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The Heine-Stieltjes correspondence and the polynomial approach to the Gaudin-Richardson models

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Abstract. The Heine-Stieltjes correspondence is extended and applied to solve Bethe ansatz equations of the Gaudin-Richardson models, from which the extended Heine-Stieltjes polynomial approach to these models is proposed. As examples for the application of this approach, exact solutions of the standard two-site Bose-Hubbard model and the standard pairing model for nuclei are formulated from the corresponding polynomials.

1. Introduction

It is well known that there are only a few problems that can be solved analytically in quantum mechanics. The one-dimensional harmonic oscillator and the hydrogen atom are such typical systems, which become important examples to be illustrated in standard textbook of quantum mechanics. As noted by Wigner in 1960, “the miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve”. But the application of mathematics to problems in physics can also, in turn, bring wider recognition to otherwise little-known mathematical concepts [1]. For example, the Bethe ansatz method for finding the exact eigenvalues and eigenvectors of the one-dimensional antiferromagnetic Heisenberg model Hamiltonian [2], the quantum inverse scattering method for solving integrable models in 1+1 dimensions introduced by Faddeev [3], and the Yang-Baxter equation for solvable problems in statistical mechanics and one-dimensional integrable systems [4, 5] finally led to formulation of quantum groups and non-commutative geometry in mathematics [6]. In the application of the Bethe ansatz method, Gaudin [7] and Richardson [8] independently found exact solutions for the spin or nuclear pairing many-body problems, which are referred to as the Gaudin-Richardson models. In this talk, we report our recent work on relation of solutions of the Bethe ansatz equations (BAE) for the Gaudin-Richardson models and zeros of the extended Heine-Stieltjes polynomials.

2. The Heine-Stieltjes correspondence

There is a large class of polynomials, called Heine-Stieltjes polynomials $y(x)$, which satisfy a second-order Fuchsian equation

$$A(x)\frac{d^2y(x)}{dx^2} + B(x)\frac{dy(x)}{dx} + V(x)y(x) = 0, \quad (1)$$

where $A(x)$ is a polynomial of degree m with $A(x) = \prod_{i=1}^m (x - a_i)$, $B(x)$ is a polynomial of degree $m - 1$ such that for a set of real positive parameters $\{\gamma_i\}$ and another set of real parameters $\{a_i\}$ is given as

$$B(x)/A(x) = \sum_{i=1}^m \frac{\gamma_i}{x - a_i}, \quad (2)$$

and $V(x)$ is an unknown, but to be determined polynomial of degree $m - 2$ that is allowed to depend on the solution $y(x)$. The latter are often called Van Vleck polynomials [9]. The case with $m = 2$ corresponds to the hypergeometric differential equation, while the case with $m = 3$ corresponds to the Heun equation [10]. Such Heine-Stieltjes polynomials and properties of their zeros have been studied extensively. If the polynomials $A(x)$ and $B(x)$ are algebraically independent, i.e. they do not satisfy any algebraic equation with integer coefficients, Heine proved that for every integer k there exist at most $\sigma(k) = (k + m - 2)! / ((m - 2)!k!)$ different Van Vleck polynomials $V(x)$ such that $y(x)$ has a polynomial solution of degree k . As summarized in Szegő's work on orthogonal polynomials [11], for every set of nonnegative integers (k_1, k_2, \dots, k_m) , there are uniquely determined real values of the parameters for the Van Vleck polynomials $V(x)$ such that $y(x)$ has a polynomial solution with k_j simple zeros in the open interval (a_{j-1}, a_j) for each $j = 1, 2, \dots, m$. The polynomial $y(x)$ is uniquely determined up to a constant factor, and has the degree $k = k_1 + \dots + k_m$. If $y(x)$ is a polynomial of degree k with simple zeros $\{x_1, x_2, \dots, x_k\}$, one may write $y(x)$ as

$$y(x) = \prod_{j=1}^k (x - x_j). \quad (3)$$

Hence, at any zero x_i , $y(x)$ satisfies

$$\frac{y''(x_i)}{y'(x_i)} = \sum_{j \neq i} \frac{2}{x_i - x_j}. \quad (4)$$

Combining (1), (2), and (4), one obtains the following important relations among the zeros:

$$\sum_{j \neq i}^k \frac{2}{x_i - x_j} + \sum_{\mu=1}^m \frac{\gamma_\mu}{x_i - a_\mu} = 0 \quad (5)$$

for $i = 1, 2, \dots, k$. It should be noted that the BAEs frequently appearing in search for exact solutions of quantum many-body problems, such as those associated with Gaudin type systems, are similar to the relations shown in (5). The link between Richardson's BCS pairing model for nuclei and the corresponding electrostatic problem was investigated in [12] based on an earlier unpublished preprint of Gaudin, which was then made clearer in [13]. A much more general approach to the pairing model was shown in [14] and [15]. Roots of the BAEs (5) simultaneously determine the eigenenergies and eigenstates of the corresponding quantum many-body problem. As an alternative, roots of the BAEs may also be calculated as zeros of the corresponding polynomial $y(x)$. In this way, a link between solutions of the Gaudin type for quantum many-body problems and the corresponding polynomials is established.

3. The standard two-site Bose-Hubbard model

As a simple extension of the Stieltjes correspondence, we revisit the Bethe ansatz solutions for the standard two-site Bose-Hubbard model studied in [16-18], for which the Hamiltonian is

$$\hat{H}_{\text{BH}} = -t(c^\dagger d + d^\dagger c) + U(c^\dagger c c^\dagger c + d^\dagger d d^\dagger d), \quad (6)$$

where c^\dagger (c) and d^\dagger (d) are boson creation (annihilation) operators. The parameters t and U in the Bose-Hubbard model are related to the Josephson coupling and the charging energy, respectively. Following [16, 18], we use the unitary transformation for the boson operators with $c = (a - ib)/\sqrt{2}$ and $d = (a + ib)/\sqrt{2}$. Then, the Hamiltonian (6) can be rewritten in terms of a - and b -boson operators as

$$\hat{H}_{\text{BH}} = t(b^\dagger b - a^\dagger a) - \frac{1}{2}UB^+(0)B^-(0) + U\hat{n}^2, \quad (8)$$

where $\hat{n} = a^\dagger a + b^\dagger b$ is the operator for the total number of bosons in the system, and

$$B^+(0) = b^{\dagger 2} + a^{\dagger 2}, \quad B^-(0) = b^2 + a^2 \quad (9)$$

are boson pairing operators. Let $|\nu_1, \nu_2\rangle = a^{\dagger\nu_1} b^{\dagger\nu_2} |0\rangle$ with $\nu_{1,2} = 0$ or 1 be the a - and b -boson pairing vacuum state satisfying

$$a^2 |\nu_1, \nu_2\rangle = b^2 |\nu_1, \nu_2\rangle = 0. \quad (10)$$

The algebraic Bethe ansatz implies that eigenvectors of (8) may be expressed as

$$|n, \zeta, \nu_1, \nu_2\rangle = B^+(x_1^{(\zeta)})B^+(x_2^{(\zeta)}) \cdots B^+(x_k^{(\zeta)}) |\nu_1, \nu_2\rangle \quad (11)$$

with $n = 2k + \nu_1 + \nu_2$, and

$$B^+(x_i^{(\zeta)}) = \frac{a^{\dagger 2}}{x_i^{(\zeta)} + 1} + \frac{b^{\dagger 2}}{x_i^{(\zeta)} - 1}, \quad (12)$$

in which $x_i^{(\zeta)}$ ($i = 1, 2, \dots, k$) are spectral parameters to be determined, and ζ is an additional quantum number for distinguishing different eigenvectors with the same quantum number k . It can then be verified by using the corresponding eigenequation that (11) is a solution when the spectral parameters $x_i^{(\zeta)}$ ($i = 1, 2, \dots, k$) satisfy the following set of BAEs:

$$U \left(\frac{2\nu_1 + 1}{x_i^{(\zeta)} + 1} + \frac{2\nu_2 + 1}{x_i^{(\zeta)} - 1} \right) + 2t + 4U \sum_{j(\neq i)} \frac{1}{x_i^{(\zeta)} - x_j^{(\zeta)}} = 0 \quad (13)$$

for $i = 1, 2, \dots, k$, with the corresponding eigenenergy given by

$$E_n^{(\zeta)} = 2t \sum_{i=1}^k x_i^{(\zeta)} + t(\nu_2 - \nu_1) + Un^2. \quad (14)$$

Hence, once the ζ -th roots $\{x_i^{(\zeta)}\}$ are obtained from (13), the eigenenergy and the corresponding eigenstate are thus determined according to (14), (11), and (12).

In order to compare to the Heine-Stieltjes correspondence, we assume $t > 0$ and $U \neq 0$, and set

$$\alpha = \nu_2 + 1/2, \quad \beta = \nu_1 + 1/2, \quad \gamma = t/U. \quad (15)$$

Then, the BAEs (13) become

$$\left(\frac{\alpha}{x_i^{(\zeta)} - 1} + \frac{\beta}{x_i^{(\zeta)} + 1} \right) + \gamma + \sum_{j(\neq i)} \frac{2}{x_i^{(\zeta)} - x_j^{(\zeta)}} = 0. \quad (16)$$

When $U > 0$, all parameters α , β and γ are always positive. When $U < 0$, we interchange the boson operator a^\dagger with b^\dagger in (12). Then, one can easily verify that this yields the BAEs (13) with the interchange of ν_1 with ν_2 and $t \rightarrow -t$. Hence, in the $U < 0$ case, one obtains the same set of final BAEs as in (16) but with $\alpha \rightleftharpoons \beta$ and $\gamma = -t/U$, thereby keeping $\alpha > 0$, $\beta > 0$, and $\gamma > 0$. Hence, it is sufficient to consider the case for $U > 0$ only with BAEs given by (16). Although the $\gamma = 0$ result is trivial, according to the Stieltjes correspondence, the polynomial corresponding to (16) in this case is the Jacobi polynomial $P_k^{(\alpha-1, \beta-1)}(x)$ satisfying the well-known differential equation

$$\frac{d^2}{dx^2} P_k^{(\alpha-1, \beta-1)}(x) + \left(\frac{\alpha}{x-1} + \frac{\beta}{x+1} \right) \frac{d}{dx} P_k^{(\alpha-1, \beta-1)}(x) - \frac{k(k+\alpha+\beta-1)}{x^2-1} P_k^{(\alpha-1, \beta-1)}(x) = 0. \quad (17)$$

In this case, the Van Vleck polynomial $V(x)$ is trivially a k -dependent constant. Hence, there is only one set of zeros of $P_k^{(\alpha-1, \beta-1)}(x_i)$ with $i = 1, 2, \dots, k$, satisfying the Bethe ansatz equation (16). The case with $\gamma \neq 0$ is non-trivial. According to (1)-(5), we write a differential equation corresponding to the Bethe ansatz equation (16) as

$$\frac{d^2 y(x)}{dx^2} + \left(\frac{\alpha}{x-1} + \frac{\beta}{x+1} + \gamma \right) \frac{dy(x)}{dx} + \frac{V(x)}{x^2-1} y(x) = 0 \quad (18)$$

with the corresponding polynomials $A(x) = x^2 - 1$ and $B(x) = \gamma x^2 + (\alpha + \beta)x + \alpha - \beta - \gamma$ shown in (1). In contrast to the Heine-Stieltjes equation (1), however, in this case the polynomial $B(x)$ is of the same degree as that of $A(x)$. Therefore, the polynomials $y(x)$ determined by (18) should be similar to but different from those of Heine-Stieltjes type.

In the search for polynomial solutions of (18), we write [19]

$$y_k(x) = \sum_{j=0}^k b_j x^j. \quad (19)$$

Substitution of (19) into (18) determines the corresponding polynomial $V(x)$ as

$$V(x) = -\gamma kx + f, \quad (20)$$

where f is an undetermined constant depending on k , and together with the unknown expansion coefficients b_j satisfy the following four-term relations:

$$(j(\alpha + \beta + j - 1) + f) b_j = (j + 2)(j + 1) b_{j+2} + (j + 1)(\beta - \alpha + \gamma) b_{j+1} + \gamma(k - j + 1) b_{j-1} \quad (21)$$

with $b_j = 0$ for $j \leq -1$ or $j \geq k + 1$. It is clear that Eq. (21) can be represented as a matrix with $k + 1$ eigenvalues labeled as $f^{(\zeta)}$ ($\zeta = 1, 2, \dots, k + 1$) and with corresponding eigenvectors $\{b_1^{(\zeta)}, \dots, b_k^{(\zeta)}\}$, which determine the polynomial $y_k^{(\zeta)}(x)$ according to (19). The number of solutions for f is indeed the same as that of the eigenstates of the standard two-site Bose-Hubbard Hamiltonian (8) when $\gamma \neq 0$. The polynomials (19) with $\gamma \neq 0$ determined by (21) are called the extended Heine-Stieltjes polynomials.

4. The standard pairing model

The Hamiltonian of the standard pairing model is given by

$$\hat{H}_{\text{SP}} = \sum_{j=j_1}^{j_n} \epsilon_j \hat{n}_j - G \sum_{jj'} S_j^+ S_{j'}^-, \quad (22)$$

where n is the total number of levels considered, $G > 0$ is the overall pairing strength, $\{\epsilon_j\}$ are unequal single-particle energies, $\hat{n}_j = \sum_m a_{jm}^\dagger a_{jm}$ is the number operator for the valence particles in the j -th level, and $S_j^+ = \sum_{m>0} (-)^{j-m} a_{jm}^\dagger a_{j-m}^\dagger$ ($S_j^- = (S_j^+)^\dagger$) are pair creation (annihilation) operators. Since the formalism for even-odd systems is similar, in the following, we only focus on the even-even seniority zero case. According to the Richardson-Gaudin method, k -pair eigenstates of (22) can be written as

$$|k; x\rangle = S^+(x_1) S^+(x_2) \cdots S^+(x_k) |0\rangle, \quad (23)$$

where $|0\rangle$ is the pairing vacuum state satisfying $S_j^- |0\rangle = 0$ for all j , and

$$S^+(x_i) = \sum_j \frac{1}{x_i - 2\epsilon_j} S_j^+, \quad (24)$$

in which x_i ($i = 1, 2, \dots, k$) are spectral parameters to be determined. It can then be verified by using the corresponding eigenequation that (23) is the eigenstates of (22) only when the spectral parameters x_i ($i = 1, 2, \dots, k$) satisfy the following set of BAEs:

$$1 - 2G \sum_j \frac{\rho_j}{x_i - 2\epsilon_j} - 2G \sum_{i'(\neq i)} \frac{1}{x_i - x_{i'}} = 0, \quad (25)$$

where $\rho_j = -(j + 1/2)/2$, with the corresponding eigenenergy given by $E_{n, k} = \sum_{i=1}^k x_i$.

According to Heine-Stieltjes correspondence, for nonzero pairing strength G , $A(x)$ and $B(x)$ of the Fuchsian equation (1), which corresponds to Eq. (25), are given by

$$A(x) = \prod_{j=j_1}^{j_n} (x - 2\epsilon_j) \quad (26)$$

and

$$B(x)/A(x) = \sum_{j=j_1}^{j_n} \frac{2\rho_j}{x - 2\epsilon_j} - \frac{1}{G}. \quad (27)$$

The corresponding Van Vleck polynomial $V(x)$ in this case is of degree $n - 1$, which needs to be determined according to Eq. (1). In the original electrostatic analogue considered by Heine and Stieltjes, the parameters $\{\rho_j\}$ acting as fixed charges should all be positive with no external

electrostatic field, $1/G \rightarrow 0$. Therefore, the polynomials $y(x)$ satisfying Eq. (1) with negative $\{\rho_j\}$ and $1/G \neq 0$ in this case are called the extended Heine-Stieltjes polynomials, which tend to be the original Heine-Stieltjes polynomials with negative $\{\rho_j\}$ in the $G \rightarrow \infty$ limit.

In the search for polynomial solutions of (1) for this case, we write [20]

$$y(x) = \sum_{j=0}^k a_j x^j, \quad V(x) = \sum_{j=0}^{n-1} b_j x^j, \quad (28)$$

where $\{a_j\}$ and $\{b_j\}$ are expansion coefficients to be determined. Substitution of Eqs. (26)-(28) into Eq. (1) yields two matrix equations. Namely, the condition that the coefficients in front of x^i ($i = 0, \dots, k$) must be zero generates a $(k+1) \times (k+1)$ matrix \mathbf{F} with $\mathbf{F}\mathbf{v} = b_0\mathbf{v}$, where the eigenvector \mathbf{v} of \mathbf{F} is given by the expansion coefficients $\mathbf{v} = \{a_0, \dots, a_k\}$, while the condition that the coefficients in front of x^i ($i = k+1, \dots, n+k-1$) must be zero generates another $(n-1) \times (k+1)$ upper-triangular matrix \mathbf{P} with $\mathbf{P}\mathbf{v} = 0$, which provides a unique solution for $\{b_1, b_2, \dots, b_{n-1}\}$ in terms of $\{a_j\}$. Entries of the two matrices are all linear in the coefficients $\{b_1, b_2, \dots, b_{n-1}\}$. Matrices \mathbf{F} and \mathbf{P} can be easily constructed, for which a simple MATHEMATICA code is now available [21]. Applications of the method to Oxygen isotopes, ^{16}O to ^{28}O , reveal a good agreement of the calculated pairing gaps as compared to the corresponding experimental values (Fig. 1).

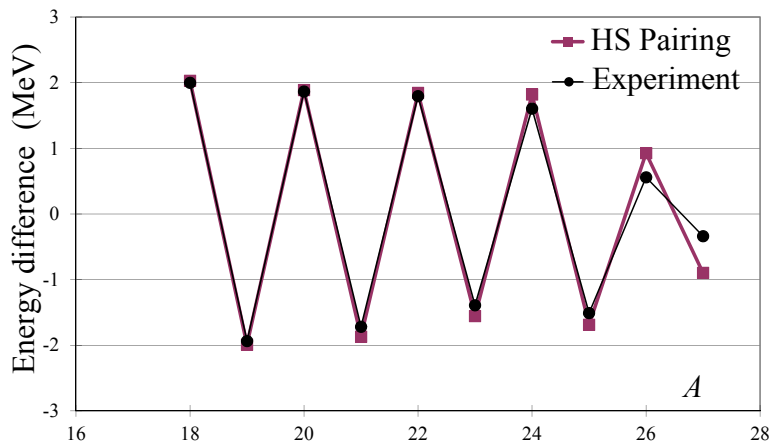


Figure 1. Energy difference, $(E(A-2) - 3E(A-1) + 3E(A) - E(A+1))/4$ (with an absolute value yielding the pairing gap) of the E binding energies for Oxygen isotopes as a function of the mass number A . Calculations (“HS pairing”) are performed for the sd shell using $G = 15/A$ MeV and single-particle energies derived from $^{16,17}\text{O}$ experimental data.

5. Summary

In this talk, the Heine-Stieltjes correspondence is extended and applied to solve Bethe ansatz equations of the Gaudin-Richardson models. From this, the extended Heine-Stieltjes polynomial approach to the Gaudin-Richardson models is proposed. Solutions of the standard two-site Bose-Hubbard model and the standard pairing model for nuclei are formulated from the corresponding extended Heine-Stieltjes polynomials. It follows that the eigenvalues and eigenstates of these models can be determined from the zeros of the corresponding polynomials. Furthermore, if these polynomials can be derived recursively, it also follows that it would be much easier to determine zeros from the polynomials of one variable than to solve a set of BAEs with a set of variables. The results by extension clearly show a new link between Bethe ansatz type solutions of a large class of Gaudin-Richardson quantum many-body problems and a class of extended Heine-Stieltjes polynomials satisfying second-order differential equations, which, in turn, opens a new path to find solutions of these BAEs from the zeros of the polynomials.

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