}x^2, -xy, -\frac{1}{2}y^2\right]$. We can endow $\mathcal{N}$ with a Poisson structure by specifying that

$$\left\{-xy, \frac{1}{2}x^2\right\} = x^2$$
$$\left\{-xy, -\frac{1}{2}y^2\right\} = y^2$$
$$\left\{\frac{1}{2}x^2, -\frac{1}{2}y^2\right\} = -xy,$$

and extending by linearity and the Liebnitz rule to the whole algebra A. This Poisson structure is simply the structure induced by considering $A \subset \mathbb{C}[x, y]$ with Poisson bracket given by $\{x, y\} = 1$. From the description given above, we see at once that $\frac{1}{2}x^2, -xy, -\frac{1}{2}y^2$ form an $\mathfrak{sl}_2$ triple and so we recover a copy of $\mathfrak{sl}_2$ inside the coordinate algebra A.

Consider the ideal $I = (x^2, xy, y^2)A$. This is a Poisson ideal of A and $Z = \text{Spec}A/I$ is the Poisson subscheme of $\mathcal{N}$ corresponding to the zero ideal in A with the reduced subscheme structure. In this case, the module category $\mathcal{O}_Z$-mod$_{\text{Poi}}$ is just equivalent to the category $\mathcal{O}_Z$-mod because every Poisson module must have trivial Poisson structure.

Other subscheme structures on the point do yield nonzero Poisson modules however. For instance, the module

$$M_n = (x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n)A/(x^{n+2}, x^{n+1}y, \ldots xy^{n+1}, y^{n+2})A$$

has a nontrivial Poisson module structure over $A/(x^{n+2}, x^{n+1}y, \ldots xy^{n+1}, y^{n+2})A$. The problem is that the module M_n is supported as a coherent sheaf on the reduced point, but not as a Poisson scheme. We can certainly give i^*M_n a natural Poisson structure, but this is not the correct notion. An easy calculation shows that i^* and i_* are not adjoint functors in the Poisson setting. Thus the structure of the subcategory of Poisson modules supported topologically on the point is not induced by the structure of the subcategory of Poisson modules supported on the reduced subscheme.

What we will do then, is to form a certain quotient category that will allow us to use the gluing mechanism as before to build a t-structure which we will call the *reduced perverse poisson t-structure*. This is somewhat unsatisfactory in that we would like to have a t-structure on the entire Poisson derived category, but this seems to be significantly more involved and does not impact our initial goal of describing the stalks of coherent sheaves which happen to come from local systems (vector bundles with a flat connection). As mentioned briefly above, another way of approaching this subject would be to generalize the idea of the Poisson enveloping algebra to schemes and adapt [Bez1] to the non-commutative setting. A reasonable statement to desire in such a theory is that the perverse coherent sheaves on some "universal Poisson scheme" correspond to what morally should be perverse Poisson sheaves on the original scheme. We do not know if such a statement is possible.

The basis for the quotient category that we will focus on will be the behavior of functors between appropriate triangulated categories. Toward this end, let $i : Z \to X$ be the inclusion of a closed subscheme and $j : U \to X$ the open complement. Two of the functors that we will need are immediately available. These are the direct image i_* and the inverse image j^*. Because these functors are exact on the abelian category, they extend to the (bounded) derived category in the usual way (one can apply them to a complex "term-wise"). We will now define the other three functors ($i^{!\text{Poi}}, i^{*\text{Poi}}$, and j_*) on the abelian category and then use our work from the previous section to construct their derived versions. The modifiers on these functors indicate that we must adapt the definition of the functors to the Poisson setting as is explained in the following definition. Most of the work in this section is adapting either classical results about functors of sheaves of $\mathcal{O}_X$-modules or results from [Bez1]. We stress, however, that the Poisson functors defined below are actually different functors from the coherent versions and require a new treatment.

Definition 4.10. Let X be a Poisson scheme. It is straightforward to generalize Definition 2.57 to the setting of sheaves. That is, we say that $\mathcal{U}$ is a *universal Poisson sheaf for* X if $\mathcal{U}$ is a sheaf of associative $\mathbb{C}$-algebras (which we make into Lie algebras via the commutator bracket) and there are morphisms $f, g : \mathcal{O}_X \to \mathcal{U}$ such that f is a morphism of sheaves of $\mathbb{C}$-algebras and g is a morphism of sheaves of complex Lie algebras satisfying the local equations

$$\begin{aligned} f(\{a,b\}) &= g(a)f(b) - f(b)g(a) \\ g(ab) &= f(a)g(b) + f(b)g(a), \end{aligned}$$

such that for any other sheaf of associative $\mathbb{C}$-algebras $\mathcal{W}$ and morphisms $h, \ell : A \to \mathcal{W}$ satisfying the same equations, there exists a unique morphism of sheaves of $\mathbb{C}$-algebras $\gamma : \mathcal{U}_X \to \mathcal{W}$ such that the diagrams

commute.

Proposition 4.11. *For any Poisson scheme X, there is a unique universal Poisson sheaf.*

Proof. The construction is simply a sheaf version of Theorem 2.58. It wlll be necessary to have two copies of the structure sheaf which we can distinguish, so let $\mathcal{R}_X$ also denote the structure sheaf of X with the identity morphism $\varphi : \mathcal{O}_X \to \mathcal{R}_X$ (we will denote the isomorphism in the other direction by φ^{-1}). Let $\mathcal{T}(\mathcal{O}_X \oplus \mathcal{R}_X)$ be the tensor algebra sheaf. That is

$$\mathcal{T}(\mathcal{O}_X \oplus \mathcal{R}_X) = \bigoplus_{n \geq 0} \underbrace{(\mathcal{O}_X \oplus \mathcal{R}_X) \otimes \cdots \otimes (\mathcal{O}_X \oplus \mathcal{R}_X)}_{n \text{ pairs}}.$$

Then $\mathcal{T}(\mathcal{O}_X)$ is a subsheaf of $\mathcal{T}(\mathcal{O}_X \oplus \mathcal{R}_X)$ and we have a canonical morphism $\psi : \mathcal{T}(\mathcal{O}_X) \to \mathcal{O}_X$. We also have morphisms

$$\begin{aligned} B &: \mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X \\ \mu &: \mathcal{O}_X \otimes \mathcal{O}_X \to \mathcal{O}_X \\ \iota_{\mathcal{O},\mathcal{R}} &: \mathcal{O}_X \otimes \mathcal{R}_X \to \mathcal{O}_X \otimes \mathcal{R}_X \\ \sigma_{\mathcal{O},\mathcal{R}} &: \mathcal{O}_X \otimes \mathcal{R}_X \to \mathcal{R}_X \otimes \mathcal{O}_X, \end{aligned}$$

where B is the Poisson bracket, μ is multiplication in the structure sheaf, ι is the identity morphism, and σ exchanges the two factors (we also have the other variations of the ι and σ). Let $\mathcal{J}$ be the ideal sheaf generated by the ideal sheaves

- $\ker \psi$
- $\operatorname{im}(\iota_{\mathcal{R},\mathcal{R}} - \sigma_{\mathcal{R},\mathcal{R}} - \varphi \circ B \circ \varphi^{-1} \otimes \varphi^{-1} : \mathcal{R}_X \otimes \mathcal{R}_X \to \mathcal{T}(\mathcal{O}_X \oplus \mathcal{R}_X))$
- $\operatorname{im}(\iota_{\mathcal{R},\mathcal{O}} - \sigma_{\mathcal{R},\mathcal{O}} - B \circ \varphi^{-1} \otimes \mathrm{id}_{\mathcal{O}_X} : \mathcal{R}_X \otimes \mathcal{O}_X \to \mathcal{T}(\mathcal{O}_X \oplus \mathcal{R}_X))$
- $\operatorname{im}(\iota_{\mathcal{O},\mathcal{R}} + \sigma_{\mathcal{O},\mathcal{R}} \circ \varphi \otimes \varphi^{-1} - \varphi \circ \mu \circ \mathrm{id}_{\mathcal{O}_X} \otimes \varphi^{-1} : \mathcal{O}_X \otimes \mathcal{R}_X \to \mathcal{T}(\mathcal{O}_X \oplus \mathcal{R}_X))$.

Though these formulas are completely opaque, they are simply the sheaf versions of the generators of the ideal J in Theorem 2.58. Define

$$\mathcal{U}_X = \mathcal{T}(\mathcal{O}_X \oplus \mathcal{R}_X)/\mathcal{J}.$$

The proof that this satisfies the universal property is completely analogous to the Poisson enveloping algebra case. □

Proposition 4.12. *We have an equivalence of categories $\mathcal{O}_X$-mod$_{\mathrm{Poi}^G} \simeq \mathcal{U}_X$-mod$_G$.*

Proof. As before, the morphisms f, g from Definition 4.10 make any $\mathcal{U}_X$-module into a Poisson $\mathcal{O}_X$-module. Conversely, the module structure morphisms ensure that for any Poisson $\mathcal{O}_X$-module $\mathcal{F}$ there is a morphism $\mathcal{U}_X \to Hom(\mathcal{F}, \mathcal{F})$ making $\mathcal{F}$ into a $\mathcal{U}_X$-module. □

Definition 4.13. Let $f : Y \to X$ be a morphism of Poisson schemes. Define the *Poisson inverse image* functor to be the functor $f^{*\mathrm{Poi}} : \mathcal{Q}_{\mathrm{Poi}^G}(X) \to \mathcal{Q}_{\mathrm{Poi}^G}(Y)$ given by

$$f^{*\mathrm{Poi}}\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{U}_X} \mathcal{U}_Y,$$

where f^{-1} denotes the topological inverse image functor.

Remark 4.14. We will only need this idea is the case where $k : Y \hookrightarrow X$ is the inclusion of a locally closed Poisson subscheme of X. In the case of a closed or open subscheme, we are able to give an explicit description of this functor. Clearly for an open subscheme, the Poisson inverse image is just the ordinary restriction. Now if $i : Z \to X$ is the embedding of a closed G-invariant Poisson subscheme with ideal sheaf $\mathcal{I}$, then $i^{*\mathrm{Poi}} : \mathcal{Q}_{\mathrm{Poi}^G}(X) \to \mathcal{Q}_{\mathrm{Poi}^G}(Z)$ can be realized as the functor

$$i^{*\mathrm{Poi}}\mathcal{F} = \mathcal{F}/(\{\mathcal{I}, \mathcal{F}\} + \mathcal{I}\mathcal{F}).$$

In particular, this shows that for any locally closed subscheme, the Poisson inverse image takes coherent sheaves to coherent sheaves since this is clearly true for closed and open subschemes, while any locally closed subscheme (of a noetherian scheme) can be realized as an open subscheme of its closure.

Definition 4.15. Dually, let $k : Y \to X$ again be a locally closed subscheme. We define the *Poisson inverse image with supports* to be the functor $k^{!\mathrm{Poi}} : \mathcal{Q}_{\mathrm{Poi}^G}(X) \to \mathcal{Q}_{\mathrm{Poi}^G}(Y)$ given by

$$k^{!\mathrm{Poi}}\mathcal{F} = \mathcal{H}om_{k^{-1}\mathcal{U}_X}(\mathcal{U}_Z, k^{-1}\mathcal{F}).$$

Remark 4.16. Similarly to the Poisson inverse image, if $i : Z \hookrightarrow X$ is a closed subscheme, then we can describe $i^{!\mathrm{Poi}}$ as a subsheaf of the usual inverse image with supports. Let $j : U \hookrightarrow X$ be the open complement of Z. If $\mathcal{F} \in \mathcal{Q}_{\mathrm{Poi}^G}(X)$ and V is an open G-invariant subscheme of Z, then $U \cup V$ is an open subscheme of X and we have

$$i^{!\mathrm{Poi}}\mathcal{F}(V) = \{s \in \mathcal{F}(V \cup U) \mid f.s = 0, \{f, s\} = 0 \text{ for all } f \in \mathcal{I}(V \cup U)\}.$$

Lemma 4.17.

(a) The functor $i_* : \mathcal{C}_{\mathrm{Poi}^G}(Z) \to \mathcal{C}_{\mathrm{Poi}^G}(X)$ is exact.

(b) The functor $j_* : \mathcal{Q}_{\mathrm{Poi}^G}(U) \to \mathcal{Q}_{\mathrm{Poi}^G}(X)$ is left exact.

Proof. Since these functors are defined exactly as in the coherent setting, the statements about exactness and the behavior on (quasi)coherent sheaves can be obtained by forgetting the Poisson structure. The claim that i_* and j_* take equivariant Poisson modules to equivariant Poisson modules is clear from the definitions of the functors. □

Lemma 4.18.

(a) The functor $i^{*\mathrm{Poi}}$ is left adjoint to the direct image i_* and hence right exact.

(b) The functor $i^{!\mathrm{Poi}}$ is right adjoint to the direct image i_* and hence left exact.

Proof. The adjunctions can be obtained in the same way as the non-Poisson setting using Remarks 4.14 and 4.16. The key here is that we are restricting to only morphisms which respect the Poisson structures. □

Definition 4.19. Let $\mathcal{F}$ and $\mathcal{G}$ be equivariant quasicoherent Poisson sheaves on X. Define a new sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ by setting $\mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \mathrm{Hom}_{\mathcal{Q}(X)}(\mathcal{F}|_U, \mathcal{G}|_U)$. Then $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is naturally a G-equivariant sheaf and can be made into a Poisson sheaf where locally the Poisson bracket is given by the formula

$$\{a, f\}_V(x) = \{\rho_{UV}(a), f_V(x)\} - f_V(\{\rho_{UV}(a), x\}),$$

for $a \in \mathcal{O}_X(U), f \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(U)$, and $x \in \mathcal{F}(V)$ (here $\rho_{UV} : \Gamma(\mathcal{O}_X, U) \to \Gamma(\mathcal{O}_X, V)$ is the restriction of sections and $f_V : \mathcal{F}(V) \to \mathcal{G}(V)$ is the induced map on sections). Then $\mathcal{H}om$ is a bifunctor and commutes with the forgetful functors (forgetting either the Poisson structure, the equivariant structure, or both).

As we would expect, we can also define a tensor product of two Poisson sheaves which has a natural Poisson structure.

Definition 4.20. Let $\mathcal{F}$ and $\mathcal{G}$ be equivariant quasicoherent Poisson sheaves on X. Define a sheaf $\mathcal{F} \otimes \mathcal{G}$ to be the sheafification of the presheaf $U \mapsto \mathcal{F}(U) \otimes \mathcal{G}(U)$. This is naturally a G-equivariant sheaf and we can give it a Poisson structure with local bracket

$$\{a, f \otimes g\} = \{a, f\} \otimes g + f \otimes \{a, g\}.$$

Lemma 4.21. *For quasicoherent equivariant Poisson sheaves $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$, we have*

$$\mathcal{H}om(\mathcal{F} \otimes \mathcal{G}, \mathcal{H}) \simeq \mathcal{H}om(\mathcal{F}, \mathcal{H}om(\mathcal{G}, \mathcal{H})).$$

Proof. This can be shown in the standard way. For instance, see [KS1, Proposition 2.2.9]. □

Remark 4.22. Let X be a Poisson scheme, and $k : Y \hookrightarrow X$ a locally closed subscheme. Then Proposition 4.4 ensures that we are able to construct the derived functors

$$Rk^{!\mathrm{Poi}} : \mathcal{D}^+_{\mathrm{Poi}^G}(X) \to \mathcal{D}^+_{\mathrm{Poi}^G}(\mathcal{Q}(Y))$$
$$Rk_* : \mathcal{D}^+_{\mathrm{Poi}^G}(Y) \to \mathcal{D}^+_{\mathrm{Poi}^G}(\mathcal{Q}(X)).$$

In fact, if we have another Poisson scheme X', we can construct the derived direct image $Rf_* : \mathcal{D}^+_{\mathrm{Poi}^G}(X) \to \mathcal{D}^+_{\mathrm{Poi}^G}(\mathcal{Q}(X'))$ for any Poisson morphism $f : X \to X'$. The fact that the target category for $Ri^{!\mathrm{Poi}}$ is $\mathcal{D}^+_{\mathrm{Poi}^G}(\mathcal{Q}(Y))$, for example, indicates that this functor might not preserve coherent cohomology (the author does not know if this is true in general). This is a consequence of the fact that we only have quasicoherent resolutions and not coherent resolutions, even if the sheaf we start with is in fact coherent. Later in the specific case of the nilpotent cone, we will be able to overcome this issue with Proposition 4.38.

If X is a locally closed Poisson subscheme of an affine Poisson scheme, we also can use Proposition 4.5 to construct the derived functors

$$Lk^{*\mathrm{Poi}} : \mathcal{D}^-_{\mathrm{Poi}^G}(X) \to \mathcal{D}^-_{\mathrm{Poi}^G}(\mathcal{Q}(Y))$$
$$R\mathcal{H}om : (\mathcal{D}^-_{\mathrm{Poi}^G}(X))^{\mathrm{op}} \times \mathcal{D}^+_{\mathrm{Poi}^G}(X) \to \mathcal{D}^+_{\mathrm{Poi}^G}(X)$$
$$\overset{L}{\otimes} : \mathcal{D}^+_{\mathrm{Poi}^G}(X) \times (\mathcal{D}^-_{\mathrm{Poi}^G}(X))^{\mathrm{op}} \to \mathcal{D}^+_{\mathrm{Poi}^G}(X),$$

where we compute $R\mathcal{H}om$ in the first variable and $\otimes^L$ in either variable with locally free sheaves. Because of this, $R\mathcal{H}om$ and $\otimes^L$ also commute with the forgetful functors (forgetting the Poisson structure, the equivariant structure, or both) since locally free resolutions may be used in these categories as well to compute $R\mathcal{H}om$ and $\otimes^L$, while For takes locally free sheaves to locally free sheaves. In particular, this shows that these functors preserve boundedness of complexes and coherence in cohomology since these are known in the non-Poisson, nonequivariant setting.

In order to make full use of the Poisson inverse image functors, we will need to see that the derived versions of these functors work well with composition. This amounts to seeing that the functors preserve the appropriate acyclic objects.

Lemma 4.23. *Let X be an affine Poisson scheme and $i : Z \hookrightarrow X$ a closed subscheme. If $k : Y \hookrightarrow Z$ is a closed subscheme of Z, then*

(a) $R(i \circ k)^{!_{\mathrm{Poi}}} = Ri^{!_{\mathrm{Poi}}} \circ Rk^{!_{\mathrm{Poi}}}$.

(b) $L(i \circ k)^{*_{\mathrm{Poi}}} = Li^{*_{\mathrm{Poi}}} \circ Lk^{*_{\mathrm{Poi}}}$.

Proof. The functor $i^{!_{\mathrm{Poi}}}$ has an exact left adjoint i_* which in turn is right adjoint to $i^{*_{\mathrm{Poi}}}$. Hence $i^{!_{\mathrm{Poi}}}$ preserves injectives and $i^{*_{\mathrm{Poi}}}$ preserves locally free objects. □

Definition 4.24. A *dualizing complex* for a Poisson scheme X is an object $\omega_X \in \mathcal{D}_{\mathrm{Poi}^G}(X)$ such that there is an isomorphism

$$\mathcal{F} \to R\mathcal{H}om(\mathcal{F}, R\mathcal{H}om(\mathcal{F}, \omega_X))$$

for all $\mathcal{F} \in \mathcal{D}_{\mathrm{Poi}^G}(X)$. We call $\mathbb{D} : \mathcal{F} \mapsto R\mathcal{H}om(\mathcal{F}, \omega_X)$ a *duality* functor.

Lemma 4.25. *If X is an affine Poisson scheme, then for an object $\omega_X \in \mathcal{D}_{\mathrm{Poi}^G}(X)$, the following are equivalent*

(a) $\omega_X \in \mathcal{D}_{\mathrm{Poi}^G}(X)$ *is a dualizing complex.*

(b) $For : \mathcal{D}_{\mathrm{Poi}^G}(X) \to \mathcal{D}_G(X)$ *takes* ω_X *to a dualizing complex.*

(c) $For : \mathcal{D}_{\mathrm{Poi}^G}(X) \to \mathcal{D}(X)$ *takes* ω_X *to a dualizing complex.*

Proof. This argument for this proof comes from [Bez1, Lemma 4]. The implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (c)$ follow from [Har1, Proposition V.2.1], which tells us that ω_X is a dualizing complex if the natural transformation $\mathcal{O}_X \to R\mathcal{H}om(R\mathcal{H}om(\mathcal{O}_X, \omega_X), \omega_X)$ is an isomorphism ($\mathcal{O}_X$ denotes the structure sheaf). Since $\mathcal{O}_X$ is certainly in $\mathcal{D}_{\mathrm{Poi}^G}(X)$ and hence in the image of (either) For, we see that $For(\omega_X)$ is a dualizing complex if ω_X is a dualizing complex. Since $R\mathcal{H}om$ commutes with (each) For, the converse implications follow from the fact that (each) For reflects isomorphisms. □

Lemma 4.26. *Any affine Gorenstein Poisson scheme admits an equivariant Poisson dualizing complex.*

Proof. For a Gorenstein scheme, the structure sheaf is a dualizing complex. Since the structure sheaf is obviously equivariant and Poisson, this lemma follows from Lemma 4.25. □

Lemma 4.27. *Let X be an affine Poisson scheme admitting a dualizing complex. For any $\mathcal{F} \in \mathcal{D}^b_{\mathrm{Poi}^G}(X)$ we have $Ri^{!_{\mathrm{Poi}}}\mathcal{F} \simeq \mathbb{D}Li^{*_{\mathrm{Poi}}}\mathbb{D}\mathcal{F}$.*

Proof. It follows immediately from the definitions that for any $\mathcal{F} \in \mathcal{Q}_{\mathrm{Poi}^G}(X)$ we have

$$i_* i^{!_{\mathrm{Poi}}}\mathcal{F} \simeq \mathcal{H}om_{\mathcal{U}_X}(i_*\mathcal{U}_Z, \mathcal{F})$$
$$i_* i^{*_{\mathrm{Poi}}}\mathcal{F} \simeq \mathcal{F} \otimes_{\mathcal{U}_X} i_*\mathcal{U}_Z.$$

Since both sides of these formulas can be computed by injective and locally free resolutions of $\mathcal{F}$, respectively, we see that this is also true for the derived versions of these functors (for

bounded complexes). Using this and the adjoint pair $(\otimes^L, R\mathcal{H}om)$, a simple computation shows that

$$i_* Ri^{!\text{Poi}} R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_X) \simeq i_* R\mathcal{H}om_{\mathcal{O}_X}(Li^{*\text{Poi}}\mathcal{F}, i^!\omega_X).$$

Now we can take the topological restriction to Z which is an exact functor (not the coherent or Poisson pullback) to get the statement to get $Ri^{!\text{Poi}}\mathbb{D} \simeq \mathbb{D}Li^{*\text{Poi}}$, which is equivalent to the claim. For the basic facts about adjunctions and $R\mathcal{H}om$ over different sheaves of rings, see [KS2, §18.2]. □

Definition 4.28. Let X be a Poisson scheme. Define a strictly full subcategory of $\mathcal{D}_{\text{Poi}^G}(X)$

$$\mathcal{S}_X = \bigcup \left\{ \mathcal{F} \in \mathcal{D}_{\text{Poi}^G} \mid j^*\mathcal{F} \simeq 0 \text{ and } i^{*\text{Poi}}\mathcal{F} \simeq 0 \right\},$$

where the union is over all pairs consisting of a reduced closed (G-invariant) subschemes $i : Z \hookrightarrow X$ along with its complementary open subscheme $j : U \hookrightarrow X$. Since we are assuming that G acts with finitely many orbits, this is a finite union.

Lemma 4.29. *The subcategory $\mathcal{S}_X$ is thick (see Definition 2.4).*

Proof. Suppose $\mathcal{F} \to \mathcal{G} \to \mathcal{H} \to \mathcal{F}[1]$ is a distinguished triangle in $\mathcal{D}_{\text{Poi}^G}$. Fix a particular reduced closed subscheme and open complement. Since j^* and $i^{*\text{Poi}}$ are triangulated functors, if any two of $\mathcal{F}, \mathcal{G}$, and $\mathcal{H}$ are in $\mathcal{S}_X$, then the third is also. Thus $\mathcal{S}_X$ is a triangulated subcategory. Clearly, $\mathcal{S}_X$ is also strictly full. Now suppose $\mathcal{F} \oplus \mathcal{G} \in \mathcal{S}_X$. Then we have a distinguished triangle

$$\mathcal{F} \to \mathcal{F} \oplus \mathcal{G} \to \mathcal{G} \xrightarrow{0} \mathcal{F}[1].$$

Applying j^* and $i^{*\text{Poi}}$ to this triangle shows that $j^*\mathcal{G}[-1] \xrightarrow{0} j^*\mathcal{F}$ and $i^{*\text{Poi}}\mathcal{G}[-1] \xrightarrow{0} \mathcal{F}$ are isomorphisms. Thus $\mathcal{F}$ and $\mathcal{G}$ are also in $\mathcal{S}_X$. Since the union of thick subcategories is again thick, we have proved the lemma. □

Definition 4.30. Define the *reduced equivariant Poisson derived category* as

$${}^{\text{red}}\mathcal{D}_{\text{Poi}^G}(X) = \mathcal{D}_{\text{Poi}^G}(X)/\mathcal{S}_X.$$

Proposition 4.31. *Let X be an affine Poisson scheme admitting a dualizing complex. For any locally closed subscheme $k : Z \hookrightarrow X$, the functors $Rk_*, Lk^{*\text{Poi}}, Rk^{!\text{Poi}}$, $R\mathcal{H}om$, and $\mathbb{D}$ descend to functors on the quotients.*

Proof. Let $\mathcal{F} \in \mathcal{S}_X$. Then there is a reduced closed subscheme $i : Z \hookrightarrow X$ with open complement $j : U \hookrightarrow X$ such that $j^*\mathcal{F} = 0$ and $Li^{*\text{Poi}}\mathcal{F} = 0$. If $Y \subset U$, then clearly $Lk^{*\text{Poi}}\mathcal{F} \simeq 0$. Otherwise let $Y' = Z \cap Y$ with the reduced subscheme structure and embedding $i' : Y' \hookrightarrow X$. Since $Li'^{*\text{Poi}}$ factors through $Li^{*\text{Poi}}$, we see that $Li'^{*\text{Poi}}\mathcal{F} \simeq 0$. Now we can also factor $i' = k \circ \ell$, where $\ell : Y' \to Y$ is the embedding of a reduced closed subscheme. Since $Li'^{*\text{Poi}} = L\ell^{*\text{Poi}} \circ Lk^{*\text{Poi}}$, we see that

$$L\ell^{*\text{Poi}}\left(Lk^{*\text{Poi}}\right) \simeq 0,$$

which shows that $Lk^{*\text{Poi}}\mathcal{F} \in \mathcal{S}_Y$ and hence is isomorphic to 0 in ${}^{\text{red}}\mathcal{D}_{\text{Poi}^G}(Y)$. The functor Rk_* can be shown to take $\mathcal{S}_Y$ to $\mathcal{S}_X$ in a similar way.

Consider $\mathcal{H}om(\mathcal{F},\mathcal{G})$, where $\mathcal{F}$ is a sheaf (not a complex of sheaves) such that $i^{*\text{Poi}}\mathcal{F}=0$ for some reduced closed subscheme $i : Z \hookrightarrow X$ and $\mathcal{F}$ is supported topologically on $\underline{Z}$. Suppose that $\mathcal{F} \simeq i'_*\mathcal{F}'$, where $i' : Z' \hookrightarrow$ is a nonreduced closed subscheme with topological space $\underline{Z}$. Then

$$i^{*\text{Poi}}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \simeq i^{*\text{Poi}}\mathcal{H}om_{\mathcal{O}_{Z'}}(\mathcal{F}', i'^!\mathcal{G}),$$

Now we already know that $i^{*\text{Poi}}\mathcal{F}'=0$ from above, and this means that $\mathcal{F}'=(\mathcal{I}\mathcal{F}'+\{\mathcal{I},\mathcal{F}'\}$. Then the definition of the Poisson structure on $\mathcal{H}om(\mathcal{F}', i'^!\mathcal{G})$ given above shows that in $i^{*\text{Poi}}\mathcal{H}om(\mathcal{F}', i'^!\mathcal{G})$, all morphisms are Poisson. Now we can use the Poisson adjunctions to see that

$$i^{*\text{Poi}}\mathcal{H}om_{\mathcal{O}_{Z'}}(\mathcal{F}', i'^!\mathcal{G}) \simeq \mathcal{H}om_{\text{Poi}}(\mathcal{F}', i'^!\mathcal{G}),$$

which is isomorphic to $\mathcal{H}om_{\text{Poi}}(i^{*\text{Poi}}\mathcal{F}', i'^!\mathcal{G})$. Since we have shown already that $i'^{*\text{Poi}}\mathcal{F}$ is 0 in ${}^{\text{red}}\mathcal{D}_{\text{Poi}^G}(Z')$, we see that $\mathcal{H}om$ descends to the quotient also. The derived version also holds since we can take an injective resolution of $\mathcal{G}$ in either situation.

Applying Lemma 4.27, we see that $Rk^{!\text{Poi}}$ also descends as long as $Lk^{*\text{Poi}}\mathcal{F} \simeq 0$ if and only if $Lk^{*\text{Poi}}\mathbb{D}\mathcal{F} \simeq 0$, which follows from the fact that $R\mathcal{H}om$ descends to the quotient. □

4.3 Perverse Poisson t-structure

Now we leave the general setting and specialize to the situation that provided the motivation for this work. Let G be a reductive algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$. Then G acts on $\mathfrak{g}^*$ with the coadjoint action and this restricts to an action on the nilpotent cone $\mathcal{N}$ with finitely many orbits. The nilpotent cone, its Poisson structure and other basic properties are reviewed in Section 2.7. In this setting we are able to say a good deal more than in the case of a general Poisson scheme. For example, the functors $i^{*\text{Poi}}$ and $i^{!\text{Poi}}$ have better properties and we are able to extend several of the basic lemmas from [Bez1] to this setting. The primary reason for this is that the G-action on $\mathcal{N}$ and the Poisson structure are intricately linked in a stronger way than requiring that the G-action to be Hamiltonian. As outlined above and used many times below, the Poisson structure on $\mathcal{N}$ is induced from the Lie bracket on $\mathfrak{g}$ while the G-action gives a $\mathfrak{g}$-action on $\mathcal{N}$. This connection has the important consequence that G-equivariant subsheaves of a Poisson sheaf are automatically Poisson. Thus any statement depending only on finding suitable G-equivariant subsheaves extends to this setting. A Hamiltonian group action is roughly an action whose corresponding infinitesimal action factors through the Poisson structure. The theory that follows can be extended to any Poisson scheme admitting a dualizing complex where the Poisson structure is determined by this factorization (see Remark 2.72).

Lemma 4.32. *Let $\mathcal{F}$ be an equivariant quasicoherent Poisson sheaf on $\mathcal{N}$. If $\mathcal{F}' \hookrightarrow \mathcal{F}$ is an equivariant subsheaf of $\mathcal{F}$, then $\mathcal{F}'$ is also a Poisson sheaf.*

Proof. Differentiating the action of G on $\mathcal{N}$, we get an action of $\mathfrak{g}$ on $\mathcal{N}$ and a corresponding action of $\mathfrak{g}$ on $\mathcal{F}$. Since $\mathcal{F}'$ is G-equivariant, we also have that $\mathcal{F}'$ is $\mathfrak{g}$-equivariant. Since the Poisson structure is induced by the Lie bracket of $\mathfrak{g}$ (see Lemma 2.69), we see that $\mathcal{F}$ is closed under the Poisson bracket. □

Lemma 4.33. *We have an equivalence of categories*

(a) ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N}) \simeq {}^{red}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{C}(\mathcal{N}))$.

(b) ${}^{\mathrm{red}}\mathcal{D}^-_{\mathrm{Poi}^G}(\mathcal{N}) \simeq {}^{red}\mathcal{D}^-_{\mathrm{Poi}^G}(\mathcal{C}(\mathcal{N}))$.

Proof. Since G-equivariant subsheaves are Poisson, this follows directly from [Bez1, Corollary 1]. □

Lemma 4.34. *If Y is a locally closed subscheme of $\mathcal{N}$, then there are enough locally free objects $\mathcal{C}_{\mathrm{Poi}^G}(Y)$.*

Proof. It is enough to prove this lemma for $\mathcal{N}$ itself since an open restriction takes locally free objects to locally free objects and the same argument works for any closed subscheme as it does for $\mathcal{N}$. For any equivariant Poisson module M over $\mathcal{O}_{\mathcal{N}}$, choose a finite dimensional G-stable subspace E of M which generates M as in Proposition 4.5. Then we can make $\mathcal{O}_{\mathcal{N}} \otimes_{\mathbb{C}} E$ an equivariant Poisson module by letting $\mathcal{O}_{\mathcal{N}}$ act on the first term, G act on both terms and defining a Poisson bracket by

$$\{a, b \otimes m\} = \{a, b\} \otimes m + b \otimes \{a, m\}.$$

We can easily check that this is a Poisson bracket, with the key fact that E being G-stable guarantees that E is also closed under the Poisson bracket on M. To see this, notice that E is also $\mathfrak{g}$-stable where the action of $\mathfrak{g}$ on M is the action corresponding to the action of G. Since the Lie bracket on $\mathfrak{g}$ induces the Poisson structure on $\mathcal{O}_{\mathcal{N}}$ (see Proposition 2.68 and Lemma 2.69), we also see that E is Poisson closed.

The map $\mathcal{O}_{\mathcal{N}} \otimes E \to M$ sending $a \otimes m \mapsto a.m$ is surjective since E generates M as a module. Since this is also equivariant and Poisson, we have shown the lemma. □

Remark 4.35. The only issue with the extending Lemma 4.34 in the general setting is that the finite dimensional G-stable subspace which generates the module might not be Poisson closed. If we are able to find such a subspace of any module which is closed under the Poisson bracket, then the proof given would go through.

Lemma 4.36. *There is an equivariant Poisson dualizing complex for $\mathcal{N}$.*

Proof. It is well known that $\mathcal{N}$ is Gorenstein (see for example [BK, Theorem 5.3.2]), so this follows immediately from Lemma 4.26. □

Lemma 4.37. *For any $\mathcal{F}$ and $\mathcal{G}$ in ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N})$ we have*

$$\mathrm{Hom}_{\mathrm{Poi}^G}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathrm{Poi}^G}(\mathbb{D}\mathcal{G}, \mathbb{D}\mathcal{F}).$$

Proof. Using the symmetry of the tensor product and Lemma 4.21, it is straightforward to see that

$$\mathrm{Hom}_{\mathrm{Poi}^G}(\mathcal{F}, \mathbb{D}\mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{Poi}^G}(\mathcal{G}, \mathbb{D}\mathcal{F})$$

and replacing $\mathcal{G}$ with $\mathbb{D}\mathcal{G}$ then gives the statement of the lemma. □

Proposition 4.38. *If $i : Z \hookrightarrow \mathcal{N}$ is any closed subscheme, the functors $Li^{*\mathrm{Poi}}$ and $Ri^{!\mathrm{Poi}}$ take bounded complexes of quasicoherent sheaves with coherent cohomology to (possibly unbounded) complexes of quasicoherent sheaves with coherent cohomology.*

Proof. By duality, it is enough to show this for $Li^{*\mathrm{Poi}}$. Since $\mathcal{F} \in {}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N})$, we can assume that $\mathcal{F}$ is actually a coherent complex and take a resolution $\mathcal{P}^\bullet$ of $\mathcal{F}$ by locally free coherent Poisson sheaves. Since quotients of coherent sheaves are clearly coherent, $Li^{*\mathrm{Poi}}$ takes $\mathcal{P}^\bullet$ to a (possibly unbounded) coherent complex of Poisson sheaves on Z. □

Lemma 4.39. *Let $i : Z \hookrightarrow \mathcal{N}$ be a closed subscheme. Then on the bounded derived category ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N})$, we have*

(a) $Ri^{!\mathrm{Poi}}\mathbb{D} \simeq \mathbb{D}Li^{*\mathrm{Poi}}$.

(b) $Li^{*\mathrm{Poi}}\mathbb{D} \simeq \mathbb{D}Ri^{!\mathrm{Poi}}$.

Proof. In the general setting (Lemma 4.27), we proved (a), and now (b) follows from Proposition 4.38. □

Lemma 4.40. *Let X be an affine Poisson scheme and $i : Z \hookrightarrow X$ a closed Poisson subscheme. If $i^!$ is the topological functor of sections supported on $\underline{Z}$, we have an isomorphism*

$$\varinjlim_{Z'} Hom_{Poi^G}(\mathcal{F}, i_* i^{!\mathrm{Poi}}\mathcal{G}) \simeq Hom_{\mathrm{Poi}^G}(\mathcal{F}, i_* i^!\mathcal{G}).$$

Proof. This follows from the same argument as in [Bez1, Lemma 3(a)], replacing "equivariant" everywhere by "equivariant Poisson," which we may do following Lemma 4.32. □

Definition 4.41. If C is a G-orbit and p is a given perversity, then define a t-structure $\left({}^p\mathcal{D}^{\leq 0}_{\mathrm{Poi}^G}(C), {}^p\mathcal{D}^{\geq 0}_{\mathrm{Poi}^G}(C)\right)$ on $\mathcal{D}_{\mathrm{Poi}^G}(C)$

$$\begin{aligned} {}^p\mathcal{D}^{\leq 0}_{\mathrm{Poi}^G}(C) &= {}^{\mathrm{std}}\mathcal{D}^{\leq 0}_{\mathrm{Poi}^G}(C)[p(C)] \\ {}^p\mathcal{D}^{\geq 0}_{\mathrm{Poi}^G}(C) &= {}^{\mathrm{std}}\mathcal{D}^{\geq 0}_{\mathrm{Poi}^G}(C)[p(C)], \end{aligned}$$

where $\left({}^{\mathrm{std}}\mathcal{D}^{\leq 0}_{\mathrm{Poi}^G}(C), {}^{\mathrm{std}}\mathcal{D}^{\geq 0}_{\mathrm{Poi}^G}(C)\right)$ is the standard t-structure on ${}^{\mathrm{red}}\mathcal{D}_{\mathrm{Poi}^G}(C)$.

Theorem 4.42. *There is a unique t-structure $\left({}^p\mathcal{D}^{\leq 0}_{\mathrm{Poi}^G}(\mathcal{N}), {}^p\mathcal{D}^{\geq 0}_{\mathrm{Poi}^G}(\mathcal{N})\right)$ on ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N})$ such that for every locally closed orbit $k : C \hookrightarrow \mathcal{N}$ we have*

$$\begin{aligned} Lk^{*\mathrm{Poi}}\left({}^p\mathcal{D}^{\leq 0}_{\mathrm{Poi}^G}(\mathcal{N})\right) &\subset {}^p\mathcal{D}^{\leq 0}_{\mathrm{Poi}^G}(C) \\ Lk^{!\mathrm{Poi}}\left({}^p\mathcal{D}^{\geq 0}_{\mathrm{Poi}^G}(\mathcal{N})\right) &\subset {}^p\mathcal{D}^{\geq 0}_{\mathrm{Poi}^G}(C). \end{aligned}$$

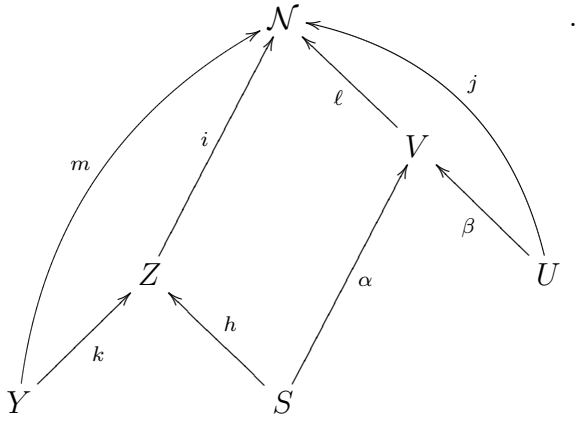
Proof. We want to apply the gluing theorem in a similar manner as we showed the construction of perverse coherent sheaves. This proof is nearly identical to the proof of Theorem 3.12, but it is necessary to restate the proof since certain arguments are different in the Poisson case. We induct on the number of orbits. There are always at least two orbits since the principal orbit is smooth, open and dense, while the point 0 is always a singular point. The

proof in the case when there are precisely two orbits (for example, $G = SL_2(\mathbb{C})$) is exactly the same as what follows, though some statements could be dramatically simplified.

In the general situation, choose an open orbit $j : U \hookrightarrow \mathcal{N}$ (say the principal orbit) and let $i : Z \hookrightarrow \mathcal{N}$ be its closed complement in $\mathcal{N}$ with the reduced subscheme structure. We assume that the t-structure on ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N})$ has been built up by iteratively gluing the t-structures on the orbits contained in Z. We want to apply the gluing theorem once more, so we need to see that the hypotheses are satisfied. For convenience of notation, we will denote ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N})$ by $\mathcal{D}$, ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(Z)$ by $\mathcal{D}_Z$ and ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(U)$ by $\mathcal{D}_U$.

The functors $i_* : \mathcal{D}_Z \to \mathcal{D}$ and $j^* : \mathcal{D} \to \mathcal{D}_U$ satisfy axiom (G1) for the same reasons as in the non-Poisson situation.

We show the existence of an induced t-structure on $\widetilde{\mathcal{D}_Z}$ required in (G2) by applying the gluing theorem within the induction argument. If Z is a single orbit, this follows from Proposition 3.2 since i_* is fully faithful in this case. Otherwise, let $h : S \hookrightarrow Z$ be an orbit which is open in Z with closed complement $k : Y \hookrightarrow Z$. Also define $\ell : V \to \mathcal{N}$ be the complement of Y in $\mathcal{N}$. For clarity, we summarize this information in the following diagram (with additional maps identified)

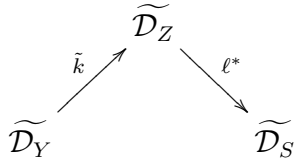


We may assume by induction that any closed subscheme $i_{Z'} : Z' \hookrightarrow \mathcal{N}$ with fewer orbits than Z satisfies (G2) with respect to its inclusion in $\mathcal{N}$. That is, if Z' has fewer orbits than Z, there is a t-structure on $\widetilde{\mathcal{D}_{Z'}} \subset \mathcal{D}_{\mathcal{N}}$ containing $i_{Z'*}(\mathcal{D}^{\leq 0}_{Z'})$ and $i_{Z'*}(\mathcal{D}^{\geq 0}_{Z'})$. The base case is clear: if there is only one orbit in a given closed subscheme, this statement follows from Proposition 3.2 since the pushforward from any orbit is fully faithful on the heart of the standard t-structure.

Therefore, since Y is a closed subscheme with fewer orbits, we get a t-structure on $\widetilde{\mathcal{D}_Y} \subset \mathcal{D}_{\mathcal{N}}$. Similarly, S is a closed subscheme of V and has fewer orbits than Z, so we get a t-structure on $\widetilde{\mathcal{D}_S} \subset \mathcal{D}_V$. Now m factors through i and so we get an inclusion

$$\tilde{k} : \widetilde{\mathcal{D}_Y} \hookrightarrow \widetilde{\mathcal{D}_Z} \subset \mathcal{D}_{\mathcal{N}}.$$

Also, $\ell^*|_{\widetilde{\mathcal{D}_Z}}$ has essential image in $\widetilde{\mathcal{D}_S}$ since any sheaf supported topologically on the intersection of Z and V must be supported topologically on S. So we have



satisfying (G1). Now (G2) is easy in this case since $\tilde{k}$ is just the inclusion of a full subcategory. For (G3), fix $\mathcal{F} \in \widetilde{\mathcal{D}_Z}$. We will use $\tau_\bullet^S$ for the truncation functor associated to the t-structure on $\widetilde{\mathcal{D}_S}$. It is easy to see by following the proof of Proposition 3.2 that this is just a shift of the standard t-structure on $\widetilde{\mathcal{D}_S}$ by $p(S)$. To define $\mathcal{F}^+$ and $\mathcal{F}^-$, consider $\overline{\ell} : \overline{V} \to \mathcal{N}$, the closure of V in $\mathcal{N}$. Let

$$\mathcal{F}^- = \tau^{\mathrm{std}}_{\leq p(S)} \overline{\ell}_* \overline{\ell}^{!\mathrm{Poi}} \mathcal{F}$$
$$\mathcal{F}^+ = \mathbb{D}\tau^{\mathrm{std}}_{<\overline{p}(S)} \overline{\ell}_* \overline{\ell}^{!\mathrm{Poi}} \mathbb{D}\mathcal{F}.$$

Then since $\ell^* = \ell^* \overline{\ell}_* \overline{\ell}^{!\mathrm{Poi}}$ and $\tau^{\mathrm{std}}_{\leq p(S)}$ is the same as $\tau^S_{\leq 0}$ on $\widetilde{\mathcal{D}_S}$, we have

$$\ell^* \mathcal{F}^- \simeq \tau^{\mathrm{std}}_{\leq p(S)} \ell^* \mathcal{F} \simeq \tau^S_{\leq 0} \ell^* \mathcal{F}.$$

Now suppose $\mathcal{G} \in \widetilde{\mathcal{D}_Y}^{\geq 1}$. Then $\mathcal{G}$ is supported on some closed subscheme $i' : Y' \hookrightarrow \mathcal{N}$ with the same topological space as Y, say $\mathcal{G} \simeq i'_* \mathcal{G}'$ and so

$$\mathrm{Hom}(\mathcal{F}^-, \mathcal{G}) \simeq \mathrm{Hom}(Li'^{*\mathrm{Poi}} \mathcal{F}^-, \mathcal{G}').$$

Since $Li'^{*\mathrm{Poi}}$ is right t-exact with respect to the standard t-structure, we see that any bounded truncation of $Li'^{*\mathrm{Poi}} \mathcal{F}^-$ is in ${}^{\mathrm{std}}\mathcal{D}_{Y'}{}^{\leq p(S)}$.The monotonicity of the perversity then guarantees that ${}^{\mathrm{std}}\mathcal{D}_{Y'}{}^{\leq p(S)} \subset \widetilde{\mathcal{D}_Y}^{\leq 0}$. The morphism $\mathcal{F}^- \to \mathcal{F}$ is just the one induced from the counit of the adjunction $(\overline{\ell}_*, \overline{\ell}^{!\mathrm{Poi}})$. For $\mathcal{F}^+$ we have the dual statements. Since $\ell^* = \ell^* \overline{\ell}_* \overline{\ell}^{!\mathrm{Poi}}$ and $\mathbb{D}\tau^{\mathrm{std}}_{<\overline{p}(S)} = \tau^{\mathrm{std}}_{\geq p(S)} \mathbb{D}$ on $\widetilde{\mathcal{D}_S}$,

$$\ell^* \mathcal{F}^+ \simeq \tau^S_{\geq 1} \ell^* \mathcal{F}.$$

The Hom vanishing statement for $\mathcal{F}^+$ follows by duality and the morphism $\mathcal{F} \to \mathcal{F}^+$ is the one induced by the dual of the counit of the adjunction $(\overline{\ell}_*, \overline{\ell}^{!\mathrm{Poi}})$.

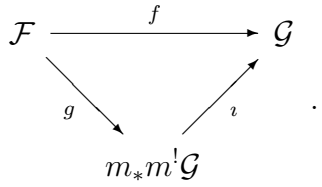
To show that the next axiom is satisfied, suppose we have a morphism $\mathcal{F} \xrightarrow{f} \mathcal{G}$ in $\widetilde{\mathcal{D}_Z}$ with $\ell^* f = 0$. Consider the distinguished triangle in $\mathcal{D}^+_G(\mathcal{Q}_\mathcal{N})$

$$m_* m^! \mathcal{G} \xrightarrow{\imath} \mathcal{G} \to \ell_* \ell^* \mathcal{G} \to m_* m^! \mathcal{G}[1],$$

which gives an exact sequence

$$\cdots \to \mathrm{Hom}(\mathcal{F}, m_* m^! \mathcal{G}) \xrightarrow{(\alpha \mapsto \alpha \circ \imath)} \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(\ell^* \mathcal{F}, \ell^* \mathcal{G}) \to \cdots.$$

Since $\ell^* f = 0$, we see that f factors through $m_* m^! \mathcal{G}$ and we get the diagram



Now if $\widetilde{\mathcal{F}}^\bullet$ and $\widetilde{\mathcal{G}}^\bullet$ are chain complexes representing $\mathcal{F}$ and $m_* m^! \mathcal{G}$, respectively and $\widetilde{g}$ is a chain map representing the morphism g, then the image of $\widetilde{\mathcal{F}}^\bullet$ under $\widetilde{g}$ is a bounded subcomplex of $\widetilde{\mathcal{G}}^\bullet$ consisting of coherent sheaves. Viewing this image in the derived category gives an object of $\widetilde{\mathcal{D}_Y}$ which satisfies (G4).

The axiom (G5) says that if $\ell^* \mathcal{F} = 0$, then $\mathcal{F}$ is in the triangulated category generated by the essential image of ${m_*}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(Z)$. We know that this is true in the non-Poisson setting, and the definition of ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N})$ guarantees that this is also true for Poisson sheaves. Let $\mathcal{I}_Y$ be the ideal sheaf corresponding to the subscheme Y. We know that we can find a sheaf $\mathcal{F}'$ and a closed subscheme $i' : Y' \hookrightarrow \mathcal{N}$ with $i'_* \mathcal{F}' \simeq \mathcal{F}$. Moreover, the proof that we can choose Y' so that the ideal sheaf $\mathcal{I}_{Y'} = \mathcal{I}_Y^n$ goes through in the Poisson setting. To see that this implies (G5), we still need to see that any sheaf supported on Y' is in the triangulated category generated by ${i_*}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(Y)$. This amounts to recognizing that in ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N})$, we have that $\mathcal{I}_Y \mathcal{F} = 0$ implies that $\{\mathcal{I}_Y, \mathcal{F}\} = 0$. Thus we can apply the gluing theorem and get a t-structure on $\widetilde{\mathcal{D}_Z}$ satisfying (G2).

Once again, we prove (G3)–(G5) with an identical argument replacing V with U and Y with Z. For (G3), fix $\mathcal{F} \in \mathcal{D}$. We will use $\tau^U_\bullet$ for the truncation functor associated to the perverse t-structure on U (the shift of the standard t-structure by $p(U)$). To define $\mathcal{F}^+$ and $\mathcal{F}^-$, consider $\overline{\jmath} : \overline{U} \to \mathcal{N}$, the closure of U in $\mathcal{N}$. Let

$$\mathcal{F}^- = \tau^{\mathrm{std}}_{\leq p(U)} \overline{\jmath}_* R\overline{\jmath}^{!_{\mathrm{Poi}}} \mathcal{F}$$
$$\mathcal{F}^+ = \mathbb{D}\tau^{\mathrm{std}}_{<\overline{p}(U)} \overline{\jmath}_* R\overline{\jmath}^{!_{\mathrm{Poi}}} \mathbb{D}\mathcal{F}.$$

Then since $j^* = j^* \overline{\jmath}_* R\overline{\jmath}^{!_{\mathrm{Poi}}}$ is exact and $\tau^{\mathrm{std}}_{\leq p(U)}$ is the same as $\tau^U_{\leq 0}$ on U, we have

$$j^* \mathcal{F}^- \simeq \tau^{\mathrm{std}}_{\leq p(U)} j^* \mathcal{F} \simeq \tau^U_{\leq 0} j^* \mathcal{F}.$$

Now suppose $\mathcal{G} \in \widetilde{\mathcal{D}_Z}^{\geq 1}$. Then $\mathcal{G}$ is supported on some closed subscheme $i' : Z' \hookrightarrow \mathcal{N}$ with the same topological space as Z, say $\mathcal{G} \simeq i'_* \mathcal{G}'$ and so

$$\mathrm{Hom}(\mathcal{F}^-, \mathcal{G}) \simeq \mathrm{Hom}(Li'^{*_{\mathrm{Poi}}} \mathcal{F}^-, \mathcal{G}').$$

Since $Li'^{*_{\mathrm{Poi}}}$ is right t-exact with respect to the standard t-structure, we see that any bounded truncation of $Li'^{*_{\mathrm{Poi}}} \mathcal{F}^-$ is in ${}^{\mathrm{std}}\mathcal{D}_{Z'}{}^{\leq p(U)}$.The monotonicity of the perversity then guarantees that ${}^{\mathrm{std}}\mathcal{D}_{Z'}{}^{\leq p(U)} \subset \widetilde{\mathcal{D}_Z}^{\leq 0}$. The morphism $\mathcal{F}^- \to \mathcal{F}$ is just the one induced from the counit of the adjunction $(\overline{\ell}_*, \overline{\ell}^{!_{\mathrm{Poi}}})$.

For $\mathcal{F}^+$ we have the dual statements. Since $j^* = j^* \overline{\jmath}_* L\overline{\jmath}^{!_{\mathrm{Poi}}}$ and $\mathbb{D}\tau^{\mathrm{std}}_{<\overline{p}(U)} = \tau^{\mathrm{std}}_{\geq p(U)} \mathbb{D}$ on U,

$$j^* \mathcal{F}^+ \simeq \tau^U_{\geq 1} j^* \mathcal{F}.$$

Just as before, the Hom vanishing statement for $\mathcal{F}^+$ follows by duality. This proves that (G3) is satisfied.

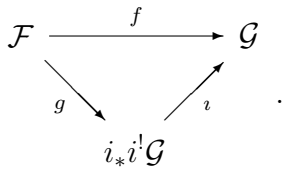
To show that the next axiom is satisfied, suppose we have a morphism $\mathcal{F} \xrightarrow{f} \mathcal{G}$ in ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N})$ with $j^*f = 0$. Consider the distinguished triangle in $\mathcal{D}^+_{\mathrm{Poi}^G}(\mathcal{Q}(\mathcal{N}))$

$$i_*i^!\mathcal{G} \xrightarrow{\imath} \mathcal{G} \to j_*j^*\mathcal{G} \to i_*i^!\mathcal{G}[1],$$

which gives an exact sequence

$$\cdots \to \mathrm{Hom}(\mathcal{F}, i_*i^!\mathcal{G}) \xrightarrow{(\alpha \mapsto \alpha \circ \imath)} \mathrm{Hom}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}(j^*\mathcal{F}, j^*\mathcal{G}) \to \cdots .$$

Since $j^*f = 0$, we see that f factors through $i_*i^!\mathcal{G}$ and we get the diagram



Now if $\widetilde{\mathcal{F}}^\bullet$ and $\widetilde{\mathcal{G}}^\bullet$ are chain complexes representing $\mathcal{F}$ and $i_*i^!\mathcal{G}$, respectively and $\widetilde{g}$ is a chain map representing the morphism g, then the image of $\widetilde{\mathcal{F}}^\bullet$ under $\widetilde{g}$ is a bounded subcomplex of $\widetilde{\mathcal{G}}^\bullet$ consisting of Poisson equivariant coherent sheaves. Viewing this image in the derived category gives an object of $\widetilde{\mathcal{D}_Z}$ which satisfies (G4).

The axiom (G5) says that if $j^*\mathcal{F} = 0$, then $\mathcal{F}$ is in the triangulated category generated by the essential image of $i_*{}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(Z)$. We know that this is true in the non-Poisson setting, and the definition of ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N})$ essentially guarantees that this is also true for Poisson sheaves. Let $\mathcal{I}_Z$ be the ideal sheaf corresponding to the subscheme Z. We know that we can find a sheaf $\mathcal{F}'$ and a closed subscheme $i' : Z' \hookrightarrow \mathcal{N}$ with $i'_*\mathcal{F}' \simeq \mathcal{F}$. Moreover, the proof that we can choose Z' so that the ideal sheaf $\mathcal{I}_{Z'} = \mathcal{I}_Z^n$ goes through in the Poisson setting. As before, to see that this implies (G5), we only need to recognize that in ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N})$, a sheaf supported on Z' as a coherent sheaf is also supported on Z' as a Poisson sheaf.

So we can apply the gluing theorem to get a t-structure $\left({}^p\mathcal{D}^{\leq 0}_{\mathrm{Poi}^G}(\mathcal{N}), {}^p\mathcal{D}^{\geq 0}_{\mathrm{Poi}^G}(\mathcal{N})\right)$ on ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N})$. The uniqueness in the statement of the theorem follows immediately from the uniqueness in Theorem 3.4.

□

Remark 4.43. Definition 4.41 makes sense for both the bounded and unbounded categories. We can summarize Theorem 4.42 as follows. We start with a shift (given by the perversity) of the standard t-structure on the bounded derived category of each orbit. We then iteratively build up a t-structure on all of $\mathcal{N}$ by adding on additional orbits one at a time using the gluing theorem. The uniqueness statement then says that the t-structure that we obtain restricts (using Poisson restrictions) to the (possibly unbounded) shift of the standard t-structure that we started with.

Definition 4.44. We call the t-structure $\left({}^p\mathcal{D}^{\leq 0}_{\mathrm{Poi}^G}(\mathcal{N}), {}^p\mathcal{D}^{\geq 0}_{\mathrm{Poi}^G}(\mathcal{N})\right)$ on ${}^{\mathrm{red}}\mathcal{D}^b_{\mathrm{Poi}^G}(\mathcal{N})$ which was constructed in theorem 4.42 the *reduced perverse Poisson* t-structure and objects in its heart $\mathcal{M}_{\mathrm{Poi}^G}(\mathcal{N})$ will be called *reduced perverse Poisson sheaves*. Note that this t-structure depends on the choice of perversity p.

4.4 Simple Perverse Poisson Sheaves

Since all of the coadjoint orbits on the nilpotent cone have even dimension, we can fix p to be the middle perversity. Recall (Definition 2.23) that the middle perversity is defined to be $p : \mathcal{N}/G \to \mathbb{Z}$ sending an orbit $C \mapsto \frac{1}{2}\dim C$. Clearly this perversity is both strictly monotone and strictly comonotone. Let $\left({}^p\mathcal{D}^{\leq 0}_{\mathrm{Poi}^G}(\mathcal{N}), {}^p\mathcal{D}^{\geq 0}_{\mathrm{Poi}^G}(\mathcal{N})\right)$ be the associated t-structure defined in the previous section and denote its heart by $\mathcal{M}_{\mathrm{Poi}^G}(\mathcal{N})$.

Proposition 4.45. *For each orbit C there is a fully faithful functor $\mathcal{IC}(C, -) : \mathcal{M}_{\mathrm{Poi}^G}(C) \to \mathcal{M}_{\mathrm{Poi}^G}(\mathcal{N})$ and every simple object in $\mathcal{M}_{\mathrm{Poi}^G}(\mathcal{N})$ is of the form $\mathcal{IC}(C, \mathcal{F})$, where $\mathcal{F}$ is a simple equivariant Poisson sheaf on C.*

Proof. This follows immediately from Proposition 3.9. □

As in the case for coherent sheaves, we have shown that simple perverse Poisson sheaves are all supported on the closure of a single orbit. In fact, we are able to give another description of simple perverse Poisson sheaves that will allow us to make a connection both with perverse coherent sheaves and with classical perverse sheaves. In order to discuss this further, we need the following definition which can be found, for example, in [Del2].

Definition 4.46. Let X be a scheme of finite type over $\mathbb{C}$ and $\mathcal{E}$ a vector bundle over X. A *connection* for $\mathcal{E}$ is an $\mathbb{C}$-linear morphism $\nabla : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ which locally satisfies the Leibnitz rule. We say that a connection for $\mathcal{E}$ is *flat* if the image of ∇ is in the kernel of the natural map $\Omega^2_X \otimes \mathcal{E}$.

Recall that given any local system, *i.e.* a locally constant sheaf of complex vector spaces, we can tensor with the structure sheaf to obtain a vector bundle. In fact, the following lemma shows that more is true.

Lemma 4.47. *On a G-orbit C, the categories of equivariant vector bundles with a flat connection and coherent sheaves of the form $\mathcal{O}_C \otimes \mathcal{L}$ for some equivariant local system $\mathcal{L}$ are equivalent.*

Proof. Since C is smooth, this is a classical result. For example, see [Del2]. □

Now we can state the connection between local systems, coherent sheaves and Poisson sheaves.

Lemma 4.48. *On a G-orbit, the category of equivariant vector bundles with a flat connection and the category of coherent equivariant Poisson sheaves are equivalent.*

Proof. Let C be a coadjoint orbit in $\mathcal{N}$. The transitivity of the G-action guarantees that any coherent sheaf on C is locally free and hence a vector bundle. It is then straightforward to see that the definition of a flat connection above is equivalent to giving a Poisson module structure. See [Pol] for more details. □

Theorem 4.49. *Let $\mathcal{F}$ be an equivariant vector bundle with a flat connection on an orbit C of $\mathcal{N}/G$ and C' another orbit contained in the closure of C. Then*

$$i_{C'}^{*\text{Poi}} H^i\left(\mathcal{IC}(C,\mathcal{F})\right) = \mathcal{O}_{C'} \otimes \mathcal{L}'$$
$$i_{C'}^{!\text{Poi}} H^i\left(\mathcal{IC}(C,\mathcal{F})\right) = \mathcal{O}_{C'} \otimes \mathcal{L}''$$

where $\mathcal{L}'$ and $\mathcal{L}''$ are equivariant local systems on C'.

Proof. From the proof of Proposition 3.7, we see that $\mathcal{IC}(C,\mathcal{F})$ is supported on the closure of C. The Poisson theory that we have developed then ensures that taking the Poisson inverse image (resp. Poisson inverse image with supports) of a cohomology sheaf to any orbit C' in the closure of C yields an equivariant Poisson sheaf on C', *i.e.* by Lemma 4.47 a vector bundle arising from a local system. □

Example 4.50. Consider the nilpotent cone $\mathcal{N} \subset \mathfrak{sl}_2^*$. Then $\mathcal{N}$ consists of two orbits, the principal open orbit C_{prin} and the closed point $\{0\}$. It is well known that there are two irreducible local systems on the principal orbit (this follows from the fact that the equivariant fundamental group in this case is $\mathbb{Z}/2\mathbb{Z}$). Let $\mathcal{L}$ be the nontrivial irreducible local system. The corresponding simple perverse sheaf $IC(C_{\text{prin}},\mathcal{L})$ (here we mean in the sense of classical perverse sheaves discussed in Section 2.2) is supported on the principal orbit. That is, the topological pullback functors i_0^{-1} and $i_0^!$ applied to $H^i IC(C_{\text{prin}},\mathcal{L})$ are zero.

If we now take the coherent sheaf $\mathcal{F} = \mathcal{O}_{C_{\text{prin}}} \otimes \mathcal{L}$, we get a different result when considering the coherent pullbacks of the simple perverse coherent sheaf $\mathcal{IC}(C_{\text{prin}},\mathcal{F})$. Here the coherent pullbacks i^* and $i^!$ of the cohomology sheaves are not zero. However, we can also consider $\mathcal{F}$ as an irreducible Poisson sheaf, and if we look at the Poisson pullbacks of the corresponding simple perverse Poisson sheaf, we see that $i^{*\text{Poi}}\mathcal{IC}(C_{\text{prin}},\mathcal{F})$ and $i^{!\text{Poi}}\mathcal{IC}(C_{\text{prin}},\mathcal{F})$ are zero. So in this case, the result from the setting of classical (constructible) perverse sheaves coincides with that from the setting of Poisson sheaves.

Remark 4.51. In general, we can not even begin to expect a correspondence as in the previous example. Indeed, in the classical setting one finds nonzero cohomology in degrees up to twice the algebraic dimension, while perverse Poisson sheaves can have nonzero cohomology only up to the algebraic dimension of the variety. This example does give us some slight indication that the connection between local systems and Perverse Poisson sheaves might give some additional information about the various restrictions of simple perverse sheaves to orbits.

Now that the basic framework for perverse Poisson sheaves is in place, Theorem 4.49 leads immediately to interesting questions. For example, we would like to be able to say precisely which local systems $\mathcal{L}'$ and $\mathcal{L}''$ can occur, and determine more exactly the relationship between the Poisson pullbacks of the cohomology sheaves and the coherent counterparts. In addition, we would also like to be able to identify other varieties which would support this theory. As mentioned above, the nilpotent cone has a very close relationship between the G-action and the Poisson structure which greatly reduces the complexity.

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Vita

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