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ABSTRACT

Let E be a Helson set in a locally compact abelian group G such that 1 is not in E. Let F be a closed subset of G that is disjoint from $E \cup E^{-1} \cup \{1\}$, and let g be element of C(E) such that $g(x^{-1}) = \overline{g(x)}$ for x, x^{-1} in E. Then for every $\epsilon > 0$ there is a positive definite function $f \in C(G)$ such that f = g on E and $|f| < \epsilon$ on F. If h denotes the Helson constant of E and fz denotes the Fatou-Zygmund constant of E, then $fz \leq (11664)(h^{12})$.

0. Introduction

Let E be a closed set in a locally compact abelian group G. $C_0(E)$ denotes the set of continuous functions on E vanishing at ∞. A function f on E is hermitian if $f(x) = \overline{f(x^{-1})}$ for all $x \in E \cap E^{-1}$. Let

$$C_0^h(E) = \{f \in C_0(E): f \text{ is hermitian}\}$$

$$A(E) = \{f|_E : f \in A(G)\}$$

$$A_+(E) = \{f|_E : f \in A(G), f \text{ is positive definite}\}$$

E is a Helson set if $C_0(E) = A(E)$. If E is a Helson set, then

$$h(E) = \sup\{\|f\|_{A(E)}: f \in C_0(E), \|f\|_{C_0(E)} \le 1\}$$

is a finite number that is called the Helson constant of E. If E is a Helson set, we refer to the number

$$h_0(E) = \sup\{\|f\|_{A(E)}: f \in C_0^h(E), \|f\|_{C_0(E)} \le 1\}$$

as the hermitian Helson constant of E. Helson sets in discrete G are called Sidon sets.

E is a Fatou-Zygmund set if $C_0^h(E) = A_+(E)$. Let $\|f\|_{A_+(E)} = \inf\{\|g\|_A : g|_E = f, g \text{ is positive definite}\}$. $\| \|_{A_+(E)}$ is in general not a norm; for instance, $\|1\|_{A_+(G)} \neq \|-1\|_{A_+(G)}$. If E is a $\mathfrak{F}-\mathfrak{Z}$ set, then

$$f\mathfrak{z}(E) = \sup\{\|f\|_{A_+(E)}: f \in C_0^h(E), \|f\|_{C_0(E)} \le 1\}$$

is a finite number [5, p. 94], that is called the Fatou-Zygmund constant of E.

E is a <u>uniform Helson set</u> if for every closed set F disjoint from E, every $\varphi \in C_0(E)$ and every $\epsilon > 0$ there exists a $g \in A(G)$ with

$$g|_E = \varphi$$
$$|g| < \epsilon \text{ on } F .$$

E is a <u>uniform $\mathfrak{I}$ - $\mathfrak{P}$ set</u> if for every closed set F disjoint from $E \cup (E)^{-1} \cup \{1\}$, every $\varphi \in C_0^h(E)$ and every $\epsilon > 0$ there exists a $g \in A_+(G)$ with

$$g|_E = \varphi$$
$$|g| < \epsilon \text{ on } F .$$

S. W. Drury [1] showed that Sidon sets are uniform Helson sets. N. Th. Varopoulos [6] generalized Drury's argument to show that Helson sets are uniform Helson sets. C. S. Herz produced an alternate proof to Varopoulos' proof. Drury [2] then showed that Sidon sets are uniform $\mathfrak{I}$ - $\mathfrak{P}$ sets. O. C. McGehee suggested that it might be possible by using Herz's techniques to extend Drury's argument to show that Helson sets are uniform $\mathfrak{I}$ - $\mathfrak{P}$ sets, and this is what we do.

We first show:

<u>Theorem I</u>. Let $f\mathfrak{P}(E) = 1$. Let F be a closed subset of G disjoint from $E \cup E^{-1} \cup \{1\}$. Let $\varphi \in C_0^h(E)$, $\|\varphi\|_{C_0(E)} \leq 1$. For every $K > 1$, $\epsilon > 0$ there exists $f \in A_+C(G)$ such that:

i) $f = \varphi$ on E

ii) $\|f\|_{C(F)} \le K^2 \epsilon$

iii) $\|f\|_A \le K^2 \dfrac{16 h_0^4}{\epsilon}$

Theorem I was proved by Varopoulos [6] for compact, totally disconnected, metrizable Kronecker sets. We prove Theorem I by making explicit an implicit proof in Herz [3]. We prove Theorem II by using Theorem I and the techniques Drury uses in showing Sidon sets are uniform $\mathfrak{I}-\mathfrak{P}$ sets. The constants in Theorem II are the same as those that Drury obtains for the case of a Sidon set E.

The outline of the paper is as follows. In Section 1 we show that in proving Theorem I or Theorem II we may suppose G is compact. We show that in proving Theorem I in the case of a compact, metrizable set we may suppose that E is also totally disconnected. Finally, we give a sufficient condition on E to yield $f\mathfrak{P}(E) = 1$. Section 2 contains the Smoothing Theorem and the Transfer Lemma which are the tools for proving Theorems I, II. Theorem I is proved in Section 3, Theorem II in Section 4. We close with a few remarks in Section 5.

1. Preliminaries

We want to show that it suffices to prove Theorems I and II in the case that G is compact. We give the reduction for Theorem I. The reduction for Theorem II is similar.

It suffices to show the reduction for an equivalent formulation of Theorem I, wherein we replace the conclusion by: For every $\xi > 0$, $K > 1$, $\epsilon > 0$ there exists a $f \in A_+(G)$ such that:

i) $\|f - \varphi\|_{C_0(E)} < \xi$

ii) $\|f\|_{C(F)} \le K^2 \epsilon$

iii) $\|f\|_A \le K^2/\epsilon$

Lemma 1. In order to prove Theorem I, it suffices to prove it in the case when E is compact.

Proof. Let us suppose Theorem I is proved for the case of compact E. Now let E be arbitrary. Consider the two convex subsets of $C_0(E)$,

$$B = \{ f \in C_0(E) : \|f - \varphi\|_{C_0(E)} < \xi \}$$

$$C = \{ f|_E : f \in A_+(G),\ \|f\|_{C(F)} \le K^2 \epsilon,\ \|f\|_A \le K^2/\epsilon \}$$

Note that B has a non-empty interior. Suppose $B \cap C = \emptyset$, we may apply the Hahn - Banach Theorem [4, p. 157 and p. 187] to the disjoint sets $B - \varphi$, $C - \varphi$. So there exists $\mu \in M(E)$, $\alpha \in R$ such that

$$\mathrm{Re}\,\langle f-\varphi,\mu\rangle \le \alpha\|\mu\| \quad \text{for} \quad f \in B$$

$$\mathrm{Re}\,\langle f-\varphi,\mu\rangle \ge \alpha\|\mu\| \quad \text{for} \quad f \in C\ .$$

But $\mathrm{Re}\,\langle f-\varphi,\mu\rangle \le \alpha\|\mu\|$ for $f \in B$ implies $\alpha \ge \xi$. We obtain that for all $f \in C$

$$\xi\|\mu\| \le \mathrm{Re}\,\langle f-\varphi,\mu\rangle \le \left|\int_E (f-\varphi)\,d\mu\right| .$$

Hence if $\xi' < \xi$ there exists a compactly supported ν such that $\|\nu\| \le \|\mu\|$,

$$\xi'\,\|\nu\| \le \left|\int_E (f-\varphi)\,d\nu\right| \quad \text{for} \quad f \in C\ ,$$

but the integral is bounded by $\|f-\varphi\|_{C(\mathrm{supp}\,\nu)}\,\|\mu\|_M$. It follows that Theorem I is false for the compact set $\mathrm{supp}(\nu)$.

<u>Lemma 2</u>. In order to prove Theorem I, it suffices to prove it in the case when F is compact.

<u>Proof</u>. Let $B = \{f \in C_0(F) : \|f\|_{C_0(F)} \le K^2\epsilon\}$.

$$C = \{f|_F : f \in A_+(G),\ \|f-\varphi\|_{C_0(E)} < \xi,\ \|f\|_A \le K^2/\epsilon\}\ .$$

Proceed as before, applying the Hahn - Banach theorem to B and C .

For E a closed subset of G let $B_{d,+}(E) = \{\hat{\mu}|_E : \mu \in M_d(\hat{G}),\ \mu \ge 0\}$. For $f \in B_{d,+}(E)$ let $\|f\|_{B_{d,+}(E)} = \inf\{\|\mu\|_M : \mu \in M_d(\hat{G}),\ \mu \ge 0,\ \hat{\mu}|_E = f\}$. If E is compact, $B_{d,+}(E) = A_+(E)$ and $\|\ \|_{B_{d,+}(E)} = \|\ \|_{A_+(E)}$.

<u>Lemma 3</u>. In order to prove Theorem I, it suffices to prove it in the case when G is compact.

Proof. Supposing the theorem is proved in the case of compact G, consider an arbitrary G. Let θ be the inclusion map of G into its Bohr compactification. We may suppose E and F are compact, so that θE and θF are compact. θ is symmetric; that is, $\theta(x^{-1}) = (\theta x)^{-1}$. So θF is disjoint from $\theta E \cup (\theta E)^{-1} \cup \{1\}$.

Since $f\mathcal{J}(E) = 1$, $B_{d,+}(E) = A_+(E)$ and $\| \|_{B_{d,+}(E)} = \| \|_{A_+(E)}$ we have $f\mathcal{J}(\theta E) = 1$. So we have the theorem for θE and θF. But $B_{d,+}(E \cup F) = A_+(E \cup F)$, $\| \|_{B_{d,+}(E \cup F)} = \| \|_{A_+(E \cup F)}$. Hence, we have the theorem for E and F.

Lemma 4. In order to prove Theorem I in the case when E is compact and metrizable, it suffices to prove it assuming that E is totally disconnected.

Proof. It suffices by the argument of Lemma 1 to show that for every $K > 1$, $\epsilon > 0$, $\xi > 0$, $\mu \in M(E)$ there is a $f \in A_+(E)$ such that

i′) $|\int f - \varphi \, d\mu| < \xi \|\mu\|$

ii) $|f| \le K^2 \epsilon$ on F

iii) $\|f\|_A \le K^2/\epsilon$.

Suppose Theorem I is true for totally disconnected subsets of E. Since E is compact and metrizable, for all $\xi' > 0$ there is a $\nu \in M(E)$ such that supp ν is totally disconnected, supp $\nu \subsetneq$ supp μ, and $\|\mu - \nu\|_{M(E)} < \xi'$.

We know there is a $f \in A_+(E)$ satisfying:
$|\int f-\varphi \, d\nu| < \xi' \|\nu\|$, ii), iii). Hence

$$|\int f-\varphi \, d\mu| = |\int (f-\varphi) \, d(\nu+\mu-\nu)| \leq$$
$$|\int f-\varphi \, d\nu| + |\int (f-\varphi) \, d(\mu-\nu)| <$$
$$\xi' \|\nu\| + (1 + K^2/_\epsilon)\xi' < \xi \|\mu\|$$

for correct ξ'.

Let $U(E) = \{f \in C(E) : |f| = 1\}$. Let $U^h(E)$ denote the hermition elements of $U(E)$.

Lemma 5. Suppose for every $\xi > 0$, every $f \in U^h(E)$ there exists $\int \in \hat{G}$ such that $|f - \int| < \xi$ on E. Then $f\mathcal{J}(E) = 1$.

Proof. It suffices to prove this lemma under the assumption that E is symmetric.

Let $A^h(E)$ denote the hermitian elements of $A(E)$. $C^h(E)$ and $A^h(E)$ are real Banach spaces under the induced norms. Let $\theta : A^h(E) \to C^h(E)$ be the natural injection. We wish to see $\|\theta\| = 1$. Let θ^* be the dual of θ. It suffices to see $\|\theta^*\| = 1$.

We know [4, p. 179] that

$$(C^h(E))^* \leq \{Re <\cdot, \mu> : \mu \in M(E)\}.$$

We call a measure, μ, hermitian if $d\mu(x) = \overline{d\mu(x^{-1})}$. We denote the set of hermitian measures in $M(E)$ by $M^h(E)$. For every $\mu \in M(E)$ the hermitian measure

$d\nu(x) = \frac{d\mu(x) + \overline{d\mu(x^{-1})}}{2}$ has the property that $\mathrm{Re}\langle f,\mu\rangle = \int f\,d\nu$ for $f \in C^h(E)$. Hence,

$$(C^h(E))^* \leq \{\langle\ ,\mu\rangle : \mu \in M^h(E)\} .$$

If $\mu \in M^h(E)$, then $\frac{d\mu}{d|\mu|}$ is hermitian a.e. So we may write $\frac{d\mu}{d|\mu|} = e^{ih}$ where $h(x) = -h(x^{-1}) \bmod (2\pi)$ a.e. By a modification of the proof of Lusin's theorem we may for every $\xi > 0$ find a continuous g such that: $g(x) = -g(x^{-1}) \bmod (2\pi)$ and $g = h$ except on a set A with $|\mu|(A) < \xi$. So for $\mu \in M^h(E)$ we have

$$\|\mu\|_{(C^h(E))^*} = \sup_{\|f\|_{C^h(E)} \leq 1} \left|\int f\,d\mu\right| = \sup_{f \in U^h(E)} \left|\int f\,d\mu\right| =$$

$$\sup_{\int \in \hat{G}} \left|\int S\,d\mu\right| = \|\mu\|_{(A^h(E))^*} .$$

So for every $\xi > 0$, every $f \in C^h(E)$ there exists $f \in L'(\hat{G})$ satisfying: $\|f\|_{L'(\hat{G})} \leq 1 + \xi$ and $\hat{f} = f$ on E . But $\mathrm{Re}\, f = f$ on E since $\mathrm{Re}\, f(x) = \frac{1}{2}[\hat{f}(x) + \overset{\Delta}{f}(x)] = \frac{1}{2}[\hat{f}(x) + \overline{\hat{f}(x^{-1})}] = \frac{1}{2}[f(x) + \overline{f(x^{-1})}] = f(x)$.

We know there exists $\int_- \in \hat{G}$ within ξ of -1 on E .

Let f^+ , f^- denote the positive and negative parts of f .

Then $(f^+ + \delta_{(\int_-)^{-1}} * f^-)^\wedge$ is within ξ of f on E .

So $f\mathscr{J}(E) = 1$.

2. The Drury-Herz Theorems

Given a Banach space B, $1 \le p < \infty$, let $L^p(G,B)$ denote the completion of the space of continuous B-valued functions with compact support on G for the norm $\|u\|_p = (\int_G \|u(x)\|_B^p \, dx)^{1/p}$ where dx is Haar measure on G. The completion of this space for the supremum norm will be $C_0(G,B)$.

The Smoothing Theorem. a) Let E be a compact Helson set in G with Helson constant h, hermitian Helson constant h_0. Let θ be a continuous map of E into a locally compact abelian group H. If C is a compact subset of $\hat{H}$, $\delta > 0$, $K > 1$, and $\eta \in \hat{H}$, then there exists $\alpha_\eta \in L^1(\hat{G})$ such that if we let $a_x(\eta) = \hat{\alpha}_\eta(x)$ for $(x,\eta) \in G \times \hat{H}$ then:

i) $|\alpha_\eta(x) - \eta(\theta x)| < \delta$ for $(x,\eta) \in E \times C$.

ii) $\hat{\alpha}_\eta \in C_0(\hat{H}, A(G))$, $\|\hat{\alpha}_\eta\|_{C_0(\hat{H},A(G))} \le K^2 h^2$

iii) $a_x \in C_0(G, A(\hat{H}))$, $\|a_x\|_{C_0(G,A(\hat{H}))} \le K^2 h^2$.

If θ is symmetric, we may replace h by h_0 in ii), iii).

b) Suppose that E is a compact symmetric Helson set and θ is symmetric. Then each α_η can be taken to be a real measure. We may write $\alpha_\eta = \alpha_\eta^{++} - \alpha_\eta^{--}$ where α_η^{++}, α_η^{--} are nonnegative measures such that if we let

$\alpha_\eta^* = \alpha_\eta^{++} + \alpha_\eta^{--}$, $a_x^*(\eta) = (\alpha_\eta^*)^\wedge(x)$ for $(x,\eta) \in G \times \hat{H}$, then

iv) $\hat{\alpha}_\eta^{++} \in C_0(\hat{H}, A(G))$, $\|\hat{\alpha}_\eta^{++}\|_{C_0(\hat{H},A(G))} \le K^2 h_0^2$.

v) $a_x^* \in C_0(G, A(\hat{H}))$, $\|a_x^*\|_{C_0(G,A(\hat{H}))} \le K^2 h_0^2$.

Proof of a) We may find a $k : \hat{H} \to [0,1]$ with compact symmetric support K such that $\|k\|_{L^2(\hat{H})} \le 1$ and

1) $|1 - (k*k)(\eta)| < {}^{\delta}/_{3}$ for $\eta \in C$.

Since θE is compact the set

$$\cup = \{\eta \in \hat{H} : |1 - \langle \theta x, \eta\rangle| < {}^{\delta}/_{3} \text{ for all } x \in E\}$$

is a neighborhood of the identity in $\hat{H}$. Since K is compact it is covered by a finite number of translated of $\cup$.

$$K \subset \cup_{i=1}^{n} \eta_i \cup .$$

Let $K_1 = K \cap \eta_1 \cup$, $K_{i+1} = (K \cap \eta_{i+1} \cup) \setminus \cup_{j=1}^{i} K_j$. Then

2) $|\langle \theta x, \eta_i\rangle - \langle \theta x, \eta\rangle| < \frac{\delta}{3}$ for $x \in E$, $\eta \in K_i$.

For every i there exists $\beta_i \in L^1(\hat{G})$ with $\|\beta_i\|_{L^1(\hat{G})} \le K h$.

3) $\hat{\beta}_i(x) = \langle \theta x, \eta_i\rangle$ for $x \in E$.

Let b be the Borel measurable, $A(G)$ valued function defined on $\hat{H}$ by

4) $$b(x,\eta) = \begin{cases} \beta_i(x) & \text{for } \eta \in K_i \\ 0 & \text{otherwise} \end{cases}$$

Then $k(\eta)\, b(x,\eta) \in L^2(\hat{H}, A(G))$ with norm bounded by $K h$.

Let

5) $$a = k\,b \underset{\hat{H}}{*} k\,b$$

$a(x,\eta) \in C_0(\hat{H}, A(G))$ and $\|a(x,\eta)\|_{C_0(\hat{H},A(G))} \le K^2 h^2$ since

$$\|k\,b \underset{\hat{H}}{*} k\,b\|_{C_0(\hat{H},A(G))} \le \|k\,b\|_{L^2(\hat{H},A(G))} \|k\,b\|_{L^2(\hat{H},A(G))}$$

We let $\alpha_\eta \in L^1(\hat{G})$ be such that $\hat{\alpha}_\eta = a(\cdot,\eta)$ ii) is immediate.

We wish to see iii). If $F, G \in L^2(\hat{H})$ then $F * G \in A(\hat{H})$ and $\|F*G\|_{A(\hat{H})} \le \|F\|_{L^2(\hat{H})} \|G\|_{L^2(\hat{H})}$. Hence to show iii) it suffices to show $k\,b \in C_0(G, L^2(\hat{H}))$ and $\|k\,b\|_{C_0(G,L^2(\hat{H}))} \le k\,h$. This follows from the norm decreasing inclusions:

$$L^2(\hat{H}, A(G)) \subseteq L^2(\hat{H}, C_0(G)) \subseteq C_0(G, L^2(\hat{H})) .$$

We now show i). Let $c(x,\eta) = \langle \theta x, \eta \rangle$. Then

$$(ck) \underset{\hat{H}}{*} (ck)(x,\eta) = \int_{\hat{H}} \langle \theta x, \eta\eta_1^{-1} \rangle\, k(\eta\eta_1^{-1})\, \langle \theta x, \eta_1 \rangle\, k(\eta_1)\, d\eta_1$$

$$= \langle \theta x, \eta \rangle\, (k * k)(\eta)$$

which for $\eta \in C$ differs from $\langle \theta x, \eta \rangle$ by no more than $\delta/3$. So for $(x,\eta) \in E \times \hat{H}$.

$$|(ck) * (ck) - (bk) * (bk)| \le |((c-b)k) * ck| + |bk * ((b-c)k)$$

$$\le \delta/3 \|k\|_{L^2(\hat{H})} \|ck\|_{L^2(\hat{H})} + \delta/3 \|bk\|_{L^2(\hat{H})} \|k\|_{L^2(\hat{H})} \le 2\delta/3 .$$

Finally, we wish to see that for the case that θ is symmetric we may replace h by h_0 in ii), iii). If θ is symmetric, $\langle\theta x,\eta_i\rangle$ is hermitian on E. So the β_i of line 3) may be chosen to have $\|\beta_i\|_{L^1(\hat{G})} \le Kh_0$.

Proof of b). We note that $(\mathrm{Re}\,\beta_i)^\wedge(x) =$

$$\frac{1}{2}[\hat{\beta}_i(x) + \bar{\hat{\beta}}_i(x)] = \frac{1}{2}[\hat{\beta}_i(x) + \overline{\hat{\beta}_i(x^{-1})}] = \frac{1}{2}[\eta_i(\theta x) + \eta_i(\overline{(\theta x)^{-1}})]$$

$$= \eta_i(\theta x) .$$

Let β^+, β^- denote the positive and negative parts of β.

Let

$$6)\quad b^+(x,\eta) = \begin{cases} \hat{\beta}_i^+(x) & \text{for } \eta \in K_i \\ 0 & \text{otherwise} \end{cases}$$

$$7)\quad b^-(x,\eta) = \begin{cases} \hat{\beta}_i^-(x) & \text{for } \eta \in K_i \\ 0 & \text{otherwise} \end{cases}$$

Let

$$8)\quad a^{++} = kb^+ \underset{\hat{H}}{*} kb^+ + kb^- \underset{\hat{H}}{*} kb^-$$

$$9)\quad a^{--} = kb^+ \underset{\hat{H}}{*} kb^- + kb^- \underset{\hat{H}}{*} kb^+$$

The same argument that gives $a(x,\eta) \in C_0(\hat{H},A(G))$ will give $a^{\pm\pm}(x,\eta) \in C_0(\hat{H},A(G))$. Let $\alpha_\eta^{\pm\pm} \in L^1(\hat{G})$ be such that $\hat{a}_\eta^{\pm\pm} = a^{\pm\pm}(\cdot,\eta)$. We see that $\alpha_\eta = \alpha_\eta^{++} - \alpha_\eta^{--}$.

We show iv).

$\|\hat{a}_\eta^{++}\|_{A(G)} =$

$$\|\int (kb^+)(\ ,\eta\eta_1^{-1})(kb^+)(\ ,\eta_1) + (kb^-)(\ ,\eta\eta_1^{-1})(kb^-)(\ ,\eta_1)\, d\eta_1\|_A$$

$$\le \int (\|(kb^+)(\ ,\eta\eta_1^{-1})\|_A + \|kb^-(\ ,\eta\eta_1^{-1})\|_A)(\|(kb^+)(\ ,\eta_1)\|_A +$$

$$\|(kb^-)(\ ,\eta_1)\|_A)\, d\eta_1 .$$

But since $(kb^{\pm})(\ ,\eta\eta_1^{-1})$ and $(kb^{\pm})(\ ,\eta_1)$ are positive definite for all $\eta_1 \in \hat{H}$ this last expression can be written

$$\int \|k(b^+ + b^-)(\ ,\eta\eta_1^{-1})\|_A\ \|k(b^+ + b^-)(\ ,\eta_1)\|_A\, d\eta_1$$

$$\le \|k(b^+ + b^-)\|_{L^2(\hat{H},A(G))} \|k(b^+ + b^-)\|_{L^2(\hat{H},A(G))} \le K^2 h_0^2 .$$

We show v). Let

$$10)\quad b^*(x,\eta) = \begin{cases} (|\beta_i|)^\wedge(x) & \text{for } \eta \in K_i \\ 0 & \text{otherwise} \end{cases}$$

Then $k(\eta)\, b^*(x,\eta) \in L^2(\hat{H},A(G))$ with norm bounded by $K h_0$. So $a^* = kb^* \underset{\hat{H}}{*} kb^*$ $\hat{a}_\eta{}^* = a^*(\cdot,\eta)$. We repeat the argument for iii).

<u>Transfer Lemma</u>. Let G,H be locally compact abelian groups. Let E be a compact subset of G . Let θ be a continuous map from E into H .

a) For every $c > 0$, every $\xi > 0$ and every $f \in L^1(\hat{G} \times \hat{H})$ there exists a compact subset C of $\hat{H}$ and a $\delta > 0$ such that if

$$\alpha_\eta \in C(\hat{H}, L^1(\hat{G})), \ \|\alpha_\eta\|_{C(\hat{H}, L^1(\hat{G}))} \leq c$$

and $|\hat{\alpha}_\eta(x) - \eta(\theta x)| < \delta$ for $(x,\eta) \in E \times C$

then $g = \int f_\eta * \alpha_{\eta^{-1}}\, d\eta$ satisfies

i) $g \in L^1(\hat{G}), \ \|g\|_{L^1(\hat{G})} \leq c\|f\|_{L^1(\hat{G} \times \hat{H})}$

ii) $|\hat{g}(x) - \hat{f}(x, \theta x)| < \xi$ for $x \in E$.

b) Let $\alpha_\eta \in C(\hat{H}, L^1(\hat{G}))$. Define $a_x(\eta)$ by

$$a_x(\eta) = \hat{\alpha}_\eta(x) \ .$$

If $a_x \in C(G, A(\hat{H})), \ \|a_x\|_{C(G, A(\hat{H}))} \leq c$, then

$g = \int f_\eta * \alpha_{\eta^{-1}}\, d\eta$ satisfies

iii) $|\hat{g}(x)| \leq c \sup_{y \in H} |\hat{f}(x,y)|$.

<u>Proof of a)</u>. Since $\|g\|_{L^1(\hat{G})} \leq$

$\int \|f_\eta\|_{L^1(\hat{G})} \|\alpha_{\eta^{-1}}\|_{L^1(\hat{G})}\, d\eta \leq c \int \|f_\eta\|_{L^1(\hat{G})}\, d\eta = c\, \|f\|_{L^1(\hat{G} \times \hat{H})}$,

we have i).

We show ii). $\hat{f}_\eta \in L^1(\hat{H}, A(G))$ and $\|\hat{f}_\eta\|_{L^1(\hat{H}, A(G))} =$

$\|f\|_{L^1(\hat{G} \times \hat{H})}$. So there is a compact symmetric set C in

$\hat{H}$ such that

$$\int_{\hat{H}\setminus C} \|\hat{f}_\eta\|_{A(G)} \, d\eta < \min(\xi/3, \xi/3c) .$$

We write $\hat{g}(x) = \int_{\hat{H}} \hat{f}_\eta \, \hat{\alpha}_{\eta^{-1}}(x) \, d\eta =$

$$\int_C \hat{f}_\eta(x) \, \hat{\alpha}_{\eta^{-1}}(x) \, d\eta + \int_{\hat{H}\setminus C} \hat{f}_\eta(x) \, \hat{\alpha}_{\eta^{-1}}(x) \, d\eta .$$

So if we choose $\delta = \dfrac{\xi}{3\|f\|_{L^1(\hat{G} \times \hat{H})}}$ we will have that for $x \in E$ $\hat{g}(x)$ differs from $\hat{f}(x, \theta x)$ by no more than ξ .

<u>Proof of b)</u>. $\hat{g}(x) = \int_{\hat{H}} \hat{f}_\eta(x) \, a_x(\eta^{-1}) \, d\eta =$

$$\int_{\hat{H}} \hat{f}_\eta(x) \left[\int_H \hat{a}_x(y) \, \eta(y) \, dy \right] d\eta =$$

$$\int_{\hat{H}} \hat{f}_\eta(x) \left[\int_H \hat{a}_x(y^{-1}) \, \overline{\eta(y)} \, dy \right] d\eta =$$

$$\int_H \int_{\hat{H}} \hat{f}_\eta(x) \, \overline{\eta(y)} \, \hat{a}_x(y^{-1}) \, d\eta \, dy = \int_H \hat{f}(x,y) \, \hat{a}_x(y^{-1}) \, dy .$$

So $|\hat{g}(x)| \le c \sup_y |\hat{f}(x,y)|$.

We refer to the δ, C given by part a) of the Transfer Lemma as the <u>c, ξ, f choice</u> .

3. Proof of Theorem I.

The Pull Back Theorem. Let E be a compact set with $f\mathcal{J}(E) = 1$. Let $\theta : E \to H$ be continuous and symmetric. Let $\pi : H \to G$ be a continuous homomorphism such that $\pi \circ \theta = \text{id}$. Let $h \in A_+(H)$. Then for every $\xi > 0$, $K > 1$, neighborhood V of 1 in G, there exists $g \in A_+(G)$ such that:

i) $|h \circ \theta - g| < \xi$ on E.

ii) $|g(x)| \leq K^2 \sup_{\pi y \in Vx} |h(y)|$

iii) $\|g\|_A \leq K^2 \|h\|_A$.

Proof. Let $k' \in A_+(G)$ be such that $\|k'\|_{A(G)} = k'(1) = 1$ $k' = 0$ off V^{-1}. Then $k'(x)h(y)$ is in $A_+(G \times H)$. Since $(x,y) \to (x\,\pi y^{-1}, y)$ is an automorphism of $G \times H$, $f(x,y) = k'(x\,\pi y^{-1})h(y) \in A_+(G \times H)$ $\|f\|_A \leq \|h\|_A$.

Let δ, C be the $K^2, \xi, \hat{f}$ choice. Since $h_0(E) = 1$ we may apply the Smoothing Theorem to δ, C, K to obtain α_η that satisfy:

i) $|\hat{\alpha}_\eta(x) - \eta(\theta x)| < \delta$ for $(x,\eta) \in E \times C$.

ii) $\hat{\alpha}_\eta \in C_0(\hat{H}, A(G))$, $\|\hat{\alpha}_\eta\|_{C_0(\hat{H},A(G))} \leq K^2$.

iii) $a_x \in C_0(G, (A(\hat{H}))$, $\|a_x\|_{C_0(G,A(\hat{H}))} \leq K^2$.

We wish to see that since $f\mathcal{J}(E) = 1$, we may choose $\alpha_\eta \geq 0$ for every η. We recall that $\hat{\alpha}_\eta(x) = a(x,\eta)$

where from line 5) $a = kb \underset{\hat{H}}{*} kb$. So we must check that for every η $k(\eta) \geq 0$ and $b(x,\eta) \in A_+(G)$. But to satisfy 1) k may be chosen to be $k = \frac{1}{(m(K))^{1/2}} \chi_K$ for a suitable compact set $K \subset \hat{H}$. Since $f\mathcal{P}(E) = 1$ and θ is symmetric we may find $\beta_i \in L^1(\hat{G})$, $\|\beta_i\|_{L^1(\hat{G})} \leq K$, $\beta_i \geq 0$ such that 3) is satisfied. And from 4)

$$b(x,\eta) = \begin{cases} \hat{\beta}_i(x) & \text{for } \eta \in K_i \\ 0 & \text{otherwise} \end{cases}$$

Therefore $g = \int_{\hat{H}} f_\eta \, \hat{\alpha}_{\eta^{-1}} \, d\eta \in A_+(G)$. g satisfies by the Transfer Lemma.

i) $|g(x) - f(x,\theta x)| < \xi$ for $x \in E$.

ii) $|g(x)| \leq K^2 \sup_{y \in H} |f(x,y)|$

iii) $\|g\|_A \leq K^2 \|f\|_A$.

But $f(x,\theta x) = k'(x \cdot x^{-1})\, h(\theta x) = h(\theta x)$ for $x \in E$, and $\sup_{y \in H} |f(x,y)| = \sup_{\pi y \in Vx} |h(y)|$.

We will suppose in the remainder of this section that G is compact. This is possible by Lemma 3 of Section 1. We may also assume $\varphi = 1$, since $f\mathcal{P}(E) = 1$.

<u>Lemma 1</u>. If E is a finite set and $f\mathcal{P}(E) = 1$, then Theorem I is true for E .

Proof. Write $E = E_0 \cup E_0' \cup e_0$ where $E_0 \cap E_0^{-1} = \emptyset$, $E_0' \subset E_0^{-1}$ and e_0 contains only elements of order 2. Suppose $|E_0| = r$, $|e_0| = s$. Let E_{r+s} denote the canonical basis of $Z^r \times (Z_2)^s$. $E_{r+s} = \{x_1, x_2, \ldots, x_r, x_{r+1}, \ldots, x_{r+s}\}$. Let $\theta : E \to E_{r+s}$ be 1-1 and symmetric. Let $\pi : Z^r \times (Z_2)^s \to G$ be the homomorphism such that $\pi \circ \theta = id$. Theorem I is true for θE since we may choose f to be transform of a suitably normalized Riesz product, $\hat{f}$

$$\hat{f} = \frac{1}{\epsilon}\left(\prod_{j=1}^{r}(1 + \epsilon(x_j + x_j^{-1}))\right)\left(\prod_{j=n+1}^{r+s}(1 + \epsilon x_j)\right) .$$

Let E be a closed subset of our compact group G. Let $\hat{\Gamma} = \hat{\Gamma}(E)$ denote the group of all continuous hermitian functions from E to T. Give $\hat{\Gamma}$ the discrete topology, and let Γ be the dual group of $\hat{\Gamma}$. We define

$$\theta : E \to \Gamma, \quad \langle \eta, \theta x \rangle = \eta(x) \quad \text{for} \quad x \in E, \quad \eta \in \hat{\Gamma}$$

$$\hat{\pi} : \hat{G} \to \hat{\Gamma}, \quad (\hat{\pi}\int)(x) = \langle x, \int \rangle \quad \text{for} \quad \int \in G, \quad x \in E$$

$$\pi : \Gamma \to G, \quad \langle \int, \pi y \rangle = \langle \hat{\pi}\int, y \rangle \quad \text{for} \quad y \in \Gamma, \quad \int \in \hat{G} .$$

Note that $\hat{\pi}$ is continuous since $\hat{G}$ is discrete.

Let E be a totally disconnected closed subset of G. Let j denote a finite collection of disjoint clopen sets with union E. Let E_j denote the finite set of equivalence classes generated by j. A function f is <u>hermitian on E_j</u>

if for every $[x_1], [x_2] \in E_j$ $f([x_1]) = \overline{f([x_2])}$ whenever there exists $x \in [x_1]$ such that $x^{-1} \in [x_2]$. $\hat{\Gamma}(E_j)$ denotes the hermitian functions from E to T . Let $\hat{\Gamma}_0(E) = \bigcup_{j \in J} \hat{\Gamma}(E_j)$ where J denotes the collection of all partitions j . Let $\hat{\Gamma}_0(E)$ have the discrete topology and let $\Gamma_0(E)$ denote its dual. We define

$$\pi_j : \Gamma_0(E) \to \Gamma(E_j), \ \pi_j y = y|_{\hat{\Gamma}(E_j)} \quad \text{for} \quad y \in \Gamma_0(E)$$

$$\theta_0 : E \to \Gamma_0(E), \ < \eta, \theta_0 x > = \eta(x) \quad \text{for} \quad x \in E, \quad \eta \in \hat{\Gamma}_0(E)$$

$$\psi : \Gamma(E) \to \Gamma_0(E), \ < \eta, \psi y > = < \eta, y > , \quad \text{for} \quad \eta \in \hat{\Gamma}_0(E) .$$

Let Δ denote the kernel of ψ .

Lemma 2 allows the reduction to a finite set situation.

Lemma 2. Suppose E is a compact, totally disconnected subset of G and F is a closed subset of $\{\theta E \cup (\theta E)^{-1} \cup \{1\}\} \cdot \Delta$ that is disjoint from $\theta E \cup (\theta E)^{-1} \cup \{1\}$. For every $\xi > 0$ there exists $k \in A_+(\Gamma(E))$ such that $|1 - k| < \xi$ on θE , $|k| < \xi$ on F and $\|k\|_A = 1$.

Proof. The map $(y_1, y) \to y_1 \cdot y$ is 1-1 and so a homeomorphism between $(\theta E \cup (\theta E)^{-1} \cup \{1\}) \times \Delta$ and $(\theta E \cup (\theta E)^{-1} \cup \{1\}) \cdot \Delta$. So there is a closed set $K \subset \Delta$ such that $1 \notin K$ and $F \subset (\theta E \cup (\theta E)^{-1} \cup \{1\}) \cdot K$. Choose $p \in A(\Gamma(E))$ such that $p(1) = 1$, $p = 0$ on K and $\|p\|_A = 1$ $p = \sum_{\eta \in \hat{\Gamma}(E)} \hat{p}(\eta)\eta$, $\hat{p} \geq 0$, $\Sigma \hat{p}(\eta) = 1$. For every

$\eta \in \hat{\Gamma}(E)$ there exists $\eta_0 \in \hat{\Gamma}_0(E)$ such that $|\eta - \eta_0| < \xi$ on E. Note this says $|\overline{\eta}_0\eta - 1| < \xi$ on E. $\eta_0(y_i\, y) = \eta_0(y_1)$ for $y = \Delta$. Let $k = \Sigma\, \hat{p}(\eta)\overline{\eta}_0\,\eta$. Then $|k(\theta x \cdot y) - p(y)| = |\Sigma \hat{p}(\eta)[\overline{\eta}_0(\theta x)\,\eta(\theta x) - 1]\,\eta(y)| < \xi$ for $x \in E$, $y \in \Delta$.

In the proofs of Lemma 3 and Theorem I we will use the fact that $f\mathcal{J}(\theta E) = 1$. This fact is a consequence of Lemma 5 of Section 1.

Lemma 3. If E is compact and totally disconnected, then Theorem I is true for θE.

Proof. Fix $K > 1$, $\xi > 0$. By Lemma 2 and the fact that $f\mathcal{J}(\theta E) = 1$, there exists $f_1 \in A_+(\Gamma(E))$ with $f_1 = 1$ on θE, $|f_1| < 2\xi$ on $F \cap ((\theta E \cup (\theta E)^{-1} \cup \{1\}) \cdot \Delta)$ and $\|f_1\|_A \le K$. Let $F_0 = F \cap \{y : |f_1(y)| \geq 2\xi\}$. Then $\psi(\theta E \cup (\theta E)^{-1} \cup \{1\}) = \theta_0 E \cup (\theta_0 E)^{-1} \cup \{1\}$ and ψF_0 are disjoint compact sets in $\Gamma_0(E)$. Hence there exists j such that $\pi_j(\theta_0 E \cup (\theta_0 E)^{-1} \cup \{1\})$ and $\pi_j \psi F_0$ are disjoint compact sets in $\Gamma(E_j)$. Since $\theta_0 E_j$ is a finite set and $f\mathcal{J}(\theta_0 E_j) = 1$, there exists by Lemma 1 $f_0 \in A_+(\Gamma(E_j))$ with $f_0 = 1$ on $\theta_0 E_j$, $|f_0| \le K\epsilon$ on $\pi_j \psi F_0$ and $\|f_0\|_A \le K/\epsilon$. Let $f = f_0 \circ \pi_j \circ \psi) \cdot f_1$. Then $f \in A_+(\Gamma(E))$, $f = 1$ on θE, and $\|f\|_{C(F)} \le \sup(K\epsilon, K \cdot 1/\epsilon \cdot 2\xi) \le K^2\epsilon$ for suitable ξ $\|f\|_A \le K^{2} \cdot 1/\epsilon$.

<u>Proof of Theorem I</u>. Let V be a neighborhood of 1 in G such that $\overline{V}F \cap \{E \cup E^{-1} \cup \{1\}\} = \emptyset$. We may select a finite number of $\int$'s from $\hat{G}$ that separate $\overline{V}F$, $E \cup E^{-1} \cup \{1\}$; that is, if x_1, x_2 are from different members of this triple we have $\int$ such that $\int(x_1) \neq \int(x_2)$. Let $\hat{G}_1$ denote the group generated by these $\int$'s. Let $\hat{G}_2$ be such that $\hat{G}_1 \times \hat{G}_2 = G$. Let $G = G_1 \times G_2$ be the corresponding decomposition. Let $p_1 : G \to G_1$, $p_2 : G \to G_2$ be the projection. Let $p_1(E) = E_1$, $p_1(\overline{V}F) = F_1$. Since G_1 is compact and metrizable E_1 is metrizable. Form $\Gamma(E_1)$ and let $\theta_1 : E_1 \to \Gamma(E_1)$, $\pi_1 : \Gamma(E_1) \to G_1$ be the corresponding maps.

Since $\theta_1 E_1$ is metrizable and $f\mathcal{J}(\theta_1 E_1) = 1$ we know by Lemma 4 of Section 1 and by Lemma 3 that there exists $f_1 \in A_+(\Gamma(E_1))$ such that:

$$f_1 = 1 \quad \text{on} \quad \theta_1 E_1$$
$$\|f_1\|_{C(\pi_1^{-1}(F_1))} \leq K^2 \epsilon$$
$$\|f_1\|_A \leq K^2/\epsilon \quad .$$

Let $f \in A_+(\Gamma(E_1) \times G_2)$ be defined by:

$$f(y, x_2) = f_1(y) \quad \text{for} \quad (y, x_2) \in \Gamma(E_1) \times G_2 \quad .$$

Then f satisfies:

$$f = 1 \quad \text{on} \quad \theta_1 E_1 \times G_2$$
$$\|f\|_{C(\pi_1^{-1}(F_1) \times G_2)} \leq K^2 \epsilon$$
$$\|f\|_A \leq K^2/\epsilon \quad .$$

Let $\theta : E \to \Gamma(E_1) \times G_2$, $\pi : \Gamma(E_1) \times G_2 \to G_1 \times G_2$ be defined by: $\theta x = (\theta, p_1 x, p_2 x)$. $\pi(y, x_2) = (\pi_1 y, x_2)$. θ is symmetric and $\pi \circ \theta = id$ on E . We now apply the Pull Back Theorem with $H = \Gamma(E_1) \times G_2$ to complete the proof.

4. Proof of Theorem II.

We now prove Theorem II under the assumption that G is compact. Form $\hat{\Gamma}(E), \Gamma(E)$. Let θ, π be the corresponding maps. Let $\text{gph}\,\theta = \{(x, \theta x) : x \in E\}$. By Lemma 5 of Section 1 $f\mathcal{Z}(\text{gph}\,\theta) = 1$. There exists $(1, \eta_-) \in \hat{G} \times \hat{\Gamma}(E)$ such that $(1, \eta_-) = -1$ on $\text{gph}\,\theta$. Let $k^o, k^e \in M_d(\hat{G} \times \hat{\Gamma}(E))$ be defined by

$$k^0 = \frac{\delta_{(1,1)} - \delta_{(1,(\eta_-)^{-1})}}{2} \quad , \quad k^e = \frac{\delta_{(1,1)} + \delta_{(1,(\eta_-)^{-1})}}{2}$$

Let N_θ be a neighborhood of $\text{gph}\,\theta$ for which $|\hat{k}^e| < \xi$. Let V be a neighborhood of 1 in G such that $E \cap V = F \cap V = \emptyset$. Let $L_0 = \{(G \backslash V) \times \Gamma(E)\} \backslash N_\theta$. Let $L = L_0 \cup \{F \times \Gamma(E)\}$. Fix $K' > 1$. By Theorem I there exists $f \in A_+(G \times \Gamma(E))$ such that:

i) $f(x, \theta x) = \varphi(x)$ for $x \in E$.

ii) $\|f\|_{C(L)} \le K'^2 \epsilon$

iii) $\|f\|_{A(G \times \Gamma(E))} \le \frac{K'^2}{\epsilon}$

We use $\bigstar$ to denote convolution on $G \times H$, $*$ to denote convolution on $\hat{G}$. Let $\sigma \in L'(\hat{G} \times \hat{\Gamma}(E)$ be such that $\hat{\sigma} = f$. We make the $K'^2 h_0^2$, ξ, $\sigma \bigstar k^0$ choice of $\delta > 0$ and of compact set $C \subset \hat{\Gamma}(E)$. We apply the Smoothing Theorem with parameters, C, δ, K', to obtain α_η, α_η^*, α_η^{--}, a_x, a_x^* for $x \in G$, $\eta \in \hat{\Gamma}(E)$. Let

1) $\tau = \int_{\hat{\Gamma}(E)} (\sigma_\eta * \alpha^{++}_{\eta-1} + (\sigma \quad \delta_{(1,(\eta_-)^{-1})})_\eta * \alpha^{--}_{\eta-1})\, d\eta$.

Then $\tau \geq 0$ and $\|\tau\|_{L^1(\hat{G})} \leq {}^2/_\epsilon\, K'^4 h_0^2$. We rewrite τ

$$\tau = \int_{\hat{\Gamma}(E)} \{(\sigma \quad k^0)_\eta * \alpha_{\eta-1} + (\sigma \quad k^e)_\eta * \alpha^*_{\eta-1}\}\, d\eta$$

$$= \tau^0 + \tau^e .$$

We use the Transfer Lemma to estimate $\hat{\tau}^0$ at various points.

For $x \in E$, $\hat{\tau}^0(x)$ is within ξ of $(\sigma \quad k^0)^\wedge(x,\theta x)$, and $(\sigma \quad k^0)^\wedge(x,\theta x) = f(x,\theta x) = \varphi(x)$.

For $x \in F$, $|\hat{\tau}^0(x)| \leq K'^2 h_0^2 \sup_{y\in\Gamma(E)} |(f\cdot\hat{k}^0)(x,y)| \leq K'^4 h_0^2 \epsilon$.

We now estimate $\hat{\tau}^e$.

For $x \in E$, $|\hat{\tau}^e(x)| \leq K'^2 h_0^2 \sup_{y\in\Gamma(E)} |(f\cdot\hat{k}^e)(x,y)| \leq K'^2 h_0^2(\max(\sup_{y\in N_\theta(x)} |(f\cdot\hat{k}^e)(x,y)|, \sup_{y\in L(x)} |(f\cdot\hat{k}^e)(x,y)|)) \leq K'^2 h_0^2 \max(\xi \frac{K'^2}{\epsilon}, K'^2\epsilon) \leq K'^4 h_0^2\epsilon$ for suitable ξ .

For $x \in F$, $|\hat{\tau}^e(x)| \leq K'^2 h_0^2 \sup_{y\in\Gamma(E)} |(f\cdot\hat{k}^e)(x,y)| \leq K'^4 h_0^2 \epsilon$.

So altogether we have

i) $\|\hat{\tau} - \varphi\|_{C(E)} \leq \xi + K'^4 h_0^2 \epsilon$.

ii) $\|\hat{\tau}\|_{C(F)} \leq 2\, K'^4 h_0^2\, \epsilon$.

iii) $\|\tau\|_{L^1(\hat{G})} \leq {}^{2}/_{\epsilon}\ K'^{4}\, h_0^2$.

i), ii), iii) and iteration give Theorem II.

5. Remarks.

a. Let E be a Helson set and suppose $1 \notin E$. Since every $\varphi \in C_0(E)$ may be written

$$\varphi(x) = \frac{\varphi(x) + \overline{\varphi(x^{-1})}}{2} + i\,\frac{\varphi(x) - \overline{\varphi(x^{-1})}}{2i}$$

we have $h(E) \le 2\, f\not{2}(E)$. And if we use Herz's estimate that $h(E \cup E^{-1}) \le 3^{3/2}(h(E))^3$ we obtain $f\not{2}(E) \le 16 \cdot 3^6 (h(E))^{12}$.

b. Let E be a closed subset of the circle group, T. Let $A^+(E) = \{\sum_0^\infty a_n e^{inx}|_E : \sum_0^\infty |a_n| < \infty\}$. E is a Carleson set if $A^+(E) = C(E)$. Let

$$\|f\|_{A^+(E)} = \inf\{\|g\|_A : g|_E = f,\ g = \Sigma a_n e^{inx}\}.$$

If E is a Carleson set, then

$$c(E) = \sup\{\|f\|_{A^+(E)} : f \in C(E),\ \|f\|_{C(E)} \le 1\}.$$

is a finite number, that is called the Carleson constant of E. I. Wik [7] showed that every Helson subset of T is a Carleson set.

Let E be a Helson subset of T such that $1 \notin E$. C. Graham asked whether $A_+^+(E) = C(E)$. The answer is yes and in fact

Theorem. Let E be a symmetric Helson subset of T such that $1 \notin E$. Let F be closed and disjoint from $E \cup \{1\}$. Let E have Carleson constant c. Let

$\varphi \in C^h(E)$, $\|\varphi\|_{C(E)} \le 1$. Then for every $K > 1$, $\epsilon > 0$ there exists $f \in A_+^+(E)$ such that:

i) $f = \varphi$ on E

ii) $|f| \le K^2\epsilon$ on F

iii) $\|f\|_A \le K^2\, 16\, c^4/\epsilon$.

<u>Proof</u>. We make the necessary modifications in the proofs of Theorems I and II. We recall from line 1) of the proof of Theorem II that the iterating function τ is given by:

$$\tau = \int_{\Gamma(E)} (\sigma_\eta * \alpha_{\eta-1}^{++} + (\sigma\ \delta_{(1,(\eta\_)^{-1})})_\eta * \alpha_{\eta^{-1}}^{--})\, d\eta\ .$$

We must adjust σ, $\alpha^{\frac{++}{--}}$ so that for all $\eta : \sigma_\eta \ge 0$, $\alpha_\eta^{\frac{++}{--}} \ge 0$ and σ_η, $\alpha_\eta^{\frac{++}{--}} \in \ell'(N)$ where $N = \{0,1,2,\ldots\}$.

Choose K', ξ so that $K > K' > 1$, $\xi > 0$. Form $\Gamma(E)$ and let $\theta_1 : E \to \Gamma(E)$, $\pi_1 : \Gamma(E) \to T$ be the corresponding maps. Let $\text{gph}\ \theta_1 = \{(x,\theta,x) : x \in E\}$. Form $\Gamma(\text{gph}\ \theta_1)$ and let $\theta_2 : \text{gph}\ \theta_1 \to \Gamma(\text{gph}\ \theta_1)$, $\pi_2 : \Gamma(\text{gph}\ \theta_1) \to T \times \Gamma(E)$ be the corresponding maps.

We select a neighborhood of 1, $W = U \times V$, in $T \times \Gamma(E)$ such that $\overline{W}\, L \cap \{\text{gph}\ \theta_1 \cup \{1\}\} \ne \emptyset$ where L is defined as in the Proof of Theorem II. Using Theorem I we find $h \in A_+(\Gamma(\text{gph}\ \theta_1))$ such that:

$$h = 1 \text{ on } \theta_2(\text{gph}\ \theta_1)$$

$$h \le K'^2 \epsilon \quad \text{on} \quad \pi_2^{-1}(\overline{W}\,L)$$
$$\|h\|_A \le K'^2 \cdot {}^1/_\epsilon \quad .$$

Choose $k_1'' \in A_+(T)$ with $\|k_1''\|_A = k_1''(1) = 1$ $k_1'' = 0$ off U^{-1}. Let k_1' be a trigonometric polynomial formed from k_1'' such that $\|k_1'\|_A = k_1'(1) = 1$ and $|k_1'| < \xi$ of U^{-1}. Choose $n \in N$ so that $e^{2\pi int} k_1'$ has only frequences in N. Let $k_1 = e^{2\pi int} k_1'$. Then $k_1 \in A_+^+(T)$, $\|k_1\|_A = k_1(1) = 1$ and $|k_1| < \xi$ of U^{-1}. Let $k_2 \in A_+(\Gamma(E))$ satisfy $\|k_2\|_A = k_2(1) = 1$ $k_2 = 0$ off V^{-1}. Let $k'(x,y,y') = k_1(x) \cdot k_2(y)$ for $x \in T$, $y \in \Gamma(E)$, $y' \in \Gamma(\text{gph}\ \theta_1)$. For all y,y' $k'(\cdot,y,y') \in A_+^+(T)$. Let $p_1 : T \times \Gamma(E) \to T$, $p_2 : T \times \Gamma(E) \to \Gamma(E)$ be the natural projections. Let

$$k(x,y,y') = k'(x \cdot p_1\, \pi_2(y')^{-1},\ y \cdot p_2\, \pi_2(y')^{-1}, y') \quad .$$

Let $\hat{k}, \hat{k}' \in \mathcal{L}'(Z \times \hat{\Gamma}(E) \times \hat{\Gamma}(\text{gph}\,\theta_1))$ denote the inverse transforms of k, k'. Then

$$\hat{k}(\int,\eta,\eta') = \begin{cases} \hat{k}'(\int,\eta,1) & \text{if} \quad \eta' = (\int p_1\, \pi_2)^{-1} \cdot (\eta\, p_2 \pi_2)^{-1} \\ 0 & \text{otherwise} \end{cases}$$

Therefore, for each y,y' $k(\ ,y,y') \in A_+^+(T)$.

So $f_2 = k \cdot h$ satisfies for each y,y' $f_2(\cdot,y,y') \in A_+^+(T)$. Let σ_2 be such that $\hat{\sigma}_2 = f_2$.

We make the K'^2, ξ, σ_2 choice of $\delta > 0$ and of compact set $C \subseteq \Gamma(\text{gph}\ \theta_1)$. We then apply the Smoothing

Theorem with parameters K', δ, C to obtain for every $\eta' \in \Gamma(\mathrm{gph}\,\theta_1)$ an $\alpha_{\eta'} \in \ell'(Z \times \hat{\Gamma}(E))$. Since $f\mathcal{J}(p_2(\mathrm{gph}\,\theta)) = 1$ we may choose $\alpha_{\eta'}$ to satisfy: $\alpha_{\eta'} \geq 0$; $\alpha_{\eta'}(\int, \eta) = 0$ unless $\int = 1$. Hence,

$$\sigma_1 = \int_{\hat{\Gamma}(\mathrm{gph}\,\theta_1)} (\sigma_2)_{\eta'} * \alpha_{(\eta')^{-1}} \, d\eta'$$

satisfies $\sigma_1 \geq 0$, $\sigma_1 \in \ell'(N \times \hat{\Gamma}(E))$. Let $\hat{\sigma}_1 = f_1 \in A(T \times \Gamma(E))$. $f_1(\cdot, y) \in A_+^+(T)$ for all $y \in \Gamma(E)$ and f_1 satisfies by the Transfer Lemma.

$$|f_1 - 1| < \xi \quad \text{on} \quad \mathrm{gph}\,\theta_1$$

$$|f_1(x,y)| \leq K'^2 \sup_{\pi_2 y' \in W\cdot(x,y)} |h(y')| + \xi K'^2$$

$$\|f_1\|_A \leq K'^2 \|h\|_A \quad .$$

Since $f\mathcal{J}(p_2(\mathrm{gph}\,\theta_1)) = 1$ we may for correct ξ obtain f such that:

$$f(\cdot, y) \in A_+^+(T) \quad \text{for} \quad y \in \Gamma(E) \quad .$$

$$f = \varphi \quad \text{on} \quad \mathrm{gph}(\theta_1)$$

$$|f| \leq K^2 \epsilon \quad \text{on} \quad L$$

$$\|f\|_A \leq K^2 \|h\|_A \quad .$$

The inverse transform of f is the required σ for the iterating function τ .

Finally, we check that we may choose for all $\eta \in \hat{\Gamma}(E)$, $\alpha_\eta^{\underline{++}} \geq 0$ and $\alpha_\eta^{\underline{++}} \in \ell'(N)$.

We recall from the proof of the Smoothing Theorem the construction of $\alpha^{\pm\pm}$. We find a function $k : \hat{\Gamma}(E) \to R^+$. with compact symmetric support K such that $\|k\|_{L^2} \le 1$ and

$$|1 - (k * k)(\eta)| < \varepsilon/3 \quad \text{for} \quad \eta \in C .$$

We choose $\eta_1, \ldots, \eta_n \in \hat{\Gamma}(E)$ and write $K = \bigcup_{j=1}^{n} K_j$ where $\eta_j \in K_j$. By Wik's theorem we may find $\beta_i \in \ell'(N)$ such that $\|\beta_i\|_{L^1} \le K\,C$ and $\hat{\beta}_i(x) = \langle \theta_1 x, \eta_i \rangle$. Take

$$b^+(x,\eta) = \begin{cases} \hat{\beta}_i^+ & \text{for} \quad \eta \in K_i \\ 0 & \text{otherwise} \end{cases}$$

$$b^-(x,\eta) = \begin{cases} \hat{\beta}_i^- & \text{for} \quad \eta \in K_i \\ 0 & \text{otherwise} \end{cases}$$

where β_i^+, β_i^- denote the positive and negative parts of β. Let

$$a^{++} = kb^+ * kb^+ + kb^- * kb^-$$
$$a^{--} = kb^+ * kb^- + kb^- * kb^+$$

where the convolutions are over $\hat{\Gamma}(E)$. Then $\alpha^{\pm\pm}$ is defined by

$$\hat{\alpha}_\eta^{++}(x) = a^{++}(x,\eta)$$
$$\hat{\alpha}_\eta^{--}(x) = a^{--}(x,\eta)$$

Hence, for all η $\alpha_\eta^{\pm\pm} \ge 0$ and $\alpha_\eta^{\pm\pm} \in \ell'(N)$.

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