Relations Invariant Under Semigroup Actions and Bounded Baer-Levi Semigroups.

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RELATIONS INVARIANT UNDER SEMIGROUP ACTIONS
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ABSTRACT

Let $X$ be a set, $S$ a semigroup and $X \times S \rightarrow X$ an action denoted by $(x,s) \rightarrow xs$. We frequently assume that this action is **effective**, i.e., that if $s,t \in S$, $s \neq t$, then there is $x \in X$ so that $xs \neq xt$. We further assume that if $S$ has an identity $1$, then $xl = x$ for each $x \in X$. A non-empty subset $A \subseteq X \times X$ is called an **invariant relation** on $X$ under $S$ if $AS \subseteq A$. We say $A$ is a **minimal invariant relation** if no proper subset of $A$ is invariant under $S$. MIR is the union of the minimal invariant relations.

Elementary properties of minimal invariant relations are given and MIR is described for semigroups with a minimal right ideal. It is shown that if $S$ acts effectively on $X$ and $S$ has a minimal left ideal, then $\text{MIR} = X \times X$ if and only if $S$ is a group. In the case where $S$ acts on itself by right multiplication, $\text{MIR} = S \times S$ and $E(S) \neq \emptyset$, then $S$ is a group; if $E(S) = \emptyset$, then $S$ is a subsemigroup of a Baer-Levi semigroup. This latter
class of semigroups is studied and called MIR semigroups.

We investigate a class of subsemigroups of Baer-Levi which are universal in the embedding sense for MIR semigroups. To describe the work, let \( X \) be a set of cardinality \( X_0 \) and let \( BL(X) \) be the Baer-Levi semigroup of one-to-one functions \( f \) from \( X \) to \( X \) so that \( X\setminus X_f \) is infinite. Let \( \mathcal{A} = \{F_a \mid a \in \Lambda\} \) be a collection of infinite subsets of \( X \) satisfying the condition that for each \( \alpha, \beta \in \Lambda \) there is \( \eta \in \Lambda \) so that \( F_\alpha \cup F_\beta \subseteq F_\eta \) and \( F_\eta \setminus (F_\alpha \cup F_\beta) \) is infinite. Let \( BBL(X) \) be the subsemigroup of \( BL(X) \) defined by \( \phi \in BBL(X) \) if there exists \( \alpha \in \Lambda \) such that \( (X)\phi \subseteq F_\alpha \). The semigroups \( BBL(X) \) are idempotent-free, right simple, left reductive, and have the property that any two elements have a common right identity. Since this implies that \( MIR = S \times S \), these semigroups are basic examples of the class of semigroups encountered above. Numerous congruences on \( BBL(X) \) are exhibited and part of the structure of the lattice of congruences on \( BBL(X) \) is given. Basically, there is a largest proper congruence \( \gamma \) and a smallest proper congruence \( \delta \) with an uncountable number of congruences between \( \delta \) and \( \gamma \).
CHAPTER I
INTRODUCTION

Let $X$ be a set, $S$ a semigroup and $X \times S \rightarrow X$ an action denoted by $(x,s) \rightarrow xs$. We frequently assume that this action is effective, i.e., that if $s,t \in S$, $s \neq t$, then there is $x \in X$ so that $xs \neq xt$. We further assume that if $S$ has an identity $1$ then $x1 = x$ for each $x \in X$. In [7], a "natural" induced action of $S$ on $X \times X$ was studied, i.e., an action $(X \times X) \times S \rightarrow X \times X$ given by $((x,y),s) \rightarrow (xs,ys)$. A non-empty subset $A \subseteq X \times X$ is called an invariant relation on $X$ under $S$ if $AS \subseteq A$. We say $A$ is a minimal invariant relation if no proper subset of $A$ is invariant under $S$.

In [7] and [8], some elementary properties of minimal invariant relations with $X$ and $S$ finite were discussed. Of primary concern there was the locally compact semialgebra generated by the incidence matrices of the minimal invariant relations on $X$ under $S$. Here we restrict our attention to the investigation of semigroup actions and their minimal
invariant relations. In particular, we will be interested in the case where \( X \times X \) is the union of minimal invariant relations.

When \( X = S \) and the action is right multiplication, a certain class of right cancellative idempotent-free semigroups arises. We call them MIR semigroups. (The designation MIR is short for "minimal invariant relation".) It has been shown that any right cancellative idempotent-free semigroup can be embedded in a Baer-Levi semigroup [1] or [3]. We investigate what we call bounded Baer-Levi semigroups, which turn out to be universal in the embedding sense for the class of MIR semigroups. We give a partial description of the lattice of congruences of a bounded Baer-Levi semigroup and raise the more general question: What lattices can be realized as the lattice of congruences for some right cancellative idempotent-free semigroup.

We will denote the set of idempotents of a semigroup \( S \) by \( E(S) \), the identity relation on \( S \) by \( \Delta \) and the universal relation by \( \uplus \). A congruence on \( S \) is a left and right compatible equivalence relation. If \( \rho \) is a congruence on \( S \), then \( S/\rho \) is a homomorphic image of \( S \) under the obvious homomorphism; and, conversely, any homomorphic image of \( S \) is isomorphic to \( S/\rho \) for some congruence \( \rho \) [2, 1.5].
If $\rho$ is any relation on $S$, then the transitive closure $\rho^t$ of $\rho$ is defined as follows:

$$\rho^t = \bigcup_{n=1}^{\infty} \rho^n = \rho \cup (\rho \circ \rho) \cup (\rho \circ \rho \circ \rho) \cup \cdots$$

The relation $\rho^t$ is the smallest transitive relation containing $\rho$.

**Proposition 1.1.** If $\rho$ is a right and left compatible, reflexive, symmetric relation on $S$, then $\rho^t$ is a congruence.

A non-empty subset $L$ of $S$ is a left ideal if $LS \subseteq L$ and a minimal left ideal if no proper subset of $L$ is a left ideal. The definitions of right ideal, minimal right ideal, (two-sided) ideal, and minimal ideal are the obvious ones. A simple semigroup has no proper ideals. A left group is a right cancellative left simple semigroup.

Let $f \in E(S)$; $f$ is primitive if whenever $e \in E(S)$ such that $ef = fe = f$, then $e = f$. $S$ is completely simple if $S$ is simple and has a primitive idempotent.

We give two useful characterizations of completely simple semigroups:

Let $X$ and $Y$ be sets, $G$ a group and $f: Y \times X \to G$ a function. Then the Rees product semigroup associated with $X, Y, G$ and $f$ is the set $X \times G \times Y$ along with the
operation \((x, g, y)(x', g', y') = (x, g, f(y, x')g', y')\).

Theorem 1.2. A semigroup is completely simple if and only if it is isomorphic with a Rees product semigroup [2, p. 94].

Proposition 1.3. S is a completely simple semigroup if and only if S is the union of its minimal right ideals and

1. \(E(S) \neq \emptyset\) or
2. S has a minimal left ideal. In this case, if R and L are minimal right and left ideals respectively, then \(R \cap L \neq \emptyset\) and \(R \cap L\) is a group [2, 2.7].
Let $X \times S \rightarrow X$ be an action of $S$ on $X$ denoted by juxtaposition, i.e., $(x,s) \rightarrow xs$. Consider the coordinate-wise action $((x,y),s) \rightarrow (x,y)s = (xs,ys)$ of $S$ on $X \times X$. Let $\Delta$ denote the diagonal of $X \times X$, $\mathcal{R}$ the set of non-empty minimal invariant relations on $X \times X$ under $S$, and $\text{MIR}$ the union of all elements of $\mathcal{R}$. Note that $\mathcal{R}$ may be empty. For example, take $X = (0,1] = S$ and let the action be right multiplication. But if $S$ has a minimal right ideal, $\mathcal{R}$ is necessarily non-empty. This section contains some elementary properties of minimal invariant relations, many of which are taken directly from [6] and [7].

**Proposition 2.1.** If $R_1, R_2 \in \mathcal{R}$, $R_1 \neq R_2$, then $R_1 \cap R_2 = \emptyset$.

**Proof.** Suppose $R_1 \cap R_2 \neq \emptyset$. Then $(R_1 \cap R_2)S \subset R_1 S \cap R_2 S \subset R_1 \cap R_2$, so $R_1 \cap R_2$ is an invariant
relation. But $R_1 \cap R_2 \subseteq R_1$ and since $R_1$ is minimal, this implies that $R_1 \cap R_2 = R_1$. Similarly, $R_1 \cap R_2 = R_2$; and therefore $R_1 = R_2$. □

Proposition 2.2. The diagonal relation $\Delta$ is invariant under $S$.

Proof. For $(a,a) \in \Delta$, and $s \in S$, $(a,a)s \in \Delta$. □

Proposition 2.3. If $R \in \mathcal{R}$ and $R^{-1} = \{(x,y) : (y,x) \in R\}$ then $R^{-1} \in \mathcal{R}$. Consequently if $R \in \mathcal{R}$, then $R = R^{-1}$ or $R \cap R^{-1} = \emptyset$.

Proof. For $(x,y) \in R^{-1}$ and $s \in S$, $(ys,xs) = (y,x)s \in R$ and hence $(xs,ys) \in R^{-1}$. The second part follows from 2.1. □

Proposition 2.4. If $(a,b) \in X \times X$, then $(a,b)S$ is invariant under $S$ and $\{(a,b)\} \cup (a,b)S$ is contained in any invariant relation containing $(a,b)$.

Proof. Certainly $(a,b)SS \subseteq (a,b)S$ and if $R$ is an invariant relation containing $(a,b)$, then $(a,b)S \subseteq R$. □

Proposition 2.5. If $(a,b) \in R \in \mathcal{R}$, then $R = (a,b)S$.

Proof. By the previous proposition, the invariant relation $(a,b)S$ is contained in $R$ and since $R$ is a minimal invariant relation, $R = (a,b)S$. □
Proposition 2.6. If \((a,b) \in X \times X\), then there is \(s \in S\) so that \((a,b)s = (a,b)\), and \((a,b)s\) is minimal invariant if and only if \((a,b) \in \text{MIR}\).

Proof. If \((a,b)s\) is minimal invariant and \((a,b) = (a,b)s\) for some \(s \in S\), then \((a,b) = (a,b)s \in (a,b)s \subseteq \text{MIR}\).

Conversely, if \((a,b) \in \text{MIR}\), then there is a minimal invariant relation \(R\) containing \((a,b)\) and by the previous proposition \(R = (a,b)s\); therefore, \((a,b) = (a,b)s\) for some \(s \in S\). □

Proposition 2.7. If \(A \subseteq X \times X\) is invariant under \(S\), then \(A \in \mathcal{R}\) if and only if \(S\) acts transitively on \(A\), i.e., if \((a,b)\) and \((c,d)\) are in \(A\), then there is \(s \in S\) so that \((a,b)s = (c,d)\).

Proof. If \(A \in \mathcal{R}\), \((c,d) \in A = (a,b)s\), so \((c,d) = (a,b)s\) for some \(s \in S\). For the other implication, suppose \(A\) is an invariant relation and \(B\) is an invariant non-empty subset of \(A\); then for \((a,b) \in B\), \((a,b)s \subseteq B\), but by the transitivity of \(S\) on \(A\), \(A \subseteq (a,b)s\) and therefore \(B = A\). This means that \(A \in \mathcal{R}\). □

Proposition 2.8. If \(R \in \mathcal{R}\) and \((a,a) \in R\) for some \(a \in S\), then \(R \subseteq \Delta\).

Proof. The minimal invariant relation \(R\) is \((a,a)s\) and
is therefore a subset of $\Delta$. □

**Proposition 2.9.** If $R \in \mathcal{R}$ and $(a,b) \in R$ with $a \neq b$, then $R \subset (X \times X) \setminus \Delta$.

**Proof.** If $R \cap \Delta \neq \emptyset$, by the previous proposition $R \subset \Delta$. □

**Proposition 2.10A.** Let $(a,b) \in X \times X$. Then $(a,b) \in \text{MIR}$ if and only if (i) there is $s \in S$ such that $(a,b)s = (a,b)$ and (ii) for each $s,t \in S$ there is $t' \in S$ such that $(a,b)st' = (a,b)t$.

**Proof.** Suppose $(a,b) \in \text{MIR}$. By 2.6, there is an $s \in S$ such that $(a,b)s = (a,b)$ and by 2.5 the minimal invariant relation containing $(a,b)$ is $(a,b)S$. Then by 2.7, for $(a,b)s$ and $(a,b)t \in (a,b)S$ there is a $t' \in S$ such that $(a,b)st' = (a,b)t$. The converse follows from 2.7 and 2.6.

**Proposition 2.10B.** Let $(a,b) \in X \times X$. Then $(a,b) \in \text{MIR}$ if and only if for each $s \in S$ there is a $t \in S$ such that $(a,b) = (a,b)st$.

**Proof.** If $(a,b) \in \text{MIR}$, $s \in S$, $(a,b) \in (a,b)S$, the minimal invariant relation containing $(a,b)$. Then since $S$ is transitive on $(a,b)S$, there is a $t \in S$ so that $(a,b) = (a,b)st$. For the other implication, if $s' \in S$, there is $t \in S$ such that $(a,b)s't = (a,b)$, so
(a,b) ∈ (a,b)S. Then for (a,b)s, (a,b)t ∈ (a,b)S there is t' ∈ S such that (a,b)st' = (a,b), and therefore (a,b)st't = (a,b)t. It now follows from the previous proposition that (a,b) ∈ MIR. □

Let S and T be semigroups, X any Y sets with f:X → Y a function and g:S → T a homomorphism. Then 

(f,g):(X,S) → (Y,T) is an act homomorphism if (xs)f = (xf)sg.

Proposition 2.11. If (f,g):(X,S) → (Y,T) is an act homomorphism and A ⊂ X × X is invariant under S, then (A)(fxf) is invariant under Sg.

Proof. Let (af,a'f) ∈ (A)(fxf) and sg ∈ Sg. Then 

(af,a'f)sg = (asf,a'sf) = (as,a's)(fxf) ∈ (A)(fxf). □

If R₁,R₂ ⊂ X × X, the composition of R₁ and R₂, 

R₁°R₂ = {(x,y) ∈ X × X : (x,z) ∈ R₁,(z,y) ∈ R₂ for some z ∈ X}.

Proposition 2.12. If R₁ and R₂ are invariant relations on X under S, then so is their composition, i.e., the invariant relations form a subsemigroup of the semigroup of relations on X.

Proof. If (x,y) ∈ R₁°R₂, then there is z ∈ X such that 

(x,z) ∈ R₁ and (z,y) ∈ R₂. Since R₁ and R₂ are invariant under S, (x,z)s ∈ R₁ and (z,y)s ∈ R₂. It
follows that \((xs, ys) \in R_1 \circ R_2\). \(\square\)

For the remainder of this section, we give some results on the nature of (minimal) invariant relations. As stated above we are assuming that \(S\) acts on the right of \(X\). If \(x \in X\) then the orbit of \(x\) is the set \(xS\). We note that a minimal invariant subset of \(X\) is an orbit and hence a minimal orbit. If \(X = S\) and the action is right multiplication, of course, a minimal orbit is a minimal right ideal. Following the notation of [3, p. 259], we will say that an orbit of \(x\), \(xS\), is strict if \(x \in xS\). The following propositions (2.13A, 2.13B, 2.14A, 2.14B and 2.15) relate minimal invariance in \(X \times X\) to minimal invariance in \(X\). It is noted that for the case of \(X = S\), where the action is right multiplication, the converses of 2.13A and 2.14A, given as 2.13B and 2.14B are partial and necessarily so as shown by Example 2.18.

**Proposition 2.13A.** If \((x,y)S\) is minimal invariant, then \(xS\) and \(yS\) are minimal orbits.

**Proof.** The relation \((x,y)S\) is minimal invariant if and only if for each \(s,t \in S\) there is \(t' \in S\) such that \((x,y)t = (x,y)st'\). It follows that \(S\) is transitive on \(xS\) and \(yS\) individually. \(\square\)

**Proposition 2.13B.** Under the assumption that \(X = S\) and
the action is right multiplication, if \( xS \) is a minimal orbit and \( y \in Sx \), then \((x,y)S\) is a minimal invariant relation.

Proof. If \( xS \) is minimal invariant and \( y = zx \) for some \( z \in S \), then for \( s,t \in S \), we have \( xsS = xS \) and hence there is a \( t' \in S \) such that \( xst' = xt \). Also \( yt = zxt = zxs = yst' \) so that \((x,y)t = (x,y)st'\) and \((x,y)S\) is minimal invariant. □

Proposition 2.14. If \((x,y) \in MIR\), then \(xS\) and \(yS\) are strict minimal orbits.

Proof. If \((x,y) \in MIR\), then, by 2.6, there is an \( s \in S \) such that \((x,y)s = (x,y)\) and \((x,y)S\) is a minimal invariant relation. It follows from 2.13A that \(xS\) and \(yS\) are minimal orbits, and since \(x = xs \in xS\) and \(y = ys \in yS\), \(xS\) and \(yS\) are strict minimal orbits. □

Proposition 2.14B. Under the assumption that \( X = S \) and the action is right multiplication, if \( xS \) is a minimal strict orbit and \( y \in Sx \), then \((x,y) \in MIR\).

Proof. Since \( xS \) is a strict orbit, there is an \( s \in S \) so that \( x = xs \). By hypothesis, there is a \( z \in S \) such that \( y = zx \); therefore, \( ys = zxs = zx = y \) or \((x,y) = (x,y)s\). Applying 2.13B, we have that \((x,y)S\) is a
minimal invariant relation. Now it follows from 2.6 that \((x,y) \in \text{MIR}\). □

**Corollary 2.15.** If \(\bar{X}\) is the union of minimal strict orbits of \(X\), then:

(i) \(\Delta \subseteq \text{MIR}\) if and only if \(X = \bar{X}\) and

(ii) \(\text{MIR} \subseteq \bigcup_{s \in S} \bar{X}_s \times \bar{X}_s \subseteq \bar{X} \times \bar{X}\).

**Proof.** If \(\Delta \subseteq \text{MIR}\) and \(x \in X\), \((x,x) \subseteq \Delta\) and therefore a minimal invariant relation. It follows that \(x \in xS\), a strict minimal orbit. For (ii) let \((x,y) \in \text{MIR}\). Then, by 2.14A, \(xS\) and \(yS\) are strict minimal orbits and, by 2.6, there is an \(s \in S\) such that \((x,y) = (x,y)s\).

Therefore, \((x,y) = (x,y)s = (xs,ys)s \subseteq xSs \times ySs \subseteq \bar{X}_s \times \bar{X}_s\).

For arbitrary \(s \in S\) and \((x,y) \in \bar{X} \times \bar{X}\), \(xs \subseteq \bar{X}\) and \(ys \subseteq \bar{X}\), so \(\bigcup_{s \in S} \bar{X}_s \times \bar{X}_s \subseteq \bar{X} \times \bar{X}\). □

**Proposition 2.16.** If \(S\) is the union of minimal right ideals and \(S\) acts on a set \(X\), then

\[\text{MIR} = \bigcup_{s \in S} (X_s \times X_s) = \bigcup_{x,y \in X} (x,y)S.\]

**Proof.** The second equality is immediate and \(\text{MIR} \subseteq \bigcup_{s \in S} (X_s \times X_s)\) by 2.15 (ii). To show that the remaining inclusion holds, let \((x_s,ys) \in X_s \times X_s\), and \(t \in S\).

Then \(stS = sS\), the minimal right ideal containing \(s\), so that \((x_s,ys) \in (x_s,ys)tS\) and \((x_s,ys) \in \text{MIR}\) by 2.10B. □
Corollary 2.17. If $S$ is a completely simple semigroup acting on itself by right multiplication, then

$$\text{MIR} = \bigcup_{e \in E(S)} Se \times Se.$$  

**Proof.** A completely simple semigroup is the union of its minimal right ideals and so $\text{MIR} = \bigcup_{s \in S} (SsxSs)$. The set $Ss$ is a minimal left ideal, contains an idempotent, $e$, and $Ss = Se$. □

Note that in this case $MIR$ is Green's relation $\mathcal{L}$.

**Example 2.18.** Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2, y_3\}$, $G = \mathbb{Z}_2 = \{0, 1\}$, and $f: Y \times X \to G$ be defined by $f(y_1, x_1) = f(y_1, x_2) = f(y_2, x_1) = f(y_3, x_1) = 0$ and $f(y_2, x_2) = f(y_3, x_2) = 1$. Let $S$ be the Rees product $[X, G, Y]_f$. If $a = (x_1, 1, y_1)$, $b = (x_2, 1, y_1)$, $c = (x_2, 1, y_2)$, $d = (x_1, 0, y_2)$ and $e = (x_2, 0, y_3)$ then since $(x_2, 0, y_1)(x_1, 1, y_1) = (x_2, 0 + f(y_1, x_1) + 1, y_1) = (x_2, 1, y_1) = b$, it follows from 2.13B that $(a, b)S$ is minimal invariant. Note that $c \not\in Sa$, as $c$ and $a$ have different third coordinates. In fact, $(a, b)S \not\subseteq (a, c)S$ so that $(a, c)S$ is not minimal invariant. However, $d \not\in Se$ and $(d, e)S$ is minimal invariant. In fact, $(d, e)S = (a, b)S$.

For $S$ acting on $X$ and $T$ a subsemigroup of $S$, we define $\text{MIR}(T) = \bigcup \{R: R \subset X \times X \text{ and } R \text{ is a minimal invariant relation under } T\}$. 
Proposition 2.19. If $T \subset S$ is any right ideal of $S$, then $\text{MIR}(T) = \text{MIR}(S)$.

Proof. We will use 2.10B for both inclusions. If $(a,b) \in \text{MIR}(T)$, $(a,b) = (a,b)t$ for some $t \in T$. Then for arbitrary $s \in S$, $ts \in T$ and hence $(a,b) = (a,b)t = (a,b)tst'$ for some $t' \in T \subset S$. If $(a,b) \in \text{MIR}(S)$ and $t_0 \in T$, then for all $t \in T$, $(a,b) \in (a,b)t_0s \subset (a,b)tT$. □

Proposition 2.20. If $S$ acts on a set $X$ and $S$ has a minimal right ideal $R$, then $\text{MIR}(S) = \bigcup_{x,y \in X} (x,y)R$.

Proof. By 2.19, $\text{MIR}(S) = \text{MIR}(R)$, and by 2.16, $\text{MIR}(R) = \bigcup_{x,y \in X} (x,y)R$. □

Remark. This means that under the hypothesis of 2.20, $(a,b) \in \text{MIR}$ if and only if there is an element $r \in R$ such that $(a,b)r = (a,b)$. It also follows that if $(a,b)r = (a,b)$ for some $r \in R$, and $R'$ is any minimal right ideal of $S$, there is an $r' \in R'$ such that $(a,b)r' = (a,b)$.

Corollary 2.21. If $S$ acts effectively on a set $X$, $S$ has a minimal right ideal $R$, and $\text{MIR} \subset \Delta$, then there is a unique minimal invariant relation.

Proof. By 2.20, $\text{MIR}(S) = \bigcup_{x,y \in X} (x,y)R$. Suppose that $\text{MIR}(S) \subset \Delta$. Then for $x,j \in X$ and $r \in R$, $xr = yr$. But this implies that $(x,x)R = (y,y)R$ or there is only one
minimal invariant relation. □

**In the following Propositions 2.22 - 2.28, we assume**

\[ X = S \] and the action is right multiplication.

**Proposition 2.22.** \( S \) is right trivial if and only if every invariant relation on \( S \) is reflexive.

**Proof.** If \( S \) is right trivial, \( R \) is an invariant relation on \( S \), and \( (a,b) \in R \), then \( \Delta = (a,b)S \subseteq R \). Hence \( R \) is reflexive.

If every invariant relation on \( S \) is reflexive, then \( \Delta \subseteq (a,b)S \) for each \( a,b \in S \). Thus if \( a,b,t \in S \) there is \( s \in S \) so that \( t = as = bs \). Using this we see that \( E(S) = S \). Let \( t \in S \). There is \( x \in S \) so that \( t = tx \) and an element \( y \in S \) so that \( ty = xy = x \). Thus \( x = ty = txy = tx = t \) and \( t = tx = tt \). We also note that \( S \) is right simple. Hence if \( s \in S \) then \( ss \cap ss \) is a group. Since \( S = E(S) \), \( ss \cap ss = \{s\} \), and since \( ss = S \), we have \( ss = \{s\} \). □

**Proposition 2.23.** \( S \) is a group where each element has order two if and only if every invariant relation on \( S \) is symmetric.

**Proof.** If \( S \) is a group with each element of order two, then \( S \) is abelian. Let \( R \) be an invariant relation on
$S$ and $(a,b) \in R$. Then $(a,b)ab = (a^2b, b^2a) = (b,a) \in R$.

Hence $R$ is symmetric on $S$.

If every invariant relation on $S$ is symmetric, $a \neq b$ in $S$, then there is $s \in S$ so that $as = b$ and $bs = a$. We first note that $s^2 = s^4$ for each $s \in S$.

If $s^2 \neq s$ then there is $t \in S$ so that $s = s^2t$ and $s^2 = st$. It follows that $s^4 = s^2$. Hence $E(S) \neq \emptyset$.

If $e \in E(S)$ then $eS = S$ and $e$ is a left unit for $S$.

Let $a \in S$. If $a \neq e$ there is $s \in S$ so that $es = a$ and $as = e$. Hence $a = es = s$ and $e = as = a^2$. Thus $E(S) = \{e\}$. Now $S$ is right simple with $E(S) = \{e\}$ and is hence a group. □

**Proposition 2.24.** $S$ is right trivial with no more than two elements if and only if every invariant relation on $S$ is transitive.

**Proof.** If every invariant relation on $S$ is transitive then for $a \neq b$ and $b \neq c$ in $S$,

$R = [(a,b)] \cup (a,b)S \cup [(b,c)] \cup (b,c)S$ is transitive.

Hence $(a,c) \in R$ and there is an element $s \in S$ so that $(a,c) = (a,b)s$ or $(a,c) = (b,c)s$. If $a^2 \neq a \in S$ then there is $t \in S$ so that $(a,a) = (a,a^2)t$ or $(a,a) = (a^2,a)t$. In any case $a = a^2t = aat = aa = a^2$. Hence $a^2 = a$ and $S = E(S)$. It further follows from above that $S$ is right simple. As in 2.22 $Ss = \{s\}$ and $S$ is right trivial.
trivial. Now if \( a \neq b \) and \( b \neq c \) we get \( s \in S \) so that
\[ a = as \text{ and } c = bs \] or \( a = bs \text{ and } c = cs \). In either case \( a = s = c \). Thus \( S \) has at most two points.

The converse is immediate. □

The three previous propositions indicate that all minimal invariant relations are equivalence relations only when \( S \) is trivial. However, \( \text{MIR} \), itself, is sometimes a congruence. In particular, if \( S \) is completely simple, \( \text{MIR} = \mathcal{I} \), and \( \mathcal{I} \) is a congruence on a completely simple semigroup.

**Proposition 2.25.** \( \text{MIR} \) is a symmetric, compatible relation.

**Proof.** If \( (a,b) \in \text{MIR} \), it follows from 2.3 that \( (b,a) \in \text{MIR} \). Further, for \( s \in S \), \( (a,b)s \in (a,b)S \subseteq \text{MIR} \). Since there is an \( e \in S \) with \( (a,b)e = (a,b) \), \( (sa,sb) = (sa,sb)e \) and by 2.16, \( (sa,sb) \in \text{MIR} \). □

**Proposition 2.26.** Let \( R = \bigcup \{ R_\alpha : R_\alpha \text{ is a minimal right ideal of } S \} \). Then \( \text{MIR} \) is transitive if and only if whenever there are \( s,t \in R \) such that \( (a,b)s = (a,b) \) and \( (b,c)t = (b,c) \) for \( a,b,c \in S \), there is a \( q \in R \) such that \( (a,c)q = (a,c) \).

**Proof.** If \( \text{MIR} \neq \emptyset \), 2.13A yields that \( R \neq \emptyset \). Then by the remark following 2.20, \( (x,y) \in \text{MIR} \) if and only if
there is an element $r \in R$ such that $(x,y)r = (x,y)$.
The proposition follows immediately. \qed

**Proposition 2.27.** The transitive closure of $MIR \cup \Delta$ is a congruence.

**Proof.** This follows from 2.25 and 1.1. \qed

**Proposition 2.28.** If $MIR$ is a congruence on $S$, then $S/MIR$ has right trivial multiplication.

**Proof.** Since $\Delta \subseteq MIR$, $S$ is the union of minimal right ideals. Then for $a,b \in S$, there is a $r \in S$ such that $br = b$ and consequently $abr = ab$. Again, this means that $(ab,b) \in MIR$ or that $S/MIR$ has right trivial multiplication. \qed

We now return to the case of a semigroup $S$ acting on a set $X$ and show that if $S$ has a minimal left ideal and the action is effective, then $MIR = X \times X$ if and only if $S$ is a group. This generalizes 3.12 of [7].

**Proposition 2.29.** If $S$ is a group then $MIR = X \times X$.

**Proof.** We use 2.10B. If $(a,b) \in X \times X$, $s \in S$ and $e$ is the identity of $S$, then $(a,b) = (a,b)e = (a,b)ss^{-1}$. (Recall that we required a semigroup identity to act like an identity function.) Hence $(a,b) \in MIR$. \qed
Definition 2.30. An element $s \in S$ is **injective** if $xs = ys$ implies $x = y$.

Proposition 2.31. If $S$ acts effectively on $X$ and $\text{MIR} = X \times X$ then every element of $S$ is injective and $S$ is right cancellative.

Proof. Let $s \in S$ and assume that $xs = ys$ for $x, y \in X$. Now $(x, y) \in \text{MIR}$ and hence by 2.10B there is $t' \in S$ so that $(x, y)t' = (x, y)$, i.e., $x = xst' = yst' = y$. Now suppose that $ts = t's$ for $t, t', s \in S$. Hence for each $x \in X$, $(xt)s = x(ts) = x(t's) = (xt')s$. Now $s$ is injective and thus $xt = xt'$ for each $x \in X$. The effectiveness of the action now gives $t = t'$. □

Proposition 2.32. If $S$ acts effectively on $X$, $\text{MIR} = X \times X$, and $e \in E(S)$ then $xe = x$ for each $x \in X$ and $e$ is a two-sided identity for $S$.

Proof. For $x \in X$, $(xe)e = x(ee) = xe$. From above $xe = x$. Then for all $s \in S$ and $x \in X$, $xse = xs = xes$; thus $se = s = e$ since the action is effective. □

Proposition 2.33. If $S$ acts effectively on $X$ and $\text{MIR} = X \times X$, then

1. $S$ is a monoid if and only if $a \in Sa$ for some $a \in S$ and then if $G$ is the group of units of $S$, $aLb$
if and only if \( b \in Ga \).

(ii) S is a group acting on X as a group of permutations if and only if S has a minimal left ideal.

Proof. If \( a \in Sa \) for some \( a \in S \), then \( a = ta \), \( t \in S \). Thus for all \( x \in X \), \( (x,xt)a \in \Delta \) so that the minimal invariant relation containing \( (x,xt) \), \( (x,xt)S \subset \Delta \). But then \( (x,xt) \in \Delta \), giving \( x = xt \) for all \( x \in X \). Since the action is effective \( t = t^2 \) and \( t \) is a two-sided identity for \( S \) by Proposition 2.32. Then if \( G \) is the group of units of \( S \) and \( aLb \), \( a = sb \) and \( b = ta \) for some \( s,t \in S \). So \( a = sta \) and \( b = tsb \), but the above argument shows that \( st = 1 = ts \) where \( 1 \) is the identity for \( S \). \( b \in Ga \) always implies \( aLb \).

If \( L \subset S \) is a minimal left ideal of \( S \), then \( L = La \) for any \( a \in L \). Thus there is an \( \ell \in L \) such that \( a = \ell a \), and again \( \ell \) is a two sided identity for \( S \). Thus \( S = L \) is a left simple monoid, and therefore a completely simple semigroup with exactly one idempotent, i.e., a group. \( \square \)
CHAPTER III
MIR SEMIGROUPS

Here, and in the remaining chapters, we restrict $X$ to $S$ and let $S$ act on $S$ by multiplication on the right. We shall frequently assume that this action is effective, i.e., $ts_1 = ts_2$ for each $t \in S$ implies $s_1 = s_2$. Note that this is just saying that $S$ is left reductive [2, p. 9]. It is then the case that $S \times S$ is the union of minimal invariant relations and $S$ has an idempotent if and only if $S$ is a group. If $S$ does not have an idempotent and $S \times S$ is the union of minimal invariant relations then we will call $S$ an MIR semigroup. To be more precise we give the following definition and results.

Definition 3.1. A semigroup $S$ is called an MIR-semigroup if (1) $E(S) = \emptyset$; (2) $S$ is the union of its minimal right ideals; (3) any two elements of $S$ have a common right identity.
We will abbreviate (2) above as S is UMRI and (3) above as S has CRIDS.

**Proposition 3.2.** Let S act on itself by multiplication on the right. Then S x S is the union of minimal invariant relations if and only if S is UMRI and S has CRIDS.

**Proof.** If S x S is the union of minimal invariant relations then for a ∈ S we have (a,a) ∈ (a,a)S which is the minimal invariant relation containing (a,a). Hence a ∈ aS and aS is a minimal right ideal. If a,b ∈ S then (a,b) ∈ (a,b)S which is the minimal invariant relation containing (a,b). Hence a and b have a common right identity.

Conversely, if (a,b) ∈ S x S then a and b have a common right identity in a minimal right ideal R. Consequently (a,b) ∈ (a,b)R and (a,b)R is a minimal invariant relation. □

**Proposition 3.3.** If a semigroup S is UMRI and has CRIDS then S is right cancellative.

**Proof.** Suppose for some x,y,s ∈ S that xs = ys. Now there is a minimal right ideal R so that s ∈ R. Since x and y have a common right identity in some minimal right ideal they have one in R, call it z. Consequently,
$z \in sR = R$ and there is $r \in R$ with $z = sr$. Hence $x = xz = xsr = ysr = yz = y$. □

**Proposition 3.4.** Let $S$ be a semigroup. If $S$ is UMRI, has CRIDS, and $E(S) \neq \emptyset$ then $S$ is left simple and hence a left group.

**Proof.** Let $x, y \in S$. There is a minimal left ideal $L$ and $z \in L$ so that $xz = x$ and $yz = y$. Consequently, $x = xz \in xL \subseteq L$ and $y = yz \in yL \subseteq L$. Hence $S = L$.
Thus $S$ is left simple and (by 3.3) right cancellative, i.e., a left group [2, p. 37].

**Proposition 3.5.** [7] Let $S$ be a semigroup. If $S$ is UMRI, has CRIDS, is left reductive, and $E(S) \neq \emptyset$, then $S$ is a group.

**Proof.** By 3.4, $S$ is left simple. If $e, f \in E(S)$ then $e$ and $f$ are both right identities for $S$ and hence, by left reductivity, equal. Then $E(S) = \{e\}$ and $S$ is a group. □

**Remark.** Let $S$ be left reductive with MIR = $S \times S$. If either (1) there is $s \in S$ with $s \in Ss$, (2) the center of $S$ is nonempty, or (3) $S$ is left cancellative, then $S$ is a group. To see this one need only observe, using the fact that $S$ is right cancellative, that $E(S) \neq \emptyset$. 
Remark. In view of 3.2 through 3.5, we note that semigroups $S$ with $S \times S$ being the union of minimal right compatible relations divide into two classes: Left groups if $E(S) \neq \emptyset$ and MIR-semigroups if $E(S) = \emptyset$.

Definition 3.6. Let $p$ and $q$ be infinite cardinal numbers with $q < p$. A semigroup $S$ is a Baer-Levi semigroup of type $(p,q)$ on a set $A$ if $\text{card}(A) = p$ and $S$ is the semigroup of one-to-one mappings $a$ of $A$ into $A$ having the property that $\text{card}(A \setminus Aa) = q$.

Baer-Levi semigroups are discussed in [1] and [3]. We note that the operation in a Baer-Levi semigroup $S$ is composition, i.e., if $a, \beta \in S$ then $a \beta$ is defined by $(x)a \beta = (xa)\beta$. As noted in [3] a Baer-Levi semigroup is right cancellative, right simple, and without idempotents. Further, a semigroup $T$ can be embedded in a Baer-Levi semigroup of type $(p,p)$ if and only if $T$ is right cancellative and $E(T) = \emptyset$ [3].

Hence we get the following.

Proposition 3.7. If $S$ is an MIR semigroup, then $S$ can be embedded in a Baer-Levi semigroup.

It is easy to find a subsemigroup of a Baer-Levi semigroup $S$ that does not have MIR $= S \times S$, e.g., $S$ itself. One way to observe this is to note that there is a pair of
elements of \( S \) which do not have a common right identity. More generally, the Croisot-Teissier generalization of the Baer-Levi semigroups [3, p. 89] also fail to have \( \text{MIR} = S \times S \). However, as the following example shows, every Baer-Levi semigroup is the union of left reductive subsemigroups \( T \) with \( \text{MIR} = T \times T \).

**Example 3.8.** Let \( S \) be a Baer-Levi semigroup of type \((p,q)\) on a set \( A \). Let \( s \in S \). Since \( \text{card}(A\setminus A_s) = q \) and \( q \) is an infinite cardinal number, we can write \( A\setminus A_s \) as the disjoint union of countably many subsets, \( A_1, A_2, \ldots \), with \( \text{card}(A_i) = q \) for each \( i = 1, 2, 3, \ldots \). Set \( A_0 = A_s \) and \( T = \{ t \in S : A_t \subseteq \bigcup_{k=0}^{\infty} A_k \) for some \( n \} \). It is easily verified that \( T \) is a left reductive subsemigroup of \( S \). To see that \( \text{MIR} = T \times T \) we use 3.2. If \( a, b \in T \), define \( s \) to be the identity on \( A_a \cup A_b \), and choosing \( A_k \) so that \( A_k \cap (A_a \cup A_b) = \emptyset \) let \( (A\setminus(A_a \cup A_b))s \subseteq A_k \). Then \( as = a \) and \( bs = b \).

Now let \( q, t \in T \). Then there is a function \( f = q^{-1}t \) on the range of \( q \) which maps \( A\setminus A_s \) into some \( A_k \) which does not intersect \( A_t \). Therefore \( qf = t \) and \( T \) is right simple. So \( T \) is a left reductive MIR semigroup.

In view of the above there are numerous left reductive semigroups of a Baer-Levi semigroup. For sake of brevity, we will refer to such a subsemigroup as one of type \( R \).
Recall that 2.10A gave two equational conditions that are sufficient for a left reductive semigroup $T$ to have $\text{MIR} = T \times T$. The first of these is the existence of a common right identity for each pair of elements of $T$, and the second is a common solution for $asx = at$ and $bsx = bt$ for arbitrary $a, s, t, b \in T$. This second condition is closely related to right simplicity for a subsemigroup of a Baer-Levi semigroup. It is conjectured that a subsemigroup of type $R$ need not be right simple but we have been unable to verify this.

**Question 3.9.** Is a type $R$ subsemigroups of a Baer-Levi semigroup right simple?

**Example 3.10.** It is easy to construct a left-reductive subsemigroup of a Baer-Levi semigroup that has CRIDS and is not right simple. In fact, there is a subsemigroup of the semigroup $T$ of Example 3.8 with these properties. For each positive integer $i$, set $(x)a_i = x$ for each $x \in \bigcup_{k=1}^{i} A_k$ and let $(X \setminus \bigcup_{k=1}^{i} A_k)a \subset A_{k+1}$. Then let $T'$ be the subsemigroup generated by $\{a_i\}_{i=1}^{\infty}$. $T'$ is not right simple since there is not $t \in T'$ such that $a_2 t = a_1$.

**Example 3.11.** Also one can find a left reductive subsemigroup of a Baer-Levi semigroup $S$ that is UMRI and not right simple. It fails to have CRIDS. Let $A = A_1 \cup A_2$
with \( A_1 \cap A_2 = \emptyset \) and \( |A_1| = |A_2| = |A| \), and let \( P_1 \) and \( P_2 \) be countably infinite partitions of \( A_1 \) and \( A_2 \) respectively into sets of the same cardinality as \( A \). Then define \( S' \) to be the collection of all elements of \( S \) whose range is contained either in \( A_1 \) or \( A_2 \) and in a finite union of partition elements of the corresponding partition.

**Example 3.12.** Finally, we give an example of a semigroup which has CRIDS, is UMRI, and is idempotent-free but is not right simple. Let \( A \) be any infinite set, \( a, b \in A \), and let \( P = \{A_i\}_{i=1}^\infty \) be a partition of \( A \setminus \{a\} \) where \( |A_i| = |A| \). Let \( R_1 = \{f : f \text{ is a one-to-one function on } A \text{ and } Af \subseteq \bigcup_{i=1}^n A_i \text{ for some } n\} \) and \( R_2 = \{f ; f \text{ is one-to-one on } A \setminus \{a, b\}, af = bf \text{ and } Af \subseteq \bigcup_{i=1}^n A_i \text{ for some } n\} \).

In \( R = R_1 \cup R_2 \), \( R_1 \) and \( R_2 \) are minimal right ideals of \( R \) and it is easily verified that \( R \) has CRIDS. \( R \) fails to be left reductive since there are functions \( x, y \in R \) such that \( sx = sy \) for each \( s \in R \) but \( (a)x \neq (a)y \). Now \( R \) is a subsemigroup of a Croissot-Tessier semigroup \([3, 8.2]\), but is right cancellative by 3.3, and can therefore be embedded in a Baer-Levi semigroup.

In the other direction we give the following results.

**Proposition 3.13.** Let \( S \) be a Baer-Levi semigroup on a set
A and let $T$ be a subsemigroup of $S$ which is maximal with respect to having a common right identity for each pair of elements. Then $T$ is right simple.

Proof. First we note that if $f \in T$ and $g \in S$ with $Ag \subseteq Af$ then $g \in T$. To see this let $T'$ be the subsemigroup of $S$ generated by $T \cup \{g\}$. Since each pair of elements of $T$ has a common right identity then so does each pair of elements of $T \cup \{g\}$. Hence $T'$ has this property and, by maximality, $T = T'$. To see that $T$ is right simple, we need to show that for each $s, t \in T$ there is an element $f \in T$ so that $f$ restricted to $As$ is $s^{-1}t$ restricted to $As$. It is clear that one can find $f \in S$ so that the restriction of $f$ to $As$ is the same as $s^{-1}t$ restricted to $As$ and $Af \subseteq At$. By the first part of the proof $f \in T$. Thus $T$ is right simple.

Proposition 3.14. Let $S$ be as in 3.13 and $T$ a subsemigroup of $S$ which is maximal with respect to $\text{MIR} = T \times T$. Then $T$ is right simple.

Proof. Again we need to extend $s^{-1}t$ restricted to $As$ for each $s, t \in T$. As in the proof of 3.13, let $g \in S$ with $g$ an extension of $s^{-1}t$ and $Ag \subseteq At$. Let $T' = \{f \in S: Af \subseteq Ah$ for some $h \in T\}$. Now $T'$ is a subsemigroup of $S$ with $\text{MIR} = T' \times T'$. Noting that $g \in T'$ and $T \subseteq T'$ we see that $T = T'$ and $g \in T$. 
CHAPTER IV

BOUNDED BAER-LEVI SEMIGROUPS

We will review the definition and properties of Baer-Levi semigroups with \( p = q \).

Let \( X \) be a set with infinite cardinal number, i.e., \( |X| = q \) where \( q \) is an infinite cardinal number. The Baer-Levi semigroup (of type \((q,q)\) in [3, Sec. 8.1]) on \( X \), noted here by \( \text{BL}(X) \), is the subsemigroup of the full transformation semigroup on \( X \) consisting of one-to-one functions \( a:X \rightarrow X \) so that \( |X \setminus Xa| = q \). It is known, [3, Th. 8.2] or [1], that \( \text{BL}(X) \) is a right cancellative, right simple semigroup without idempotents. (We point out that here functions are composed left to right.) In fact, the Baer-Levi semigroups contain as subsemigroups any right cancellative, idempotent-free semigroup [3, Th. 8.8]. Considerable information about the lattice of congruences on a Baer-Levi semigroup has been obtained by E. G. Sutov [9] and B. W. Mielke, [4], [5] and [6].

Here we will be interested in certain subsemigroups
of $\text{BL}(X)$. Let $\mathcal{J}$ be a nonempty set of subsets of $X$ satisfying

(B1) For each $F \in \mathcal{J}$, $|F| = q$,

(B2) For each pair $F_1, F_2 \in \mathcal{J}$ there is $F \in \mathcal{J}$ so that $F_1 \cup F_2 \subseteq F$ and $|F \setminus (F_1 \cup F_2)| = q$.

Such a collection will be called a bounding collection for $X$. We define the bounded Baer-Levi semigroup on $X$ with respect to $\mathcal{J}$, noted $\text{BBL}(X, \mathcal{J})$, to be the subsemigroup of $\text{BL}(X)$ consisting of functions $a : X \to X$ so that there is $F \in \mathcal{J}$ with $X_a \subseteq F$. A set $A \subseteq X$ will be designated bounded if there is $F \in \mathcal{J}$ with $A \subseteq F$. A function will be bounded if its range is bounded. If the $\mathcal{J}$ is not clear, we will say bounded with respect to $\mathcal{J}$. Note that bounded Baer-Levi semigroups are a generalization of Example 3.8.

Proposition 4.1. Let $S = \text{BBL}(X, \mathcal{J})$ be a bounded Baer-Levi semigroup. Then $S$ is (a) right cancellative, (b) idempotent-free, and (c) right simple, and (d) $S$ has common right identities (CRIDS).

Proof. Statements (a) and (b) follow because $S$ is a subsemigroup of $\text{BL}(X)$, and (c) and (d) are routine consequences of conditions (B1) and (B2) on $\mathcal{J}$. □

Proposition 4.2. A subsemigroup $T$ of $\text{BL}(X)$ is left reductive provided $\bigcup \{X_a : a \in T\} = X$. 
Proof. Let $a, b \in T$ and $ta = tb$ for each $t \in T$. If $x \in X$ let $t \in T$ so that there is $y \in X$ with $yt = x$. Then $xa = yta = ytb = xb$. Consequently, $b = a$. □

Proposition 4.3. The semigroup $BBL(X, \mathcal{F})$ is left reductive if and only if $\cup \{F : F \in \mathcal{F}\} = X$.

Proof. If $\cup \{F : F \in \mathcal{F}\} \neq X$ there is $x_0 \in X$ so that $x_0$ is not in the range of any $z \in BBL(X, \mathcal{F})$. Define $a, b \in BBL(X, \mathcal{F})$ so that $xa = xb$ if $x \neq x_0$ and $x_0a \neq x_0b$. Now $ta = tb$ for each $t \in BBL(X, \mathcal{F})$ but $b \neq a$. The reverse implication follows from 4.2. □

Proposition 4.4. If $S$ is an MIR-semigroup then it is isomorphic to a subsemigroup of a bounded Baer-Levi semigroup.

Proof. By 3.3, $S$ is right cancellative. It is known [3, Th. 8.8] that there is a set $X$, $|X| = |S|$, with $S$ embedded in $BL(X)$. Consider $S$ as a subsemigroup of $BL(X)$ and let $\mathcal{F} = \{Xa : a \in S\}$. Now for each $F \in \mathcal{F}$, $|F| = |X|$, and if $F_1, F_2 \in \mathcal{F}$ let $Xa = F_1$ and $Xb = F_2$. If $c$ is a common right identity for $a$ and $b$ then $c$ is the identity function when restricted to $F_1 \cup F_2$. Since $Xc = F \in \mathcal{F}$ we must have $|F \setminus F_1 \cup F_2| = |X|$. Consequently $\mathcal{F}$ is a bounding collection. Now by design $S \subseteq BBL(X, \mathcal{F})$. □
The following result characterizes the left ideals of \( \text{BBL}(X,\mathcal{J}) \). Let us write \( S = \text{BBL}(X,\mathcal{J}) \). It \( a, b \in S \), then saying \( ba^{-1} \in S \) means that \( Xb \subseteq Xa \) and \( Xba^{-1} \) is a bounded set. The functional value \( (x)ba^{-1} \) is the obvious one.

A nonempty subset \( A \) of \( S \) is said to have \((*)\) if \( a \in A, s \in S \), and \( sa^{-1} \in S \) implies \( s \in A \).

**Proposition 4.5.** A nonempty subset \( L \) of \( S = \text{BBL}(X,\mathcal{J}) \) is a left ideal of \( S \) if and only if \( L \) has \((*)\).

**Proof.** If \( L \) is a left ideal choose \( a \in L, s \in S \) so that \( sa^{-1} \in S \). Then \( (sa^{-1})a = s \) is in \( L \). Consequently, \( L \) has \((*)\).

Conversely, assume that \( L \) is a subset of \( S \) and \( L \) has \((*)\). Let \( a \in L \) and \( t \in S \). We show \( ta \in L \), i.e., \( L \) is a left ideal of \( S \). Choose \( F_1 \in \mathcal{J} \) so that \( Xt \subseteq F_1 \). Choose \( F_2 \in \mathcal{J} \) so that \( |F_2 \setminus F_1| = |X| \). Let \( z \) be a one-to-one mapping of \( X \) onto \( F_2a \) with \( F_1z = Xta \). Now \( z \in S \) and since \( Xza^{-1} = F_2 \) we have \( za^{-1} \in S \). By \((*)\), \( z \in L \). Since \( Xta^{-1} = F_2 \) we have \( taz^{-1} \in S \). Consequently, by \((*)\), \( ta \in L \). □

Let \( \mathcal{J} \) and \( \mathcal{K} \) be bounding collections for \( X \). Clearly, if \( \mathcal{J} \subseteq \mathcal{K} \), then \( \text{BBL}(X,\mathcal{J}) \subseteq \text{BBL}(X,\mathcal{K}) \); in fact, if for \( F \in \mathcal{J} \) there is \( G \in \mathcal{K} \) so that \( F \subseteq G \), then \( \text{BBL}(X,\mathcal{J}) \subseteq \text{BBL}(X,\mathcal{K}) \).
If, in addition, for each $G \in \mathcal{J}$, there is an $F \in \mathcal{J}$ such that $G \subseteq F$, the two bounding collections $\mathcal{J}$ and $\mathcal{J}$ determine the same bounded Baer-Levi semigroup and will be called **interlaced**.

**Remark.** Among all bounding collections interlaced with $\mathcal{J}$, there exists a greatest one $\overline{\mathcal{J}}$ which is the set of ranges of elements of $\text{BBL}(X,\mathcal{J})$. A bounding collection $\mathcal{J}$ is greatest (i.e., $\mathcal{J} = \overline{\mathcal{J}}$) if and only if it has the property: if $F \in \mathcal{J}$ and $Y \subseteq F$ with $|Y| = |F \setminus Y|$, then $Y \in \mathcal{J}$.

A one-to-one function $\pi$ from $X$ onto $X$, i.e., a permutation of $X$, induces an isomorphism $\hat{\pi} : \text{BL}(X) \rightarrow \text{BL}(X)$ given by $a^{\hat{\pi}} = \pi^{-1}a\pi$. Now if $\mathcal{J}$ is a bounding collection for $X$ then $\mathcal{J}^{\hat{\pi}} = \{F^{\hat{\pi}} : F \in \mathcal{J}\}$ is a bounding collection for $X$ and $\text{BBL}(X,\mathcal{J})^{\hat{\pi}} = \text{BBL}(X,\mathcal{J}^{\hat{\pi}})$, i.e., the conjugate by permutations of bounded Baer-Levi semigroups are bounded Baer-Levi semigroups. Whether or not every pair of isomorphic bounded Baer-Levi subsemigroups of $\text{BL}(X)$ are conjugates is unknown to the authors. A related problem is whether or not every automorphism of $\text{BL}(X)$ is inner, i.e., induced by a permutation of $X$. We give below two non-isomorphic bounded Baer-Levi subsemigroups of $\text{BL}(X)$ where $X$ is a countable set.

**Example 4.6.** Let $X$ be the set $\{(m,n) : m$ and $n$ are
positive integers). Let $\mathcal{F}$ be any countable tower that satisfies the conditions (Bl) and (B2) of a bounding collection, say $\mathcal{F} = \{F_1, F_2, \ldots\}$ where $F_i \subseteq F_{i+1}$ for each $i$. Now there is a sequence, $e_1, e_2, \ldots$, of elements of $\text{BBL}(X, \mathcal{F})$ so that $e_i$ is the identity when restricted to $F_i$. Consequently if $a \in \text{BBL}(X, \mathcal{F})$ there is an $n$ so that $ae_n = a$. Clearly any isomorphic copy of $\text{BBL}(X, \mathcal{F})$ contains such a sequence. Conversely, if $\text{BBL}(X, \mathcal{F})$ is a bounded subsemigroup of $\text{BL}(X)$ and there is such a sequence, say $f_1, f_2, \ldots$, let $\mathcal{G}' = \{G_n \in \mathcal{G} : Xf_n \subseteq G_n \text{ for } n = 1, 2, 3, \ldots\}$. It is easy to see that $\mathcal{G}'$ is a bounding collection for $X$, and since $\mathcal{G} \subseteq \mathcal{F}$, $\text{BBL}(X, \mathcal{G}') \subseteq \text{BBL}(X, \mathcal{F})$. If $a \in \text{BBL}(X, \mathcal{G})$ then there is $n$ so that $af_n = a$ and consequently $Xa \subseteq G_n \in \mathcal{G}'$. Thus $\text{BBL}(X, \mathcal{G}') = \text{BBL}(X, \mathcal{G})$. In particular, we have seen that $\mathcal{F}$ contains a subset $\mathcal{G}'$ which is a tower and if $G \in \mathcal{F}$, there is $G' \in \mathcal{G}'$ so that $G \subseteq G'$. Now let $\mathcal{H}$ be the collection of subsets of $X$ that are finite unions of graphs of functions from the positive integers to the positive integers. Clearly $\mathcal{H}$ is a bounding collection that does not contain a tower as above. Consequently, $\text{BBL}(X, \mathcal{F})$ and $\text{BBL}(X, \mathcal{H})$ are not isomorphic.
CHAPTER V
CONGRUENCES ON BOUNDED BAER-LEVI SEMIGROUPS

In this chapter, the lattice of congruences on bounded Baer-Levi semigroups is discussed. If $X$ is an infinite set and $\mathcal{F}$ is a bounding collection for $X$, then we will assume that $\cup \mathcal{F} = X$ if not otherwise specified. The following somewhat justifies this assumption.

Proposition 5.1. Let $S = \text{BL}(X, \mathcal{F})$ with $\cup \mathcal{F}$ not necessarily all of $X$. Let $\rho = \{(a, b) \in S \times S : sa = sb \text{ for each } s \in S\}$. Then $\rho$ is a congruence on $S$ and $S/\rho$ is isomorphic to $\text{BL}(\cup \mathcal{F}, \mathcal{F})$.

Proof. It is well known and routine to verify that $\rho$ is a congruence. For $a \in S$ let $a\rho$ denote the $\rho$-class of $a$. Define $\hat{\phi} : S/\rho \to \text{BL}(\cup \mathcal{F}, \mathcal{F})$ by letting $(a\rho)\hat{\phi}$ be the restriction of $a$ to $\cup \mathcal{F}$. If $a\rho = b\rho$ and $x \in \cup \mathcal{F}$ there is $d \in \text{BL}(\cup \mathcal{F}, \mathcal{F})$ and $y \in \cup \mathcal{F}$ so that $(y)d = x$. Hence $(x)a = (y)da = (y)db = (x)b$. Thus $\hat{\phi}$ is a function. That $\hat{\phi}$ is a homomorphism follows from the fact
that if \( b \in S \) then \( Xb \subseteq U\mathcal{F} \). This also immediately implies that \( \mathcal{F} \) is one-to-one. The property (B2) for \( \mathcal{F} \) yields that \( \mathcal{F} \) is onto. □

For the remainder of this chapter, we will assume that \( X \) is a countably infinite set, i.e., \( |X| = \aleph_0 \), \( \mathcal{F} \) is a bounding collection for \( X \) with \( U\mathcal{F} = X \), and \( S = BBL(X,\mathcal{F}) \). The results given here have analogs in the cases where \( X \) has larger cardinal numbers.

If \( a \) and \( b \) are transformations of \( X \) then the difference set of \( a \) and \( b \), noted \( D(a,b) \), is the set \( \{x \in X : xa \neq xb\} \). The following can be routinely verified.

**Lemma 5.2.** If \( a,b \) and \( c \) are transformations of \( X \) then

(i) \( D(a,c) \subseteq D(a,b) \cup D(b,c) \);

(ii) \( D(ca,cb) = (D(a,b))c^{-1} \); and

(iii) if \( c \) is one-to-one, then \( D(ac,bc) = D(a,b) \). □

If \( T \) is a semigroup of transformations on \( X \), define the finite difference relation

\[ \delta = \{(a,b) \in T \times T : D(a,b) \text{ is finite}\} \]. If \( T \) is a semigroup of finite-to-one transformations, then \( \delta \) is a congruence on \( T \). Sutov [9] has shown that \( \delta \) is the only proper congruence on \( BL(X) \); in fact, the following proofs are similar to proofs of some of Sutov's results.

We will say that a subset \( A \) of \( X \) is scattered in
a bounding collection $\mathcal{F}$ if $A \cap F$ is finite for each $F \in \mathcal{F}$. Then let $\text{Sc}(\mathcal{F})$ denote the set of subsets of $X$ which are scattered in $\mathcal{F}$. Now let

$$\gamma = \{ (a,b) \in S \times S : D(a,b) \in \text{Sc}(\mathcal{F}) \}.$$

**Proposition 5.3.** The relation $\gamma$ is a congruence on $S$.

**Proof.** It is clear that $\gamma$ is reflexive and symmetric. Transitivity and right compatibility follow from 5.2. To see that $\gamma$ is left compatible, let $(a,b) \in \gamma$, $s \in S$, and $F \in \mathcal{F}$. Then there is $F' \in \mathcal{F}$ so that $X_s \subseteq F'$ and $D(sa,sb) \cap F = [D(a,b)]s^{-1} \cap F$. Thus $([D(a,b)]s^{-1} \cap F)s = D(a,b) \cap Fs \subseteq D(a,b) \cap F'$, and the latter set is finite by the definition of $\gamma$. Since $s$ is one-to-one it follows that $D(sa,sb)$ is finite. □

**Lemma 5.4.** If $a,b \in S$ and $D$ is an infinite subset of $D(a,b)$, then there is an infinite set $Y \subseteq D$ so that $Ya \cap Yb = \emptyset$.

**Proof.** Let $\alpha$ be the collection of subsets $Z$ of $D$ so that $Z_a \cap Z_b = \emptyset$. If $x \in D$, then $\{x\} \in \alpha$ and thus $\alpha \neq \emptyset$. Let $C$ be a tower in $\alpha$ and set $C = \cup C$. Then $C \in \alpha$ and by Zorn's lemma $\alpha$ has a maximal element, say $Y$. Suppose $Y$ is finite. Then $D \setminus Y$ is infinite and for $x \in D \setminus Y$, $(Y \cup \{x\})a \cap (Y \cup \{x\})b \neq \emptyset$. So either (1) $xa = yb$ for some $y \in Y$ or (2) $xb = za$ for some
There is a resulting function \( \alpha : D \setminus Y \to Y \) defined as follows: \( (x)\alpha = y \) in the case of (1), and if (1) fails then \( (x)\alpha = z \) as in (2). This function is finite-to-one. To see this let \( (x)\alpha = (x_1)\alpha = (x_2)\alpha = y \). Then

\[
\begin{align*}
xa &= yb \quad \text{or} \quad xb = ya, \\
x_1b &= yb \quad \text{or} \quad x_1b = ya_1, \quad \text{and} \\
x_2a &= yb \quad \text{or} \quad x_2b = ya.
\end{align*}
\]

In each resulting case \( \{x, x_1, x_2\} \) contains at most two elements. This is a contradiction. Hence \( Y \) must be infinite. □

**Lemma 5.5.** If \( \rho \) is a congruence on a semigroup \( T \), \( (a,b) \in \rho \), and for some \( s,t,q \in T \), \( at = aq \) and \( bs = bq \), then \( (as, bt) \in \rho \).

**Proof.** By compatibility, \( (as, bs) \) and \( (at, bt) \) are in \( \rho \). Further \( (bs, at) = (bq, aq) \in \rho \). Using transitivity \( (as, bt) \in \rho \). □

For elements \( a \) and \( b \) of a semigroup we will denote by \( \langle (a,b) \rangle \) the smallest congruence containing the pair \( (a,b) \).

**Lemma 5.6.** If \( a, b \in S \) and \( Xa \cap Xb = \emptyset \), then \( \langle (a,b) \rangle = S \times S \).
Proof. Let $c \in S$. Choose $F \in \mathcal{F}$ so that $X_a \cup X_b \cup X_c \subseteq F$. Choose $F', F'' \in \mathcal{F}$ so that $F \subseteq F' \subseteq F''$ and $|F'/F| = |F''/F'| = k_0$. Define $s \in S$ so that $as = a$ and $(X\setminus X_a)s \subseteq F'/F$. Define $t \in S$ so that $bt = c$ and $(X\setminus X_b)t \subseteq F''/F'$. There is a function $q \in S$ so that $q$ agrees with $s$ on $X_b$ and with $t$ on $X_a$, i.e., $bq = bs$ and $aq = at$. By 5.5, $(a,c) = (as, bt) \in \langle(a,b)\rangle$. Since $c$ was arbitrary, $\langle(a,b)\rangle = S \times S$. □

**Theorem 5.7.** The congruence $\gamma$ is the unique maximal one on $S$.

**Proof.** It suffices to show if $(a,b) \notin \gamma$ for some $a, b \in S$ then $\langle(a,b)\rangle = S \times S$. If $(a,b) \notin \gamma$, there is $F \in \mathcal{F}$ so that $D(a,b) \cap F$ is infinite. By 5.4, there is an infinite subset $Y$ of $D(a,b) \cap F$ so that $Ya \cap Yb = \emptyset$. Let $s \in S$ so that $X_s \subseteq Y$. Hence $Xsa \cap Xsb = \emptyset$ and consequently, by 5.6, $\langle(sa,sb)\rangle = S \times S$. Then $\langle(a,b)\rangle = S \times S$ since $\langle(sa,sb)\rangle \subseteq \langle(a,b)\rangle$. □

**Proposition 5.8.** If $T$ is a right simple, right cancellative semigroup, $\rho$ is a congruence on $T$, and $a \rho \neq \{a\}$ for some $a \in T$, then $b \rho \neq \{b\}$ for each $b \in T$.

**Proof.** If $(a,c) \in \rho$, $a \neq c$, there is $s \in T$ so that $as = b$. Thus $(b, cs) = (as, cs) \in \rho$ and $b = as \neq cs$. □
Recall that $\delta = \{(a,b) \in S \times S : D(a,b) \text{ is finite}\}$.

**Lemma 5.9.** If $s,t \in S$ and $(s,t) \in \delta$, then there is a sequence of elements $\{a_i\}_{i=1}^n$ of $S$ such that $a_1 = s$, $a_n = t$, and $|D(a_i,a_{i+1})| = 1$.

**Proof.** Let $D = D(s,t) = \{x_1,x_2,\ldots,x_k\}$. If $Ds \cap Dt = \emptyset$ let $a_1 = s$, $a_{k+1} = t$, and $a_i = s$ on $X \setminus \{x_1,\ldots,x_{i-1}\}$ and $a_i = t$ on $\{x_1,\ldots,x_{i-1}\}$ for $2 \leq i \leq k$. If $Ds \cap Dt \neq \emptyset$ we define $q \in S$ by $q = s = t$ on $X \setminus D$ and so that $Dq \cap (Ds \cup Dt) = \emptyset$. Now as in the first part define $a_1 = s$, $a_{k+1} = q$, $a_{2k+1} = t$ with $|D(a_i,a_{i+1})| = 1$ for $1 < i < 2k+1$. □

**Lemma 5.10.** If $\rho$ is a congruence on $S$, $\rho \subseteq \delta$ and $c,f \in S$ so that $c \rho \neq \{c\}$ and $|D(c,f)| = 1$, then $(c,f) \in \rho$.

**Proof.** Choose $d \in c \rho$ with $d \neq c$. Let $D(c,f) = \{x\}$. Define $s \in S$ so that $(x)s \in D(c,d)$ and $(X \setminus \{x\})s \subseteq X \setminus D(c,d)$. Since $D(c,y) = \{x\}$ one notes that $(x)f \notin Xc$ and $(x)c \notin Xf$. Consequently, there is $t \in S$ so that $sc = c$ and $sd \neq f$. Then $(sc,sd)t = (st,dt) = (c,f)$ and $(c,f) \in \rho$. □

**Theorem 5.11.** The congruence $\delta$ is the unique minimal one on $S$.

**Proof.** Let $\rho$ be a congruence on $S$, $\rho \neq \Delta = \{(s,s) \mid s \in S\}$ and $\rho \subseteq \delta$. If $(c,d) \in \delta$, by 5.9 there is a sequence
\([a_1, a_2, \ldots, a_n] \subseteq S\) so that \(a_1 = c\), \(a_n = d\), and
\(|D(a_i, a_{i+1})| = 1\) for \(i = 1, \ldots, n-1\). By 5.8, \(\mathcal{C} \neq \{c\}\)
and by 6.10, \((c, a_2) \in \rho\). Similarly, \((a_2, a_3), \ldots, (a_{n-1}, d) \in \rho\)
and thus \((c, a) \in \rho\). Hence \(\rho = \delta\) and \(\delta\) is minimal. To
see that \(\delta\) is unique minimal let \(\sigma\) be a congruence on \(S\),
\(\sigma \neq \Delta\), \((a, b) \in \sigma\) with \(a \neq b\). If \((a, b) \notin \gamma\) then by 5.7,
\(\sigma = S \times S\) and hence \(\delta \subseteq \sigma\). If \((a, b) \in \gamma\), choose \(F \in \mathcal{F}\)
so that \(D(a, b) \cap F \neq \emptyset\) and let \(s \in S\) so that \(X_s \subseteq F\) and
\(X_s \cap D(a, b) \neq \emptyset\). Thus \(D(a, b) s^{-1} = D(sa, sb)\) is finite and
nonempty, and \((sa, sb) \in \delta \cap \sigma\). Then \(\delta \cap \sigma \neq \Delta\) and hence
\(\delta \cap \sigma = \delta\), i.e., \(\delta \subseteq \sigma\). \(\square\)

Theorem 5.12. If \(\rho\) is a congruence on \(S\), \(\Delta \neq \rho \neq S \times S\), then

(i) \(E(S/\rho) = \emptyset\),

(ii) \(S/\rho\) is right simple

(iii) \(S/\rho\) has common right identities (CRIDS)

(iv) \(S/\rho\) is right cancellative, and

(v) \(S/\rho\) is left reductive if and only if \(\rho = \gamma\).

Proof. (i) If \(S/\rho\) has an idempotent then so does \(S/\gamma\), but
\(D(a, a^2) = X\{x:xa = x\}\) and \(\{x:xa = x\} \subseteq X_a\) is
bounded. Hence \((a, a^2) \notin \mathcal{F}\) and \(E(S/\rho) = \emptyset\). (ii) and
(iii) are trivial. (iv) \(S/\rho\) is right cancellative by
4.3 since (i), (ii) and (iii) hold. (v) If \(S/\rho\) is left
reductive and \((a, b) \in \gamma\setminus \rho\), \(a_\rho \neq b_\rho\) but \(sa = sb\) for
each \( s \in S \). Therefore \( \rho = \gamma \). Conversely, if \( \rho = \gamma \), since \( \gamma \) is maximum, \( S/\gamma \) is congruence free. But if \( e = \{(a, b) \in S/\rho \times S/\rho : sa = sb \text{ for each } s \in S/\rho \} \), then \( e \) is a congruence on \( S/\rho \) and so \( e = \Delta \) or \( e = S/\rho \times S/\rho \). But if \( e = S/\rho \times S/\rho \), \( ss^2 = ss \) for each \( s \in S/\rho \), and \( s^4 = s(ss^2) = s(ss) = s^2 \) or \( S/\rho \) has an idempotent, contradicting (i). This implies that \( e = \Delta \) or \( S/\gamma \) is left reductive. □

**Corollary 5.13.** A congruence-free, idempotent-free semigroup is reductive.

**Proof.** This is a corollary to the last part of the proof of 5.12. □

**Corollary 5.14.** Some bounded Baer-Levi semigroups contain subsemigroups that are congruence-free.

**Proof.** We note that since \( \gamma \) is a maximal congruence on \( S \) then \( S/\gamma \) is a congruence-free MIR-semigroup (Definition 3.1) and by 4.4, is embeddable in a bounded Baer-Levi semigroup. □

**Remark.** One can by the analogous argument show that some Baer-Levi semigroups contain congruence-free subsemigroups. In fact, if \( |X| > \aleph_0 \) one can argue that \( BL(X) \) and \( BBL(X, \mathcal{F}) \) have congruence-free subsemigroups. The author
does not know whether $\mathbf{BL}(X)$ or $\mathbf{BBL}(X,3)$ have congruence-free subsemigroups if $|X| = \aleph_0$.

A congruence $\rho$ on a semigroup $T$ will be called monogenic if $\rho = \langle (a,b) \rangle$ for some $a,b \in T$. Clearly the collection of monogenic congruences on $T$ is a basis for all congruences on $T$, i.e., every congruence is the supremum of a collection of monogenic congruences. For the remainder of this chapter we will give a partial description of the monogenic congruences on $S$. We recall [3, Sec. 10.1] that if $\mathcal{J}$ is a symmetric reflexive relation on a semigroup $T$ then the unique minimal congruence containing $\mathcal{J}$ is the transitive closure of $\{(a,b) : a = su \cdot t, b = sv \cdot t \text{ for some } (u,v) \in \mathcal{J}, s,t \in T\}$. ($T^1$ is $T$ if $T$ has an identity and is $T$ with an identity 1 adjoined otherwise.)

Proposition 5.15. If $(a,b) \in \gamma$ and $(c,d) \in \langle (a,b) \rangle$ then $D(c,d) \setminus D(a,b)$ is finite.

Proof. We first note that if $s,t \in S^1$ then $D(sat,sbt)$ is either finite or equal to $D(a,b)$. Consequently if $(c,d) \in \langle (a,b) \rangle$ there are elements $x_1, \ldots, x_n \in S$ so that $(c,x_1), (x_1,x_2), \ldots, (x_{n-1},x_n), (x_n,d)$ are all of the form $(sat,sbt)$ as above. Hence $D(c,d)$ is contained in the union of $D(a,b)$ and some finite set. Hence $D(c,d) \setminus D(a,b)$ is finite. □
We note that as a consequence of the above it is easy to see that \( \gamma \) is not monogenic on \( S \). Of course, \( \delta \), being minimal is monogenic, and \( \gamma \) being maximal implies that \( S \times S \) is monogenic.

If \( C \) and \( D \) are subsets of \( X \) we say \( C \) is almost contained in \( D \), written \( C \subseteq_a D \), if \( C \setminus D \) is finite. We say \( C \) and \( D \) are almost equal, written \( C =_a D \), if \( C \subseteq_a D \) and \( D \subseteq_a C \).

If \( A \subseteq X \) then we define \( \sigma_A = \{(a,b) \in \gamma : D(a,b) \subseteq_a A \} \).

**Proposition 5.16.** The relation \( \sigma_A \) is a congruence on \( S \).

**Proof.** The proof is similar to that of 5.3 for \( \gamma \) and is omitted. \( \square \)

We note that \( \sigma_\emptyset = \delta \). If \( (a,b) \in S \times S \) where \( D(a,b) = A \) is scattered in \( \emptyset \) and \( Aa \cap Ab = \emptyset \), then the next two propositions show that \( \sigma_A = \langle (a,b) \rangle \), i.e., \( \sigma_A \) is monogenic.

**Proposition 5.17.** Let \( a,b \), and \( A \) be as above. If \( c,d \in S \) and \( D(c,d) \subseteq A \), then \( (c,d) \in \langle (a,b) \rangle \).

**Proof.** There are functions \( s,t \in S \) so that \( as = c \), \( bt = d \), \( Abs \cap Aat = \emptyset \), and \( xs = xt \) for \( x \in X \setminus (Xa \cup Xb) \). Further, there is a function \( q \in S \) which agrees with \( s \) on \( Xb \), agrees with \( t \) on \( Xa \), and assumes the common
value of $s$ and $t$ on $X \setminus (Aa \cup Ab)$. Hence $bs = bq$ and
$at = aq$. By 5.5, $(c,d) = (as, bt) \in (a, b)$. □

**Proposition 5.18.** If $A$ is scattered in $\mathfrak{F}$, the congruence
$\sigma_A$ is monogenic on $S$.

**Proof.** Given $S c(\mathfrak{F})$, we can find $a, b$ in $S$ such
that $D(a, b) = A$ and $Aa \cap Ab = \emptyset$. Let $(c,d) \in \sigma_A$, i.e.,
$D(c,d) \subset A$ and let $c',d' \in S$ where $c' = c$ except on
$D(c,d) \setminus A$, $d' = d$ except on $D(c,d) \setminus A$, and $c' = d'$ on
$D(c,d) \setminus A$. Then $D(c',d') \subset A$ and by 5.17, $(c',d') \in (a, b)$.
Further, $(c,c')$, $(d,d') \in \delta \subset (a, b)$, and by transitivity,
$(c,d) \in (a, b)$. Hence $\sigma_A = (a, b)$. □

**Theorem 5.19.** If $A$ is scattered in $\mathfrak{F}$, then in the
lattice of congruences on $S$, $\sigma_A \land \sigma_B = \sigma_{A \cap B}$ and
$\sigma_A \lor \sigma_B = \sigma_{A \cup B}$.

**Proof.** First, $\sigma_A \land \sigma_B = \sigma_A \cap \sigma_B = \sigma_{A \cap B}$ is obtained with
a set-theoretic argument. Since $\sigma_{A \cup B}$ is a congruence and
$\sigma_A \cup \sigma_B \subset \sigma_{A \cup B}$ it follows that $\sigma_A \lor \sigma_B \subset \sigma_{A \cup B}$. Let
$(a,b) \in \sigma_{A \cup B}$ so that $(AUB)a \cap (AUB)b = \emptyset$. Define $x \in S$
by $(z)x = (z)a$ for $z \in A$, $(z)x = (z)b$ for $z \in B \setminus A$, and
$(z)x = (z)a = (z)b$ for $z \in X \setminus A \cup B$. Then $D(a,x) \subset B$
and $D(x,b) = A$. Thus $(a,x) \in \sigma_B$, $(x,b) \in \sigma_A$, and
hence $(a,b) \in \sigma_A \lor \sigma_B$. It follows by 3.17 that
$\sigma_{A \cup B} = (a, b) \subset \sigma_A \lor \sigma_B$. □
Corollary 5.20. The set \( \{ \sigma_A : A \subseteq X \text{ and } A \text{ is scattered in } \mathcal{F} \} \) is a distributive sublattice of the lattice of congruences on \( S \). □

If \( Y \) is any set we will note by \( \mathcal{J}(2^Y) \) the collection of infinite subsets of \( Y \).

We now define another collection of congruences on \( S \). If \( \mathcal{B} \subseteq \mathcal{J}(2^X) \) we define \( \rho_\mathcal{B} = \{(a,b) \in \mathcal{F} : \text{Ba = a, Bb for each } B \in \mathcal{B}\} \). If \( \mathcal{A} = \{B' \subseteq \mathcal{J}(2^X) : \rho_{B'} = \rho_B, \rho_{\mathcal{A}} = \rho_B \} \) then \( \rho_{\mathcal{A}} \) is the unique maximal collection \( \mathcal{C} \) so that \( \rho_{\mathcal{C}} = \rho_B \). We will thus assume that if we speak of a \( \rho_\mathcal{B} \) that \( \mathcal{B} \) is this unique maximal collection.

Proposition 5.21. If \( \mathcal{B} \subseteq \mathcal{J}(2^X) \) then \( \rho_\mathcal{B} \) is a congruence on \( S \).

Proof. That \( \rho_\mathcal{B} \) is a right compatible equivalence relation is clear. If \( B \in \mathcal{B}, s \in S \), and \((a,b) \in \rho_\mathcal{B} \subseteq \mathcal{F}, Bs \subseteq F \) for some \( F \in \mathcal{F}, \text{ and } D(a,b) \cap F \) is finite. Consequently \( Bs_a \text{ = } a, Bs_b \), and hence \((sa, sb) \in \rho_\mathcal{B} \). □

Proposition 5.22. For congruences \( \rho_\mathcal{B} \) and \( \rho_{\mathcal{B}'} \), \( \rho_\mathcal{B} \subseteq \rho_{\mathcal{B}'} \) if and only if \( \mathcal{B}' \subseteq \mathcal{B} \).

Proof. If \( \mathcal{B}' \subseteq \mathcal{B} \) then clearly \( \rho_{\mathcal{B}'} \subseteq \rho_\mathcal{B} \). If \( B' \in \mathcal{B}' \setminus \mathcal{B} \), since \( \mathcal{B} \) is maximal, there is a pair \((a,b) \in S \times S\) so
that $B_a = B_b$ for each $B \in \mathcal{B}$ but $B_a' \neq B_b'$. Consequently if $B' \notin \mathcal{B}$ then $\rho_{B'} \notin \rho_{B}$. □

**Theorem 5.23.** The collection $\{\rho_{\mathcal{B}} : \mathcal{B} \subseteq \mathcal{J}(2^X)\}$ is a distributive lattice of congruences on $S$.

**Proof.** If $\mathcal{B}, \mathcal{C} \subseteq \mathcal{J}(2^X)$ then it follows easily that

$$\rho_{\mathcal{B}} \land \rho_{\mathcal{C}} = \rho_{\mathcal{B} \cap \mathcal{C}}, \quad \rho_{\mathcal{B}} \lor \rho_{\mathcal{C}} = \rho_{\mathcal{B} \lor \mathcal{C}}.$$ 

It is also clear that $\rho_{\mathcal{B} \lor \mathcal{C}} \subseteq \rho_{\mathcal{B} \land \mathcal{C}}$. Moreover, if $\rho_{\mathcal{B}} \lor \rho_{\mathcal{C}} \subseteq \rho_{\mathcal{D}}$ for any $\mathcal{D} \subseteq \mathcal{J}(2^X)$ then by 5.22, $\mathcal{D} \subseteq \mathcal{B} \cap \mathcal{C}$. Thus $\rho_{\mathcal{B} \lor \mathcal{C}} \subseteq \rho_{\mathcal{B}} \lor \rho_{\mathcal{C}}$.

It follows that $\rho_{\mathcal{B}} \lor \rho_{\mathcal{C}} = \rho_{\mathcal{B} \lor \mathcal{C}}$. The distributivity follows. (We note that it is not claimed that this supremum $\rho_{\mathcal{B}} \lor \rho_{\mathcal{C}}$ is $\rho_{\mathcal{B} \lor \mathcal{C}}$ in the lattice of all congruences on $S$.) □

The next two results compare the $\rho_{\mathcal{B}}$'s and the $\sigma_A$'s. Following this we will consider congruences of the form $\rho_{\mathcal{B}} \cap \sigma_A$.

**Proposition 5.24.** Let $\mathcal{B} \subseteq \mathcal{J}(2^X)$ and $A \in \mathcal{S}(\mathcal{J})$. Then $\rho_{\mathcal{B}} \subseteq \sigma_A$ if and only if $\mathcal{J}(2^X \setminus A) \subseteq \mathcal{B}$.

**Proof.** Suppose $\rho_{\mathcal{B}} \subseteq \sigma_A$. Let $(a,b) \in \rho_{\mathcal{B}}$ and $J \in \mathcal{J}(2^X \setminus A)$. Then since $D(a,b) \subseteq_A A$ it follows that $Ja = aJb$ and hence $J \in \mathcal{B}$ since $\mathcal{B}$ is maximal.

Suppose $\mathcal{J}(2^X \setminus A) \subseteq \mathcal{B}$. If there is an infinite set $J \subseteq X \setminus A$ so that $J \subseteq D(a,b)$, by 5.3 there is an infinite set $Y \subseteq J$ so that $Ya \cap Yb = \emptyset$. Thus $Y \notin \mathcal{B}$. This
shows that \( D(a, b) \subseteq A \) . □

**Proposition 5.25.** If \( \sigma_A \) and \( \rho_B \) are as above then
\( \sigma_A \subseteq \rho_B \) if and only if for each \( B \in \mathcal{B}, B \subseteq A \).

**Proof.** Suppose \( B \in \mathcal{B} \) and \( B \cap A \) is infinite. Take \( a, b \in S \) so that \((B \cap A)a \cap (B \cap A)b = \emptyset\) and let \( a \) equal \( b \) otherwise. Hence \((a, b) \in \sigma_A \) and \((a, b) \notin \rho_B \). Consequently, \( \sigma_A \subseteq \rho_B \) implies for each \( B \in \mathcal{B}, B \subseteq A \).

Conversely, if \((a, b) \in \sigma_A\), then for each \( B \in \mathcal{B} \), \( Ba = aBb \) since \( B \subseteq A \). Hence \((a, b) \in \rho_B \). □

We now merge the \( \sigma_A \)'s and \( \rho_B \)'s into a single description. If \( \mathcal{B} \subseteq \mathcal{P}(2^X) \) and \( A \subseteq X \) define \( \tau_{A, \mathcal{B}} = \sigma_A \cap \rho_B \) and agree that \( \mathcal{B} \) is the unique maximal collection of subsets of \( X \) with respect to \( B \in \mathcal{B} \) implying that \( Ba = aBb \) for every \((a, b) \in \tau_{A, \mathcal{B}}, \) i.e., \( \mathcal{B} \) is the maximum \( C \) so that \( \tau_{A, \mathcal{B}} = \tau_{A, C} \). With the results of 5.24 in mind, we define \( S(\mathcal{B}) = \{ T \in \mathcal{P}(2^X) : \mathcal{P}(2^T) \subseteq \mathcal{B} \} \). We observe that since \( \mathcal{B} \) is maximal and \( \rho_B \subseteq \gamma \), then \( \mathcal{B} \subseteq S(\mathcal{B}) \). If we note by \( \mathcal{M} \) the \( \bigcup \{ \mathcal{P}(2^F) : F \in \mathcal{B} \} \) then for each representation \( \tau_{A, \mathcal{B}} \) we have \( \mathcal{M} \subseteq \mathcal{B} \). Then one can write \( \gamma = \tau_{X, \mathcal{M}} \) and \( \delta = \tau_{\emptyset, \mathcal{P}(2^X)} \).

**Proposition 5.26.** Let \( \tau_{A, \mathcal{B}} \) and \( \tau'_{A', \mathcal{B}'} \) be congruences as described above. Then \( \tau_{A, \mathcal{B}} \subseteq \tau'_{A', \mathcal{B}'} \) if and only if
(i) there exists $K \in S(\mathcal{B})$ so that $A \setminus K \subseteq aA'$, and
(ii) if $B' \in \mathcal{B}$ there is $B \in \mathcal{B}$ so that $B = aB' \cap A'$.

**Proof.** If (i) does not hold, for each $K \in S(\mathcal{B})$ the set $(A \setminus A') \setminus K$ is infinite. (One needs to recall that $\mathcal{J} \subseteq S(\mathcal{B})$ and the properties of $\mathcal{J}$.) Thus there is an infinite set $J \subseteq A \setminus A'$ so that $J \not\subseteq \mathcal{B}$. Consequently there is $(a,b) \in \tau_{A,\mathcal{B}}$ so that $Ja \neq_a Jb$. If $(a,b) \in \tau_{A',\mathcal{B}}$, then $D(a,b) \subseteq aA'$ and thus $Ja = aJb$. Thus $(a,b) \not\in \tau_{A',\mathcal{B}}$.

If (ii) does not hold there is $B' \in \mathcal{B}$ so that for each $B \in \mathcal{B}$, $B' \cap A' \neq B$. By maximality of $\mathcal{B}'$ there is $(a,b) \in \tau_{A,\mathcal{B}}$ so that $(B' \cap A)a \neq_a (B' \cap A)b$. Since $D(a,b) \subseteq aA$ we must have $B' \neq aB'$.

Conversely, let $(a,b) \in \tau_{A,\mathcal{B}}$. Hence $D(a,b) \subseteq aA$ and there is $K \in S(\mathcal{B})$ so that $A \setminus K \subseteq aA'$. It follows that $\mathcal{J}(2^X \setminus (A \setminus K)) \subseteq \mathcal{B}$ and by 5.24, $D(a,b) \subseteq aA \setminus K \subseteq aA'$. Now if $B' \in \mathcal{B}$, $(B' \cap A)a = a(B' \cap A)b$ and thus $B'a = aB'b$. Consequently, $(a,b) \in \tau_{A',\mathcal{B}}$. □

**Remark.** We do not know if the $\tau_{A,\mathcal{B}}$'s above yield all proper congruences on $S$, or even if they form a basis. One missing ingredient is the determination of the monogenic congruences $<(a,b)>$ where there is an infinite subset $B \subseteq D(a,b)$ so that $Ba = Bb$. As one sees by comparing
5.20 with 5.23 the fact that the \( \sigma_A \)'s are monogenic is helpful. Let \( a, b \in S \) and set \( D = D(a, b) \). Technically one can describe the maximal set \( D' \subset D \) so that \( D'a = D'b \).

To do this we let \((ba^{-1})^n\) and \((ab^{-1})^n\) be the \( n \)-fold compositions of \( ba^{-1} \) and \( ab^{-1} \) where all compositions are taken in the semigroup of partial transformations on \( X \).

Set \( D_n = D(ba^{-1})^n \cap D(ab^{-1})^n \) and \( D' = \bigcap_{n=1}^{\infty} D_n \).

**Proposition 5.27.** For \( a \) and \( b \) elements of \( S \), \( D'a = D'b \).

**Proof.** For \( d \in D' \), we will show that \( da \in D'b \). Note that for each \( n \in \mathbb{N} \), \( D(ba^{-1})^n a = D(ba^{-1})^{n-1} ba^{-1} a \subseteq D(ba^{-1})^{n-1} b' \).

Using this, we have \( da \in \bigcap_{n=1}^{\infty} D(ba^{-1})^n a \cap D(ab^{-1})^n a \subseteq \bigcap_{n=1}^{\infty} D(ba^{-1})^{n-1} b \cap D(ab^{-1})^{n-1} a \). So \( da = d'b \) for some \( n \in \mathbb{N} \).

\( d' \in \bigcap_{n=1}^{\infty} D(ba^{-1})^{n-1} \) and for each \( n \in \mathbb{N} \), \( d'b \in \bigcap_{n=1}^{\infty} D(ab^{-1})^n a \) or \( d' \in \bigcap_{n=1}^{\infty} D(ab^{-1})^n ab^{-1} \). It follows that \( d' \in D' \).

**Proposition 5.28.** If \( A \subseteq D \) and \( Aa = Ab \), then \( A \subseteq D' = \bigcap_{n=1}^{\infty} D_n \).

**Proof.** If \( x \in A \), \( xa = yb \) for some \( y \in A \) or \( x = yba^{-1} \); also \( xb = za \) for some \( z \in A \) or \( x = zab^{-1} \). Since \( y, z \in A \subseteq D_1 \), this implies \( x \in D_1 \). Now assume \( A \subseteq D_{n-1} \) for some \( n \in \mathbb{N} \) and \( x \in A \); then \( xa = db \) for \( d \in D_{n-1} \) so that \( xa \in D_{n-1}b \) or \( x \in D_{n-1}ba^{-1} \). Also \( xb = d'a \).
for some \( d^* \in D_{n-1} \) so \( xb \in D_{n-1} \) or \( x \in D_{n-1}ab^{-1} \). We have \( x \in D_{n-1}ab^{-1} \cap D_{n-1}ab^{-1} = D(ba^{-1})^{n-1}ba^{-1} \cap D(ab^{-1})^{n-1}ab^{-1} = D_n \).

**Proposition 5.29.** If \( D_n \) is finite for some \( n \), then \( \langle (a,b) \rangle = \sigma_D(a,b) \).

**Proof.** Suppose \( D(ba^{-1})^n \cap D(ab^{-1})^n \) is finite for some \( n \). Let \( K \) and \( M \) be infinite subsets of \( X \) such that \( K \cap M = \emptyset \), \( (Xa \cup Xb) \cap (K \cup M) = \emptyset \) and \( K \cup M \subseteq F \) for some \( F \in \mathcal{F} \). Let \( K = \bigvee_{i=1}^n K_i \), \( M = \bigvee_{i=1}^n M_i \), \( |K_i| = |M_i| = Y_0 \) for each \( 1 \leq i \leq n \).

There are elements \( s_1, s_2, \ldots, s_n \in S \) such that \( as_1 = b \) and \( (X \setminus X_a)s_1 \subseteq K_1 \), \( as_2 = bs_2 \), \( (X \setminus X_a)s_2 \subseteq K_2 \), \( as_n = bs_{n-1} \) and \( (X \setminus X_a)s_n \subseteq K_n \). Similarly, there are elements \( t_1, t_2, \ldots, t_n \in S \) such that \( at_1 = b \), \( (X \setminus X_b)t_1 \subseteq M_1 \), \( at_2 = bt_2 \), \( (X \setminus X_b)t_2 \subseteq M_2 \), \( \ldots \) and \( at_n = bt_n \), \( (X \setminus X_b)t_n \subseteq M_n \). We have \( (b, bs_1) = (a,b)s_1 \in \langle (a,b) \rangle \), \( (bs_1, bs_2) = (a,b)s_2 \in \langle (a,b) \rangle \) \( \ldots (bs_{n-1}, bs_n) = (a,b)s_n \in \langle (a,b) \rangle \) so that \( (b, bs_n) \in \langle (a,b) \rangle \). Similarly \( (a, at_n) \in \langle (a,b) \rangle \).

We need to show that \( D = D(at_n, bs_n) \). It suffices to show that \( D \cap \cap Dbs_n \) is finite.

If \( x \in D \), either \( xbs_n \in K_i \) for some \( 1 \leq i \leq n \) or \( xbs_n \in x(a^{-1}b)b^n = x(ba^{-1})^{nb} \). Also \( xat_n \in M_i \) for some \( 1 \leq i \leq m \) or \( xat_n \in x(ab^{-1})^{na} \). But \( L = D(ba^{-1})^{nb} \cap D(ab^{-1})^{na} \) is finite, since \( L \subseteq Xa \) and \( L^{-1} = (D(ba^{-1})^{nb} \cap D(ab^{-1})^{na})a^{-1} = \).
= D(ba^{-1})^n ba^{-1} \cap D(ab^{-1})^n a^{-1} \leq D(ba^{-1})^{n+1} \cap D(ab^{-1})^n \subseteq D(ba^{-1})^n \cap D(ab^{-1})^n, \text{ which is finite by hypothesis.} \qed
CHAPTER VI

SOME EXAMPLES OF LATTICES OF CONGRUENCES
ON RIGHT-CANCELLATIVE, IDEMPOTENT-FREE SEMIGROUPS

In Chapter V, we investigated the lattice of congruences on a particular class of subsemigroups of the Baer-Levi semi-group on a countable set. A more general question arises: If $C$ is the class of subsemigroups of all Baer-Levi semigroups or equivalently the class of right cancellative idempotent-free semigroups, then which lattices can be realized as the lattice of congruences of some element of $C$? It was noted in Chapter V that for $|X| > X_0$, $BL(X)$ has a congruence-free subsemigroup whose lattice of congruences is, of course, a two element chain. In this chapter, we will again assume that $|X| = X_0$ and examine the lattices of congruences for some other subsemigroups of $BL(X)$.

For each $s \in BL(X)$, the subsemigroup generated by $s$ is isomorphic to the semigroup $N$ of the positive integers under addition. It is easy to completely describe the lattice of congruences on $N$. For $n \in N$, let $\iota$
denote the ideal congruence $\triangle_N \cup [n, \infty) \times [n, \infty)$. Then for $k \in \mathbb{N}$, let $\mu_k$ denote the congruence

$$\{(a, b) \in \mathbb{N} \times \mathbb{N} : a = b \text{ modulo } k\}.$$

Finally, for $n, k \in \mathbb{N}$, let $\gamma_{n,k} = \iota_n \cap \mu_k$.

In what follows for integers $n$ and $m$, $n \lor m = \max\{n, m\}$ and $n \land m = \inf\{n, m\}$. The least common multiple of two positive integers $n$ and $n'$ is noted by $\text{lcm}(n, n')$ and the greatest common factor by $\text{gcf}(n, n')$. The proofs of the following two propositions are routine:

**Proposition 6.1.** Let $\mathcal{L} = \{\iota_n : n \in \mathbb{N}\}$. Then for $n, n' \in \mathbb{N}$,

$$\iota_n \leq \iota_{n'} \text{ if and only if } n \geq n'.$$

Consequently, $\iota_n \land \iota_{n'} = \iota_{n \land n'}$ and $\iota_n \lor \iota_{n'} = \iota_{n \lor n'}$.

**Proposition 6.2.** Let $\mathcal{L}' = \{\mu_k : k \in \mathbb{N}\}$. Then, if $k, k' \in \mathbb{N}$, $\mu_k \leq \mu_{k'}$ if and only if $k'$ divides $k$. Consequently

$$\mu_k \land \mu_{k'} = \mu_{\text{lcm}(k, k')} \text{ and } \mu_k \lor \mu_{k'} = \mu_{\text{gcf}(k, k')}.$$

**Proposition 6.3.** The set $\mathcal{Y} = \{\gamma_{n,k} : n, k \in \mathbb{N}\} \cup \{\triangle\}$ is the set of all congruences on $\mathbb{N}$.

**Proof.** For each pair of positive integers $n$ and $k$, $\gamma_{n,k}$ is the intersection of two congruences and is therefore a congruence. Conversely, let $\sigma \neq \triangle$ be a congruence on $\mathbb{N}$, and let $n$ be the smallest element of $\mathbb{N}$ such
that there is a positive integer \( q, q \neq n, (n,q) \in \sigma \).

Further, assume \( q \) is the smallest element of \( N \) so that \((n,q) \in \sigma \). Now, let \( k = q-n \) and it is routine to verify that \( \sigma = \gamma_{n,k} \). □

**Proposition 6.4.** In the lattice of congruences on \( N \),

\[ \gamma_{n,k} \leq \gamma_{n',k} \]

if and only if \( k' \) divides \( k \) and \( n \geq n' \).

**Proof.** If \( \gamma_{n,k} \leq \gamma_{n',k} \), then \( \iota_n \leq \iota_{n'} \), i.e., \( n \geq n' \).

Also \( \mu_k \leq \mu_{k'} \) and \( k' \) must divide \( k \). The converse is easy. □

Since \( \mathcal{L} \) and \( \mathcal{L}' \) are lattices, we can form the product lattice \( \mathcal{L} \times \mathcal{L}' \). Further, we will adjoin a universal lower bound or zero to \( \mathcal{L} \times \mathcal{L}' \) and use the notation \( \mathcal{L} \times \mathcal{L}'^0 \) to represent \( \mathcal{L} \times \mathcal{L}' \) with a zero, 0, adjoined. As shown below, this lattice \( \mathcal{L} \times \mathcal{L}'^0 \) is the lattice of congruences on \( N \). It follows from 6.1 and 6.2, that in the lattice \( \mathcal{L} \times \mathcal{L}'^0 \),

\[ (\iota_n,\mu_k) \wedge (\iota_{n'},\mu_{k'}) = (\iota_{n \wedge n'},\mu_{\text{lcm}(k,k')}) \]

and

\[ (\iota_n,\mu_k) \vee (\iota_{n'},\mu_{k'}) = (\iota_{n \vee n'},\mu_{\text{gcf}(k,k')}) \].

**Proposition 6.5.** The lattice of congruences \( \mathcal{L} \) on \( N \) is isomorphic to \( \mathcal{L} \times \mathcal{L}'^0 \).
Proof. Define \( \hat{\psi} : \mathcal{L} \times \mathcal{L}^{0} \to \mathcal{L} \times \mathcal{L}^{0} \) as follows: \( (\gamma_{n,k})^{\hat{\psi}} = (\iota_{n}, \mu_{k}) \) and \( (\Delta)^{\hat{\psi}} = 0 \). Clearly \( \hat{\psi} \) is one-to-one and onto \( \mathcal{L} \times \mathcal{L}^{0} \). If \( \gamma_{n,k} \leq \gamma'_{n',k'} \), then by 6.4, \( k' \) divides \( k \) and \( n' \geq n \). It follows from 6.1 and 6.2, that \( \iota_{n} \leq \iota'_{n} \) and \( \mu_{k} \leq \mu'_{k} \); therefore, \( (\iota_{n}, \mu_{k}) \leq (\iota'_{n}, \mu'_{k}) \). The diagonal relation \( \Delta \) is below every other element of \( \mathcal{L} \), and \( (\Delta)^{\hat{\psi}} = 0 \) which is below every element of \( \mathcal{L} \times \mathcal{L}^{0} \). So \( \hat{\psi} \) is order preserving. \( \square \)

If \( Y \) is any infinite subset of \( X \), then \( S_{Y} = \{ f \in \text{BL}(X) : (X)f \subset Y \} \) is a subsemigroup of \( \text{BL}(X) \). We want to investigate the lattice of congruences on \( S_{Y} \). Let \( e = \{ (a,b) \in S_{Y} \times S_{Y} : \text{for each } s \in S_{Y}, sa = sb \} \). This relation \( e \) is a congruence and identifies pairs of functions which agree on \( Y \).

**Proposition 6.6.** The semigroup \( S_{Y}/e \) is isomorphic to \( \text{BL}(Y) \).

**Proof.** For \( se \in S_{Y}/e \), define \( se \hat{=} Y | s \), the function which assumes the value of \( s \) on the set \( Y \). Then \( \hat{\psi} : S_{Y}/e \to \text{BL}(Y) \) is the required isomorphism as shown in 5.1.

**Proposition 6.7.** In the lattice of congruences on \( S_{Y} \), there is exactly one congruence \( \delta_{Y} \) such that \( e \not\subset \delta_{Y} \not\subset w \).

**Proof.** There is a one-to-one correspondence between
congruences on $S_Y/e$ and congruences on $S_Y$ which contain $e$ [2, 1.5]. By 6.6, $S_Y/e$ is isomorphic to $BL(Y)$, which has a three-element chain, $\{\Delta, \delta, \omega\}$ as its lattice of congruences. □

Note that $\delta_Y = \{(a,b) \in S_Y \times S_Y: D(a,b) \cap Y$ is finite$\}$.

Proposition 6.8. For $A \subset X\setminus Y$, set $\sigma_A = \{(a,b) \in S_Y \times S_Y: D(a,b) \subset A\}$. Then $\sigma_A$ is a congruence.

Proof. It follows easily, as in the proofs of Chapter V, that $\sigma_A$ is a right compatible equivalence relation. Then for $(a,b) \in \sigma_A$, $D(sa, sb) = [D(a,b)]_{s^{-1}} = \emptyset$, so $(sa, sb) \in \Delta \subset \sigma_A$. □

Proposition 6.9. The set $\{\sigma_A: A \subset X\setminus Y\}$ is a distributive lattice.

Proof. Note that $\sigma_A \leq \sigma_B$ if and only if $A \subset B$. Therefore, for $A, B \subset X\setminus Y$, $\sigma_A \wedge \sigma_B = \sigma_{A \cap B}$ and $\sigma_A \vee \sigma_B = \sigma_{A \cup B}$. Distributivity follows immediately. □

As on $BBL(X)$, there are several other simple types of congruences on $S_Y$. For $B \subset X\setminus Y$, we can define

$\beta_B = \{(a,b) \in S_Y \times S_Y: Ba = Bb\}$ and $\sigma'_B = \{(a,b): D(a,b) \subset B$ and $Y \setminus (Xa \cup Xb' : : infinite)\}$. Each of these relations is a congruence.

Proposition 6.10. For each $x \in X\setminus Y$, $\sigma'_{\{x\}}$ is a minimal
element in the lattice of congruences on $S_Y$.

**Proof.** It suffices to show that for $(a,b) \in \sigma'_x$, $\langle (a,b) \rangle = \sigma'_x$. Let $(c,d) \in \sigma'_x$. Then there is a function $s \in S_Y$ such that $as = c$ and $(x)bs = (x)d$; therefore, $(c,d) \in \langle (a,b) \rangle$. □

We now consider a few simple examples where $X \setminus Y$ is finite. Note that if $X \setminus Y$ is finite, then for each $B \subset X \setminus Y$, $\sigma'_B = \sigma_B$. As illustrated below, it is possible to construct numerous finite lattices as the lattice of congruences for some subsemigroup $S_Y$ of $BL(X)$.

**Example 6.11.** If $X \setminus Y = \{x\}$, then the lattice of congruences on $S_Y$ is a four-element chain.

**Proof.** From 6.7 we get a three-element chain $\{u, \delta_y, e\}$. As noted before, since $X \setminus Y$ is finite, $\sigma'_x = \sigma_x$. And since $\sigma_x = e$, $e$ is minimal by 6.10. □

**Example 6.12.** If $X \setminus Y = \{x,y\}$, the lattice of congruences on $S_Y$ has six elements, $\{w, \delta_y, e, \sigma(x), \sigma(y), \Delta\}$. It is the union of two chains $\{w, \delta_y, e, \sigma(y), \Delta\}$ and $\{w, \delta_y, e, \sigma(x), \Delta\}$ with $\sigma(x) \land \sigma_y = \Delta$ and $\sigma_x \lor \sigma_y = e$.

**Proof.** It suffices to show that $\sigma(x) \lor \sigma(y) = \sigma(x,y)$. Let $a, b, c \in S_Y$ such that $(a, b) \in \sigma(x)$, $(b, c) \in \sigma(y)$ and $xa \neq yc$. Then $(a, c) \in \sigma(x) \lor \sigma(y)$, $D(a, c) = \{x, y\}$,
and \( \{x,y\}_a \cap \{x,y\}_b = \emptyset \). Using the techniques of Chapter V, it can be shown that \( \langle (a,c) \rangle = \sigma(x,y) \), which implies that \( \sigma(x,y) \subseteq \sigma(x) \lor \sigma(y) \).

BIBLIOGRAPHY


VITA

Diana Lindsey was born in Jacksonville, Florida, on Friday, February 13, 1942. She received her elementary and secondary education in the public schools there, graduating in 1959. She received a Bachelor of Science degree from Louisiana State University in 1969, and a Master of Science degree in 1971. She is presently a candidate for the degree of Doctor of Philosophy in mathematics.
EXAMINATION AND THESIS REPORT

Candidate: Diana Lindsey
Major Field: Mathematics
Title of Thesis: Relations invariant under semigroup actions and bounded Baer-Levi semigroups

Approved:

Bernard Madison
Major Professor and Chairman

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Date of Examination: July 17, 1975