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A characterization of near outer-planar graphs

Tanya Allen Lueder
Louisiana State University and Agricultural and Mechanical College

A CHARACTERIZATION OF NEAR OUTER-PLANAR GRAPHS

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Tanya Allen Lueder
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Abstract

This thesis focuses on graphs containing an edge whose removal results in an outer-planar graph. We present partial results towards the larger goal of describing the class of all such graphs in terms of a finite list of excluded graphs. Specifically, we give a complete description of those members of this list that are not 2-connected or do not contain a subdivision of a three-spoke wheel. We also show that no members of the list contain a five-spoke wheel.

Chapter 1
Introduction

In short, a graph in this thesis may contain parallel edges, but not loops. More specifically, a graph G is a triple $(V, E, \mathscr{I})$ where V is a finite set whose elements are called *vertices*; E is a finite set disjoint from V whose elements are called *edges*; and $\mathscr{I}$, called the *incidence relation*, is a subset of $V \times E$ in which each edge is in relation with exactly two vertices, u and v, called its *endpoints*. Any two vertices connected by an edge are *adjacent*. The *degree* of a vertex is the number of edges incident to the vertex. The number of vertices of a graph G is the *order* of G and is indicated by $|G|$. If two edges are incident to the same pair of vertices, then we call them *parallel edges*. A graph without parallel edges is called a *simple graph*. A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and $\mathscr{I}(H) \subseteq \mathscr{I}(G)$.

A *trail* is a sequence $v_0, e_0, v_1, e_1, \ldots, e_n, v_n$ where each edge, e_i, is incident with vertices, v_{i-1} and v_i, and no edge is repeated. A *path* is a trail with no repeated vertices. The length of a path is the number of edges it contains. The first and the last vertices of a path are its *endpoints*. All other vertices of a path are its *internal vertices*. Two paths are *independent* if no vertex of one is an internal vertex of the other. An *isomorphism* between two graphs G and H is a pair of bijections, φ and ψ, such that $\varphi : V(G) \to V(H)$ and $\psi : E(G) \to E(H)$, where $(u, e) \in \mathscr{I}(G)$ if and only if $(\varphi(u), \psi(e)) \in \mathscr{I}(H)$.

We call a graph *connected* if every pair of its vertices is connected by a path, and *disconnected*, otherwise. The maximal connected subgraphs of a graph are its *components*. A *cut vertex* in a graph is a vertex whose removal results in an increase in the number of components. The *connectivity* of a graph G is zero if a graph is disconnected, $|G| - 1$ if G is connected but has no pair of distinct non-adjacent vertices, or the size of the smallest set of vertices that disconnects G if G is connected and has a pair of non-adjacent vertices.

An acyclic graph is called a *forest*. A *tree* is a connected forest. A vertex of degree one or zero of a tree is called a *leaf*.

There are several classes of graphs that we will use in this thesis. A *complete graph* is a simple graph in which every pair of vertices is connected by an edge. We denote complete graphs by K^n, where n is the number of vertices. A *bipartite graph* is composed of two disjoint sets of vertices such that each edge is incident to one vertex in each set. A *complete bipartite graph* is a simple bipartite graph in which each vertex is adjacent to every vertex in the other set. We denote a complete bipartite graph by $K_{r,s}$ where r and s denote the number of vertices in the disjoint sets. A *cycle* on n vertices, denoted C_n, is a trail of n vertices in which no vertices are repeated except the first equals the last. A *wheel*, W_n, is obtained from C_n by adding a new vertex, called the *hub* and joining every vertex of C_n to the new vertex. The cycle, viewed as a subgraph of W_n is called the *rim*. The vertices of the rim are the *rim vertices*. The edges that connect the hub to the rim vertices are called *spokes*. Three examples of graphs listed in Figure 1.1 will play important roles in this paper.

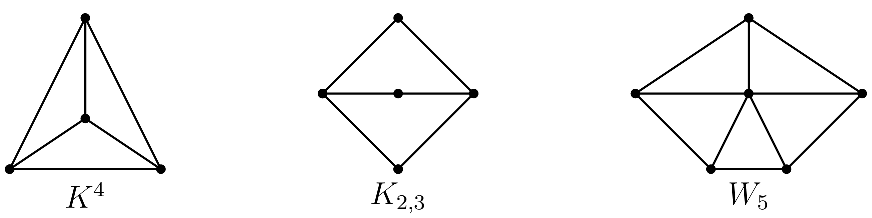


FIGURE 1.1. Examples of a complete graph, complete bipartite graph, and wheel.

A graph is *planar* if it can be drawn on the plane so that its edges only intersect at common vertices. The embedding a planar graph in the plane divides the plane into regions called *faces*. One face is unbounded; we call this the *outer face*. A graph is called *outer-planar* (OP) if it has a plane embedding in which all of the vertices lie on the boundary of the outer face. The focus of this thesis is to investigate graphs one edge away from being outer-planar graphs.

Definition 1.1. A graph is *near outer-planar*, or NOP, if it is edgeless or has an edge whose deletion results in an outer-planar graph.

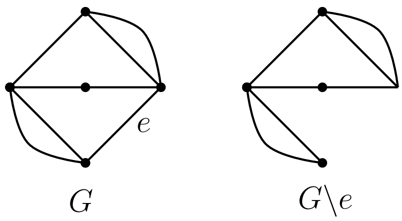


FIGURE 1.2. An example of a graph that is NOP.

The graph G, shown in Figure 1.2, is NOP, but $G\backslash e$ is OP. To describe the class of NOP graphs, we will provide a list of graphs which are not NOP and are minimal, in a sense that we will describe later. We will call the members of this list *excluded near outer-planar* or XNOP. First, we shall define some relations on graphs. *Edge contraction* is an operation where an edge e, and all edges parallel to it are removed from a graph and the two endpoints are identified to form a new vertex v. Any edges not parallel to e, but adjacent to e before the contraction are incident to v after the contraction. We denote an edge contraction of G by G/e. *Edge deletion* is an operation in which an edge is removed from a graph and *vertex deletion* is an operation in which a vertex and its incident edges are removed, denoted by $G\backslash e$ and $G - v$, respectively. A graph H is a *minor* of G if a graph isomorphic to H can be obtained from G by a sequence (possibly null) of operations, each of which is one of the following three operations: contracting an edge, deleting an edge, or deleting a vertex. We denote that a graph H is a minor of G by $G \geqslant_{\mathrm{m}} H$ or $H \leqslant_{\mathrm{m}} G$. Similarly, a *topological minor* is obtained by a sequence (possibly null) of operations each of which is one of the following: contracting an edge incident to a vertex of degree two, deleting an edge, or deleting a vertex. An edge, uv, is *subdivided* if it is replaced with a path, uwv of length two through a new vertex, w. A graph G is a subdivision of another graph H, if a graph isomorphic to G can be obtained by a sequence of subdivisions (possibly zero) of edges of H. An alternate way to describe H as a topological minor of G is to say that G contains a subdivision of H as a subgraph.

We are motivated by the following well-known theorem and corollary.

Theorem 1.2. *(Kuratowski 1930) A graph is planar if and only if it does not contain K^5 or $K_{3,3}$ as a topological minor.*

The following corollary can be easily derived from this theorem.

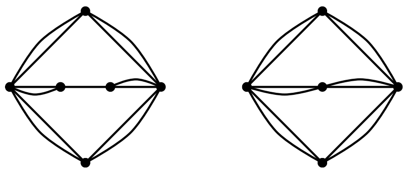
Corollary 1.3. *A graph is outer-planar if and only if it does not contain K^4 or $K_{2,3}$ as a topological minor.*

The corollary gives us a starting point for some XNOP graphs. The graphs K^4 and $K_{2,3}$ are NOP since each contains an edge whose removal results in a graph that is OP. (In fact, the removal of any edge from these two graphs results in a graph that is OP). We want to describe the class of NOP graphs in a finite way. Theorem 1.2 and Corollary 1.3 both describe infinite classes by excluding a finite list of graphs. We would like to give a similar description of NOP graphs. If we could use minors, we could be sure that our list is finite because of the following theorem.

Theorem 1.4. *(Robertson, Seymour 2004) Every infinite set of finite graphs contains two graphs, such that one is a minor of the other.*

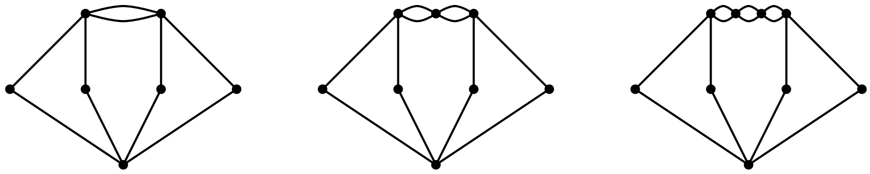
But, the excluded list of NOP graphs cannot be formed by the taking of minors, since the class of NOP graphs is not closed under the taking of minors. For example the graph DE_1' shown in Figure 1.3 is NOP, but the graph DE_1, a minor of DE_1' is not NOP.

Although the class of NOP graphs is closed by the taking of topological minors, the resulting list of XNOP graphs would not be finite. The graphs in Figure 1.4, which can be extended to form an infinite set, should be on the list of XNOP graphs since they are not NOP and not topological minors of one another. The graphs are very similar to one another



(a) DE_1' (b) DE_1

FIGURE 1.3. DE_1' is NOP, but its minor, DE_1 is not NOP.



S_1

FIGURE 1.4. Examples of graphs that have a sequence of mutiple edges.

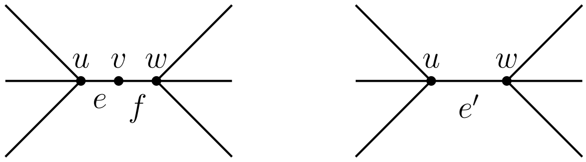
except the sequences of parallel edges have different lengths. To make our list finite, we would like to represent all graphs in this infinite set by S_1. We achieve this by introducing a new operation and a new relation on graphs.

Definition 1.5. Suppose v is a vertex of G with exactly two neighbors u and w, which may or may not be adjacent to each other. Let n denote the minimum of the number of uv edges and the number of vw edges in G. *Suppressing* the vertex v in G is the operation of replacing v and all its incident edges with n new uw edges. An example is given in Figure 1.5.

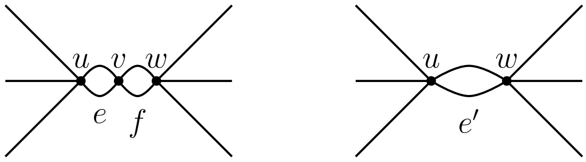
With this operation, we can define another relation.

Definition 1.6. A graph H *dominates* a graph G, written $G \preceq H$, if G can be obtained from H by a sequence of operations each of which is one of the following:

- deleting an edge,
- deleting a vertex and all its incident edges, and
- suppressing a vertex with exactly two neighbors.



(a) A suppression of v incident with single edges.



(b) A suppression of v incident with parallel edges.

FIGURE 1.5. Suppression of v.

If H dominates G and is not isomorphic to G, then we say that it *properly dominates* G and write $G \prec H$. Note that if G is a topological minor of H, then $G \preceq H$.

The following proposition establishes that domination may be used in defining XNOP graphs.

Proposition 1.7. *The class of NOP graphs is closed under domination.*

Proof. Let G and H be graphs such that $H \preceq G$ and G is NOP. We want to show that H is also NOP. Since G is NOP, it has an edge e such that $G \backslash e$ is OP. If H is isomorphic to G, then the conclusion follows. We assume $H \prec G$.

We examine edge- and vertex-deletions first. If a series of edge- or vertex- deletions of H result in a graph isomorphic to H, then either e is removed or not. If e is removed in the series of deletions, then $H \backslash e$ is OP since $G \backslash e$ is OP. Hence, we assume that e is an edge of H. Then $M \backslash e$ is not OP and the conclusion follows. If in the process of obtaining H from G, the edge was deleted, then H is OP and the conclusion follows.

It remains to consider the case where e is an edge incident to a vertex of G that is suppressed in the process of obtaining H from G. In the suppression process, e is replaced by f and $H \backslash f$ is OP, hence H is NOP. □

We have already used the term XNOP, but in an informal way. The following defines the term XNOP precisely, and from now on, we will refer to an XNOP graph as described below.

Definition 1.8. A graph H is *excluded near outer-planar* or *XNOP* if it is not NOP, but every graph properly dominated by H is NOP.

In Appendix A, we list the 43 known graphs that are XNOP. Verifying that each of these is XNOP is tedious, so for the sake of brevity, we will only show the technique of verifying that a graph is XNOP with one example. Appendix B details the verification that WT_4 is XNOP. We have verified the other 42 graphs, but do not present the details here.

The set of all XNOP graphs may be divided into these sets (with the known graphs listed in parentheses):

I. Nonplanar graphs ($K_{3,3}$)

II. Disconnected graphs (D_1, D_2, D_3)

III. Graphs with a cut vertex (CV_1, CV_2, CV_3, CV_4, CV_5, CV_6)

IV. 2–connected graphs that do not dominate W_3 (DE_1, $K_{2,4}$, S_1, S_2, S_3, S_4, S_5, S_6)

V. 2–connected graphs that dominate W_3, but not W_4 (WT_1, WT_2, WT_3, WT_4, WT_5, WT_6, WT_7, WT_8, WT_9, WT_{10}, WT_{11}, WT_{12}, WT_{13}, WT_{14}, WT_{15}, WT_{16}, WT_{17}, DE_2, $CUBE$)

VI. 2–connected graphs that dominate W_4 (WF_1, WF_2, WF_3, WF_4, $K_5\backslash e$, WF_5).

We will devote a chapter to each of I–IV to show that each set is finite and to present the complete list of its elements. Although we strongly believe that the sets of V and VI are finite as well, we do not prove this here, although we do provide the known members of those sets, since we will use some of them in Chapters 5 and 6. The Chapters 5 and 6 present partial evidence that the sets V and VI are finite.

A common construction that we will need is that of *bridges*.

Definition 1.9. Let G be a graph and let J be a subgraph of G. A *bridge* of J in G is a subgraph B of G that satisfies the following:

- B is not a subgraph of J.
- Every vertex of B that is incident in G with an edge not in B lies in J.
- No proper subgraph of B satisfies both (1) and (2).

Throughout this thesis, we will look at subdivisions of $K_{2,3}$ and K^4. Observe that $K_{2,3}$ and K^4 both have vertices of degree three. We refer to these vertices in $K_{2,3}$ and K^4 and the corresponding vertices in the subdivisions as *branch vertices*. In $K_{2,3}$, the branch vertices are the endpoints of three distinct paths, which we call the *legs* of $K_{2,3}$ or of the subdivision of $K_{2,3}$. A vertex on a leg of H that is not a branch vertex is called an *internal vertex*.

The following lemma is easy to verify.

Lemma 1.10. *A subdivision of $K_{2,3}$ or K^4 has three pairwise independent paths between any two branch vertices.*

Chapter 2
Nonplanar Graphs

Theorem 2.1. *If G is XNOP and nonplanar, then G is $K_{3,3}$.*

Proof. Since G is nonplanar, by Theorem 1.2, G must contain $K_{3,3}$ or K^5 as a topological minor. But G cannot contain K^5 since $K^5\backslash e$ is XNOP. Then G must contain $K_{3,3}$ as a topological minor and hence must dominate it. But $K_{3,3}$ is XNOP, so G must be isomorphic to $K_{3,3}$. □

Chapter 3
Disconnected Graphs

Theorem 3.1. *If G is XNOP and disconnected, then G consists of 2 components, each of which is $K_{2,3}$ or K^4, as in D_1, D_2, and D_3.*

Proof. If G is disconnected, then it contains at least two components. We begin by proving the following.

(1) *G cannot contain any components that are OP.*

Suppose C_1 is an OP component. Let e be an edge of C_1. Since G is XNOP, it follows that there is an edge f that makes for which $G \backslash e \backslash f$ is an OP graph. Since G is XNOP, it follows that $G \backslash f$ is also NOP. But, $G \backslash f$ is OP, a contradiction. This proves (1). We now focus on the number of components.

(2) *G consists of exactly two components, each of which is not OP.*

Suppose G has at least three components C_1, C_2, and C_3, each of which is not OP by (1). Clearly the removal of only two edges from G always leaves a component that is not OP; a contradiction. This proves (2).

By (1) and (2), the two components of G are not OP and, by Corollary 1.3 each of the two components must have $K_{2,3}$ or K^4 as a topological minor. Hence G must dominate D_1, D_2, or D_3 as topological minors, and the conclusion follows. □

Chapter 4
Graphs with a Cut Vertex

Theorem 4.1. *If G is XNOP and contains a cut vertex, then G must be isomorphic to one of the following graphs: CV_1, CV_2, CV_3, CV_4, CV_5, and CV_6.*

Proof. If G has a cut vertex v, then the removal of v and its incident edges results in at least two components. Let n be the number of components that results when v is removed, and let A_i, for $i = 1, 2, \ldots, n$, be the components of $G - v$. Let P_i, for $i = 1, 2, \ldots, n$, be the subgraph induced by A_i and v. We begin by observing the following fact.

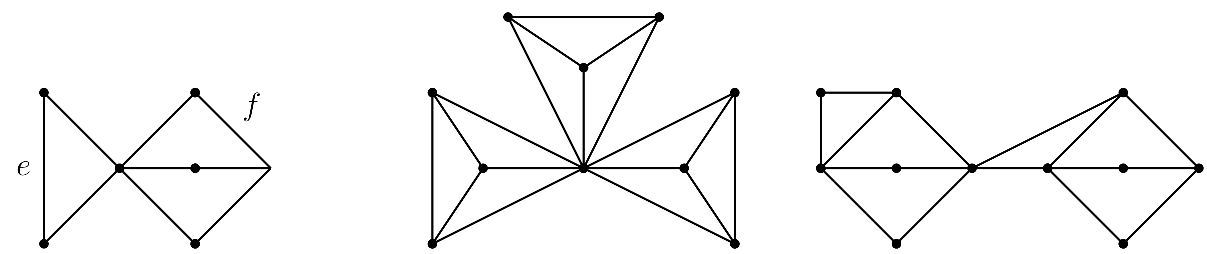
(1) *The graphs $K_{2,3}$ and K^4 do not have cut vertices.*

This implies that if $K_{2,3}$ or K^4 is a topological minor of G, then $K_{2,3}$ or K^4 is a topological minor of one of the $P_i's$. We can use this fact to prove the following.

(2) *If any P_i is OP, then G is not XNOP.*

Suppose P_1 is OP. Let e be an edge of P_1. Since G is XNOP, it follows $G\backslash e$ is NOP. Since P_1 is OP, then $\bigcup_{i \neq 1} P_i$ must be NOP. So there is an edge f in $\bigcup_{i \neq 1} P_i$ such that $\bigcup_{i \neq 1} P_i \backslash f$ is OP. Since G is XNOP, it follows that $G\backslash f$ is also NOP. But $\bigcup_{i \neq 1} P_i \backslash f$ is OP and P_1 is OP. So, $G\backslash f$ is OP; a contradiction. This proves (2). We now focus on the number of components.

(3) *The graph $G - v$ contains exactly two components.*



(a) OP P_1 and non-OP P_2 (b) Three P_i. (c) P_i with bridges

FIGURE 4.1. Examples of graphs with cut vertices that are not XNOP

Suppose G consists of at least three $P_i's$, say P_1, P_2, and P_3, each of which is not OP by (2). Clearly, the removal of only two edges from G always leaves a P_i that is not OP; a contradiction. This proves (3).

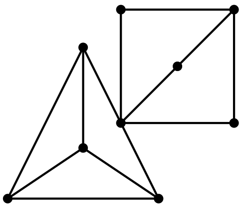
So G is the union of P_1 and P_2, each of which is not OP, such that P_1 and P_2 share a single vertex v. By Corollary 1.3, each of P_1 and P_2 must have $K_{2,3}$ or K^4 as a topological minor. In fact, we can prove the following.

(4) *Each of P_1 and P_2 is a subdivision of $K_{2,3}$ or K^4.*

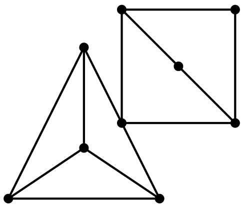
Suppose not. By (3), each of P_1 and P_2 contains a subdivision of $K_{2,3}$ or K^4, and so, at least one of P_1 or P_2 must properly contain a subdivision of $K_{2,3}$ or K^4. Let K_1 and K_2 be the subdivisions of $K_{2,3}$ or K^4, which are subgraphs of P_1 and P_2, respectively. If all of the edges and vertices that are not of K_1 or K_2 are removed, then the resulting graph G' is either connected or disconnected. If G' is disconnected, G must dominate D_1, D_2, or D_3, a contradiction. We now assume that G' is connected, and K_1 and K_2 share a common vertex v. But, then G' dominates CV_1, CV_2, CV_3, CV_4, CV_5, or CV_6; a contradiction. See Figure 4.2. This proves (4).

So G consists of two non-OP graphs, P_1 and P_2, each of which is a subdivision of $K_{2,3}$ or K^4, and P_1 and P_2 share a common vertex v. We have two types of vertices in P_1 and P_2 – vertices that are $K_{2,3}$ or K^4 without suppression and vertices that may be suppressed to leave a $K_{2,3}$ or K^4. Let the former be the K-vertices and the latter be the S-vertices. If v is a K-vertex in both P_1 and P_2, then G dominates CV_1, CV_2, CV_3, CV_4, CV_5, or CV_6. We assume that v must be an S-vertex in at least one of P_1 or P_2, say P_1. If P_1 is a subdivision of K^4, then G dominates CV_3 or CV_4 since every nontrivial subdivision of K^4 dominates a subdivision of $K_{2,3}$.

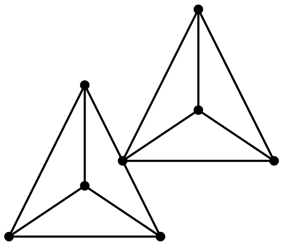
We assume that P_1 is a subdivision of $K_{2,3}$. Since v is an S-vertex, it must be adjacent to another vertex of degree 2 in P_1. So G dominates CV_3 or CV_4. Hence, the only XNOP graphs with a cut vertex are CV_1, CV_2, CV_3, CV_4, CV_5, or CV_6. □



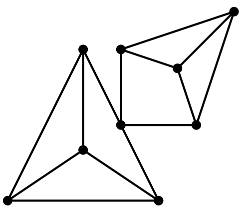
(a) This graph dominates CV_4.



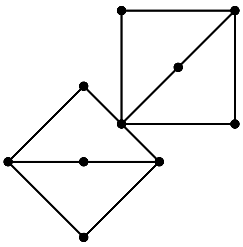
(b) This graph dominates CV_3.



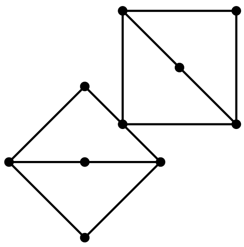
(c) This graph dominates CV_3.



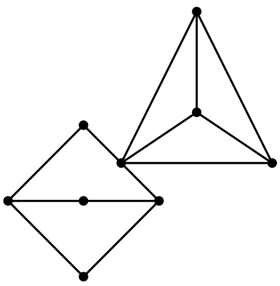
(d) This graph dominates CV_3.



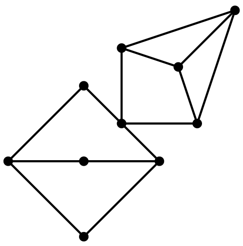
(e) This graph dominates CV_4.



(f) This graph dominates CV_3.



(g) This graph dominates CV_3.



(g) This graph dominates CV_3.

FIGURE 4.2. Examples of graphs with cut vertices with S-vertices.

Chapter 5

2-Connected Graphs that Do Not Dominate W_3

In this chapter, we will prove a result about 2-connected graphs that do not dominate W_3, using the following well-known theorem.

Corollary 5.1. *(Menger 1927) If G is a 2-connected graph and A, B are two disjoint subsets of $V(G)$, each containing at least two elements, then G contains two disjoint A–B paths.*

In Chapters 2–4, we have exhibited complete graphs that are not 2-connected. The following theorem presents the complete list of XNOP graphs that are 2-connected, but do not dominate W_3.

Theorem 5.2. *If G is XNOP, is 2-connected and does not dominate W_3, then G is isomorphic to one of DE_1, $K_{2,4}$, S_1, S_2, S_3, S_4, S_5, and S_6.*

Proof. To make the following proof more understandable, we will break it into 7 smaller parts, numbered (1)–(7). Since $K_{3,3}$ dominates W_3, but G does not, Theorem 2.1 implies that G may be assumed to be a plane graph. It is easy to verify, following the method outlined in Appendix B, that each of DE_1, $K_{2,4}$, S_1, S_2, S_3, S_4, S_5, and S_6 is indeed XNOP.

Since G is not OP and does not dominate K^4, it follows from Corollary 1.3 that G must dominate $K_{2,3}$. Let H be a subgraph of G such that H is a subdivision of $K_{2,3}$. We prove the following.

(1) *Every bridge of H in G has all vertices of attachment on a single leg of H.*

Suppose not. Let B be a bridge of H, and let L_1, L_2, and L_3 be the legs of H. If B has at least two vertices of attachment that are internal vertices of two distinct legs, say L_1 and L_2 of H, then the union of B, L_1, L_2, and L_3 contains a subdivision of K^4; a contradiction. This proves (1).

(2) *Every bridge, B, of H in G has exactly two vertices of attachment.*

Suppose not. If B has fewer than two vertices of attachment, then G is not 2-connected. So B must have at least three vertices of attachment. We know from (1) that B has all vertices of attachment on a single leg, say L_1, of H. But then the union of B, L_1, and one of the other legs of H contains a subdivision of K^4; a contradiction. This proves (2).

Assume n is an integer exceeding 2 and $K_{2,n} \leqslant_{\mathrm{m}} G$. The *skeleton* of $K_{2,n}$ in G is a minimal subgraph of G that contains $K_{2,n}$ as a minor. Let F be a skeleton of $K_{2,n}$ in G, and let D be a minimal set of edges of F such that F/D is a subdivision of $K_{2,n}$. Then the edges of F that are not in D form n paths, each of length at least two, which are called the *legs* of the skeleton. Since $K_{2,n}$ has only two vertices of degree exceeding two, each of these vertices corresponds either to a vertex of degree n in F, or to a connected acyclic subgraph of F induced by some edges in D. Hence each of the two degree-n vertices of $K_{2,n}$ corresponds to a subtree of F, which is called a *branch tree*. If a branch tree consists of a single vertex only, it is called *trivial*. Otherwise, it is called *nontrivial.* Recall that a vertex of degree one or zero of a branch tree is called a *leaf*. A vertex of a branch tree that is incident to legs of the skeleton of F is called a *toe.*

(3) *Every leaf of a branch tree is adjacent to at least two legs.*

Let the branch trees of F be BT_1 and BT_2. If BT_1 and BT_2 are trivial, then each leaf of F is incident with at least three legs. We may assume that at least one branch tree, say BT_1, is nontrivial. Suppose one leaf, w_l, of BT_1 is incident with exactly one leg, L_1. Let w_m be the vertex of BT_1 that is a toe and is nearest in BT_1 to w_l. Since there are no toes on the path of BT_1 between w_m and w_l, it follows that L_1 contradicts the minimality of D in the definition of a skeleton. This proves (3). See Figure 5.1.

With (3), we can also observe the following statement about leaves.

(4) *If F has 5 or fewer legs, each of its branch trees has at most 2 leaves.*

Now we can prove the following statement about legs and their toes.

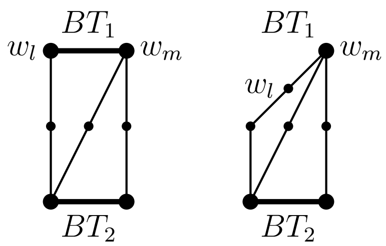


FIGURE 5.1. A contradiction of the minimality of the branch tree.

(5) *If F has 5 or fewer legs and two legs share the same toe of one nontrivial branch tree, then they share a toe on the other branch tree.*

If F is a skeleton of $K_{2,5}$, and BT_1 and BT_2 are the two branch trees, then by (3) each of BT_1 and BT_2 is one of the following:

(a) a single vertex;

(b) a nontrivial path with exactly 2 toes;

(c) a nontrivial path with exactly 3 toes.

Now we will investigate all of the combinations of (a)–(c) for each of BT_1 and BT_2. Since one branch tree is nontrivial by assumption, we need not look at the case where both branch trees are of type (a).

Case (i) Suppose one branch tree, BT_1, is of the type (a) and the other branch tree, BT_2, is of the type (b). It is obvious that if the legs of F share toes at BT_2, then the legs share toes at BT_1.

Case (ii) Suppose both branch trees are of type (b). Then, all of the toes are leaves and each is incident to at least two legs. Let w_l and w_r be toes of BT_1, such that w_l is incident to legs L_1 and L_2. Suppose further that L_1 and L_2 are incident to two different toes of BT_2, respectively, x_l and x_r. The toe w_r is incident to at least two legs, L_3 and L_4, whose other endpoints are contained in the set $\{x_l, x_r\}$ of BT_2. If L_3 and L_4 are not incident to the same toe of BT_2, then G dominates K^4. If L_3 and L_4 are incident to the same toe of BT_2, then

L_5 must be incident to the other toe of BT_2 and G dominates S_4. So, (5) holds in Case (*ii*). See Figure 5.2.

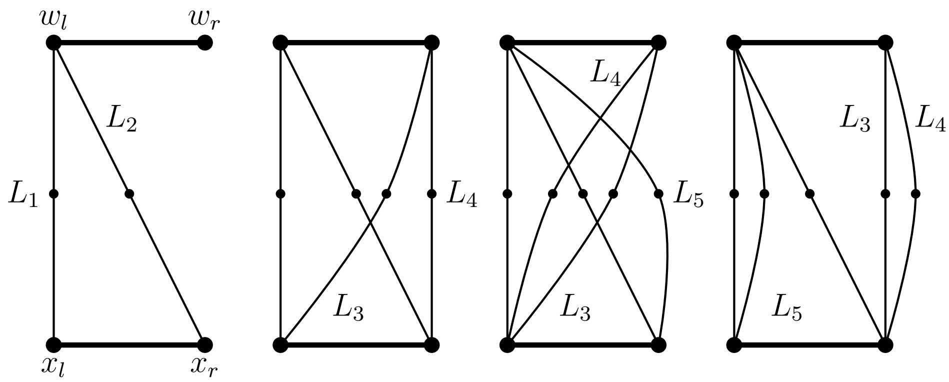


FIGURE 5.2. Subgraphs of G arising in Case (*ii*) in the proof of (5).

Case (*iii*) Suppose BT_1 is of type (*b*) and BT_2 is of type (*c*). As above, suppose that w_l is incident to legs L_1 and L_2 such that L_1 and L_2 are also incident to different toes of BT_2. Since BT_2 has three toes, two of which are leaves, then the legs L_1 and L_2 have toes that are both leaves or that is one leaf and one internal vertex of BT_2. If the incident toes of both L_1 and L_2 are the leaves, x_l and x_r, respectively, then, by (3), the vertex w_r, a leaf of BT_2, is incident to two adjacent legs, L_3 and L_4, which also are also incident to two different toes of BT_2. The toes incident to L_3 and L_4 of BT_2 are two of x_l, x_m, or x_l. But, in each of the cases, the union of L_3, L_4, BT_1, and BT_2, results in a graph that dominates K^4. See Figure 5.3.

So, in Case (*iii*), one of L_1 or L_2, say L_2, is incident to w_m, an internal vertex of BT_2, and L_1 is incident to a leaf of BT_2, say x_l. It follows that the toe w_r has two incident legs, L_3 and L_4. If both of these legs are incident to different toes of BT_2, then G dominates K^4. But, if both L_3 and L_4 are incident to the same toe, x_r, then G dominates S_5. So, we assume that BT_1 has three toes. See Figure 5.4.

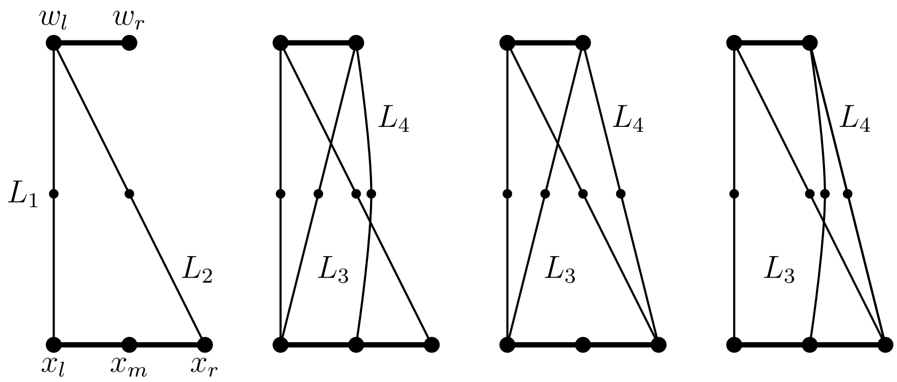


FIGURE 5.3. Graphs of Case (*iii*) with toes of L_1 and L_2 at leaves of BT_2.

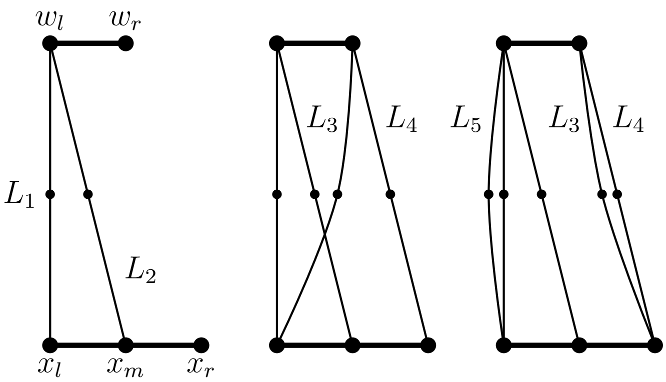


FIGURE 5.4. Graphs of Case(*iii*) with the toe of L_2 at an internal vertex of BT_2.

Case (*iv*) Suppose that BT_1 is of type (c) and BT_2 is of type (b). Then, BT_2 has exactly two vertices, x_l and x_r, incident to legs. Since w_l is incident to two legs which are incident to different toes of BT_2, then w_m, an internal vertex of BT_1, is incident to one leg, L_3, and w_r is incident to two legs, L_4 and L_5. The leg L_3 is incident to either x_l or x_r. It follows from (3), that L_3 and L_4 are not adjacent. The two possible cases that arise result in graphs, each of which dominate K^4. See Figure 5.5.

Case (v) Suppose BT_1 and BT_2 are both of type (c). So, w_l is incident to legs, L_1 and L_2, which are also incident to distinct elements of $\{x_l, x_m, \text{ or } x_r\}$. If the toes of both L_1 and L_2

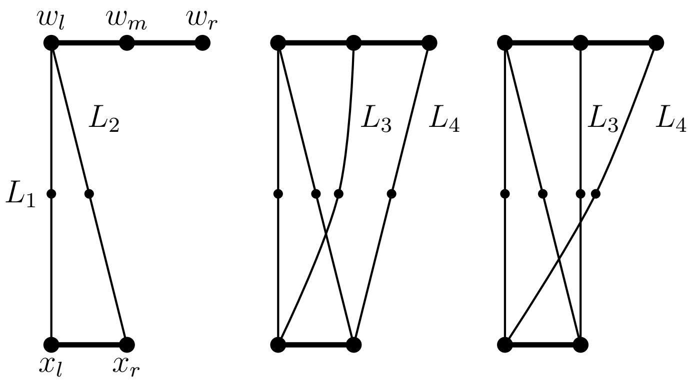


FIGURE 5.5. Graphs of Case (iv).

are the leaves of BT_2, then the cases that arise results in graphs that dominate a K^4. See Figure 5.6.

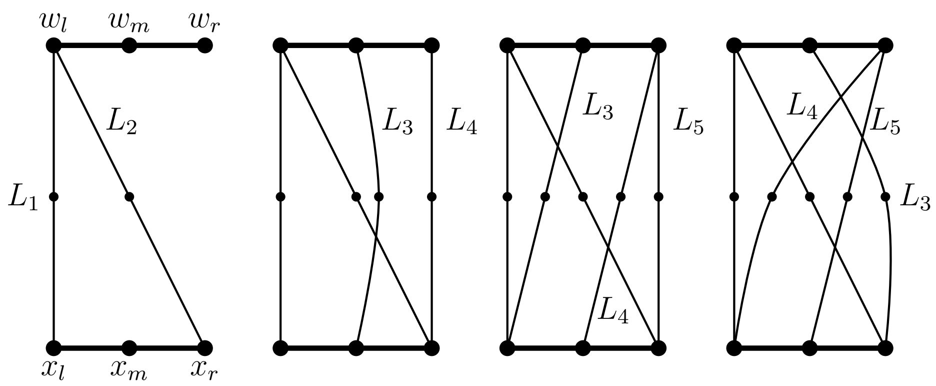


FIGURE 5.6. Graphs of Case (v) with toes of L_1 and L_2 at leaves of BT_2.

So, we assume that one of L_1 and L_2, say L_2, is incident to x_m. It follows from (3) that x_l is incident to a leg L_3 which is also incident to either w_m or w_r. In either case, w_r is incident to another leg, L_4, which is incident to x_r, and G dominates K^4. See Figure 5.7 for examples of the above graphs. This proves (5).

(6) $G \not\geq_m K_{2,5}$.

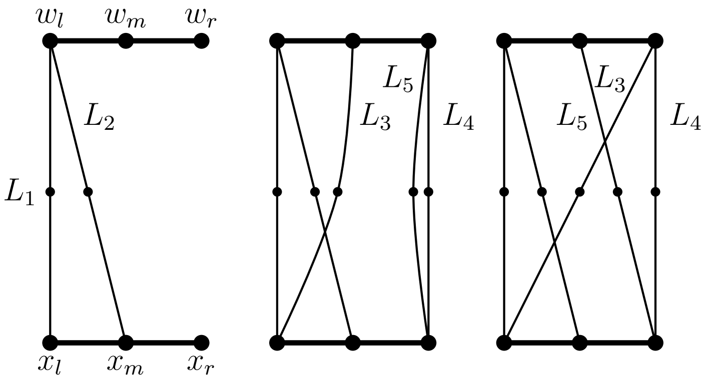


FIGURE 5.7. Graphs of Case (v) with the toe of L_2 on an internal branch vertex.

Suppose $G \geqslant_{\mathrm{m}} K_{2,5}$. Let F be a skeleton of $K_{2,5}$, and let BT_1 and BT_2 be the two branch trees. It follows from (3) that BT_1 and BT_2 each can be one of (a)–(c) from the proof of statement (5). We will examine each of the cases and show that each of these is a contradiction.

Case (i) Suppose both BT_1 and BT_2 are as described in (a). it follows that $G \succ K_{2,4}$; a contradiction showing that Case (i) cannot occur.

Case (ii) Suppose BT_1 is of the type (a) and BT_2 is of the type (b). It follows from (3) that BT_2 has two toes, x_l and x_r, which are incident to two and three legs, respectively. Let one of the legs incident to x_l be L_2. Since G is XNOP, it follows that if we delete an internal vertex from F of L_2, then the resulting graph F' must be NOP since G is XNOP. But, G and G' properly dominate F', which dominates $K_{2,4}$. So, in fact, Case (ii) cannot occur. See Figure 5.8.

Case (iii) Suppose BT_1 is of the type (a) and BT_2 is of the type (c). It follows that BT_2 has three toes, two of which are leaves which are incident to two legs and one internal vertex which is incident to one leg as depicted in Figure 5.9. Let the leg incident to the internal toe of BT_2 be L_3. Since G is XNOP, it follows that the graph obtained from F by suppressing an internal vertex of L_3 is NOP. But $F' \succ S_3$ as depicted in Figure 5.9.

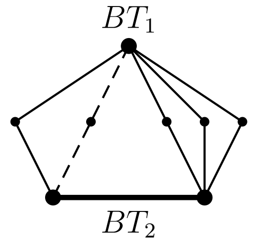


FIGURE 5.8. An example of F for Case (ii).

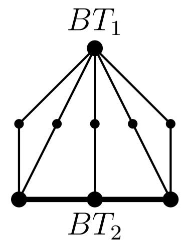


FIGURE 5.9. An example of F for Case (iii).

By symmetry, we have the same results if BT_1 and BT_2 were reversed in Cases (i)–(iii). Now we may assume that neither BT_1 nor BT_2 is trivial.

Case (iv) Suppose both BT_1 and BT_2 are as described in (b). Let w_l and w_r be the toes of BT_1, and let x_l and x_r be the toes of BT_2. It follows from (5) that at least three legs are incident to the same toes of each branch tree, w_l and x_l, and at least two other legs are incident to the two other toes, w_r and x_r. Let one of the legs incident to both w_r and x_r be L_2. Since G is XNOP, it follows that if we suppress an internal vertex of L_2 then the new graph G' is NOP. But F' dominates $K_{2,4}$ as depicted in Figure 5.10.

Case (v) Suppose one of the branch trees is of type (b) and the other is of type (c). It follows from (5) that this case does not arise.

Case (vi) Suppose both branch trees are as described in (c). It follows from (3) that the two leaves of each branch tree are incident to two legs each and that the internal toe from each branch tree, w_m and x_m, is incident to one leg, and from (5), the legs are incident to

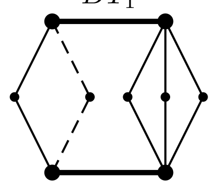


FIGURE 5.10. An example of F in Case (iv)

the same toes on each branch tree as shown in Figure 5.11. Let the leg which is incident to toes w_m and x_m be L_3. Since G is XNOP, it follows that if we suppress an internal vertex of L_3 in F, then the new graph G' is NOP. But F' dominates S_6. This proves (6). If n is the largest value for which $K_{2,n}$ is a minor of G, then n is 3 or 4.

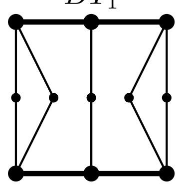


FIGURE 5.11. An example of F in Case (vi)

(7) *If* $G \geqslant_m K_{2,4}$*, then* G *is isomorphic to one of* $K_{2,4}$*,* S_1*,* S_2*,* S_3*,* S_4*,* S_5*, and* S_6*.*

Let F be a skeleton of $K_{2,4}$ with branch trees of BT_1 and BT_2. It follows from (4) that each of BT_1 and BT_2 is described as in (a) or (b), as listed in the proof of (5). This gives us three cases.

Case (i) Suppose both BT_1 and BT_2 are of type (a). Then $G \succ K_{2,4}$.

So at least one of BT_1 or BT_2 has two vertices. It follows from (1) and (2) above, that bridges are attached in one of the following three ways: from BT_1 to BT_2, on a single leg, or only on one branch tree.

Case (ii) Suppose BT_1 is of the type (a) and BT_2 is of the type (b). If F has a bridge from BT_1 to an internal vertex of BT_2, then G dominates S_3 as shown in Figure 5.12.

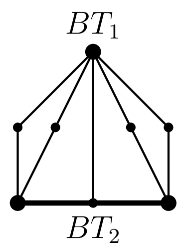


FIGURE 5.12. An example of F in Case (ii).

Now we consider the subgraph, Z, which consists of BT_2 and all bridges of F in G whose vertices of attachment lie only in BT_2. From (4) and (5), we know that the graph G is a union of three graphs, R_1, R_2, and Z, such that R_1 and R_2 have two legs of $K_{2,4}$ each and share BT_1.

Suppose there is an edge z in Z whose deletion separates the endpoints of BT_2 in Z into Z_1 and Z_2, where Z_i has a vertex in common with R_i for $i = 1, 2$. Consider $G\backslash z$. It is the union of two graphs R_1' and R_2', where R_i' is the union of R_i and Z_i. Since G is XNOP, $G\backslash z$ is not OP. Since $G \not\succeq W_3$, $G\backslash z$ dominates $K_{2,3}$, and a subdivision of $K_{2,3}$ is in either R_1' or R_2'. Let K be the subdivision of $K_{2,3}$. Without loss of generality, suppose K is in R_1'. Since G is 2-connected, Theorem 5.1 implies that there are two disjoint paths in G from R_2 to K. One of these paths is through BT_1 and the other through the endpoint of BT_2 shared by R_2. Consider the ends of these two paths in K. By (1), these endpoints must lie on the same leg of K. If the endpoints of the paths are the branch vertices of K, then $G \succ K_{2,4}$. So, at least one path's endpoint must be an internal vertex of a leg of K. Since G is 2-connected, the endpoints of both paths cannot be at the same vertex, or else G has a cut-vertex. If both of the endpoints of the paths are two different internal vertices of a leg of K, then $G \succ S_6$. We assume that one endpoint must be a branch vertex and the other an internal vertex. But $G \succ S_5$. See Figure 5.13. Hence, $G\backslash z$ does not dominate $K_{2,3}$.

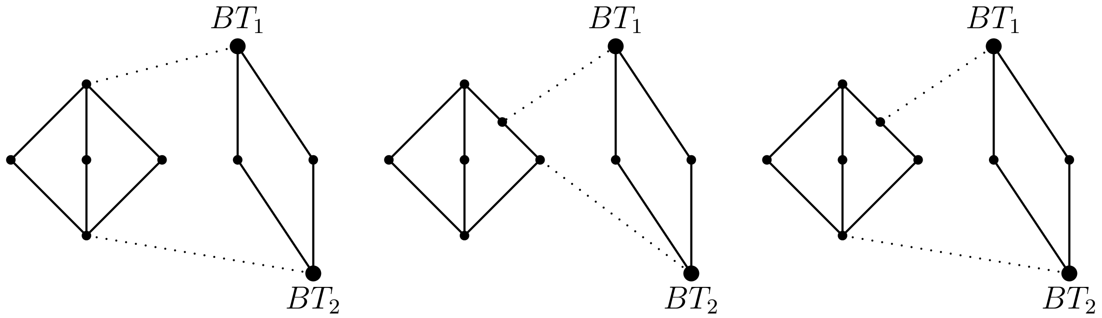


FIGURE 5.13. Examples of Case (ii) with $G\backslash z$

It suffices to consider the case where no edge separates Z. So, there are two distinct paths, P_1 and P_2, in Z, which do not share edges, from one endpoint of BT_1, x_l, to the other endpoint, x_r. It follows that $G \succeq S_1$. Hence, neither BT_1 or BT_2 is trivial.

Case (iii) Suppose both BT_1 and BT_2 are both of type (b). Bridges are attached in one of the following three ways: from BT_1 to BT_2, on a single leg, or on a single branch tree. If a bridge B has vertices of attachment on both BT_1 and BT_2, then the vertices of attachment can either be internal vertices or endpoints of the branch trees. We can eliminate some graphs with certain types of bridges. If both of the vertices of attachment of B are each internal vertices of both branch trees, then G dominates S_6. If one of the vertices of attachment of B is an endpoint of a branch tree and the other is an internal vertex of the other branch tree, then G dominates S_5. If both vertices of attachment are endpoints of each branch tree, but do not share any adjacent legs, then G dominates S_4. See Figure 5.14.

So, a bridge B is attached in one of three ways: to a single leg or to a single branch tree. Let A be the union of BT_1 and all bridges of F in G which lie only on BT_1 and Z be the subgraph BT_2 and all bridges which lie only on BT_2. Then G is the union of A, Z, R_1 and R_2, where R_1 and R_2 have two legs of $K_{2,4}$ each. Suppose there is an edge z in Z whose deletion

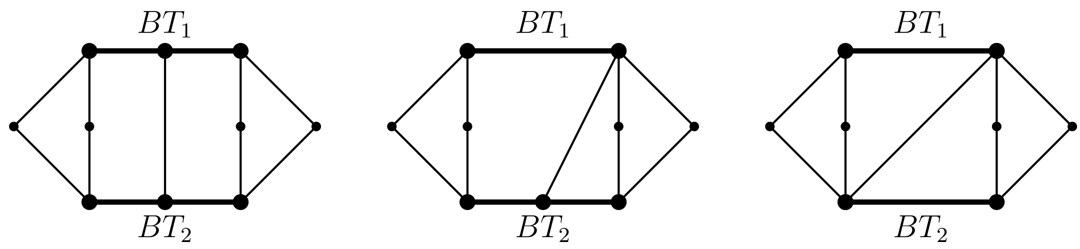


FIGURE 5.14. Case (*iii*) graphs that dominate S_4, S_5, or S_6

separates the endpoints of BT_2 into two subgraphs Z_1 and Z_2 such that Z_i has a vertex in common with R_i for $i = 1, 2$. Consider $G\backslash z$. It is the union of three graphs A, R_1' and R_2', where R_i' is the union of R_i and Z_i. Since G is XNOP, $G\backslash z$ is not OP and dominates $K_{2,3}$. So, a subdivision of $K_{2,3}$ is in either R_1 or R_2. Let K be the subdivision of $K_{2,3}$. Without loss of generality, suppose K is in R_1. Since G is 2-connected, Theorem 5.1 implies that there are two disjoint paths from R_2 to K. One of these paths is through BT_1 and the other through BT_2. Consider the ends of these two paths on K. By (1), these endpoints must lie on the same leg of K. If the endpoints of the paths are the branch vertices of K, then $G \succ K_{2,4}$. So, at least one path's endpoint must be an internal vertex of a leg of K. If both of the endpoints of the paths are internal vertices of a leg of K, then $G \succ S_6$. So, one endpoint must be a branch vertex and the other an internal vertex. But $G \succ S_5$. See Figure 5.15. Hence, $G\backslash z$ does not dominate $K_{2,3}$ and because of symmetry, $G\backslash a$ does not dominate $K_{2,3}$, where a is an edge whose deletion separates the endpoints of BT_1.

We may assume that no single edge separates A and no single edge separates Z. So, each branch tree has two distinct paths from one of its endpoints to the other, and $G \succeq S_2$. This exhausts the cases of $G \geqslant_{\mathrm{m}} K_{2,4}$ and proves (7).

We assume that $G \not\geqslant K_{2,4}$. By (3) and (4) above, both branch trees of G are trivial, and there is a skeleton F which is a subdivision of $K_{2,3}$. Let the legs of the subdivision of $K_{2,3}$ be L_1, L_2, and L_3. By (1) and (2), every bridge of F has 2 vertices of attachment on a single

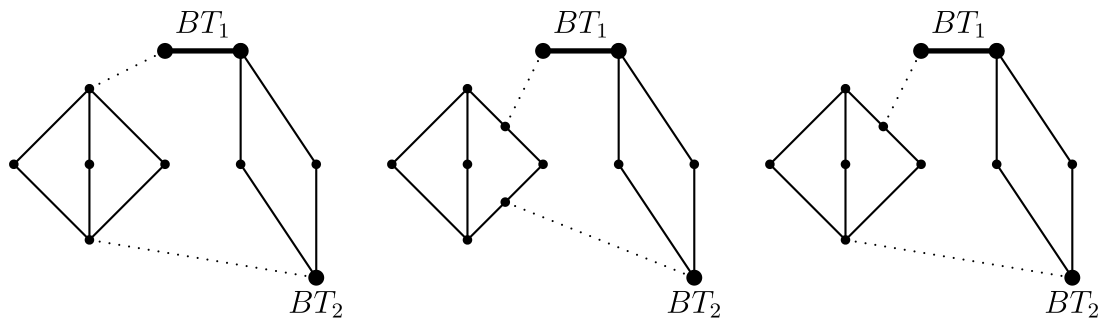


FIGURE 5.15. Graphs of Case (iii) with $G\backslash j_2$.

leg of $K_{2,3}$. So, G is the union of 3 graphs, M_1, M_2, and M_3, such that M_i is L_i along with any of its bridges. Any two of the subgraphs M_1, M_2, and M_3 meet only at BT_1 and BT_2.

Suppose there is an edge m of M_1 such that $M_1\backslash m$ separates BT_1 from BT_2 in M_1. Consider $G\backslash m$. Since G is XNOP, $G\backslash m$ is not OP and dominates $K_{2,3}$. So a subdivision of $K_{2,3}$ is in the union of M_2 and M_3. Let K be the subdivision of $K_{2,3}$. By Menger's, Theorem 5.1, there are 2 disjoint paths to K through BT_1 and BT_2. So, K is either entirely in M_2, entirely in M_3, or meets edges in both. But, if K is entirely in M_2 or M_3, then $G \geqslant_{\mathrm{m}} K_{2,5}$. So, K has at least one leg each in M_2 and M_3. The third leg of K cannot be in M_1, so it must be a bridge of M_2 or M_3. But, then $G \geqslant_{\mathrm{m}} K_{2,4}$.

We may assume that no single edge separates BT_1 from BT_2 in M_1. Similarly, no single edge separates BT_1 from BT_2 in M_2 or M_3. So, there are two paths from BT_1 to BT_2 in each of M_1, M_2, and M_3. Each of these paths must share a vertex in each of M_1, M_2, and M_3 since $G \not\geqslant K_{2,4}$. But, then $G \succ DE_1$. □

Chapter 6
Graphs that Dominate W_5

Theorem 6.1. *No XNOP graph dominates* W_5*.*

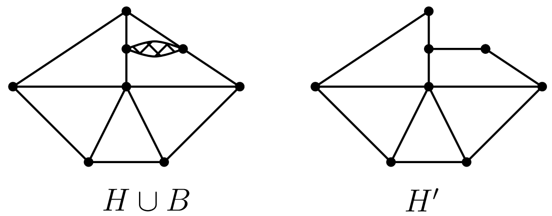
Proof. To make the following proof more understandable, we will break it into 7 smaller parts, numbered (1)–(7). If G is XNOP and nonplanar, then Theorem 2.1 implies that G is isomorphic to $K_{3,3}$, and $G \not\succ W_5$. Hence we assume G is a plane graph and that is G is XNOP. The graph H is a subgraph of G that is a subdivision of W_n, for $n \geq 5$, such that G contains no subdivision of W_{n+1}. The spokes and the rim edges of W_n correspond to *spoke paths* or *rim paths* in H. The hub of H corresponds to the hub of W_n. A vertex of H that corresponds to a vertex on W_n and is incident to both the rim and a spoke is called a *corner*. The union of all rim paths of H is the *rim*. It is easy to see that, without loss of generality, we may assume the hub of H lies in the finite region R of the plane that is homeomorphic to a disk and bounded by the rim. Moreover, we chose H so that no other subdivision of W_n is contained in R. The plane embedding induces a cyclic order on the spoke paths. Two spoke paths are *consecutive* if they are adjacent in that cyclic order.

(1) *If a bridge B of H in G has one vertex of attachment that is an internal vertex of a spoke path of H, then all other vertices of attachment of B are on the same spoke path, possibly including the hub and the corner.*

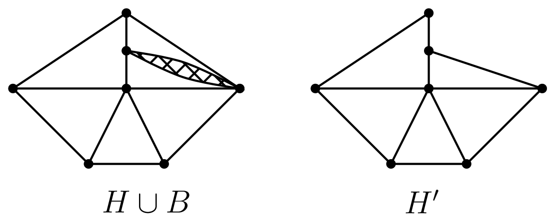
Suppose not. If B has vertices of attachment one of which is a vertex of spoke path and the other of which is an internal vertex of a rim path, then this contradicts the choice of H. See Figure 6.1 (a).

If B has vertices of attachment one of which is an internal vertex of a spoke path and the other of which is a non-adjacent corner vertex on an adjacent rim path, then H is not a minimal subdivision of W_n. See Figure 6.1 (b).

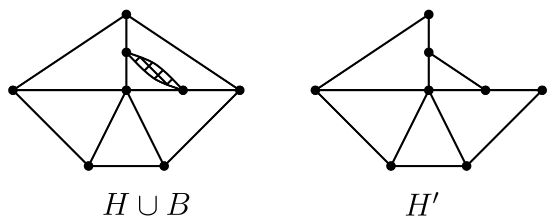
Similarly, if B has vertices of attachment of internal vertices on two distinct spoke paths, then H is not a minimal subdivision of W_5. See Figure 6.1 (c) for an example. So, B has vertices of attachment of a internal vertex on the spoke path, a corner on the spoke path, or the hub. This proves (1).



(a) B is attached from spoke path to rim path



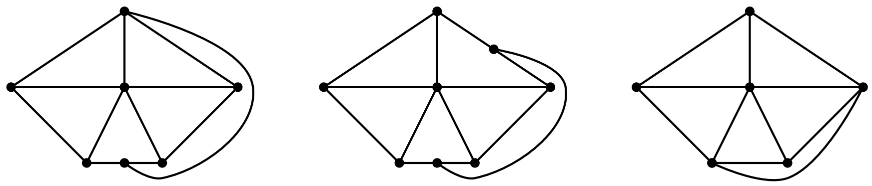
(b) Bridge from spoke path to non-adjacent corner



(c) Bridge from spoke path to spoke path

FIGURE 6.1. $H \cup B$ contains H', which contradicts the choice of H.

The *span* of a path, P, which is a subgraph of the rim, between two vertices, u and v, is a pair of numbers of which the first is the number of corners other than u and v in P and the second is the number of the vertices, u and v, that are corners. Spans are ordered lexicographically. Since the rim of H is a cycle, it contains two independent u–v paths. The



(a) Span of $(2,1)$ (b) Span of $(2,0)$ (c) Span of $(1,2)$

FIGURE 6.2. $H \cup B$ with spans of B greater than or equal to $(1,2)$.

span of two vertices of the rim is the minimum span of the two independent u–v paths contained in the rim. A bridge all of whose vertices of attachment are on the rim of H is called an *outer bridge*, and an *inner bridge*, otherwise. The span of an outer bridge of the rim is the maximum span of all of the pairs of vertices of attachment. We prove the following.

(2) *An outer bridge B of H in G has span less than* $(1,2)$.

Let k be the number of corners of H. The outer bridge B, with span (m,n), where $0 \leq m \leq k$ and $0 \leq n \leq k$, has corresponding vertices of attachment u and v. There are two disjoint paths on the rim of H from u to v. One path has m corners and the other path has m' corners, where $m \leq m' \leq k-m$. Clearly $n \leq 2$. If B has a span of $(q,1)$ or $(q,0)$, where $q \geq 2$, then $G \succeq WF_6$. See Figure 6.2 (a) and (b). If B has a bridge span of $(p,2)$, where $p \geq 1$, then $H \succeq K_5 \backslash e$. See Figure 6.2 ($c$). So, the span of B is smaller than $(1,2)$. This proves (2).

It follows from (2) that all vertices of attachment of a bridge B lie on a path whose span is less than $(1,2)$. The minimal such path is the *base* of the bridge. Without loss of generality, we assume that the embedding of G on the plane is such a way that the vertices of the rim that do not lie on the base of B are on the boundary of the outer face of $H \cup B$. A bridge *spans a corner* if the corner is a vertex of the base. A bridge *spans an edge* is it is an edge of the base. A bridge is *weak* if it consists of a single edge, and *strong* otherwise.

Let c_i for $i = 1, 2, \ldots, n$ be the corners of W_n of H. We can prove the following.

(3) *There are three consecutive corners c_1, c_2, and c_3 of H that satisfy the following conditions:*

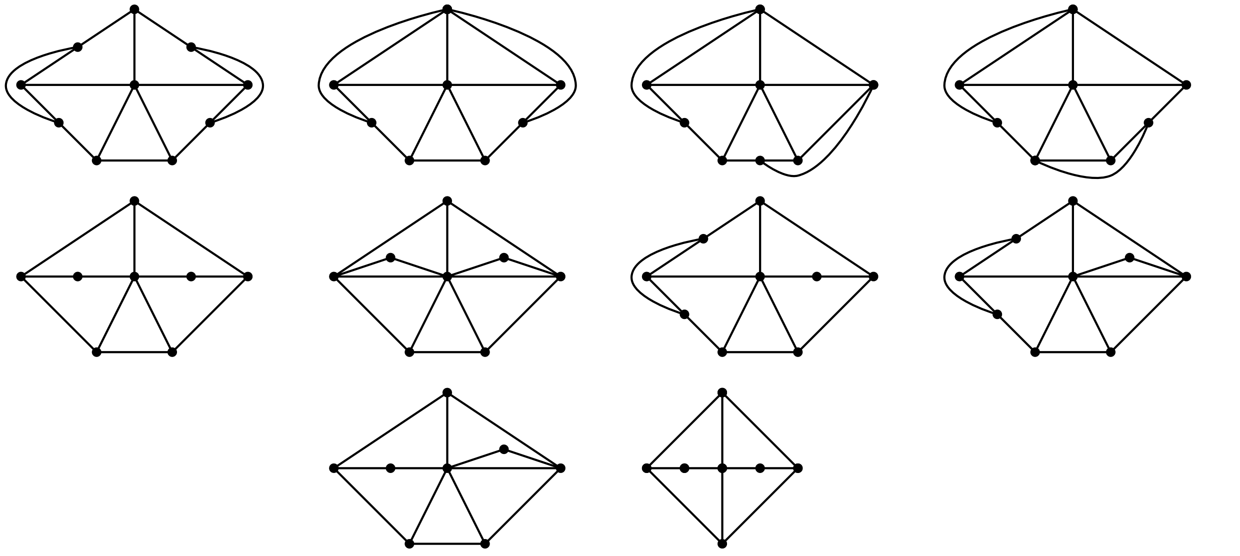


FIGURE 6.3. Graphs of Statement (3) that dominate WF_1.

(*a*) *none of* c_1*,* c_2*, or* c_3 *is spanned by an outer bridge of* H*;*

(*b*) *each of* c_1*,* c_2*, and* c_3 *are adjacent to the hub; and*

(*c*) *none of* c_1*,* c_2*, or* c_3 *is a vertex of attachment of a strong inner bridge of* H*.*

If two or more non-adjacent corners each violate one of the conditions, then clearly $G \succ WF_1$. If exactly one corner or two adjacent corners violate any of these conditions, then there are three remaining consecutive corners that do not violate the conditions. See Figure 6.3. This proves (3).

So, H has 3 consecutive corners which lie on the boundary of the outer face whose corresponding spoke paths of length 1 are SP_1, SP_2, SP_3. Of the other two corners, c_4 and c_5, only one can be spanned. So, one of c_4 or c_5 must be on the boundary of the outer face. The corresponding spoke paths may be subdivided or not.

Since G is XNOP, every graph that G properly dominates is NOP. Since SP_2 is a single edge, it follows that $G \backslash SP_2$ is NOP. So there is an edge f such that $G \backslash SP_2 \backslash f$ is OP; a contradiction.

(4) *The edge* f *is on the rim of* H*.*

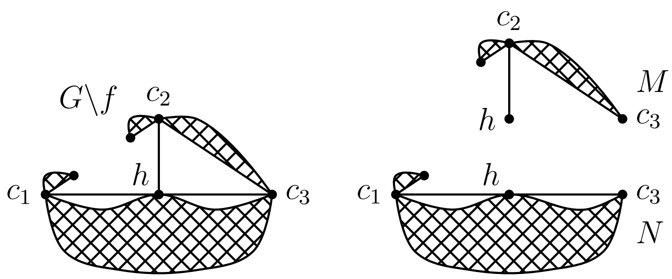


FIGURE 6.4. A representation of $G\backslash f$ and the subgraphs M and N of $G\backslash f$.

If f is not an edge of the rim, then $G\backslash SP_2\backslash f$ has at least 3 spoke paths, which, together with the rim of H, form K^4; a contradiction. This proves (4).

(5) *No bridge spans f.*

Suppose a bridge spans f. By (2), the bridge must have vertices of attachment on the same rim path or on two adjacent rim paths. It follows that $G\backslash f \succ W_4$. But, $G\backslash f\backslash SP_2$ is not OP since $G\backslash f\backslash SP_2 \succ K^4$; a contradiction. This proves (5).

We now focus on the rim paths of H. Let $P_{i,i+1}$ be the rim path between c_i and c_{i+1} for $i = 1, 2, \ldots 4$ and let $P_{5,1}$ be the rim path between c_5 and c_1.

(6) *The edge f is not an edge of $P_{1,2} \cup P_{2,3}$.*

Suppose f is an edge of $P_{1,2}$. Consider $G\backslash f$. By (3) and (5), $G\backslash f$ is a graph with the hub h and three corners, c_1, c_2, and c_3, on the boundary of the outer face. The graph $G\backslash f$ has two subgraphs, M and N, each of which is connected, and such that $M \cap N = \{h, c_3\}$, $M \cup N = G\backslash f$, $c_2 \in V(M)$, and all edges between h and c_3 lie in N. See Figure 6.4. Since G is XNOP, it follows that $G\backslash f$ is not OP and so contains K, a subdivision of $K_{2,3}$ or K^4. Since $G\backslash SP_2\backslash f$ is OP, the edge SP_2 must be an edge of K.

We will now examine the location of the branch vertices of K in relation to M and N. Suppose two of the branch vertices of K are h and c_3. Observe that $K_{2,3}$ with a path between two internal vertices of two different legs of $K_{2,3}$ is a subdivision of K^4. By the structure of W_n and Lemma 1.10, there are two independent paths in N from c_3 to h. By (3), SP_3 cannot have a strong bridge with vertices of attachment at h and c_3. See Figure 6.5 (a). So,

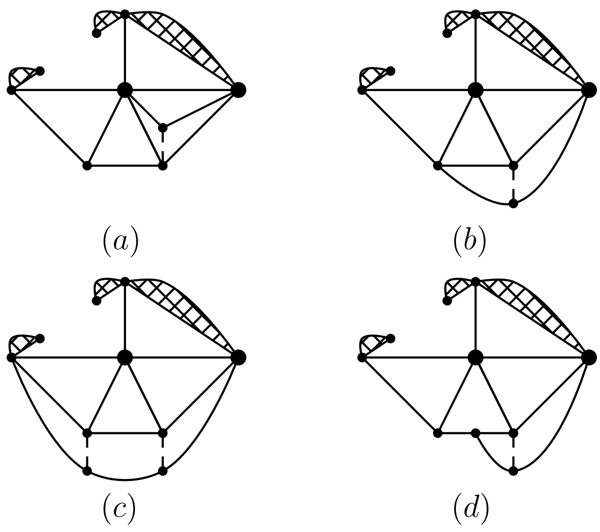


FIGURE 6.5. The subgraph K of $G\backslash f$ cannot have branch vertices at h and c_3.

at least one leg of K contains an edge in an outer bridge that spans c_4. But, then $N \succ K^4$; a contradiction. See Figure 6.5 (b)–(d). Thus, we assume that not both h and c_3 are branch vertices.

Suppose at least one branch vertex of K is in $N - \{c_3, h\}$. It follows from Lemma 1.10 that upon replacing the path of K between c_3 and h containing SP_2 with SP_3, the subgraph N contains another subdivision of $K_{2,3}$ or K^4, which does not involve SP_2 and $G\backslash f\backslash SP_2$ is not OP; a contradiction.

Hence, $M - \{h, c_3\}$ contains at least one of the branch vertices of K. Let M' be the subgraph of M composed of $P_{2,3}$ together with the bridges of H in G that attach to $P_{2,3}$, and let $K \cap M' = K^-$. It follows that M contains two of the legs of K. So, K^- is a subdivision of $K_{2,3}$ minus a leg, or a subdivision of K^4 minus an edge. See Figure 6.6.

Suppose M' contains an edge g that separates c_2 from c_3 in M'. The graph $G\backslash g$ is not OP, so it contains K'', a subdivison of $K_{2,3}$ or K^4, such that K'' meets K^- only possibly at c_2 or c_3. Since G is 2-connected, it contains two disjoint paths, one of which may be edgeless and the other of which contains g, from K^- to K''. It follows that G dominates one of $K_{2,4}$, S_3, S_4, S_5, S_6, WT_{10}, WT_{11}, WT_{12}, and WT_{13}; a contradiction. See Figure 6.7.

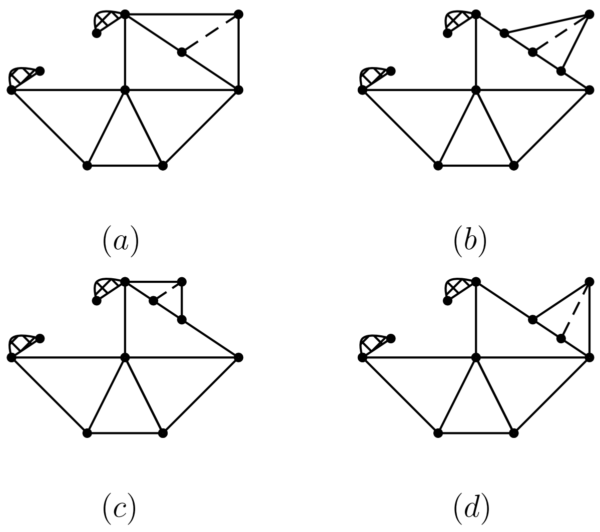


FIGURE 6.6. The subgraph K^- of K .

Hence, we now assume that no single edge of M' separates c_2 from c_3 in M'. But, then G dominates $K_{2,4}$, S_1, or S_2; a contradiction. This shows that f cannot lie on $P_{1,2}$. By symmetry, f cannot lie on $P_{2,3}$ either. This proves (6).

We now consider the case where f is an edge of one of $P_{3,4}$, $P_{4,5}$, and $P_{5,1}$. By (1) and (6), no bridge of H has vertices of attachment that are internal vertices of two spoke paths or internal vertices of rims, and no bridge spans f.

Since G is XNOP, it follows that $G\backslash f$ is not OP and so contains K, a subdivision of $K_{2,3}$ or K^4. Since $G\backslash SP_2\backslash f$ is OP, the edge SP_2 must be an edge of K. Let $P_{3,1}$ be the union of $P_{3,4}$, $P_{4,5}$, and $P_{5,1}$. Then, f separates the endpoints of $P_{3,1}$ in $P_{3,1}$ and therefore the hub h is incident to the boundary of the outer face of $G\backslash f$.

The graph $G\backslash f$ has two subgraphs, M and N, each of which is connected, and such that $M \cap N = \{h, c_2\}$, $M \cup N = G\backslash f$, $c_1 \in V(M)$, $c_3 \in V(N)$, and all edges between h and c_2 lie in N. See Figure 6.8. So, by Lemma 1.10 K must lie either entirely in N or entirely in $M \cup SP_2$. First we assume that K is in N. But, then $G\backslash SP_2$ is not OP, since the union of $K\backslash SP_2$, SP_1, and $P_{1,2}$ is also a subdivision of $K_{2,3}$ or K^4; a contradiction. now, if K is in $M \cup SP_2$, then $G\backslash SP_2$ is not OP since the union of $K\backslash SP_2$, SP_3, and $P_{2,3}$ is also a subdivision

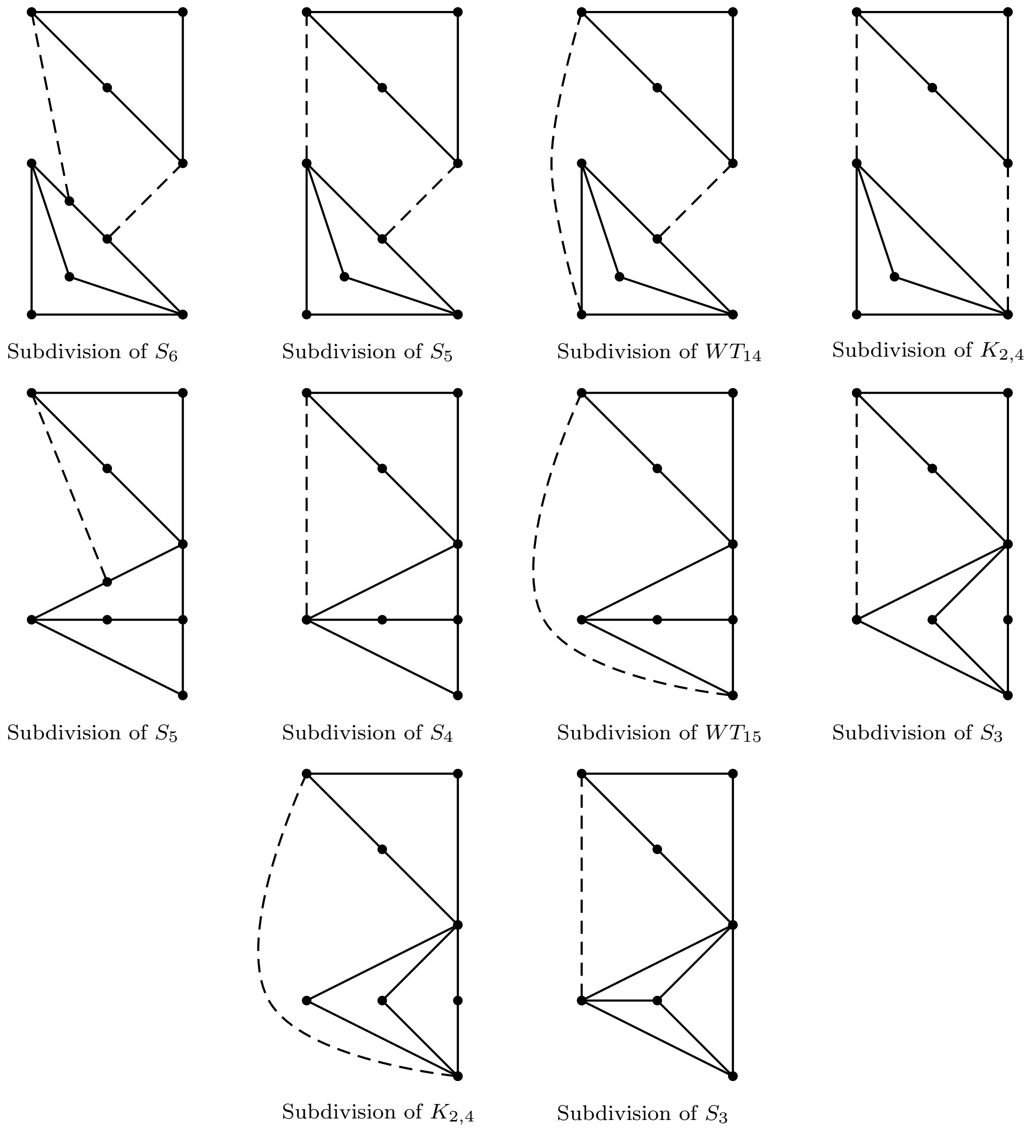


FIGURE 6.7. No single edge in M' separates c_2 from c_3 in H.

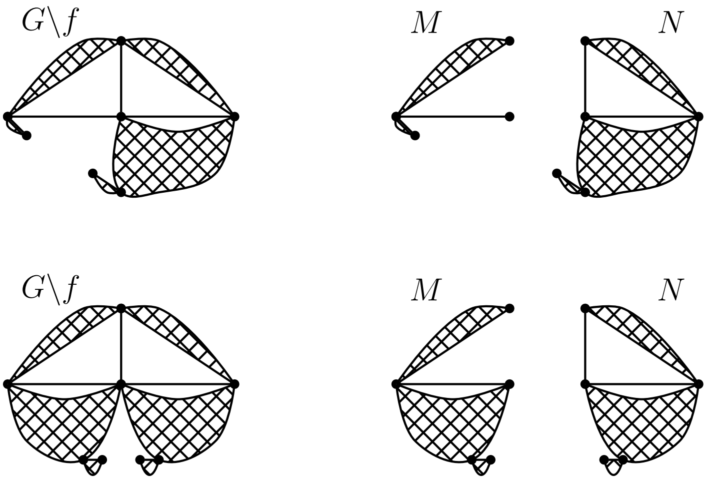


FIGURE 6.8. Graphs of $G\backslash f$, where f is an edge of $P_{3,1}$, and the subgraphs M and N.

of $K_{2,3}$ or K^4; a contradiction. Therefore, f is not an edge of the rim contradicting (4) and so $G \not\succcurlyeq W_5$. □

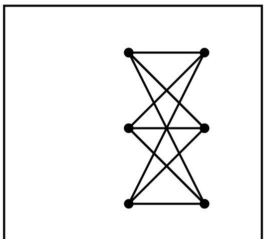
References

[1] R. Diestel, *Graph Theory, 3rd Edition*, Graduate Texts in Mathematics, Springer-Verlag, Heidelberg 2006.

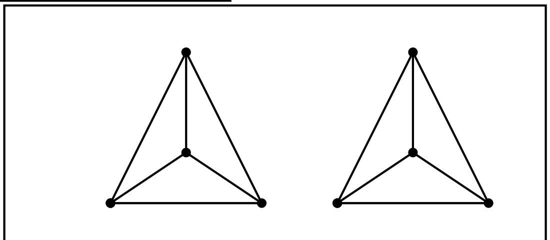
[2] D. West, *Introduction to Graph Theory, 1st Edition*, Prentice-Hall, Inc., New Jersey, 1996.

[3] N. Robertson, P. D. Seymour, *Graph Minors. XX. Wagner's Conjecture*, J. Combin. Theory, Ser. B **92** (2004): 325-357.

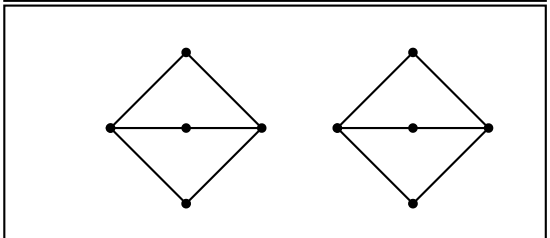
Appendix A: List of XNOP Graphs



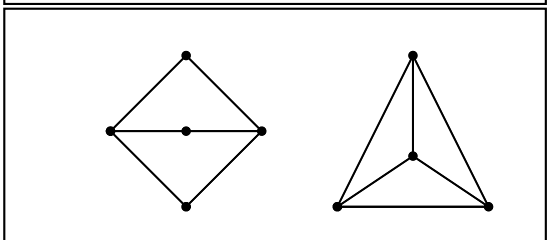
$K_{3,3}$



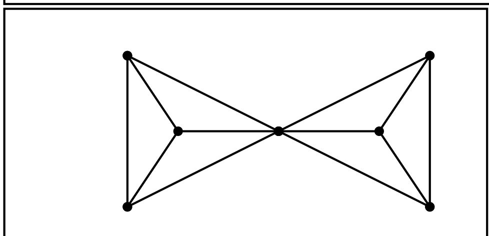
D_1



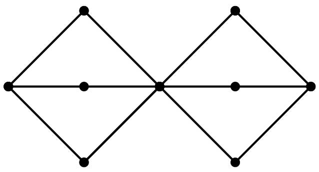
D_2



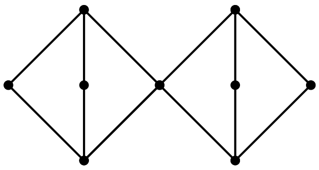
D_3



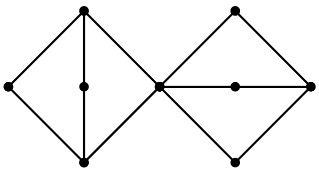
CV_1



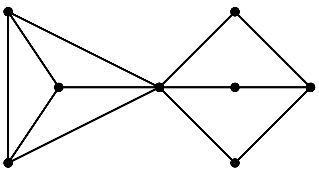
CV_2



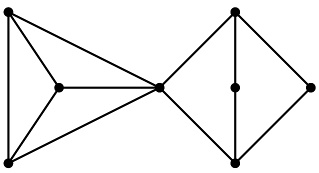
CV_3



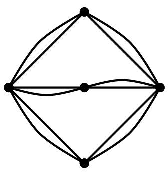
CV_4



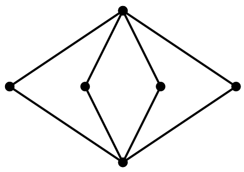
CV_5



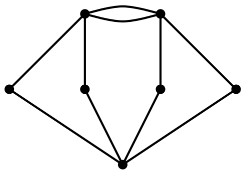
CV_6



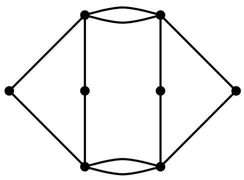
DE_1



$K_{2,4}$



S_1



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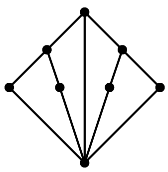

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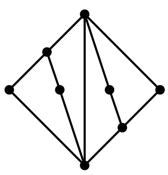

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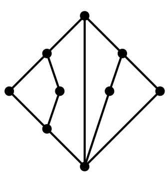

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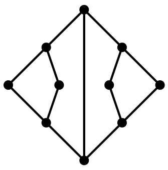

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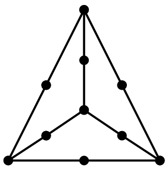
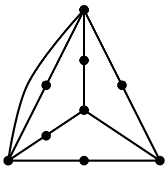


WT_1



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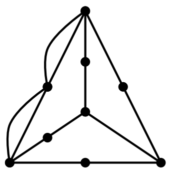

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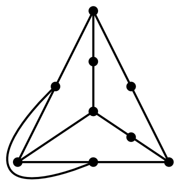

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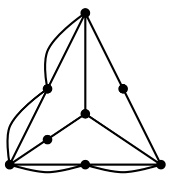
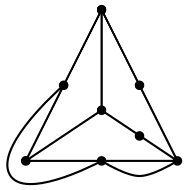


WT_5



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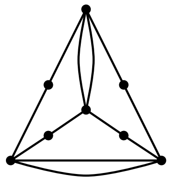

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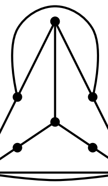

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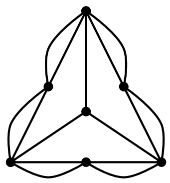

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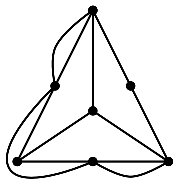
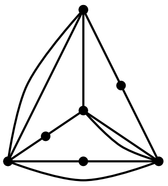
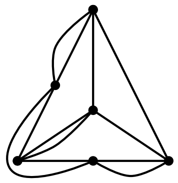


WT_{10}



WT_{11}



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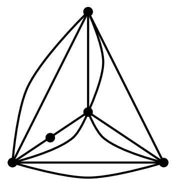

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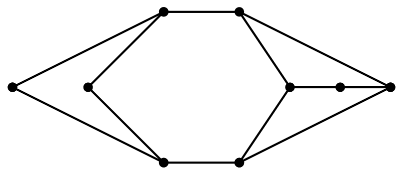

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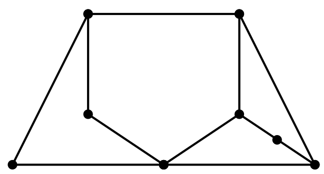

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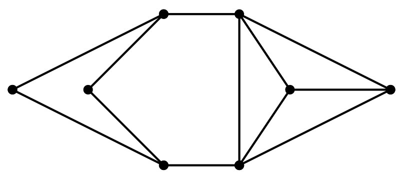

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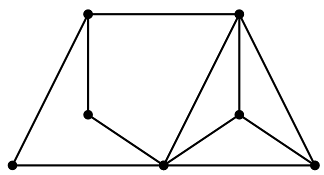

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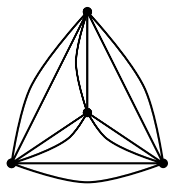

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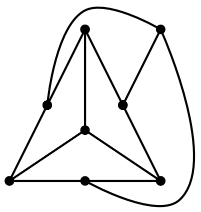

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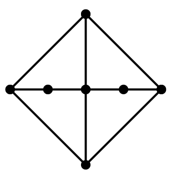

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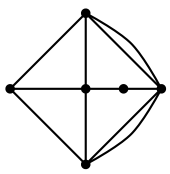

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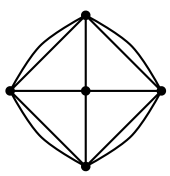

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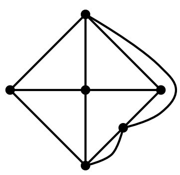

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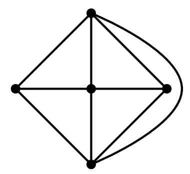
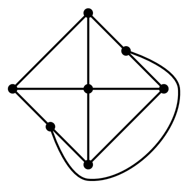


$K_5 \backslash e$



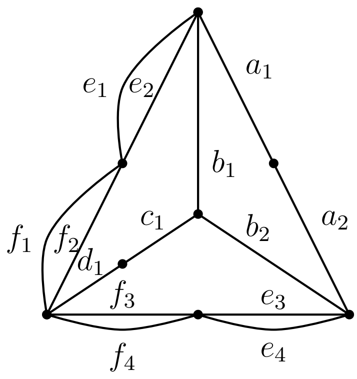
WF_6

Appendix B: Verification of WT_4

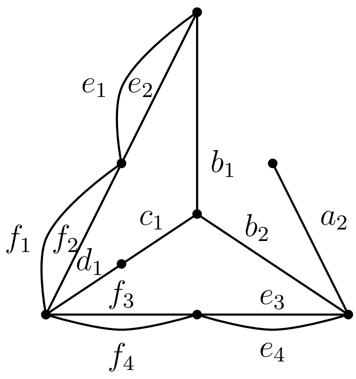
To verify that a graph is XNOP, we must first check that the graph is not NOP and then check that every graph that G properly dominates is NOP. The list of graphs is long, so for brevity, we only show one example, WT_4, here. The verification of the others is left to the curious reader.

We begin by looking at the graphs that result when one edge is removed and verifying that the resulting graph is not OP. Since WT_4 has many symmetries, we will look at the edge deletions only for edges which are in different orbits of the automorphism. The edge orbits as determined by action of the automorphism group on WT_4 are indicated by different letters, while indices just enumerate the element of each orbit as in Figure 6.9 (a). We want to look at $WT_4 \backslash k$ where k is one of the edges in each of the automorphic groups. If k is a_1 or a_2, then $WT_4 \backslash a_1 = A'$ has a subgraph of $K_{2,3}$ (see edges d_2, e_2, e_3, d_3, b_2, b_1, c_1, and c_2). If b_1 is removed, then $WT_4 \backslash b_1 = B'$ has a subgraph of $K_{2,3}$ using the edges d_2, e_2, e_3, d_3, a_1, a_2, b_2, c_1, and c_2. If c_1 is removed, then $WT_4 \backslash c_1 = C'$ has a subgraph of $K_{2,3}$ using the edges d_2, e_2, e_3, d_3, a_1, a_2, b_1, and b_2. When d_1 or e_1 is removed, then $WT_4 \backslash d_1 = D'$ and $WT_4 \backslash e_1 = E'$ each have a subgraph of K_4 using the edges d_2, e_2, e_3, d_3, a_2, a_1, b_1, b_2, c_1, and c_2. Hence, if any edge of WT_4 is deleted, $WT_4 \backslash k$ is NOP.

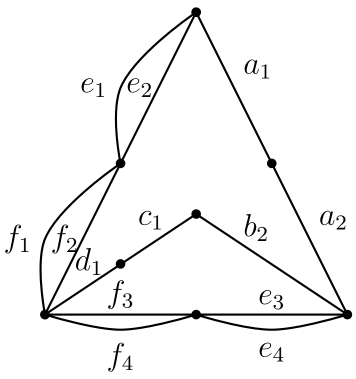
Now, we want to confirm that WT_4 is minimal XNOP by domination, that is we want to show that all graphs that are properly dominated by WT_4 are NOP. The graphs that WT_4 properly dominates are found by edge deletions, vertex deletions, and vertex suppressions. We begin with edge deletions. If k is a_1 or a_2, then the subsequent removal of b_1, b_2, c_1, or c_2 gives a graph that is OP. If b_1 is removed, then the subsequent removal of a_1, a_2, b_2, c_1, or c_2 gives a graph that is OP. If c_1 is removed, then the subsequent removal of a_1, a_2, b_1, or b_2 gives a graph that is OP. When d_1 or e_1 is removed, the subsequent removal of a_1, a_2, b_1, b_2, c_1, c_2, d_2 (in D'), or e_2 (in E') gives a graph that is OP. Hence, if any edge of WT_4 is deleted, $WT_4 \backslash k$ is NOP. All vertices in WT_4 are of degree 2 or greater, so any vertex deletion also means at least two edge deletions. Since one edge deletion results in graphs that are NOP, a vertex deletion with at least two edge deletions must also be NOP. So, we should look at vertex suppression. There are four vertices in WT_4 with exactly 2 neighbors. Label the vertex whose only incident edges are a_1 and a_2 as v_a. The vertex v_{cd} is the one incident to only c_1 and d_1. The vertex with only e_1, e_2, f_1, and f_2 as incident edges is v_{ef1}. Similarly, v_{ef2} has e_3, e_4, f_3, and f_4 as its only incident edges and is in the same orbit as v_{ef1}. We will only consider v_{ef1}. The graph WT_4 with the suppressible vertices labeled is shown in Figure 6.10 (a) and the results of suppressing these vertices are shown in Figure 6.10 (b)-(d). The graph $WT_4 \cdot v_a$ is NOP since the deletion of c_1 or d_1 results in a graph that is OP. Similarly, the deletion of a_1 or a_2 from $WT_4 \cdot v_{cd}$ and the deletion of b_2 from $WT_4 \cdot v_{ef1}$ result in graphs that are OP. So all graphs that are properly dominated by WT_4 are NOP and WT_4 is minimal XNOP by domination.



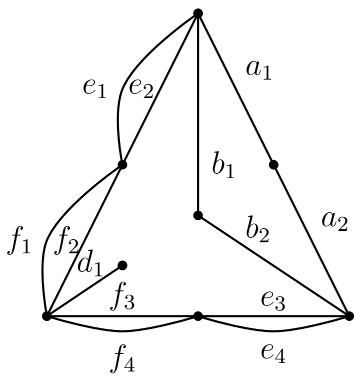
(a) WT_4



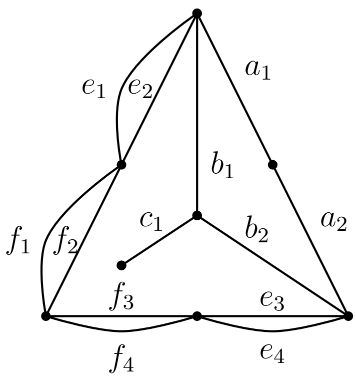
(b) $WT_4 \backslash a_1$



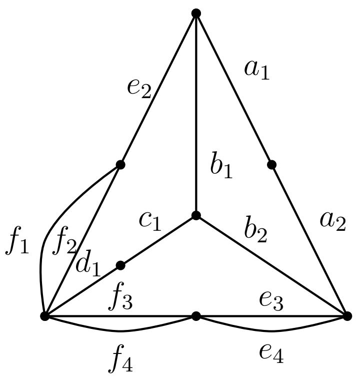
(c) $WT_4 \backslash b_1$



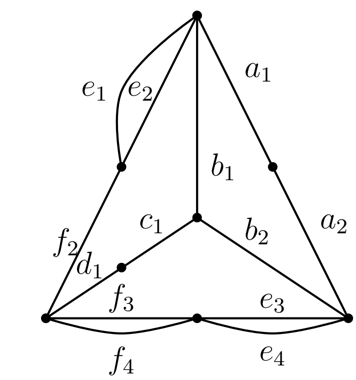
(d) $WT_4 \backslash c_1$



(e) $WT_4 \backslash d_1$

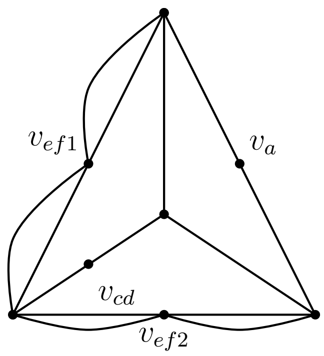


(f) $WT_4 \backslash e_1$

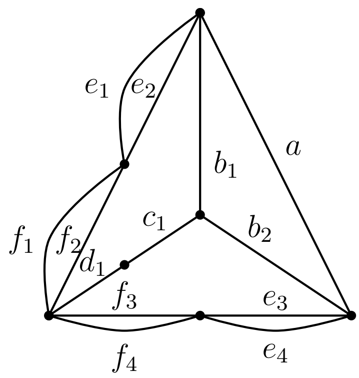


(g) $WT_4 \backslash f_1$

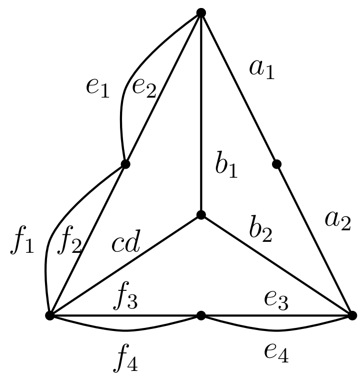
FIGURE 6.9. WT_4 and WT_4 with one edge removed.



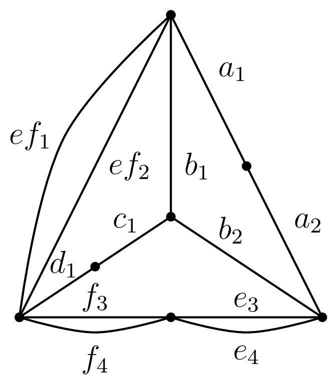
(a) Suppressible vertices of WT_4



(b) $WT_4 \cdot v_a$



(c) $WT_4 \cdot v_{cd}$



(d) $WT_4 \cdot v_{ef1}$

FIGURE 6.10. Suppression of vertices of WT_4

Vita

Tanya Allen Lueder genannt Luehr was born on August 31, 1974, in Lafayette, Louisiana. She finished her undergraduate studies in chemical engineering at Louisiana State University in December 1996. She worked for six years as an engineer for International Paper Company. In August 2005 she began work at Louisiana State University in mathematics. She is currently a candidate for the degree of Master of Science in mathematics, which will be awarded in May 2010.