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ABSTRACT

Let G be a compact abelian group whose dual group Γ has a finite torsion subgroup. Let $\mu \in M(G)$ such that $|\mu|$ assigns no mass to any coset of any closed subgroup of G whose index is infinite. Then there is $d > 0$, dependent only on $\|\mu\|$, such that if for each $\gamma \in \Gamma$, $|\hat{\mu}(\gamma)| \geq 1$ or $|\hat{\mu}(\gamma)| \leq d$, then the set $\{\gamma : |\hat{\mu}(\gamma)| \geq 1\}$ is finite. An upper bound on the cardinality of this set is obtained in terms of $\|\mu\|$ and the cardinality of the torsion subgroup of Γ.

SECTION I

INTRODUCTION

G will denote a compact abelian group, Γ its dual group and $M(G)$ the measure algebra of finite Borel measures on G. We shall assume that Γ has a finite torsion subgroup. Let $\mu \in M(G)$ such that $|\mu|$ assigns no mass to any coset of any closed subgroup of G whose index is infinite. We prove that there is a number $d > 0$, dependent only on $\|\mu\|$, such that if for each $\gamma \in \Gamma$, $|\hat{\mu}(\gamma)| \geq 1$ or $|\hat{\mu}(\gamma)| \leq d$ then the set of γ such that $|\hat{\mu}(\gamma)| \geq 1$ is finite. We obtain an upper bound on the cardinality of this set in terms of $\|\mu\|$ and the cardinality of the torsion subgroup of Γ.

De Leeuw and Katznelson first proved this theorem for the circle group T [1, Lemma 2]. They proved that, for any $C > 0$, there is $d = d(C) < 10^{-2}$ satisfying the following: Suppose that $\mu \in M(T)$ is a continuous measure with $\|\mu\| \leq C$ and, for $|n|$ sufficiently large, $|\hat{\mu}(n)| < d$ or $\mathrm{Re}(\hat{\mu}(n)) > 1-d$; then $\{n : |\hat{\mu}(n)| \geq d\}$ is finite. Without a numerical bound on the cardinality of $\{n : |\hat{\mu}(n)| \geq d\}$

their method does not seem to generalize. Such a bound can be obtained by imitating Davenport's procedure in [2], if α is small enough and $|\hat{\mu}(n)| < \alpha$ or $|\hat{\mu}(n)| > 1-\alpha$ for all integers n. In [2] Davenport proves that if a trigonometric polynomial $p(x) = \Sigma\, \alpha(n)\exp(2\pi inx)$ has N coefficients of modulus at least one and all other coefficients equal to zero, then the L^1-norm of p is at least $8^{-1}(\log N)^{1/4}(\log\log N)^{-1/4}$.

For an arbitrary locally compact abelian group G, Glicksberg proved in [3] that if $\mu \in M(G)$ and 0 is isolated in $\{0\} \cup \hat{\mu}(\hat{G})$ then there is a compact subgroup H of G for which μ_H, the part of μ carried by the cosets of H, is the convolution of a non-zero idempotent and an invertible. He proved that $\mu_H = [(\sum_{\gamma\in\Lambda} \gamma)m_H] * \lambda$ where Λ is a finite subset of $\hat{G}$, m_H is the Haar measure on H, and $\lambda \in M(G)^{-1}$. For measures μ such that $\mu_H = 0$ when H is a closed subgroup of G of infinite index, the hypothesis that 0 is isolated in $\{0\} \cup \hat{\mu}(\hat{G})$ yields the conclusion that μ is a trigonometric polynomial. When G is compact and $\hat{G}$ has a finite torsion subgroup, one can then use Hewitt and Zuckerman's generalization [4] of Davenport's result [2] to estimate the cardinality of the set of $\gamma \in \Gamma$ such that $|\hat{\mu}(\gamma)| > 0$.

SECTION II

THEOREMS

In this section we state our theorems precisely and prove Theorem 2 assuming Theorem 1. In what follows $B = B(\mu) = \{\gamma \in \Gamma : |\hat{\mu}(\gamma)| \geq 1\}$.

Theorem 1. Let G be a compact abelian group whose dual group Γ has at most K torsion elements. Let $\mu \in M(G)$ such that $|\mu|$ assigns no mass to any coset of any closed subgroup of G whose index is infinite. If $\mathrm{card}(B(\mu)) > K(r+1)^{3r^2}$, then there exist γ_0 and $\gamma_{k,j}$, $1 \leq k \leq r^2$, $1 \leq j \leq r$, in $B(\mu)$ such that if $P_0 = \{\gamma_0\}$ and

$$P_{k+1} = P_k \cup \{\gamma_{k+1,j} : 1 \leq j \leq r\} \cup [\bigcup_{i<j} P_k + \gamma_{k+1,i} - \gamma_{k+1,j}] ,$$

then

$$\text{for } \gamma \in P_{k-1} \text{ and } i < j , \text{ we have } \gamma + \gamma_{k,i} - \gamma_{k,j} \notin B . \quad (1)$$

The proof of Theorem 1 requires several reductions which will be postponed to later sections. Theorem 1 was suggested by Theorem 1' which for the case of $G = T$ can be found in Davenport's paper [2].

Theorem 1'. Let G be a compact abelian group whose dual group Γ is an ordered group. Suppose that $\mu \in M(G)$ such that $(r+1)^{3r^2} < \text{card}(B(\mu)) < \infty$. Then there exist γ_0 and $\gamma_{k,j}$, $1 \leq k \leq r^2$, $1 \leq j \leq r$, in $B(\mu)$ such that if $P_0 = \{\gamma_0\}$ and

$$P_{k+1} = P_k \cup \{\gamma_{k+1,j} : 1 \leq j \leq r\} \cup [\bigcup_{i<j} P_k + \gamma_{k+1,i} - \gamma_{k+1,j}] ,$$

then

for $\gamma \in P_{k-1}$ and $i < j$ we have $\gamma + \gamma_{k,i} - \gamma_{k,j} \notin B$.

Since the proof of Theorem 1' found in [2] for $G = T$ works without change for the general case, we omit the proof of Theorem 1' .

Theorem 2. Let G be a compact abelian group whose dual group Γ has at most K torsion elements. Let $\mu \in M(G)$ such that $|\mu|$ assigns no mass to any coset of any closed subgroup of G whose index is infinite. Let r be a positive integer greater than 2 such that $4^{-1}(1-e^{-2})r^{1/2} > \|\mu\|$. If

$$|\hat{\mu}(\gamma)| \geq 1 \quad \text{or} \quad |\hat{\mu}(\gamma)| \leq 2^{-1} r^{3/2} r^{-2r^2} \qquad (2).$$

for all $\gamma \in \Gamma$, then the cardinality of $B(\mu)$ is at most $K(r+1)^{3r^2}$.

The proof of Theorem 2 is adapted from [2]. Originally condition (2) read

$$|\hat{\mu}(\gamma)| \geq 1 \quad \text{or} \quad |\hat{\mu}(\gamma)| \leq 2^{-1} r^{3/2} (r+1)^{-3r^2} .$$

Gordon Woodward suggested the improvement.

Proof of Theorem 2. Suppose $\text{card}(B(\mu)) > K(r+1)^{3r^2}$. Using γ_0 and $\gamma_{k,j}$, $1 \leq k \leq r^2$, $1 \leq j \leq r$, as given by Theorem 1, we define trigonometric polynomials $\varphi_0, \cdots, \varphi_{r^2}$ inductively as follows:

$$\varphi_0 = \overline{\sigma}(\hat{\mu}(\gamma_0))(\gamma_0, \cdot)$$

where $\sigma(x) = x|x|^{-1}$ for $x \neq 0$ and $\overline{\sigma}(x) = \overline{x}|x|^{-1}$.

$$\varphi_k = \varphi_{k-1}\{1 - 2r^{-2} - r^{-3} \sum_{i<j} \overline{\sigma}(\hat{\mu}(\gamma_{k,i}))\sigma(\hat{\mu}(\gamma_{k,j}))(\gamma_{k,i} - \gamma_{k,j}, \cdot)\}$$

$$+ r^{-5/2} \sum_j \overline{\sigma}(\hat{\mu}(\gamma_{k,j}))(\gamma_{k,j}, \cdot) .$$

Note that if $P_0, \cdots, P_{r^2}$ are defined as in the statement of

Theorem 1, each φ_k is a P_k-polynomial. By [2, Lemmas 1 and 2], $|\varphi_k(g)| \leq 1$ for all $g \in G$. Let $I_k = \int_G \varphi_k(g)d\mu(-g)$. Then $I_0 = |\hat{\mu}(\gamma_0)| \geq 1$. Moreover

$$\mathrm{Re}(I_k) \geq (1-2r^{-2})\mathrm{Re}(I_{k-1}) + \frac{1}{2} r^{-3/2} . \tag{3}$$

To compute (3) we write

$$I_k = (1-2r^{-2})I_{k-1} + r^{-5/2} \sum_j |\hat{\mu}(\gamma_{k,j})|$$

$$- r^{-3} \sum_{\gamma \in P_{k-1}} \sum_{i<j} \hat{\varphi}_{k-1}(\gamma)\overline{\sigma}(\hat{\mu}(\gamma_{k,i}))\sigma(\hat{\mu}(\gamma_{k,j}))\hat{\mu}(\gamma + \gamma_{k,i} - \gamma_{k,j})$$

$$= (1-2r^{-2})I_{k-1} + r^{-5/2} \sum_j |\hat{\mu}(\gamma_{k,j})| - r^{-3}A .$$

Thus,

$$\mathrm{Re}(I_k) \geq (1-2r^{-2})\mathrm{Re}(I_{k-1}) + r^{-3/2} - r^{-3}|A| .$$

Observe that each term of A is bounded in modulus by $2^{-1}r^{3/2}r^{-2r^2}$ by (1) and (2) and that the number of terms in A is at most $\frac{1}{2} r(r-1)\cdot \mathrm{card}(P_{k-1}) \leq r^2 \,\mathrm{card}\, P_{k-1} \leq r^{2k}$. Note that $\mathrm{card}(P_0) = 1 \leq r^{2\cdot 0}$ and that

$$\begin{aligned}
&\mathrm{card}(P_{k+1}) \\
&= \mathrm{card}(P_k \cup \{\gamma_{k+1,j} : 1 \leq j \leq r\} \cup [\bigcup_{i<j} P_k + \gamma_{k+1,i} - \gamma_{k+1,j}]) \\
&\leq \mathrm{card}\, P_k + r + \frac{1}{2} r(r-1)\mathrm{card}\, P_k \leq (1+r+\frac{1}{2}(r)(r-1))\mathrm{card}(P_k) \\
&\leq r^2 \,\mathrm{card}(P_k) ,
\end{aligned}$$

hence $\mathrm{card}(P_k) \leq r^{2k}$. It follows from (3) using induction that

$$\mathrm{Re}(I_k) \geq \frac{1}{4} r^{1/2} - (1-r^{-2})^k (\frac{1}{4} r^{1/2}-1) .$$

For $k = r^2$ we conclude that

$$|I_k| \geq \mathrm{Re}(I_k) \geq \frac{1}{4} r^{1/2} - (1-r^{-2})^{r^2} (\frac{1}{4} r^{1/2}-1)$$

$$\geq \frac{1}{4} r^{1/2} - e^{-2}(\frac{1}{4} r^{1/2}-1) \geq \frac{1}{4} r^{1/2}(1-e^{-2}) > \|\mu\|$$

although $|\varphi_k(g)| \leq 1$ for all $g \in G$. This contradiction establishes Theorem 1.

SECTION III

THEOREM 1 FOR $G = T$

In this section we prove Theorem 1 for $G = T$.

<u>Proof</u>. Let $\mu \in M(T)$ and $\mathrm{card}(B(\mu)) > (r+1)^{3r^2}$. We must exhibit γ_0 and $\gamma_{k,j}$, $1 \leq k \leq r^2$, $1 \leq j \leq r$, in $B(\mu)$ such that if $P_0 = \{\gamma_0\}$ and

$$P_{k+1} = P_k \cup \{\gamma_{k+1,j} : 1 \leq j \leq r\} \cup [\bigcup_{i<j} P_k + \gamma_{k+1,i} - \gamma_{k+1,j}] ,$$

then for $\gamma \in P_{k-1}$ and $i < j$ we have $\gamma + \gamma_{k,i} - \gamma_{k,j} \notin B$. By Theorem 1' we may assume $\mathrm{card}(B(\mu)) = \infty$. We suppose that $B(\mu) \cap Z^+$ is infinite.

Let γ_0 be any member of $B(\mu)$. Suppose that $\gamma_{k,j}$ in $B(\mu)$ have been chosen for $1 \leq j \leq r$, $1 \leq k \leq m-1$ $(m \geq 1)$ consistent with (1). Let $\gamma_{m,r}$ be any element of B such that

$$\gamma_{m,r} > |\gamma| \quad \text{for} \quad \gamma \in P_{m-1} . \qquad (4)$$

We suppose that $\gamma_{m,j}$ have been chosen in $B(\mu)$ for

$i+1 \leq j \leq r$ consistent with (1) and satisfying (4) in the place of $\gamma_{m,r}$. Suppose that no $\rho \in B$ can be chosen as $\gamma_{m,i}$ to satisfy (4) in the role of $\gamma_{m,r}$. Then for large $\rho \in B$ there are $\gamma \in P_{m-1}$ and $i+1 \leq j \leq r$ such that $\rho+\gamma-\gamma_{m,j} \in B$. If ρ is large enough $\rho+\gamma-\gamma_{m,j}$ will satisfy (4) in the place of $\gamma_{m,r}$. There exist $\gamma' \in P_{m-1}$ and $i+1 \leq j' \leq r$ so that $(\rho+\gamma-\gamma_{m,j}) + \gamma' - \gamma_{m,j'} \in B$. Let M be $2 \max\{\gamma_{m,j} : i+1 \leq j \leq r\}$. If $LM \leq \rho < (L+1)M$, then $(L-1)M < \rho+\gamma-\gamma_{m,j} < \rho$; thus there are at least L points in $B \cap [M,(L+1)M)$. We conclude that

$$\lim_{R\to\infty} \inf(2R+1)^{-1} \sum_{|n|\leq R} |\hat{\mu}(n)|^2 \geq (2M)^{-1} > 0$$

which implies that μ is not continuous, a contradiction. Thus some $\rho \in B$ satisfying (4) in the place of $\gamma_{m,r}$ can be chosen as $\gamma_{m,i}$. Inductively we obtain γ_0 and $\gamma_{k,j}$, $1 \leq k \leq r^2$, $1 \leq j \leq r$, as required.

SECTION IV

RANDOM WALKS IN Z^n

We shall prove Theorem 1 for groups $G = T^n$, $n > 1$, by induction on n . We require some geometrical lemmas concerning random walks in Z^n . In what follows, a hyperplane H in R^n will be called rational if for some $z \in Z^n$, $z+H$ is a subspace of R^n containing $n-1$ linearly independent vectors from Z^n . This is equivalent to saying that for some z in Z^n , $(z+H) \cap Z^n$ is isomorphic to Z^{n-1} .

Lemma 1. Let $n > 1$, $\{p_i\}$ be a sequence in Z^n and S be a finite subset of Z^n such that $p_{i+1}-p_i \in S$ for all i . Then for each positive integer N there are N integers j and a rational hyperplane H such that $p_j \in H$.

Before we prove Lemma 1, consider an example in Z^2 . We assume that $p_i \neq 0$ for all i and that $\theta = (\theta_1,\theta_2)$ is a cluster point of $\{\|p_i\|^{-1}p_i\}$ such that θ_1 and θ_2 are rational. Let H be the line through 0 and θ . Since θ_1 and θ_2 are rational there is a minimum distance $d > 0$

between translates of H by elements of Z^2. We can enumerate such translates of H as H_i so that H_i is a distance $d|i|$ from H. Suppose the lemma false for some N. Fix a point p_J. Among the first $(2k+1)(N-1)+1$ successors of p_J at least one, say p_j, occurs on an H_i with $|i| > k$. Let M be the maximum of $|\langle s,\theta\rangle|$ for $s \in S$, where $\langle\cdot,\cdot\rangle$ denotes the usual inner product in R^2. Consider the angle A formed between two lines, H and the line through 0 and p_j. We have

$$|\tan(A)| \geq (kd)(|\langle p_J,\theta\rangle| + (2k+1)(N-1)M)^{-1}$$

If k is large enough,

$$|\tan(A)| \geq \frac{1}{2} d(2MN)^{-1} = d(4MN)^{-1} .$$

Let H' and H'' be lines through 0 with rational slopes forming angles with H that are less than $\arctan d(4MN)^{-1}$, but on opposite sides of H. Since a subsequence of $\{\|p_i\|^{-1}p_i\}$ converges to θ, we have infinitely many choices for p_J in the same region between H' and H'' as H is. For each such p_J there is a successor p_j on the opposite side of H' or H''. We conclude that the broken-line path traced by the sequence $\{p_i\}$ crosses H' or H'' infinitely often. Since H' and H'' have rational slopes, a finite number of translates of them cover all the points in Z^n within a certain fixed distance. If we choose that distance to be the maximum of $\|s\|$ for $s \in S$, one of the translates contains p_i for infinitely many i.

This example suggested how to handle the general case. When θ_1 and θ_2 could not both be rational, we chose θ_1' and θ_2' close to θ_1 and θ_2 and attempted a similar argument. It became important to control the least common denominator Q' of θ_1' and θ_2' because our lower estimate for d was $(Q')^{-1}$. We were led to invoke the diophantine approximations given by Theorem VII of [5, p. 14]: If $\theta_1, \cdots, \theta_n$ are real numbers, then there are integers Q, $q_1, \cdots, q_n$ with Q arbitrarily large such that

$$Q^{1/n}\max\{|Q\theta_i - q_i| : 1 \leq i \leq n\} < n/(n+1) .$$

<u>Proof of Lemma 1</u>. We shall argue by contradiction to obtain a rational hyperplane H which the broken-line path traced by the sequence $\{p_i\}$ crosses infinitely often. A finite number of translates of any rational hyperplane H covers all the points in Z^n whose distance from H is bounded by a certain number. In our case, if we choose that number to be the maximum of $\|s\|$ for $s \in S$, some translate of H will contain p_i for infinitely many integers i, because there will be that many points p_i for which p_i and p_{i+1} are on

opposite sides of H .

Let $\theta = (\theta_1, \cdots, \theta_n)$ be a cluster point of $\{\|p_i\|^{-1} p_i\}$. Note that if $p_i = 0$ infinitely often, the lemma follows. We may therefore assume that $p_i \neq 0$ for all i . Since $\theta \neq 0$, we may assume $\theta_1 \neq 0$. By Theorem VII of [5, p. 14] there are integers $Q > 0$ and $q_1, \cdots, q_n$ such that

(a) $Q^{1/n} |Q\theta_i - q_i| < 1$ for $1 \leq i \leq n$;

(b) $Q^{1/n} > 64MNn^{1/2}$, where $M > 1 + \|s\|$ for all $s \in S$;

(c) $|q_i| \geq (1/2)|\theta_i Q|$ for $1 \leq i \leq n$;

(d) $Q^{-1}(q_1^2 + q_2^2)^{1/2} \leq 2\|\theta\| = 2$.

Let q be the vector $(q_1, \cdots, q_n)$ and w the vector $(-q_2, q_1, 0, \cdots, 0)$. Choose a rational number r so that $16n^{1/2}Q^{-(n+1)/n} < r < (4MNQ)^{-1}$, by (b). Let H' and H'' be the subspaces of R^n orthogonal to $rq-w$ and $rq+w$, respectively.

Assuming the lemma false, we shall show that the path traced by the sequence $\{p_i\}$ crosses either H' or H'' infinitely often. We shall estimate the ratio $|\langle p,w\rangle \langle p,q\rangle^{-1}|$ for some points p from the sequence. We shall show that the inequalities $|\langle p_i,w\rangle \langle p_i,q\rangle^{-1}| < r$ and $|\langle p_i,w\rangle \langle p_i,q\rangle^{-1}| \geq (4MNQ)^{-1}$ each have infinitely many solutions for the index. It then suffices to show that points satisfying the first inequality are separated from points satisfying the second by H' or H'' . Note that H' and H'' are the points where that ratio is r . For example, suppose that

$|\langle p_i,w\rangle\langle p_i,q\rangle^{-1}| < r$ and that $(\langle p_i,q\rangle) > 0$, but that $\langle p_j,w\rangle\langle p_j,q\rangle^{-1} \geq (4MNQ)^{-1}$ and $(\langle p_j,q\rangle) > 0$. Then

$$\langle p_i,rq-w\rangle = r\langle p_i,q\rangle - \langle p_i,w\rangle = \langle p_i,q\rangle(r - \langle p_i,w\rangle\langle p_i,q\rangle^{-1}) > 0$$

but

$$\langle p_j,rq-w\rangle = \langle p_j,q\rangle(r - \langle p_j,w\rangle\langle p_j,q\rangle^{-1}) < 0 .$$

Thus p_i and p_j are on opposite sides of H'. The other cases are handled similarly.

To see that $|\langle p_i,w\rangle\langle p_i,q\rangle^{-1}| < r$ infinitely often, we need only show that $|\langle\theta,w\rangle\langle\theta,q\rangle^{-1}| \leq 16n^{1/2}Q^{-(n+1)/n}$, since a subsequence of $\{\|p_i\|^{-1}p_i\}$ converges to θ and $16n^{1/2}Q^{-(n+1)/n} < r$. Since q is orthogonal to w,

$$|\langle\theta,w\rangle| = |\langle\theta-Q^{-1}q,w\rangle| \leq \|\theta-Q^{-1}q\|\|w\|$$

$$\leq n^{1/2}Q^{-(n+1)/n}(q_1^2+q_2^2)^{1/2} \leq 2n^{1/2}Q^{-1/n} .$$

On the other hand,

$$|\langle\theta,q\rangle| = Q^{-1}|\langle Q\theta,q\rangle| = Q^{-1}|\langle Q\theta-q,q\rangle + \langle q,q\rangle|$$

$$\geq Q^{-1}(\|q\|^2 - \|Q\theta-q\|\|q\|) \geq Q^{-1}\|q\|(\|q\| - n^{1/2}Q^{-1/n}) .$$

Since $|q_i| \geq \frac{1}{2}|\theta_i Q|$ for all i, $\|q\| \geq \frac{1}{2}Q$. Moreover, by (b), $\frac{1}{4}Q > n^{1/2}Q^{-1/n}$. Thus

$$Q^{-1}\|q\|(\|q\| - n^{1/2}Q^{-1/n}) \geq 1/2(\tfrac{1}{2}Q - n^{1/2}Q^{-1/n}) \geq (1/8)Q .$$

Thus

$$|\langle\theta,w\rangle\langle\theta,q\rangle^{-1}| \leq 2n^{1/2}Q^{-1/n}[(1/8)\cdot Q]^{-1} = 16n^{1/2}Q^{-(n+1)/n} .$$

To argue that $|\langle p_i,w\rangle\langle p_i,q\rangle^{-1}| \geq (4MNQ)^{-1}$ infinitely often we need to assume that the lemma is false. Let F be the subspace of R^n generated by the vector q and the vectors $e_i = (\delta_{k,i})$, for $3 \leq i \leq n$. Note that F is a rational hyperplane and that there is a minimum distance d between translates of F by elements of Z^n. Enumerate these translates as F_i for $i \in Z$ in such a manner that F_i is in distance $|i|d$ from F. Fix some integer J. If the lemma is false, among the points p_{J+i}, $1 \leq i \leq (2k+1)(N-1) + 1$, at least one, say p_{J+j}, occurs on an F_i with $|i| > k$. Since w is orthogonal to F and F is n-1-dimensional, the distance from p_{J+j} to F is equal to $|\langle p_{J+j},\|w\|^{-1}w\rangle|$. Thus

$$|\langle p_{J+j},w\rangle| > kd\|w\| .$$

Estimating $|\langle p_{J+j},q\rangle|$, we obtain

$$|\langle p_{J+j},q\rangle| \leq |\langle p_{J,q}\rangle| + [(2k+1)(N-1)+1]\max\{|\langle s,q\rangle| : s \in S\} .$$

Since

$$|\langle s,q\rangle| \leq |\langle s,q-Q\theta\rangle| + |\langle s,Q\theta\rangle|$$

$$\leq \|s\|\|q-Q\theta\| + Q\|s\| \leq \|s\|n^{1/2}Q^{-(n+1)/n} + Q\|s\|$$

$$\leq 1 + Q\|s\| < QM , \quad \text{by (b)} ,$$

we have

$$|\langle p_{J+j},q\rangle| \leq |\langle p_J,q\rangle| + (2k+1)NQM .$$

Thus

$$|\langle p_{J+j},w\rangle\langle p_{J+j},q\rangle^{-1}| \geq k\alpha\|w\|(|\langle p_J,q\rangle| + (2k+1)NQM)^{-1} .$$

If k is large enough

$$|\langle p_{J+j},w\rangle\langle p_{J+j},q\rangle^{-1}| \geq \alpha\|w\|(4NQM)^{-1} .$$

All we have left to show is that $\alpha\|w\| \geq 1$.

Let $u = (q_1,q_2,0,\cdots,0)$. For $z = (z_1,z_2,\cdots,z_n) \in Z^n$ let $Q(z)$ be the set of vectors v in R^n such that

(i) $z_i < (\langle v,e_i\rangle) \leq z_i + 1$ for $3 \leq i \leq n$;

(ii) $z_2\|u\|^2 < (\langle v,u\rangle) \leq (z_2+1)\|u\|^2$;

(iii) $z_1\|w\|^2 < (\langle v,w\rangle) \leq (z_1+1)\|w\|^2$.

The sets $Q(z)$ partition R^n and each has Lebesgue measure $\|w\|\|u\| = q_1^2 + q_2^2$. Since the cardinality of $Q(z) \cap Z^n$ is independent of z , each $Q(z)$ has $q_1^2 + q_2^2$ points of Z^n . For every hyperplane $z+F$ with $z \in Z^n$ and $0 < (\langle z,w\rangle) \leq \|w\|^2$ there is a point z' in $Z^n \cap Q(0,0,0,\cdots,0)$ such that $z'+F = z+F$. For example, if z is in $Q(0,v_2,\cdots,v_n)$, then $z' = z - (v_2u + v_3e_3 + \cdots + v_ne_n)$ is in $Q(0,\cdots,0)$. Since $z'-z \in F$, $z'+F = z+F$. Thus the cardinality of $Z^n \cap Q(0,\cdots,0)$ is an upper bound on the number of hyperplanes $z+F$ such that $0 < (\langle z,w\rangle) \leq \|w\|^2$ and $z \in Z^n$. The distance of a point v in $Q(0,\cdots,0)$ from F is given by $|\langle v,\|w\|^{-1}w\rangle|$. Since $|\langle v,w\rangle| \leq \|w\|^2$, that distance is bounded by $\|w\|$. Since the distance between two translates of F by elements of Z^n is always an integer multiple of the minimum distance d , we have

$$d(q_1^2 + q_2^2) \geq \|w\|$$

but since $\|w\|^2 = q_1^2 + q_2^2$, we have $d\|w\| \geq 1$, and we are done.

Lemma 2 is a restatement of Lemma 1 in the form that will be used in the next section.

<u>Lemma 2</u>. Let $n > 1$ and S a finite subset of Z^n . Let $N \in Z^+$ be given. Then there is an $N' \in Z^+$ such that if

$\{p_i\}$ is a finite sequence in Z^n of length N' and $p_{i+1} - p_i \in S$ for all $i < N'$, then there are N distinct integers j and a hyperplane H such that $p_j \in H$.

Proof. Suppose Lemma 2 is false for some N; then choose for each $N' \in Z^+$ a sequence $\{p_{N',i}\}$ of length N' which meets no rational hyperplane more than $N-1$ times. We shall inductively define an infinite sequence $\{p_i\}$ such that $p_{i+1} - p_i \in S$ for all i and such that $\{p_i\}$ meets no hyperplane more than $N-1$ times, a contradiction of Lemma 1. We may assume that $p_{N',1} = 0$ for all N'; let $p_1 = 0$. Then $p_{N',2} \in S$ for all N'. Since S is finite there is a $p_2 \in S$ such that for infinitely many N', $p_{N',1} = p_1$ and $p_{N',2} = p_2$. Suppose $p_1,\cdots,p_k$ have been chosen so that for arbitrarily large choices of N' we have $p_{N',i} = p_i$ for $1 \leq i \leq k$. Then among those sequences we have $p_{N',k+1} \in p_k + S$. Since $p_k + S$ is finite, there is a $p_{k+1} \in p_k + S$ such that for infinitely many N', $p_{N',i} = p_i$ for $1 \leq i \leq k+1$. The infinite sequence $\{p_i\}$ meets no rational hyperplane more than $N-1$ times because no initial segment of it can.

SECTION V

THEOREM 1 FOR $G = T^n$

In this section we prove Theorem 1 for $G = T^n$ by induction on n .

<u>Proof</u>. We may assume that $n > 1$ and that the theorem is true for $G = T^{n-1}$. If $\mathrm{card}(B(\mu)) < \infty$ we are done by Theorem 1', since Z^n , the dual group of T^n , is an ordered group under the lexicographic ordering. We therefore suppose that $\mathrm{card}(B(\mu)) = \infty$. Let π be the projection onto a coordinate such that $\pi(B(\mu))$ is unbounded. We may suppose that there exists $\{\gamma_i\} \subseteq B(\mu)$ such that $\lim \pi(\gamma_i) = +\infty$. Let $\gamma_0 \in B(\mu)$ be arbitrary and suppose that $\gamma_{k,j}$, $1 \leq j \leq r$, $1 \leq k \leq m-1$ $(m \geq 1)$, have been found in $B(\mu)$. It is consistent with (1) to let $\gamma_{m,r}$ be any member of $B(\mu)$ such that

$$\pi(\gamma_{m,r}) > \pi(\gamma) \quad \text{for all} \quad \gamma \in P_{m-1} . \tag{5}$$

We may suppose that $\gamma_{m,j}$ for $i+1 \leq j \leq r$ have been found to satisfy (5) in the role of $\gamma_{m,r}$ and (1), but that no $\gamma_{m,i}$ can be found to satisfy (5) in the role of $\gamma_{m,r}$. Let M be $2\max\{\pi(\gamma_{m,j}) : i+1 \leq j \leq r\}$. Let N' be the integer given by Lemma 2 for $S = \{\gamma-\gamma_{m,j} : \gamma' \in P_{m-1}$ and $i+1 \leq j \leq r\}$ and $N = (r+1)^{3r^2} + 1$. Consider any $\rho \in B$ such that $\pi(\rho) > N'M$. Since ρ satisfies (5) in the place $\gamma_{m,r}$, the reason ρ cannot serve as a $\gamma_{m,i}$ satisfying (5) is that for some $\gamma \in P_{m-1}$ and $i+1 \leq j \leq r$ we have $\rho+\gamma-\gamma_{m,j} \in B$. Note that $(N'-1)M < \pi(\rho+\gamma-\gamma_{m,j}) < \pi(\rho)$. Since $\rho' = \rho+\gamma-\gamma_{m,j}$ satisfies (5) also, there must be $\gamma' \in P_{m-1}$ and $i+1 \leq j' \leq r$ such that $\rho'+\gamma'-\gamma_{m,j'} \in B$. Also $(N'-2)M < \pi(\rho'+\gamma'-\gamma_{m,j'}) < \pi(\rho')$. If we let $\rho = \rho_1$, and $\rho' = \rho_2$, we can continue in this way to obtain a sequence $\{\rho_i\}$ of length N' such that $\rho_{i+1} - \rho_i \in S$ for $i < N'$. Note that the sequence is composed of distinct points. By Lemma 2 there are a rational hyperplane H and $N = (r+1)^{3r^2}+1$ integers j such that $\rho_j \in H$. Thus, since the ρ_i's are distinct, $\operatorname{card}(H \cap B(\mu)) > (r+1)^{3r^2}$.

Let $z \in Z^n$ such that $(z+H) \cap Z^n$ is isomorphic to Z^{n-1}. Let ψ be the quotient map from T^n to $T^n/[(z+H) \cap Z^n]^\perp$. Let $\nu = \psi(z\cdot u)$ be the measure on $\psi(T^n)$ such that $\nu(E) = (z\cdot u)(\psi^{-1}(E))$ for all Borel sets $E \subseteq \psi(T^n)$. Equivalently, ν is that measure in $M(\psi(T^n))$ such that for $\gamma \in (z+H) \cap Z^n$

$$\hat{\nu}(\gamma) = [\psi(z\cdot u)]^\wedge(\gamma) = (z\cdot u)^\wedge(\gamma) = \hat{\mu}(\gamma-z) . \qquad (6)$$

Then ν satisfies the hypotheses of Theorem 1 for $\psi(T^n)$ isomorphic to T^{n-1}. Let γ_0', $\gamma_{k,j}'$, $1 \leq k \leq r^2$, $1 \leq j \leq r$, be given in $B(\nu) = z + (B(\mu) \cap H)$ satisfying condition (1) of Theorem 1. Then redefine γ_0 to be $\gamma_0' - z$, $\gamma_{k,j}$ to be $\gamma_{k,j}' - z$ for $1 \leq k \leq m-1$, $1 \leq j \leq r$, and $\gamma_{m,j}$ to be $\gamma_{m,j}' - z$ for $i+1 \leq j \leq r$. If we let $\gamma_{k,j} = \gamma_{k,j}' - z$ for the remaining indices, we are done.

SECTION VI

THE GENERAL CASE

In this section we finish the proof of Theorem 1.

Proof. Let us assume that Γ is finitely generated. By [6, p. 49], $\Gamma = \Lambda \oplus Z^n$ for some non-negative integer n, where Λ is the torsion subgroup of Γ and hence by assumption $\text{card}(\Lambda) \leq K$. Since $\text{card}(B(\mu)) > K(r+1)^{3r^2}$ there is a $\lambda \in \Lambda$ such that $\text{card}((\lambda+Z^n) \cap B(\mu)) > (r+1)^{3r^2}$. Let ψ be the quotient map G to $G/(Z^n)^{\perp}$. Then $\nu = \psi(\overline{\lambda}\cdot\mu)$ is the measure on $\psi(G)$ satisfying

$$\hat{\nu}(z) = (\overline{\lambda}\mu)^{\wedge}(z) = \hat{\mu}(\lambda+z) \ , \quad \text{for} \quad z \in Z^n \ .$$

Since $\psi(G)$ is isomorphic to T^n and since ν satisfies the hypotheses of Theorem 1, there exist γ_0' and $\gamma_{k,j}'$, $1 \leq k \leq r^2$, $1 \leq j \leq r$, in $B(\nu) = -\lambda + [B(\mu) \cap (\lambda+Z^n)]$ satisfying (1), with γ_0', $\gamma_{k,j}'$ replacing γ_0 and $\gamma_{k,j}$, respectively. Then $\gamma_0 = \lambda+\gamma_0'$, $\gamma_{k,j} = \lambda+\gamma_{k,j}'$, $1 \leq k \leq r^2$, $1 \leq j \leq r$, satisfy (1) for μ .

In the fully general case, let S be a subset of $B(\mu)$ of cardinality $K(r+1)^{3r^2} + 1$. Let Λ be the subgroup of Γ generated by S . Let ψ be the quotient map from G to $G/\Lambda^{\perp}$. Let $\nu = \psi(\mu)$ be the measure on $G/\Lambda^{\perp}$ such that $\hat{\nu}(\lambda) = \hat{\mu}(\lambda)$ for all $\lambda \in \Lambda$. ν satisfies the hypotheses of Theorem 1 and $[\psi(G)]^{\wedge} = \Lambda$ which is finitely generated. As we have already proven, for ν there are γ_0 , $\gamma_{k,j}$, $1 \leq k \leq r^2$, $1 \leq j \leq r$, satisfying (1). Since $B(\mu) \cap \Lambda = B(\nu)$ and Λ is a subgroup, the same γ_0 and $\gamma_{k,j}$'s will work for μ .

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