

2005

Wavelet sets with and without groups and multiresolution analysis

Mihaela Dobrescu

Louisiana State University and Agricultural and Mechanical College

Follow this and additional works at: https://repository.lsu.edu/gradschool_dissertations



Part of the [Applied Mathematics Commons](#)

Recommended Citation

Dobrescu, Mihaela, "Wavelet sets with and without groups and multiresolution analysis" (2005). *LSU Doctoral Dissertations*. 3017.

https://repository.lsu.edu/gradschool_dissertations/3017

This Dissertation is brought to you for free and open access by the Graduate School at LSU Scholarly Repository. It has been accepted for inclusion in LSU Doctoral Dissertations by an authorized graduate school editor of LSU Scholarly Repository. For more information, please contact gradetd@lsu.edu.

WAVELET SETS WITH AND WITHOUT GROUPS AND
MULTIRESOLUTION ANALYSIS

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

Mihaela Dobrescu

B.S. in Math., University of Bucharest, 1995

M.S., Louisiana State University, 2002

August, 2005

Acknowledgments

This dissertation would not be possible without several contributions.

First and foremost I want to thank my advisor, Gestur Ólafsson, for his intellectual and material support.

It is a pleasure to thank Professors Mark Davidson, Ray Fabec, Jimmie Lawson, Richard Litherland, Frank Neubrandner, James Oxley, and Robert Perlis, for their valuable instructions, priceless advice, and professional help provided during my graduate studies.

I want to thank my family and to all my friends who ever helped me throughout my graduate studies.

Last but not least, I thank my husband. Without his endless patience, love, and support of all kind, intellectual, technical and moral, this dissertation would not be what it is now. Thank you, Michael.

Table of Contents

| | |
|--|-----------|
| Acknowledgments | ii |
| Abstract | v |
| 1 Introduction | 1 |
| 2 Integral Transforms | 4 |
| 2.1 Function Spaces | 4 |
| 2.2 Convolution in \mathbb{R}^n | 5 |
| 2.3 Fourier Transform | 6 |
| 2.4 Fourier Series | 7 |
| 2.5 Heisenberg Uncertainty Principle | 9 |
| 2.6 The Shannon Sampling Theorem | 10 |
| 2.7 Windowed Fourier Transform | 11 |
| 2.8 Continuous Wavelet Transform | 12 |
| 3 Wavelets | 14 |
| 3.1 Multiresolution Analysis | 14 |
| 3.2 Construction of Wavelets from a MRA | 15 |
| 3.3 Band-limited Wavelets | 18 |
| 3.4 The Haar System | 19 |
| 3.5 The Shannon Wavelet | 20 |
| 3.6 The Daubechies Wavelet | 21 |
| 3.7 Minimally Supported Frequency Wavelets | 22 |
| 4 Wavelet Sets in \mathbb{R}^2 | 23 |
| 4.1 Spectral Sets | 23 |
| 4.2 Tilings | 23 |
| 4.3 Fuglede's Conjecture | 24 |
| 4.4 Benedetto and Sumetkijakan's Construction | 25 |
| 4.5 Existence of Subspace Wavelet Sets | 28 |
| 4.6 Rotations | 34 |
| 4.7 Hyperbolic Rotation | 37 |
| 4.8 Example of Dilations Which Are Not Groups | 38 |
| 5 Coxeter Groups | 42 |
| 5.1 Roots | 44 |
| 5.2 Fundamental Domain | 45 |
| 5.3 Wavelet Sets in \mathbb{R}^n | 46 |
| 5.4 Wavelet Sets and SMRA | 48 |
| 5.5 Multiresolution and Coxeter Groups | 51 |

| | |
|---|-----------|
| 5.6 Wavelet Sets and Coxeter Groups | 53 |
| References | 56 |
| Vita | 58 |

Abstract

In this dissertation we study a special kind of wavelets, the so-called minimally supported frequency wavelets and the associated wavelet sets. Most of the examples of wavelet sets are for dilation sets which are groups. In this work we construct wavelet sets for which the dilation set, \mathcal{D} , is of the form $\mathcal{D} = \mathcal{M}\mathcal{N}$, where the product is direct, and so \mathcal{D} is not necessarily group. In the second part of this dissertation we construct multiwavelets associated with MRA's and we generalize the rotations in the dilation sets to Coxeter groups.

1. Introduction

Wavelets have been used in mathematics, physics, and signal or image processing long time before they were given the name and the importance they have now.

Wavelets are an extension of Fourier analysis. The periodic exponentials, $e^{-2\pi it\lambda}$, which are used as basis in Fourier analysis, are replaced in wavelet analysis by translates and dilates of a single function, called mother wavelet.

One of the obvious question is the construction of wavelets with given properties.

The classical wavelet system on the line, is a function $\psi \in L^2(\mathbb{R})$, such that the dyadic dilates and integer translates of ψ form a orthonormal basis for $L^2(\mathbb{R})$. Thus, $\{\psi_{j,n}\}_{j,n \in \mathbb{Z}}$ with $\psi_{j,n}(t) = 2^{j/2}\psi(2^j t + n)$, is an orthonormal basis for $L^2(\mathbb{R})$. There are several obvious generalization: One can replace 2 by any integer N ; one can allow several functions ψ^1, \dots, ψ^L ; and one can consider orthonormal basis for a closed subspace of $L^2(\mathbb{R})$. There have also been several publications of wavelets in higher dimensions, cf. [1, 2, 3, 4, 10, 12, 13, 23, 5, 20, 21] to name few. One of the difference in higher dimensions is, that we now have much more choices in the sets of dilations and translations.

An important way of constructing wavelets involves the concept of a *multiresolution analysis* or MRA. This method is completely recursive and therefore it is ideal for computations; a signal f_0 is split into a blurred version f_1 at a coarser resolution and a detail version, d_1 . By repeating this process, one gets a sequence of coarser and coarser versions f_0, f_1, \dots of the signal together with the details d_1, d_2, \dots . The interesting thing is that each detail is a linear combination of dilations and translations of a mother wavelet.

So, to fix the notation, let $\mathcal{D} \subseteq GL(n, \mathbb{R})$ and $\mathcal{T} \subseteq \mathbb{R}^n$ countable sets. A $(\mathcal{D}, \mathcal{T})$ -*wavelet* is a square integrable function ψ with the property that the set

$$\{|\det d|^{\frac{1}{2}}\varphi(dx + t) \mid d \in \mathcal{D}, t \in \mathcal{T}\} \quad (1.1)$$

forms an orthonormal basis for $L^2(\mathbb{R}^n)$. The set \mathcal{D} is then called the dilation set and the set \mathcal{T} is called the translation set. If we replace $L^2(\mathbb{R}^n)$ in the above definition by

$$L^2_M(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) \mid \text{supp } \hat{f} \subseteq M\}$$

for some measurable subset $M \subseteq \mathbb{R}^n$, $|M| > 0$, we get a $(\mathcal{D}, \mathcal{T})$ -subspace wavelet. Here \mathcal{F} stands for the Fourier transform

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i \langle \lambda, x \rangle} dt.$$

The most natural starting point is to consider groups of dilations and full rank lattices as translation sets. The simplest examples would then be groups generated by one element $\mathcal{D} = \{a^k \mid k \in \mathbb{Z}\}$, see [24] and the reference therein. In [20, 21] more general sets of dilations were considered, and in general those dilations do not form a group. Even more general constructions can be found in [1].

In this dissertation, we consider a special class of wavelets corresponding to *wavelet sets*. Those are functions ψ such that $\mathcal{F}(\psi) = \chi_\Omega$, and Ω is a measurable subset of \mathbb{R}^n . The wavelet property is then closely related to geometric properties of the set Ω , in particular spectral and tiling properties of Ω . The study of wavelet sets then becomes an interplay between group theory, geometry, operator theory and analysis, cf. [7, 9]. One of our results is an existence Theorem for such wavelets for some special dilation sets \mathcal{D} which are *not* necessarily groups, see Theorem 4.16 and Theorem 4.18.

The first chapter reviews some elements of Fourier transform, Fourier series, the windowed Fourier transform and the continuous wavelet transform that are essential to a proper understanding of wavelet analysis..

Chapter 3 provides an exposition of the general notion of multiresolution analysis in one dimension, followed by a recipe for constructing wavelet orthonormal bases, and some examples of such wavelets.

Chapter 4 and Chapter 5 contain our main results. In chapter 4 we give a constructive existence theorem for wavelet sets with dilation sets \mathcal{D} of the form

$$\mathcal{D} = \mathcal{M}\mathcal{N},$$

where the product is direct, and it ends with three examples of such wavelet sets.

In Chapter 5 we present a method of constructing scaling sets and the associated multiresolution analysis and wavelets, corresponding to Coxeter groups as dilations. We end Chapter 5 and this dissertation with several examples which describe very nicely our main results.corresponding

2. Integral Transforms

We start this chapter by describing several function spaces which are well suited for the Fourier transform and the wavelet transform.

2.1 Function Spaces

Let

$$L^p(\mathbb{R}^n) = \{f : \mathbb{R}^n \rightarrow \mathbb{C} \mid f \text{ measurable, } [\int_{\mathbb{R}^n} |f(x)|^p dx]^{\frac{1}{p}} < \infty\},$$

be the space of p -Lebesgue integrable functions.

In particular, we will mostly use the space of integrable functions on \mathbb{R}^n , $L^1(\mathbb{R}^n)$, and the space of square integrable functions, or finite energy functions, $L^2(\mathbb{R}^n)$.

Another important vector space of functions is the space of rapidly decreasing smooth functions on \mathbb{R}^n , which consists of the smooth functions f , satisfying

$$\sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N |D^\alpha f(x)| < \infty,$$

for all $N \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. This space with seminorms $|\cdot|_{N,\alpha}$ given by

$$|f|_{N,\alpha} = \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^N |D^\alpha f(x)|$$

is the Schwartz space of rapidly decreasing smooth functions on \mathbb{R}^n and is denoted by $\mathcal{S}(\mathbb{R}^n)$. It contains $C_c^\infty(\mathbb{R}^n)$, the space of smooth functions with compact support, as a vector subspace. Note that other norms and seminorms, in particular the p -norms, are continuous in the Schwartz topology.

Proposition 2.1. *The space $C_c^\infty(\mathbb{R}^n)$ is dense in $\mathcal{S}(\mathbb{R}^n)$. Moreover, $C_c^\infty(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, for $1 \leq p < \infty$.*

2.2 Convolution in \mathbb{R}^n

Assume that f and g are functions on \mathbb{R}^n such that the function $y \rightarrow f(y)g(x-y)$ is integrable for almost all x . Then the function

$$f * g(x) := \int f(y)g(x-y)dy$$

is defined almost everywhere and it is called *the convolution* of f and g . The convolution has some very nice properties.

- i) If f, g are complex valued measurable functions on \mathbb{R}^n , then $f * g(x)$ exists iff $g * f(x)$ exists and then $f * g(x) = g * f(x)$;
- ii) Let $1 \leq p \leq q \leq \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g(x)$ exists, $f * g$ is continuous and $|f * g(x)| \leq |f|_p |g|_q$.
- iii) Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, where $1 \leq p < \infty$. Then $f * g \in L^p(\mathbb{R}^n)$ and $|f * g|_p \leq |f|_1 |g|_p$. In particular, the mapping $g \rightarrow f * g$ is a bounded linear operator on $L^p(\mathbb{R}^n)$.
- iv) Let $f \in L^1(\mathbb{R}^n)$ and $g \in \mathcal{S}(\mathbb{R}^n)$. Then $f * g$ is smooth and

$$p(D)(f * g) = f * (p(D)g)$$

for any differential operator of the form $p(D) = \sum a_\alpha D^\alpha$ with constant coefficients a_α .

- v) If $f, g \in \mathcal{S}(\mathbb{R}^n)$, then $f * g \in \mathcal{S}(\mathbb{R}^n)$ and the bilinear map

$$\mathcal{S}(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n), (f, g) \rightarrow f * g$$

is continuous.

2.3 Fourier Transform

We start by defining the Fourier transform of an integrable function.

Let

$$e_\lambda : \mathbb{R}^n \rightarrow \mathbb{C}, e_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle}.$$

For $f \in L^1(\mathbb{R}^n)$, define

$$\hat{f}(\lambda) = \mathcal{F}(f)(\lambda) = \int_{\mathbb{R}^n} f(x) e_\lambda(-x) dx = \int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle \lambda, x \rangle} dx$$

exists for each $\lambda \in \mathbb{R}^n$. The function \hat{f} is called *the Fourier transform* of the L^1 function f and is continuous at each point $\lambda \in \mathbb{R}^n$.

We define now the following linear transformations:

$$\lambda(y)f(x) = f_{y,0} = f(x - y)$$

$$\delta(a)f(x) = f_{0,a^{-1}} = a^{-\frac{n}{2}} f(a^{-1}x)$$

and

$$\tau(y)f(x) = e^{-2\pi i \langle x, y \rangle} f(x).$$

We will list some of the most important properties of the Fourier transform in the following lemma.

Lemma 2.2. *Let $f \in L^1(\mathbb{R}^n)$. Then the following holds:*

i) $\hat{f} \in C(\mathbb{R}^n)$ and $|\hat{f}|_\infty \leq |f|_1$

ii) $\widehat{\lambda(y)f} = \tau(y)\hat{f}$

iii) $\widehat{\tau(y)f} = \lambda(y)\hat{f}$

iv) $\widehat{\delta(a)f} = \delta(a^{-1})\hat{f}$

Note that \hat{f} may not be in $L^1(\mathbb{R}^n)$ and $\hat{\hat{f}}$ may not exist. That is why one needs a nicer space to work with. For that we consider the Schwartz space, $\mathcal{S}(\mathbb{R}^n)$.

Proposition 2.3. *If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\hat{f} \in L^1(\mathbb{R}^n)$. Moreover, $\hat{\hat{f}}$ exists and*

$$\hat{\hat{f}}(x) = f(-x).$$

The fact that $\mathcal{S}(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$ leads to the Plancherel theorem.

Theorem 2.4 (Plancherel Theorem). *The mapping $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n)$ extends to an unitary isometry of $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$. We denote this extension again by \mathcal{F} or \hat{f} and we have that $\mathcal{F}^2 f(x) = f(-x)$.*

Note that if $f \in L^2(\mathbb{R}^n)$, then \hat{f} is the L^2 limit of any sequence $\{\hat{f}_k\}_k$, where f_k are Schwartz functions converging in L^2 to f . The Fourier transform \hat{f} may not be given by the integral

$$\int_{\mathbb{R}^n} f(x) e^{-2\pi i \langle \lambda, x \rangle} dx,$$

since this integral might not exist.

2.4 Fourier Series

Harmonic analysis attempts to understand complicated periodic functions in terms of simple ones. It was Jean Baptiste Joseph Fourier who developed the idea that a periodic function f , of period 1, can be written as an infinite sum of harmonics

$$\sum \hat{f}(n) e^{2\pi i n \theta}.$$

Definition 2.5. *A measurable function $f : \mathbb{R} \rightarrow \mathbb{C}$ is periodic if there exists a positive number L , called the period, such that $f(x + L) = f(x)$ for almost all $x \in \mathbb{R}$.*

When the period is 2π , the domain is the unit circle.

For $f \in L^1([0, 1])$, define $\hat{f} : \mathbb{Z} \rightarrow \mathbb{C}$ by

$$\hat{f}(n) = \int_0^1 f(t) e^{-2\pi i n t} dt.$$

Definition 2.6. The Fourier series of a function $f \in L^1[0, 1]$, is an expression of the form

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi in\theta}.$$

Proposition 2.7. If $f \in L^1([0, 1])$ and $\hat{f}(n) = 0$ for all n , then $f = 0$ a.e.

Let l^2 be the space of square summable sequences $\{a_n\}_{n \in \mathbb{N}}$, i.e.

$$l^2 = \{\{a_n\}_{n \in \mathbb{N}} \mid \sum_{n=1}^{\infty} |a_n|^2 < \infty\}.$$

This is a Hilbert space and more than that, l^2 is isomorphic with $L^2([0, 1])$.

Theorem 2.8. (Plancherel Theorem) The Fourier transform $\mathcal{F} : L^2([0, 1]) \rightarrow l^2$ is an isomorphism of Hilbert spaces.

Theorem 2.9. (Poisson summation formula) Let f be a smooth function such that $f(x)(1+x^2)^N$ is bounded for all $N \in \mathbb{N}$. Then

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{2\pi inx}.$$

The interesting question about the Fourier series is if they converge, and if the answer is positive, then how do they converge?

Theorem 2.10. If f is a C^∞ periodic function of period 1, then its Fourier series $\sum \hat{f}(n)e^{2\pi in\theta}$ converges uniformly to f and the derivatives of these series converge uniformly to the derivatives of f .

What if the function $f(x)$ is not even once differentiable?

Definition 2.11. A function f on a finite interval I is called piecewise differentiable on I if it is piecewise continuous with only jump discontinuities if any, if f' exists at all points in I but finitely many and if f' is piecewise continuous with only jump discontinuities if any.

A function which is piecewise differentiable has, as Gustave Lejeune Dirichlet showed, a pointwise convergent Fourier series.

Theorem 2.12. *Let f be a periodic, piecewise differentiable function, of period 1.*

Then the sequence of partial sums of the Fourier series of f , $\{S_N(x)\}$, where

$$S_N(x) = \sum_{n=-N}^N \hat{f}(n)e^{2\pi inx},$$

converges pointwise to $\tilde{f}(x) = \frac{1}{2}[\lim_{t \rightarrow x^-} f(t) + \lim_{t \rightarrow x^+} f(t)]$.

If we assume that $f(x)$ is continuous, then we get a stronger version of the above theorem.

Theorem 2.13. *Let $f(x)$ be a continuous, periodic, piecewise differentiable function, of period 1. Then the sequence of partial sums of the Fourier series of f , $\{S_N\}$, converges uniformly to f .*

2.5 Heisenberg Uncertainty Principle

A signal f and its Fourier transform \hat{f} cannot be simultaneously localized in a small domain. If the signal has compact support, then its Fourier transform spreads out to infinity. This phenomenon is described quantitatively in the famous Heisenberg Uncertainty Principle.

Theorem 2.14. *(Heisenberg Uncertainty Principle) Let $f \in L^2(\mathbb{R}^n)$. Then*

$$|xf|_2 \cdot |\lambda\hat{f}|_2 \geq \frac{1}{2}|f|_2^2.$$

This problem appears in any method one would use to decompose a signal simultaneously into time and frequency. Precise information about time can be obtained only by accepting a certain vagueness about frequency and the other way around.

2.6 The Shannon Sampling Theorem

One of the interesting questions about a signal, i.e. a measurable function on \mathbb{R}^n , is if it is possible to reconstruct it from discrete values completely. Most of the time the answer to this question is no. But with enough assumptions on the signal f , Shannon sampling theorem is one of the theorems which gives an affirmative answer to that questions.

Before stating this theorem, we need to define what *bandlimited functions* are.

Definition 2.15. *An integrable function f is called bandlimited, if its Fourier transform vanishes outside a compact set.*

For a measurable set $M \subset \mathbb{R}^n$, $|M| > 0$, set

$$L_M^2(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) \mid \text{supp} \hat{f} \subseteq M\}.$$

Theorem 2.16. *(The Shannon sampling theorem) Let $f \in L^2(\mathbb{R}^n)$ be a bandlimited function with $\text{supp} \hat{f} \subseteq [-\Omega, \Omega]$. Then f can be reconstructed from its values at a discrete number of points in its domain, by the following formula:*

$$f(t) = \sum_{n \in \mathbb{Z}} \frac{\sin[2\pi\Omega(t - nT)]}{2\pi\Omega(t - nT)} f(nT),$$

where $T = \frac{1}{2\Omega}$.

Note, however, that the function $t \rightarrow \text{sinc}(t) := \frac{\sin t}{t}$ decays very slowly.

2.7 Windowed Fourier Transform

One way to get around Heisenberg Uncertainty Principle is to use a windowed Fourier Transform. The Windowed Fourier transform, abbreviated WFT, uses a *window function* g , to give a better localization with respect to time and frequency.

Let $f, \psi \in L^2(\mathbb{R}^n)$. Then, for $b, \omega \in \mathbb{R}^n$, the function $x \rightarrow \psi(x - b)e^{2\pi i \langle \omega, x \rangle}$ is also square integrable and so, we can define the following operator in $L^2(\mathbb{R}^n)$:

$$S_\psi f(b, \omega) := \int_{\mathbb{R}^n} f(x) \overline{\psi(x - b)} e^{-2\pi i \langle \omega, x \rangle} dx.$$

This is a well defined linear operator. Some of the properties of the WFT are listed in the proposition below:

Proposition 2.17. *Let $f, \psi \in L^2(\mathbb{R}^n)$. Then the following holds:*

- i) $|S_\psi f(b, \omega)| \leq \|f\|_2 \|\psi\|_2$. In particular, the mapping $S_\psi f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}$ is continuous and bounded.
- ii) $S_\psi : L^2(\mathbb{R}^n) \rightarrow C_0(\mathbb{R}^n \times \mathbb{R}^n)$ is a linear map.
- iii) $S_\psi f(b, \omega) = e^{2\pi i \langle \omega, x \rangle} S_{\hat{\psi}} \hat{f}(\omega, -b)$

Theorem 2.18. (Plancherel) *Let $f, g, \psi, \phi \in L^2(\mathbb{R}^n)$. Then*

$$\langle S_\psi f, S_\phi g \rangle = \langle f, g \rangle \langle \psi, \phi \rangle.$$

Let $\phi_b^\omega(x) = \phi(x - b)e^{2\pi i \langle \omega, x \rangle}$.

Theorem 2.19. (Inversion) *Let $\psi, \phi \in L^2(\mathbb{R}^n)$. Then for all $f \in L^2(\mathbb{R}^n)$, the function*

$$(b, \omega) \rightarrow S_\psi f(b, \omega) \phi_b^\omega$$

is weakly integrable and

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} S_\psi f(b, \omega) \phi_b^\omega db d\omega = \langle \phi, \psi \rangle f.$$

2.8 Continuous Wavelet Transform

The Windowed Fourier transform combines the exponential functions with a *window function* to localize a signal in both time and frequency domain. To get rid completely of the exponential functions, we can dilate and translate enough the *window function* to get all the information about the signal.

We consider now the group of affine linear transformations on \mathbb{R}^n , denoted by $\text{Aff}(\mathbb{R}^n)$, see [10], [22]. $\text{Aff}(\mathbb{R}^n)$ consists of pairs (a, b) with $a \in GL(n, \mathbb{R})$ and $b \in \mathbb{R}^n$. The action of $(a, b) \in \text{Aff}(\mathbb{R}^n)$ on \mathbb{R}^n is given by

$$(a, b) \cdot x = ax + b$$

with $x \in \mathbb{R}^n$ and the product of group elements is given by

$$(a, b)(a', b') = (aa', ab' + b).$$

Define a measure on $G := \text{Aff}(\mathbb{R}^n)$, $d\mu(a, b) = \frac{dad b}{|\det a|^2}$. This measure is left invariant.

Let $f, \psi \in \mathbb{R}^n$. Define $W_\psi : L^2(\mathbb{R}^n) \rightarrow L^2(G)$ by

$$W_\psi f(a, b) = |\det a|^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(\omega) \overline{\psi[a^{-1}(\omega - b)]} d\omega.$$

Then

$$\|W_\psi f\|_{L^2(G)}^2 = \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 \int_H |\hat{\psi}(h^T \xi)|^2 \frac{dh}{|\det h|} d\xi.$$

Hence, W_ψ is an isometry into $L^2(G)$ if and only if

$$\int_H |\hat{\psi}(h^T \xi)|^2 \frac{dh}{|\det h|} = 1.$$

Note, however, that for any $\xi \neq 0$, $\xi \in \mathbb{R}^n$, there exist $g \in GL(n, \mathbb{R})$ such that $g^T e_1 = \xi$, where $e_1 = (1, 0, \dots, 0)$. Thus

$$\begin{aligned} \int_H |\hat{\psi}(h^T \xi)|^2 \frac{dh}{|\det h|} &= \int_H |\hat{\psi}(h^T g^T e_1)|^2 \frac{dh}{|\det h|} \\ &= \int_H |\hat{\psi}((gh)^T e_1)|^2 \frac{dh}{|\det h|} \\ &= \int_H |\hat{\psi}(h^T e_1)|^2 \frac{dh}{|\det h|} \end{aligned}$$

and so, W_ψ is an isometry into $L^2(G)$ if and only if

$$\int_H |\hat{\psi}(h^T e_1)|^2 \frac{dh}{|\det h|} = 1.$$

Theorem 2.20. (Plancherel Theorem) *Let $\psi \in \mathbb{R}^n$ such that*

$$\int_H |\hat{\psi}(h^T e_1)|^2 \frac{dh}{|\det h|} = 1. \text{ Then}$$

$$\langle W_\psi f, W_\psi g \rangle_{L^2(G)} = \langle f, g \rangle_{L^2(\mathbb{R}^n)}$$

for all $f, g \in L^2(\mathbb{R}^n)$. In particular the following holds:

- i) $W_\psi : L^2(\mathbb{R}^n) \rightarrow L^2(G, \mu)$ is continuous.
- ii) For all $f \in \mathbb{R}^n$, $f = W_\psi^* W_\psi f$.
- iii) $W_\psi : L^2(\mathbb{R}^n) \rightarrow \text{Im}(W_\psi)$ is an unitary isomorphism.

Plancherel theorem gives an inversion formula using W_ψ^* , but f can be recovered as a weak integral only from W_ψ .

Theorem 2.21. *Let $\psi \in \mathbb{R}^n$ such that $\int_H |\hat{\psi}(h^T e_1)|^2 \frac{dh}{|\det h|} = 1$. Then*

$$f = \int_G W_\psi f(a, b) \psi_{a,b} d\mu(a, b),$$

for all $f \in \mathbb{R}^n$.

3. Wavelets

3.1 Multiresolution Analysis

The classical definition of a Multiresolution Analysis or a MRA is as follows, see [14].

Definition 3.1. *A multiresolution analysis on \mathbb{R} is a sequence of subspaces $\{V_j\}_{j \in \mathbb{Z}}$ of functions in $L^2(\mathbb{R})$ satisfying the following properties:*

- i) For all $j \in \mathbb{Z}$, $V_j \subseteq V_{j+1}$*
- ii) If $f(\cdot) \in V_j$, then $f(2\cdot) \in V_{j+1}$*
- iii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$*
- iv) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$*
- v) There exists a function $\phi(x) \in L^2(\mathbb{R})$ such that $\{\phi(\cdot - k) | k \in \mathbb{Z}\}$ is an orthonormal basis of V_0 .*

The function ϕ is called a *scaling function*. There have been done some generalizations of this definition. One can replace the diadic dilation by any integer dilation and in higher dimensions the dilation becomes a matrix with certain properties.

Definition 3.2. *A family $(f_i)_{i \in I}$ in a Hilbert space H is called a Riesz basis if there exist constants $0 < A \leq B$ such that*

$$A \sum_{i \in I} |a_i|^2 \leq \left\| \sum_{i \in I} a_i f_i \right\|^2 \leq B \sum_{i \in I} |a_i|^2,$$

for any $(a_i)_{i \in I} \in l^2(I)$, and if $\overline{\text{span}(f_i)} = H$.

Definition 3.3. A family $(f_i)_{i \in I}$ in a Hilbert space H is called a frame if there exist constants $0 < A \leq B$ such that for any $f \in H$, the following inequality holds

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2.$$

One can weaken condition $v)$ in the definition of MRA by replacing the orthonormal basis with a Riesz basis or a frame. That is usually called a Generalized Multiresolution Analysis or a GMRA. One also can allow more than one scaling function, say d , and then the MRA or GMRA has multiplicity d . Even though the traditional definition of a MRA has five properties, it was shown in Weiss that they are dependent.

Theorem 3.4. Conditions $i)$, $ii)$, and $v)$ imply $iii)$ even if in $v)$ we only assume that $\{\psi(\cdot - n)\}$ is a Riesz basis.

Theorem 3.5. Assume that $\{V_j\}_{j \in \mathbb{Z}}$ is a sequence of closed subspaces of $L^2(\mathbb{R})$ satisfying conditions $i)$, $ii)$, and $v)$. If the scaling function ϕ is such that $|\hat{\phi}|$ is continuous at 0, then

$$\hat{\phi}(0) \neq 0 \quad \text{if and only if} \quad \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

The following proposition gives a method for finding scaling functions.

Proposition 3.6. If $f \in L^2(\mathbb{R})$, then $\{f(\cdot - k) : k \in \mathbb{Z}\}$ is an orthonormal system if and only if

$$\sum_{k \in \mathbb{Z}} |\hat{f}(\omega + k)|^2 = 1$$

for almost all $\omega \in \mathbb{R}$.

3.2 Construction of Wavelets from a MRA

We are discussing now the construction of wavelets from MRA. For any $i \in \mathbb{Z}$, let W_i be the orthogonal complement of V_i in V_{i+1} ; that is, $V_{i+1} = V_i \oplus W_i$. It is easy

to see that

$$V_j = \bigoplus_{l=-\infty}^j W_l$$

and so

$$L^2(\mathbb{R}) = \bigoplus_{l=-\infty}^{\infty} W_l.$$

For $j \in \mathbb{Z}$ and $k \in \mathbb{Z}$, set

$$\psi_{j,k}(x) := 2^{j/2} \psi(2^j x - k)$$

If there exists a function $\psi \in W_0$ such that $\{\psi(\cdot - k) | k \in \mathbb{Z}\}$ is an orthonormal basis for W_0 , then $\{\psi_{j,k} | k \in \mathbb{Z}\}$ is an orthonormal basis for W_j , and $\{\psi_{j,k} | k, j \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$. Such a function ψ is called an *orthonormal wavelet associated with the given MRA*.

Since $\{\phi_k | k \in \mathbb{Z}\}$ is an orthonormal basis for V_0 , we obtain

$$\phi(x/2) = \sum_{k \in \mathbb{Z}} \alpha_k \phi(x - k).$$

Taking Fourier transforms, we get

$$\hat{\phi}(2\xi) = \hat{\phi}(\xi) m_0(\xi),$$

where

$$m_0(\xi) = \sum_{k \in \mathbb{Z}} \alpha_k e^{2\pi i k \xi}$$

is a periodic function. The periodic function m_0 is called the *low pass filter* associated with the scaling function ϕ . An important property of the low pass filter is

$$|m_0(\xi)|^2 + |m_0(\xi + 1/2)|^2 = 1$$

almost everywhere. There is a characterization of all orthonormal wavelets in W_0 given by the following proposition.

Proposition 3.7. *If ϕ is a scaling function for an MRA and m_0 is the associated low-pass filter, then a function $\psi \in W_0$ is an orthonormal vector for $L^2(\mathbb{R}^n)$ if and only if*

$$\hat{\psi}(2\xi) = e^{2\pi i \xi} \overline{m_0(\xi + \frac{1}{2})} \hat{\phi}(\xi)$$

almost everywhere, for some 1-periodic function s such that $|s(\xi)| = 1$ almost everywhere.

In particular, if we define ϕ by

$$\hat{\psi}(2\xi) = e^{2\pi i \xi} \overline{m_0(\xi + \frac{1}{2})} \hat{\phi}(\xi)$$

then we get an orthonormal wavelet in W_0 . Using also the fact that

$$\hat{\phi}(2\xi) = \hat{\phi}(\xi) m_0(\xi),$$

we get that

$$\hat{\psi}(2\xi) = \sum_{k \in \mathbb{Z}} (-1)^k \overline{\alpha_k} e^{-2\pi i (k-1)\xi} \hat{\phi}(\xi).$$

Therefore

$$\hat{\psi}(\xi) = \sum_{k \in \mathbb{Z}} (-1)^k \overline{\alpha_k} e^{-2\pi i (k-1)\frac{\xi}{2}} \hat{\phi}(\frac{\xi}{2})$$

and by taking the inverse Fourier transform, we get

$$\psi(x) = 2 \sum_{k \in \mathbb{Z}} (-1)^k \overline{\alpha_k} \phi(2x - (k-1)).$$

We will show next how to obtain $|\hat{\phi}|$ from $|\hat{\psi}|$. We have that

$$|\hat{\phi}(2\xi)|^2 + |\hat{\psi}(2\xi)|^2 = |\hat{\phi}(\xi)|^2 \{ |m_0(\xi)|^2 + |m_0(\xi + \frac{1}{2})|^2 \} = |\hat{\phi}(\xi)|^2$$

and so

$$|\hat{\phi}(\xi)|^2 = |\hat{\phi}(2^p \xi)|^2 + \sum_{j=1}^p |\hat{\psi}(2^j \xi)|^2 \quad \text{for all } p \geq 1.$$

Moreover,

$$\int_{\mathbb{R}} |\hat{\phi}(2^p \xi)|^2 d\xi = \frac{1}{2^p} \int_{\mathbb{R}} |\hat{\phi}(\xi)|^2 d\xi \rightarrow 0 \text{ as } p \rightarrow \infty$$

and so, by Fatou's lemma we get that

$$\lim_{p \rightarrow \infty} |\hat{\phi}(2^p \xi)|^2 = 0$$

which shows that

$$|\hat{\phi}(\xi)|^2 = \sum_{j=1}^{\infty} |\hat{\psi}(2^j \xi)|^2 \text{ a.e..}$$

3.3 Band-limited Wavelets

One of the conditions that characterize the completeness of an orthonormal system $\psi_{j,k}$ is given by the following theorem, see [14].

Theorem 3.8. *If ψ is a band-limited orthonormal wavelet, then*

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = 1$$

almost everywhere.

Before we get to the second condition, we need the following propositions:

Proposition 3.9. *If ψ is band-limited, $|\hat{\psi}|$ is continuous at zero and $\{\psi_{j,k} | j, k \in \mathbb{Z}\}$ is an orthonormal system, then $\hat{\psi}(0) = 0$.*

A stronger result holds if ψ is an orthonormal wavelet.

Proposition 3.10. *If ψ is a band-limited orthonormal wavelet such that $|\hat{\psi}|$ is continuous at zero, then $\hat{\psi} = 0$ almost everywhere in an open neighborhood of the origin.*

We are ready now to state the second condition on the completion of the system $\{\psi_{j,k} | j, k \in \mathbb{Z}\}$.

Theorem 3.11. *If ψ is a band-limited orthonormal wavelet such that $|\hat{\psi}|$ is continuous at zero, then for each odd integer p we have*

$$\sum_{j=0}^{\infty} \hat{\psi}(2^j \xi) \overline{\hat{\psi}(2^j(\xi + p))} = 0 \quad \text{a.e. on } \mathbb{R}.$$

The following theorem gives an necessary and sufficient condition for the completeness of a system.

Theorem 3.12. *If $\psi \in L^2(\mathbb{R})$ is a bandlimited function such that $|\hat{\psi}|$ is zero in a neighborhood of the origin and $\{\psi_{j,k} | j, k \in \mathbb{Z}\}$ is an orthonormal system, then the system is complete if and only if*

$$\sum_{j \in \mathbb{Z}} |\hat{\psi}(2^j \xi)|^2 = 1 \quad \text{a.e. on } \mathbb{R}$$

and

$$\sum_{j=0}^{\infty} \hat{\psi}(2^j \xi) \overline{\hat{\psi}(2^j(\xi + p))} = 0 \quad \text{a.e. on } \mathbb{R}, \quad p \in 2\mathbb{Z} + 1.$$

3.4 The Haar System

The simplest wavelet was introduced by the hungarian mathematician, Alfred Haar, in 1909, long before wavelets came in the attention of the mathematicians. The *Haar wavelet* is constructed from the MRA generated by the scaling function $\phi(x) = \chi_{[-1,0]}$. Then V_j is the space of $L^2(\mathbb{R})$ functions which are constant on intervals of the form $[2^{-j}k, 2^{-j}(k+1)]$, $k \in \mathbb{Z}$. Since

$$\frac{1}{2}\phi\left(\frac{1}{2}x\right) = \frac{1}{2}\chi_{[-2,0]}(x) = \frac{1}{2}\phi(x) + \frac{1}{2}\phi(x+1),$$

we get that

$$\begin{aligned} \psi(x) &= \phi(2x+1) - \phi(2x) = \chi_{[-1, -\frac{1}{2}]} - \chi_{[-\frac{1}{2}, 0]}, \\ \hat{\phi}(\xi) &= \frac{e^{2\pi i \xi} - 1}{2\pi i \xi}, \quad m_0(\xi) = \frac{e^{2\pi i \xi} + 1}{2}. \end{aligned}$$

3.5 The Shannon Wavelet

Another well known wavelet is the Shannon wavelet. Let $I = [-1, -\frac{1}{2}) \cup [\frac{1}{2}, 1)$ and

$$\hat{\psi}(\xi) = e^{\pi i \xi} \chi_I(\xi).$$

Then the function ψ is called the *Shannon wavelet*. To find a scaling function of the MRA associated with this wavelet, we compute $\hat{\psi}(2^j \xi)$,

$$\hat{\psi}(2^j \xi) = e^{2^j \pi i \xi} \chi_{I_j}(\xi),$$

where

$$I_j = [-2^{-j}, -2^{-j-1}) \cup [2^{-j-1}, 2^{-j}).$$

The intervals I_j 's are disjoint and their union for $j \geq 1$ is, up to measure zero, the interval $[-\frac{1}{2}, \frac{1}{2}]$ and so, we can take $\hat{\phi}(\xi) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}$. Let $V_j = \overline{\text{span}\{\phi_{j,k} | k \in \mathbb{Z}\}}$ for all $j \in \mathbb{Z}$. Then $\{V_j | j \in \mathbb{Z}\}$ is a multiresolution analysis and the low pass filter is a 1-periodic function defined by the equation

$$m_o(\xi) = \begin{cases} 1 & \text{if } -\frac{1}{4} \leq \xi \leq \frac{1}{4} \\ 0 & \text{if } -\frac{1}{2} \leq \xi \leq -\frac{1}{4} \text{ or } \frac{1}{4} \leq \xi \leq \frac{1}{2} \end{cases}$$

extended periodically from $[-\frac{1}{2}, \frac{1}{2}]$ to \mathbb{R} . The graph of the Shannon wavelet ψ is given in Figure 3.1.

We say that a wavelet ψ has k *vanishing moments* if

$$\int_{\mathbb{R}} x^k \psi(x) dx = 0.$$

It was shown that if $\{\psi_{j,k} | j, k \in \mathbb{Z}\}$ is an orthonormal system on \mathbb{R} and if ψ is smooth, then it will have vanishing moments.

Theorem 3.13. *If $\{\psi_{j,k} | j, k \in \mathbb{Z}\}$ is an orthonormal system on \mathbb{R} , and if $x^N \psi(x)$ and $\xi^{N+1} \hat{\psi}(\xi)$ are both integrable, then*

$$\int_{\mathbb{R}} x^m \psi(x) dx = 0 \text{ for } 0 \leq m \leq N.$$

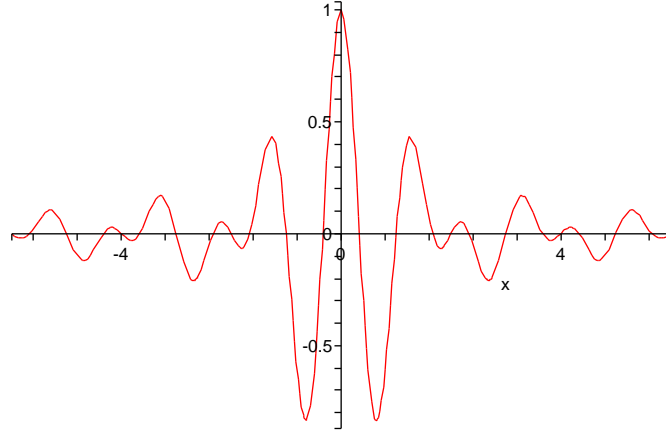


FIGURE 3.1. The wavelet $\psi = \frac{1}{\pi x}(\sin 2\pi x - \sin \pi x)$.

Remark 3.14. *Since $\xi^{N+1}\hat{\psi}(\xi) \in L^1(\mathbb{R})$, it follows that $\psi \in C^{N+1}(\mathbb{R})$, so this assumption can be viewed as a smoothness assumption.*

3.6 The Daubechies Wavelet

Ingrid Daubechies was the first to give a general construction of orthonormal wavelets with compact support and with a given degree of smoothness. She noticed that if the wavelet ψ and the scaling function ϕ are both in $C^{N-1}(\mathbb{R})$, then the low pass filter m_0 is of the form

$$m_0(\xi) = \left(\frac{1 + e^{-2\pi i \xi}}{2}\right)^N g(\xi)$$

with g being 1-periodic and $g \in C^{N-1}(\mathbb{R})$.

Example 3.15. *The low pass filters for $N = 1, 2$ are as follows:*

$N=1$ *Then the periodic function $g(\xi) = 1$ and so*

$$\frac{1 + e^{-2\pi i \xi}}{2}.$$

$N=2$ *Then the periodic function is*

$$g(\xi) = \frac{1 + \sqrt{3}}{2} + \frac{1 - \sqrt{3}}{2} e^{-2\pi i \xi}$$

and so, the low pass filter is

$$m_0(\xi) = \sum_{k \in \mathbb{Z}} \alpha_k e^{2\pi i k \xi}$$

with

$$\alpha_0 = \frac{1 - \sqrt{3}}{8}, \alpha_{-1} = \frac{3 - \sqrt{3}}{8}, \alpha_{-2} = \frac{3 + \sqrt{3}}{8}, \alpha_{-3} = \frac{1 + \sqrt{3}}{8},$$

and the rest of coefficients zero.

3.7 Minimally Supported Frequency Wavelets

A special kind of band-limited wavelets are the MSF, *Minimally Supported Frequency* wavelets. These wavelets have the form

$$\mathcal{F}^{-1} \chi_{\Omega},$$

for some measurable set Ω . The sets Ω which are the support of the MSF wavelets are called *wavelet sets*. In the next chapter we study the relation between wavelets sets and *tilings and spectral sets*.

4. Wavelet Sets in \mathbb{R}^2

4.1 Spectral Sets

Spectral sets have been intensely studied lately also because of the well known Fuglede's conjecture. Set $e_\lambda(\xi) = e^{2\pi i \langle \lambda, \xi \rangle}$. If Ω is a Lebesgue measurable subset of \mathbb{R}^n , then

$$|\Omega| = \int \chi_\Omega(t_1, \dots, t_n) dt_1 \dots dt_n$$

denotes the measure of Ω with respect to the standard Lebesgue measure on \mathbb{R}^n .

Definition 4.1. *A set $\Omega \subseteq \mathbb{R}^n$ with $0 < \mu(\Omega) < \infty$ is a spectral set if there exists a set $\mathcal{T} \subseteq \mathbb{R}^n$ such that*

$$\{e_\lambda \mid \lambda \in \mathcal{T}\}$$

is an orthogonal basis for $L^2(\Omega)$. The set \mathcal{T} is called a spectrum of Ω , and (Ω, \mathcal{T}) is said to be a spectral pair.

4.2 Tilings

Definition 4.2. *A measurable tiling of a measure space (M, μ) is a countable collection of subsets $\{\Omega_j\}$ of M , such that*

$$\mu(\Omega_i \cap \Omega_j) = 0$$

and

$$\mu(M \setminus \bigcup_j \Omega_j) = 0.$$

Definition 4.3. *Let $M \subseteq \mathbb{R}^n$ be a measurable set with $|M| > 0$. Let $\mathcal{D} \subseteq \text{GL}(n, \mathbb{R})$ and $\mathcal{T} \subseteq \mathbb{R}^n$.*

1.) We call \mathcal{D} a multiplicative tiling set of M if there exists a measurable set $\Omega \subseteq \mathbb{R}^n$, $|\Omega| > 0$, such that $\{d\Omega \mid d \in \mathcal{D}\}$ is a measurable tiling of M . The set Ω is called a multiplicative tile.

2.) We call \mathcal{T} an additive tiling set of \mathbb{R}^n if there exists a measurable set $\Omega \subseteq \mathbb{R}^n$, $|\Omega| > 0$, such that $\{\Omega + t \mid t \in \mathcal{T}\}$ is a measurable tiling of \mathbb{R}^n . The set Ω is called an additive tile.

3.) A set Ω is called a $(\mathcal{D}, \mathcal{T})$ -tile if it is a \mathcal{D} -multiplicative tile and a \mathcal{T} -additive tile.

Remark 4.4. Note, that we do not assume that $\mathcal{D}M \subseteq M$ or even

$$|(M \setminus \mathcal{D}M) \cup ((\mathcal{D}M) \setminus M)| = 0. \quad (4.1)$$

Neither do we assume that there exists a zero set Z , such that $\Omega \subseteq M \cup Z$. That will always be the case if $\text{id} \in \mathcal{D}$. But we can always, without loss of generality assume that $\text{id} \in \mathcal{D}$ and $\Omega \subseteq M$.

Dai and Larson proved that a measurable set $\Omega \subseteq \mathbb{R}^n$ is a wavelet set with respect to the translation set \mathbb{Z}^n and dilation set $\mathcal{D} = \{2^j \text{id} \mid j \in \mathbb{Z}\}$ if and only if

- i) $\{\Omega + z \mid z \in \mathbb{Z}\}$ is an additive tiling of \mathbb{R}^n and
- ii) $\{2^j \Omega \mid j \in \mathbb{Z}\}$ is an multiplicative tiling \mathbb{R}^n .

4.3 Fuglede's Conjecture

The spectral property of a set is closely related to the tiling property, in particular if the spectrum is a lattice. This was first noticed by Fuglede in [11]. For a non-empty subset $\mathcal{T} \subset \mathbb{R}^n$, set

$$\mathcal{T}^* := \{t \in \mathbb{R}^n \mid \langle t, s \rangle \in \mathbb{Z}, \text{ for all } s \in \mathcal{T}\}.$$

If \mathcal{T} is a lattice, then so is \mathcal{T}^* . In that case \mathcal{T}^* is called *the dual lattice* of \mathcal{T} .

Theorem 4.5 (Fuglede [11]). *Assume that \mathcal{T} is a lattice. Then Ω is a spectral set with spectrum \mathcal{T} if and only if $\{\Omega + t \mid t \in \mathcal{T}^*\}$ is a measurable tiling of \mathbb{R}^n .*

This result and several examples led Fuglede to the conjecture:

Conjecture 1 (The Spectral-Tile Conjecture). *A measurable set Ω , with positive and finite measure, is a spectral set if and only if it is an additive tile.*

Several people worked on this conjecture and derived important results and validated the conjecture for some special cases, see [15, 17, 18, 19, 26] and the references therein. However, in 2003, T. Tao [25] showed that the conjecture is false in dimension 5 and higher, if the lattice hypothesis is dropped. But even now, after the Spectral-Tiling conjecture have been proven to fail in higher dimension, it is still interesting and important to understand better the connection between spectral properties and tilings, in particular because of the connection to wavelet sets.

Theorem 4.6 (Tao, 2003). *Let $d \geq 5$ be an integer. Then there exists a compact set $\Omega \subset \mathbb{R}^d$ of positive measure such that $L^2(\Omega)$ admits an orthonormal basis of exponentials $\{e_\lambda \mid \lambda \in \mathcal{T}\}$ for some $\mathcal{T} \subset \mathbb{R}^d$, but such that $\{\Omega\}$ does not tile \mathbb{R}^d by translation. In particular, Fuglede's conjecture is false in \mathbb{R}^d for $d \geq 5$.*

In June 2004, M. Kolountzakis and M. Matolcsi show that the direction *tile* \rightarrow *spectral* is also false for dimension 5 or higher.

Theorem 4.7 (Kolountzakis, Matolcsi). *In each of the groups \mathbb{Z}^d and \mathbb{R}^d , $d \geq 5$, there exists a set which tiles the group by translation but is not spectral.*

In October 2004, M. Kolountzakis and M. Matolcsi improve the last two results by showing that the direction *spectral* \rightarrow *tile* is false already in dimension 3.

4.4 Benedetto and Sumetkijakan's Construction

Benedetto and Leon have given a general construction of wavelet sets in \mathbb{R}^d , with respect to the translation set \mathbb{Z}^n and dilation set $\mathcal{D} = \{2^j \text{id} \mid j \in \mathbb{Z}\}$, see [16]. They

consider a bounded neighborhood of the origin K_0 and a fixed number N such that $K_0 \subseteq [-N, N]^d$ and $K_0 \sim_{\mathbb{Z}} Q$, where $Q := [-\frac{1}{2}, \frac{1}{2}]^d$. Define an injective integer translated map,

$$T : K_0 \rightarrow \overline{[-2N, 2N]^d \setminus [-N, N]^d}$$

$$T(x) = x + n_x, \text{ for some } n_x \in \mathbb{Z}^d,$$

as follows:

$$A_0 := K_0 \cap \left[\bigcup_{j \geq 1} 2^{-j} K_0 \right] \text{ and } K_1 := [K_0 \setminus A_0] \cup T(A_0).$$

Note that if $K_n = K_n^- \cup K_n^+$, where

$$K_n^- \subseteq [-N, N]^d,$$

$$K_n^+ \subseteq \overline{[-2N, 2N]^d \setminus [-N, N]^d},$$

then we have that

$$K_1^- = K_0 \setminus A_0 \subseteq K_0 \subseteq [-N, N]^d,$$

and

$$K_n^+ = T(A_0) \subseteq \overline{[-2N, 2N]^d \setminus [-N, N]^d}.$$

In general, define

$$A_n := K_n \cap \left[\bigcup_{j \geq 1} 2^{-j} K_n \right]$$

and

$$K_{n+1} := [K_n^- \setminus A_n] \cup [K_n^+ \cup T(A_n)] = K_{n+1}^- \cup K_{n+1}^+.$$

Then we have that

$$K_{n+1} = \left[K_0 \setminus \bigcup_{k=0}^n A_k \right] \cup \left[\bigcup_{k=0}^n T(A_k) \right]$$

and then finally define

$$K = \left[K_0 \setminus \bigcup_{k=0}^{\infty} A_k \right] \cup \left[\bigcup_{k=0}^{\infty} T(A_k) \right].$$

It can be verified easily that

$$\{K + z | z \in \mathbb{Z}\}$$

is an additive tiling of \mathbb{R}^d and

$$\{2^j K | j \in \mathbb{Z}\}$$

is an multiplicative tiling of \mathbb{R}^d , and so, K is a wavelet set.

Example 4.8. Let $T : [-\frac{1}{2}, \frac{1}{2}]^2 \rightarrow [-1, 1]^2 \setminus [-\frac{1}{2}, \frac{1}{2}]^2$ defined by

$$T(x, y) = (x, y) - (\text{sgn } x, \text{sgn } y).$$

Then the wavelet set obtained using the above method is a generalization to \mathbb{R}^2 of the wavelet set $[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$ associated with the Shannon wavelet.

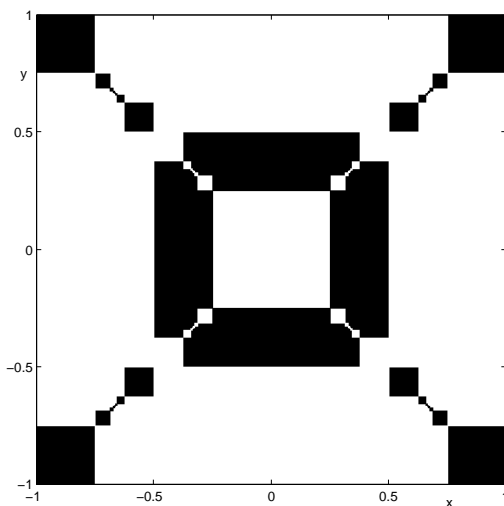


FIGURE 4.1. The set K_7

4.5 Existence of Subspace Wavelet Sets

We discuss now briefly the some results by Dai, Larson, and Speegle [7]. For that assume that G is a countable group acting on a measure space (M, μ) by measurable transformations. Two sets E and F are said to be \mathcal{G} -dilation congruent, $E \sim_{\mathcal{G}} F$, if there exists measurable partitions $\{E_i\}$ and $\{F_i\}$ of E and F , respectively, such that $F_i = g_i E_i$ for some $g_i \in \mathcal{G}$. Similarly, two sets E and F are said to be \mathcal{T} -translation congruent, $E \sim_{\mathcal{T}} F$, if there exists measurable partitions $\{E_i\}$ and $\{F_i\}$ of E and F , respectively, such that $F_i = t_i + E_i$ for some $t_i \in \mathcal{T}$.

In 1996, the above three authors showed in [7], that wavelet sets exists for groups of dilations. They first introduce the notion of *abstract dilation-translation pair*.

Definition 4.9 (Dai-Larson-Speegle). *Let X be a metric space and \mathcal{D} and \mathcal{T} discrete groups of automorphisms of X . A pair $(\mathcal{D}, \mathcal{T})$ is called an abstract dilation-translation pair if the following holds:*

- i) For each bounded set E and each open set F there exist $d \in \mathcal{D}$ and $t \in \mathcal{T}$ such that $t(E) \subseteq d(F)$.*
- ii) There is a fixed point θ for \mathcal{D} such that for any neighborhood N of θ and for any bounded set E , there is an element $d \in \mathcal{D}$ such that $d(E) \subseteq N$.*

In [7] the following is proved:

Theorem 4.10. *Let X be a metric space and $(\mathcal{D}, \mathcal{T})$ an abstract dilation-translation pair with θ as fixed point for \mathcal{D} . If E and F are bounded measurable sets in M such that E contains a neighborhood of θ and F has nonempty interior and is bounded away from θ , then there exists a measurable set $W \subseteq M$, $W \subseteq \cup_{d \in \mathcal{D}} d(F)$ which is \mathcal{D} -congruent to F and \mathcal{T} -congruent to E .*

If $d \in \text{GL}(n, \mathbb{R})$, $\gamma \in \mathbb{R}^n$, and $\psi : \mathbb{R}^n \rightarrow \mathbb{C}$ set $\psi_{d,\gamma}(x) = |\det d|^{1/2}\psi(dx + \gamma)$.

Note that the Fourier transform of $\psi_{d,\gamma}$ is given by

$$\widehat{\psi_{d,\gamma}}(\lambda) = e^{2\pi i \langle \gamma, d^{-T}\lambda \rangle} \widehat{\psi}(d^{-T}\lambda). \quad (4.2)$$

Definition 4.11. *Let $M \subseteq \mathbb{R}^n$ be measurable, $|M| > 0$, and $\mathcal{D} \subset \text{GL}(n, \mathbb{R})$. Let $\mathcal{T} \subset \mathbb{R}^n$ be discrete. Then a measurable set $\Omega \subseteq M$ is called a M -subspace $(\mathcal{D}, \mathcal{T})$ -wavelet set if the set of function $\{\psi_{d,\gamma}\}_{(d,\gamma) \in \mathcal{D} \times \mathcal{T}}$ is a orthogonal basis for $L^2_M(\mathbb{R}^n)$, where $\psi = \mathcal{F}^{-1}\chi_\Omega$.*

Remark 4.12. *Note again, that we do not assume that $\Omega \subset M$ nor that $\mathcal{D}M = M$ up to set of measure zero. But this will follows if $\text{id} \in \mathcal{D}$. As in Remark 4.4 one can always assume this be replacing \mathcal{D} by $d^{-1}\mathcal{D}$ and Ω by $d^T\Omega$ for a fixed $d \in \mathcal{D}$.*

We get from Theorem 4.9:

Theorem 4.13 (Dai-Larson-Speegle). *Let a be an expansive matrix, and let $M \subseteq \mathbb{R}^n$ be a measurable set of positive measure such that $a^T M = M$. Let $\mathcal{D} = \{a^k \mid k \in \mathbb{Z}\}$ and let \mathcal{T} be a full rank lattice. Then there exists a $(\mathcal{D}, \mathcal{T})$ subspace wavelet set for $L^2(M)$.*

There are several generalization of this Theorem. We refer to [21] for discussion and references. We will only mention two important result here.

A matrix with all eigenvalues greater than 1 is called an expansive matrix.

Theorem 4.14 (Dai, Diao, Gu, Han [9]). *Let M be a measurable subset of \mathbb{R}^n , with positive measure satisfying $a^T M = M$, for some expansive matrix a and let \mathcal{T} be a full rank lattice. Then there exists a set $E \subseteq M$ such that $\{E + t \mid t \in \mathcal{T}\}$ is a measurable tiling of \mathbb{R}^n and $\{(a^T)^k E \mid k \in \mathbb{Z}\}$ is a measurable tiling of M . In particular W is a subspace wavelet set for the space $L^2_M(\mathbb{R}^n)$.*

Theorem 4.15 (Wang [26]). *Let $\mathcal{D} \subseteq GL(n, \mathbb{R})$ and $\mathcal{T} \subseteq \mathbb{R}^n$. Let $\Omega \subseteq \mathbb{R}^n$ be measurable, with positive and finite measure. If Ω is a measurable \mathcal{D}^T -tile and (Ω, \mathcal{T}) is a spectral pair, then Ω is a $(\mathcal{D}, \mathcal{T})$ -wavelet set. Conversely, if Ω is a $(\mathcal{D}, \mathcal{T})$ -wavelet set and $0 \in \mathcal{T}$, then Ω is a measurable \mathcal{D}^T -tile and (Ω, \mathcal{T}) is a spectral pair.*

Let us sketch some of the ideas of the proof to underline the connection between spectral properties, tilings, and wavelet sets.

Let $\psi = \mathcal{F}^{-1}\chi_\Omega$. As the Fourier transform is an unitary isomorphism, it follows, that the set $\{\psi_{d,t} \mid d \in \mathcal{D}, t \in \mathcal{T}\}$ is an orthogonal basis for $L^2(\mathbb{R}^n)$ if and only if the set $\{\widehat{\psi_{d,t}} \mid d \in \mathcal{D}, t \in \mathcal{T}\}$ is an orthogonal bases for $L^2(\mathbb{R}^n)$. Here, as before, we have set

$$\psi_{d,t}(x) = |\det d|^{1/2}\psi(dx + t).$$

A simple calculation shows that

$$\widehat{\psi_{d,t}}(\lambda) = |\det d|^{-\frac{1}{2}}e^{2\pi i\langle d^{-1}t, \lambda \rangle}\chi_{d^T\Omega}(\lambda) = |\det d|^{-\frac{1}{2}}e^{2\pi i\langle t, d^{-T}\lambda \rangle}\chi_{d^T\Omega}(\lambda).$$

The fact, that $d^T\Omega$ is a measurable tiling of \mathbb{R}^n implies that

$$L^2(\mathbb{R}^n) \simeq \bigoplus_{d \in \mathcal{D}} L^2(d^T\Omega).$$

The orthogonal projection onto $L^2(d^T\Omega)$ is given by $f \mapsto f\chi_{d^T\Omega}$ and

$$f = \sum_{d \in \mathcal{D}} f\chi_{d^T\Omega}.$$

The spectral property implies that $\{e_t\}_{t \in \mathcal{T}}$ is an orthogonal basis for $L^2(\Omega)$. As $f \mapsto |\det d|^{-1/2}f(d^{-T}\cdot)$ is a unitary isomorphism $L^2(\Omega) \simeq L^2(d^T\Omega)$ it follows, that the set of functions $\{|\det d|^{-1/2}e^{2\pi i\langle t, d^{-T}\cdot \rangle} = |\det d|^{-1/2}e_{d^{-1}t} \mid t \in \mathcal{T}\}$ is an orthogonal basis for $L^2(d^T\Omega)$. Putting those two things together, we get that $\{|\det d|^{-1/2}e_{d^{-1}t} \mid d \in \mathcal{D}, t \in \mathcal{T}\}$ is an orthogonal basis for $L^2(\mathbb{R}^n)$.

In this section we discuss how to construct subspace wavelet sets using kind of “induction” process, i.e., using well known facts discussed in the previous section on smaller dilation sets acting on a smaller frequency set and then extending those to our bigger dilation set and frequency set. We start with two simple, but important, observations. For $\mathcal{A}, \mathcal{B} \subset \text{GL}(n, \mathbb{R})$ we say that the product $\mathcal{AB} = \{ab \mid a \in \mathcal{A}, b \in \mathcal{B}\}$ is *direct* if $a_1b_1 = a_2b_2$, $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$, implies that $a_1 = a_2$ and $b_1 = b_2$.

We state the following simple Lemma, but note, that we will be using the proof more than the actual statement.

Lemma 4.16. *Let $M \subseteq \mathbb{R}^n$ be measurable. Let $\mathcal{A}, \mathcal{B} \subset \text{GL}(n, \mathbb{R})$ be two non-empty sets and let $\mathcal{D} = \mathcal{AB} = \{ab \mid a \in \mathcal{A}, b \in \mathcal{B}\}$, such that the product \mathcal{AB} is direct. Then there exists a \mathcal{D} -tile Ω for M if and only if there exists a measurable set $N \subseteq \mathbb{R}^n$, such that \mathcal{AN} is a measurable tiling of M , and a \mathcal{B} -tile Ω for N .*

Proof. Assume that Ω is a \mathcal{D} -tile for M . Set $N := \mathcal{B}\Omega = \bigcup_{b \in \mathcal{B}} b\Omega$. Assume, that there are $b_1, b_2 \in \mathcal{B}$ such that $|b_1\Omega \cap b_2\Omega| > 0$. Then $|(ab_1\Omega) \cap (ab_2\Omega)| > 0$ for all $a \in \mathcal{A}$, which contradicts our assumption, that $\mathcal{D}\Omega$ is a measurable tiling of M . Hence $\mathcal{B}\Omega$ is a measurable tiling of N . We have, up to set of measure zero:

$$\mathcal{AN} = \bigcup_{a \in \mathcal{A}} an = \bigcup_{a \in \mathcal{A}, b \in \mathcal{B}} ab\Omega = M.$$

Assume, that there are $a_1, a_2 \in \mathcal{A}$ such that $|a_1N \cap a_2N| > 0$. Then we can find $b_1, b_2 \in \mathcal{B}$ such that $|a_1b_1\Omega \cap a_2b_2\Omega| > 0$. As the product \mathcal{AB} is direct, and $\mathcal{D}\Omega$ is a measurable tiling of M , it follows that $a_1 = a_2$. Hence \mathcal{AN} is a measurable tiling of M .

For the other direction, assume that \mathcal{AN} is a measurable tiling of M and $\mathcal{B}\Omega$ is a measurable tiling of N . Then, up to sets of measure zero,

$$\bigcup_{d \in \mathcal{D}} d\Omega = \bigcup_{a \in \mathcal{A}} \bigcup_{b \in \mathcal{B}} ab\Omega = \bigcup_{a \in \mathcal{A}} a \bigcup_{b \in \mathcal{B}} b\Omega = \bigcup_{a \in \mathcal{A}} an = M.$$

Assume that $|d_1\Omega \cap d_2\Omega| > 0$. Then there are unique $a_1, a_2 \in \mathcal{A}$, and $b_1, b_2 \in \mathcal{B}$ such that $d_1 = a_1b_1$ and $d_2 = a_2b_2$. Hence $|a_1N \cap a_2N| > 0$ which implies that $a_1 = a_2$, as $\mathcal{A}N$ is a measurable tiling for M . But then $|b_1\Omega \cap b_2\Omega| > 0$, which implies that $b_1 = b_2$. Hence $d_1 = d_2$. This shows, that $\mathcal{D}\Omega$ is a measurable tiling of M . \square

Remark 4.17. *We would like to remark at this point that we do not assume that $\Omega \subseteq M$, nor that $N \subseteq M$. This will in fact be the case in most applications because \mathcal{D} will contain the identity matrix. Recall also from Remark 4.4 and Remark 4.12 that we can always assume that $\text{id} \in \mathcal{D}$ and $\Omega \subseteq M$ up to set of measure zero. The same remarks hold for the following Theorems.*

Theorem 4.18 (Construction of wavelet sets by steps I). *Let $\mathcal{M}, \mathcal{N} \subset \text{GL}(n, \mathbb{R})$ and let $\mathcal{L} = \mathcal{M}\mathcal{N}$ such that the product $\mathcal{M}\mathcal{N}$ is direct.*

Assume that $M \subseteq \mathbb{R}^n$ with $|M| > 0$, is measurable. Let $\mathcal{T} \subset \mathbb{R}^n$ be discrete. Then there exists a $(\mathcal{L}, \mathcal{T})$ -wavelet set $\Omega \subset M$ for M if and only if there exists a \mathcal{N}^T -tiling set $N \subset M$ and a $(\mathcal{M}, \mathcal{T})$ -wavelet set Ω_1 for N .

Proof. Set $\mathcal{A} = \mathcal{N}^T$ and $\mathcal{B} = \mathcal{M}^T$. Then the conditions in Lemma 4.16 are satisfied.

Assume, that $\Omega \subset M$ is a $(\mathcal{L}, \mathcal{T})$ -wavelet set for M . As above we set

$$N := \mathcal{B}\Omega := \bigcup_{b \in \mathcal{M}} b^T\Omega.$$

Then, as above, we see that $\mathcal{A}N$ is a measurable tiling of M . As Ω is a spectral set, it follows from Theorem 4.15 that Ω is a $(\mathcal{M}, \mathcal{T})$ -wavelet set for N .

Assume now that N is a \mathcal{N}^T -tiling for M , and that Ω_1 is a $(\mathcal{M}, \mathcal{T})$ -wavelet set for N . Then, in particular Ω_1 is a \mathcal{B} -tile for N . As $\mathcal{A}N$ is a measurable tiling of M , it follows that $\mathcal{L}^T\Omega_1$ is a measurable tiling of M . As Ω_1 is a spectral set it follows from Theorem 4.15 that Ω_1 is a $(\mathcal{L}, \mathcal{T})$ -wavelet set for M . \square

Recall that if $\mathcal{D} \subseteq \text{GL}(n, \mathbb{R})$, and $\mathcal{G} \subset \text{GL}(n, \mathbb{R})$ is a group that acts on \mathcal{D} from the right, then there exists a subset $\mathcal{D}_1 \subseteq \mathcal{D}$, such that $\mathcal{D} = \mathcal{D}_1\mathcal{G}$ and the product is direct. Note, that we do not assume that $\mathcal{G} \subset \mathcal{D}$.

Theorem 4.19 (Construction of wavelet sets by steps II). *Let $\mathcal{D} \subset \text{GL}(n, \mathbb{R})$ and $M \subseteq \mathbb{R}^n$ measurable with $|M| > 0$. Let $\mathcal{T} \subset \mathbb{R}^n$ be discrete. Assume that $\mathcal{G} \subset \text{GL}(n, \mathbb{R})$ is a group that acts on \mathcal{D} from the right. Let $\mathcal{D}_1 \subseteq \mathcal{D}$ be such that $\mathcal{D} = \mathcal{D}_1\mathcal{G}$ as a direct product. Then there exists a $(\mathcal{D}, \mathcal{T})$ -wavelet set Ω for M if and only if there exists a \mathcal{G}^T -tiling set N for M and a $(\mathcal{D}_1, \mathcal{T})$ -wavelet set Ω_1 for N .*

Proof. This follows directly from Theorem 4.18 with $\mathcal{M} = \mathcal{D}_1$ and $\mathcal{N} = \mathcal{G}$. \square

The question is how to obtain a wavelet set for the starting subset N . The following gives one way to do that.

Theorem 4.20 (Existence of subspace wavelet sets). *Let $M \subseteq \mathbb{R}^n$ be a measurable set, $|M| > 0$. Let $a \in \text{GL}(n, \mathbb{R})$ be an expansive matrix and $\emptyset \neq \mathcal{D} \subset \text{GL}(n, \mathbb{R})$. Assume that \mathcal{D}^T is a multiplicative tiling of M , $a\mathcal{D} = \mathcal{D}$ and $a^T M = M$. If \mathcal{T} is a lattice, then there exists a measurable set $\Omega \subseteq M$ such that $\Omega + \mathcal{T}$ is a measurable tiling of \mathbb{R}^n and $\mathcal{D}^T\Omega$ is a measurable tiling of M . In particular, Ω is a $(\mathcal{D}, \mathcal{T})$ wavelet set.*

Proof. Let $b = a^T$ and $\mathcal{B} = \{b^k \mid k \in \mathbb{Z}\}$. Then \mathcal{B} is an abelian group that acts on \mathcal{D}^T from the right. Hence, there exists a set $\mathcal{A} \subset \mathcal{D}$ such that $\mathcal{D}^T = \mathcal{A}\mathcal{B}$ and the product is direct. Thus, the conditions in the previous Theorems are satisfied.

Let $E \subset M$ be such that $\mathcal{D}^T E$ is a measurable tiling of M . Set $N := \mathcal{B}E \subseteq M$. Then N is \mathcal{B} invariant. By Theorem 4.13 there exists a $(\mathcal{B}, \mathcal{T})$ -wavelet set Ω for

N . Set

$$N := \bigcup_{k \in \mathbb{Z}} b^k \Omega.$$

Then, as before, we see that Ω is a $(\mathcal{D}, \mathcal{T})$ -wavelet □

4.6 Rotations

We will start this section by constructing a wavelet set for the dilation group $\mathcal{D}_{2, \pi/2} := \{2^n R_{\pi/2}^k | n \in \mathbb{Z}, k = 0, \dots, 3\}$, where $R_{\pi/2}$ represents the rotation in \mathbb{R}^2 by $\pi/2$.

Example 4.21. We use first Benedetto's construction described in section 4.4. Then the wavelet set is

$$W' := W \cap \left([0, \frac{1}{2}]^2 \cup [-1, -\frac{1}{2}]^2 \right),$$

where W is the wavelet set described in section 4.4.

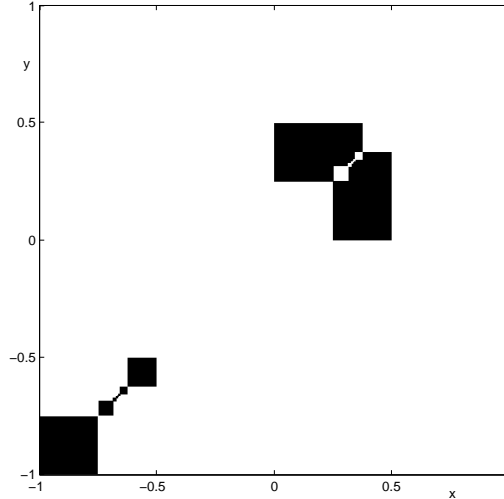


FIGURE 4.2. The set K'_7

Benedetto's construction works only for the rotations in \mathbb{R}^2 by $\pi/2$ or π . We give next a construction of wavelet sets which works for rotations in \mathbb{R}^2 by any angle of the form $2\pi/k$ with $k \in \mathbb{Z}$. We need this condition on the angle to get a finite group as dilation set.

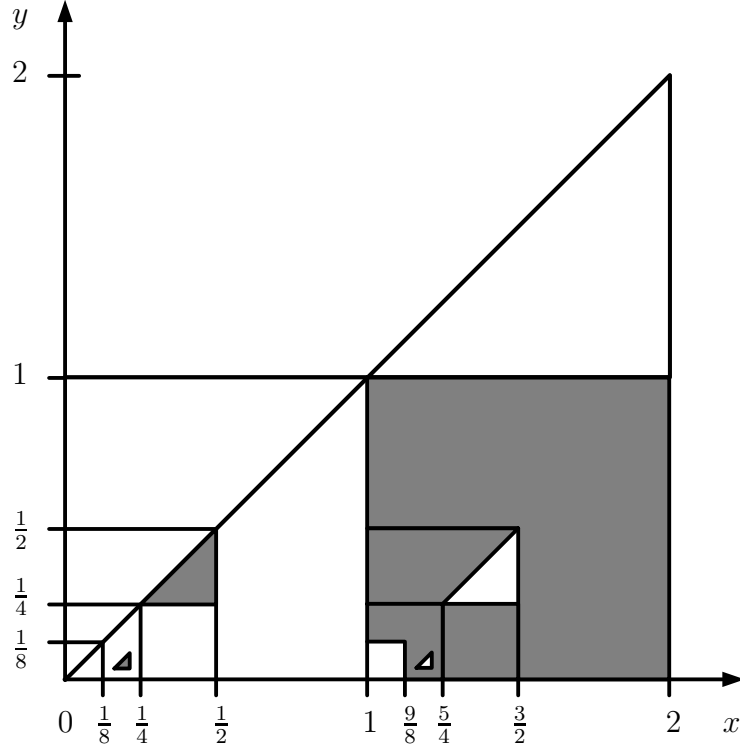


FIGURE 4.3. A $(\mathcal{D}_{2,4}, \mathcal{T})$ wavelet set.

Example 4.22. For $\theta \in \mathbb{R}$, let

$$R_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$$

denote the rotation in \mathbb{R}^2 by the angle θ . Let $a > 1$. For any integer $m \geq 2$ let

$$\mathcal{D}_{a,m} := \{a^n R_{2\pi/m}^k \mid n \in \mathbb{Z}, k = 0, \dots, m-1\}.$$

Note, that $R_{2\pi/m}^k = R_{2\pi k/m}$ and that $\mathcal{D}_{a,m}$ is a group. Let $\mathcal{T} = \mathbb{Z}^2$ Let

$$\mathbb{R}_{2\pi/m}^2 = \{r(\cos \psi, \sin \psi)^T \mid 0 \leq \psi \leq 2\pi/m, r > 0\}.$$

Then $\mathbb{R}_{2\pi/m}^2$ is a tiling set for the finite group $\{R_{2\pi/m}^k \mid k = 0, 1, \dots, m-1\}$. As aid is expansive, it follows from Theorem 4.13 that there exists a $(\mathcal{A} := \{a^j \text{id} \mid k \in \mathbb{Z}\}, \mathcal{T})$ -wavelet set Ω for $\mathbb{R}_{2\pi/m}^2$ and hence a $(\mathcal{D}_{a,m}, \mathcal{T})$ -wavelet set for \mathbb{R}^2 , see also Theorem 4.20. We show here how to construct such a wavelet set. Note that

we only have to construct a $(\mathcal{A}, \mathcal{T}_\theta)$ -wavelet set for $\mathbb{R}_{2\pi/m}^2$. For that, let

$$E = [0, 1] \times [0, \tan(2\pi/m)] \text{ if } m \neq 2, 4$$

$$E = [0, 1]^2 \text{ if } m = 4$$

$$E = [-1, 1] \times [0, 1] \text{ if } m = 2$$

$$F = \{(x, y) \in \mathbb{R}_{2\pi/m}^2 \mid 1 < x < a\}$$

There are infinitely many choices for E , F and \mathcal{T}_θ , we just thought these are the most convenient ones. The wavelet set W has the form

$$W = \bigcup_{i=1}^2 \bigcup_{j=1}^{\infty} W_{i,j},$$

see figure 4.3. The description of the $W_{i,j}$ is as follows

$$W_{1,1} = (E \setminus a^{-1}E) + (1, 0)$$

$$W_{2,1} = a^{-2}(F \setminus (E + (0, 1)))$$

$$W_{1,2} = [(a^{-1}E \setminus a^{-2}E) \setminus W_{2,1}] + (1, 0)$$

$$W_{2,2} = a^{-3}[W_{2,1} + (0, 1)]$$

For $j \geq 3$, we have the following formulas

$$W_{1,j} = [(a^{-n+1}E \setminus a^{-n}E) \setminus W_{2,j-1}] + (1, 0)$$

and

$$W_{2,j} = a^{-n-1}[W_{2,j-1} + (0, 1)].$$

From the construction, it is clear that W and E are \mathcal{T} -translation congruent, and W and F are \mathcal{D} -dilation congruent. On the other hand, F is a $\mathcal{D}_{a,m}$ -multiplicative tile and $\{E, \mathcal{T}\}$ is a spectral pair. It follows that W is a $\mathcal{D}_{a,m}$ -multiplicative tile and $\{W, \mathcal{T}\}$ is a spectral set. Thus, by Theorem 4.15 W is a $(\mathcal{D}_{a,m}, \mathcal{T})$ wavelet set.

4.7 Hyperbolic Rotation

The hyperbolic rotations case is very similar to the rotations described above. For $\theta \in \mathbb{R}$, let

$$Rh_\theta = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix}$$

denote a hyperbolic rotation in \mathbb{R}^2 by the angle θ . Let $a > 1$ and

$$\mathcal{D}h_{a,m} := \{a^n Rh_k | n \in \mathbb{Z}, k \in \mathbb{Z}, a > 1\}.$$

Note that $\mathcal{D}h_{a,m}$ is a group. Let

$$\mathcal{T} = \mathbb{Z} \times (\tanh 1)\mathbb{Z}.$$

Then \mathcal{T} is a full rank lattice in \mathbb{R}^2 . Let

$$E = [0, 1] \times [0, \tanh 1]$$

$$F = \{(l \cosh t, l \sinh t) | t \in (0, 1), l \in (2, 4), \}$$

The sets $W_{i,j}$ are constructed as above, with small differences: Let

$$F'_{2,1} := \{(l \cosh t, l \sinh t) | t \in (0, 1), l \in (2, 4) \cosh t < 4\},$$

$$F''_{2,1} := \{(l \cosh t, l \sinh t) | t \in (0, 1), l \in (2, 4) \cosh t > 4\}$$

and $s = 2 \cosh(\sinh^{-1}((\tanh 1)/2))$. Then

$$W_{1,1} = (E \setminus 2^{-1}E) + (s, 0)$$

$$W_{2,1} = 2^{-3}F'_{2,1} \cup 2^{-4}F''_{2,1}$$

$$W_{1,2} = [(2^{-1}E \setminus 2^{-2}E) \setminus W_{2,1}] + (s, 0)$$

$$W_{2,2} = 2^{-4}[W_{2,1} + (s, 0)]$$

For $j \geq 3$, we have the following formulas

$$W_{1,j} = [(a^{-n+1}E \setminus a^{-n}E) \setminus W_{2,j-1}] + (s, 0)$$

and

$$W_{2,j} = a^{-n-2}[W_{2,j-1} + (s, 0)].$$

The wavelet set W has again the form

$$W = \bigcup_{i=1}^2 \bigcup_{j=1}^{\infty} W_{i,j}.$$

From the construction, it is clear that W and E are \mathcal{T}_θ -translation congruent, and W and F are \mathcal{D} -dilation congruent. On the other hand, F is a $\mathcal{D}_{a,m}$ -multiplicative tile and $\{E, \mathcal{T}_\theta\}$ is a spectral pair. It follows that W is a $\mathcal{D}_{a,m}$ -multiplicative tile and $\{W, \mathcal{T}_\theta\}$ is a spectral set. Thus, by Theorem 4.15 W is a $(\mathcal{D}_{a,m}, \mathcal{T}_\theta)$ wavelet set.

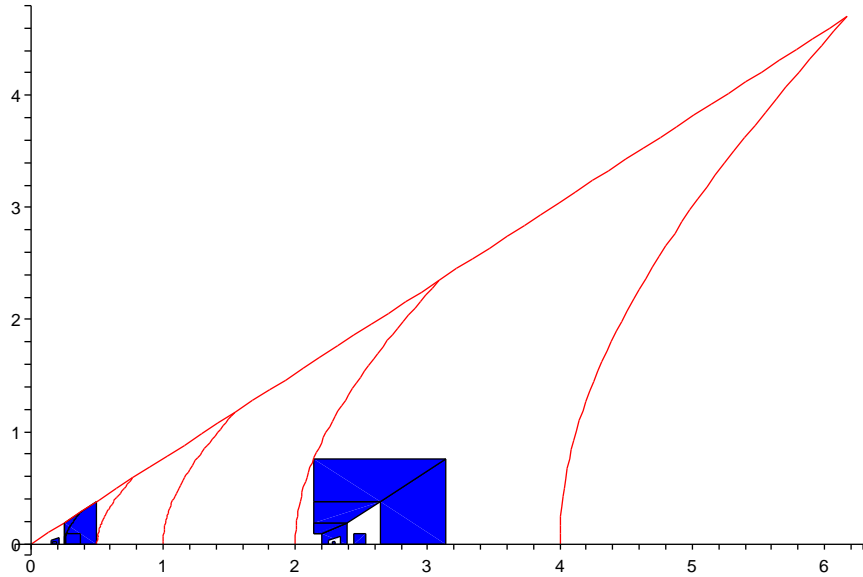


FIGURE 4.4. A $(\mathcal{D}_{2,4}, \mathcal{T}_1)$ wavelet set.

4.8 Example of Dilations Which Are Not Groups

Example 4.23. Let $a = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, $\mathcal{D} = \{R_{\pi/4}^k a^n \mid n \in \mathbb{Z}\}$ and $\mathcal{T} = \mathbb{Z}^2$. We consider now the action of the group \mathcal{D} on the first quadrant of the \mathbb{R}^2 , $[0, \infty)^2$. Let $E = [0, 1]^2$

and $F = ([0, 2] \times [0, 3]) \setminus [1, 1]^2$. Note that E is a \mathbb{Z}^2 -tile and that F is a a -tile for $[0, \infty)^2$. Following the same procedure, we get a set W such that W and E are \mathbb{Z}^2 -translation congruent, W and F are \mathcal{D} -dilation congruent and so, it follows that W is a \mathcal{D} -multiplicative tile and $\{W, \mathbb{Z}^2\}$ is a spectral set. Thus, W is a $(\mathcal{D}, \mathbb{Z}^2)$ wavelet set. The wavelet set W has the form

$$W = \bigcup_{i=1}^2 \bigcup_{j=1}^{\infty} W_{i,j},$$

see figure 4.6. The description of the $W_{i,j}$ is as follows

$$W_{1,1} = (E \setminus a^{-1}E) + (1, 0)$$

$$W_{2,1} = a^{-2}[(0, 1) \times (1, 3)]$$

$$W_{1,2} = [(a^{-1}E \setminus a^{-2}E) \setminus W_{2,1}] + (1, 0)$$

$$W_{2,2} = a^{-3}[W_{2,1} + (0, 1)]$$

For $j \geq 3$, we have the following formulas

$$W_{1,j} = [(a^{-n+1}E \setminus a^{-n}E) \setminus W_{2,j-1}] + (1, 0)$$

and

$$W_{2,j} = a^{-n-1}[W_{2,j-1} + (0, 1)].$$

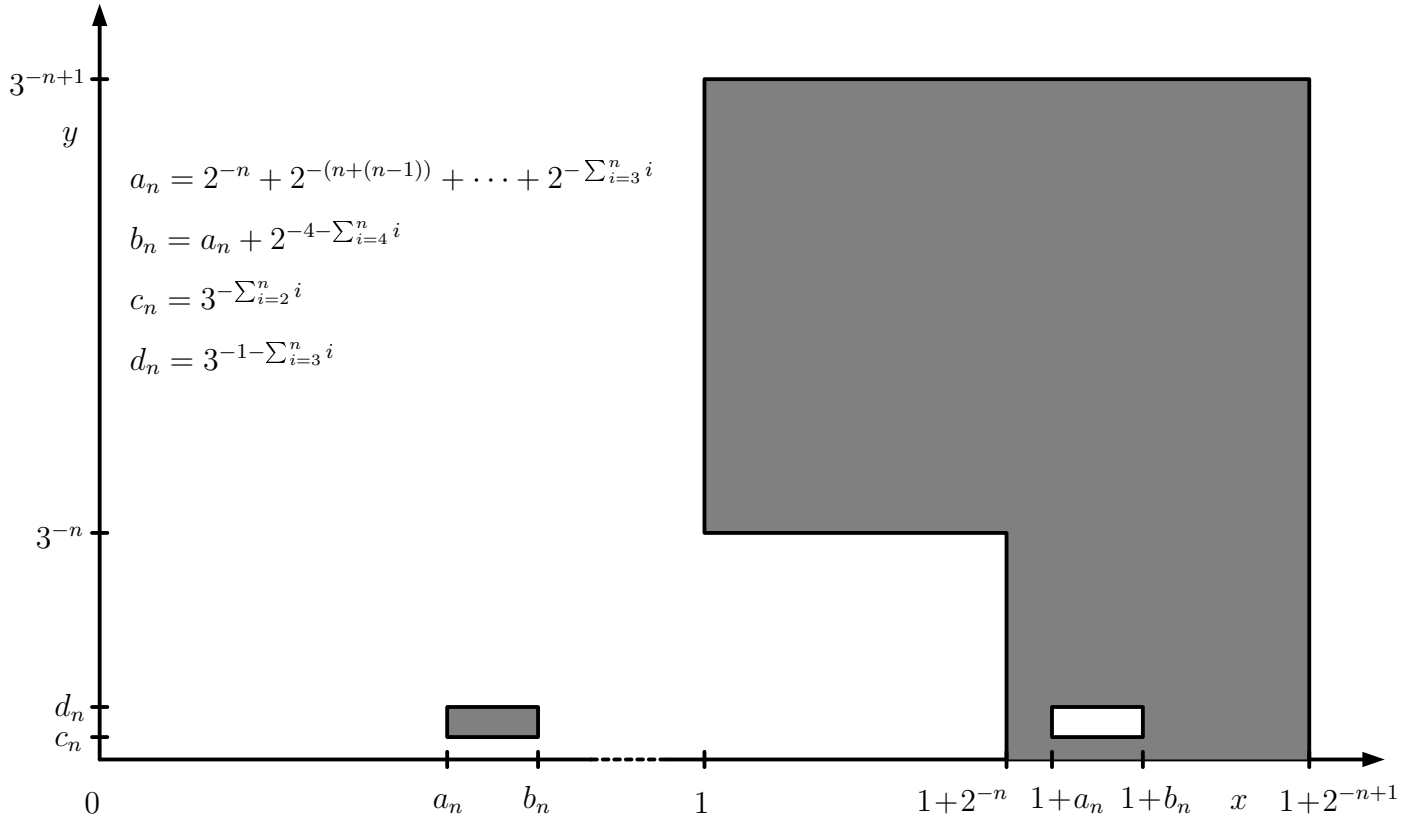


FIGURE 4.5. The set $W_{i,j}$ for $j \geq 3$.

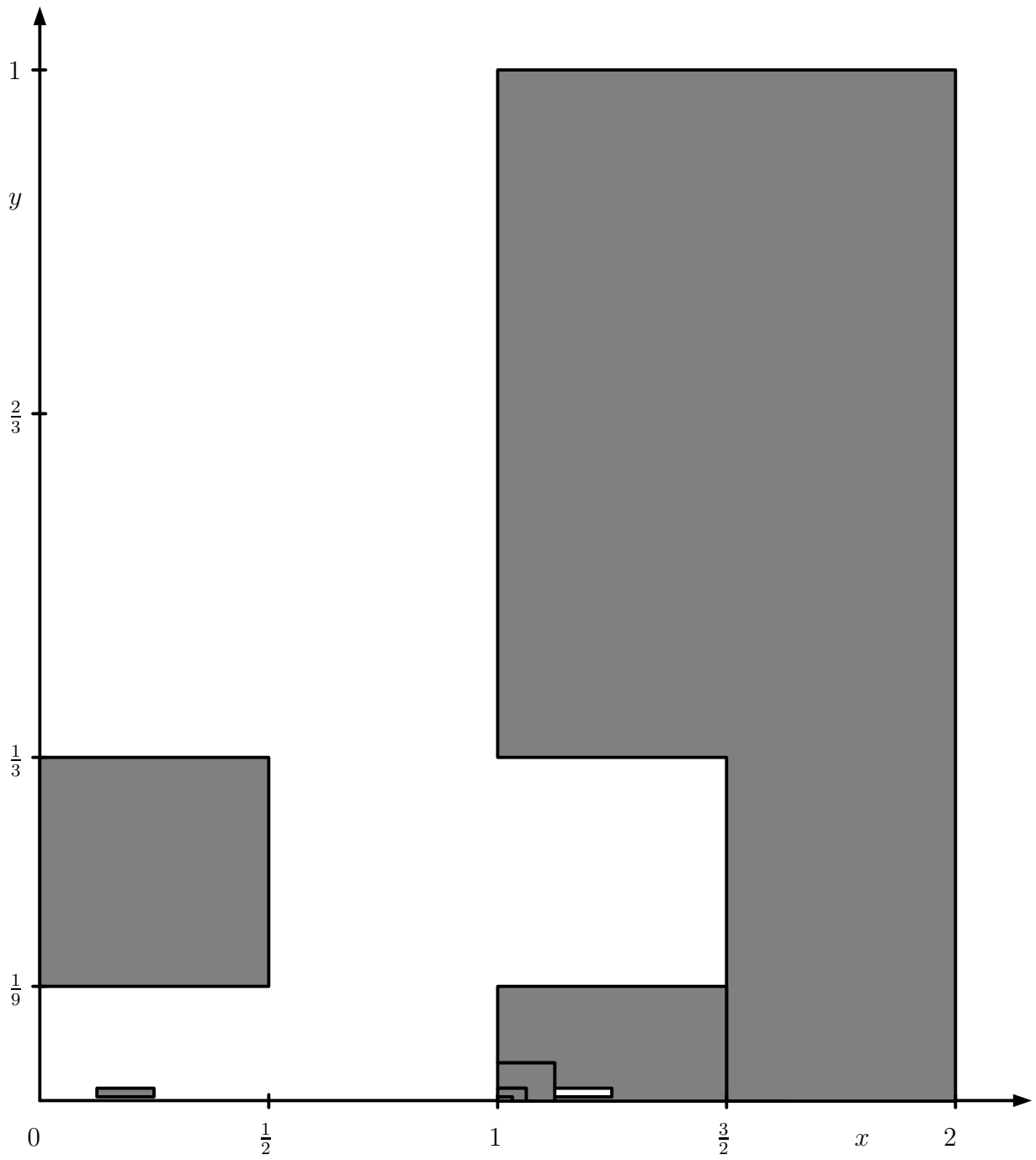


FIGURE 4.6. The sets W .

5. Coxeter Groups

A Coxeter group W is an abstract group with certain properties, but one thinks of W as a motion group generated by reflections through hyperplanes with respect to a symmetric bilinear form, (\cdot, \cdot) . If the bilinear form is positive definite, then the result is a finite Coxeter group.

Coxeter groups first appear as symmetry groups of regular geometric objects.

A *reflection* is a linear operator r on \mathbb{R}^n which sends some nonzero vector α to its negative and fixes pointwise the hyperplane H_α orthogonal to α . There is a simple formula

$$r_\alpha \lambda = \lambda - \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \alpha.$$

It is clear that $r_\alpha^2 = 1$ and r_α is an orthogonal transformation, so r_α has order 2 in the group of orthogonal transformations on \mathbb{R}^n . In this chapter, we will describe a special kind of finite subgroups of the group of orthogonal transformations, those finite groups generated by reflections, or *finite reflection groups*, for short.

Dihedral Group. Let \mathbb{R}^2 be the euclidian plane, and let \mathcal{D}_m be the dihedral group of order $2m$, consisting of the orthogonal transformations which preserve a regular m -sided polygon centered at the origin.

\mathcal{D}_m contains m rotations through multiples of $2\pi/m$, and m reflections about the diagonals of the polygon. By 'diagonal', we mean a line joining two vertices or the midpoints of opposite sides if m is even, or joining a vertex to the midpoint of the opposite side if m is odd.

The group \mathcal{D}_m is actually generated by reflections, since a rotation through $2\pi/m$ is a product of two reflections relative to a pair of adjacent diagonals which meet at an angle of $\theta = \pi/m$, see Figure 5.1.

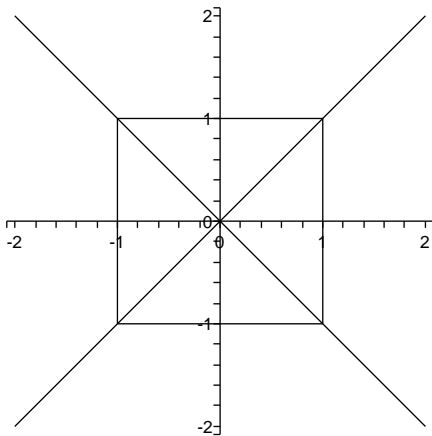


FIGURE 5.1. The dihedral group \mathcal{D}_4

The three dimensional case is more interesting. Let a, b, c be three linearly independent vectors such that the corresponding reflections lie in a finite group. That is only possible if $\angle(a, b), \angle(a, c), \angle(b, c)$ are rational multiple of π . This can be obtained by choosing $\angle(a, b)$ to be an arbitrary multiple of π and then choosing c such that $\angle(a, c) = \angle(b, c) = \pi/2$. In that case, the group generated by r_a and r_b , $\langle r_a, r_b \rangle$, is a dihedral group and $\langle r_a, r_b, r_c \rangle$ is the direct product of the dihedral group $\langle r_a, r_b \rangle$ and the cyclic group of order 2 generated by r_c .

Except these direct products, there are only three 3-dimensional Euclidian reflection groups, the groups of symmetries of a regular tetrahedron, a cube, and a regular dodecahedron.

Example 5.1. For each tetrahedron centered, there is a 'dual' tetrahedron which is congruent to the given one, and has the property that each edge of the given tetrahedron is perpendicularly bisected by an edge of the dual. Together, the vertices of the two tetrahedra give the vertices of a cube. Let a and c be the position vectors of the midpoints of a pair of parallel but not opposite edges e_1 and e_2 . Let b be the position vector of the midpoint of one of the edges on the opposite face to

that determined by e_1 and e_2 , which are not parallel to e_1 and e_2 . Then we have the following:

$$\sphericalangle(a, b) = 2\pi/3, \quad r_a r_b \text{ has order } 3;$$

$$\sphericalangle(a, c) = \pi/2, \quad r_a r_c \text{ has order } 2;$$

$$\sphericalangle(b, c) = 2\pi/3, \quad r_b r_c \text{ has order } 3;$$

r_a, r_b, r_c are symmetries of the tetrahedron, and so these three reflections generate the group of all symmetries of the tetrahedron, which is just $\text{Sym}(4)$.

5.1 Roots

Let Δ be a finite set of nonzero vectors in \mathbb{R}^n satisfying the conditions:

- i) $\Delta \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Delta$;
- ii) $r_\alpha \Delta = \Delta$ for all $\alpha \in \Delta$.

Let W be the group generated by all reflections r_α , $\alpha \in \Delta$. We call Δ a *root system* with associated reflection group W . The elements of Δ are called *roots*.

Suppose W is finite.

Recall that a *total ordering* of a real vector space V is a transitive relation on V (denoted $<$) satisfying the following axioms:

- 1) For each pair $\mu, \nu \in V$, exactly one of $\mu < \nu, \mu = \nu, \mu > \nu$ holds.
- 2) Let $\mu, \nu, \eta \in V$. If $\mu < \nu$, then $\mu + \eta < \nu + \eta$.
- 3) If $\mu < \nu$ and c is a nonzero real number, then $c\mu > c\nu$ if $c < 0$ and $c\mu < c\nu$ if $c > 0$.

Given such a total ordering, we say that $\nu \in V$ is *positive* if $0 < \nu$.

Let $\Delta^+ = \{\nu \in \Delta \mid 0 < \nu\}$ be the set of all positive roots in Δ .

A subset Π of Δ is a *simple system* if Π is a vector space basis for the \mathbb{R} -span of Δ in V and if each $\nu \in \Delta$ is a linear combination of Π with coefficients all of the same sign. It is easy to see that if Π is a simple system, then $w\Pi$ is also a simple system, for any $w \in W$.

Theorem 5.2. *Every positive system contains a unique simple system.*

The group W is actually generated by simple reflections.

Theorem 5.3. *If a subset Π of Δ is a simple system, then*

$$W = \langle r_\alpha \mid \alpha \in \Pi \rangle .$$

It turns out that the group W is completely characterized by the following relations:

$$(r_\alpha r_\beta)^{m(\alpha, \beta)} = 1, \quad \alpha, \beta \in \Pi,$$

where $m(\alpha, \beta)$ is the order of $r_\alpha r_\beta$ in W . Any group having such a representation is called a *Coxeter group* .

5.2 Fundamental Domain

In this section we describe the fundamental domain for the action of the Coxeter group W on \mathbb{R}^n . Assume that $\mathbb{R}^n = \text{span}\Delta$. To describe the action of the group W on \mathbb{R}^n , we need to describe the orbits. Fix a simple system Π . Associated with each hyperplane H_α , there are the open half-spaces V_α and V'_α , where

$$V_\alpha := \{\lambda \in \mathbb{R}^n \mid (\lambda, \alpha) > 0\}$$

and

$$V'_\alpha := -V_\alpha.$$

Definition 5.4. *Let G be a group acting on \mathbb{R}^n . Then a closed subset D of \mathbb{R}^n is called a fundamental domain of G on \mathbb{R}^n , if*

$$\mathbb{R}^n = \bigcup_{g \in G} gD,$$

and $gD \cap hD$ has empty interior for all $g, h \in G$.

Definition 5.5. A subset C of a vector space V is a cone if $\lambda C \subseteq C$, for any real $\lambda > 0$.

Definition 5.6. A subset C of a vector space V is convex if for any vectors $u, v \in C$, the vector $(1 - t)u + tv$ is also in C for all $t \in [0, 1]$.

Let $C := \bigcap_{\alpha \in \Pi} V_\alpha$. Then C is an open convex cone. Let D be the closure of C . Then

$$D := \{\lambda \in \mathbb{R}^n \mid (\lambda, \alpha) \geq 0 \text{ for all } \alpha \in \Pi\}$$

is a closed convex cone which is actually a fundamental domain for the action of W on \mathbb{R}^n .

Theorem 5.7. Let Π be a simple root system. Then D is a fundamental domain for the action of W on V .

So, we have associated to a simple system Π an open convex cone C . If we replace Π by $w\Pi$, with $w \in W$, then we replace C by wC . All of these open convex cones are called *chambers* and they are the connected components of the complement of $\bigcup_{\alpha \in \Pi} H_\alpha$ in \mathbb{R}^n . Given a chamber C associated with a simple system Π , its *walls* are defined to be the hyperplanes H_α , with $\alpha \in \Pi$. The angle between any two walls is an angle of the form π/k , for some positive integer $k > 1$

5.3 Wavelet Sets in \mathbb{R}^n

Let A be an expansive matrix and \mathcal{T} be a full rank lattice. The definition of a Multiresolution Analysis on higher dimensions than 1 is as follows.

Definition 5.8. A multiresolution analysis on \mathbb{R}^n is a sequence of subspaces $\{V_j\}_{j \in \mathbb{Z}}$ of functions in $L^2(\mathbb{R}^n)$ satisfying the following properties:

- i) For all $j \in \mathbb{Z}$, $V_j \subseteq V_{j+1}$
- ii) If $f(\cdot) \in V_j$, then $f(A\cdot) \in V_{j+1}$

$$iii) \bigcap_{j \in \mathbb{Z}} V_j = \{0\}$$

$$iv) \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}^n)$$

v) There exists a function $\phi \in L^2(\mathbb{R}^n)$ such that $\{\phi(\cdot + t) | t \in \mathcal{T}\}$ is an orthonormal basis of V_0 .

The function ϕ is called a *scaling function*. One can allow more than one scaling function, say m , and then the MRA has multiplicity m .

Lemma 5.9. Let $\{\Omega + t | t \in \mathcal{T}\}$ be a measurable tiling of \mathbb{R}^n . If $f \in L^2(\mathbb{R}^n)$, then $\{f(\cdot + t)\}_{t \in \mathcal{T}}$ is an orthonormal system if and only if

$$\sum_{t \in \mathcal{T}} |\hat{f}(\xi + t)|^2 = 1,$$

for almost all $\xi \in \mathbb{R}^n$.

Proof. Let $\{f(\cdot + t)\}_{t \in \mathcal{T}}$ be an orthonormal system. If $s \in \mathcal{T}$, then

$$\begin{aligned} \delta_{o,s} &= \int_{\mathbb{R}^n} f(x) \overline{f(x+s)} dx \\ &= \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 e^{2\pi i \langle s, \xi \rangle} d\xi \\ &= \sum_{t \in \mathcal{T}} \int_{\Omega - t} |\hat{f}(\xi)|^2 e^{2\pi i \langle s, \xi \rangle} d\xi \\ &= \int_{\Omega} \left(\sum_{t \in \mathcal{T}} |\hat{f}(\xi + t)|^2 \right) e^{2\pi i \langle s, \xi \rangle} d\xi \end{aligned}$$

Thus, the periodic function $\sum_{t \in \mathcal{T}} |\hat{f}(\xi + t)|^2$ is 1, since its Fourier coefficient at frequency 0 is 1 and the rest of coefficients are zero. Note that the $\int_{\Omega - t} |\hat{f}(\xi)|^2 e^{2\pi i \langle s, \xi \rangle} d\xi$ is bounded, so by the Lebesgue Dominated Convergence, we can interchange the summation and integration in the last two equalities. The other direction is immediate. \square

If we change condition iv) in the definition of the multiresolution analysis into iv') $\overline{\bigcup_{j \in \mathcal{T}} V_j} = L^2_M(\mathbb{R}^n)$, for some subset $M \subseteq \mathbb{R}^n$, then we get a *subspace multiresolution analysis*, SMRA.

We are discussing now the construction of wavelets from MRA. Let W_0 be the orthogonal complement of V_0 in V_1 , that is, $V_1 = V_0 \oplus W_0$. In general, let $W_i = V_{i+1} \ominus V_i$, for each $j \in \mathbb{Z}$,

$$V_j = \bigoplus_{l=-\infty}^j W_l$$

and so

$$L^2(\mathbb{R}^n) = \bigoplus_{l=-\infty}^{\infty} W_l.$$

If there exists a function $\psi \in W_0$ such that $\{\psi(\cdot + t) | t \in \mathcal{T}\}$ is an orthonormal basis for W_0 , then $\{\psi_{j,t} | t \in \mathcal{T}\}$ is an orthonormal basis for W_j , and $\{\psi_{j,t} | t \in \mathcal{T}, j \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$, which means that ψ is an orthonormal wavelet associated with the given MRA..

Since $\{\phi_t | t \in \mathcal{T}\}$ is an orthonormal basis for V_0 , we obtain

$$\hat{\phi}(B\xi) = \hat{\phi}(\xi)m_0(\xi),$$

with low pass filter

$$m_0(\xi) = \sum_{t \in \mathcal{T}} e^{2\pi i \langle t, \xi \rangle}.$$

5.4 Wavelet Sets and SMRA

Let \mathcal{T} be a full rank lattice and let A be an expansive matrix. Set $B = A^T$. Suppose $B\mathcal{T} \subseteq \mathcal{T}$ and then let $\mathcal{T}/B\mathcal{T}$ be the quotient group, where we identify its elements with their representative vectors in \mathbb{R}^n , v_0, v_1, \dots, v_{q-1} , where $q = |\det(B)|$.

Lemma 5.10. *Let K be a \mathcal{T} -tile such that $B^{-1}K \subset K$. Let*

$$K_i = (B^{-1}K + B^{-1}v_i + \mathcal{T}) \cap K.$$

Then

i) $K = \bigcup_{i=0}^{q-1} K_i$ up to measure zero and $K_i \cap K_j = \emptyset$ up to measure zero for $i \neq j$.

ii) $BK_i \sim_{\mathcal{T}} K$.

Proof. Let $x \in K_i \cap K_j$. Then there exist $u_1, u_2 \in K$, $v_i, v_j \in \mathcal{T}/B\mathcal{T}$ and $t_1, t_2 \in \mathcal{T}$, such that

$$x = B^{-1}u_1 + B^{-1}v_i + t_1 = B^{-1}u_2 + B^{-1}v_j + t_2,$$

and so $u_1 - u_2 \in \mathcal{T}$. But K is a \mathcal{T} tile, so $u_1 = u_2$, and then $v_i - v_j = B(t_1 - t_2)$. Thus $v_i = v_j$ and so $i = j$. Let the notations be as before and let

$$K_{i,t} = B^{-1}K \cap (K - B^{-1}v_i - t).$$

Then $\bigcup_{t \in \mathcal{T}} (K_{i,t} + B^{-1}v_i + t) = K_i$.

Since K is a \mathcal{T} tile, it follows that $K - B^{-1}v_i$ is also a \mathcal{T} tile. Thus $K_{i,t}$ are measurewise disjoint and $\bigcup_{t \in \mathcal{T}} K_{i,t} = B^{-1}K$.

By definition, $K_i \subseteq K$, and

$$|K_i| = \sum_{t \in \mathcal{T}} |K_{i,t}| = |B^{-1}K| = q^{-1}|K|$$

for $i = 0, \dots, q-1$, and $|K_i \cap K_j| = 0$ for $i \neq j$. Therefore

$$K = \bigcup_{i=0}^{q-1} K_i$$

up to measure zero. Moreover,

$$B(K_i) = \bigcup_{t \in \mathcal{T}} B(K_{i,t} + B^{-1}v_i + t) = K + v_i + Bt$$

and thus

$$B(K_i) \sim_{\mathcal{T}} K.$$

□

Let $K \subset \mathbb{R}^n$ be a measurable set. Set $V_0 = L_K^2(\mathbb{R}^n)$ and $V_j = \{f(A^j \cdot) | f(\cdot) \in V_0\}$.

Definition 5.11. A set $K \subset \mathbb{R}^n$, $|K| = 1$, is a scaling set, if the sequence described above, $\{V_j\}$, is a multiresolution analysis with scaling function $\phi = \mathcal{F}^{-1}\chi_K$.

Theorem 5.12. A subset $K \subset \mathbb{R}^n$ is a scaling set if and only if $B^{-1}K \subseteq K$ and K is a \mathcal{T} -tile.

Proof. Suppose first that $K \subset \mathbb{R}^n$ is a scaling set. Then $\phi = \mathcal{F}^{-1}\chi_K$ is a scaling function and so

$$\{\phi_{0,t}\}_{t \in \mathcal{T}}$$

is an orthonormal basis for V_0 . This implies that

$$\{\hat{\phi}_{0,t}\}_{t \in \mathcal{T}}$$

is an orthonormal basis for \hat{V}_0 . It follows that \hat{V}_0 has a orthonormal basis of the form

$$\{e^{-2\pi i \langle t, \cdot \rangle} \chi_K\}_{t \in \mathcal{T}}.$$

From this we get two things. The first is that (K, \mathcal{T}) is a spectral pair and thus K is a \mathcal{T} -tile. The second is that $\hat{V}_0 = L^2(K)$ and by the SMRA structure, we get that

$$\hat{V}_{-1} \subset \hat{V}_0$$

which implies that

$$B^{-1}K \subset K.$$

Assume now that $B^{-1}K \subset K$ and that K is a \mathcal{T} -tile.

Set $\hat{V}_0 = L^2(K)$ and $\hat{V}_j = L^2(B^j K)$. Since $B^{-1}K \subset K$, it follows that $\hat{V}_j \subset \hat{V}_{j+1}$.

The other conditions are easy to verify. Thus $K \subset \mathbb{R}^n$ is a scaling set. \square

The next theorem gives in a constructive way, the existence of SMRA wavelets.

Theorem 5.13. If $K \subset \mathbb{R}^n$ is a scaling set, then

i) $\widehat{V}_0 = L^2(K)$ and $\widehat{V}_j = L^2(B^j K)$

ii) $\{\psi^i = \check{\chi}_{\Omega_i}\}_{i=1}^{q-1}$ is a SMRA multiwavelet, where $\Omega_i = BK_i$.

Proof. i) follows from the theorem above.

ii) By lemma 1, $\Omega_i \sim K$. This implies that (Ω_i, \mathcal{T}) is a spectral pair. Then

$$\{\widehat{\psi}_{\alpha,t}^i\}_{t \in \mathcal{T}}$$

is an orthonormal basis for $L^2(\Omega_i)$.

Set $\widehat{W}_{0,i} = L^2(\Omega_i)$ for $i = 1, \dots, q-1$. By construction,

$$BK = K \cup \bigcup_{i=1}^{q-1} \Omega_i.$$

Therefore

$$\widehat{V}_1 = L^2(BK) = L^2(K) \oplus \bigoplus_{i=1}^{q-1} L^2(\Omega_i) = \widehat{V}_0 \oplus \widehat{W}_{0,1} \oplus \dots \oplus \widehat{W}_{0,q-1}.$$

So

$$V_1 = V_0 \oplus \bigoplus_{i=1}^{q-1} W_{0,i}$$

and for any $j \in \mathcal{Z}$, we have

$$V_{j+1} = V_j \oplus \bigoplus_{i=1}^{q-1} W_{j,i}.$$

Thus, $\{\psi^i = \mathcal{F}^{-1} \chi_{\Omega_i}\}_{i=1}^{q-1}$ is a SMRA multiwavelet. \square

5.5 Multiresolution and Coxeter Groups

Let $W = \langle r_{\alpha_i} | \alpha_i \in \Pi \rangle$ be a finite Coxeter group, where $\Pi = \{\alpha_i | i = 1, \dots, n\}$ is a simple root system, and let D be the fundamental domain for the action of W on \mathbb{R}^n . Let $\Pi^* = \{\alpha_i^* | (\alpha_i, \alpha_j^*) = \delta_{i,j}\}$ be the dual basis and let $R = (\alpha_i^*)_{\alpha_i^* \in \Pi^*}$. Let A be an expansive, diagonal matrix with respect to the basis Π^* , and let $B = A^T$.

$$P = \left\{ \sum_{i=1}^n t_i \alpha_i^* \mid 0 < t_i \leq s_i \right\},$$

where s_i are such that $|P| = 1$. Note that P is a n dimensional parallelepiped and a $R\mathbb{Z}^n$ tile.

Indeed, if $z \in \mathbb{Z}^n$, then

$$\begin{aligned} P + Rz &= \left\{ \sum_{i=1}^n t_i \alpha_i^* + \sum_{i=1}^n n_i \alpha_i^* \mid 0 < t_i \leq s_i, n_i \in \mathbb{Z} \right\} \\ &= \left\{ \sum_{i=1}^n (t_i + n_i) \alpha_i^* \mid 0 < t_i \leq s_i, n_i \in \mathbb{Z} \right\} \end{aligned}$$

so

$$|P \cap (P + Rz)| = 0.$$

Let d_1, \dots, d_n be the eigenvalues of $B = A^T$. Then

$$B^{-1}P = \left\{ \sum_{i=1}^n d_i^{-1} t_i \alpha_i^* \mid 0 < t_i \leq s_i \right\} \subset P,$$

since $0 < d_i^{-1} t_i \leq s_i$. Moreover,

$$\left(\sum_{i=1}^n t_i \alpha_i^*, \alpha_m \right) = \sum_{i=1}^n t_i (\alpha_i^*, \alpha_m) = t_m > 0.$$

Thus,

$$B^{-1}P \subset P \subset D.$$

Theorem 5.14. *Let P, B be as above. Then $\{\psi^i = \check{\chi}_{\Omega_i}\}_{i=1}^{q-1}$ is a SMRA multiwavelet, where $\Omega_i = BP_i$ and $P_i = B^{-1}P + B^{-1}v_i$.*

Proof. As shown above, P is a $R\mathbb{Z}^n$ -tile and $B^{-1}P \subset P$ and so P is a scaling set.

Thus, by theorem 4.8, $\{\psi^i = \check{\chi}_{\Omega_i}\}_{i=1}^{q-1}$ is a SMRA multiwavelet. \square

Example 5.15. Let the group $\mathcal{D} = \{R_{\frac{2\pi}{m}}^k\}_{k=0}^{m-1}$ act on \mathbb{R}^2 and let $D = \{t_1(1, 0) + t_2(\cot 2\pi/m, 1), 0 < t_{1,2}\}$ be the fundamental domain of this action. Let $B = 2id_2$ and $P = \{t_1(1, 0) + t_2(\cot 2\pi/m, 1), t_{1,2} \in [0, 1]\}$. Let

$$\Omega_1 = P + (1, 0),$$

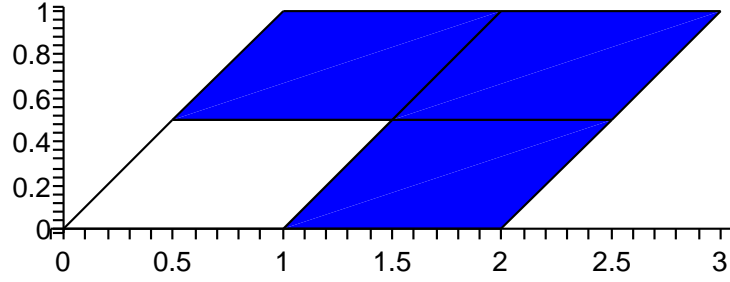


FIGURE 5.2. SMRA wavelet sets in \mathbb{R}^2

$$\Omega_2 = P + (1 + \cot 2\pi/m, 1),$$

$$\Omega_3 = P + (\cot 2\pi/m, 1).$$

Then $\{\psi^i = \tilde{\chi}_{\Omega_i}\}_{i=1}^3$ is a SMRA multiwavelet, see Figure 5.2.

Example 5.16. Let $W = \langle r_a, r_b, r_c \rangle$ be a Coxeter group, where a, b, c are as described in Example 5.1. Then the fundamental domain for the action of W on \mathbb{R}^3 is

$$D = \{t_a a^* + t_b b^* + t_c c^* | 0 < t_a, t_b, t_c\}.$$

Let $P = \{t_a a^* + t_b b^* + t_c c^* | 0 < t_a < s_a, 0 < t_b < s_b, 0 < t_c < s_c\}$, such that $|P| = 1$.

Let $B = 2id_3$. Then $\det(B) = 2^3 = 8$ and so there are 7 MRA wavelet sets, see Figure 5.3.

5.6 Wavelet Sets and Coxeter Groups

The construction of wavelet sets given in chapter 3., can also be generalized to higher dimensions using Coxeter groups.

Theorem 5.17. *Let P be as above and let $F = BP \setminus P$. Define*

$$W_{1,1} = (P \setminus B^{-1}P) + \alpha_i^*$$

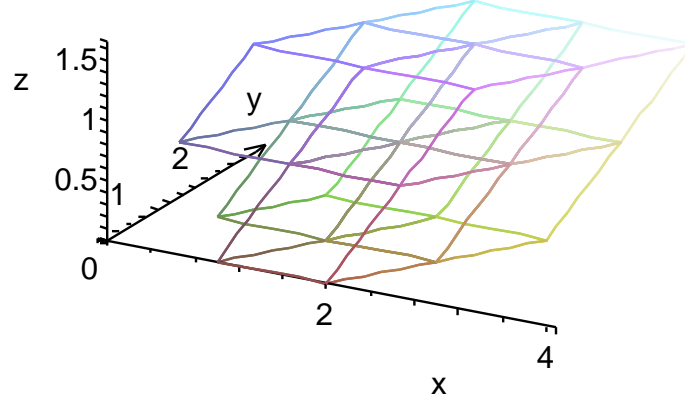


FIGURE 5.3. SMRA wavelet sets in \mathbb{R}^3

$$W_{2,1} = B^{-2}[F \setminus (P + \alpha_i^*)]$$

...

$$W_{1,n} = [(B^{-n+1}P \setminus B^{-n}P) \setminus W_{2,n-1}] + \alpha_i^*$$

$$W_{2,n} = B^{-n-1}\{[(B^{-n+1}P \setminus B^{-n}P) + \alpha_i^*] \setminus W_{1,n}\}.$$

Then

$$P = \bigcup_n^\infty W_{2,n} \bigcup_n^\infty (W_{1,n} - \alpha_i^*)$$

$$F = \bigcup_n^\infty W_{1,n} \bigcup_n^\infty B^{n+1}W_{1,n}.$$

Moreover, if we let

$$W = \bigcup_{j=1,2} \bigcup_n^\infty W_{j,n},$$

then W is a wavelet set.

Proof. We have shown above that P is a $R\mathbb{Z}^n$ -tile. On the other hand,

$$BF \cap F = \emptyset,$$

and

$$\bigcup_{n \in \mathbb{Z}} B^n F = D,$$

so F is a multiplicative tiling. By definition,

$$W \sim_{R\mathbb{Z}^n} P,$$

and

$$W \sim_B F.$$

Thus, W is a wavelet set. □

References

- [1] A. Aldroubi, C. Cabrelli, U. Molter, *Wavelet on irregular grids with arbitrary dilation matrices, and frame atoms for $l^2(r^d)$* , Preprint, 2003.
- [2] P. Aniello, G. Cassinelli, E. De Vito, A. Levrero, *Wavelet transforms and discrete frames associated to semidirect products.*, J. Math. Phys. **39** (1998), 3965–3973.
- [3] L. Baggett, A. Carey, W. Moran, P. Ohring, *General Existence Theorem for Orthonormal Wavelet. An Abstract Approach.*, Publ. Res. Inst. Mth. Sci. (1) **31** (1995), 95–111.
- [4] D. Bernier, K. F. Taylor, *Wavelets from square-integrable representations.*, SIAM J. Math. Anal. **27** (1996), 594–608.
- [5] C.E. Heil, and D.F. Walnut, *Continuous and discrete wavelet transform.*, J. Math. Phys. **39** (1998), 3974–3986.
- [6] X. Dai, D.R. Larson, *Wandering vectors for unitary systems and orthogonal wavelets.*, Mem. Amer. Math. Soc. **134** (1998), no. 640.
- [7] X. Dai, D.R. Larson, D. Speegle, *Wavelet sets in \mathbb{R}^n* , J. Fourier Anal. **3** (1997), no. 4, 451–456.
- [8] X. Dai, D.R. Larson, D. Speegle, *Wavelet sets in \mathbb{R}^n II*, Wavelets, multi-wavelets and their applications (San Diego, CA, 1997), Contemporary Mathematics, vol. 216, AMS, Providence, RI, 1998, pp. 15–40.
- [9] X. Dai, Y. Diao, Q. Gu, D. Han, *The existence of subspace wavelet sets.*, Preprint, 2003.
- [10] R. Fabec, G. Ólafsson, *The Continuous Wavelet Transform and Symmetric Spaces.*, Acta Applicandae Math. **77** (2003), 41–69.
- [11] B. Fuglede, *Commuting self-adjoint partial differential operators and a group theoretical problem.*, J. Func Anal. **16** (1974), 101–121.
- [12] H. Führ, *Wavelet frames and admissibility in higher dimensions.*, J. Math. Phys. **37** (1996), 6353–6366.
- [13] H. Führ, *Continuous wavelet transforms with Abelian dilation groups.*, J. Math. Phys. **39** (1998), 3974–3986.
- [14] E. Hernandez, G. Weiss, *A first course on wavelets*, CRC Press, 1996.
- [15] A. Iosevich, N.H. Katz, T. Tao, *Convex bodies with a point of curvature do not have fourier bases.*, Amer. J. Math. **123** (2001), 115–120.

- [16] S Sumetkijakan J. Benedetto, *A fractal set constructed from a class of wavelet sets*, Contemporary Math. **313** (2002), 19–35.
- [17] P. Jorgensen, S. Pedersen, *Orthogonal harmonic analysis of fractal measures.*, Elec. Res. Announc. Amer. Math. Soc. **4** (1998), 35–42.
- [18] P. Jorgensen, S. Pedersen, *Spectral pairs in cartesian coordinates.*, Fourier. Anal. Appl. **5** (1999), 285–302.
- [19] J. Langarias, Y. Wang, *Spectral sets and factorizations of finite abelian groups.*, J. Funct. Anal. **145** (1997), 73–98.
- [20] G. Ólafsson, *Continuous action of Lie groups on \mathbb{R}^n and frames.*, Preprint, 2002.
- [21] G. Ólafsson, D. Speegle, *Wavelets, wavelet sets, and linear actions on \mathbb{R}^n* , Wavelets, Frames and Operator Theory (College Park, MD, 2003) (C. Heil, P.E.T. Jorgensen, D.R. Larson, ed.), Contemporary Mathematics, vol. 345, AMS, Providence, RI, 2003, pp. 253–282.
- [22] G. Ólafsson R. Fabec, *Noncommutative Harmonic Analysis*, Lecture Notes, Louisiana State University, 2002.
- [23] R.S.Laugesen, N. Weaver, G.L. Weiss, E.N. Wilson, *A characterization of the higher dimensional groups associated with continuous wavelets.*, J. Geom. Anal. **12** (2002), 89–102.
- [24] D. Speegle, *On the existence of wavelets for non-expansive dilations*, Collect. Math. **54** (2003), no. 2, 163–179.
- [25] T. Tao, *Fuglede’s conjecture is false in 5 and higher dimensions.*, Math. Res. Letters **11** (2004).
- [26] Y. Wang, *Wavelets, tiling, and spectral sets.*, Duke Math. J. **114** (2002), no. 1.

Vita

Mihaela Dobrescu was born on April 30 1972, in Bumbesti Jiu, Gorj, România. She finished her undergraduate studies at University of Bucharest, May 1995. In August 2000 she came to Louisiana State University to pursue graduate studies in mathematics. She is currently a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2005.