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Constructive habituation as an educational approach to process-object reification in mathematics

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CONSTRUCTIVE HABITUATION AS AN EDUCATIONAL APPROACH TO PROCESS-OBJECT REIFICATION IN MATHEMATICS

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DEDICATION

This effort is dedicated to my grandmother Lula, my mother Gloria, and my daughters, Aycia and Caila. They are the inspiration and motivation for the things that I have accomplished in my life. This dissertation is a much theirs as mine. I thank them for their love, prayers, and support during this long process.

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ABSTRACT

Sfard and Thompson (1994) state that what matters most is that educators develop ways of thinking, teaching, and learning mathematics. This study introduced constructive habituation, a new strategy developed to aid both students and teachers in the thinking, teaching, and learning of mathematics. Constructive habituation attempts to unite constructivist teaching methods aimed at supporting students' conceptual understanding of content and habituationist teaching method aimed at establishing routine responses to routine tasks. This study is exploratory in nature, designed to investigate if constructive habituation is a more effective means than a traditional teaching method in helping students reach process-object reification as evidenced by higher levels of student achievement.

The study primarily addressed introductory function concepts and symmetry and transformations of functions. The subjects were university students enrolled in a precalculus I course. The results indicated that constructive habituation was not a more effective means in helping students reach process-object reification than a traditional teaching method. No significant differences were found for any of the variables examined. However, some promising practical results were revealed. The students taught using the experimental method averaged more than nine points higher than the students taught using a more traditional teaching method on an examination that evaluated their understandings of the relationship between changes made to the graph of a function and changes made to its formula. Explanations on why constructive habituation may not have reached its intended goal are given. A discussion is presented of the developmental stage

at which constructive habituation may become an effective pedagogical method. Study also includes a brief history of the major pedagogical movements over the last half century and the psychological perspectives that influenced each.

CHAPTER 1

BACKGROUND OF THE PROBLEM

Introduction

The proposed study grows from the efforts of the calculus reform movement of the 1980's and 1990's. In January 1986, Ron Douglas of SUNY at Stony Brook organized a working conference sponsored by the Sloan foundation at Tulane University in New Orleans, LA. The purpose of the conference was to discuss and rethink calculus instruction in the United States. The participants agreed that calculus was neither meeting the needs of its students nor those of its client disciplines and subsequent mathematics courses (Tucker & Leitzel, 1994). There were five main concerns that were repeatedly echoed:

1. too few students successfully completed calculus;
2. students were mindlessly implementing symbolic algorithms with no understanding and little facility at using calculus in subsequent mathematics courses;
3. faculty were frustrated at the need to work so hard to help poorly prepared, poorly motivated students learn material that was a shadow of the calculus they had learned;
4. calculus was being required as an unmotivated and unnecessary filter by some disciplines that made little use of it in their own courses; and
5. mathematics was lagging behind other disciplines in the use of technology.

The conference had workshops on content, instructional methods, and implementation. According to Tucker and Leitzel (1994), in the content and methods workshops, there evolved an agreement that greater emphasis should be placed on conceptual understanding. This could be accomplished through a variety of approaches including using calculators and computers for applications and explorations. This conference was followed in 1987 by a national colloquium on *Calculus for a New Century*, sponsored by the National Academy of Sciences and the National Academy of Engineering. A request for proposals in 1988 from the National Science Foundation (NSF) was the catalyst for action and lead to real and fundamental changes in pedagogy and curricula throughout the nation. The large number of students taking and failing calculus was of great concern to many throughout the nation. Smith (2000) reports some 700,000 students are enrolled in college-level calculus courses in the United States in any given year. The attrition (failure and withdrawal) rate for these students is very high. There are many reasons for such high attrition, one of which is that students are not adequately prepared for college-level calculus (Gordon, 2000; Schattschneider, 1996).

There has been much written in very recent years about calculus reform efforts and their effects (Ganter; 2000; Ganter, 2001; Haver, 1998; Schoenfeld, 1997; Tucker, 1990; Tucker & Leitzel, 1995). Several researchers (Fife, 1994; Gordon & Hughes-Hallet, 1994; Gordon, 2000; Knoebel, Krutz & Pengelley, 1994; Rodi & Gordon, 1994; Schattschneider, 1996) feel that calculus reform can not be maximally effective if the pedagogy and curricula of the precursor subjects (i.e. pre-calculus) are not rethought and reevaluated. Many of the same problems and concerns dealing with attrition, understanding, and student academic preparation debated in calculus could be found in pre-calculus. Schattschneider (1996) reports the attrition rate of students in precalculus at Moravian College was extremely high. She adds that the attrition rate at larger institutions was even higher. The solution at Moravain College was to drop the precalculus course altogether and integrate precalculus topics within the calculus course as needed. Other schools like New Mexico State began using novel writing assignments in precalculus as well as calculus courses that they named ‘student research projects’ for a two-week assignment involving problem solving and writing (Knoebel, Kurtz, & Pengelley, 1994). Still other colleges like Lincoln University in Pennsylvania are attempting to revise their precalculus sequence, drawing examples from real-world situations, and teaching algebra in the context of solving real-world problems.

There were several reasons as to why the failure rate is high for calculus and pre-calculus courses as suggested by Fife (1994) and Schattschneider (1996):

1. Students do not take the course seriously. Many students take the subjects because they have to. Moreover, students give these mathematics courses low priority since they are not math majors.
2. Many students do not see the relevance in these mathematics courses. "What does this have to do with …." or "when will I ever use this in real life" are frequent cries.
3. Some students are bored at being required to take a course that they already took in high school. Traditional precalculus courses merely attempt to reteach algebra and trigonometry in the same manner as they were taught in high school. Hence, the students never realize any new material and the old mathematics they see is essentially a repeat of what they did in high school.
4. Students are led to believe that algebra is merely a collection of arbitrary rules and procedures to be memorized.
5. Textbook word problems are highly unrealistic and artificial and fail to stimulate students' interest.
6. The notion of function which is critical in calculus is presented in precalculus as yet another algebraic topic. Functions should be shown in several different contexts as to insure the student get as fuller and more meaningful understanding of it.

Several of these problems are concerned directly with pedagogy. As previously mentioned students are being taught the same material over and over in the same traditional manner. This method is partially based on the metaphor of learning as habituation that is informed by behaviorist and information processing theories (Kirshner, 2002). More concisely, Kirshner's (2002) position is that traditional instruction aims for a blend of habituation and conceptual understanding. However, he asserts, that because lecture is effective as a support for conceptual understanding for only the top, most students only gain the habituationist part.

Habituation and Traditional Instruction

Kirshner (2002) contends that whether for rote recall of facts or for skillful performance of algorithms or word problems, the basic premise of habituation is that repeated practice of routine problems leads to gradual adjustment to task constraints. The topically organized traditional practice delivered primarily by lecture, that presumably aims for a blend of repetitive practice of routine problems and conceptual mastery by verbal explanation results in little more than habituation. Because of the very nature of traditional practice, students see mathematics as purely algorithmic and memorized procedures. They are intimidated by the fact that if they “miss one step” their problem will be wrong and they will have to start all over. They become easily confused and frustrated. Furthermore, they become bored and tend to day-dream. The concentration needed to develop meaningful understanding and make important connections is nonexistent. Many never really get the type of conceptual understandings needed to be successful in their current course and future mathematics courses. Still the most popular method of mathematics instruction is the lecture.

One significant reason why habituation fails students is because it promotes, sustains, and is irreversibly linked to procedural knowledge. Hiebert and Lefevre (1986) claim procedural knowledge is made up of two distinct parts. One part is composed of the formal language, or symbolic system of mathematics. The second part consists of algorithms, or rules for completing mathematical tasks. This second part is what limits many students. They more fully describe it as the step by step instructions that prescribe how to complete a task. A key figure of procedures, they suggests, is that they are performed in a somewhat linear sequence. The linear nature of procedures propounded by habituation is what clearly sets it apart from other forms of knowledge. The linearity of procedural knowledge and the habituation of this linearity can be crippling to students. This linearity leaves no way out for students. It is essentially one-way in and one-way out. Students have no means to critique their work because habituation does not provide alternate paths for students. That is to say that when students come to the end of a task and find that it is incorrect, many times they have to return to the beginning of the problem and go through the procedures again.

Habituation aids in easing this process in that it breeds familiarity. As students do more and more of the same type of problem, teachers hope that the familiarity of the process will cut down on the errors. This does not always happen. Many times it is carelessness that causes the errors. Probably more often it is the lack of complete understanding that causes the errors and impedes the student's progress in finding and fixing the errors. In short, knowledge without understanding is meaningless. The authors of the calculus reform movement concurred. Tucker and Leitzel's (1994) concern is that "students were mindlessly implementing symbolic algorithms with no understanding…" Two of the themes for calculus the reformers proposed were:

1. To focus on a conceptual understanding that used a variety of intuitive graphical and numerical approaches and gear this to the needs of the average student.
2. Emphasize the importance of changing the modes of instruction and the use of technology to engage students as active learners (Tucker and Leitzel, 1994).

Calculus Reform and Constructivism

The nation was primed for calculus reform. Because of the general dissatisfaction of current instruction, the introduction of graphing calculators, the 1989 publication of the NCTM Standards, and major support by the National Science Foundation, calculus reform began to spread. In addition to major curricular changes, several reform oriented pedagogical strategies began to be implemented. Included in this list of strategies are: discovery learning, cooperative learning, extended–time projects, laboratory experience, alternative assessment methods, oral student presentations, and technical writing (Ganter, 2001). Laboratory experience which mostly involved computers and other technology was the most popularly used strategy. This was followed closely by discovery learning which evolved from constructivism.

Constructivism suggests that students are always constructing understanding and meaning from their experiences. Furthermore, these student constructions are generally weak. Constructivism therefore commits the teacher, among other things, to teach students how to create more powerful constructions. Constructivism had stirring implications for the mathematics classroom. Confrey (1990) states that the teacher's goal for constructivist instruction should be to promote and encourage the development for each individual within his/her class a repertoire for powerful mathematical constructions for posing, constructing, exploring, solving, and justifying mathematical problems and concepts and should seek to develop in students the capacity to reflect on and evaluate the quality of their constructions. She continues that this goal suggests acceptance of three fundamental assumptions:

1. Teachers must build models of student's understanding of mathematics. To do this, teachers need to create as many and as varied ways of gathering evidence for judging the strength of a student's constructions as possible. The result will be that a teacher creates a "case study" of each student.

2. Instruction is inherently interactive; through their interactions with students regarding their knowledge of subject matter, teachers construct a tentative path

> upon which students may move to construct a mathematical idea more consonant with accepted mathematical knowledge. Teachers, however, must already be prepared for the likelihood that the students' constructions will not coincide with their own beliefs or to negotiate with the student to find a mutually acceptable alternative (which may or may not endorse the conventions of mathematical practice). If the student advocates a solution that is clearly lacking adequate argument, teachers will need to signal firmly that, their judgment, the student's position lacks legitimacy.
>
> 3. Ultimately, the student must decide on the adequacy of his/her construction. (p.112)

For the many teachers who had been teaching and been taught in a traditional manner all of their academic careers, the first two assumptions were very daunting tasks. Some teachers found the move from dispenser of knowledge to facilitator of knowledge, and the students increasing mathematical autonomy, very satisfying. Cobb, Wood, and Yackel (1990) discuss a teacher's reconceptualization of her classroom role. The teacher commented:

> My teaching role is pleasantly different. Rather than being the "person with all of the answers," the children have been given the opportunity to count on themselves and each other… Giving them the responsibilities gives them the feeling that they are needed and are important in our classroom, they do have ownership in what they are learning. (p. 137)

The reaction for others was quite different. Some teachers questioned and rebelled against the reform movement and constructivism. Ganter (2001) described the opposition as the "backlash" to calculus reform. This nationwide group of faculty believed the major components of the reform movement, which included technology, new and diverse pedagogical methods, and an emphasis on real-life application problems failed to actually teach students mathematics.

In response to the three aforementioned assumptions proposed by Confrey (1990), those against reform (and even some in favor) complained that many teachers did not have the expertise nor time to build "case studies" of each student that they taught. As to the second assumption, faculty complained that the time needed to facilitate students' learning in a discovery or constructivist atmosphere was enormous. Only a very few topics could be thoroughly discussed in this manner and many important ideas were being left out. Reform courses were labeled as "fluff", "soft", and "watered-down" mathematics (Ganter, 2001). As to the final assumption, "ultimately, students must decide on the adequacy of his/her construction,"

Erlwanger's Benny (1973) was used in both those who supported and those who opposed constructivism. Benny had a systematic method that he could (in his mind) logically explain. Converting 3/2 to 0.5 was carried out by adding the 2 and the 3, then prefixing the decimal. That this rule made it possible to convert 2/3 to 0.5 did not seem to bother Benny (Davis, Maher, and Noddings, 1990). The constructivists pointed to Benny to demonstrate what happens when students' learning do not include an emphasis on meaning and understanding. The traditionalist countered, that this is what happens when a student is left to rationalize and validate his/her own mathematical constructions.

Constructivism in the Classroom (Features and Faults)

Constructivism frees students from the procedural driven curriculums of habituation. Another feature of constructivism is that it is student oriented and student driven. Teachers generally act as facilitators or coaches, viewing the students as thinkers with emerging ideas about mathematics. Students therefore take on more responsibility for their own learning.

The constructivist teaching method relies heavily on discovery of concepts. Students may be presented a scenario or task and asked to explore the task in an effort to uncover certain concepts. As students construct their own individual understanding of the concepts, differing views and perspective emerge. Teachers must assess each of these perspectives to insure that concept is being properly developed and understood by each student. This process takes a great amount of skill and time.

Teachers must therefore be very confident in their own mathematical knowledge and be very flexible and adaptive in response to the differing student points of view. Not all teachers possess this ability. This is one of the pitfalls of constructivism. Another problem area in constructivism is that it requires significantly more time to explore, uncover, and develop a concept in the classroom. Current accountability requirements, characterized by high stakes standardized test, do not allow for the amount of time required to fully develop a concept under constructivism. Teachers are forced to get through the curriculum to have at least covered all of the required test material.

Constructive Habituation as a Solution

In this study I introduce constructive habituation to incorporate the multiple representational benefit of constructivism and the direct instructional linearity of habituation. Concepts are presented symbolically, numerically, and graphically. Those ideas are then habituated through a series of multi representational examples (multi-Reps). The term “Reps” has a multiple meaning (Representations and Repetitions). Hence, multi-Reps are multiple representations with multiple repetitions. The habituation of the multi-Reps insures that the students are familiar with several representations of a concept and helps them move from one representation to another with little effort. Hence, students are provided procedural resources to contribute to linkages that can be a source for making meaning. That is to say, students are exposed to the “big picture” from the onset. As they view the “big picture” they are challenged to make sense of and understand how the “pixels[1]” (the multiple representations) are connected and relate to each other. Understanding and meaning are therefore constructed as the relationship between the “pixels” is recognized. So how does constructive habituation differ from constructivism? The emphasis and interest of constructive habituation is effectively, efficiently, and expeditiously getting student to see the “big picture”. Constructive habituation is a more direct route to helping students seeing the “big picture”. Constructive habituation does not suggest that students do not construct their own knowledge but it is designed that the constructions are more directly guided by the teacher. Multi-Reps may be employed to the point of cognitive saturation, therefore minimizing or eliminating “weak” constructions that students can often form. Constructive habituation does not suggest that skills are unimportant. In fact, constructive habituation was initially conceived and developed as a pedagogical tool to promote reification of procedures and algorithms in order that they could be used more efficiently in higher order concepts. Most teachers recognize that good skills are important in constructing certain concepts.

In order to be effective, mathematics teachers have a need to understand how, why, and what their students are thinking. Noddings (1990) suggests that because of its very nature, pedagogical constructivism offers sophisticated diagnostic tools that uncover patterns of thinking, systematic errors, and persistent misconceptions. The sophistication of constructive habituation is in its simplicity. When teachers teach a particular subject

[1] A pixel is a basic unit of programmable color on a computer display or computer image. Screen image sharpness is sometimes expressed as dpi (dots per inch). In this usage the term dot means pixel. Thus on a computer screen an image will be more resolute the greater the number of pixels. In this spirit, my usage of “pixels” suggests the more representations that a student experiences with a target concept, the clearer, sharper, and more resolute his/her “picture” or understanding of the target concept.

more than once, they consciously or subconsciously build case studies. Over time (and in relatively short time), good teachers begin to notice certain tendencies, common errors and frequent misconceptions in their students work and thinking. In my experience as a teacher, I have found that, in general, from semester to semester different students have the same misunderstandings and different teachers have the same complaints about their students relative to these misunderstandings. These reoccurring misconceptions are what constructive habituation addresses best. Teachers, by experience, have forewarning that certain a trouble spot is on the horizon. He/she can then devise a strategy, employing multi-Reps that will intercept these problems before they have a chance to fully manifest and cause havoc in students thinking.

The Nature of Constructive Habituation

Skemp (1987) argues that concepts of higher order than what people already have cannot be effectively communicated by a definition, but only introduced by a suitable collection of examples. Good teachers, he continues, intuitively support a definition with examples. Constructive habituation takes rich and effective conceptual examples (that may or may not have been developed through constructivism) and habituates them. The student is enriched with these robust examples until the point of cognitive saturation. More than just a meeting ground, constructive habituation is a healthy and powerful marriage of habituation and constructivism.

When properly and continuously emphasized, these conceptually rich examples should eventually lead to understanding. Skemp (1987) gives a good picture of what I believe the true nature of these examples should be. He states that examples must have in common the properties which form the concept but no others. Further, he notes the examples must be alike in the ways which are to be abstracted and different enough otherwise for irrelevant properties to this particular concept to cancel out. In other words, the examples should be clear, concise, and address the particular concept that the teacher is trying to develop. The examples should be such that the student is almost forced to make its connection.

Constructive habituation does not suggest that students should not struggle and sometimes . . . fail. Failure can, in some cases, bring about resilience and a certain amount of success in its' own right. Constructivist models by their very nature tend lull the student into a zone of comfort and then inject a perturbation, a cognitive road block, throwing the student out of his or her comfort zone, leaving him or her to rethink and reconstruct intrinsic conceptual meanings and understandings. So, even within constructive habituation, the cognitive struggles and exercises are not eliminated but persevere in a somewhat constructivist spirit.

Constructive habituation could serve as a tool to cut down the length of time associated with the internal construction of ideas. Further, it may help to eliminate some of the frustrations, hopelessness, and helplessness that weaker math students feel when trying to make sense of difficult mathematical concepts. In contrast to constructivism, constructive habituation does not depend so much on the skill and creativity of the individual teacher in trying to figure out why a student is having trouble making a particular connection. Thousands of teachers are "stuck in the rut of rote". They believe that children can only learn by drill, drill, drill. Moreover they are convinced that children must be engaged in the memorization of facts, processes, and procedures so they can "do the math". Constructive habituation offers the comfort of habituation with the wonderful experience of conceptually meaningful models. But of course the teacher is not totally relieved of all his or her responsibilities. The teacher must have a clear and present appreciation of the concept to be conveyed. This idea was communicated by Skemp (1987) in his discussion of the sequencing of mathematical topics within a curriculum. He writes:

> By careful analysis of the mathematical structure to be acquired, we can sequence the presentations of new material in such a way that it can always be assimilated to a conceptual structure, and not just memorized in terms of symbolic manipulations (p. 182).

The teacher must take the time to so fully understand the mathematical concept that the examples he or she presents can do nothing but lend themselves to cognitive digestion and eventual reification. Some educators may complain that constructive habituation requires full understanding of the mathematical concept but does not address the teacher's responsibility to understanding the needs of the student. I will answer this critique by making two points. First, it requires a sufficient amount of skill, experience, and a little bit of luck for teachers to be able to successfully implement a constructivist agenda. In as much, the teacher has to figure what each student needs, why they do not understand, and shape an experience where the student finally "gets it". Not all teachers have this skill. And as a result of mandatory high stakes testing many teachers have neither the desire nor time to develop these skills. Their main objective is to get through the curriculum, and cover the required testing material. Creativity, inventiveness, and the time required to birth and mature this sort of student oriented constructive adeptness is essentially forfeited by the teacher. Constructive habituation's feature of requiring less skill on the part of the teacher is a decided advantage. Secondly, constructive habituation's main concern is the student. It is concerned with getting the students to make connections from one concept to another and developing foundations so that mathematical structures can be built and higher level mathematical thinking can occur. Constructive habituation forces the student to focus keenly on the desired concept by bombarding him or her with conceptually rich examples and models that do not easily lend themselves to misinterpretation and all lead to the development of the desired concept.

As a tried and rigorously tested educational strategy, I cannot say that constructive habituation works. The proposed study that follows will help us determine this. With time and input from other mathematics educators, constructive habituation may develop into a most useful teaching strategy. By no means do I claim that constructive habituation will be the savior of mathematics education or its children. But, I echo Sfard and Thompson (1994) in that, what matters most, is that educators develop ways of thinking, teaching, and learning mathematics.

Why Precalculus?

Calculus is a "gate-keeper" course for the science and engineering discipline. Many in these disciplines were instrumental in calling for these changes. They argued that the calculus instruction and curriculum should reflect the ever changing needs of the client disciplines. The calculus that was being taught was static and non-conceptually based. Calculus, they argued, needed to be more dynamic, including different representations, applications, and more transferability from one discipline to another. With the encouragement of groups like the Mathematics Association of America (MAA) and National Council of Teachers of Mathematics (NCTM) and the support of the

National Science Foundation (NSF) several new pedagogical models aimed at improving calculus instruction and increasing student achievement and retention have been developed.

Significance of the study is enhanced by the fact that in the same spirit that calculus is the considered the “gate-keeper” course for science and engineering, precalculus (whether by design or default) is the “gate-keeper” of calculus. It has been clear to many of the reformers that, at least in institutions with large populations of under prepared students, any successful reform must begin with precalculus (Fife, 1994). We cannot logically reform and teach calculus using innovative conceptual techniques while continuing to teach the precursor courses in a traditional manner. The transition from precalculus to calculus should be effortless and seamless for students. The procedures and concepts found in precalculus should establish a solid foundation in which to build calculus concepts.

Local Context

Student mathematics course enrollment data obtained from the Southern University Strengthening Minority Access to Research and Training (SMART) Program reveals distressing facts concerning the passage rate in several mathematics courses. The courses in question are College Algebra, Pre-Calculus (Pre-Calculus I and Pre-Calculus II), Calculus I, and Calculus II. These courses are the “gate-keepers” or foundation and prerequisite mathematics courses for many of the science and engineering disciplines taught at the university.

The data reveals that of the 2878 students enrolled in Pre-Calculus from the Fall 1996 semester through the Summer 2002 semester only 1054 students successfully completed the courses. Successful completing of a course is defined as a passing grade of "C" or better. The successful completion/course enrollment ratio reveals an approximate 36.62% passage rate for the Pre-Calculus courses. The university found this number to be unacceptable.

During the fall 2002 semester the SMART Program organized a weekend long conference in Lafayette, LA with various university officials, SMART Program administrators and staff, and Department of Mathematics faculty. The objective of the conference was to determine the root of the problem and devise strategies of improving the low passage rates in the "gate-keeping" courses. Several promising suggestions and plans came from that conference including the SMART Program's charge to the Department of Mathematics faculty to develop new and innovative teaching strategies to meet the needs of the Southern University mathematics students by improving conceptual understanding of pertinent mathematics topics and showing the relevance and importance of the mathematics being taught to the science and engineering disciplines and other sequential mathematics courses.

Nature of the Problem

Sfard and Thompson (1994) suggest that what matters most [in education] is that educators develop useful ways of thinking about aspects of teaching, learning, and experiencing mathematics. There have been few recent studies (Confrey & Smith, 1991; Schwarz & Bruckheimer, 1988; Schwarz, Dreyfus, & Bruckheimer, 1990) of the effects of multiple representations of mathematics concepts on the understanding and achievement of students. The theory of process-object reification gives us a means of gauging students' understanding of these mathematical concepts. But this theory does not give us any direction on how to get to concept structuralization. Reification does not offer any method, teaching technique, or educational philosophy that can be used to guide students through its vicious cycle.

Purpose of the Study

The purpose of this study is to determine if constructive habituation is more effective means of helping students reach process-object reification than a traditional teaching pedagogy as evidenced by higher levels of student achievement. The following research questions are posed:

1. Is there any difference in the achievement between students taught using constructive habituation (CH) and students taught using a traditional method (TRAD) as measured by the Graphs, Functions, and Models regular examination?

2. Is there any difference in the achievement between students taught using constructive habituation (CH) and students taught using a traditional method (TRAD) as measured by the Symmetry, and Transformation of Functions regular examination?

3. Is there any difference in the achievement between students taught using constructive habituation (CH) and students taught using a traditional method (TRAD) as measured by the departmental final examination?

4. Is there any difference in the conceptual understanding of functions and their applications between students taught using constructive habituation (CH) and students taught using a traditional method (TRAD) as measured by the target questions from the two regular examinations?

a. Is there a difference in their ability to define functions and explain their applications?

b. Is there a difference in their ability to graph and model functions and their applications?

c. Is there a difference in their ability to reify functions and their applications?

The following general research question is posed:

Is constructive habituation a more effective means of helping students reach process-object reification than the traditional method as evidenced by higher levels of student achievement?

Significance of the Study

There is a strong desire and need to change the way we teach calculus. The importance of a study that introduces and measures the achievement outcomes of a new teaching strategy is enhanced by the fact that for more than twenty years efforts have been on the way to change curriculums and instructional methods to positively affect students' experiences in calculus.

Currently there is no pedagogical strategy that combines and utilizes two pointedly different theories of learning like habituation and constructivism in an effort to improve student mathematical understandings. More exactly, there is no pedagogical strategy that attempts conceptual restructuring (the pedagogical objective of constructivism) through repetitive practice (the pedagogical focus of habituation).

This study is important to the mathematics education community. The study proposes to develop a teaching strategy designed at increasing the learning and conceptual understandings of undergraduate mathematics students. Further, the study will provide valuable information to the researchers and mathematics instructors concerning effective strategies of teaching and engaging students in conceptually challenging mathematical topics.

After the study the results will be made available to the educational community including educational policy makers, pre-service and in-service teacher programs. Because of the nature of the problem, the study may have ramifications in other undergraduate mathematics programs as well as secondary and elementary mathematics programs.

Definitions

Constructive Habituation - teaching strategy designed to encourage strong conceptual understandings of mathematical topics by uniting constructivist teaching strategies aimed at supporting students' conceptual understanding of content and habituationist teaching strategies aimed at establishing routine responses to routine tasks.

Multi-Reps **-** term referring to the idea of presenting multiple repetitions of multiple representations of target concepts.

Reification **-** cognitive process where mathematical concepts and processes come to be viewed structurally or as objects, allowing the learner to connect multiple representations of concepts as well as use these mathematical objects in higher order mathematical concepts and processes.

Transformation of Functions **–** defined as any translation, reflection, stretch, or shrink of a defined function.

CH students – refers to students in the experimental group who are being taught using constructive habituation.

TRAD students – refers to students in the control group who are being taught using a traditional teaching method.

CHAPTER 2

REVIEW OF THE LITERATURE

This chapter describes at some length, the major pedagogical movements in mathematics
education during the past half century and the psychological perspectives that informed each of these movements. This chapter then discusses Sfard's process-object theory of reification and its inherent difficulties. Hiebert and Lefevre's procedural-conceptual knowledge theory is used to give us direction and set the stage for introduction of the process-object theory. It begins by discussing the importance and usefulness of procedural knowledge and its connection with conceptual knowledge. Next, the fundamental differences between reification and other procedural-conceptual knowledge theories are addressed. Finally, a discussion of the inherent difficulties with reification including: the process-object duality theory and the discontinuous nature of reification is presented.

Introduction

One of the major hurdles faced by mathematics educators for many years is the varying levels of knowledge, understanding, and ability of their students. Mathematics educators have tried to compensate for this by developing a host of different curriculums and pedagogical methods in hope of reaching the vast majority of students. Early on, the job of classroom education was left to the teachers and school officials. But outside interests and developments including politics, military conflicts, economic issues and issues of race and gender equity have made a substantial difference in "who" has become interested in mathematics education. Politicians, philosophers, and psychologists have become more integral and permanent fixtures in the policy, curricular, and pedagogical decision making of the mathematics classroom. Of this group over the past half century, it has been psychologist who may have had the most substantial influence on the way we teach and think about how learning takes place within the mathematics classroom and within our mathematics students.

Some psychologists have become deeply involved in the diverse and ever changing mathematics classroom. Psychologists have proposed fresh perspectives on the way humans (students in particular) behave, understand, and learn. Their research and collaborations have caused math educators to reevaluate, reconstruct, and created new and innovative teaching techniques and methods. This section attempts to discuss some of the foremost pedagogical developments that have occurred over the last half-century and the psychological perspectives that informed them. Several pedagogical developments were shaped within the same general psychological influence but with varying interpretations, and points of view. Further, a few of these developments were being discussed, studied, and put into practice near the same general time period. Although it is clear the psychological perspectives endure as influences in education over a long period of time, mathematics educators generally point to distinct pedagogical movements with distinct beginnings and endings. Hence, psychological influences discussed here are indexed to a timeline of pedagogical innovations.

The “New Math””

The first major development of mathematics education in this half century that this paper will discuss is the “modern math” or “new math” movement. This movement was initiated by mathematicians, scientists, and politicians. This group recognized a trending shortage of mathematicians and scientists. The "new math" movement was fueled by the fact the Soviet Union was to put a man in outer space. Some people felt that the United States could lose its place as world super power. America became highly driven to produce more mathematically sophisticated citizens.

According to Howson (1983), this movement was founded on the belief that the existing syllabi were not appropriate. He maintains that this movement was essentially a content oriented movement that showed very little concern for pedagogical matters (p. 25). Proponents (like Bruner and Dienes) of this movement might disagree with Howson's claim that this movement was not interested in pedagogy. Some argued that the new math's model of instruction was the discovery method supported by a curricular design called the "spiral curriculum". Within the spiral curriculum, concepts could first be presented in an exploratory or discovery manner and then revisted in greater depth through opportunities to use these concepts in their appropriate mathematical contexts. Pedagogic methods that emphasized discovery techniques were recognized as extremely valuable, but time consuming. Proponents further believed that new content would result in attainment of greater, newer, and more relevant objectives. Osborne and Kasten (1992) discussed the intentions of the "new mathematics". They proposed that the "new math" was aimed at:

> Bringing the mathematics taught in schools into line with that of colleges and universities and used in industry. In some cases the language used had to be change to reflect what was used in industry;
>
> Exploiting the fact children could learn faster than they had previously been expected to do;

> Attending to the needs of students of higher ability by providing a richer, more demanding experience in mathematics, thereby extending the pool of talented people who understood and used mathematics; and
>
> Giving students an experience of honest mathematics, wherein every new idea can be justified or built on ideas previously established. This rational approach increased the emphases on arguments and proofs.

Hill (1975) reports the content innovations K-12, the emphasis on student understanding of mathematical methods, the judicious use of powerful unifying concepts and structures, and increased precision of mathematical expression have made substantial improvement in the school mathematics program. Two men that played important parts improving the school mathematics program under the "new math" agenda were Jerome Bruner and Zoltan P. Dienes. They proposed an idea that is referred to as structuralism.

New Math and the Structuralists

From mathematicians, cognitive psychologists like Jerome Bruner, and mathematics educators like Z.P. Dienes came the suggestion that children could have more meaningful experiences in mathematics if they were taught the structures of mathematics. Howson (1983) claims that structuralists' movement origins may have been discerned through a quotation from Bagley from the 1923 of the National Committee on Mathematical Requirements, The Reorganization of Mathematics in Secondary Education. He writes, "Bagley expressed the view that transfer could take place if the teaching made the students conscious of procedures and of the value of general procedures: The theory of transfer through 'concepts of method' and 'ideals of procedure' furnishes a definite suggestion for teaching."

The proponents of this initiative believed that instruction that involved teaching the structures of math, using concrete manipulable materials, could bring about deeper understandings of mathematical ideas than traditional approaches. This approach could help students understand the fundamental structures of mathematics (beginning in kindergarten) by making clear how concepts were related and why certain mathematical operations worked as they did. Students should then have more meaningful understandings of mathematics, retain concepts, and easily transfer these concepts to other situations. This could be accomplished in part by getting students to understand that the larger body of mathematics is an interrelationship of many distinct areas like arithmetic, algebra, geometry, and calculus. Each subject is different, yet they share some of the same basic characteristics and essentially build on each other.

Hence, to understand the structures of mathematics requires the student mentally connect these interrelationships, and reformulate and reorganize them to discover new patterns and make new generalizations. Resnick and Ford (1981) offered that Bruner believed that instruction should and could be responsive to the learner's cognitive processes. They continue that he developed a cognitive theory of conceptual development that implies that instruction be sequenced. They describe this sequence as the three modes of representation: enactive, iconic, and symbolic.

Enactive is the mode associated with representing past events through appropriate motor response. An example would be when children tap their fingers on a table or desk as they count. The enactive mode also emphasizes the use of concrete materials and other objects for learner manipulation in a hands-on approach (Ediger, 1999). Iconic is the mode where children make mental images or "pictures" of a mathematical operation in order to remember and recall it when necessary. The symbolic mode is another way capturing mental images. A symbol as defined by the authors is a word or mark that stands for something but in no way resembles that thing. It is completely abstract.

Bruner believed that if intellect developed in the enactive, iconic, symbolic sequence, then mathematics should be presented in such a manner that corresponded directly to aforementioned modes. Contrary to Piagetian stage theory that proposed that the curriculum be age appropriate, Bruner believed that complicated topics or ideas could be simplified enough that any student could understand it relative to their intellectual capabilities. Bruner further advocated the use of inductive methods of teaching in which students discover structural or major concepts of a topic.

Dienes, a mathematics educator, drew heavily on Piagetian theory and worked closely with Bruner. Yet, Dienes focused on the use of manipulatives in the classroom to help aid in meaningful learning. Dienes proposed instruction that would take into account mathematical structures as well as the cognitive abilities of the learner. Dienes designed his own set of mathematics manipulatives called multi-base arithmetic blocks or Dienes blocks. The Dienes blocks are sets of wooden blocks that are used to represent the different base systems. Dienes blocks were not only used to develop multiple ideas in arithmetic, but were also used to demonstrate the factoring of quadratic equations. Dienes (1973) proposed a four stage learning cycle that he suggested be considered when teaching mathematics.

The first stage of learning is play. This, he says, is the spontaneous adaptation to the environment for which we find ourselves. Children need a time of exploration or discovery of their environment (in our case manipulatives) before they can form an opinion on them. Following this time of exploration, the child's experiences can be made more structured and concrete by adding properties or rules to the use of the manipulatives. The third stage is a period when the child begins to deal with the abstract nature of concepts being represented by the manipulatives. The child begins to compare the common things in his concrete activity. The child begins to learn the "mathematics" and ceases to play with the concrete.

Once the child is through the first three stages it is now time to have him talk about his experience and findings. At this final stage the child is encouraged to abstract his learning further away from the concrete materials by drawing simple pictures, graphs, or maps and eventually attaching mathematical symbols (Resnick and Ford, 1981). The parallels of Bruner's modes and Diene's learning cycle become fairly apparent. The "new math" was greeted with enthusiasm and the hope that it could accomplish its rich goals. But the reforms sought in this movement failed to manifest.

Demise of New Math

As more students aspired to go to college individual differences in learning became evident. Schools continued to be integrated and the enormous gap in mathematics preparation between minority and majority students became more and more apparent. Opponents of this movement forged several complaints. Gardner (2001) in discussing the new math's Bruner curriculum reports, "the curriculum worked best with well-prepared teachers working in schools with advantaged students." He further reports, the curriculum was judged to be "elitist" and "relativistic".

As suggested by Gardner, one major complaint was that the "new math" curriculum was deficient in serving the needs of low achieving and many average students. In fact, it was believed that the "new math" curriculum failed to meet even their basic mathematical needs. The teaching and learning of mathematics became too abstract. One explanation for this belief is put forward by Hill. Hill (1975) suggests that critics held that the reform curriculum produced programs that were excessively formal, deductively structured, and theoretical. This complaint was countered by results from the Madison Project.

During the 1957-58 school-year, Robert Davis began exploratory work with disadvantaged students with low-IQ scores at the Madison Junior High School in Syracuse, New York. Efforts to teach arithmetic to these students had previously failed. Davis decided to teach these students algebra and analytical geometry instead. Davis found success in teaching these particular subjects and became convinced that "advanced" mathematical ideas could be taught to any child.

A second complaint was that mathematics teachers who did not possess the rigor in their own mathematics background failed to develop the positive experiences for students needed to meet the "new math's" agenda. In referencing the Madison Project, Hayden (1981) reports that many teachers had great difficulty encouraging students' creativity, exploration, and discovery. On the surface, this criticism seemed to have the most validity.

The problem with this criticism is that we are to be reminded that the "new math" told teachers what to do but did not specify how to do it. That is to say that the "new math" was primarily concerned with and chiefly brought about changes in the mathematics curriculum. It brought little direction and focus to the "pedagogy table". Hence there existed a big difference in what the psychologists and pedagogy experts were recommending and what was actually going on in the classroom.

Hayden (1981) suggests that the "new math" movement came to an end for much simpler reasons than the aforementioned. He contends that the movement ended when the forces that brought it into being were dissipated. He suggests that these forces included the mathematics revolution of the nineteenth century, progressivism in education, World War II and Americans awareness of the important roles of science and technology in modern society and in particular modern warfare, and a substantial decrease in government funding. Moreover, Hayden suggests that by the late 1960's and 1970's the United States found itself in a bitter civil protest against an unpopular war in Viet Nam. Mathematics, and the science and technology it birth, was regarded by some as detrimental to the peace and humanity. The interest and the fervor for mathematics that greeted "new math" began to decline.

"Back to Basics": Behaviorism

Thorndike and S-R

Under a barrage of criticism, social pressures, and decreasing students' standardized test scores, many teachers and parents became disenchanted with the "New Math" movement. Mathematics teachers reverted to habituation and began to emphasize facts and skills. A "back to basics" cry filled the mathematics education environment.

Educational psychology, birth from the minds of behavioral psychologists, led the nation into the behaviorist or habituationist period. The foundations for behaviorist instruction had already been laid in the early 1900's by E.L. Thorndike, the "founding father" of educational psychology. Resnick and Ford (1981) suggest that Thorndike is best known in psychology for his statement of the *law of effect* (Thorndike, 1924) more presently know as *the principles of reinforcement.* Thorndike's law of effect essentially stated that when a modifiable connection between a situation and a response is made and is accompanied by a successful or pleasing result, then the connection or bond between the situation and the response is strengthened. Likewise if the situation and response is accompanied by an unsuccessful or displeasing result, the connection is weakened.

Thorndike believed that this learning principle could and should apply to humans even though it was developed through his work with animals. Gardener (1987) stated that the behaviorists believed that all psychological activity could be explained without resorting to topics that involved mental constructs like symbols, ideas, schemas, or other forms of mental representation. Resnick and Ford (1981) further suggest, that Thorndike, along with other psychologists of that era called “connectionists” or “associationist” believed that all human behavior could be explored through two simple ideas; stimuli and response.

Stimuli in this case is defined as events external to or not controlled by the person. Response is defined as what the person did in reply to the external event. Resnick and Ford (1981) continue that when a certain response was given to a certain stimulus and was followed by praise or reward then the stimulus-response bond was strengthened. The more frequent the reward, the stronger the bond. Hence, the *law of effect* suggested that practice followed by praise or reward was an important part of human learning.

Thorndike (1922) reasoned that the chore of curriculum and instruction design and implementation was to form the necessary bonds and habits that would allow children to easily perform arithmetic computations and solve problems. His first task was to designate the bonds to be formed. Once the appropriate bonds that made up the particular mathematics subject were chosen then they were to be organized so that the more simple ones were strengthened first and then used as a foundation for the more difficult ones that followed. Finally, the appropriate levels of drill and practice were implemented. More important bonds were strengthened while less important bonds were not practiced as much.

Thorndike and his contemporaries became deeply vested in changing education. They were not only involved in developing theories of learning, but they were also intensely involved in the practical applications of those theories. In addition to writing the Psychology of Arithmetic in 1922, which laid the foundations of human learning and the *law of effect*, he was also involved in developing several arithmetic textbooks that supported his theories on mathematics instruction and learning.

Gagne and the Analysis of Subject-Matter

The psychologists who led this renewed effort after the collapse of the “new math” era were behaviorists who had recently prepared successful training programs for the U.S. military during World War II. These psychologists believed that the programs they developed and implemented, often in conjunction with technology, could be used to positively effect instruction and learning. The developments of mathematics education, during this postwar era, centered on the pragmatic uses of mathematics. It was under the influence of psychologist like Robert Gagne that behavorism began to be concerned with the analysis of subject matter.

Gagne proposed in his *cumulative learning theory* that subject-matter be analyzed thoroughly and be broken down into individual objectives or tasks. Any given objective or task could then be broken down into smaller, simpler components. These components can be organized into a hierarchy (Ediger, 1999; Gagne, 1983; Resnick and Ford, 1981). Because cumulative learning has a built in capacity for transfer (Fields, 1996), one could expect transfer of learning from lower levels of the hierarchy to higher ones (Resnick and Ford, 1981). Fields (1981), further elaborates that Gagne maintains that using this model, which begins with stimulus-response associations, and proceeds through concept and simple rules and ends with complex rules, will in enhance positive transfer. Positive transfer occurs when learning one task (usually a simple task) assists in learning a complex one or enhances the performance of another task. Transfer could be observed in two different dimensions: horizontal or lateral and vertical. Horizontal transfer involves the ability to use school-acquired abilities in ones practical or everyday life. Vertical transfer involves students being able to learn more advanced and complex skills within the subject-matter based on their mastery of subordinate skills.

Opponents of cumulative learning questioned whether it was always necessary to proceed through small incremental steps. Many children are able to gain “whole” or more complex concepts without teaching every step in between. It would appear that Gagne, himself, would later come to believe that some of his ideas might eventually become obsolete and be replaced. Ediger (1999) writes that when Gagne was questioned if he thought instructional design would eventually transition entirely to from behaviorism to cognitive psychology. Gagne replied he believed that the cognitive approach will come to dominate. The reason he suggested was that designers who work with cognitive learning theory in mind really incorporate the important parts of behavioral theory. Behaviorism endured and still endures today. But a cognitive revolution was already on the way. With the publication of the National Council of Teachers of Mathematics, “Agenda for Action”, mathematics education began to experience a change in direction and witnessed the rise of the formative era from which evolved constructivism.

An Agenda for Action

In 1980 the National Council of Teachers of Mathematics (NCTM) flatly rejected the "back to basics" notion that math is primarily skills and computation. The NCTM felt a special obligation to present a responsible and knowledgeable viewpoint of the directions that mathematics in the 1980's should go. In it's 1980 publication, An Agenda for Action: Recommendations for School Mathematics for the 1980's it suggested eight recommendations for the future of mathematics education in the United States. One of these recommendations was that problem solving should be the focus of school mathematics.

The NCTM believed that performance in problem solving would be the measuring stick of the effectiveness of our personal and national possession of mathematical competence. In the "Agenda for Action" the NCTM suggested that problem solving included a host of routine and commonplace as well as non-routine functions that were essential to the day-to-day living of every citizen. Further, problem solving involved "applying mathematics to the real world, serving the theory and practice of current and emerging sciences, an dissolving issues that extend the frontiers of the mathematical sciences themselves" (NCTM 1980). The need for problem solving in a ever-changing society did not suggest the elimination of skills, but sought to incorporate them in with other knowledge to insure their usefulness in the everyday life of the student. The "Agenda for Action" states:

> "The current organization of the curriculum emphasizes component computational skills apart from their application. These skills are necessary tools but should not determine the scope and sequence of the curriculum. The need of the student to deal with the personal, professional, and daily experiences of life requires a curriculum that emphasizes the selection and use of these skills in unexpected, unplanned settings" (NCTM, 1980).

The NCTM suggested that computation and skills should not be overemphasized in a changing world. It suggested that what was appropriate for the seventies was not necessarily appropriate for the eighties and beyond. The "Agenda for Action" states:

> "It is dangerous to assume that skills from on era will suffice for another. Skills are tools. Their importance rests in the needs of the times. Skills once considered essential become obsolete, and this is likely to increase in pace and scope as advances in technology revolutionize our individual, social, and economic lives" (NCTM, 1980).

The "Agenda for Action" suggested that higher-order mental processes of logical reasoning, information processing, and decision making should be considered when designing mathematics curriculums. "Mathematics curricula and teachers should set as objectives the development of logical processes, concepts and language. . ." (NCTM, 1980). Moreover the NCTM suggested that teachers should put some value on a students' thoughtful and productive approach to a problem and not solely on a single correct answer. The influence of cognitive science and constructivism came into play with the advent higher-order mental processes and the conceptual understanding of mathematical content.

Constructivism – A Pedagogy, A Psychology, An Epistemology

The formative (constructivist) movement differed from the behaviorist movement in many ways. One of the main differences was that the constructivists did not believe in rote-learning. E. von Glasersfeld (1994) had very little patience for instruction that did not take into consideration the cognitive process of the learner and was only concerned with performance. He writes, ". . . if we want to teach arithmetic, we have to pay a great deal of attention to the mental operations of our students. Teaching has to be concerned with understanding rather than performance and the rote-learning of say, the multiplication table, or training the mechanical performance of algorithms-because training is suitable only for animals whom one does not credit with a thinking mind." Constructivists saw the pre-formed lessons of structuralism and the stimulus-response notions of behaviorism as an inadequate means of promoting meaningful learning.

These ideas, they felt, did not meet the cognitive, affective, motivational, nor social needs of the child. Moreover, they also felt that these theories did not take into account the experiences of the child and their cognitive development and maturation. Learning and meaning, they felt, is developed intrinsically. Understanding is not a product of external stimuli or sequenced or structured subject-matter. Understanding is formed and organized in the minds of the child and then evidenced by their ability to speak, write, generalize, and transfer acquired concepts from one domain to another. To that end, the works of Piaget and his followers provide the foundation for constructivist epistemology and pedagogy.

Piaget and Learning Readiness

One of the main thrusts of Piaget's theory was that of development and readiness. Developmental approaches to "learning readiness", according to Orton (1987), are likely to state that a child is only ready when the quality of thinking and processing skills available matches the demands of the subject matter. Piaget's theory of children's intellectual development was based on years of experiments using the clinical or individual interviewing methods. Many of these methods, although they may be modified, are currently used (see Ginsburg, 1997). Piaget suggested that there were several different stages of intellectual development. They are:

a. the sensori-motor stage
b. the pre-operational stage

c. the concrete operational stage
d. the formal operational stage

I offer a characterization of these stages based on the descriptions given by Ediger (1999) and Bell, Costello, and Kuchemann (1983).

The sensori-motor stage occurs from birth to about age two. Children at this stage need plenty of objects or toys to experience and manipulate. Listening to sounds made by those objects is also salient to sensori-motor learning. The pre-operational stage occurs from ages two to seven years. The child at this stage goes through an initial period where he or she fails to conserve. Eventually, according to Orton (1987), after a period of time, a more mature view is finally expressed and conservation is consistently admitted. The concrete operations stage, (ages seven through eleven), is a critical stage. Here, the child has matured enough to recognize commutativity with two to three addends. He may also be able to recognize and identify the identity elements for addition and multiplication. At about twelve, children enter the stage of formal operations. At this stage children may be able to begin think inductively, deductively, and abstractly about mathematics.

According to Piagetian theory, all children pass through these stages and in that order. Therefore, a child known to be operating in a particular stage will not normally approach mathematical tasks that require a higher level of maturation in a uniform and systematic manner. For example, when asked to determine the number of ways one could arrange a blue chip, a red chip, black chip, and a green chip the child who is still in the concrete stage begins by randomly moving the chips in an attempt to determine all the arrangements. On the other hand, the child in who is in the formal stage would develop some approach or scheme by which all chip combinations are arranged in turn or some sequential order. Further, there is no magical jump or leap from one stage to the next but in fact a slow transition. It may seem that a child is spending more time in transition than in the actually stage itself.

According to Ediger (1999), there are definite factors that effect children's intellectual development. These are biological maturation (previously mentioned), experiences in the natural environment, social activities and interactions, and homeostasis, a balance between self and experiences in the physical environment. All of these factors combine to aid in what Piaget calls "constructive learning" through which meaningful understanding has taken place.

Meaningful understanding requires that the student build for himself the understandings of mathematical concepts and ideas. Constructive learning implies that the student develop his or her own approaches and mental routes to a solving a particular problem. Further, constructive learning requires exploration and discovery, and a learning environment that is conducive to providing feedback and direction to the students. To fairly discuss Piaget's constructivism, I must mention that his model of constructivism stems heavily from his epistemology.

Piaget's model of constructivism promoted the idea that all reality and knowledge is internally constructed by the individual. This model of constructivism suggests that meaning is not realized and learning does not take place until the designated information is absorbed and reshaped into a form that is useful to and usable by the individual. Furthermore, the exact same information can result in totally different meanings from individual to individual. The presentation of knowledge, events, information, etc. is futile to the individual until they are mentally adapted and organized by the individual.

Boudourides (1998) explains that adaptation is a process of assimilation and accommodation where, on the one hand, external events are assimilated into thoughts and on the other new and unusual mental structures are accommodated into the new mental environment. The process of organization refers to the structuring of the adapted mental material. Boudourides (1998) comments that "knowledge for Piaget is never (and can never be) a representation of the real world. Instead it is a collection of conceptual structures that turn out to be adopted . . ."

This version of constructivism is sometimes called "trivial" constructivism. This "trivial" constructivism is further defined by Steffe and Kieren (1994) as a form of constructivism that asserts that children gradually build up their cognitive structures while maintaining that the structures being built up are reflections of an ontological reality. Ernest von Glasersfeld introduced a more "radical" version of constructivism that abandons the traditional philosophy of constructivism that infers that knowledge lives outside the individual until it is experienced by the individual and that this knowledge only reflects or is "mirror image" of this outside world.

von Glasersfeld and Radical Constructivism

Radical constructivism states that individuals actively construct their knowledge. Steffe and Kieren (1994) suggest that concepts of reality are not mirror images but individuals construct their own reality through actions and reflections on actions. I likened the difference between the trivial version of constructivism and the radical version to that of a king and an explorer. The explorer has received his charter or commission from the king, just as radical constructivism has its commission from its trivial predecessor. The explorer (radical) sets out to actively and aggressively seek and discover the “treasures” of the world. The explorer is constantly adapting and changing with his circumstances and environment to survive and remain viable while on this exploration. The king (trivial) on the other hand, sits on his throne and waits for the treasures to be brought to him. He is the faced with the chore of making sense of the treasures, deciding what is valuable and precious and what is not. He must determine the importance or use of say some sacred tablet or some golden chalice. The explorer is not faced with same “objective-subjective” experience. His experience is “subjective-subjective”. It is based on his lived experiences, actions, interactions, and reflections on his actions and interactions with his explored world. He has first-hand experienced knowledge of the importance and use of the “treasures”. Steffe and Kieren (1994) further discriminate the two constructivist ideas in expounding on the radical version:

> “In this construction, although there may be well-defined tasks or spaces for experience, there are no pregiven prescribed ends toward which this construction strives. There is no optimal selection of the individual’s actions or ideas by the environment, nor is some perfect internal representation or match against an external environment the test of the constructed ‘reality’” (p. 721).

Hence in von Glasersfeld’s radical constructivism one is not studying reality but the real-time construction of realty. For mathematics educators this means that we are studying the real-time goings-on of our students learning processes and the creation of the mathematical reality.

Teacher’s Role

In radical constructivism the teacher is not subject to waiting until the student has reached a particular stage before she can introduce new concepts. The teacher has the responsibility to consistently influence important aspects of the student’s intellectual development. The job of the teacher then is twofold. One is to always provide problems or tasks that are slightly above the student's current capability but not unreasonable, unreachable, or completely removed from his current intellectual stage of development. This draws on Piaget's idea of cognitive conflict, "perturbations" (Steffe and Tzur, 1994) or as I have termed cognitive "roadblocks". It is these "roadblocks" that interrupt the student's state of homeostasis and forces the student to reflect, struggle, and to press beyond his current thinking.

Secondly, the teacher is a facilitator of learning. Pure constructivist teaching is consistently being criticized about the amount of time required to develop a concept. While the student struggles with his ideas, the teacher insures that the student's work remains focused and does not veer too far from the learning goal. This helps in the time management aspect of constructivism. The teacher may offer a different perspective or ask the student to consider another route in developing understanding of a particular concept. In other words, the teacher must act as a guide or "shepherd" so to speak, always keeping the "sheep" and the curriculum moving forward, looking ahead to greener pastures. This does not remove the student's responsibility for developing his own understandings but allows the teacher to play a more significant role and provide expert guidance to the learning experience.

The Standards

In 1989 the NCTM published the "Curriculum and Evaluation Standards for School Mathematics" more commonly referred to as "The Standards". The Standards broadened the focus of the "Agenda for Action". In that the "Agenda for Action" to move mathematics education from emphasis on skills, computation, and mindless rote learning to problem solving and higher-order thinking. The Standards set out to expand mathematics education to include problem solving, communication, reasoning, connections between and within different types of knowledge and topics. To its' credit the Standards addressed not only curricular, and cognition concerns but also touched on pedagogical issues. The Standards specifically noted that students should be provided with opportunities to work both individually and in small and large group arrangements. This helped to ignite strong interest in research on collaborative learning in the classroom.

Collaborative learning is increasingly considered one of the most effective methods of engaging students in mathematics topics and concepts. Collaborative learning is believed to encourage social responsibility in learning, and increase students' performance and enjoyment of mathematics. Collaborative learning within the classroom positions itself as one of the most effective pedagogies that is usable and can effect change outside the classroom. Bosworth and Hamilton (1994) discuss the importance of collaborative learning. They write:

> "Collaborative learning may well be the most significant pedagogical shift of the century for teaching and learning in higher education. It has the potential to transform learners' and instructors' views of learning, knowing, and understanding as it acquaints students with the skills needed to cooperate, negotiate, and formulate productive responses to the changing demands of this increasingly complex world " (p. 2).

Collaborative learning is believed to be so useful and effective because it is based on the idea that all learning is a social act. Most people talk to each other. As we talk, we exchange ideas and beliefs. Mathematics concepts are more easily understood within the collaborative setting. The interaction and immediate feed back from others allows students to more efficiently and effectively construct, reflect, and reconstruct notions on mathematical ideas. In a collaborative setting, it is quite easy to pick-up on for example, who has completed and understood their mathematics homework. Normally, the student who seems confident in his discussion of the topic and is able to fully elaborate on the topic is usually the one who has completed his assignment and fully understands. Students' speech can sometimes indicate whether they fully understand a particular concept. Further, a student's speech can give an attentive teacher information concerning that student's cognitive processes and mathematical maturity and sophistication. Gerlach (1994) suggests that collaborative learning promotes cognitive development and is such a strong pedagogical method because it is based on several fundamental principals. He explains:

> "First, learning is an active, constructive process in which students integrate new material with prior knowledge to create new ideas and new meaning. Second, learning depends on rich contexts that ask students to collaborate with peers to identify and solve problems by engaging in higher order reasoning and problem-solving skills. Third, learners are diverse and have different backgrounds and experiences. The various perspectives that emerge during collaborative work clarify and illuminate learning for all involved -- the student, the members of the collaborative group, and the teacher. Fourth, learning is a social act in which students talk to learn. This social interaction often improves the participants' understanding of the topic under consideration. Fifth, learning has affective and subjective dimensions. Collaborative activities are both socially and emotionally demanding and most often require students not only to articulate their own points of view but also to listen to the views of others"(pp. 8-9).

Collaborative learning has distinct advantages over many individualized pedagogical methods. The immediacy of ones' peers accepting or rejecting a conceptual notion, prompting the student to reconsider and re-evaluate his understanding is invaluable. Further the social interactions can tend to strengthen a student's confidence in writing and speaking the language of mathematics. In some instances students feel more comfortable with talking with their peers than the teacher. Asking ones teacher a seemingly "dumb" question in front of the entire class, as opposed to a small group of ones peers, can be a very intimidating ordeal. The idea of collaboration has its foundations in enculturationist theory.

Cognition and Enculturation

This theory, which we will briefly discuss, matches learning and enculturation. This idea bonds learning and the culture in which the learning takes place. The major pedagogical movement that came from this idea was cognitive apprenticeship. This pedagogic theory was inspired by cognitive psychologists and cognitive theorists. This group proposed that all learning was situated. This psychological perspective from which this idea was born was termed situated cognition. First let us discuss the idea of situated cognition.

One the main requisites of situated cognition is that it separate itself from the idea that learning is uniquely and chiefly constructed by the individual as proposed the constructivists and other cognitive psychologists. Kirshner and Whitson (1997) write, “the critical strategic requirement for situated cognition theory is to shift the focus from the individual as the unit of analysis toward the socioculture setting in which activities are embedded. A central claim of situated cognition is that action [learning] is situationally grounded (Anderson, Reder, & Simon (1996). They explain that this means that the prospects for action cannot be fully described independently of the specific situation. Situated cognition advocates, what I refer to as the “criminal defense attorney theory”: all learning (good or bad) and actions are inseparably interwoven with and are uniquely a result of ones environment, circumstance, and situation where the learning took place.

Apprenticeship

From this psychological perspective comes the pedagogical strategy of apprentice learning in education, in particular cognitive apprenticeship. Lave (1990) explains that cognitive apprenticeship implies that it might be possible to learn math by doing what mathematicians do. By this he means that mathematics students could be able to understand mathematics by working directly with their math instructors to solve problems and engaging in the same solid mathematical practices and problem solving techniques as expert mathematicians do. In this the situated character, of say, problem solving is emphasized while the focus is on doing.

Situated cognitionists believe that apprenticeship can be successful because it involves the natural tendencies of students. Brown, Collins, and Dugid (1989) explain, “from a very early age and throughout their lives, people consciously or unconsciously, adopt the behavior and belief systems of new social groups.” They continue, “. . . given the opportunity to observe and practice them [behaviors in the appropriate situation] people adopt them with great success.” Proponents of cognitive apprenticeship argue that this idea goes beyond giving students problem solving strategies and techniques. Cognitive apprenticeship reveals the seemingly mysterious and secret nature mathematical practices.

Concluding Remarks on Pedagogical Movements

There are a whole host of competing psychological theories and pedagogical strategies claiming to be the one perfect answer to our students’ and teachers’ mathematical woes. This section addressed several of the more prominent ideas of our past half-century. Each carried within it some very good approaches to understanding students learning processes and knowledge acquisition and teaching strategies. I am reasonably sure that there does not exist and will never exist a "one size fits all" pedagogical strategy. Students have to be taught to problem solve by reason and reflection early on in the school years. Teachers have to be taught to listen to and closely observe their students and to catalog what they consider good and bad mathematical behaviors. This will allow them to develop better pedagogies and design stronger and more promising mathematics curriculums.

Sfard's Process-Object Reification

When a student is presented with an equation such as, **2x - 1 = 5x + 8**, what is it that he sees? In many cases, if not most, the student will see this as an equation to be solved for the unknown letter **x**. "In our attempts to explain student learning of algebra, we often find ourselves dealing with student interpretations that are based in arithmetic notions" (Goodson-Espy, 1998). In fact there are several interpretations of this equations that are seldom initially realized by students: **a)** this equality could be interpreted as two separate expressions balanced by the known value of **x** and the equal sign, **b**) this equality could be interpreted as two linear functions (if we first defined and understood structure of linear equations) equal to each other at some value for **x**, **c**) having an understanding of the meaning of linear . . . this equality could be interpreted as two lines on a graph, intersecting at a common point **x** or **d**) two equal functions at **x**. Why then, if so many different interpretations are possible is the first the most common?

One reason could be that students are introduced to algebra in what Sfard and Linchevski (1994) refer to as a "computational process". In fact the NCTM in its

Curriculum and Evaluation Standards for School Mathematics (1989) has emphasized using variables as a computational process for teaching mathematics in grades 5-8.

A second reason could be that the multi-uses of variable within algebra as previously described are a source of confusion and frustration for many students. A third reason could be an incomplete conceptual base. Silver (1986) argues that some errors are easy to eradicate while others persist for years. These errors may persist, despite attempts to correct them because of an underlying conceptual network of partially flawed ideas related to the procedures. This flawed conceptual network acts as a support system for error. The error is continually fed and thus will not die. A fourth and certainly powerful reason that students are not able to see multiple meanings in algebra is because of compartmentalization. Things learned in a certain context are tied to that context (Hiebert, 1984). Students do not look for relationships or connections between old and new knowledge.

During the latter part of the fall 2001 semester, I introduced equations of lines to my college algebra I class. We developed and defined the idea of intercepts, slope, and line. We established the formula for slope, standard equation of the line, slope-intercept form of the line and the point slope formula. One of the most frustrating incidences occurred when I asked my students to find the equation of the line passing through a set of points and write the equation in slope intercept form. First I gave them the following: **m = 5 and b = 4.** We had already established (or so I thought) that **b** represented the **y-intercept.** Nearly all students were able to substitute the formula and write **y = 5x + 4.** Next, I gave them the following: **m = 2/5 and y-intercept = (0, -1).** One young lady informed me that this could not be done because there was no **"b".** Why was this incident so much more terribly frustrating to me as a teacher than any other during that particular week?

I felt that I had clearly established the fact that a **y-intercept** is a location where a graph crosses the y-axis representing a point **(0, b).** Even if one did not catch that fact, previously during that same class period we had worked problems that asked us to find the equation of the line (in slope intercept form) having some slope and passing through some point, for example, **m = -2 and through (7, 4)**. Several of my students failed to realize that they could actually solve the first problem using the point-slope formula as we had just done within that same class session. Some of my students could not see **(0, -1)** as a point that the line passed through like **(7, 4).** The point **(0 -1)** was just as it was stated, the **y-intercept.** The **y-intercept** was that other thing you found after you found the **x-intercept** and they were simply places (locations, if you will) where your line crossed the axes**.** For several students a problem that gave the slope and a point required one type of procedure, and a problem that gave the slope and an intercept required another. For these students, absolutely no relationship was found nor connection made between the slope-intercept of the line which can be derived from the point-slope formula which can be derived from the formula for slope.

The second reason may be that a move from beyond the realm of procedural computational processes to a realm that is more conceptually rich requires more than just superficial understandings of the processes themselves. Hiebert (1984) states some children can demonstrate flawless computations with virtually no idea of the logic of the algorithm that they are employing. Silver (1984) concurs by stating, “It should be reasonably clear . . . that one can demonstrate procedural fluency without conceptual knowledge". But Silver adds that it is reasonable and appropriate to note that procedural knowledge can be quite limited unless it is connected to a conceptual knowledge base. Students need to be able to make the transition from the procedural computational processes of the aforementioned equality to a conceptual understanding of the host of interpretations to this equality. It is not to say that procedural knowledge is bad.

Silver (1984) finds it plausible that conceptual knowledge can be based on procedural knowledge. Slavit (1995) in discussing his "growth conception of function" states, "through an intensive look at elementary functions, students may understand function to be a related set of procedures . . ." "In essence" he continues, "the procedures performed on functions give rise to understanding . . ." Hiebert (1984) states that procedures can in fact facilitate the application of conceptual knowledge because highly routinized procedures can reduce the mental effort required in solving a problem and thereby make possible the solution of complex tasks. He suggests that meaningful learning should accompany procedural learning. Procedures can be learned with meanings. Hiebert proposes that procedures that are learned with meaning are procedures that are linked to conceptual knowledge.

How are these procedures linked or transitioned to conceptual knowledge? Several authors, (Breidenbach et. al, 1992; Goodson-Espy, 1998; Sfard, 1992; Sfard and Linchevski, 1994; Slavit 1997; Slavit, 1995), argue that the transition is made primarily when processes [procedures, algorithms, rules] are reified (Breidenbach in authorship with Dubinsky uses the term encapsulated). In other words when processes are realized as products of themselves, then conceptual understandings have manifested. Piaget's theory of reflective abstraction directs us to the basic agreement about the roles of (in this particular case) algebraic processes and objects. Sfard's (1991) explanation for why concepts and process as an object understanding are agreeable rather than oppositional to one another proposes the fundamental difference between reification and Hiebert's efforts to reconcile procedures and concepts. She writes:

> "In order to see a function as an object, one must try to manipulate it as a whole: there is no reason to turn process into object unless we have some high-level processes performed on this simpler process. But here is a vicious circle: on one hand, with an attempt at the higher-level interiorization, the reification will not occur; on the other hand, existence of objects on which the higher-level processes are performed seems indispensable for the interiorization-without such objects the processes must appear quite meaningless. In other words: the lower-level reification and the higher-level interiorization are prerequisites for each other!"

There are two important points that we can take from this statement. Sfard argues that reification's operational and structural idea is distinguishably different from ideas like: Piaget's conceptual understanding and successful action; Tulving's semantic memory and episodic memory; Anderson's declarative knowledge and procedural knowledge; and Hiebert and Lefevre's procedural knowledge and conceptual knowledge. These approaches attempted to link two dichotomous interpretations of learning and understanding. The operational and structural approach proposed by Sfard is complementary to each other. They are fused and indispensable to each other. The second point that can be taken from this statement is that reification is cyclic in nature.

Where does Sfard's reification differ from the procedural-conceptual notions of Hiebert and Lefevre (1985) and Silver (1985) and other dichotomous notions of understandings? First these two forms of knowledge have been historically viewed as separate entities . . . coexisting as disjoint neighbors . . . (Hiebert and Lefevre, 1985).

These and other similar ideas suggest the linking of the two distinct, separate, dichotomous forms of knowledge namely procedural knowledge and conceptual knowledge. The linking of these dichotomous forms of knowledge is orchestrated to increase the usefulness of one with the other. It is proposed by Hiebert and Lefevre (1985) that linking conceptual knowledge with symbols creates a meaningful representation system. This, they suggest is a fundamental necessity for sound mathematical learning and performance. Further, they suggest, that linking conceptual knowledge with rules, algorithms, or procedures, reduces the number of procedures that must be memorized therefore increasing the probability that the appropriate procedure can be recalled and used both effectively and efficiently. Sfard's classification attempts to eliminate the dichotomous nature of the two forms of knowledge.

Sfard's approach emphasizes unity, wholeness, or oneness. "Let me stress once more: unlike 'conceptual' and 'procedural', or 'algorithmic' and 'abstract', the terms 'operational' and 'structural' refer to inseparable, though dramatically different, facets of the same thing" (Sfard, 1991). The operational and structural elements of reification cannot be separated from each other. The two elements are mutually dependent upon each other. Sfard's approach is essentially dealing with a duality rather than a dichotomy. Secondly, her approach addresses ontological as well as psychological issues. It focuses on "the nature of mathematical entities" as perceived by a thinker. Sfard's theory of reification is composed of a three-component cycle that she suggests is found at almost every turning point in mathematical history and in the process of learning. Sfard (1991) summarized that the historical development of number was a cyclic process of which the same sequence could be observed time and time again. She condensed this historical process into three phases:

> (1) The preconceptual stage, at which mathematicians were getting used to certain operations on already known numbers (or, as in the case of counting - on concrete objects); at this point, the routine manipulations were treated as they were: as processes, and nothing else (there was no need for new objects, since all the computations were still restricted to those procedures which produce the previously accepted numbers).
>
> (2) A long period of predominantly operational approach, during which a new kind of number begun to emerge out of the familiar processes (what triggered this shift were certain uncommon operations, previously regarded as totally forbidden, but now accepted as useful, if strange); at his stage, the just introduced name of the new number served as a cryptonym for certain operations rather than as a signifier of a 'real' object; the idea of a new abstract construct, although already in wide use, would still evoke strong objections and heated philosophical discussions;
>
> (3) The structural phase, when the number in question has eventually been recognized as a full-fledged mathematical object. From now on, different processes would be performed on this new number, thus giving birth to even more advanced kinds of numbers. (p. 14)

So, if Sfard's theory on the cyclic objectification of mathematics in history is true then three things must be true. First processes including procedures, algorithms, and rules must be conducted on already known and established objects. Secondly, there must be a sort of dissatisfaction with the process and the idea to turn this process into an independent structure must surface. And finally, the ability to see this independent structure as an object within itself must manifest. Sfard's cyclic theory of how students learn is developed from this idea and is summarized by Goodson-Espy (1998). She reports the three stages in concept development are interiorization, condensation, and reification. Interiorization is described as the stage where the learner performs operations on lower level mathematical objects. As the learner becomes more familiar with the process she no longer has to actually write out the steps in order to think about the process. I offer this example from my College Algebra I class.

I introduced the idea of subtraction of signed numbers to my class. I began asking my students to give me the solution to the problem **8 - 5.** All were able to give me the correct solution of **3**. I then asked my students to give me the solution of **5 - 8**. Immediately many answered, **-3.** No, I responded. Many were shocked and began reworking the problem on paper and using calculators, while others began guessing, **13, -13**. “There is no solution to this problem,” I said. When we learned the concept of subtraction in second or third grade, our teachers taught us (**for 8 - 5**) eight “take away” five. This is the correct idea behind subtraction. I asked my students to hold up five fingers and physically “take away” eight fingers. Needless to say no one was able to complete this task. I then asked them why are we able to accept **-3** as a correct answer to the problem **5 - 8** when obviously this could not be physically done. No one could offer a reasonable explanation. I suggested to them that this was an arithmetic problem that could not be physically solved. But we could redefine this arithmetic problem into an algebra problem with ideas with which we were already familiar (i.e. addition of like and unlike signs). I expressed to them that we could redefine a subtraction problem in the form **a - b** to **a + (-b)** where both **a** and **b** are real numbers. Now the problem **5 - 8** could be solved in its new form **5 + (-8)** where we could subtract the absolute values and keep the sign of the number with the larger absolute value. I allowed the students to work many more problems on the board and at their seats. Several students were content and happy with simply going through the steps of this new definition and as they worked their assigned problems. They changed their subtraction into algebra on their papers and proceeded to follow the basic algebraic addition rules.

One student told me that he now remembered being previously taught the new definition and fully understood why **5 - 8** worked. He asked me if he had to show the steps involved in this new definition because that's what he was doing in his head anyway. I suggested to him that if he could articulate exactly what he was thinking and if it was logical, then I would accept that he fully understood the idea. He was able to fully and completely explain this process. I was satisfied that he understood. This student, like several others in the class, probably had long ago interiorized this process. He no longer needed to show the steps.

Sfard uses the term "interiorization" in the same sense as Breidenbach et al (1992): when the total action [any repeatable physical or mental manipulation that transforms objects (numbers, geometric figures, sets) to obtain objects] can take place entirely in the mind of the subject, or just be imagined as taking place, without necessarily running through all of the specific steps, we say that the action has been interiorized.

One of my concerns with this process of interiorization is that over time the process itself gets lost. Students forget why things work and without prompts or hints may not be able to articulate the process. Teachers must encourage students to articulate their thinking. A student in a previous College Algebra I class approach to this same problem **(5 - 8)**, was to view the problem as subtraction with two positive numbers. If the second number was larger then the answer would be negative. This process works, but the logical and conceptually rich notion that **a - b = a + (-b)** is lost. She [the student] could not explain why her process worked. She was not writing any of the steps. She was doing them in her head. If I had not asked her to explain what she was doing, I may have assumed that she completely understood how to correctly subtract signed numbers. I would not have realized that she had interiorized a faulty process that handicapped her when we began to subtract numbers with different signs.

Without discourse (having the students articulate their ideas) teachers may be allowing students to interiorize incorrect methods. When students are allowed to interiorize faulty processes without correction, they become limited in the number of new ideas that can be successfully constructed from the previous idea and its processes. As students get acquainted with the processes (correct processes), they begin realize new concepts. For example, long division can bring about a realization of fractions, decimals, and percents and eventually the "big picture" of rational numbers.

Condensation is a lengthy period during the second of the two operational phases of the reification process. It is described as the stage where complicated processes are condensed into a form that becomes easier to think about. Sfard (1991, states that condensation is the period of "squezzing" lengthy sequences of operations into more manageable units. At this stage the learner becomes increasing more adept at considering a given process as a whole. The learner can image the outcome or "bigger picture" of a procedure without necessarily having to go through each step. Sfard (1991) states "it is like turning a recurrent part a computer program into an autonomous procedure: from now on the learner would refer to the process in terms of input-output relations rather than by indicating any operations."

She suggests that any processes that cause perturbations in the mind of the learner, like subtracting a number from a smaller number when only unsigned numbers are known (essentially a transition from arithmetic to algebra), will serve as a spark or igniter for the idea for the new mathematical entity. Condensation represents a rather significant change in the cognitive approach to a concept. In fact, at this stage a new concept is actually formed. But this new concept is still dependent on an algorithmic process. Condensation appears to be the stage where much of the cognitive effort to fully grasp an understanding of a concept is put forth. But the question remains, how can educators determine when or if a student is in the condensation stage.

Sfard (1991) states that “in the case of negative numbers condensation may be assessed through the student’s proficiency in combining the underlying processes with other computational operations; or in other words, in his or her ability to perform such arithmetic manipulations as adding or multiplying negative numbers.” When considering the concept of function she suggests, “the more capable a person becomes of playing with a mapping as a whole, without actually looking into its specific values, the more advanced in the process of condensation he or she should be regarded.” One of the more notable characteristics of condensation is that the learner becomes increasingly able to alternate between different representations of a concept. For example the learner would become increasingly able to alternate between the graphical and algebraic representations of the linear equation given by **2x – 1 = 5x + 8.** Furthermore, the learner becomes more able to generalize, make comparisons, and combine mathematical processes. In the case of functions, Sfard (l991) suggest the learner becomes more able to investigate functions, draw graphs, compose and decompose function. Again, the learner remains in condensation as long as the new concept is still dependent on a particular process.

Reification as further reported by Goodson-Espy is the stage where the learner can conceive of the mathematical concept as a “fully-blown object” (Sfard, 1992). Concepts that have been reified can be thought of in relation to the categories to which they belong. The characteristics of these categories may be compared to others. In the case of our aforementioned example, the learner thoroughly understands the different representations that our linear equation can take. Further, the learner can talk about other linear equations like when the coefficient of the variable is negative or one of the coefficients is zero. Let us consider again **2x - 1 = 5x + 8**.

From our understandings of reification, for a student to make the transition from say, seeing the plus on either sign of the equality symbol just as a key to “do something” or “action” symbol, to being able to interpret our problem as either a formula, an equation to solve, an identity, a property, or an equation of a function of direct variation (Usiskin, 1988) or all of the above, process reification must occur. As the student gradually moves from interiorization to condensation, his cognitive abilities to view mathematical expressions or structures become increasingly more and more flexible. For example, not only does he see **2x - 1** as something to be subtracted but also as a function, as a linear equation, or as line on a graph that has a shifted one unit down, increasing from left right, with a steepness factor (slope) of positive two or even an expression with a range of values (Kieran, 1991). Then, as he suddenly jolts from condensation to reification (a light pops on) all expressions of the form **mx + b** are viewed as complete entities. The student is not thrown off for example, by **m**, **x**, or **b** being represented as a rational number (which is another trouble spot for many students).

"Reification is an instantaneous quantum leap: a process solidifies into object, into a static structure. Various representations of the concept become semantically unified by the abstract, purely imaginary construct. The new entity is soon detached from the process which produced it and begins to draw its meaning from the fact that it is a member of a certain category" (Sfard, 1991).

The learner can then deduce generalities concerning this category. This entity can then be used as inputs to other processes. The third stage of reification now cycles back to the first stage-interiorization. The lower level reification and the higher level interiorization are prerequisites for each other (Sfard, 1991). Reification of a given process occurs simultaneously with the interiorization of the higher-level processes. New mathematical objects begin to be created from the present one. This leads us to the inherent difficulties of reification.

The ability to visualize something old and established as something new is usually very difficult to do. My former pastor used to tell our congregation of his years before he became a Christian and a minister. He told us of the times that he sold drugs, smoked, drank, and partied all night at night clubs. Most in our congregation had never known him in this way so it was not a problem to respect him in his new office as pastor. But, this was not the case for his friends and acquaintances who knew him before. They continued to call him by his "nickname", "Sweet Mo". He said that they found it extremely difficult, (based on their past experiences with him) to believe and accept that he had actually given up his previous life and had become a minister. Although he insisted on being addressed as Pastor Moore, some of his old buddies simply refused to see him in this new light and continued to address him as “Sweet Mo”. One of the difficulties with reification is the same.

It can sometimes be a great struggle to see a familiar step by step process in the bigger picture of an autonomous structure or object. The student must, in many cases, must first labor to grasp and understand algorithms, rules and procedures. Then they must compete with these established procedural understandings to move these processes into a structured form. Finally, after this terrible struggle they must start the process over again and see this structure as a process in a higher level mathematical concept. Hence within each phase the student must eventually be able to see the duality of the process. One must be able to move freely in ones mind from the abstract to the real. Sfard and Linchevski (1994) explain this idea as follows:

> “The ability to perceive mathematics in the dual way make the universe of abstract ideas into the image of the material world: like in real life, the actions performed here have their ‘raw materials’ and their products in the form of entities that are treated as genuine, permanent objects. Unlike in real life, however, a closer look at these entities will reveal that they cannot be separated from the processes themselves as self-sustained beings. Such abstract objects like $\sqrt{-1}$ (square root of -1), -2, or the function 3(x + 5) + 1 are the result of a different way of looking on the procedures of extracting the square root from -1, of subtracting 2, and of mapping the real numbers onto themselves through a linear transformation, respectively. Thus, mathematical objects are an outcome of reification - of our mind’s eye’s ability to envision the result of a process as permanent entities in their own right.” (pp.193-194)

I see it as being similar to an interior decorator, who envisions in his mind’s eye what a room would look like with a flower or plant placed in a particular corner, a painting hung on a particular wall, movement of furniture, new light or light fixtures installed to bring about a certain atmosphere, or new wallpaper or boarder being hung. Each idea is distinct, individual, and separate and must be accomplished step by step. But when all is brought together and the project is complete you have essentially a brand-new room that looks and smells and feels just as the decorator envisioned. In other words it is sort of like having all of the materials laid out on the floor and at same time having to see them in their place in the finished product of the room. The decorator can now take all of his combined experiences of color schemes, what to hang, when to hang, what nails or glue to use, what plant or flower to place, etc. as a whole with him and use it as moves to the next larger or more complicated project that might require his creative touch. Elaborating further on the difficulties of abstraction and objectification, Sfard (1994) writes:

> The present treatment of the issue of understanding [as an operational schema] sheds new light on the inherent difficulty of reification. The frequent problem with new abstract ideas is that they have no counterpart in the physical world or, worse than that, that they may openly contradict our experiential knowledge. Obviously in the latter case no metaphor is available to support these abstractions . . . In fact, the very idea of reification contradicts our bodily experience: we are talking here about creation of something out of nothing. Or about treating a process as its own product. There is nothing like that in the world of tangible entities, where an object is an "added value" of the action, where processes and objects are separate, ontologically different entities which cannot be substituted one for the other. Our whole nature rebels against the ostensibly parallel idea of, say, regarding a recipe for a cake as the cake itself.

This idea of having the vision (and in our case the understanding) to see the finished product before it is finished, to see the "big picture" when all you have before are the materials, is extremely difficult for an average person in everyday life and more so in mathematics.

The final comments on the difficulties with reification concern its discontinuous nature. It would appear that the transitions from interiorization to condensation are smooth, seemingly logical, and continuous. But then there is a long and arduous battle within condensation. Ideas and perceptions are consciously and subconsciously campaigning for their correct and rightful place in the “full” and “complete” understandings of the particular concept. I liken it to a country, currently like Afghanistan, engaged essentially in, civil war. Each faction in the country skirmishing, competing, bumping into each other trying to stake a claim to their legitimate place within the country before complete reorganization occurs and the new government (in our case reification) is formed and put into place. After condensation, there occurs a sudden ontological shift, a sudden illumination when all those competing ideas seem to be connected and can be viewed as one complete structure.

These ideas, all independent and unconnected in ones mind, seem to find themselves at family reunion, strangers, not knowing each other. Then as they begin to talk and interact with each other . . . “oh you’re Lula’s grandson” . . . “bam”, in an instant, they realize that they are related, cousins if you will, part of the same family. Skemp (1987) claims all new learning in mathematics that involves building concepts consists of individuals forming new ideas in their own minds, from their own points of view. He describes the ontological shift as conscious awakening; a point where the student is fully conscious of the nature and structure of the desired concept.

Sfard (1994) submits that the issue of discontinuities in the process of understanding seems to be of the utmost importance, and at same time is not easily observed. Even for teachers and researchers, because of its instantaneous and immediate nature, this phenomenon is not easily detected or recognized in the learning process. But in spite of the difficulties associated with reification there are some benefits.

Reification is important for developing higher level concepts. As previously mentioned the nature of reification is that lower lever structures are formed and are used as the foundation for developing higher level concepts. Reification acts as a tool or mechanism for future learning by integrating and organizing existing knowledge.

Consider for example, a parabola given in the form $\mathbf{ax^2 + bx + c}$. For students to have a better understanding of the nature of the transformations of this parabola, they need to rewrite it in the form of $\mathbf{a(x - h)^2 + k}$. In the latter form it is easier to visualize the shrinks or stretches, shifts, and reflections without having to actually graph the equation. Well then, how is prior or existing knowledge used in this case? This question may be answered by considering the following function: $\mathbf{f(x) = 2x^2 + 4x - 5}$. In order to put this function in the general form of a parabola, the following must be completed:

$$\begin{aligned} \mathbf{f(x)} &= \mathbf{2x^2 + 4x - 5} \\ &= \mathbf{2(x^2 + 2x) - 5} \\ &= \mathbf{2(x^2 + 2x + 1 - 1) - 5} \\ &= \mathbf{2(x^2 + 2x + 1) - 2 - 5} \\ &= \mathbf{2(x + 1)^2 - 7.} \end{aligned}$$

In order to successfully complete this process a student must be able to factor the greatest common factor. Next, the student must have a reasonable understanding of completing the square. The student must also understand how to use the distributive property. The student must understand how to factor trinomials and in particular perfect trinomials. And finally the student must know how to add and subtract signed numbers. All of these competencies are required to transform the quadratic equation into the general form of a parabola. The concepts are all integrated and organized to give a form that is more easily understood and visualized when referenced to a graph. This is not to suggest that integrated knowledge implies reified knowledge but merely to emphasis the volume of concepts that may need to be understood and incorporated in order to complete the reification process.

The reification process may serve to strengthen ones understandings of existing knowledge. I am not saying that a student could not complete the transformation from quadratic form to standard parabolic form by only memorizing the necessary steps. But it would seem that steps and rote procedures can get lost and cause confusion for many students when, for instance, negative numbers or fractions are required to complete the transformation. A student must have an idea of the “big picture” and have some reasonable understandings of the aforementioned concepts. Some of these concepts may have only been interiorized by the student. While others may have been condensed and then fully reified. In either case, these understandings are strengthened by removing them from their isolated contexts and integrating them with other understandings in a multi-conceptual context. Reification invariably strengthens the students’ existing understandings by helping him to realize how his previous knowledge and understandings are related to each other.

Reification makes routine processes effortless. In fact, its intent is to it help the student achieve “full consciousness” of concepts and eliminate the processes all together. It is not to say that the processes no longer exist or that some processes are neither important nor necessary. But reification helps the student condense and “package” his knowledge. He does not have to continually revisit procedures each time he wishes to use an idea to help him develop and understand a different or higher level idea. When a concept has been fully realized as a whole concept and “stored away”, the student no longer has to go through the arduous and time-consuming task of a step by step process. The student merely needs to call up the reified concept, add the necessary information, and use it to build a higher level conceptual structure.

Sfard and her theory of reification give us a process of gauging students' understandings. But the theory does not give us any direction on how to get to concept structuralization. Reification does not offer any method, teaching technique, or educational philosophy that can be used to ultimately lead students through that “vicious” reification “cycle”.

I am convinced that people (students in particular) learn by doing something over and over again. We learn to play instruments, sing, dance, and become experts in our particular fields by practicing and applying and practicing again. There is an old saying that practice makes perfect. There is a modification to this saying that states, that perfect practice makes perfect. I don't believe I would find too many rational people who would argue that simply practicing makes perfect nor would I find many who would disagree that consistent “perfect” practice is extremely difficult to achieve. I would like to settle somewhere in the middle and suggest that good, sound practice makes perfect.

One of the world's best golfer, Tiger Woods, did not become the best by simply practicing or consistently practicing perfectly, I would argue that it is consistently sound practice that helped him maintain his number one world ranking for such a long period. Such is the case with constructive habituation. Concepts developed by sound examples, multiple repetitions of these examples, and multiple representations of these examples may provide a pathway for teachers to travel in effort to lead students through the reification process.

CHAPTER 3

DESIGN OF THE STUDY

This study applied a mixed methods research design, employing the data collection and analysis associated with both quantitative and qualitative research. It can be more accurately characterized as a sequential explanatory design. According to Creswell (2003), it is the most straightforward of the six major mixed methods approaches. The priority is typically given to the quantitative data. It is characterized by the collection and analysis of quantitative data followed by qualitative data. The purpose of this design is to use the qualitative results to assist in explaining and interpreting the findings of a primarily quantitative study. This section contains a description of the design of this experiment. First, it presents the quantitative elements of the study, and then it explains the qualitative features. These are followed by a brief discussion of the treatments and the instruments.

Quantitative

The quantitative portion of this experiment can be characterized as a quasi-experimental design. This portion of the study is divided into four sections. These sections were crafted from the four research questions. The first two sections examined if constructive habituation is a more effective means of helping students reach process-object reification by developing richer and more meaningful understanding functions, inverses of functions and transformation of functions (research questions 1 and 2) than the traditional mathematics pedagogy.

Two sections of the precalculus I students were involved in this part of the study. The experimental group was instructed using constructive habituation. The control group was instructed using a traditional teaching method. The instruments used in this part of the investigation are the regular examination designed to evaluate the conceptual understanding of graphs, and functions, and the regular examination designed to evaluate the conceptual understanding of the symmetry and transformation of functions.

The third section compared the general achievement of conceptual understanding and procedural proficiency of both the constructive habituation students and the traditional students as reflected by the departmental final examination. The idea here is to examine whether constructive habituation students would perform as well as traditional students on an examination that was primarily constructed under a traditional teaching philosophy and that emphasized procedural proficiency. The differences here were expected to be only slight, because the content of the departmental final examination includes much material taught identically to the two groups. The departmental final examination is given to all sections of the Precalculus I course.

The final section of the quantitative portion of this study addressed the conceptual understandings and the abilities of the students to define and explain functions, model and graph functions, and reify functions. These components were extracted from target questions found within the two regular examinations.

Qualitative

This study also collected qualitative data. The purpose of the qualitative portion is to elucidate the results of the quantitative portion and examine more fully the students' understanding of the target concepts. This portion of the study involved interviewing six students (three from each class). The subjects were chosen based on their willingness to participate in the interviews, class grades, and gender. The selection process was guided by a balance in terms range of abilities.

The qualitative portion was conducted after the symmetry regular examination but before the final examination. The qualitative portion contained several non-routine, non-traditional, and applications problems. The problems were designed to get a clearer picture of the students understanding of functions, their abilities to connect one representation to another, and their understanding of transformations of functions. Moreover, the qualitative portion helped in strengthening the results from the quantitative portion of this study.

Treatments

The treatments in this study are two teaching strategies: Constructive habituation and a traditional method.

Constructive Habituation

Constructive habituation is a newly developed teaching strategy in its infancy stage. Brooks (1993) suggest that becoming a constructivist teacher requires a paradigm shift. They suggest that becoming such a teacher requires a willing abandonment of familiar perspectives and practices and adoption of new ones. The habituationist practices found in traditional teaching are representative of the practices to which they refer. Constructive habituation does not require the teacher to totally abandon those practices that long typified American classrooms. Constructive habituation attempts to unite constructivist teaching methods aimed at supporting students' conceptual understanding of content and habituationist teaching method aimed at establishing routine responses to routine tasks. Furthermore, constructive habituation has the potential to be considered both a pedagogical and curricular strategy. It can be used to address either or both the "how" and the "what" one teaches: the "how" relative to the method of instruction and the "what" relative to the content of the instruction. There are several other characteristics of constructive habituation.

The first is that it presents concepts as whole with emphasis on the big picture. Second, students are ultimately responsible for their own understanding but the leash (the amount of latitude) they are given in exploring a concept is much shorter than in discovery oriented constructivist classroom. Thirdly, students' points of view are valued and sought after in an effort to understand their present conceptions for development of future lessons.

Reoccurring misconceptions or problems are addressed for present and future students by developing examples that can be connected with other representations of the target concept. In that spirit, constructive habituation can be fundamentally characterized by its emphasis in introducing a topic or idea through multi-Reps (multiple representations with multiple repetitions). These multi-Reps include but are not limited to symbolic, numerical, tabular, and graphical representations.

The idea is to present these concept-representations in concert in order to help insure a satisfactory level of understanding on the part of the student. Furthermore, constructive habituation recognizes and appreciates the need for procedural and algorithmic learning. In fact, constructive habituation aims to promote reification of procedures within concepts in order that they are used at an increasingly higher level of abstraction. Since constructive habituation is such a new idea, I find it necessary to give an example of a typical lesson that uses this teaching strategy.

A typical lesson using constructive habituation begins with introducing a new concept to the whole class either by definition (symbolic), several examples (numeric) or graph with possible reference to tables. The instructor decides which representation is the most appropriate or reasonable for his or her students. Consider the following example:

> Suppose the teacher was introducing transformation of functions to his or her class. Schwarz, Dreyfus & Bruckheimer (1990) report that students have been found to have difficulty relating graphs to formula when presented with tasks of function transformation such as shifts, $f(x) \rightarrow f(x) + d$ or $f(x) \rightarrow f(x + c)$, and stretches $f(x) \rightarrow af(x)$ or $f(x) \rightarrow f(bx)$. One option may be to begin with a specific function, say the squaring function where $f(x) = x^2$.
>
> The graph and the algebraic representation are presented together using the TI-83 graphing utility. The students are then given several examples with this particular function that involve vertical shifts (both on the positive and negative y-axis). The students are then asked to make a general conjecture about their observation. The teacher may then want to introduce the cubing function, having the students graph the examples as before and make a general conjecture about this function and any function in the form $f(x) \pm d$ **(d > 0)** which represents a vertical shift d units up or down. At this point the teacher has reinforced this idea of vertical translations through three different perspectives. Next the teacher

might consider introducing a function like the square root function using an example like ***f(x)*** **=** $\sqrt{x+4}$**.**

The students would then be asked to make a statement about the graph of this function with respect to the reference function ***f(x)*** **=** $\sqrt{x}$. A typical student response is that the graph shifts up four units. The teacher would then ask the students to graph the function. Cognitive perturbations occur at this point. Not only does the graph <u>not</u> shift up but it moves <u>horizontally</u> four units to the <u>left</u>.

Many students will struggle with the idea that the function has been changed and some constant say ***c***, shifts the graph left ***c*** units while subtraction shifts the graph to the right. This appears to go against all that is commonsensical to the average student. At this point the teacher may want to introduce tabular data in conjunction with the graph and the numerical example. The tabular data allows the student to see the pattern that is developed when ***c*** is added or subtracted to domain of the function. Several different functions may be given and a general conjecture should be made by the students.

A host of ideas are exposed and solidified after these examples including the ideas of domain and range and their algebraic and graphical connections. Moreover, students have now been exposed to multiple concepts using multiple representations including abstract definitions, algebraic examples, graphs, and tables.

One of the main objectives is to get the student to have a reasonable understanding of the effects of any appropriate ***a,b,c, or d*** on any function and easily move from one representation to another of any function in the general form

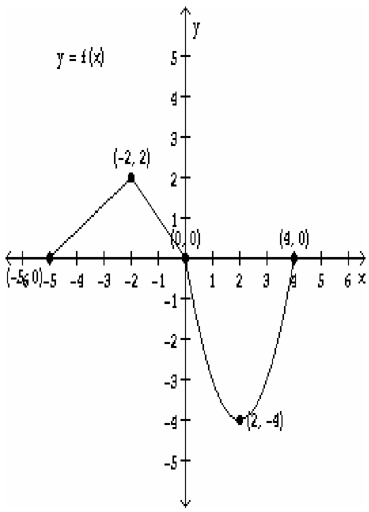
a(f(bx) $\pm$ ***c)*** $\pm$ ***d***. This can be verified in the end by challenging the students with non-routine and non-traditional functions and their graphs and have them to graph certain transformations or presenting them with the graphs and have them to produce either orally and write the equation that fits the graphs. In having the students produce the equations orally, the teacher can get a real sense of the student's understanding and reasoning. Consider figure 1 adapted from Bittenger, Beecher, Ellenbogen & Penna (2000). A description of the introductory lesson on functions is presented in Appendix L.

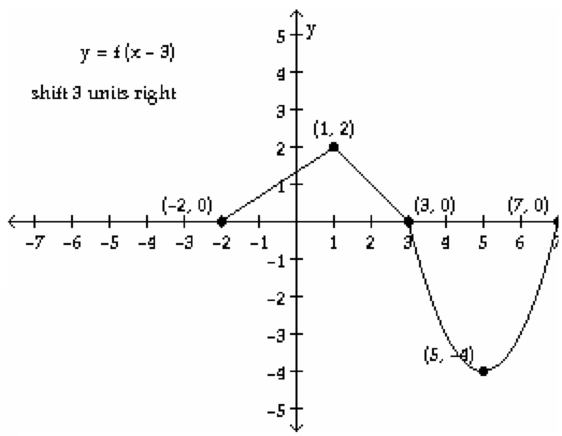
Traditional Teaching Method

The traditional teaching is lecture and teacher oriented. Teachers adhere to a fixed curriculum. This curriculum is presented as parts to a whole. There is an emphasis on basic skills and efficient student performance of these skills. The typical classroom routine begins with a review of the previous day's work and answering students' homework questions. No effort is normally put forth to reteach a topic but merely show students how to work problems. Students are then encouraged to go back and rework these or similar problems. Hence the idea behind a traditional teaching method is the more times a student does a problem the more likely he or she will be able to perform a similar problem on the quiz or test. The quiz or the test reveals to the teacher if the student really understood the concept.

The problem with this train of thought is that the students may not truly understand the central ideas of the topic nor may not be able to transfer the idea from one domain to another but are able to memorize steps and procedures to get to the correct answer. This period of question and answer is followed by the presentation of new material. This usually involves introducing a definition and presenting some examples of problems to help the students make sense of the definition and to give them a guide to assist them with their upcoming homework assignment. There are usually questions

Use the graph of y = f(x) to make a graph of y = -2f(x – 3) + 1





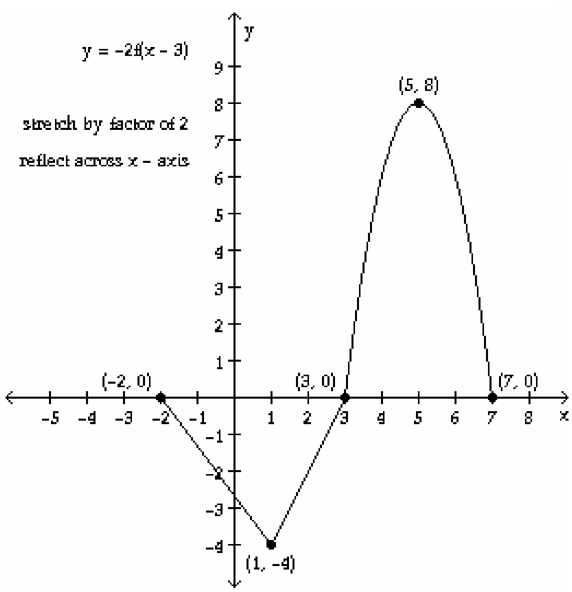
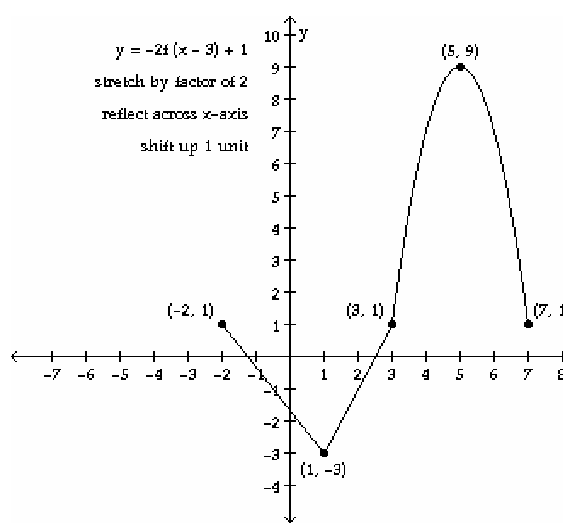


Figure1: Transformation of a general function



asked by the teacher and the students but student-teacher interaction is very limited and student-student interaction is almost non-existent, resulting in a very book-board-teacher oriented classroom.

The Precalculus I Course

- **Intended Audience**: This course is designed for students in the business, scientific, and engineering programs and prospective teachers of mathematics. The major purpose is to provide students with the essential concepts and skills in precalculus which are needed to successfully complete a trigonometry course and a calculus course.
- **Topics**: Topics include a review of the real numbers and their properties; operations with complex numbers; equations and inequalities; polynomial, rational, exponential, and logarithmic functions and their graphs; and systems of equations and inequalities. Modeling is introduced and applications are emphasized.
- **Textbook**: Precalculus, 2nd ed. by Bittinger, Beecher, et al. Addison Wesley Publisher, 2001.

The Participants

The overall scope of the study included all of the students in enrolled in the Precalculus I course (Math 135) at Southern University-Baton Rouge in the spring semester, 2004. Southern University and A&M College is a comprehensive institution offering four-year, graduate, professional and doctoral degree programs. The University today is part of the only historically black Land Grant university system in the United

States. It offers bachelors degrees in 42 areas, master's degrees in 19 areas, and doctoral degrees in five areas and associate degrees in two areas. An average of 9,000 students is enrolled each year at the Baton Rouge campus. Most of the students are from lower-middle to middle socio-economic status. More than ninety-percent of the students receive federally funded pell-grant assistance.

A total of 19 classes with an initial combined enrollment of 502 students were involved in the analysis of this project. All of the classes were required to participate in the departmental final examination at the end of the semester. Their results were used to evaluate research question 3.

The primary focus of this study consisted of 72 undergraduate students enrolled in Precalculus I during the spring semester 2004. This represents about 14% of the total enrollment in Precalculus I. The 72 students are enrolled in sections one and two as offered in the university class scheduling booklet. Section one which was the experimental class initially enrolled 37 students. Section two which was the experimental class initially enrolled 35 students. The classes were filled by normal online and telephone registration procedures. The sample consisted of undergraduate students who were enrolled in the researcher's class. Hence the sample will be conveniently selected.

Instruments

The Functions, Graphs, and Models Regular Examination

This test is an achievement test. A central theme in concepts of functions and its applications is the search for patterns. Mathematics offers powerful tools in its ability to recognize and define these patterns succinctly through multiple representations. The term function is used to describe a particular relationship between two variables. The functions regular examination focused on patterns of change between independent and dependent variables. The functions regular examination was developed to determine if the student is able to:

1. define a function
2. recognize from a when one variable (with respect to another) is increasing, decreasing, constant and when it reaches it's maximum or minimum.
3. identify independent and dependent variables, domain and range.
4. determine whether a correspondence or relation is a function.
5. determine whether a graph is that of a function.
6. solve applied problems using functions.

The Symmetry and Transformation of Functions Regular Examination

This is an achievement test. The central theme of this test is to measure the students understanding of the relationship between changes made to the graph of a function and changes made to the formula defining the function. Knowledge of symmetry in mathematics helps us graph and analyze equations and functions. This test considers symmetry, algebraic test to determine if a function is even or odd, vertical and horizontal shifts, reflections, and stretches and shrinks of the graphs and the corresponding effects on their formulas.

The Departmental Final Examination

This is an achievement test. This test is given to all students enrolled in the Precalculus I course. This is a two hour comprehensive examination developed by the Precalculus committee in Department of Mathematics at Southern University. The finalexamination primarily stresses the solution of routine problems.

The test is divided into two main parts: multiple-choice and extended response. The multiple choice section primarily focuses on the solution to routine problems and serves as the department's test of algebraic knowledge and proficiency as required by the state's Board of Regents. The second portion of the exam serves as a test of both procedures and concepts by using both routine and non-routine problems.

Interviews

The interviews were intended to explore more deeply into the students conceptual understanding of functions and their transformation and to substantiate the findings from the quantitative portion of the study. There were one set of interviews at the end of the semester. All interviews were conducted in the researcher's office. The interviews lasted approximately forty-five minutes and were audio-taped for later analysis.

The questionnaire used during the interview session consisted of a mixture of questions developed by the researcher and an adaptation of O'Callaghan (1994) interview protocols. The problems on the questionnaire were chiefly non-routine and application problems. The benefit of the interview format was that it allowed not only an inspection of the student's methods but also of his/her reasoning in using the particular method.

Weaknesses

Every potential weakness in this study could not be predicted but one apparent potential weakness is the fact that the researcher functioned as the experimenter. The researcher was the instructor for both classes and also conducted and analyzed the interviews. This introduced the possibility of the experimenter's biases and expectations affecting the results in terms of behavior of the participants and behavior of the researcher. A second mathematics educator was invited to observe several classes as a safeguard to protect against this threat. Readers are reminded that constructive habituation is a new experimental teaching strategy. This study represents its first experimental test. Hence this study was more investigative in nature intending more to explore and uncover results rather than prove them.

Data Analysis

Descriptive statistics which are the procedures and their associated numerical indices that help clarify data for samples were used for the quantitative portion of this study. The numerical data in this study were analyzed by several different methods involving parametric and non-parametric tests. The students in this study enrolled in this course prior to the beginning of the spring 2004 school semester. Since they were not randomly assigned to each class by the researcher, the classes were regarded as intact prior to the beginning of the study. Therefore, in a study such as this, it is necessary to eliminate systematic bias and within group variance caused by the non-randomization of the classes.

A typical procedure in dealing with these concerns is to introduce a covariate that will help correct for these two non-randomization concerns. The mathematics ACT was chosen as that covariate. Cognitive variables from high-quality standardized tests are considered very reliable covariates. In order for covariates to be appropriately used, two important assumptions had to be met. First, the covariate had to demonstrate significant correlation with the dependent variable. Second, dependent variable and the covariate had to have normal distributions. A correlations matrix was run for the covariate and the three examinations to test for the first assumption. The Kolomogorov-Smirnov test was used to test for normality.

The functions examination met all of the requirements for analysis of variance with mathematics ACT as its covariate. An ANCOVA was used to analyze these scores.

The symmetry examination failed to meet the assumption of linearity with mathematics ACT as the covariate. Therefore an ANOVA was used to analyze the scores on this examination.

In order for any analysis of variance procedure to be appropriately used to evaluate the differences in means of two or more samples, their variances must be assumed equal. The Levene's test for homogeneity of variances was conducted on each examination. The final examination scores for the two target classes failed the test for homogeneity. Therefore the non-parametric Mann-Whitley U-test was conducted to evaluate the difference in means of the scores on this examination. An ANOCVA was conducted on the final examination scores of all precalculus classes. A MANCOVA was conducted on the three component variables found in research question 4. Descriptive statistics were produce for each of the quantitative analyses in the study.

The interviews were analyzed by carefully reviewing the audiotape, the students' worksheets, and the researchers' notes. Each interview was evaluated in three categories. These categories included each of the three components areas: defining and explaining functions, modeling and graphing functions and their applications, and reifying functions and their applications. The researcher considered these areas essential in evaluating the students' ability to negotiate the reification process. These individual evaluations were analyzed to produce a summary of each groups' responses.

CHAPTER 4

ANALYSIS OF THE DATA

The purpose of this study was to determine if constructive habituation is a more effective means of helping students reach process-object reification than a traditional teaching pedagogy as evidenced by higher levels of student achievement. The study contains both quantitative and qualitative data in an attempt to explore this question as thoroughly as possible. This chapter presents the data and analysis for the quantitative and qualitative components.

Quantitative Analysis

This section provides an examination of the descriptive statistics and the tests that were used to evaluate the quantitative data that were collected. The data analyses are divided into four sections which derived from the four research questions presented in chapter one.

The following research questions were posed:

1. Is there any difference in the achievement between students taught using constructive habituation (CH) and students taught using a traditional method (TRAD) as measured by the Graphs, Functions, and Models regular examination?

2. Is there any difference in the achievement between students taught using constructive habituation (CH) and students taught using a traditional method (TRAD) as measured by the Symmetry, and Transformation of Functions regular examination?

3. Is there any difference in the achievement between students taught using constructive habituation (CH) and students taught using a traditional method (TRAD) as measured by the departmental final examination?

4. Is there any difference in the conceptual understanding of functions and their applications between students taught using constructive habituation (CH) and students taught using a traditional method (TRAD) as measured by the target questions from the two regular examinations?

a. Is there a difference in their ability to define functions and explain their applications?

b. Is there a difference in their ability to graph and model functions and their applications?

c. Is there a difference in their ability to reify functions and their applications?

The overall scope of this study included all students enrolled in the Precalculus I course (Math 135) at Southern University-Baton Rouge (SUBR) during the spring 2004 school semester. This involved a total of 18 classes with total initial enrollment of 440 students. The average initial class size was approximately 26 students with the smallest class having as few as 10 students and the largest class containing as many as 37 students.

The primary focus of this study was on two classes taught by the researcher: the experimental class (CH) and the traditional class (TRAD). The CH class was the largest class having 37 students while the TRAD class was the second largest (one of two classes) containing 35 students. These two classes accounted for 16.4% of the total number of students enrolled in the Precalculus I course.

The overall withdrawal and failure rate for all sections was 48.4 % (23.6% -withdrawal and 24.8%-failure). The CH students' withdrawal and failure rate of 48.6% (24.3%-withdrawal and 24.3%-failure) was similar to that of the overall withdrawal and failure rate. The TRAD students had a much higher withdrawal and failure rate (62.9%). The withdrawal rate of 48.6% was double that of the CH students but the failure rate of those who remained in the class was slightly lower at 14.3%. Absenteeism may have been a cause of this high withdrawal and failure rate. The TRAD class experienced regular absenteeism of a number of its students early in the semester. The researcher attempted to address this by requiring certain homework assignments to be handed in and giving announced quizzes. These incentives had minimal effect. Absenteeism was not as much a problem for the CH class. Interestingly, many CH students who received failing grades, opted to stay in the class until the end of the semester. This suggested to the researcher that the CH students may have found the class more interesting or more accessible than the TRAD students. The university has begun aggressive measures to lower the withdrawal and failure rates in this and several other mathematics courses.

Students enrolled in the Precalculus I course via telephone and online registration. Therefore the groups were considered intact prior to the beginning of the study. Hence, in comparing the means for the variables in the study there was a need to eliminate systematic bias and reduce within group or error variance. According to Stevens (1999), one way to deal with these two concerns is to introduce a covariate. The mathematics ACT score was chosen as the covariate used in many of the analyses of this study. The mathematics ACT was chosen for two primary reasons. First, cognitive variables from good standardized test (like the ACT) are considered very reliable covariates. Second, the university was able to provide mathematics ACT scores for nearly all of the students enrolled in the Precalculus I course. The mean mathematics ACT score for all students enrolled in the Precalculus I course was 16.46. The mean mathematics ACT score for the CH students was 16.83 (sd = 2.81) and the mean mathematics ACT score for the TRAD students was 16.86 (sd = 2.26). The mean mathematics ACT score for CH students who received a passing grade ("D" or higher) was 17.84 (sd = 3.13). The mean mathematics ACT score for TRAD students who received a passing grade was 16.31 (sd = 2.46).

The Department of Mathematics requires that certain fundamental or prerequisite skills be assessed and reviewed with all Precalulus I students. A host of basic algebra topics are assessed including but not limited to: integers, order of operations, polynomials, factoring, and rational expressions. The purpose of the assessment is to determine which specific areas the students are most deficient and should receive assistance. The students in this study were given an examination on their prerequisite skills (pre-test) during the first week of the school semester. An ANCOVA with mathematics ACT as the covariate was used to analysis the scores.

The ANCOVA revealed that the main effect for SECTION was not significant ($F(1, 46) = 3.00$ $p > .05$) with the scores of the CH students not being significantly higher ($m = 36.18$ $sd = 17.18$) than the scores of the TRAD students ($m = 30.53$, $sd = 14.06$) with respect to their fundamental algebraic skills. Typically, in studies of this nature, the pre-test is used as the covariate.

The researcher determined that this pre-test could not be considered an appropriate covariate for the three reasons a) each teacher developed his\her own pre-test, therefore the precalulus students were given different examinations with emphasis on different areas and hence there was no standard throughout the department b) only 54% of the students originally enrolled in the TRAD class took the pretest whereas 84% of the CH students took the pre-test and c) a correlations matrix showed that the pretest and the mathematics ACT are highly correlated. This correlation indicates that they were measuring the same variability. Multiple covariates are used that are not highly correlated to reduce error. The pretest is mentioned in this chapter only to present evidence of the students' deficiencies in fundamental algebraic skills for later discussion.

The study had four main research questions. The first question asked if there was any significant difference in the achievement between students taught using constructive habituation (CH) and students taught using a traditional method (TRAD) as measured by the Graphs, Functions, and Models regular examination.

Table 1: Descriptive Statistics for Pretest

	SECTION	N	Mean	Std. Deviation	Std. Error Mean
Pretest	CH Students	31	36.48	17.553	3.153
	TRAD Students	19	30.53	14.065	3.227

Table 2: T-Test for Pretest Scores

		Levene's Test for Equality of Variances		t-test for Equality of Means				
		F	Sig.	t	df	Sig. (2-taile)	Mean Difference	Std. Error Difference
Pretest	Equal variances assumed	1.319	.256	1.252	48	.217	5.96	4.759
	Equal variances not assumed			1.321	44.460	.193	5.96	4.511

Research Question 1

The use of intact groups of students did not allow for the random assignment of students to groups. As a result, an analysis of covariance with mathematics ACT as a covariant was used to statistically control for any initial differences in students' performance. The following covariance assumptions were analyzed in relation to the scores of the Functions, Graphs, and Models regular examination:

1. Statistical tests were administered on the assumption of a linear relationship between the dependent variable and the covariate, i.e. the covariate and the dependent variable should be significantly related. The assumption of a linear relationship between the covariate and the dependent variable was met ($F(1, 51) = 5.643$, $p < .05$). A Pearson's correlation coefficient was conducted to verify the results of the F-test and to examine the strength of the linear relationship between the covariate and the dependent variable. A moderate positive correlation was found ($r(52) = .313$, $p < .05$), indicating a significant linear relationship between the covariate and the dependent variable. Table 3 displays the summary of assumption of linear relationship for the Functions, Graphs, and Models examination (Functions Exam) and Math ACT. Table 4 displays the correlation matrix for the covariate and the dependent variable.
2. A Kolomgorov-Smirnov Z was conducted on the covariate and the dependent variable in each of the independent groups on the assumption of normality between the covariate and the dependent variable. No significant deviation from the normal distribution was found for either the dependent variable ($Z = .704$, $p > .05$) or the covariate ($Z = 1.197$, $p > .05$) for the experimental group (CH). No significant deviation from the normal distribution was found for either the dependent variable ($Z = .837$, $p > .05$) or the covariate ($Z = .640$,

p > .05) for the control group (TRAD). Table 5 displays the Kolmogorov-Smirnov analysis.

A one-way between subjects ANCOVA was calculated to examine the effect of teaching method (SECTION) on the scores of the Functions, Graphs, and Models examination, covarying out the effect of Math ACT. The main effect for SECTION was not significant ($F(1,51) = .250$, $p > .05$) with the scores of the CH students not being significantly higher ($m = 61.30$, $sd = 14.75$) than the score of the TRAD students ($m = 59.71$, $sd = 15.45$). Table 6 displays the descriptive statistics for the Functions, Graphs, and Models examination. Table 7 displays a summary of the ANCOVA.

Table 3: Linear Relationship for Covariate (Math ACT) and Dependent Variable

Dependent Variable: functions

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	1186.187[a]	2	593.093	2.904	.064
Intercept	1234.624	1	1234.624	6.046	.017
Math ACT	1152.408	1	1152.408	5.643	.021
SECTION	51.074	1	51.074	.250	.619
Error	10414.850	51	204.213		
Total	209860.000	54			
Corrected Total	11601.037	53			

a. R Squared = .102 (Adjusted R Squared = .067)

Research Question 2

The second main research hypothesis was that there is no significant difference in the achievement between students taught using constructive habituation (CH) and students taught using a traditional teaching method (TRAD) as measured by the Symmetry and Transformation of Functions regular examination (Symmetry Exam).

Table 4: Correlations for Covariate (Math ACT) and Main Dependent Variables

		Math ACT	Functions Exam	Symmetry Exam	Final Exam
Math ACT	Pearson Correlation	1	.313*	.265	.516**
	Sig. (2-tailed)	.	.021	.075	.002
	N	72	54	46	34
Functions Exam	Pearson Correlation	.313*	1	.276	.591**
	Sig. (2-tailed)	.021	.	.058	.000
	N	54	58	48	34
Symmetry Exam	Pearson Correlation	.265	.276	1	.450**
	Sig. (2-tailed)	.075	.058	.	.007
	N	46	48	51	35
Final Exam	Pearson Correlation	.516**	.591**	.450**	1
	Sig. (2-tailed)	.002	.000	.007	.
	N	34	34	35	36

*. Correlation is significant at the 0.05 level (2-tailed).

**. Correlation is significant at the 0.01 level (2-tailed).

Table 5: Normality Test for Covariate (Math ACT) and Main Dependent Variables

SECTION			Functions Exam	Symmetry Exam	Final Exam	Math ACT
CH Students	N		33	33	24	37
	Normal Parameters [a,b]	Mean	60.55	55.58	54.92	16.73
		Std. Deviation	15.734	21.455	21.984	2.815
	Most Extreme Differences	Absolute	.123	.097	.115	.197
		Positive	.075	.068	.109	.197
		Negative	-.123	-.097	-.115	-.161
	Kolmogorov-Smirnov Z		.704	.558	.562	1.197
	Asymp. Sig. (2-tailed)		.705	.915	.910	.114
TRAD Students	N		25	18	12	35
	Normal Parameters [a,b]	Mean	59.20	45.89	54.67	16.86
		Std. Deviation	15.338	20.554	13.076	2.264
	Most Extreme Differences	Absolute	.167	.109	.168	.108
		Positive	.106	.109	.168	.108
		Negative	-.167	-.094	-.161	-.097
	Kolmogorov-Smirnov Z		.837	.464	.581	.640
	Asymp. Sig. (2-tailed)		.486	.983	.889	.807

a. Test distribution is Normal.

b. Calculated from data.

Table 6: Descriptive Statistics for Functions Exam

Dependent Variable: functions

SECTION	Mean	Std. Deviation	N
CH Students	61.30	14.475	30
TRAD Students	59.71	15.451	24
Total	60.59	14.795	54

Table 7: Main Effect for Teaching Method (SECTION) Functions Exam

Dependent Variable: functions

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	1186.187[a]	2	593.093	2.904	.064
Intercept	1234.624	1	1234.624	6.046	.017
Math ACT	1152.408	1	1152.408	5.643	.021
SECTION	51.074	1	51.074	.250	.619
Error	10414.850	51	204.213		
Total	209860.000	54			
Corrected Total	11601.037	53			

a. R Squared = .102 (Adjusted R Squared = .067)

Again the use of intact groups did not allow for the random assignment of students to groups. As a result, an analysis of covariance with mathematics ACT as a covariate was used to statistically control for any initial differences in students' performance. The following covariance assumptions were analyzed in relation to the scores of the Symmetry and Transformation of Functions regular examination.

Statistical tests were administered on the assumption of a linear relationship between the dependent variable and the covariate, i.e. the covariate and the dependent variable should be significantly related. The assumption of a linear relationship between the covariate and the dependent variable was not met ($F(1, 43) = 3.023$, $p > .05$). A

Pearson's correlation coefficient was conducted to verify the results of the F-test and to examine the strength of the linear relationship between the covariate and the dependent variable. A weak positive correlation was found ($r(46) = .265$, $p > .05$), indicating that there was no significant linear relationship between the covariate and the dependent variable. Table 8 displays the statistical analysis which addresses the assumption of linear relationship. Table 4 displays the correlation matrix.

Table 8: Linear Relationship Between Math ACT and Symmetry Exam

Dependent Variable: SYMMETRY

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	2930.275[a]	2	1465.138	3.446	.041
Intercept	466.000	1	466.000	1.096	.301
Math ACT	1285.171	1	1285.171	3.023	.089
SECTION	1437.181	1	1437.181	3.380	.073
Error	18282.942	43	425.185		
Total	149792.000	46			
Corrected Total	21213.217	45			

a. R Squared = .138 (Adjusted R Squared = .098)

Failure of the assumption of linear relationship between Mathematics ACT and Symmetry and Transformation of Function regular examination indicated that the Analysis of Covariance with Math ACT as the covariate was not the appropriate statistical analysis procedure. A one-way analysis of variance was conducted to evaluate the main effect of teaching method (SECTION) on the students' scores for the Symmetry and Transformation of Functions regular examination. The main effect for SECTION was not significant ($F(1,49) = 2.44$, $p > .05$) with the scores of the CH students not being significantly higher ($m = 55.58$, $sd = 21.45$) than the scores of the TRAD students (m = 45.89, sd = 20.55) on the Symmetry and Transformation of Functions regular examination. Although there was no significance the results have some encouraging practical significance. The mean difference between the CH class and the TRAD class for this particular examination was 9.69 points. These classes were graded using a standard ten-point grading scale. The mean difference of 9.69 is practically significant because it represents almost a full letter grade increase for the CH class. Table 9 displays a summary of the descriptive statistics. Table 10 displays a summary of the ANOVA.

Table 9: Descriptive Statistics for Symmetry Exam

SYMMETRY

	N	Mean	Std. Deviation	Std. Error	95% Confidence Interval for Mean	
					Lower Bound	Upper Bound
CH Students	33	55.58	21.455	3.735	47.97	63.18
TRAD Students	18	45.89	20.554	4.845	35.67	56.11
Total	51	52.16	21.450	3.004	46.12	58.19

Table 10: Main Effect of Teaching Method on Symmetry Exam

SYMMETRY

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	1092.907	1	1092.907	2.444	.124
Within Groups	21911.838	49	447.180		
Total	23004.745	50			

Research Question 3

The third main research hypotheses was that there was no significant difference in the achievement of student taught using constructive habituation (CH) and students taught using a traditional method (TRAD) as measured by the departmental final examination. The use of intact groups of students did not allow for the random assignment of students to groups. As a result, an analysis of covariance with mathematics ACT as a covariant was initially considered to statistically control for any initial differences in students' performance. Levene's Test for homogeneity of variances indicated the variances of the scores of two classes for this examination were not equal. The test showed that there was a significant difference ($F(2, 34) = 4.44$, $p < .05$) in the variances of the scores of the CH students and the TRAD students. Therefore, parametric statistical tests to determine the differences in means were not appropriate. A non-parametric statistical test which could evaluate the differences in the means of two treatments was needed. The Mann-Whitney U-Test was implemented. The main effect for teaching method (SECTION) was not significant ($Z = -.168$, $p > .05$), with the mean rank for the CH students ($m = 18.71$) not being significant higher than the mean rank of the TRAD students ($m = 18.08$). Table 11 displays a summary of descriptive statistics for the main effect of teaching method on the final examinations scores. Table 10 displays results of Levene's test for homogeneity of variances for the final examination scores. Table 13 displays a summary of the mean rank of the main effect of teaching method on the final examination scores. Table 14 displays a summary of the main effect of teaching method on the final examination scores.

Table 11: Descriptive Statistics for Final Exam

	SECTION	N	Mean	Std. Deviation	Std. Error Mean
Final Exam	CH Students	24	54.92	21.984	4.487
	TRAD Students	12	54.67	13.076	3.775

Table 12: Levene's Test for Homogeniety of Variances for Final Exam

		Levene's Test for Equality of Variances		
		F	Sig.	df
Final Exam	Equal variances assumed	4.442	.043	34
	Equal variances not assumed			32.765

Table 13: Mean Ranks for Main Effect of Teaching Method on Final Exam

	SECTION	N	Mean Rank	Sum of Ranks
Final Exam	CH Students	24	18.71	449.00
	TRAD Students	12	18.08	217.00
	Total	36		

Table 14: Main Effect of Teaching Method on Final Exam

b

	Final Exam
Mann-Whitney U	139.000
Wilcoxon W	217.000
Z	-.168
Asymp. Sig. (2-tailed)	.867
Exact Sig. [2*(1-tailed Sig.)]	.882 [a]

a. Not corrected for ties.

b. Grouping Variable: SECTION

An ANCOVA was calculated to compare the main effect of teaching method on the final examination scores of all students enrolled in Precalculus I at the university who took the department final examination, covarying out the effect of Math ACT. The main effect for teaching method was not significant ($F(1, 212) = .411$, $p > .05$) with the main effect for the CH students (m = 55.05, sd = 22.74) not being significantly higher then all TRAD students (m = 54.54, sd = 18.83). Table 15 displays a summary of the descriptive statistics for the final examination for all Precalculus I students who took the departmental final examination. Table 16 displays a summary of the main effect of teaching method on the final examination scores.

Table 15: Descriptive Statistics for Final Exam (All Classes)

Dependent Variable: FINALEX

METHOD	Mean	Std. Deviation	N
CH	55.0476	22.73648	21
TRAD	54.5361	18.83446	194
Total	54.5860	19.19016	215

Table 16: Main Effect of Teaching Method on Final Exam (All Classes)

Dependent Variable: FINALEX

Source	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	14090.767[a]	2	7045.383	23.079	.000
Intercept	12.269	1	12.269	.040	.841
Math ACT	14085.808	1	14085.808	46.142	.000
SECTION	125.536	1	125.536	.411	.522
Error	64717.391	212	305.271		
Total	719430.000	215			
Corrected Total	78808.158	214			

a. R Squared = .179 (Adjusted R Squared = .171)

Research Question 4

The final research question asked if there was any difference in the conceptual understanding of functions between students taught using constructive habituation (CH) and students taught using a traditional method (TRAD). This question consisted of three component questions. The data for the component questions came from target questions found in both the functions examination and the symmetry examination. These were questions that specifically sought to evaluate the students' ability to understand functions including definitions, examples, and applications, understand, interpret, and model applications of functions and their graphs, and to reify functions.

The first component question asked if there was any difference in the student's ability to define functions and explain their applications. A MANCOVA was calculated covarying out the effect for Math ACT. The assumption of linear correlation of the covariate and dependent variable was not met for either of the three component research questions. Failure of the assumption of linear relationship between Mathematics ACT and defining functions and explaining their applications (dfinexp), graphing and modeling functions and their applications (gphmodel), and reifying functions (reify) indicated that Math ACT as the covariate was not the appropriate statistical analysis procedure. A MANOVA was conducted to evaluate the main effect of teaching method (SECTION) on dfinexp, gphmodel, and reify.

Dfinexp

This component was designed to evaluate the students' ability to define functions and to understand and explain their real-life applications. Students were encouraged to use either their own definitions that demonstrated their unique understanding of functions, or the definition presented in the book. Further, original and innovative real life examples were encouraged that demonstrated their appreciation and knowledge of the applications of functions. This was by far the strongest component for the students with each class correctly answering almost 60% of the target questions. The main effect for

SECTION was not significant ($F(1,47) = .026$, $p > .05$). The scores of the CH students was not significantly higher ($m = 58.91$, $sd = 32.17$) than the scores of the TRAD students ($m = 58.78$, $sd = 27.98$) for defining functions and explaining their applications (dfinexp).

Gphmodel

The graphing and modeling area were combined into one category. The problems that involved modeling also involved either graphing a function or at least interpreting the graph of a function. The problems that made up this component were intended to assess the students' ability to graph functions, interpret graphs of functions and their applications, and model functions and their applications. With respect to the graphing aspect of the problems, the CH students tended to do slightly better with graphing applications of functions than did the TRAD students. But overall, all students tended to do better in the graphing area than in modeling. Modeling the applications of functions proved to be an obstacle for students in both classes. The main effect for SECTION was not significant ($F(1, 47) = .470$, $p > .05$) with the scores of the CH students not being significantly higher ($m = 27.33$, $sd = 21.66$) than the scores of the TRAD students ($m = 22.19$, $sd = 22.34$) for graphing and modeling functions and their applications (gphmodel).

Reify

This component was constructed to measure the students' ability to build on one level of understanding of a mathematical idea and to incorporate it into a higher level mathematical idea. Although not proven, the researcher believed that a student who was able to reify functions probably possessed an exceptional understanding of the concept, its' applications, and its' different representations. As with the previous component, this component also proved difficult for both classes. Neither class was able to correctly score more than 30% on the target questions. In fact, even students who passed the course with an "A" or "B" did poorly on these target questions. The main effect for SECTION was not significant (F(1, 47) = .069, p > .05) with the scores of the CH students not being significantly higher (m = 28.57, sd = 34.50) than the scores of the TRAD students (m = 25.00, sd = 25.72) for reifying functions (reify). Table 17 displays the descriptive statistics for the components of conceptual understanding of functions and their applications. Table 18 displays the summary of the main effect for teaching method (SECTION) on the components of conceptual understanding of functions and their applications.

Table 17: Descriptive Statistics for Component Concepts

	SECTION	Mean	Std. Deviation	N
DFINEXP	MATH 135-01	58.9107	32.17229	28
	MATH 135-02	58.7778	27.98173	18
	Total	58.8587	30.27919	46
GPHMODEL	MATH 135-01	27.3393	21.66199	28
	MATH 135-02	22.1944	22.34630	18
	Total	25.3261	21.83199	46
REIFY	MATH 135-01	28.5714	34.50328	28
	MATH 135-02	25.0000	25.72479	18
	Total	27.1739	31.10291	46

Table 18: Main Effects of Teaching Method on the Component Concepts

Source	Dependent Variable	Type III Sum of Squares	df	Mean Square	F	Sig.
Corrected Model	DFINEXP	23.349 [a]	1	23.349	.026	.873
	GPHMODEL	219.086 [b]	1	219.086	.470	.497
	REIFY	66.656 [c]	1	66.656	.069	.794
Intercept	DFINEXP	161228.0	1	161228.0	178.326	.000
	GPHMODEL	27091.576	1	27091.576	58.058	.000
	REIFY	31291.145	1	31291.145	32.363	.000
SECTION	DFINEXP	23.349	1	23.349	.026	.873
	GPHMODEL	219.086	1	219.086	.470	.497
	REIFY	66.656	1	66.656	.069	.794
Error	DFINEXP	42493.498	47	904.117		
	GPHMODEL	21931.618	47	466.630		
	REIFY	45443.548	47	966.884		
Total	DFINEXP	217061.8	49			
	GPHMODEL	52700.750	49			
	REIFY	80000.000	49			
Corrected Total	DFINEXP	42516.847	48			
	GPHMODEL	22150.704	48			
	REIFY	45510.204	48			

a. R Squared = .001 (Adjusted R Squared = -.021)

b. R Squared = .010 (Adjusted R Squared = -.011)

c. R Squared = .001 (Adjusted R Squared = -.020)

Qualitative Analysis

Qualitative data were also collected in this study. The collection of qualitative data served to clarify, strengthened, and augment the results found in the quantitative portion of the study. The qualitative data were collected by the researcher in the form of interviews with students from both classes. This section concludes with a summary of the qualitative data. Integration of qualitative and quantitative results is taken up in the next chapter.

Subjects

There were six students interviewed for this section: three from each class. The students from the constructive habituation class are labeled CH1-CH3. Those from the traditional class are labeled TRAD1-TRAD3. An attempt was made to balance the students based on achievement, ability, and gender. The mean of the Math ACT for the CH students was slightly higher than those of the TRAD students: the mean of the CH students was 18.33 while the mean of the TRAD students was 16.00. Further, the gender make up was not balanced. In particular, there were two males and one female in the CH group while there were two females and one male in the TRAD group. The class grades were balanced with two students receiving a “C” grade and one student receiving a “B” grade in both classes.

The students’ ages ranged from 19 years old to 21 years old. All students reported that they were in their sophomore year of college. Since each of these students received passing grades in their respective class, they do not represent the full complement of students (or their grades) that completed the course. The students who participated in the interviews were volunteers. No student could be forced to participate in the interview process and hence the researcher was limited in his choices for this part of the study. It is worth noting that other students were asked to participate in the interviews but were either unwilling or unable to take part. The Table 19 gives some important information concerning the students.

Table 19: Student Descriptions

Student	Gender	Age	Math ACT	Grade in Class
CH1	Male	19	15	C
CH2	Female	19	19	B
CH3	Male	19	18	C
TRAD1	Male	19	18	B
TRAD2	Female	19	15	C
TRAD3	Female	21	15	C

Procedures

Every effort was made by the researcher to be impartial and to treat the students equally. Care was taken not to influence their responses or otherwise give any verbal or physical signal, hint, or indication concerning the correctness or incorrectness of their responses. The researcher attempted to stay on target with the questioning and within the 30 to 45 minute time frame. Students were challenged to verify the correctness of their responses. Only when the students had resigned to or were otherwise satisfied with their responses, did the questioning proceed. No student was discouraged from returning to a previous response if he or she later decided to reconsider it.

The interviews were audio taped for later analysis and to help insure that the interview analysis were accurate and captured the important element of the interview session. The researcher also took notes during the interview to capture certain observations that could not be captured on audiotape. The interviews were selectively transcribed and can be found in the appendix. The responses of each student to each interview question were carefully examined. These responses were compared to and balanced against the responses of the other interviewees and later, the quantitative data.

Each interview began with the students being asked to read and sign an interview consent form. Next the researcher read the introduction to the interview to the student. This was followed by the interview questions. There were three primary questions. The first two questions were complemented by a set of probes designed to give better insight into the three targeted areas.

Interviews

This study used the standardized open ended interview format. Patton (1990) suggests that when it is desirable to have the same information from each person interviewed, this interview format may be used in which each person is asked essentially the same question. The basic purpose of this type of interview is to minimize interviewer effects by asking the same questions of each subject. The interviews in this study consisted of a set of questions carefully worded and arranged with the intention of taking each subject through the same sequence and with essentially the same words.

All interviews were conducted within a three day period toward the end of the semester. Each class had completed the Functions, Graphs, and Models regular examination and the Symmetry and Transformation of Functions regular examination. The interview questions targeted three specific areas: defining and explaining functions and their real life applications, graphing and modeling functions and their applications, and reifying functions and their applications. These targeted areas were found in questions in the two regular examinations. The researcher believed that these areas were fundamental in evaluating the students' overall understanding of functions and hence were the best candidates to help strengthen and support the quantitative results.

The following is an inventory of the interview questions and the details of the information gathered from these questions. See Appendix C for the complete description and listing of the interview protocols, questions, and probes.

Question 1

> I am sure you are aware that after the prerequisites skills examination that we have worked a lot with functions. Can you tell me what a function is in your opinion?

Probes:

a. Can you give me a real life or everyday example of a function?

b. In everyday life do think that functions are important? Why?

c. How can functions be represented?

d. Ask about domain and range if necessary.

Question 1 and its probes were given verbally to the students. This question was designed to explore the students’ formal and informal understanding of functions, and their real life applications. Bolstering the quantitative results, this area was the strongest area for each class.

CH Students

Each of the CH students was able to give a definition of function. CH1 and CH2 recited the definition that was given in class and in the book. This played a part in helping the researcher determine if they understood the function concept. Only CH3 made an effort to give a definition that was somewhat original. He struggled to organize his thoughts and to make his definition fit that which was given in class but with some sense of it being in his own words. He finally gave a definition that was a mixture of his opinion and the formal definition.

Because the students could not or did not give a definition of function in the own words, it was difficult to determine if they really understood the idea of function. Therefore, the first probe played a very important role. It asked the students to give a real life or everyday example of a function. The probe served to help the researcher determine if the students really had a grasp of the function concept.

The CH students gave various real life examples. CH1 gave an example that matched one toothbrush to one person's teeth. Here is a portion of his interview that addressed this question. The researcher is identified by the abbreviation RE.

RE: Can you give me a real life or everyday example of a function?

CH1: [Repeats the question]. A real life example?

RE: Yes, something that you might see everyday. Remember at the beginning of the semester we talked about real life functions and I had you guys give some examples?

CH1: Yeah, I remember. One example could be a person teeth and a toothbrush.

RE: What do you mean?

CH1: Everyone has their own toothbrush. So, there is one toothbrush to one person's teeth.

RE: Okay, well what would be the domain and the range in your example?

CH1: The domain would be the toothbrush and the range would be the person's teeth.

RE: In everyday life do you think that functions are important.

The example shows that CH1 has some appreciation for the one to one nature of functions. It is not clear if he understood, in the spirit of his example, that one toothbrush could not be used by multiple persons or multiple toothbrushes could be used by only one person. CH3 gave an example that showed that he had a similar understanding of the definition. But his example was more specific than CH1's example. The following is a portion of the interview.

RE: I am sure you are aware that after the prerequisites skills examination that we worked a lot with functions. Can you tell me what a function is in your opinion?

CH3: Yes sir, a function is [pauses and thinks] a given set . . . [frowns].

RE: Sounds like you're trying to remember the definition from class. Tell me what you think a function is.

CH3: A function is a set of two intervals where each element in the first set is matched to exactly one element in the second set.

RE: In your definition you said that a function is a set of intervals. Which is it, a set, interval, or a combination?

CH3: [Confidently] It's a combination of both.

RE: Can you give me a real life or everyday example of a function?

CH3: Say for instance when I get up to go to church and I'm getting dressed I always match my dress shirt with the same color tie. I can't match my dress shirt with two different [colored] ties.

CH3 had a little difficulty remembering the exact definition. He attempted to give his own opinion but muddled it with the formal definition. His example, however, demonstrates that he understood the essence of the definition. He understood the nature of the one-to-one relationship of the elements of the two sets that was stressed in the formal definition. His final comment indicates he understood the one-to-many relationship, implied in the formal definition, which does not represent a function.

In class, as an everyday example of a function, the researcher matched different brands of corn to different prices. Those same brands were then matched to the same price. As a counter-example, the researcher matched one can of corn to two distinct prices. The researcher had also discussed the idea of functions as two variables where was one dependent on the other and had given a real-life example of such.

CH2 gave the most detail. She gave both an example of a function and a counterexample. She recited the function example that was given in class. Her counter-example was to give a set of ordered pairs and show an element from the domain being matched to more than one element in the range. The counter-example was more of what one might see in the mathematics classroom. It did not fit the everyday criterion that was requested in the question, yet she had more explicitly demonstrated her understanding.

The students were most comfortable using the book definition which involved matching each element in the domain to exactly one element in the range or they found it easy to memorize and apply. The students used this idea of matching in their real life examples. Although the examples were correct they maintained a convenient closeness to the example presented in class and did not reveal any original or creative thinking on the part of the students.

The next probe asked if the student felt functions were important in our everyday lives. All of the CH students felt that functions were important in our daily lives. CH3 suggested that functions were important in our everyday lives because they help us maintain control. He felt that functions help us maintain some order and organization in our day to day living.

Probe C examined the students' ability to identify different representations of functions. The CH students were able to talk about functions in more than one representation. The CH students agreed that functions could be represented in more than symbolic notation. The importance of being able to understand different representations of functions had been stressed to the students during the course of the semester. The CH students talked about functions as graphs, charts, numbers, and sets of ordered pairs. This was a crucial development for both the students and the teaching method. The students had to be aware and able to talk about the different representations in order to make the connections and to demonstrate the influence of the teaching method.

The final probe examined the students' understanding of domain and range. The students were able to give reasonable answers when asked about the domain and range. CH1 talked about the domain and range in terms of his real life example. CH2 first defined an ordered pair and explained that the x coordinate was the domain and y coordinate was the range. CH3 describe the domain and the range as independent and dependent variables and in terms of the real life example he had given. The CH students varied discussions of domain and range showed the extent of knowledge concerning this topic. It was important that the students be able to relate domain and range to their own example. The CH students were convincing in the knowledge and understanding of domain and range.

TRAD Students

The TRAD students performed similarly to the CH students. Each of the TRAD interviewees gave the definition of function that was given in the book. Although the question asked them to tell what a function was in their own words, none attempted to give an alternative definition. Likewise, when asked to give a real life example of a function TRAD2 and TRAD3 gave examples that were very similar to the ones presented in class. Only TRAD3 gave a different example in which he matched different pairs of shoes to different sizes.

When asked if functions were important in everyday life, all of the TRAD students agreed that functions were important. Interestingly, both TRAD1 and TRAD2 agreed that functions help us keep order in our lives. TRAD1 proposed that if we did not have functions, there would be confusion in our lives. This idea that functions help us maintain daily order appears to reflect the students' fixation with the one to one characteristic of function.

The TRAD students were limited in their response to the different representations of functions. TRAD1 mentioned that functions could be represented as graphs, percentages, and ratios. When asked to explain and give an example of a ratio or percentage, he could not. TRAD1 was not very sure of his answers. He was comfortable with giving graphs as an answer because we spent a great deal of time with graphs and the graphing calculator in class. TRAD2 appeared not to have a firm grasp of the question. She answered that functions could be represented in everyday life using different products. Further, she said that functions could be represented by definitions and examples with numbers and variables. Yet she was hesitant and appeared unsure in her responses. TRAD3 could only give graphs as a representation of functions.

The TRAD students' inability to express multiple representations of functions may be a direct consequence of the teaching method. Traditional teaching does not typically emphasize the different representations and their connections to the overall concept. This type of information is presented in different phases and the students are expected to make the connections.

The students were also questioned about the domain and range. TRAD1 and TRAD3 talked about the domain and range of their particular real life examples. TRAD2 opted to talk about domain and range in terms of the different interpretations like independent variables and dependent variables, input and output, and x and y coordinates. The TRAD students showed that they had a good handle on the idea of domain and range. The TRAD students were exposed to the different interpretations of domain and range. Their responses demonstrated that they could move fairly seamlessly from one interpretation to another and that they had a good understanding of their applications.

Summary of Question 1

All of the students gave or attempted to give the definition of functions that was presented in the book. Only one student, CH3, attempted to give a definition in his own words. He struggled with his own definition and after several failed attempts to coordinate his thoughts with his words, finally gave a definition that combined his own words with the book definition. Each student was asked if they could give a real life example of a function. The students paired shoes and sizes, clothing or outfit combinations and accessories, and grocery items and pricing.

As an in-class example of an everyday function, the researcher presented a real life example of a function. For clarity, the researcher also gave a counter-example of a function. Unlike the definition, several of the students, particularly in the CH class, were not opposed to branching out and attempting to develop their own example. Several of the examples had a product-price relationship, similar to what was presented in class. But the students looked to offer examples that were either original or personal.

When asked did they think that functions were important in everyday life, all of the students said yes. When asked why, the most common response was that functions help us maintain order or control in our everyday lives. The students preferred to give the formal definition of functions which involved one element being matched to exactly one other element. This idea of one to oneness, matching, and order was prevalent in the students' real life examples.

When asked how functions can be represented, all of the CH students said they could be represented in multiple ways including graphics, charts, tables, numerically or as set of ordered pairs. The TRAD students struggled in their responses. TRAD2 said that they could be represented as a definition and in examples with variables. TRAD3 could only give graphs as an additional representation. This was a very weak area for the TRAD students but a convincingly strong area for the CH students.

The CH students and the TRAD students were able to discuss and give examples of domain and range with some authority. The students talked about domain and range in terms of the x and y coordinate in an ordered pair, independent and dependent variables, and inputs and outputs of functions. Further, several of the students could describe the domain and range of their particular real life example. This was a strong area for both sets of students.

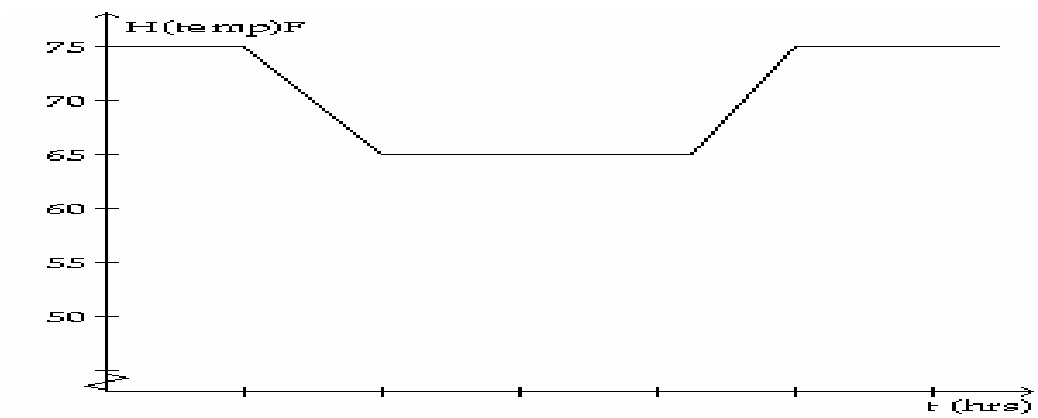
The two groups of students were similar in most areas pertaining to this question. But, overall, the CH students had a slightly broader perspective of functions. The TRAD students showed weakness in expressing the different representations of functions. The CH students were more able to discuss the different representations. Recognizing and understanding the many "faces" of functions, played a crucial role in evaluating the strength of the students' understanding.

Both groups of students were exposed to multiple representations of functions. This difference between the two groups may be directly attributed to the difference in teaching methods. Most of the examples and applications experienced by the CH students in this area were supplemented and reinforced with graphs and/or tables. The amount and type of exposure received by the CH group may have helped distinguish the two groups in an area where, for the most part, they were very similar.

Although the CH students were more adept at articulating the different representations of functions, they fell short of possessing the firm grasp that was expected. Their inability to give a more personalized definition and more creative real-life examples provided evidence of a lack of real depth and comprehensive understanding of functions. The students were given a sheet with a graph and asked to read question 2 aloud. This question was intended to help evaluate the students understanding and ability to model and graph functions and their applications.

Question 2

Question 2 reads as follows:



Note: time scale-0 hrs to 24hrs.

A graph of the university's heating schedule, showing temperature **(H)** in Fahrenheit as a function of time **(t)** in hours is given. The initial time t = 0 represents midnight.

a. Can you write a general formula to represent this graph?

b. Between what times is the building the coolest?

Probes:

a. Graph the function **H(t) – 5**. If the university decides its heating schedule according to this function, what has the university decided to do?

b. Graph the function **H(t – 2**), what has the university decided to do?

c. When you get to school at 8am, will the classroom be cooler under the **H(t)** schedule, the **H(t) – 5** schedule, or the **H(t – 2)** schedule? What will the temperature be?

<u>CH Students</u>

The CH students had difficultly producing a formula that modeled the graph. Only CH2 was able to correctly model the graph. CH3 first expressed time as the dependent variable and temperature as the independent variable. He wrote the following formula: t(H). CH3 was either interpreting the graph or the information provided in the question incorrectly. The question clearly stated the temperature was a function of time. His formula indicated that the time was dependent on the temperature. After thinking about what he had written, he changed his mind. CH3 then accurately articulated the domain and the range for this particular graph and wrote the formula as H = f(t). This formula presented evidence of his understanding. It indicated that he had a sound understanding of the definition of functions and that the temperature was dependent on the time. CH1 had no idea how to write this formula and could not respond to the question.

The problem in the subsequent question involved simply interpreting the graph. The students found this task to be much easier than the previous. Each of the students was able to read the graph and correctly determine the time in which the building would be the coolest. The ensuing probe gave two of the CH students some trouble.

Probe A asked the students to graph the function **H(t) – 5**. If the university decided its heating schedule according to this function, what had the university decided to do? CH1 knew that there was a five unit shift but did not quite know where to shift the graph. He not only shifted the graph five units down but also five units to the right. Clearly, CH1 had difficulty interpreting the formula graphically. He had confused and combined the horizontal and vertical translation formulas. CH2 also had a little difficulty with settling on a graph. But to her credit she graphed, re-graph, and reflected on her drawings and the function until she believed she had the correct illustration. Though this was still not the correct illustration, she later corrected it when presented with probe C. CH3 had no problems connecting the symbols and the graph. He correctly drew the graph without indecision. Despite the difficulty that students experienced interpreting and drawing the graph, each of them was certain that the formula and the graph implied that the university had decided to lower the temperature five degrees.

Probe B examined the students understanding of horizontal shifts. The probe asked the students to graph the function **H(t – 2**). The students had to decide what had the university decided to do with respect to the heating schedule? Again each of the students was able to explain what the function meant. They expressed that the university had decided to begin the heating schedule two hours later. But all had difficulty constructing a graph. CH3 was the only student who could correctly draw the graph. It took him a while to really realize that what he was saying did not agree with what he was drawing.

A portion of the interview in presented here. The researcher is identified by the abbreviation RE.

RE: Okay, number b.

CH3: [Moves the graph four units right]

RE: What has the university decided to do based on your graph?

CH3: They decided . . .they decided to um. . . start the heating schedule two hours later.

RE: Okay when does the heating schedule begin.

CH3: It begins a 4:00.

RE: So if it begins at 4:00 where would two hours later be?

CH3: It should be at 6:00. But it doesn't get cooler until 8:00.

RE: Does your graph show it getting cooler at 6:00.

CH3: Oh, I should have shifted the whole graph over two units. [He renumbers the graph and talks himself through while drawing the correct graph].

CH3 had the instinct to question several of his initial answers throughout the interview, and the insight to correct them. CH1 did not show as much intuition. CH1 read the question and graphed the function. He shifted his previously incorrectly drawn graph two units to the right and then five units down. He erased the five-unit vertical shift and adjusted it so that it was only two units down. From his graph he declared, that the university moved the schedule to two hours later. When questioned if his graph represented what he has said, he emphatically responds, "yes". Once more he had combined the horizontal and vertical translation formulas. CH2 appeared to have the same trouble as CH3. Her graph was shifted two units up. A portion of her graph was shifted two units to the right with respect to the reference graph. CH2 gave an indication that she knew that officials started the heating schedule two hours later because of the minus sign. She stated, "anytime you have a minus here [referring to the formula] it causes a shift to the right". Yet she could not accurately show this horizontal shift graphically. This probe was effective in exposing the students' weakness in this area. Both of these students had memorized the algorithm for the general formula

af(bx $\pm$ c) $\pm$ d but did have a clear conceptual appreciation of its use.

The students were asked in the final probe, under which of the three different heating schedules the classroom would be the coolest at 8:00 a.m. Since the graphs of CH1 and CH2 were incorrect, their understanding of the general formula for translations of functions helped them in determining the correct answer. It was clear that this probe, along with her understanding of the general formula, helped CH2. She was able to figure out that the graph she gave in response to probe A was faulty. She stated the coolest temperature 60^{o}. She observed that her graph did not reflect this fact. She then went back and corrected the graph.

TRAD Students

The TRAD students did better at developing a formula for the reference graph than did the CH students. Each one of the TRAD students was able to give a correct formula. TRAD1 talked through his solution. He brought into play everything he knew about domain and range, and independent and dependent variables, as he convinced himself that he had an acceptable response. TRAD3 took a similar path. She first determined that time was the domain and temperature was the range. She then gave the correct formula.

The TRAD students were also successful when it came to determining when the building would be the coolest. Both classes showed they were proficient in reading and interpreting this particular graph. Two of the TRAD students had problems with probe A. TRAD1 appeared to have no problems understanding the probe and drawing the correct graph. Immediately after reading the question, he confidently drew the graph. TRAD2 appeared to be on her way to giving a correct graph. Only the beginning and ending of her graph were correct. She correctly shifted the graph down between 12:00 a.m. and about 4:00 a.m. and from 8:00 p.m. till midnight. But the 4:00 a.m. - 8:00 p.m. time frame was incorrect. She only shifted the graph down about two units. Oddly enough TRAD3 did just the opposite. The only part of her graph that was shifted down five units was the 8:00 a.m. to 5:00 p.m. time frame. She had assumed that the temperature changed only during "normal" working hours. She had not taken into consideration that the entire graph was representative of the function. All the TRAD students however, were able to explain that the function showed that the temperature was decreased five units.

Probe B proved trouble-free for the TRAD students. They were all able to shift the entire graph two units to the right. TRAD1 and TRAD3 both concluded this meant that the university had begun the heating schedule two hours later. TRAD2 stated that the university decided to cool the building before the employees came to work.

TRAD2 was the only student who was unsuccessful with probe C. She decided that the H(t - 2) heating schedule would provide the coolest classroom. She looked at her graph and reasoned that the H(t - 2) schedule started earlier, even though her graph was clearly shifted to the right. She stated that the reason she chose H(t - 2) was because it began a 6:00 a.m. At this point her previous statement became clear to the researcher.

She had decided that the point where her graph began to change directions was the beginning of the graph. She had not considered the 12:00 a.m. to 6:00 a.m. portion of the graph.

Summary of Question 2

The qualitative data revealed that the students were generally weaker in graphing and interpreting graphs than what was previously demonstrated. In particular, they were weakest in representing formulas graphically. However, the students were strong when it came to understanding and explaining translations in symbolic notation. Modeling proved to be generally difficult for all of the CH students. However, it was shown to be a strong point for the TRAD students.

The students were asked to write a general formula that represented the graph. All of the TRAD students and CH2 were able to give a formula. TRAD1 reasoned his formula through by determining which variable was independent and which was dependent. CH3 correctly reasoned that time was the independent variable and that temperature was the dependent variable but still was unable to produce a correct formula. The students were then instructed to determine when the classroom would be the coolest based on the graph.

All of the students were successful in reading the graph and determining a correct response to this question. The first probe asked the students to graph the function H(t) when the graph was shifted five units down and explain what this meant in terms of the heating schedule. All of the students correctly responded that the university had decided to lower the temperature five degrees in the classroom. TRAD2, TRAD3 and CH1, however, had difficulty producing a correct graph. Each of these students chose to shift only parts of their graph. All seemed to believe the temperature changed only during a certain time of the day. Neither took into consideration the function was represented by the entire graph.

Next the students were asked to graph the function H(t - 2) and explain what it meant in relation to the heating schedule. All of the students understood and clearly stated that the heating schedule changed to two hours later. CH1 and CH2 were the only students who did not ultimately make the connection between their statements and their graphs. Their graphs are shown in the Appendices.

The final probe of question 2 asked the students to determine under which of the three heating schedules the classroom would be the coolest when they arrived at school. Only TRAD2 chose an incorrect schedule. She chose the H(t – 2) graph. Although she produced a correct graph for the function, she produced faulty reasoning in why she chose it. All of the others, including CH1, correctly chose the H(t) – 5 heating schedule. CH1's graph showed that at 8:00 a.m. the temperature would be 70^o. But he, like the rest of the students, correctly stated 60^o. Again, it appeared for CH1 that the symbolic notation and the graph were not connected and shared no relationship. He used his algebraic skills and algorithmic knowledge of translations to produce the correct answers.

Overall the TRAD students demonstrated a better understanding of modeling and graphing then did the CH students. The CH students possessed obvious deficiencies in modeling despite having been exposed to it during the course of the semester. The TRAD students were able to model the original function. They applied their knowledge of domain and range to construct the formula. In particular, they appeared to use an algorithm that identified the independent and dependent variables within any general function. They were able to substitute in the algorithm to produce the correct model.

Although the TRAD students did a better job, both groups had some difficulty interpreting the symbols graphically and interpreting the graphs symbolically. Yet, the students had little difficulty interpreting and explaining what the symbols implied. The excerpt from the interview with CH3 and the comment from CH2 revealed why the students were so efficient in the latter.

Both students correctly stated the university had decided to start the heating schedule two hours later. Yet CH3 shifted his graph four units to the right. After being questioned by the researcher and comparing his verbal response with his graph, he changed his answer. CH2 was more direct in her response. She stated that she knew the heating schedule started two hours later because of the minus sign.

It seemed that for most of these students, CH and TRAD, the appearance of their conceptual understanding was an illustration borne of their skill at applying the **af(bx $\pm$ c) $\pm$ d** algorithm. This conclusion seems more logical when evidenced by their difficulties in reconciling the symbols and graph.

Question 3

Finally the students were presented a problem that examined their ability to reify and model functions. Question 3 asked:

> Suppose the original heating schedule is represented by the formula, H = f(t). Suppose the graph is shifted 5 units upward. This new schedule is represented by the formula H = q(t). How are the formulas f(t) and q(t) related? Can you write this relationship algebraically?

CH Students

Crucial to examining the students' ability to reify was their understanding that the two functions were in fact related. The reification process could only begin with the students' recognition of the relationship between the two functions. CH1 believed that there was no relationship between the two functions. Hence, he did not attempt to write an algebraic relationship and ended his interview.

CH2 observed that the two functions were related. She set H = f(t) and H = q(t) + 5. She stated that she thought the functions were related because the graph was simply shifted up five units. As previously mentioned, CH2 understood translations because she had memorized the algorithm for the general equation for the translation of functions. She did not have a clear conceptual appreciation for the formula. She set H = f(t) and H = q(t) + 5. She did not proceed to show the connection between the two formulas.

CH3 had excellent appreciation for the problem and a strategy to work through it. He used the law of transitivity to complete the problem. CH3 was very deliberate in his thinking. The following excerpt from his interview provides some insight into his reasoning and comprehension of the task.

RE: Okay, let's move on to the next question.

CH3: [Reads question and begins to write formulas]. Okay, ah, the formulas f(t) and q(t) are related because f(t) is the basic function. The heating schedule is given by f(t) and it changes from f(t). Making f(t) hotter 5 degrees gives you q(t).

RE: On your paper I see you wrote H = f(t) + 5 = H = q(t). Why did you write the H's.

CH3: Because H is the heating schedule and H = f(t) and they want to change the heating schedule by moving it higher 5 degrees. And so in order to do

> that you would have to add 5 to the function, to the outside of the function cause that would cause a vertical shift. And that would give you H = f(t) + 5. And they said they wanted that new formula to be represented by q(t). So q(t) = f(t) + 5.

RE: But that's not exactly what you wrote down on your paper.

CH3: Okay, f(t) + 5 and q(t). The first part f(t) is the reference function so you should go from there. [Works through his paper for a while and finally writes f(t) + 5 = q(t)].

This type of reasoning was scarce throughout the interviews. Reification involves a particular level of abstraction. The modeling area of the previous exercise also involved a certain level of abstract thinking though not as involved as reification. Because this group of TRAD students had done so well in the previous modeling exercise the researcher anticipated that they might be successful in this task as well. This was not quite the case. Only one TRAD student was able to reify the function and give a convincing explanation suggesting a genuine understanding.

TRAD Students

The TRAD students' results of this portion of the interview were as mixed as those of the CH students. TRAD1 spent a great deal of time pondering this question. He wrote down several different solutions he thought were plausible. He knew that the functions were related and attempted to combine them. In his initial attempts he tried to add the two functions. He eventually produced the correct response. But he was not convinced it was correct. He attempted another formula after having given the correct one. In the end he was still not sure which formula was correct.

TRAD2 took the same view as CH2. She decided the two functions were unrelated. She did attempt to create an equation. She combined the two functions in a manner similar to the initial attempts by TRAD1. She wrote H = f(t) + g(t) and H = f(t) +

g(t) + 5. She offered that the two functions had the same input (referring to time (t)), but had different outputs. She appeared to be more concerned that she could not logical combine the two functions. She concluded that the two were not related.

TRAD3 established the H = f(t) + 5 and H = q(t). After several failed attempts at formulas she felt were possibilities, she decided to return to her initial set of equalities. She determined that since both functions were equal to H, then they should be equal to each other.

Summary of Question 3

This question presented the greatest challenge to the students. The students spent more time reading, re-reading, and analyzing this question than any of the other questions or probes. The students' recognition that the two functions were in fact related was essential to interpreting their ability to reify functions and shaping their response to the second question. A negative response to the first question demonstrated to the researcher that the students had very little conceptual understanding of functions and probably had been able to manage the course by memorization of facts and algorithms. Although this may have been true for most of the students it was clearly evident in the response for both CH1 and TRAD2.

CH1 and TRAD2 both believed that there was no relationship between the functions. However, TRAD2 did attempt to write the functions algebraically. CH1 was satisfied the functions were not related and elected not to attempt an algebraic solution. CH2 believed the two functions were related but could not manage to find a proper representation for the combined function. TRAD1 believed the two functions were related and spent a great deal of time writing down possible solutions and crossing out those that did not seem reasonable to him. TRAD1 eventually obtained the right response, but still questioned whether it was correct.

CH3 and TRAD3 were the only students who were both confident that there was a relationship between the two functions and in their solutions. Interestingly, both students used the same idea of mathematical transitivity to solve the problem. Both established a new function, H = f(t) + 5, and equated that new function to the given function H = q(t). Each then deduced that f(t) + 5 was equal to q(t). Again, both students were extremely confident that they had found the correct relationship.

Overall, there was no obvious difference between the two groups of students when it came to reifying functions. Reification requires a rather high level of abstraction. Further, it requires a more comprehensive understanding of the underlying concepts. It appears from the two previous questions that the students did not possess this type of understanding of functions. It seemed they were more involved with memorization of formulas and algebraic manipulation. Reification requires much more.

Summary of Qualitative Data

Question 1 and its probes revealed of a lack of real depth and comprehensive understanding of functions by the both groups of student. Although requested to do so, none of the students gave their own interpretation or opinion of a function. Several of the students attempted to be innovative in giving a real life example of a function. Though most of them gave examples that possessed a product-price relationship, similar to what was presented in class, efforts to be original were visible. The CH students were more adept to discussing multiple representations of functions. The TRAD students struggled to demonstrate they understood that there were actually relationships between the different representations.

The results for question 2 which involved the modeling and interpretation of graphs revealed mixed results. Overall the students were not as strong in the graphing and interpretation of graphs as previous data suggested. Most of the students were able to adequately interpret the reference graph and explain what it represented. However, while all of the TRAD students were able to model the original graph, only one of the CH students could.

All of the students were able to adequately interpret formulas that accompanied the application problems. The researcher observed that several students had memorized algorithms which helped produce some success in this area. This was probably the case with more students. The fact that students from both groups had a great deal of difficulty producing graphs that correctly matched the models provides some support for this argument.

The final question involved evaluating the students' ability to reify. On the whole this was the most difficult area for the students. Two students, one from each class, were totally unable to reify the given functions. Likewise, an equal number of students from the two classes suggested that the two given functions were related but was either not sure or could not give a correct graph. Similarly, the remaining two students were able to reify the functions and give a proper solution. Interestingly, both of these students used the same logic in reifying the functions.

The students' apparent reliance on their ability to memorize procedures and algorithms was ineffective in this area. Traditionally, students in this course are exposed to more concrete ideas and examples. The fact that reification requires a certain level of abstraction and that this level can be at times very difficult for students to achieve may provide a partial reason for the students’ apparent limitations in this area.

CHAPTER 5

DISCUSSION AND CONCLUSIONS

Summary of Research Objectives and Design

The purpose of this study was to examine if constructive habituation was a more effective means of helping students reach process-object reification than a traditional teaching method as evidenced by achievements levels on in-class examinations. This goal was addressed through four research questions. Each question played an indispensable role in the evaluation of the general research question.

Questions 1 and 2 evaluated the data from the two regular examinations. These data presented a wide picture of the students' overall understanding of functions. The data from these examinations covered a spectrum of implicit and explicit ideas that shed light on the students' understanding of functions, their transformations, and their applications.

Question 4 probed more deeply the students' understanding. The data for question 4 were extracted from target questions contained within the two regular examinations. As previously mentioned, the two regular examinations gave us an overall look at the students' knowledge and understandings related to different notions of functions. The target questions from research question 4 were more focused. These questions strictly addressed concepts the researcher considered crucial in evaluating the students' understandings and appreciations of functions relevant to the reification process. These ideas included: defining and understanding functions their applications, modeling and graphing functions and their applications, and reifying functions.

Question 3 evaluated the students' achievement on the final examination. This question served to determine if the CH students could achieve comparably to the TRAD students and to all other Precalculus I students on an examination that was developed under a more traditional teaching philosophy. Finally, qualitative data were used to support and clarify the results of the quantitative data. Though the qualitative data had implications for questions 1 and 2, the interview questions were somewhat more oriented to supporting data about students' understanding of function concepts that were directly related to categories analyzed in question 4.

The study included all students who were enrolled in Precalculus I at Southern University at Baton Rouge during the spring 2004 school semester. However, two classes taught by the researcher were the central focus. One class was taught using constructive habituation and the other class was taught using a traditional teaching method. The constructive habituation class used a new idea called "multi-Reps" (multiple repetitions of multiple representations of concepts) as its' pedagogical device. Mathematics ACT scores were used to determine if any significant differences existed between the two classes as well as all of the Precalculus I students, prior to the beginning of the study. The following is a discussion of the relevant findings and implications of this study.

Conclusions

Overall constructive habituation was not a more effective means of helping students reach process-object reification than a traditional teaching method. The results of the analyses revealed that there was no statistically significant difference in achievement between the two groups of students. The teaching methods appeared to have little consequence to the outcomes of the students' understandings of functions.

Through the analysis of the aforementioned four research questions, four explanations or themes have emerged as to why the constructive habituation did not achieve its intended effects on the students' learning. These explanations are:

a) Students' concept definition was restricted by their concept image brought on by a lack of understanding of the conceptual underpinnings of definitions. Students were able to give definitions but could not apply them in meaningful ways;

b) Students were predisposed to memorization of procedures and algorithms. Students failed to develop meaningful understandings because they concentrated more on the mechanics of algorithms of formulas than on the underlying meanings of the formulas;

c) Students compartmentalized and failed to integrate knowledge. Students failed to make the connections between the different representations. In particular, students found it difficult to transition from an application to a symbolic representation and a graphical representation;

d) Reification is inherently difficult. Students may not have objectified lower level process which is a prerequisite for higher level processes.

The following is a discussion of these conclusions.

Conceptual Underpinnings of Definitions

There was no significant difference found between the two groups of students when the target questions evaluating the ability to define and explain functions were analyzed. The qualitative data supported this result. This was the strongest area for both groups of students. The majority of the students were able to define functions and determine if certain correspondences or relations were functions. A central issue important to teaching and learning functions involves the definition used by teachers and textbooks.

The textbook in this course presented a standard formal definition which defined a function in the following manner: Given two sets A and B. Each element in set A is matched to exactly one element in set B. Some educators claim modern definitions like this one are too formal and abstract for students (Tall & Vinner, 1981). Drefus and Vinner (1982) suggest that the historical definition, functions as a relationship between variables, is more relevant to students as it takes advantage of their prior intuitive notions on functions.

The quantitative data showed that students in both groups were able to recite the formal definition and use it to give real life examples. The students were correct about 60% of the time on questions that pertained to defining and explaining functions. The qualitative results support this finding. Almost all of the students who participated in the qualitative portion were comfortable with the formal definition and able to give a real life example. These findings demonstrated the students had developed an adequate but limited understanding of the formal definition of functions regardless of the teaching method used.

However, there are subtleties in this particular definition which students did not grasp. The definition appears to state and stress the idea of one-to-oneness; each (one) element in the domain matched to exactly one element in the range. However, the definition actually includes many-to-one relations, as well. That is to say, different elements in the domain can be match to the same element in the range; in fact, every element in the domain can be matched to the same element in the range. Students tended not to pick up on these details. Students were inclined to interpret the definition as implying that each element in the domain was match to its own separate and distinct element in the range. Their inability to grasp this detail was revealed during the qualitative analysis. They were very much dependent on the matching aspect of the definition to develop their real-life examples. Understandings and descriptions of functions as relationships between variables, dependent relations involving input and output, and as cause-and-effect situations were not produced by the students.

When asked to give real life examples of functions, students were much more original and creative in their responses on the examination than in the interviews. The interviewed students did make an effort to give original examples. Yet, the majority produced product-price related examples similar to that which was presented in class. The responses given on the examination revealed a wide range of real life applications and personal experiences for the students including sports, automobiles, and relationships. But most of these examples still remained in the "correspondence" category as suggested by Vinner and Dreyfus (1989). Many students were fixated on the one-to-one aspect of the definition and failed to acknowledge the implicit many-to-one characteristic of the definition.

The quantitative data indicated that both groups of students appeared to have some understanding of the definition of domain and range. The students could give the domain and range of an ordered pair, and could name the domain and range of an example that clearly illustrated a one-to-one correspondence. The qualitative data supported this conclusion. In her interview, CH2 stated, "in ordered pairs the domain is *x* and the range is *y*", when asked about the domain and range. CH3 talked about domain and range in terms of independent and dependent variables as well as naming the domain and range of his particular real-life example. What was not made initially clear through the data analysis was that although the students could in essence define and name the domain and range, they could not apply the idea in real life examples and give numerical answers.

In the application questions that required the students to determine a physical or numerical domain and use this domain to obtain the range, the students were extremely unsuccessful. One possible reason for this lack of success for some students was that they could not produce a formula for the application. Therefore it would have been impossible to produce the accurate domain and range. But, even when students were explicitly given the formula and domain of an application problem, problem # 14 for example, many students still failed to produce a correct range.

It appeared that students had developed a more one dimensional interpretation of this definition just as they did with the function definition. Students were able to recite the definitions and use them in limited context. Meanings and understanding of these definitions were to a certain extent superficial and shallow. Tall and Vinner (1981) suggests this type of problem exist because students total cognitive structure associated with a concept - "concept image" may not be coherently related to definition of the concept - "concept definition". They suggest the concept image may develop into a more restricted notion. At the same time the concept definition may largely be inactive in the cognitive structure. Initially students are able to work well within this limited context but incur significant problems when the concept is defined or presented in a broader context. Deeper understanding of the concept definition is required to deal with this problem.

Edwards and Ward (2004) suggest that instructors generally assume that if a student can accurately state and explain a definition, then they understand it. Their research proved otherwise. Based on the nature of the examples of functions the students gave in this study and on their inability to deal with applications of domain and range it is apparent that the students in this study failed to develop deeper understandings of the conceptual underpinnings of their definitions.

Memorization of Algorithms

Much of the quantitative data in support of the second conclusion that students focused on the mechanical application of skills were obtained from the symmetry examination. It may be helpful to give a brief review of this examination. The symmetry examination measured the students' understanding of the relationships between changes made to the graph of a function and changes made to its formula. The quantitative results showed that there was no significant difference in achievement between the two classes as measured by this examination.

The symmetry and transformation topics presented the liveliest discussion and greatest interest in both classes. The interviews reflected similar responses of the CH and TRAD students on the examination. The CH and TRAD students experienced the full array of representations of functions and their transformations. They were expected to make the connections between the algebraic representation and the graphical representation. Many students, especially the TRAD students, were particularly concerned with knowing the mechanics of the transformations. For example given the equation $f(x) = 4(x - 3)^2$, these students wanted to know that 4 caused the function to stretch vertically, the "minus" caused a horizontal shift right, and the 3 indicated that horizontal shift was three units. The qualitative data indicated that the CH students also focused on these processes. This emphasis on mechanics led to memorization of facts and not understanding of concepts.

Two general observations can be made concerning these students who relied so heavily on mechanics: a) students from both groups could explain in words how the transformations affected its graph, but many could not actually correctly graph the new transformation and b) some students became confused when the problem consisted of both vertical and horizontal translations.

When given reference formulas students were extremely successful when asked to explain how a particular transformation was obtained from the reference formula. CH2 correctly concluded that the temperature dropped five degrees when given the formula $H(t) - 5$ and time was changed two hours later when given the formula $H(t - 2)$. However, she acknowledged that she knew this because she had memorized the transformation algorithm. TRAD2 stated that she knew her answer to the transformation problem was correct because she remembered it from class, an indication that she had also memorized the transformation algorithm. The students could explain transformations presented symbolically. But, they had difficulty correctly graphing transformations of symbolically presented functions.

The students had particular problems graphing transformations when the reference graph was of the general nature and formula was given by, say, $f(x)$.

Problem #9 on the symmetry examination (Appendix B) is an example of this type of problem. This difficulty was also reflected in the qualitative data. Each of the students except TRAD1 initially failed to draw the graph that correctly represented the given formula. CH2 and CH3 were later able to correct their graphs. Both the quantitative and qualitative data showed that the students were able to describe transformations given as formulas but were not able to accurately graph these formulas. These results support the conclusion that the students had memorized the transformation algorithm. A problem with memorization without understanding is that facts are more easily forgotten or confused. The following paragraph discusses this observation.

The second observation made of students who relied on mechanics was that they became confused when the problem consisted of both vertical and horizontal translations. Sometimes, recalling the symbol or value that caused a particular action was problematic for these students. When asked to draw the function $H(t) - 5$, CH1 shifted his graph five units down and five units right. He then shifted that same graph two units right and two units down when asked to draw $H(t - 2)$. He knew the minus sign caused a vertical shift down and a horizontal shift right. But he made no distinction between $H(t) - 5$ and $H(t - 5)$, and $H(t - 2)$ and $H(t) - 2$. CH2 had relied on memorization of the transformation algorithm. His lack of understanding left him with no way to critique or to correct his imperfect recall.

Algorithmic manipulation is an area that receives a substantial amount of attention in traditional teaching. Students under traditional pedagogies spend a great deal of time learning symbol manipulation skills. Yet, many students never master these skills. Furthermore, many develop neither an appreciation for, nor a practical understanding of, the concepts that underlie these skills. Tucker and Leitzel (1994) maintain that students mindlessly implement symbolic algorithms with no understanding. The ability to manipulate the transformation algorithm is important. Still, a conceptual understanding of transformations may have led to less confusion on the part of the students.

Integration of Knowledge

Both the CH and TRAD students were presented the ideas related to functions from several different perspectives. The CH students received this as a sort of “whole concept” package. They were given definitions and multiple examples using different representations. They were also given application problems in class in concert with tables and graphs that were intended to make their learning experience clearer and less confusing. However, overall, they did not perform any better than the TRAD students who experienced a more sequential and compartmentalized traditional approach to their in-class lessons.

Early in the semester some of the CH students express some concern with the new teaching strategy. A number of students found having to deal with the different representations all at one time, slightly overwhelming. Some students found it difficult to make the conceptual transition from one representation to another. Other researchers have noted this problem as well. Schwartz, Dreyfus, & Bruckheimer (1990) suggested that educational research has identified the transition between tabular, algebraic, and graphical representations as an example of what makes the function concept difficult for students to fully understand. Several students in this study had been so programmed by traditional teaching that they asked the researcher to just show them “how to do the problem". They did not realize that they were being shown how do the problem but with a method that gave them several options and avenues to approaching the problem.

Overall, the students' understanding of function concepts was poor. The mean scores of functions examination for the CH students and the TRAD students was 61.30 and 59.71 respectively, and 55.58 and 45.89 respectively for the symmetry examination. These scores indicated that neither class had a good overall grasp of the introductory function concepts or the symmetry and transformations concepts. Students had problems in many areas of these examinations. In particular, students from both classes had difficulty with the application problems.

The students could not smoothly transition from an application to any of the other representations. Students performed extremely poorly, for example on question #11 of the functions examination (Appendix A). The problem involved representing the given application as a formula, determining independent and dependent variables, and determining domain and range. No student, in either class, was able to successful complete all aspects of the problem.

The quantitative data also showed that students in both classes had difficulty with application problems that involved modeling functions. Many students left these question unanswered on their examination forms and moved on to parts of the question that did not require the formula to answer. Students were quite simply unable to interpret the application problems and model them with formulas. The qualitative analysis suggests that this must have been especially true for the CH students. Only one of the CH students was able to produce a correct model of the "heating schedule" problem when interviewed.

As previously discussed, several students had admittedly memorized algorithms. The TRAD students were particularly concerned with the algorithmic features of the formula. They wanted to know what symbol in the general formula caused a specific translation. This attention to the algorithmic nature of the formula suggests an explanation as to why they were more successful at producing formulas of graphs and their translations in the qualitative section.

In general, mathematics students have conceptual difficulties with different representations of functions. Furthermore, as supported by these results, they have trouble with the transition and connections between these different representations. Schwarz et al. (1990) suggests that the process of transferring knowledge from one representation to another is beyond most students. They suggest that students exhibit a "lack of integration" of knowledge. Included in this is a compartmentalization of knowledge, a lack of transfer between representations, and a dissociation of procedural and conceptual knowledge. The results suggest that the students in this study exhibited similar characteristics.

Inherent Difficulties of Reification

Reification involves the construction and objectification of abstract, symbolic, or conceptual entities from algorithms, procedures, or other lower level mathematical processes. These new mathematical objects can then be used to operate at higher levels of mathematical understandings. This question was intended to examine the students' levels of abstraction in dealing with functions as objects of higher level processes. The quantitative analysis revealed that there was no significant difference between the CH and TRAD students' ability to reify functions and their applications. The qualitative data supported this finding. The analyses suggested that this was by far the most difficult component for both the CH and the TRAD students.

The students averaged below thirty percent on the questions pertaining to reification. Very few of the students could use one or more functions to construct a new function, an activity that would have demonstrated a higher level of understanding of the abstract concepts. This result is characteristic of students' challenges with reification. Sfard (1989) suggests that reifying is an extremely difficult process and that very few students ever obtain this conceptualization of functions. Both classes had been exposed to abstract ideas in class and as homework problems. Students were given generalized graphs and formulas in the class periods that symmetry was discussed. They were given exercises where they matched the different generalized graphs to the correct formulas.

Difficulties with reification reported in the quantitative section were also observed in the qualitative data. Only two of the students, one CH student and one TRAD student, were able to successfully complete and justify their responses to the question that involved reification. It appears that the teaching method had no effect on the students' ability to reify functions. The problems that dealt with reification required that the students see lower level processes as objects so that they could be used to objectify higher level processes.

Symbolically represented algorithms and processes that are performed take on dual roles. These roles suggest either the process itself or the product of the process. I will use two examples from chapter two to more fully explain. The equation $2x - 1$ represents both the process "subtract *1* from the product of *2* and *x*" and the product of the process. The function $f(x) = 2(x + 1)^2 - 7$ tells both how to calculate the value of the function for a particular value of *x* and in effect summarizes the complete concept of the function for the general value of *x*. Tall (1992) suggests what makes mathematical thinking so powerful is the flexible way in which this conceptual structure is used. By using symbols to evoke a process, it can be used to compute a result. By thinking of this lower level process as a mathematical object it can be used as higher level process. This is the essence of reification.

Sfard (1991) suggested the lower-level reification and the higher level-interiorization are prerequisites for each other. This is the inherit difficulty of the reification process: the transition from the operational stages to the structural stage. This transition requires students to make an ontological leap that for many students can be impossible. Those students who are able to reach certain conceptual levels, at some point, reify mathematics processes. Students, who are able to make this ontological leap and reify processes, actually simplify their mathematical understandings. These students are probably more likely to move with more confidence into higher levels of mathematics. Obviously, the process is very difficult to achieve. The fact that this process is so difficult may account somewhat for the lack of statistical significance among the two groups of students in this component of the study.

Discussion

The discussion of the results of this study is focused around an additional explanation of constructive habituations' failure to reach its intended goal. This discussion puts into context constructive habituation's place within present and future literature concerning strategies for teaching mathematics and process-object reification.

The next explanation that will be considered concerns the location of constructive habituation within the reification process. To fully understand this issue it is necessary to revisit the discussion of the students' abilities before the implementation of the study and the discussion of the stages of the reification process.

Recall that the students' mathematics ACT scores revealed that these students were generally weak mathematics students. The mean mathematics ACT score for the CH students was 16.83 (sd = 2.81). The students were also given a pretest that evaluated their fundamental algebra skills. The purpose of the pretest was to determine in what specific areas the students were most deficient and should receive assistance. The pretest covered basic algebra topics including: integers, order of operations, polynomials, factoring, and rational expressions. The pretest analysis revealed a mean score of 36.48 (sd = 17.55) for the CH students. A correlation matrix revealed that the mathematics ACT scores and the pretest scores were highly correlated. This gave us some assurance that we were looking at an accurate picture of the students' pre-study mathematical abilities.

Next, it is important to review the three stages of process-object reification. Sfard (1991) defined these three stages as interiorization, condensation, and reification. Interiorization is the stage where students' perform operations on lower level mathematical objects. As students become familiar with the process then they can think of the outcome of the process without actually carrying out the process. Condensation is the stage were complicated processes become easier to think about. In this stage, the new concept is actually born. As long as this new concept is connected to an algorithmic process the student is in condensation. Goodson-Espy (1998) suggests that one important facet of the condensation phase is the learner's increasing ability to alternate between different representations of a concept. Reification is the stage where the student can conceive a mathematical concept as complete object with its own characteristics.

The analysis of the results of this study shed new light on the developmental stage at which constructive habituation may become effective as an instructional method. The point of departure for constructive habituation within the reification process is condensation. An important precondition for constructive habituation is that the student is no longer in the interiorization stage. Furthermore, constructive habituation assumes that the student is not in the early phases of condensation: the phase where the new concept is actually born. Constructive habituation does however assume that the targeted concept is still connected to an algorithmic process, but that the student is at point were he/she is cognitively ready to think about and work with different representations of the concept.

Within the condensation stage, students become more and more able to alternate between the different representations of concepts. If the students in this study were already in the condensation stage they should have been cognitively ready to integrate or bring together the different representations of functions. The results of this study show that though some of the students may have met this criterion, many more were still negotiating their path through interiorization.

Further, the mathematics ACT scores and the results of the pretest indicated that these students lacked basic skills that were the foundation for some of the procedures, algorithms, and concepts presented in the course. This demonstrates that a number of the students may have still been in the interiorization stage of reification. The researcher assumed that most, if not all, of the students were products of traditional teaching. The students' over reliance on procedures and algorithms as previously addressed supported this assumption. The researcher may have been in error in assuming that the students had already passed through the interiorization stage. This assumption was based on the years of habituation and procedural learning of which the students were likely exposed.

One of aims of constructive habituation is to help students understand and connect the different representations of a concept by presenting the "big picture" and employing multi-Reps to a point of cognitive saturation. Constructive habituation emphasizes multiple representations. Condensation is the stage where students have an ever-increasing ability to transition between the different representations. Students must therefore be in condensation for constructive habituation to be implemented.

The goal of this process is objectification of the target concept. Hence, constructive habituation, in effect, is bridge builder of sorts from in the latter phases of condensation to reification. This study demonstrated that constructive habituation is ineffective for students who are in either the interiorization stage or the early phases of condensation.

Validity

There are threats to internal and external validity in any type of research. This section contains a discussion of the major threats to both of these types of validity for the results reported in this study. The discussion is based on internal and external threats as suggested by Slavin (1992). The discussion is divided into two parts. The first part addresses the extent to which extraneous factors have been controlled. The second part discusses the extent two which these findings can be generalized or have meaning to other samples, populations, and settings.

This study applied a mixed methods research design. It employed the data collection and analysis associated with both quantitative and qualitative research. It can be more accurately characterized as a sequential explanatory design. It is the most straightforward of the six major mixed methods approaches, according to Creswell (2003). It is characterized by the collection and analysis of quantitative data followed by qualitative data. The quantitative portion of this experiment can be characterized as a quasi-experimental design.

One of the distinguishing features of the quasi-experimental design is non-random assignment of subjects to groups. It was not possible to randomly assign the students to the classes. The students pre-registered for the classes the previous semester through telephone and online registration processes. The classes were therefore considered intact before the beginning of the experiment. The researcher was however able to stipulate a limit on the number of students who could enroll in each of the two classes.

All students were advised and signed consent form acknowledging that they would be involved in a research experiment. They were not told the exact nature of the experimental process or what other classes would be involved. They were merely advised that the researcher and other university officials were seeking ways to better prepare students for future mathematics courses and decrease the failure rate for this and other mathematics courses. The students did not appear to be overly interested in the nature of the experiment. They were however, eager to begin the semester and were chiefly concerned with passing the class.

One of the main threats to internal validity was the differential selection of the participants. The use of intact groups introduced the possibility the results were not due to the treatment but to pre-existing differences among the two classes. According to Charles & Mertler (2002), this threat is always inherent is quasi-experimental research.

The mathematics ACT scores were used to help deal with this threat. The university was able to provide mathematics ACT scores for nearly all students enrolled in Precalculus I at the university. The evaluation of the mathematics ACT scores revealed there was no significant difference between the two classes. The mean mathematics ACT score for the CH students was 16.83 (sd = 2.81) and the mean mathematics ACT score for the TRAD students was 16.86 (sd = 2.26). Additionally, the mathematics ACT scores were found to be positively correlated with the two of the examinations administered in study. The mathematics ACT scores were therefore introduced as a covariate.

The analysis of covariance is the recommended statistical method in this type of quantitative design. Hinkle, Wiersma, & Jurs (1988) and Stevens (2002) suggest that ANCOVA provides a post hoc statistical procedure to adjust for preexisting differences among intact groups. The ANCOVA was done for the functions regular examination and the final examination scores that included all Precalculus I students, with the corresponding mathematics ACT scores as the covariate. The MANCOVA was performed on the component variables. This process helped lessen within group or error variance and reduce the effects of preexisting differences among the two groups.

One of the most common causes for low internal validity in experiments is selection bias or the fact that the groups being compared may not be equivalent. This threat existed in this experiment because the groups of students were not randomly selected by the experimenter. Again, the groups were considered intact before the initiation of the experiment.

To test the effects of non-randomization on the two classes, mathematics ACT scores were analyzed using Levene's test of homogeneity. The test revealed that the two classes were equivalent. This greatly reduced the possibility of selection bias as a significant threat to the study. Next, mathematics ACT scores were used a covariate throughout the study to help control for any preexisting biases or differences between the two classes as a consequences of non-randomization.

Experimenter bias can also be problematic when the data being collected are subjective in nature. Therefore, a great amount of time and care was taken in constructing the instruments. The interview protocols were followed closely. Yet, the interviews were flexible enough to accommodate any changes or important developments brought on by the students that may have needed to be more fully explored. In an attempt to document this effort of the research, a complete record of the interview protocols, along with selective transcriptions from all six interviews can be found in Appendices C - I.

A final threat to the internal validity of this study that will be addressed is attrition. Over the course of the semester there was a loss of students in both classes due to absences and eventual withdrawal. From the time the functions examination was given to the final examination, the traditionally taught class experienced a 52% attrition rate. The constructive habituation class experienced a 27% attrition rate. Early in the semester, before the functions examination, the traditionally taught class experienced high absenteeism. The researcher began requiring certain homework problems to be turned in for credit and gave announced and unannounced quizzes as motivation for the students to attend class. The incentives were ineffective and the withdrawal rate continually increased.

Kolomogorov-Smirnov Test for Normality and Levene’s Test for homogeneity of variance were conducted on the classes for each examination. These tests helped deal with the attrition threat. The Kolomogorov-Smirnov test indicated that the scores of the three examinations were normally distributed. The Levene’s test determined if the variances of the two classes are approximately equal on each examination. If the two classes were found to have equal variances then we could assume that attrition was not a significant factor and we could proceed with the parametric test to evaluate the research data. If the variances were found to be unequal while the scores were normally distributed then attrition could have been factor. We then would have to evaluate the research data using a non-parametric test.

Each of the examinations was normally distributed. The scores for the functions regular examination and the symmetry regular examination were found to have variances that were approximately equal. More than 50% of the TRAD students who had taken the functions examinations were no longer in the class by the final examination period. Attrition was an obvious concern. The final examination failed the test for homogeneity. We could not rule out attrition as a factor affecting the out-come of the study if we used a parametric test. Hence, the Mann-Whitney U test (a non-parametric test) was used to evaluate the means of the scores on the final examination.

Limitations

The threats to the external validity of an experiment influence the limitations. External validity determines the extent to which the results of a research study can be generalized to individuals and situations beyond those involved in the study. In this study the most evident threat was experimenter bias. Experimenter bias is also a threat to internal validity and was previously addressed in that section. The researcher also functioned as the interviewer (experimenter). It is conceivable that his expectations could have affected the data collection and analysis, and the students' performance.

An independent observer was invited to several sessions of both classes to address this issue in the study. His report serves a verification of the fair and impartial treatment of the students in each of the two primary classes. His report is found in Appendix J.

Though great care was taken, the researcher's expectations about what would or should occur during the class sessions and interviews may have been unintentionally transmitted to the students so that their behavior was affected. As previously mentioned, all students were aware that they were involved in a research study that could potentially help the university in preparing future students for certain undergraduate mathematics courses. However, they were not given any significant details on the procedures or other participants in the study. The students did not appear very interested in the details of the study. They were concerned with getting the information about tests, quizzes, homework, and other subjects needed to pass the course.

The students in this study were enrolled at Southern University at Baton Rouge during the spring semester 2004. These students were experimentally accessible to the researcher. Southern University is a historically black land grant institution. Though students of several ethnicities were included in the overall study, it is worth noting that only one out of the initial seventy-one students in the primary classes was of an ethnicity other than African-American.

The foregoing discussion concerning the generalizability of the results of this study refers to students enrolled at this university as its target population. Any efforts to generalize these results beyond this population to other institutions or settings should be approached with caution. Careful consideration should be given to the characteristics of each group to determine if they are similar to this population.

General Observations

Constructive habituation is new teaching method and this study was the first to formally explore its potential strengths and weaknesses. It was the methodological aim of this ambitious project to establish a set of conditions such that the observed differences could be qualified as a consequence of the experiment treatment and not to other variables. The argument has been presented that this was achieved. Furthermore, it was accomplished despite two potential hazards: the high attrition rate in one of the classes, and the dual role of the researcher who also served as both instructor and interviewer for this study.

The overall purpose of this study was to examine if constructive habituation was a more effective means of helping students reach process object reification than a more traditional teaching method as evidenced by achievement levels on in-class examinations. Overall, the results of this study revealed no significant quantitative or qualitative evidence that suggests that constructive habituation was in fact more effect in this effort than a traditional teaching method.

In general, the constructive habituation teaching strategy yielded results that were similar to the traditional teaching method in each quantitative category. This suggests to some extent, that this new strategy was at least not detrimental to the students learning process. Furthermore, there were some promising practically significant results that were revealed. The most encouraging was observed in the results of the symmetry regular examination. The CH students averaged more than nine points higher than there TRAD student counterparts. Though these differences were not statistically significant, they suggest that constructive habituation may possess the potential to be beneficial to the development of students' conceptual understanding of functions and its symbolic notation.

One of the most important points drawn from this study is that constructive habituation is a teaching strategy that requires students to possess a certain level of fundamental skills prior to implementation. Constructive habituation emphasizes multiple representations of concepts. It attempts to help students understand the relationships and transition between the different representations of the target concept. Constructive habituation is therefore more useful to students who have moved from the interiorization stage of reification. That is to say, constructive habituation may be more effective if the students possess the basic skills relative to the target concept and already able to manipulate lower level mathematical processes as objects.

Implications

This study has several implications for mathematics education and mathematics teaching. Implications for practice and future research are discussed in the following paragraphs.

Practice

As addressed in chapter one, this study grew out of the calculus reform movements of the 1980's and 1990's. Several of the concerns educators expressed about the state of calculus curriculum and instruction were also some of the major concerns of precalculus. The major criticisms with precalculus involved overemphasis of procedures and algorithms, symbols, and the lack of conceptual development of fundamental concepts.

Many reformers at institutions with large populations of under-prepared students suggested any successful reform efforts of calculus must begin with precalculus (Fife, 1994). Teaching calculus using innovative conceptual techniques while continuing to teach the precursor courses in a traditional manner is not sensible. The transition from precalculus to calculus should be natural for students. This can be accomplished if educators have a reasonable amount of pedagogical tools at their disposal.

Constructive habituation can potentially be one of those tools when properly implemented. The point here is that all pedagogical tools don't work for all circumstances. Traditional teaching has been promoted over the years as a one size fits all teaching strategy that has handicapped students in their conceptual understanding of certain mathematics topics. Teachers, by experience, know what topics, concepts, or areas will give students the most difficulty. Teachers can devise strategies, employing constructive habituation that lessens these trouble areas before they have a chance to fully manifest themselves. The suggestion here is that constructive habituation could not only be used throughout an entire course, but also selectively in areas were teachers have traditionally or by experience seen a need for strategy that can either supplement or replace the method they are currently using.

Multiple definitions of functions should be introduced and used throughout the course. The formal definition which is more static in nature serves a great purpose in the introduction of function concepts. This study demonstrates that students can have great success with this definition in an introductory lesson. But as the course evolves, so should the definition. The more dynamic view that suggests functions as relationships between variables (quantities that change) should be promoted. Real life examples that bring alive this definition to the students should be explored.

Research

Definition Use

The primary focus of this study was to compare constructive habituations' effect on students' ability to reify mathematical processes, in particular processes on functions. While analyzing the students' ability to define and explain functions the researcher observed that although students had been exposed to different interpretations of functions, like functions as a dependence relationship between two variables, nearly all students chose to state the formal definition and give a real life example that fit a superficial interpretation of this definition. In particular students gave examples that reflected what Vinner and Dreyfus (1989) refer to as the "one-valuedness" of functions. In other words, if a correspondence assigns exactly one value to every element in its domain, then it is a function. For instance, in their real life examples many students tended to match one product to its own separate and distinct price. What many students (not all) missed was that this formal definition also implies that multiple members of a domain could be matched to a single member of the range and some members of the range may not have a corresponding partner at all.

On the surface the students seemed to have at least an adequate understanding of the formal definition, though several other interpretations were at their disposal. But after further consideration of this matter it is quite possible the students really did not have a complete understanding of the formal definition and were merely repeating what they had seen presented in class and in the text. This phenomenon is not all that unusual. Edwards and Ward (2004) suggest some students might profess a seemingly adequate understanding of the role of formal definitions in mathematics without really understanding this role. They further suggest it is not uncommon for students or any person to repeat something they do not fully understand.

There is a need to further develop and test theories and ideas on how students use definitions in mathematics. Not only do students need to know definitions but also need to more fully understand the function the definition plays in the mathematics. The goal of helping students reach deep conceptual understandings of mathematical ideas goes hand in hand with understanding how they make sense of and use mathematical definitions.

Increased Access to Reification

The study revealed that students were very reliant on algorithms and symbol manipulation in interpreting formulas as well as interpreting and a modeling some graphs. This appeared to hinder students when they had to actually reify two functions. Since students have been trained to rely so heavily on algorithms and symbols, getting students to understand symbols as representing both processes and products may help in reification efforts.

Mathematicians and students with a predisposition to succeed in mathematics develop this ability almost instinctively. There is a need to develop and examine new and innovative theories and ideas on how students reach process-object reification. Moreover, it is the responsibility of the mathematics education community to develop new ideas on how teachers can most effectively and efficiently make the reification process available to a wider range of students with differing abilities.

Constructive habituation is essentially in its infancy stages as a viable pedagogical strategy. Further research and dialogue is necessary to explore the possible efficacy of this approach. Therefore, additional studies on students' conceptual gains as a result of constructive habituation as a pedagogical method are recommended. Replication of this study should be made with several modifications and improvements.

Replication of Study

Focused Examinations

Replication of this study should be done with more focused examinations. The examinations in this study included material that was not necessarily the target of the investigation but were required by the curriculum. Examinations that evaluate the students' abilities to construct mathematics objects and the components that lead up to and are necessary for this type of construction are recommended. These examinations should involve more abstract, conceptually based, and applications based problems that can be more effectively used to determine and evaluate the students' current stage of the reification process namely interiorization, condensation, and reification. These stages are described thoroughly in chapter two and briefly reviewed in this chapter. As recognition of the stages is made then assessments on how the stages are reached and what pedagogical tools can help future students reach this stage can be made.

Increased Number of Interviews

Secondly, replication of the study should be done with more interviews throughout the semester and a larger number of students participating in the interviews. The study was limited by the small number of students who participated in the interviews. Each of the students who participated in interviews received passing grades in the course. Interviews with students with a wider range of achievement levels and abilities could be helpful in comparing the students' reification stage based on abilities, current achievement levels, sex, and age.

An increased number of interviews may be helpful in determining the students' conceptual gains from one topic to another. Additionally, more interviews can help assess how different students operate within their current reification stage. Researchers may benefit from a more informal and open-ended structure in the early interviews to help establish essential questions, characteristics, and categories for future interviews.

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APPENDIX A

FUNCTIONS REGULAR EXAMINATION

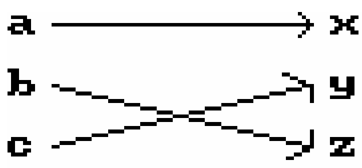
This appendix contains the Graphs, Functions, and Models regular examination. It was given to both the constructive habituation class and the traditional class. Students were given fifty minutes to complete this examination.

1) What is a function? Give a definition, not an example.

2) Give an example of function that one might encounter in real life (outside the mathematics classroom).

Is the following correspondence a function?

3)



A) Yes B) No

Tell whether or not the relation is a function.

4) {(-1, 2), (2, 8), (4, -1), (7, -9), (10, -8)}

A) Yes B) No

Determine the domain and range of the relation.

5) {(-2, 7), (7, 4), (8, -8), (8, -6)}

A) D = { -2, 8, 7, 8}; R = { 7, -8, 4, -6} B) D = { -2, 8, 7}; R = { 7, -8, 4, -6}

C) D = { 7, -8, 4, -6}; R = { -2, 8, 7} D) D = { -2, 8, 7, -8}; R = { 7, -8, 4, -6}

Evaluate as requested.

6) Find f (k - 1) for $f(x) = 4x^2 - 4x - 1$

A) $f(k - 1) = 4k^2 - 12k - 1$ B) $f(k - 1) = 4k^2 - 8k - 1$

C) $f(k - 1) = 4k^2 - 12k + 7$ D) $f(k - 1) = -12k^2 + 4k + 7$

7) Find f(-4) for $f(x) = x^2 - 3x - 6$

A) f(-4) = 34 B) f(-4) = 22 C) f(-4) = 10 D) f(-4) = -2

Find the domain of the function.

8) $f(x) = \frac{x}{x - 6}$

A) $(0, \infty)$ B) $(-\infty, -6) \cup (-6, \infty)$

C) $(-\infty, 6) \cup (6, \infty)$ D) $(-\infty, 0)$

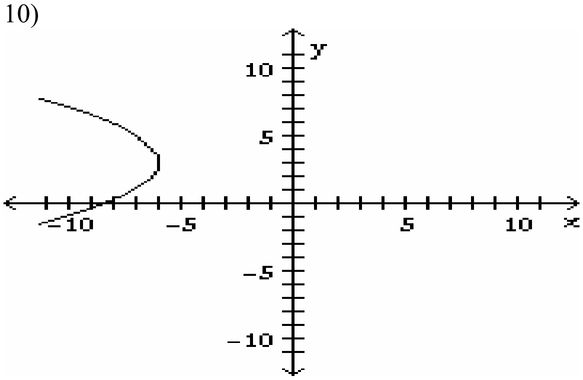
9) $f(x) = \frac{1}{x^2 + 6x - 16}$

A) $(-\infty, -8) \cup (-8, \infty)$ B) $(-\infty, 2) \cup (2, \infty)$

C) $(-\infty, -8) \cup (-8, 2) \cup (2, \infty)$ D) $(-\infty, \infty)$

Determine whether the graph is the graph of a function.

10)



A) Yes B) No

11) For persons who earn less than $20,000 a year, income tax is 16% of their income.
A) Give a general formula that decribes income tax in terms of income.
B) What are the independent and dependent variables?
C) Does your formula represent a function? Briefly Explain.
D) What is the domain and range?

12) A person's blood sugar level at a particular time of the day is partially determined by the time of the most recent meal. After a meal, blood sugar level increases rapidly, then slowly comes back down to a normal level. Sketch a graph showing a person's blood sugar level as a function of time over the course of a day. Label the axes to indicate normal blood sugar level and the time of each meal. Use only two meals, say breakfast and dinner.

Graph each function using the given viewing window. Using the graph, find any relative extrema. Change viewing windows, if it seems appropriate for further analysis.

13) $f(x) = x^2 - 2$; [-4, 4, - 3, 4]
A) Relative maximum of - 2 at x = 0 B) Relative minimum of - 2 at x = 0
C) No relative extrema D) Relative minimum of - 2 at x = 1

Answer the questions.

14) A manufacturing company estimates that it will have revenue of $R if it produces x units of its product, where $R(x) = -0.001x^2 + 16x$ for $0 \le x \le 16{,}000$. Graph the function using a grapher. Then find the relative maximum. How many units should be produced to obtain the maximum revenue? What is the maximum revenue?
A) 8000 units; $61,440 B) 6400 units; $64,000
C) 6400 units; $61,440 D) 8000 units; $64,000

Solve.

15) Elissa wants to set up a rectangular dog run in her backyard. She has 32 feet of fencing to work with and wants to use it all. Suppose the dog run is to be x feet long.
a) Express the area of the dog run as a function of x.
b) Find the domain of the function.
c) What dimensions yield the maximum area? (you may have to graph the function)

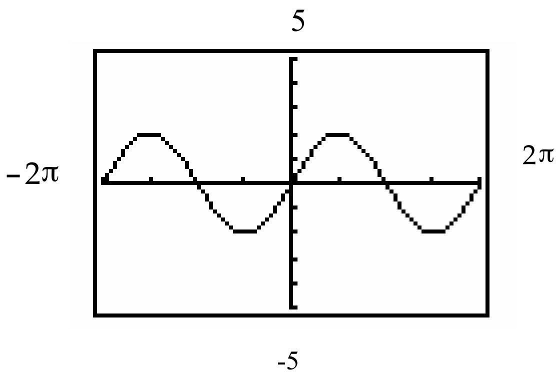
APPENDIX B

SYMMETRY REGULAR EXAMINATION

This appendix contains the Symmetry and Transformation of Functions regular examination. This test was given to both the constructive habituation class and the traditional class. Students were given fifty minutes to complete this examination.

Determine if the graph is symmetric with respect to x-axis, y-axis, and origin.

1)



A) no symmetry B) origin C) y-axis D) x-axis

Use your graphing calculator to determine if the equation is symmetric with respect to the x-axis, the y-axis, and the origin.

2) $y = 2x^2 - 1$

Determine algebraically whether the graph is symmetric with respect to the x-axis, the y-axis, and the origin.

3) $y = |18x|$

Determine algebraically whether the function is even, odd, or neither even nor odd.

4) $f(x) = \sqrt{x^2 + 16}$

A) Even B) Neither C) Odd

5) $f(x) = x^3 - x^2 + 1$

A) Even B) Odd C) Neither

Answer the question.

6) How can the graph of $f(x) = 1/2\ (x + 12)^2 - 3$ be obtained from the graph of $y = x^2$?

7) How can the graph of $f(x) = -(x - 3)^2 + 3$ be obtained from the graph of $y = x^2$?

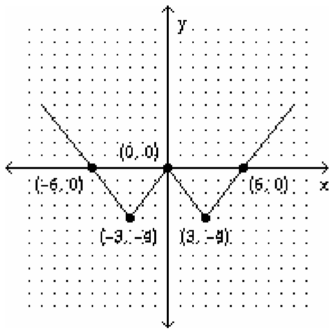
Write an equation for a function that has a graph with the given characteristics.

8) The shape of $y = \sqrt{x}$ is shifted 10 units to the left. Then the graph is shifted 4 units upward.

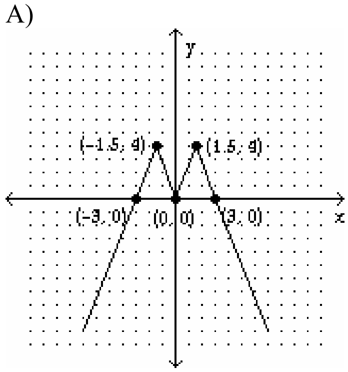
A) $f(x) = \sqrt{x+4} + 10$ B) $f(x) = 4\sqrt{x + 10}$

C) $f(x) = \sqrt{x + 10} + 4$ D) $f(x) = \sqrt{x - 10} + 4$

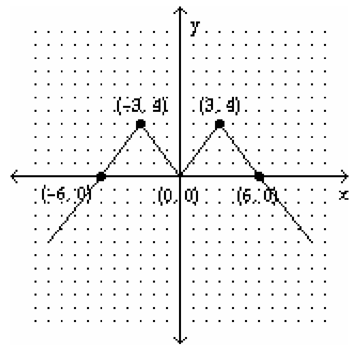
A graph of y = f(x) follows. No formula for f is given. Make a hand-drawn graph of the equation.



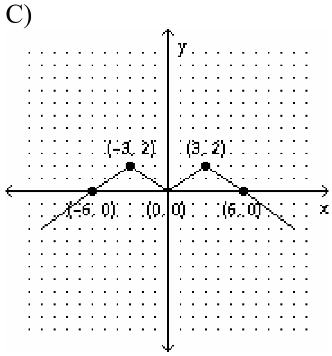
A)



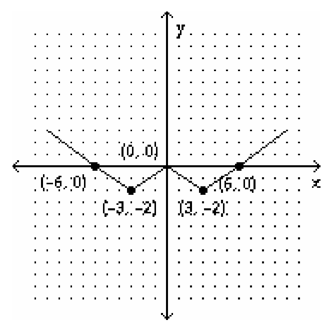
B)



C)

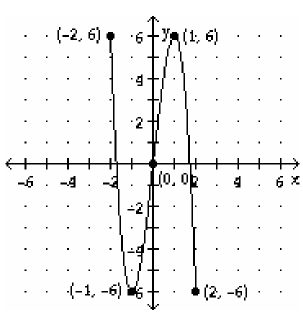


D)

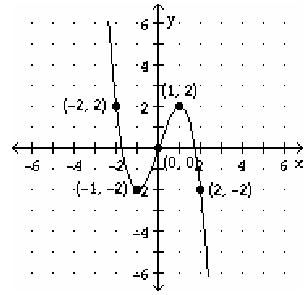


Given the graph of the function f(x) = $-x^3 + 3x$; find a formula for g(x) .

10) f(x) = $-x^3 + 3x$



g (x) =



A) g(x) = 3f(x) B) g(x) = f (x + 3) C) g(x) = 1/3f(x) D) g(x) = f(x) + 3

Answer the question.

11) How can the graph of f(x) = . 9□-x□ - 7 be obtained from the graph of y = □x□?

12) Suppose S(d) gives the height of high tide in Seattle on a specific day, d, of the year. Use a translation of S(d) to describe each of the following functions.

(a) T(d), the height of high tide in Tacoma on day d, given that the high tide in is always one foot higher that high tide in Seattle.

(b) Give a formula for the high tide in Portland given that the high tide in Portland is the same as the high tide in Seattle on the previous day.

13) A graph showing temperature (H) as a function of time (t) in a certain office building is given by the figure. Let y = H(t) be the heating schedule formula.
Let r be a transformation of H defined by the equation r(t) = H(t - 2) - 5.
a. Sketch a graph of r(t).
b. Describe in words the heating schedule determined by r.

APPENDIX C

INTERVIEW PROTOCOLS

This appendix contains the protocol for the interview section of this study.

It includes an introduction to the interview and the questions, probes, and graphs that were used.

Introduction to the Interview

In the next 30 to 45 minutes, I am you some questions about some of the material we have covered in the class. This is not intended to be any kind of test, and you will not be graded in any kind of way. I am trying to understand how you reason and go about solving problems. I will audiotape our conversation so that I can listen to your answers without having to write everything down.

Some of the questions I will ask verbally and some I will have written down on a piece of paper. I would like for you to read the question aloud before you start working it. When you are working, I would like for you to think out loud as much as possible. So I will probably be asking you questions like: How did you get this? Why did you do that? Can you explain that? etc. When I ask question like these, it doesn't mean that you done anything right or wrong. It only means that I am interested in how you are thinking, and how you are going about solving a problem.

Do you have any questions so far? Are you ready to begin?

Questions and Probes

1. Am I sure you are aware that after the prerequisites skills examination that we have worked a lot with functions. Can you tell me what a function is in your opinion?

Probes:

a. Can you give me a real life or everyday example of a function?

b. In everyday life do think that functions are important? Why ?

c. How can functions be represented?

d. Ask about domain and range.

2. A graph of the university's heating schedule, showing temperature **(H)** in Fahrenheit as a function of time **(t)** in hours is given. The initial time t = 0 represents midnight.

a. Can you write a general formula to represent this graph?

b. Between what times is the building the warmest?

Probes:

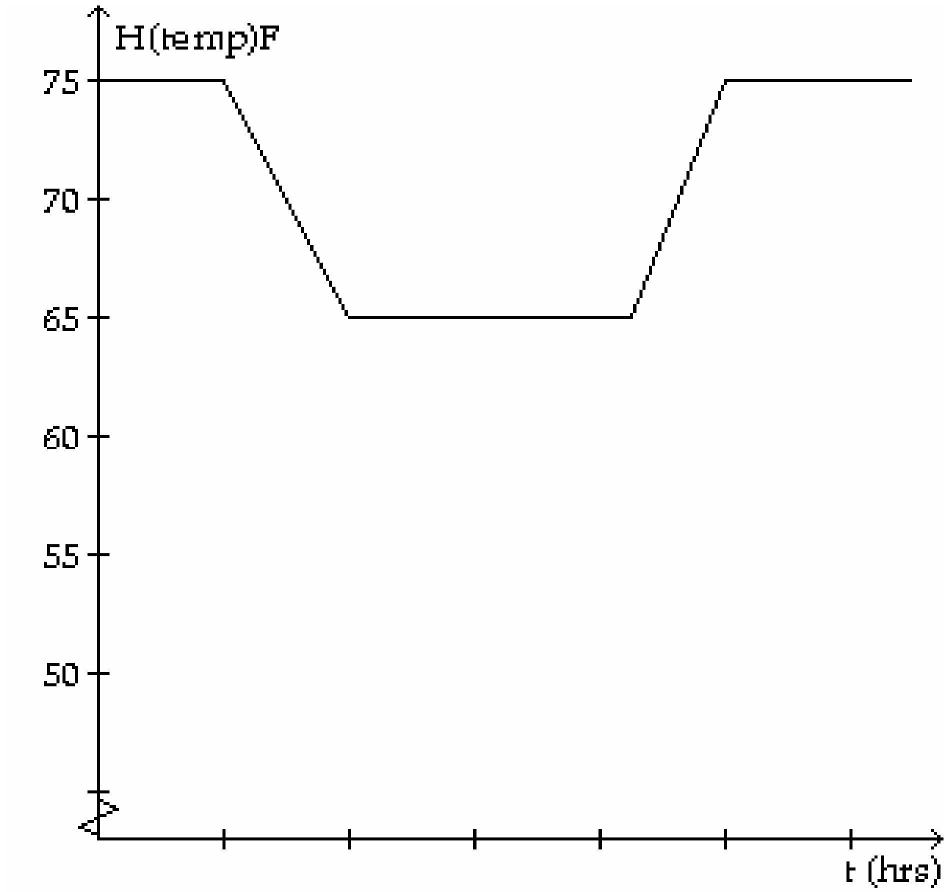
a. Graph the function **H(t) – 5**. If the university decides its heating schedule according to this function, what has the company decided to do?

b. Graph the function **H(t – 2**), what has the university decided to do?

c. When you get to school at 8am, will the classroom be warmer under the **H(t)** schedule, the **H(t) – 5** schedule, or the **H(t – 2)** schedule? What will the temperature be?

3. Suppose the original heating schedule is represented by the formula, **H = f(t**). Suppose this graph is shifted 5 units upward. This new schedule is represented by the formula **H = q(t**). How are the formulas **f(t)** and **q(t)** related? Can you write this relationship algebraically?

Graph of the university heating schedule.



APPENDIX D

INTERVIEW WITH TRAD1

This appendix contains the selective interview with TRAD1. TRAD1 was a 19 year old male student. His mathematics ACT score was 18. He earned a "B" grade in the class.

RE: [Reads the introduction to the interview] Are you ready to begin?

TR1: Yes.

RE: I am sure you are aware that after the prerequisites skills examination that we worked a lot with functions. Can you tell me what a function is in your opinion?

TR1: Well can I write it and read it to you?

RE: Sure.

TR1: A function is a rule of correspondence that states on element in the domain, A, must be matched to exactly one element in the range, B.

RE: Okay. Can you give me a real life example of a function?

TR1: [Begins to draw a diagram using letters and numbers.] The letters represents the names of shoes and the numbers are their shoe sizes. [He draws A (Adidas) matching to 10 and 10.5, R (Reebok) matching to 10.5, and P (Puma) matching to 9]. One pair of shoes can be in different sizes [referring to the Adidas shoe]

RE: What is your domain and range?

TR1: Shoes are the domain and the range is the sizes.

RE: Okay. In everyday life do you think that functions are important?

TR1: Umm, yeah I think they're important. If we didn't have functions there would be confusion. It gives us some order or organization.

RE: Well okay. Let's move on. How can functions be represented?

TR1: Umm by graphs…ah ratios and percentages.

RE: [TR1 is then presented with a graph and is instructed to read question two aloud] Can you write a general formula to represent this graph?

TR1: [Thinks a while and then writes f(x) = H(t)]. [He then attempts to explain] H(t) because temperature is a function of time Her graph is shifted two units up and two units to the right with respect to the reference graph. Time is dependent on temperature. Temperature is the domain …the independent variable. [He stops and ponders his response]. As the day progresses the temperature rises. It gets hotter. [He is convinced of his answer].

RE: Well can you tell what's going on with this graph.

TR1: As the day progresses the temperature rises. It gets hotter.

RE: Okay, now between what times is the building the warmest?

TR1: Between 8 and 16 hours.

RE: Read probe A aloud.

TR1: [Reads probe and proceeds to correctly draw graph]. The university has decided make it five degrees cooler. . .drop the temperature down.

RE: Okay. Read probe B.

TR1: [Reads probe and again draws graph correctly]. They are gonna start cooling two hours later.

RE: Read probe c aloud.

TR1: H(t) – 5 will be warmer because it's less than the other two. The temperature will 60 degrees.

RE: Read probe three for me please, aloud.

TR1: [Reads probe and begins to ponder]

RE: Take your time and think about the questions.

TR1: Um, q(t) are the five units that are shifted upward.

RE: Okay, can you write the relationship between q(t) and f(t)?

TR1: [He writes correct q(t) = f(t) + 5]

RE: Okay, can you tell me the domain and range of this new function?

TR1: The domain is the x values. . .ah the domain is t. The range is the y values which is f(t) + 5.

RE: Okay, well what is q(t)?

TR1: [Begins to think, shrugs his shoulders and shakes his head] I don't know.

RE: Okay, you did good. That wasn't to bad was it?

APPENDIX E

INTERVIEW WITH TRAD 2

This appendix contains the selective interview with TRAD 2. TRAD 2 was a 19 year old female student. Her mathematics ACT score was 15. He earned a “C” grade in the class.

RE: [Reads the introduction to the interview] Are you ready to begin?

TR2: Yes.

RE: I am sure you are aware that after the prerequisites skills examination that we worked a lot with functions. Can you tell me what a function is in your opinion?

TR2: [Thinks and then recites the formal definition presented in class] A function is a rule of correspondence, given a set A and a set B, that say each element is A is match to exactly one element in B.

RE: Can you give me a real life or everyday example of a function?

TR2: Yes, products and prices. If you go to the mall to buy a shirt each shirt is matched to its own price.

RE: In everyday life do you think that functions are important? And why?

TR2: Yes I think they are important.

RE: Why do you think so?

TR2: Because you have to make decisions. And functions help you do that.

RE: Alright. Can you tell me how can functions be represented?

TR2: Everyday – using different products. Mathematically by definitions, examples, examples [she emphasizes] with numbers and variables.

RE: Tell me about domain and range. Can you tell me what they are?

TR2: Domain is the x-axis, range is the y-axis, or the independent variable and the dependent variable or the input and output.

RE: [Gives her a sheet with questions and a graph]. Read question two for me. [She begins to read silently]. Oh, I'm sorry . . . aloud for me if you don't mind.

TR2: [Reads the question]. H(t) is the output. [Then writes H(t) = ___ and stops]

RE: Question 2b.

TR2: [Looks at graph]. The building is the warmest between 7 and 4.

RE: Read probe a.

TR2: [Reads question and draws graph. Only the first and last portions of her graph are properly shifted five units down. The middle portion is only shifted about two units]. I remember this type from class.

RE: Alright, probe b, for me now.

TR2: [Reads question and correctly graphs the function]. They decided to let it get cooler before people came to work.

RE: Okay, next question.

TR2: [Reads question – probe 2c]. I think its H(t – 2) [begins to look at her graph] because it shifted two hours the right. It starts a 6:00.

RE: Okay final question. Read number three for me.

TR2: [Reads question]. Ah, H = f(t) + g(t) and H = f(t) + g(t) + 5

RE: Do you think there's any relationship between f(t) and g(t).

TR: No! They have the same input and different outputs [obviously referencing the definition of functions]. But they are not related.

APPENDIX F

INTERVIEW WITH TRAD 3

This appendix contains the selective interview with TRAD 3. TRAD 3 was a 21 year old female student. Her mathematics ACT score was 15. He earned a "C" grade in the class.

RE: [Reads the introduction to the interview] Are you ready to begin?

TR3: [Nods her head in the affirmative yes].

RE: You seem a little nervous, relax. This isn't a test, it to help me improve my teaching, alright?

TR3: [Laughs] Alright.

RE: I am sure you are aware that after the prerequisites skills examination that we worked a lot with functions. Can you tell me what a function is in your opinion?

TR3: A function is a rule between two sets where every element in one set can only be matched to one element in the other set.

RE: Can you give me a real life or everyday example of a function?

TR3: I can give you the example that you gave us in class.

RE: But you're not supposed to us my example or anything close. You were supposed to get your own. Didn't I say that in class.

TR3: Yeah, but that's the only thing I can think of right now. And I understand it.

RE: Okay, go ahead since it makes sense to you.

TR3: Two cans of corn of the same kind at two different, um no the same store. [She struggles to remember and recite the example] Mmm, the cans of corn can have different prices.

RE: Well, that's not quite the example that I gave in class.

TR3: But that's as close as I can remember. And I really did understand it when you did it in class.

RE: [Laughs] Alright sister, tell me about the domain and range of your example. What represents the domain? What represents the range?

TR3: The price is the range and the corn is the domain. [Frowns]

RE: You sure?

TR3: Well the domain is a constant and the range is a variable. And prices change so that has to be the range.

RE: Okay, read problem number two aloud please. [Gives her problem and graph]

TR3: [Reads problem and write H = f(t)] The domain is time, hmm, and the range is temperature. [Writes H(t) =].

RE: Okay, read 2b.

TR3: [Reads problem]. The building is the warmest between 8 o'clock and 4. It's 65 degrees.

RE: Okay, next question [probe 2a].

TR3: [Draws graph but only draws decrease 60 degrees between 8:00 and 4:00]. They have decided to lower the temperature 5 degrees between 8:00 and 4:00.

RE: Alright, question b.

TR3: [Reads probe 2b and correctly draws graph shifted two units to the right]. The University will turn the heaters on two hours later.

RE: Okay, next question.

TR3: [Reads probe 2c]. The classroom will be warmer with the H(t) – 5 schedule. It will be 60 degrees.

RE: We're almost finished.

TR3: [Reads question 3]. Okay, H = f(t) and if it is shifted five units up it becomes H = q(t). So f(t) and q(t) are related because . . . well they are the same root formula but it changes because of the vertical shift so H = f(t) + 5 = H = g(t).

RE: Well if H = f(t) + 5 and H also = g(t), what can you say about these two equations.

TR3: Oh yeah, f(t) + 5 must be equal to g(t). [She then writes f(t) + 5 = g(t).

APPENDIX G

INTERVIEW WITH CH1

This appendix contains the selective interview with CH1. CH1 was a 19 year old male student. His mathematics ACT score was 15. He earned a "C" grade in the class.

RE: [Reads the introduction to the interview] Are you ready to begin?

CH1: [Apprehensively] Yes, I'm ready.

RE: I am sure you are aware that after the prerequisites skills examination that we worked a lot with functions. Can you tell me what a function is in your opinion?

CH1: [Thinks a little while, begins to write]. A function is where every element in a set A corresponds to only one in set B.

RE: Can you give me a real life or everyday example of a function?

CH1: [Repeats the question]. A real life example?

RE: Yes, something that you might see everyday. Remember at the beginning of the semester we talked about real life functions and I had you guys give some examples?

CH1: Yeah, I remember. One example could be a person teeth and a toothbrush.

RE: What do you mean?

CH1: Everyone has their own toothbrush. So, there is one toothbrush to one persons teeth.

RE: Okay, well what would be the domain and the range in your example?

CH1: The domain would be the toothbrush and the range would be the person's teeth.

RE: In everyday life do you think that functions are important.

CH1: Ah . . .yeah.

RE: Why?

CH1: Hmm, I don't know but I know that they're important.

RE: How can functions be represented?

CH1: In mathematics or everyday life?

RE: In mathematics.

CH1: Well, in math as numbers, variables . . .[pauses].

RE: Is there any relationship between these numbers and variables.

CH1: I….I don't think so. Oh and [functions can be expressed] graphically and numerically.

RE: [Gives CH1 a graph and sheet with questions]. Read question number 2 aloud and answer it for me.

CH1: [Reads question]. [Thinks about question and writes. . . H = t]. [Shakes his head]. I don't know.

RE: That's alright. This isn't a test. You're okay. Read question [2] b.

CH1: [Reads question and looks over graph]. Between 4:00a.m and 4:00p.m. and between 8:00a.m. and 4:00p.m.

RE: Read the next question probe a.

CH1: [Reads question and graphs the function]. [He shifts the graph to the right approximately five units and then down five units]. They have dropped the temperature five degrees.

RE: Okay, question b.

CH1: [Reads questions and graphs function]. [He shifts the previously drawn graph two units to the right and then five units down. He erases the five-unit vertical shift and adjusts it so that it is only two units down]. They moved the schedule to two hours later.

RE: Does your graph represent what you have just said?

CH1: [Looks at graph]. Yes.

RE: Next question.

CH1: [Reads probe 2c]. The classroom will be the coolest at H(t) – 5 and it will be 60 degrees.

RE: Okay, question number 3.

CH1: [Reads question and thinks a long time].

RE: How are H = f(t) and H = q(t) related?

CH1: [Seems perplexed and again ponders the question for a long period].

RE: Well, let me ask you this then. Do you think that the two functions are related at all?.

CH1: No. I don't think they're related to each other.

APPENDIX H

INTERVIEW WITH CH2

This appendix contains the selective interview with CH2. CH2 was a 19 year old female student. Her mathematics ACT score was 19. She earned a "B" grade in the class.

RE: [Reads the introduction to the interview] Are you ready to begin?

CH2: Yes.

RE: I am sure you are aware that after the prerequisites skills examination that we worked a lot with functions. Can you tell me what a function is in your opinion?

CH2: Given a set A and B. A function is a rule of correspondence that says each element in A can be matched to exactly one element in B.

RE: Can you give me a real life or everyday example of a function?

CH2: [Draws an example of what is not a function by matching a set of ordered pairs and draws an example of a function similar to that which was presented in class- three different brands of canned corn. She explains both scenarios]. Suppose you have these number 5, 2, 3 in one over here [she list them in a column] and 1, 4, 6 on this side. [She has drawn 5 being matched to both 1 and 4, 2 being matched to 4, and 3 being matched to 6.] This is not a function because 5 is matched to both 1 and 4. [She then explains the second example] You have three different cans of corn say the Albertson's brand, Del-Monte, and Green Giant. Each can of corn has its own price, say the Albertson's brand is $0.33, the Del-Monte brand is $0.45 and the Green Giant brand is $0.67. This is a function, each can with its own separate price.

RE: In everyday life do you think functions are important.

CH2: I don't know, I never really thought about until I got in this class. Yeah, I guess so!

RE: How can functions be represented?

CH2: [Frowns].

RE: Well let me ask it this way. Can functions be represented in different ways?

CH2: Oh yea, f(x) is a representation, as graphs, and as ordered pairs.

RE: Hmm, okay, well tell me about domain and range of a function.

CH2: Well [pauses] in ordered pairs [pauses] the domain is the x, and the range is the y.

RE: Okay take a look at this [gives CH2 graph and question sheet]. Read question two aloud and answer it.

CH2: [Reads the question and writes H(t) = 0.]

RE: Okay, look at the graph between what times do you think the building will be the coolest.

CH2: [Looks at graph] Between 8 and 17 hours.

RE: Read probe [2]a aloud for me please.

CH2: [Reads question]. They drop the temperature 5 degrees between 12:00 and 4:00. [She struggles in drawing the graph, making several corrections, and then settles on an incorrect version].

RE: Okay here is the next question [probe 2b].

CH2: [Reads question]. They change the time to two hours later because any time you have a minus here [referring to the formula] it causes a shift to the right. They turned on the air at 6:00 instead of 4:00. [She could not accurately show this horizontal shift graphically]. [Her graph is shifted two units up and two units to the right with respect to the reference graph].

RE: Now look at the next question [probe 2c].

CH2: H(t) – 5 is when the classroom will be coolest. It will be 60 degrees. H(t) is already 70 degrees. Then the lowest temperature will be 60 degrees on my graph. My graph is wrong. This is wrong. [Begins to erase incorrect graph from probe 2a and draws correct graph].

RE: Okay that’s good. Now let’s look at question 3.

CH2: [Reads question]. I believe they are related because H = f(t) and H = g(t) + 5. [The both are equal to H] the graph shifted up five units.

RE: You think that’s it.

CH2: Yeah, I think so.

APPENDIX I

INTERVIEW WITH CH3

This appendix contains the selective interview with CH3. CH3 was a 19 year old male student. His mathematics ACT score was 18. He earned a "C" grade in the class.

RE: [Reads the introduction to the interview] Are you ready to begin?

CH3: Yes, I'm ready.

RE: I am sure you are aware that after the prerequisites skills examination that we worked a lot with functions. Can you tell me what a function is in your opinion?

CH3: Yes sir, a function is [pauses and thinks] a given set . . . [frowns].

RE: Sounds like you're trying to remember the definition from class. Tell me what you think a function is.

CH3: A function is a set of two intervals where each element in the first set is matched to exactly one element in the second set.

RE: In your definition you said that a function is a set of intervals. Which is it, a set, interval, or a combination.

CH3: [Confidently] It's a combination of both.

RE: Can you give me a real life or everyday example of a function?

CH3: Say for instance when I get up to go to church and I'm getting dressed I always match my dress shirt with the same color tie. I can't match my dress shirt with two different [colored] ties.

RE: In everyday life do you think that functions are important.

CH3: Yes sir.

RE: Why?

CH3: I guess cause the help you maintain control.

RE: Tell me about the domain and range in your example.

CH3: The domain would be my tie and my shirt would be my range.

RE: How can functions be represented?

CH3: Ah, interval notation.

RE: Are there more was to represent functions. Remember in class we talked about multiple representations?

CH3: Ah yes, I guess in interval notation, something like a chart, a graph, set notation.

RE: [Gives CH3 a graph and question sheet] Read number 2a.

CH3: [Begins to read silently].

RE: Read it aloud.

CH3: [Begins to talk himself through the variables]. The temperature is the domain and the time is the range. [He writes t(H)]. No temperature is the range. [He writes H = f(t)].

RE: Read question 2b.

CH3: [Reads question]. Between 8:00 and 4:00.

RE: Okay, read the probe a.

CH3: [Reads question and begins to correctly draw graph]. The university has decided to lower the temperature by 5 degrees.

RE: Okay, number b.

CH3: [Moves the graph four units right]

RE: What has the university decided to do based on your graph.

CH3: They decided . . .they decided to um. . . start the heating schedule two hours later.

RE: Okay when does the heating schedule begin.

CH3: It begins a 4:00.

RE: So if it begins at 4:00 where would two hours later be?

CH3: It should be at 6:00. But it doesn't get cooler until 8:00.

RE: Does your graph show it getting cooler at 6:00.

CH3: Oh, I should have shifted the whole graph over two units. [He renumbers the graph and talks himself through while drawing the correct graph].

RE: Okay, good, question [probe 2] c.

CH3: [Reads question and looks at graph]. It will be cooler under the H(t) – 5 schedule 'cause they lowered the temperature 5 degrees. And then they did change the time when the temperature changes like in the other scale [pauses and thinks].

RE: Okay, you happy with your answer. What will the temperature be?

CH3: Oh, the temperature will be 60 degrees.

RE: Okay, let's move on to the next question.

CH3: [Reads question and begins to write formulas]. Okay, ah, the formulas f(t) and q(t) are related because f(t) is the basic function. The heating schedule is given by f(t) and it changes from f(t). Making f(t) hotter 5 degrees gives you q(t).

RE: On your paper I see you wrote H = f(t) + 5 = H = q(t). Why did you write the H's.

CH3: Because H is the heating schedule and H = f(t) and they what to change the heating schedule by moving it higher 5 degrees. And so in order to do that you would have to add 5 to the function, to the outside of the function cause that would cause a vertical shift. And that would give you H = f(t) + 5. And they said they wanted that new formula to be represented by q(t). So q(t) = f(t) + 5.

RE: But that's not exactly what you wrote down on your paper.

CH3: Okay, f(t) + 5 and q(t). The first part f(t) is the reference function so you should go from there. [Works through his paper for a while and finally writes f(t) + 5 = q(t).

APPENDIX J

OBSERVER NOTES

I, John McGee, an Assistant Professor of Mathematics at Southern University, acted as an independent observer of Mr. Alonzo Peterson's classes during the spring semester 2004.

The first class I observed was the experimental class. This class met in room 322 T.T. Allain Hall. The class was Math 135 – precalculus I. I observed this class on February 18, 2004 at 8:00 a.m.

Mr. Peterson was excited and enthusiastic about the introduction of functions to his class. After returning examinations from a previous chapter, answering questions, and other class administrative duties, he presented a problem pertaining to distance to the class. After a short discussion Mr. Peterson then proceeded to in leading the students in deriving a table and graph of the problem. The lesson ended with the students developing a formula for the graph.

On February 20, 2004 at 8:00 a.m., he began the class with the same enthusiasm he had in the previous session. He continued the lesson on functions. He reviewed the previous lesson and formulas. He introduced the function notation and the definition of functions. The students seemed very attentive responsive to his teaching style. He did a very good job of related the graphs, tables, and definition. The classes ended with the students having to give their own real-life example of a function.

The second class I observed was the control class. This class met in room 315 T.T. Allain Hall at 9:00 a.m. Again, Mr. Peterson was very energetic and enthusiastic during the class. He began this class the same as he began the experimental class. He returned a previous examination and answer questions about the test. He then began an introductory lesson on functions. He began by having students copy the definition of functions. He gave the students examples and had the students do problems at their seats. He gave the students a real-life example of a function and had the students give their own real-life examples.

On February 20, 2004, Mr. Peterson began the class by asking for questions on the previous days' example. Several students asked questions and he did a very good job in making sure they understood. After taking questions he discussed functions as sets of ordered pairs. He emphasized things like domain-range, and input-output. The students seemed to be responsive and interested in the lesson. He then gave several examples and had the students do seat-work with more problems. He asked the student to determine if a set of ordered pairs was a function and explain their answer. After some discussion, he introduced the functions notation. The students work on several more examples before the class end.

Mr. Peterson was enthusiastic and energetic with both classes. He spent a great deal of time answering questions in both classes and making sure the students understood the topic. He was very courteous and professional to all the students. I did not observe any favoritism or negative treatment of any student in either class.

Submitted: October 25, 2004

APPENDIX K

CONSENT FORMS

This appendix contains the interview consent form for student involved in the qualitative analysis. It also contains the general consent form used for all students enrolled in the two target classes.

A STUDY TO EXAMINE IF CONSTRUCTIVE HABITUATION IS A MORE EFFECTIVE MEANS OF HELPING STUDENTS REACHPROCESS OBJECT REIFICATION THAN A TRADITIONAL TEACHING METHOD

Interview Consent Form

I, ___________________________________, am an adult and do consent to this interview with Alonzo Peterson who is a graduate student of Louisiana State University and A&M College in Baton Rouge, LA.

I also understand the following statements.

(1) This interview is for research purposes, and that the information I share could become part of some publication in a refereed journal or book.

(2) The focus of the interview is on the student's conceptual understandings of certain mathematical topics and concepts.

(3) Should I become uncomfortable about answering any question, I am not obligated to answer it.

(4) I have the option of terminating this interview should I deem it necessary.

(5) Pseudonyms will be used if reference to a name should be necessary for composition and/or content purposes.

(6) The information I share with Mr. Peterson will be collected, organized, analyzed and interpreted by him under the direction of his major professor Dr. David Kirshner.

(7) I release any rights or obligations of the interviewer to the information I share.

(8) There is no financial exchange or obligation associated with this interview.

(9) I have agreed to this time and place of this one time interview for at least one hour. Should I or the interviewer find it necessary to follow-up on what was shared in this interview in the near future; contact can be made with the interviewer via apeter2@lsu.edu or with me at _________________________, respectively.

____
Interviewee

____
Interviewer

A STUDY TO EXAMINE IF CONSTRUCTIVE HABITUATION IS A MORE EFFECTIVE MEANS OF HELPING STUDENTS REACH PROCESS OBJECT REIFICATION THAN A TRADITIONAL TEACHING METHOD

General Consent Form

Performance Site:	Southern University and A&M College, Baton Rouge, LA
Investigators:	The following investigators are available for questions concerning this study, M-F, 9:00am – 4:30pm. Dr. David Kirshner 225-578-6867 Alonzo Peterson 225-771-5180
Purpose of Study:	The purpose of this study is to examine if constructive habituation is a more effective means of helping students reach process object reification than a traditional teaching method.
Subject Inclusion:	Undergraduate mathematics students in enrolled in Alonzo Peterson's precalculus I courses at Southern University-BR.
Number of Subjects:	Quantitative portion 71 Qualitative (interview portion) 6
Study Procedures:	The study will be conducted in two portions. The quantitative portion will involve the normal classroom instruction throughout the semester. One class will be instructed using traditional teaching methods while the other will be instructed using the experimental method. In the qualitative portion of the study six student volunteers will complete a series of mathematical questions aimed at more fully understanding their conceptual knowledge and understandings of the target concepts. These interviews will last approximately 45-60 minutes.
Benefits:	The study may yield valuable information about students' understandings of certain mathematical concepts.
Risks:	The risks are minimal. The students scores will be reported collectively thus eliminating any possibility of tracing a particular score to a particular student. No names will be used in the interview portion thus eliminating any possibility of tracing a particular response to a particular student.
Right to Refuse:	Students may choose not to participate or to withdraw from the study at any time without penalty or loss of any benefit to which t hey might otherwise be entitled.

Privacy: Results of the study may be published, but no names or identifying information will be used.

The study has been discussed with me and all my questions have been answered. I may direct additional questions regarding study specifics to the investigators. If I have questions about subjects' rights or other concerns, I can contact Robert C. Mathews, Institutional Review Board at 225-578-8692. I agree to participate in the study described above and acknowledge the investigators obligation to provide me with a signed copy of this consent form.

Signature of Student: Date:

__________________ ___________________

APPENDIX L
CONSTRUCTIVE HABITUATION LESSON

This appendix describes the introduction of functions lesson presented to the constructive habituation students on February 18, 2004. This description is presented to demonstrate the nature of the constructive habituation teaching strategy.

Constructive Habituation Class Session

Introduction to Functions

The following is description of a lesson on the introduction of functions in the constructive habituation classroom. This description is given to give the reader a better understanding of the type of pedagogical structure that exists in the constructive habituation classroom. The text used for all Precalculus classes at the university is Algebra and Trigonometry 2nd editon by Bittenger, Beecher, Ellenbogen and Penna.

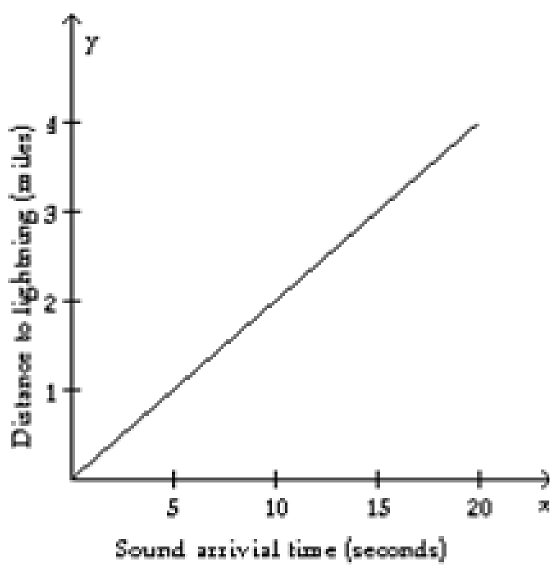
1. The functions concept was initially introduced with a real life application.

   The Thunder Time Related to Lightning Distance Example: During a thunderstorm, it is possible to calculate how far away, y (in miles), lightning is when the sound of thunder arrives x seconds after the lightening has been sighted. It is known that the distance, in miles, is 1/5 of the time, in seconds. If we hear the sound of thunder 10 seconds after we've seen the lightning, we know the lightning is (1/5)(10) = 2 miles away.

2. Students were then instructed to make a table to find the distance of lightning (y) given six different times (x -in seconds). After the students found the distances that were asked to write them in the table as a set of ordered pairs. An example of the final table follows.

x	y	Ordered Pairs (x, y)
0	**0**	**(0, 0)**
1	**1/5**	**(1, 1/5)**
2	**2/5**	**(2, 2/5)**
5	**1**	**(5, 1)**
10	**2**	**(10, 2)**
15	**3**	**(15, 3)**

3. The students were then instructed to make graph using the newly formed ordered pairs. An example of a correct graph is shown.



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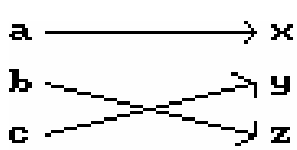
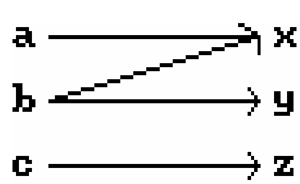
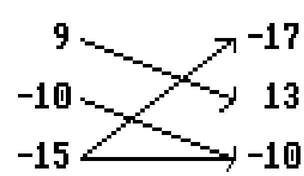
4. Students were informed the ordered pairs described a relationship or correspondence between the x and y coordinate. Students were instructed to observe this relationship in the graph as well.

5. Based on the given information and table we derived the equation y = (1/5)x as the equation that describes the relationship.
6. Students were then told that this relationship is an example of a function.
7. Reflecting on the table and equation we emphasized the y's dependence on x.
8. The function notation f(x) was then introduced.
9. Using this information from the table and emphasis place on y's dependence on x, the ideas of domain and range were introduced.
10. At this stage the students had been exposed to the idea of functions through applications, tables, as a set of ordered pairs, graphically, as a formula, and with abstract notation. We then entered a discussion on the how these different representations were related to each other.
11. The formal definition of functions was introduced:

    A **function** is a correspondence between a first set, called the **domain**, and a second set, called the **range**, such that each member of the domain corresponds to exactly one member of the range.
12. Students were then presented examples and counter examples from the text based on this definition. They were asked to determine if the following correspondence as a function and explain.







13. Students were then given a real-life examples and counter examples based on this definition. The example was product-price related. The cans of corn example as discussed in chapter 4.
14. Students were instructed to give their own real-life example of a function using any of the definitions or information we had covered. Examples could not be similar to the product-price example presented by the instructor.
15. The following class period the real life examples given by the students were discussed by the whole class to determine if they fit the description of a function relative to the definition.
16. The instructor then gave the students a table of x and y coordinates that fit some application problem. The students were then led through steps 2 – 10. This put in place the multi-Reps requirement of constructive habituation.
17. Students were given homework that reinforced the two days instruction.

VITA

Alonzo F. Peterson was born in the city of Pineville, LA, the son of Wilbert Hudson and Gloria J. Peterson. After graduating with honors from Pineville High School, he accepted an academic scholarship to Southern University at Baton Rouge. He left Southern University and attended Louisiana State University for several years. He returned to Southern University and received a B.S. in mathematics. He was awarded a graduate assistantship at Southern University and soon obtained an M.S. degree in computational and applied mathematics. At the urging of several of his professors at Southern University, he applied for the doctoral program in the Department of Curriculum and Instruction at Louisiana State University. He was accepted and awarded a Louisiana Board of Regents fellowship.

In the second year of his doctoral program, Alonzo began his professional career as an instructor of mathematics at Southern University. He was a member of numerous committees within the mathematics department. In particular, he was chair of the Recruiting Committee, and the Black History Program Committee. In service to the university, he served as a senator on the faculty senate, a committee member for the Louisiana High School District Rally Association, and as a member of the university's Reseacher of the Year selection committee.

He has held membership in several professional, service, and social organizations including: Phi Delta Kappa, Mathematics Association of America, Pi Mu Epsilon-National Honor Society for Mathematics, Alpha Phi Omega-National Service Fraternity, and Alpha Phi Alpha Fraternity, Inc.