Local Evolution Systems in General Banach Spaces.

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in

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by

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ABSTRACT

This dissertation is concerned with evolution systems and solutions to the evolution equation

\[ u'(t) = Atu(t), \quad u(0) = x, \]

where \((A(t))\) is a family of nonlinear, multi-valued operators with common domain \(D\), on a real Banach space \(X\).

We show that under certain conditions on \((A(t))\), a local evolution system can be constructed by using the method of product integration.

If \(X\) is uniformly convex we prove two existence and uniqueness theorems for the above evolution equation. The first is a generalization of a known result, and the second is a theorem about operators which have their domain decomposed into a non-decreasing sequence of sets.
INTRODUCTION

In this paper we define and construct a local evolution system \( U(t,s) \) from a family of nonlinear, multivalued operators \( \{ A(t) \} \) with common domain \( D \), in a real Banach space \( X \). In particular, in Chapter II we show that there is a family of operators \( \{ U(t,s) \} \) with domains \( \{ D(t,s) \} \) satisfying:

\[
U(t,s): D(t,s) \rightarrow \overline{D},
\]

\[
D \subseteq \bigcup_{s \leq t} D(t,s) \quad \text{for each } s,
\]

\[
D(t,r) \subseteq D(s,r) \quad \text{for } r \leq s \leq t,
\]

\[
U(t,t)x = x \quad \text{for } x \in D(t,t) \supseteq D, \quad \text{and}
\]

\[
U(s,r)D(t,r) \subseteq D(t,s) \quad \text{and } U(t,s)U(s,r) \supseteq U(t,r).
\]

We prove the existence of \( \{ U(t,s) \} \) by showing that

\[
\lim_{s \to t} \prod_{t} (I - \Delta t A(t))^{-1}x \quad \text{exists for } x \in D, \quad \text{where } "\lim" \text{ denotes the refinement limit. When this limit exists it is called the product integral, and } U(t,s)x \text{ is defined to be this product integral.} \]
Related problems have been studied by numerous authors, and we mention only a few. In 1963, Kato [13] dealt with the case of linear \( \{A(t)\} \). In 1967, he extended this work to the nonlinear case, but required that \( X \) be uniformly convex. Webb [14] in 1970 treated the nonlinear case continuous \( \{A(t)\} \). In 1971 Crandall and Pazy [6], and in 1973 Plant [8] considered the case of nonlinear, multivalued \( \{A(t)\} \) in a general Banach space. Both Crandall and Pazy, and Plant assume that \( X \subset X(1 - \lambda A(t)) \) for small positive \( \lambda \), whereas we assume the local condition that for each \( x \in X \) there is a ball \( B(x, r) \) so that \( B(x, r) \cap D \) is contained in \( X(1 - \lambda A(t)) \) for small \( \lambda \). This is called condition \( \mathcal{A} \) in the paper.

In Chapter II we consider the time dependent evolution equation

\[ u'(t) \in A(t)u(t), \quad u(s) = x, \]

and prove that when \( X \) is uniformly convex, a strong solution exists on \([s, T]\) if \( \{A(t)\} \) satisfies certain conditions. This result is a generalization of the work of Kreiss and Pazy [1] to the time dependent case.

In Chapter IV we define what it means to say that \( u \) is a solution of

\[ u'(t) \in A(t)u(t), \quad u(0) = x, \]

with respect to \( \{D_n\} \), where \( \{D_n\} \) is a non-decreasing
sequence of sets whose union is \( D \). We show that such solutions are unique. We also prove that if \( X \) is uniformly convex, then there exists solutions with respect to \( (\Omega_n) \) under certain assumptions on \( (A(t)) \). Solutions to the evolution equation in this chapter are local solutions.

Chapter I contains known results, preliminary definitions, and notations which are used in the sequel. The Appendix contains the proofs of some algebraic identities needed in Chapter II.
In this chapter we collect some basic definitions, elementary facts, known theorems, and some notations which will be used in the sequel. Most of the items found in this chapter are standard and appear in the existing literature. See, for example, [1], [4], [5], [10], [14], [18], and [23].

Throughout the paper X will denote a real Banach space with norm \( \| \cdot \| \).

**Multi-valued Operators**

**Definition 1.1:** If A assigns to each \( x \in X \), a subset \( Ax \), contained in \( X \), then A will be called a *multi-valued operator in X*. The *domain* of A, \( D(A) \), is the set \( \{ x \in X : Ax \neq \emptyset \} \). The *range* of A, \( R(A) \), is the set \( \bigcup_{x \in D(A)} Ax \).

We write \( A_C \) or \( A(S) \) for \( \bigcup_{x \in S} Ax \), \( S \subset X \). For subsets
1.1. \( \{x, y \mid x \in \mathcal{A}, y \in \mathcal{B}\} \) denotes the set \( \{x, y : x \in \mathcal{A}, y \in \mathcal{B}\} \).

1.2. \( \mathcal{C} \) or \( \mathcal{C}_0 \) then \( \mathcal{C} - \mathcal{C}_0 = \emptyset \). For a scalar \( \lambda \) and a subset \( \mathcal{A} \), \( \lambda \mathcal{A} \) denotes the set \( \{\lambda x : x \in \mathcal{A}\} \).

**Definition 1.1:** For two multi-valued operators, \( A \) and \( B \), we define their sum, product, and multiplication by a scalar respectively as follows:

\[
A + B)x = A x + B x, \quad D(A + B) = D(A) \cup D(B).
\]

\[
A \cdot B)x = A x \cdot B x, \quad D(A \cdot B) = \{x \in D(A): x \in D(B)\}.
\]

\[
\lambda A)x = \lambda A x, \quad D(\lambda A) = D(A).
\]

**Definition 1.3:** If \( A \) and \( B \) are two multi-valued operators in \( \mathcal{X} \), \( B \) is an extension of \( A \), denoted by \( A \subseteq B \), if \( A x \subseteq B x \) for each \( x \in \mathcal{X} \). If \( \mathcal{S} \subseteq \mathcal{X} \) then the restriction of \( A \) to \( \mathcal{S} \), denoted by \( A|_\mathcal{S} \), is defined by \( A|_\mathcal{S} x = A x \) if \( x \in \mathcal{S} \) and \( D(A|_\mathcal{S}) = D(A) \cap \mathcal{S} \).

**Definition 1.4:** If \( A \) is a multi-valued operator in \( \mathcal{X} \) then \( B \) is called the closure of \( A \), denoted by \( \overline{B} = A \), if the graph of \( B \) is identical to the closure of the graph of \( A \), where the graph of \( A \) is the set \( \{(x, y) : x \in D(A) \text{ and } y \in A x\} \).

**The Duality Mapping**

**Definition 1.5:** Let \( \mathcal{X}^* \) denote the dual space of \( \mathcal{X} \). Let \( <x, f> \) denote the value of \( f \) at \( x \), for \( x \in \mathcal{X} \) and \( f \in \mathcal{X}^* \). The duality mapping \( F \) is the mapping of \( \mathcal{X} \) into \( \mathcal{X}^* \).
defined by
\[ F(x) = \{ f \in X^* : \langle x, f \rangle - \|x\|^2 = \|f\|^2 \}. \]

**Proposition 1.6:** Let \( F \) denote the duality mapping of \( X \) into \( X^* \). If \( X^* \) is uniformly convex then \( F \) is single-valued and uniformly continuous on bounded sets.

**Proof:** See Kato's paper, [13].

**Proposition 1.7:** Let \( F \) denote the duality mapping of \( X \) into \( X^* \). Let \( x, y \in X \). Then \( \|x\| \leq \|x + ay\| \) for every \( a > 0 \) if and only if there is an \( f \in F(x) \) so that \( \langle y, f \rangle \geq 0 \).

**Proof:** See Kato's paper, [13].

**Proposition 1.8:** Let \( u \) be an \( X \)-valued function on an interval of real numbers. Suppose \( u \) has a weak derivative \( u'(s) \in X \) at \( s \). If \( \|u(\cdot)\| \) is also differentiable at \( s \), then
\[ \|u(s)\| \frac{d}{ds} \|u(s)\| = \langle u'(s), f \rangle \]
for every \( f \in F(u(s)) \).

**Proof:** See Kato's paper [13].

**\( \alpha \)-Dissipative Operators**

**Definition 1.9:** A multi-valued operator \( A \) in \( X \) is said to be \( \alpha \)-dissipative if \( \alpha \) is a non-negative real number and
whenever \(x, y, A, y^\bot Ax_1, y_1 Ax_1\), and \(\lambda > 0\).

Multi-valued \(\omega\)-dissipative operators are studied by numerous authors. See, for example, [5], [7], [19], [16], and [10].

**Proposition 1.10:** Suppose \(A\) is a multi-valued \(\omega\)-dissipative operator and \(0 < \lambda \omega < 1\), then

1. \((I-\lambda A)^{-1}\) is a function, and
   \[
   \|(I-\lambda A)^{-1}x - (I-\lambda A)^{-1}y\| \leq (1-\lambda \omega)^{-1}\|x-y\|
   \]
   for \(x, y \in R(I-\lambda A)\).

2. \(\|(I-\lambda A)^{-1}x - x\| \leq \lambda (1-\lambda \omega)^{-1}|Ax|\)
   for \(x \in R(I-\lambda A) \cap D(A)\), where \(|Ax| = \inf_{y \in Ax} \|y\|\).

3. \((I-\lambda A)^{-1}x - x \in \lambda A(I-\lambda A)^{-1}x\)
   for \(x \in R(I-\lambda A)\).

4. If \(\lambda > 0\) and \(\mu\) is a real number, then
   \[
   \frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} (I-\lambda A)^{-1}x \in R(I-\mu A),\] and
   \[
   (I-\lambda A)^{-1}x = (I-\mu A)^{-1}\left(\frac{\mu}{\lambda} x + \frac{\lambda - \mu}{\lambda} (I-\lambda A)^{-1}x\right)
   \]
   for \(x \in R(I-\lambda A)\).

**Proof:** See Crandall and Liggett, [4].
Time dependent families

Definition 1.11: Let \( \{A(t) : 0 \leq t \leq T\} \) be a family of multi-valued operators with common domain \( D \). \( \{A(t)\} \) is said to satisfy \textit{Condition X} if there is a non-decreasing function \( \mathcal{X} : [0, \infty) \to [0, \infty) \) such that

\[
|A(t)x| \leq |A(s)x| + |t-s| \mathcal{X}(|x|)(1 + |A(s)x|)
\]

for \( x \in D \) and \( 0 \leq s, t \leq T \).

Families of multi-valued operators which satisfy condition X are studied in [4], [5], [11], and [10].

Definition 1.12: Let \( \{A(t) : 0 \leq t \leq T\} \) be a family of multi-valued operators with common domain \( D \). \( \{A(t)\} \) is said to satisfy \textit{Condition C} if there is a non-decreasing function \( \mathcal{X} : [0, \infty) \to [0, \infty) \) and a \( \lambda_0 > 0 \) such that

\[
\|(-\lambda A(t)^{\frac{1}{2}} - \lambda A(s)^{\frac{1}{2}})^{\frac{1}{2}}x\| \leq \lambda |t-s| \mathcal{X}(|x|)(1 + |A(s)x|)
\]

whenever \( 0 \leq s, t \leq T \), \( 0 \leq \lambda \leq \lambda_0 \), and \( x \in D \).

Families of operators which satisfy Condition C are studied in [4], [5], and [10].

Definition 1.13: Let \( \{A(t) : 0 \leq t \leq T\} \) be a family of multi-valued operators with common domain \( D \). \( \{A(t)\} \) is said to satisfy \textit{Condition \( \mathcal{F} \) if whenever \( n \in (0, T) \) and \( (x_n, y_n) \in A(t_n), n=1,2,3, \ldots \), and \( t_n \to t, x_n \to x, \) and \( y_n \to y, \) then \( (x,y) \in A(t) \).
Multi-valued operators satisfying this type of condition are considered in [17].

**Product Integrals**

**Definition 1.14:** If \( \{T_i\} \) is any collection of functions on \( T \), then

\[
\prod_{i=1}^{k+1} T_i x = \prod_{i=1}^{k} T_i x \prod_{i=j}^{k} T_i x
\]

where

\[
\prod_{i=j}^{k} T_i x = x \text{ if } k < j.
\]

**Definition 1.15:** If \([a,b]\) is a finite interval, then a set of points

\[P = \{r_0, r_1, \ldots, r_n\}\]

satisfying the inequalities \( a = r_0 < r_1 < \ldots < r_{n-1} < r_n = b \) is called a partition of \([a,b]\). The collection of all possible partitions of \([a,b]\) will be denoted by \( \Theta[a,b] \).

**Definition 1.16:** Let \( F \) be a mapping from \([a,b] \times [0,\lambda_0]\) into the set of mappings on \( X \). For each partition \( R = \{r_i\}_{i=0}^{n} \) belonging to \( \Theta[a,b] \), with \( \|R\| < \lambda_0 \), let \( \prod_{i=1}^{n} F(r_i, \Delta r_i) x \) denote the product \( \prod_{i=1}^{n} F(r_i, \Delta r_i) x \), where \( \Delta r_i = r_i - r_{i-1} \). If there is a \( \omega \in X \) such that for every \( \epsilon > 0 \) there exists \( P_0 \in \Theta[a,b] \) so that
whenever $P$ is a refinement of $P_0$, then we write
\[ P^w_{\infty} \rightarrow P x \quad w \] in this case, $w$ is called the product integral of $P$ from $a$ to $b$ with respect to $x$.

Some theorems about product integrals are found in [9], [23], [24], and [22].

Miscellaneous

Finally we list some notations which are used in the paper.

(1) Let \( \{x_n\} \) be a sequence in $X$, then \( x_n \rightarrow x \) means that \( \{x_n\} \) converges to $x$ in the norm topology, whereas \( x_n \rightharpoonup x \) means that \( \{x_n\} \) converges to $x$ in the weak topology.

(2) $P(x,r) = \{y \in X : \|x-y\| \leq r\}$.

(3) $\sigma(u,n) = \sum_{i=1}^{n} \mu_i$ if \( \{\mu_i\} \) is a sequence of real numbers.

(4) $|S| = \inf\{|\|x\| : x \in S\}$ if $S \subseteq X$ and $S \neq \emptyset$.

(5) If \( \{A(t) : 0 \leq t \leq T\} \) is a family of multi-valued operators in $X$ with common domain $D$, then

(a) \( M(x) = \sup_{0 \leq t \leq T} |A(t)x| \) for $x \in D$,

(b) \( J(t, \lambda) = \left( I - \lambda A(t) \right)^{-1} \),
(c) $u(a, \lambda, \ell) \prod_{i=1}^{l} (a \circ (\lambda, 1), \lambda_{i})$ if $|\lambda_{i}|$ is a sequence of numbers so that $\sigma(\lambda, \ell) < T$.

**Definition 1.17:** Let $\{a_{k}\}$, $\{b_{k}\}$ be sequences of non-negative real numbers so that $a_{k} + b_{k} = 1$, $k = 1, 2, \ldots$.

Let $r$, $s$, and $t$ be non-negative integers. For $r \leq s$ and $0 \leq t \leq s-r+1$ let

$$A(r, s, t) = \{(x_{r}, x_{r+1}, \ldots, x_{s}) \in \mathbb{R}^{s-r+1} : \text{exactly } t \text{ of the components are 1 and the remaining components are 0}\}.$$  

Let $f : A(r, s, t) \to \mathbb{R}$ be given by

$$f(x_{r}, x_{r+1}, \ldots, x_{s}) = \prod_{i=1}^{s} \eta_{i} \text{ where } \begin{cases} \eta_{i} = a_{i} & \text{if } x_{i} = 1 \\ \eta_{i} = b_{i} & \text{if } x_{i} = 0 \end{cases}.$$  

Finally, define $[a, b]_{t} = \sum_{y \in A(r, s, t)} f(y)$.

For notational convenience if $s < r$ and $t > 0$ define

$$[a, b]_{t} = 1.$$  

As an illustration of the above definition, let

$$a_{1} \cdot b_{1} = 1 \text{ for } i = 1, 2, 3.$$  

Then

$$[a, b]_{1}^{l} = a_{1}b_{2} + b_{1}a_{2},$$

$$[a, b]_{1}^{3} = a_{1}b_{2}b_{3} + b_{1}a_{2}b_{3} + b_{1}b_{2}a_{3}, \text{ and}$$

$$[a, b]_{1}^{3} = a_{1}a_{2}a_{3} + a_{1}b_{2}a_{3} + b_{1}a_{2}a_{3}.$$
CHAPTER 11
LOCAL EVOLUTION SYSTEM

In this chapter we give the definition of a local evolution system and give conditions on a family
\[ A(t) : 0 \leq t \leq T \] of multi-valued operators with common domain \( D \) which will guarantee the existence of a local evolution system on \( D \).

The results in this chapter are similar to those found in [6], [11], and [22]. These papers are not concerned with the local problem however. The main result of this chapter shows the existence of the product integral as defined in Chapter 1. Crandall and Pazy in [6] have proven the existence of the product in a weaker sense, for example, they consider only regular refinements. A. T. Plant in [22] proves the existence in a stronger sense but his proofs use probabilistic methods which are not as elementary as those found herein.

Before we state our main results we need the following definitions.
Definition 2.1: Let \((A_t) : 0 \leq t \leq T\) be a family of multi-valued operators in \(X\). The family \((A_t)\) will be called \(\alpha\)-dissipative if \(A_t\) is \(\alpha\)-dissipative for each \(t \in [0, T]\).

\[\text{Condition } \mathcal{A}\text{ on the family } A(t) \text{ implies Condition } \mathcal{A}\text{ of [11]. See also [16].}\]

\[\text{Condition } \mathcal{A}\text{ on the family } A(t) \text{ implies Condition } \mathcal{A}\text{ of [11]. See also [16].}\]

Definition 2.3: Let \(X, 0 < T,\) and let \(D(t,s) \subseteq X\) for \(0 < t \leq T\). A family of operators \(U(t,s) : D(t,s) \to X\) is called a local evolution system on \(X\) if

1. \(D(t,s) \subseteq \bigcup_{s < t} D(t,s)\) for each \(s \in [0, t)\)
2. \(D(t,r) \subseteq D(s,r)\) for \(0 \leq r < s \leq t \leq T\)
3. \(U(t,t)x = x\) for \(x \in D(t,t)\)
4. \(U(s,r)D(t,r) \subseteq D(t,s)\) and \(U(t,s)U(s,r) = U(t,r)\) for \(0 \leq r < s \leq t \leq T\).

Remark: In [7], Dorroh gives the definition of a local transformation semi-group. Definition 2.3 may be
viewed as a generalization of that definition. Also, in a manner analogous to that in [7], we can show that a natural way for local evolution systems to arise is from solutions of time dependent nonlinear evolution equations.

**Theorem 2.4:** Let \( \{A(t) : 0 \leq t \leq T\} \) be a family of \( \rho \)-dissipative, multi-valued operators with common domain \( \mathbb{D} \), which satisfies Conditions \( B \), \( \chi \), and \( C \). Let \( x \in \mathbb{D} \), \( (\lambda_0, r) \in A_x \), and \( 0 \leq s \leq t \leq T \). Then there exists a real number \( b = b(s, x, r) \) such that if \( 0 < t - s < b \), then

\[
\mathbb{N}_s J(I, dI)x \text{ exists.}
\]

The proof of this theorem will be given after the proof of Theorem 2.5.

**Theorem 2.5:** Let \( \{A(t) : 0 \leq t \leq T\} \) be a family of \( \rho \)-dissipative, multi-valued operators with common domain \( \mathbb{D} \), which satisfies Conditions \( B \), \( \chi \), and \( C \). Let \( E(t, s) \) denote the subset of \( \mathbb{D} \) to which \( x \) belongs if and only if it is true that if \( s < t' < t \) and \( \varepsilon > 0 \) then there exists \( P \in \Theta[s, t'] \) such that (i) if \( P' \) is a refinement of \( P \) then \( \mathbb{N}_s J(P')x \) is defined, and (ii) if \( P' \) and \( P'' \) are refinements of \( P \), then \( \|\mathbb{N}_s J(P')x - \mathbb{N}_s J(P'')x\| < \varepsilon \). Then

\[
\|\mathbb{N}_s J(I, dI)x - \mathbb{N}_s J(I, dI)y\| \leq \exp(u(t-s))\|x-y\|
\]

for \( x, y \in E(t, s) \) and if we define \( U(t, s) : E(t, s) \to \mathbb{D} \) by
Let $U(t,s)x = \begin{cases} \pi^+_s(j,dl)x, & x \in E(t,s) \\ \lim_{n \to \infty} \pi^+_s(j,dl)x_n, & x \in E(t,s), \ x_n \to x \in E(t,s). \end{cases}$

Then $U(t,s)$ is a local evolution system on $\Theta$ with $E(t,s) = \overline{E(t,s)}$.

**Proof:** First note that for $x, y \in E(t,s)$ there exists $P \in \Theta[s,t]$ such that if $0 < ||x|| < 1$, and $P' = \{r_1, r_2, \ldots, r_n\}$ is a refinement of $P$, then

$$||\pi^+_s(j,dl)x - \pi^+_s(j,dl)y|| \leq \exp(||t-s||)||x - y||,$$

where $u_i = r_i - r_{i-1}$. Therefore, for $x, y \in E(t,s)$

$$||\pi^+_s(j,dl)x - \pi^+_s(j,dl)y|| \leq \exp(||t-s||)||x - y||,$$

and the first statement of the theorem is proved.

Now we let $D(t,s) = E(t,s)$ and check the four properties of a local evolution system.

**Proof of (i):** Let $s \in [0,T)$ and $x \in D$. Choose $t-s < b(s,x,r)$ (see Theorem 2.4) for some $r > 0$. By Theorem 2.4 it follows that $x \in E(t,s)$, and hence $D \subset \bigcup_{s \leq t} D(t,s)$.

**Proof of (ii):** Choose $x \in E(t,r)$ and suppose that $r < s < t$, $s' \in [r,s]$, and $e > 0$. Because $x \in E(t,r)$, there exists a $P \in \Theta[r,s]$ such that if $P'$ is a refinement of $P$,
then $\pi_{P'}^{S'}(P')x$ is defined, and also if $P'$ and $P''$ are refinements of $P$, then

$$\|\pi_{P'}^{S'}(P')x - \pi_{P''}^{S''}(P'')x\| < \epsilon.$$

Hence $x \in E(s, r)$, and it follows that $D(t, r) \subseteq D(s, r)$.

**Proof of (iii):** $U(t, t)x = \pi_{\mathcal{U}}^{(\mathcal{U})}(t)x = \{ x : x \in D(t, t) \} \subseteq \mathcal{U}$, hence also for $x \in D(t, t \cdot)$.

**Proof of (iv):** We prove that $U(s, r)x \in E(t, r) \subseteq E(t, s)$. Then by definition of $U(s, r)$ the result will follow for $E(t, r)$. Let $x \in E(t, r)$. Choose a sequence $\{R_n\} \subseteq [r, t]$ such that $x \in R_n$, and so that if $P_n = R_n \times [r, s]$, then any refinement $\pi'_n$ of $P_n$ and any refinement $P'_n$ of $P_n$ have the property that

$$\|\pi_{P_n}^{(\mathcal{U})}(R'_n) - U(t, r)x\| < \frac{1}{n}, \text{ and}$$

$$\|\pi_{P_n}^{(\mathcal{U})}(P'_n) - U(s, r)x\| < \frac{1}{n}.$$  

Let $P'_n$ be a refinement of $P_n$ for $n=1, 2, \ldots$, and $y_n = \pi_{P'_n}^{(\mathcal{U})}(P'_n)x$, $n=1, 2, \ldots$. We show that $y_n \in E(t, s)$ for large $n$, and noting that $y_n = U(s, r)x$, it will follow that $U(s, r)x \in E(t, s)$. Let $\epsilon > 0$ be given, and choose $n$ large enough so that $\frac{1}{n} < \epsilon$. Let $Q_n = R_n \times [s, t]$. Suppose that $Q'_n$ is a refinement of $Q_n$, then $\pi_{Q_n}^{(\mathcal{U})}(Q'_n) x$ is defined since $Q'_n \subseteq Q_n$ is a refinement of $R_n$. However,
Thus $\pi_{s}^{t}(Q_{n}'')y_{n}$ is defined for each refinement $Q_{n}'$ of $Q_{n}$.

Next, let $Q_{n}'$ and $Q_{n}'''$ be refinements of $Q_{n}$, then

$$\|\pi_{s}^{t}(Q_{n}')y_{n} - \pi_{s}^{t}(Q_{n}'')y_{n}\|$$

$$\leq \|\pi_{s}^{t}(Q_{n}'')y_{n} - U(t,r)x\| + \|U(t,r)x - \pi_{s}^{t}(Q_{n}'')y_{n}\|$$

$$\leq \frac{1}{n} + \frac{1}{n}$$

because $Q_{n}'$ and $Q_{n}''$ are refinements of $K_n$. Hence, by definition of $E(t,s)$, $y_{n} \in E(t,s)$, and thus it follows that $U(s,r)x \in \overline{E(t,s)}$.

Now we prove that $U(t,s)U(s,r)x = U(t,r)x$ for $x \in E(t,r)$. Let $y_{n}$, $R_{n}$, $P_{n}$, and $Q_{n}$ be as above, then by definition of $U(t,s)$ we know that $U(t,s)y_{n} \to U(t,s)U(s,r)x$.

We prove that $U(t,s)y_{n} \to U(t,r)x$. Let $\epsilon > 0$ be given. Choose $n$ large enough so that $\frac{1}{n} < \frac{\epsilon}{8}$. Choose $Q_{n}' \in \Theta(s,t)$ so that $\|\pi_{s}^{t}(Q_{n}')y_{n} - U(t,s)y_{n}\| < \frac{\epsilon}{8}$. (We may do this since $y_{n} \in E(t,s)$.) Now for each such $n$, we have that
\[ \|u(t,s)y_n - \bar{t},r\| + \|u(t,s)y_n - \pi_t^i \gamma_i y_n\| \]
\[ \leq \|\pi_t^i ((\bar{t},r)^{\partial E})x - \bar{t},r\| \]

where $\bar{q}_n = q_n \cup \bar{q}_i$. This is the desired result.

In order to prove Theorem 2.4, we need the following six lemmas.

**Lemma 2.5:** Let $\{A_t: 0 \leq t \leq T\}$ be a family of $\lambda$-dissipative, multi-valued operators with common domain $I$, satisfying conditions $\beta$ and $\chi$. Let $x_0\in \sigma(0,1], \lambda_0, r\in A_x, \lambda_0 r\in A_x$, $\delta(s,x,r) = \min\{T-s, r(\exp(s(T-s))M(x))^{-1}\}$, and let $r_i \downarrow 0$ be a partition of $[s,s+\delta]$, and let $u_1 r_i = r_{i+1}$ for $i = 1,2,\ldots,n$. Suppose that $0 \leq u_1 \leq 1$ and that $0 < u_1 \leq \frac{1}{\exp}$ for $i = 1,2,\ldots,n$. If $\sigma(u,t) \downarrow b$ for $t = 1,2,\ldots,n$ then $u(s,u,t)x$ is defined, $u(s,u,t)x : = \delta(x,r)\cap D$, and
\[ \|u(s,u,t)x - x\| = \sigma(u,t)x \|
\]

**Proof:** The proof will be by induction on $i$. Let $i = 1$. Since $(\lambda_0, r)\in A_x, \delta(x,r,0) \subset R(1-u_1 A(r_1))$, so $u(s,u,1)x$ is defined. Since $\{A_t: 0 \leq t \leq T\}$ is $\lambda$-dissipative and satisfies condition $\chi$ we have that
\[ \|u(s,u,1)x - x\| \leq u_1(1-u_1)^{-1}|A(r_1)x| \]
\[ \leq u_1 \exp(2u_1)M(x) \]
\begin{align*}
&b(s, x, r) \exp (\omega_{l+1} M(x)) \\
&\leq r.
\end{align*}

Noting that \( \sigma(1, u_1) \in \mathcal{D}(A(1)) = \Omega \), we conclude that

\( u(s, u, 1)x = (x, r \in \Omega) \). Next assume the statement is true for \( l \) and show that it is true for \( l+1 \). Suppose that

\[ a(s, u, l+1) = b(s, x, r) \].

The induction hypothesis guarantees that \( u(s, u, l)x = (x, r \in \Omega) \) and that

\[ \|u(s, u, l)x - x\| \leq a(s, u, l)M(x) \].

Thus \( (s_0, s_1) \in \mathcal{A} \) implies that \( u(s, u, l)x \in \mathcal{K}(1-u_{l+1}A(r_{l+1})) \)

and \( u(s, u, l+1)x \) is defined. Again using the fact that

\( (s_0, s_1) \in \mathcal{A} \) we observe that \( u(s, u, l)x \) is defined.

Therefore,

\[ \begin{align*}
\|u(s, u, l+1)x - x\| &\leq \|u(s, u, l)x - x\| + \|u(s, u, l+1)x - x\| \\
&\leq (1-u_{l+1}u_{l+1})^{-1}\|u(s, u, l)x - x\| + u_{l+1}(1-u_{l+1}u_{l+1})^{-1}M(x) \\
&\leq \exp (2\omega_{l+1}a(s, u, l)M(x)) + \mu_{l+1}\exp (2\omega_{l+1}M(x)) \\
&= \sigma(s, u, l+1)\exp (2\omega_{l+1}M(x)) \\
&\leq b(s, x, r)\exp (2\omega_{l+1}M(x)) \\
&\leq r.
\end{align*} \]

Finally, noting that \( u(s, u, l+1)x = J(r_{l+1}, u_{l+1})u(s, u, l)x \)
belongs to $\mathcal{D}$, we conclude that $u(s, \mu, \ell+1) x \in \mathcal{D}(x, r) \cap \mathcal{D}$, and all the statements of the lemma have been proved.

Lemma 2.7: Let the hypotheses of Lemma 2.6 be satisfied. If $\sigma(\mu, \ell) < b$ for $\ell = 1, 2, \ldots, n$, then

$$|A(r_\ell)u(s, \mu, \ell)x| \leq \exp(\omega \sigma(\mu, \ell)) M_\ell,$$

where

$$M_\ell = M(x) \prod_{i=1}^{\ell} (1 + \mu_i L) + \sum_{i=1}^{\ell} \mu_i L \prod_{k=i+1}^{\ell} (1 + \mu_k L),$$

and $L = \mathcal{E}(r^+\|x\|)$. Furthermore, there is a constant $R = R(s, x, r)$ so that if $\sigma(\mu, \ell) < b$, then

$$|A(r_\ell)u(s, \mu, \ell)x| \leq R$$

for $\ell = 1, 2, \ldots, n$.

Proof: We prove the statement by induction on $\ell$. For $\ell = 1$ we have from Lemma 2.6 and Proposition 1.10 that

$$|A(r_1)u(s, \mu, 1)x|$$

$$= |A(r_1)x|$$

$$\leq \exp(\omega \mu_1)|A(r_1)x|$$

$$\leq \exp(\omega \mu_1)\{ |A(r_1)x| + \mu_1 L \cdot |A(r_0)x| \}$$

$$\leq \exp(\omega \mu_1)\{ M(x) + \mu_1 L \}$$

$$= \exp(\omega \mu_1) M_1.$$

Assume that the statement is true for the natural number $\ell$, and show that it is true for $\ell+1$. 
To prove the existence of the constant $R$, choose $\ell \leq n$ and assume that $c(u, \ell) < b(s, x, r)$. Then

$$|A(r_{\ell+1})u(s, \mu, \ell+1)x|$$

$$\leq \mu_{\ell+1}^{-1} \exp \left( 2u \right) (\mu_{\ell+1}) u(s, \mu, \ell) x - u(s, \mu, \ell) x$$

$$\leq (1 - \mu_{\ell+1}w)^{-1} |A(r_{\ell+1})u(s, \mu, \ell) x|$$

$$\leq \exp(2u\mu_{\ell+1}) \left( |A(r_{\ell})u(s, \mu, \ell) x| + \mu_{\ell+1} L (1 + |A(r_{\ell})u(s, \mu, \ell) x|) \right)$$

$$\leq \exp(2u\mu_{\ell+1}) \left( \exp(2\sigma(u, \ell)) M \right)$$

$$\leq \exp(2u\mu_{\ell+1}) \left( \exp(2\sigma(u, \ell)) M + \mu_{\ell+1} L \right)$$

$$\leq \exp(2u\mu_{\ell+1}) \exp(2\sigma(u, \ell)) M_{\ell+1}$$

$$= \exp(2\omega(u, \ell + 1)) M_{\ell+1}.$$
Lemma 1.6: Let the hypotheses of Lemma 1.6 be satisfied. Let $u_1, u_2, \ldots, u_n = \lambda$ and $0 < \lambda < \lambda_0$. In addition, let $a_k = \frac{u_k}{x}$, $b_k = 1 - a_k$, $s_k = k\lambda$, $m = b$, for $k = 1, 2, \ldots, n$, and

$$d_{m,n} = \| \prod_{k=1}^{m} J(r_k, u_k)x - \prod_{k=1}^{n} J(s_k, \lambda)x \|.$$  

Then (i)

$$d_{k,j} = \exp(2au_j)[a_jd_{k-1,j-1} + b_jd_{k,j-1} + e_{k,j}],$$

where

$$d_{k,j} = \| (s_k, u_j) \prod_{i=1}^{j-1} J(r_i, u_i)x - J(r_j, u_j) \prod_{i=1}^{j-1} J(r_i, u_i)x \|,$$

and (ii)

$$d_{m,n} = \exp(\delta_{au})(\sum_{j=0}^{m} [a,b]_{J}^{n-m-j,0} + \sum_{j=m}^{n} a_{n-j+1}[a,b]_{m-l,0, n-j}^{m-l, n-j+2} + \sum_{k=0}^{m-1} \sum_{i=1}^{n-k} [a,b]_{k, e_{m-k, i}}^{m-1, n-k}),$$

for $1 \leq m \leq n$.

Proof of (i):

$$d_{k,j} = \| \prod_{l=1}^{k} J(s_l, \lambda)x - \prod_{i=1}^{j} J(r_i, u_i)x \|$$

$$= \| J(s_k, \lambda) \prod_{l=1}^{k-1} J(s_l, \lambda)x - J(r_j, u_j) \prod_{i=1}^{j-1} J(r_i, u_i)x \|.$$
\[
\begin{align*}
&= \|j(s_k^j, \lambda) \prod_{\ell=1}^{k-1} j(s_\ell, \lambda)x - j(s_k^j, \mu_j^i) \prod_{\ell=1}^{j-1} j(r_\ell^i, \mu_j^i)x\|

&= \|j(s_k^j, \mu_j^i) \prod_{\ell=1}^{j-1} j(r_\ell^i, \mu_j^i)x - j(s_k^j, \mu_j^i) \prod_{\ell=1}^{j-1} j(r_\ell^i, \mu_j^i)x\|

&= \frac{1}{1-\mu_j^i} \|j(s_k^j, \mu_j^i) \prod_{\ell=1}^{j-1} j(s_\ell, \lambda)x - \prod_{\ell=1}^{j-1} j(r_\ell^i, \mu_j^i)x\|

&\leq \exp(-\omega(u, n)) a_j b_{j-1} c_{j-1} d_{j-1} + b_j d_{j-1} + c_j b_{j-1}.
\end{align*}
\]

**Proof of (ii):** Let \(c_{m, n} = \exp(-\omega(u, n)) d_{m, n}.\) Then

\[
\begin{align*}
c_{m, n} &\leq \exp(-\omega(u, n)) d_{m-1, n-1} + b_n c_{m-1, n-1} + e_{m, n}
\end{align*}
\]

So by A.7 in the Appendix, we have that

\[
\begin{align*}
c_{m, n} &\leq \sum_{j=0}^{m} \sum_{i=0}^{n} [a, b] c_{m-j, n-i} + \sum_{j=0}^{m} a_{j} d_{m-j, n-j} + \sum_{j=0}^{m} b_{n-j} e_{m-j, i}
\end{align*}
\]
Therefore,

\[ d_{m,n} = \exp\left(\sum_{|a|,|b|} |\lambda|^{m-j} c_{1}\right) \]

\( \sum_{j=0}^{n} a_{n-j} \int_{|a|,|b|}^{m-j,0} \]

\( \sum_{j=m}^{n} a_{n-j+1} \int_{|a|,|b|}^{n-j+1,0,n-j} \]

\( \sum_{k=0}^{m-1} \sum_{i=1}^{n-k} \left[ a_{i,b}^{k,m-k,i} \right] \).

This is the desired result.

**Lemma 2.9**: Let the hypotheses of Lemma 2.8 be satisfied. Then there are constants \( c_{1} \) and \( c_{2} \) so that

\[ \lambda^{m-n} \sum_{j=0}^{n} a_{n-j} \int_{|a|,|b|}^{m-j,0} \]

\( \lambda \sum_{j=m}^{n} a_{n-j+1} \int_{|a|,|b|}^{n-j+1,0,n-j} \]

\( \lambda^{2} \sum_{k=0}^{m-1} \sum_{i=1}^{n-k} \left[ a_{i,b}^{k,m-k,i} \right] \)

for \( 1 \leq m \leq n \).

**Proof**: Let \( c_{1} = M(x) \). Using Lemma 2.6, we obtain

\[ d_{m-j,0} = \|u(s,\mu,0)x - u(s,\lambda,m-j)x\| \]

\[ = \|x - u(s,\lambda,m-j)x\| \]

\[ \leq \sigma(\lambda,m-j)M(x), \quad j=0,1,\ldots,m \]

\[ = \sigma(\lambda,m-j)c_{1}. \]

Similarly,

\[ d_{0,n-j} \leq \sigma(\mu,n-j)c_{1} = \lambda \sigma(a,n-j)c_{1}. \]
Let \( c_1, c_2 \geq 0 \) for \((x,y,z)\leq b\), (see Lemma 2.7). Then we have

From Condition C that

\[
\lambda^m \prod_{i=1}^{m} (m-k) \lambda - \sigma(u,1) \prod (1+k(x,y,z))
\]

\[
= \lambda^m a_1 (m-k) - \sigma(a,i) c_1.
\]

Now from Lemma 2.8 and the above we have that

\[
d_{m,n} = \exp(2\omega(u,n)) \frac{m}{\lambda} \sum_{j=0}^{m} \left[ a, b \right] (m-j) c_1
\]

\[
\leq \lambda^m \sum_{j=0}^{m} \sum_{k=1}^{n} a_k \left[ a, b \right] (m-k) \sigma(a,i) c_2.
\]

**Lemma 3.10:** Let the hypotheses of Lemma 2.8 be satisfied. If \( n(m,n) = m \), then

\[
\sum_{j=0}^{m} \sum_{k=1}^{n} a_k \left[ a, b \right] (m-j) = \sum_{j=0}^{m} \sum_{k=1}^{n} a_k \left[ a, b \right] (m-j) \sigma(a,j) c_2.
\]

and so

\[
d_{m,n} \leq \exp(2\omega(u,n)) \frac{m}{\lambda} \sum_{j=0}^{m} \left[ a, b \right] (m-j)
\]

\[
\leq \lambda^m \sum_{j=0}^{m} \sum_{k=1}^{n} a_k \left[ a, b \right] (m-k) \sigma(a,i) c_2
\]

where \( \sigma = \max(c_1, c_2) \).

**Proof:** The first statement is A.12 in the Appendix. To prove the second statement use Lemma 2.9 to obtain that
\[ d_{m,n} \leq \exp(\sum_{j=0}^{m} \lambda \sum_{i=1}^{n} a_i [a,b] [a,b]) (m-j) \]

\[ \lambda \sum_{j=0}^{m-1} \sum_{i=1}^{n-k} a_i [a,b]^2 \left( (m-k) - \sigma(a,i) \right) \in C. \]

**Lemma 2.11:** Let the hypotheses of Lemma 2.8 be satisfied. If \( \sigma(a,n) = m \), then

(i) \( \sum_{j=0}^{m} \sum_{i=1}^{n} a_i [a,b] [a,b] \leq \lambda \sum_{j=0}^{m} \sum_{i=1}^{n} a_i [a,b] [a,b] \left( (m-k) - \sigma(a,i) \right) \in C. \)

(ii) \( \sum_{j=0}^{m} \sum_{i=1}^{n} a_i [a,b] [a,b] \left( (m-k) - \sigma(a,i) \right) \in C. \)

(iii) \( d_{m,n} \leq \exp(\sum_{j=0}^{m} \lambda \sum_{i=1}^{n} a_i [a,b] [a,b]) (m-j) \)

**Proof of (i):**

\[ m \sum_{j=0}^{n} [a,b] [a,b] \]

\[ \leq \sum_{j=0}^{n} [m-j] [a,b] [a,b] \]

\[ \leq (m+1) \sum_{j=0}^{n} [m-j] [a,b] [a,b] \]

\[ = n \sum_{j=0}^{n} [m-j] [a,b] [a,b] \]

\[ - n \sum_{j=0}^{n} \sum_{i=1}^{j} a_i \sum_{j=0}^{n} [a,b] [a,b] \]

\[ \leq \sum_{j=0}^{n} \sum_{i=1}^{j} a_i \sum_{j=0}^{n} [a,b] [a,b] \]

\[ \leq m^2 - \sum_{j=0}^{n} \sum_{i=1}^{j} a_i \sum_{j=0}^{n} [a,b] [a,b] \]

\[ \leq \sum_{i=1}^{n} a_i \sum_{j=0}^{n} [a,b] [a,b] \]

\[ \leq \sum_{i=1}^{n} a_i \sum_{j=0}^{n} [a,b] [a,b] \]

\[ \leq \sum_{i=1}^{n} a_i (1-a_i) \]
This completes the proof of (i). Before proving (ii), we prove the three numbered relations below by using A.1, A.2, and A.4 from the Appendix.

Let \( a'_1 = a_{i+j-1} \) and \( b'_1 = b_{i+j-1} \). Then

\[
\begin{align*}
  a'_1 + b'_1 &= a_{i+j-1} + b_{i+j-1} = 1, \\
  n-j+1, j &= \left\lfloor a, b \right\rfloor_k - \left\lfloor a, b \right\rfloor_k.
\end{align*}
\]

Therefore,

\[
\begin{align*}
(1) & \sum_{k=0}^{n} \left\lfloor a, b \right\rfloor_k = \sum_{k=0}^{n} \left\lfloor a, b \right\rfloor_k = 1, \\
(2) & \sum_{k=1}^{n-j+1} k \left\lfloor a, b \right\rfloor_k = \sum_{k=1}^{n-j+1} k \left\lfloor a, b \right\rfloor_k \\
& \quad = \sum_{k=1}^{n-j+1} a_k \\
& \quad = \sum_{k=j}^{n} a_k, \text{ and} \\
(3) & \sum_{k=1}^{n-j+1} k^2 \left\lfloor a, b \right\rfloor_k = \sum_{k=1}^{n-j+1} k^2 \left\lfloor a, b \right\rfloor_k \\
& \quad = \sum_{k=1}^{n-j+1} a_k^2 - \sum_{k=1}^{n-j+1} a_k^2 + \sum_{k=1}^{n-j+1} a_k \\
& \quad = \left( \sum_{k=j}^{n} a_k \right)^2 - \sum_{k=j}^{n} a_k^2 + \sum_{k=j}^{n} a_k.
\end{align*}
\]
Therefore,

\[ \sum_{k=0}^{m-1} \sum_{i=0}^{n-k} a_i [a, b]_k (m-k) - c(a, i) \]

\[ \leq \sum_{i=1}^{n} a_i \sqrt{m} \]

\[ = m \sqrt{m}. \]

This completes the proof of (ii).

**Proof of (iii):** Using Lemma 2.10 and (i) and (ii) above, we obtain immediately that

\[ d_{m,n} \leq \exp(2\lambda \chi) \left\{ 2\lambda \sqrt{m} + \lambda^2 \frac{m}{\sqrt{m}} \right\}, \]

which is the desired result.

Now we are ready to prove Theorem 2.4.

**Proof of Theorem 2.4:** Let \( \varepsilon > 0 \) be given. Let

\[ P_m = \{ s + \frac{t-s}{m} \}_{i=0}^{m}, m=1, 2, \ldots . \]

Choose \( m_0 \) large enough so that

\[ \exp(2\lambda (t-s)) \left\{ 2(t-s) + (t-s)^2 \right\} \frac{C}{\sqrt{m_0}} < \frac{\varepsilon}{2}, \]

where \( C \) is chosen as in Lemma 2.10. Let \( P' = \{ r_i' \}_{i=1}^{n} \) and \( P'' = \{ r_i'' \}_{i=1}^{k} \) be refinements of \( P_{m_0} \) satisfying Lemma 2.6. Then,
Thus \( \Pi^t_s(I) x \) exists, and Theorem 2.4 is proved.

**Remark:** Notice that if \( D \subset R(I-\lambda A(t)) \), for \( t \in [0, T] \) and \( 0 < \lambda < \lambda_0 \), then we may choose \( r = \infty \) and \( b(s, x, r) = T-s \), so that the condition \( 0 \leq t-s < b(s, x, r) \) becomes \( s \leq t \leq T \). Hence Theorem 2.4 implies Theorem 2.1 of Crandall and Pazy in [6].
CHAPTER 11

TIME DEPENDENT EVOLUTION EQUATIONS

In this chapter we prove some theorems about time
dependent evolution equations. The results found herein
are similar to those of Crandall and Pazy in [6] and
Brezis and Lady in [1].

Definition 3.1: Let \( \{A(t) : 0 \leq t \leq T\} \) be a family
of multi-valued operators with common domain \( D \). Let \( x \in D \)
and \( s \in [0,T] \). By a strong solution to the problem

\[ u'(t) \in A(t)u(t), \quad u(s) = x \]

we mean a function \( u:[s,T] \to X \) which satisfies the
following conditions:

1. \( u \) is lipschitz continuous,
2. \( u'(t) \) exists a.e. on \( (s,T) \), and
3. \( u(t) \in D \) a.e. on \( (s,T) \), \( u(s) = x \), and
   \( u'(t) \in A(t)u(t) \) a.e. on \( (s,T) \).
Theorem 3.2: Let \( \{A(t) : 0 \leq t \leq T\} \) be a family of \( \alpha \)-dissipative, multi-valued operators with common domain \( D \), satisfying Conditions 9, \( \mathcal{K} \), and \( \mathcal{O} \). Let \( x \in D \), \( s \in [0,T) \), and \( (\lambda_0,r) \in \Lambda_x \). If the problem

\[
 u'(t) \in A(t)u(t), \ u(s) = x
\]

has a strong solution \( u \) on \([s,T]\), then

\[
 U(t,s)x = u(t) \text{ on } [s,s+b),
\]

where \( b \) is chosen as in Theorem 2.4.

Proof: The idea of this proof is due to Crandall and Pazy. Let \( \{\lambda_n\} = \{\frac{b}{n}\} \) and \( \{r_{i,n}\} = \{s+i\lambda_n\} \)

for \( i = 0,1,\ldots,n \). Let

\[
 u_n(t) = \begin{cases} 
 u(s,\lambda,k-1)x & \text{for } r_{k-1,n} \leq t < r_{k,n} \\
 x & \text{for } t < s. 
\end{cases} 
\]

Theorem 2.4 shows that \( u_n(t) \to U(t,s)x \) for each \( t \in [s,s+b) \).

Note that \( u_n(t) = J(r_{k-1,n},\lambda_n)u_n(t-\lambda_n) \)

if \( r_{k-1,n} \leq t < r_{k,n} \) and \( 1 \leq k \leq n \).

Letting \( g_n(t) = \frac{u(t) - u(t-\lambda_n)}{\lambda_n} - A(t)u(t) \), it follows that

\[
 u(t) = J(t,\lambda_n)(u(t-\lambda_n) + \lambda_n g_n(t)).
\]

Now if we extend \( u(t) \) as \( u(t) = x \) if \( 0 \leq t < s \), then
Using condition (2), we obtain

\[ \| u_n(t) - u(t) \| \leq \| f(t, \lambda_n)(u(t-\lambda_n) + \lambda_ng_n(t)) - f(t, \lambda_n)u_n(t-\lambda_n) \| + \| f(t, \lambda_n)u_n(t-\lambda_n) - f(r_{k-1}, \lambda_n)u_n(t-\lambda_n) \| \]

if \( r_{k-1,n} \leq t < r_k,n \). Thus, if \( t \in [s, s+1) \), then

\[ \| u_n(t) - u(t) \| \leq (1-\lambda_nw)^{-1}\| u(t-\lambda_n) - u_n(t-\lambda_n) \| + \lambda_n(1-\lambda_nw)^{-1}\| g_n(t) \| + \lambda_n^2K \]

for some constant \( K \). Integrating, we obtain

\[ \int_s^t \| u_n(\tau) - u(\tau) \| d\tau \leq \int_s^{t-\lambda_n} (1-\lambda_nw)^{-1} \| u_n(\tau) - u(\tau) \| d\tau \]

\[ + \lambda_n(1-\lambda_nw)^{-1} \int_s^t \| g_n(\tau) \| d\tau + \lambda_n^2|t-s|K \]

or

\[ \frac{1}{\lambda_n} \int_{t-\lambda_n}^{t} \| u_n(\tau) - u(\tau) \| d\tau \leq \int_s^{t-\lambda_n} (1-\lambda_nw)^{-1} \| u_n(\tau) - u(\tau) \| d\tau + \lambda_n|t-s|K \]

\[ + (1-\lambda_nw)^{-1} \int_s^t \| g_n(\tau) \| d\tau \]

Letting \( n \to \infty \), we obtain
\[ \|U(t,s) - u(t)\| \leq \int_s^t \|U(s,\tau) - u(\tau)\| d\tau, \]

which implies that

\[ U(t,s) = u(t). \]

The existence theorem which follows and its proof are generalizations of the work of Brezis and Pazy in [1] to the time dependent case.

**Theorem 3.3:** Let \( X^* \) be uniformly convex. Let \([A(t) : 0 \leq t \leq T]\) be a family of \( \omega \)-dissipative, multi-valued operators with common domain \( D \), satisfying conditions \( B, X, C, \) and \( M \). Let \( x \in D \) and \( s \in [0, T) \), then there exists a unique strong solution to the initial value problem

\[ u'(t) \in A(t)u(t), \ u(s) = x \]

on the interval \([s, T]\).

For the proof of this theorem we need the three lemmas below and the following definition.

**Definition 3.4:** Let \([A(t) : 0 \leq t \leq T]\) be a family of multi-valued operators. \([A(t)]\) is said to satisfy condition \( P \) if whenever \( t_n \in [0, T] \) and \((x_n, y_n) \in A(t_n)\) \( n=1, 2, \ldots \) and \( x_n \to x, \ y_n \to y, \ t_n \to t \), then \((x, y) \in A(t)\).
**Lemma 3.3**: Let $X$ be uniformly convex. Let 
\[ \{A(t) : 0 \leq t \leq T\} \] be a family of $u$-dissipative, multi-
valued operators with common domain $D$, satisfying
conditions $\mathcal{D}$, $\mathcal{X}$, and $\mathcal{M}$. Let $x \in D$, $(\lambda_0, r) \in \Lambda_x$, $s \in [0, T)$
and $b = h(s, x, r)$. If the sequence of functions $\{u_n\}$
defined in Theorem 3.2 converges pointwise to a function
$u$ on $[s, s+b)$, then $u(t) \in D$ for $t \in [s, s+b)$.

**Proof**: Define a sequence $v_n : [s, s+b) \to X$ as follows:

if $\lambda_n = \frac{b}{n}$ and $r_{1,n} = s + 1\lambda_n$, $i=1,2,...,n$, then

$$v_n(t) = u(s, \lambda, k-1)x + \frac{1}{\lambda_n}[u(s, \lambda, k)x - u(s, \lambda, k-1)(t - r_{k,n})]$$

for $r_{k-1,n} \leq t < r_{k,n}$.

Let $t \in [s, s+b)$ and choose a sequence of non-
negative integers \{k_n\} so that $t_n = s + k_n\lambda_n$, $t_n < t$, and
t - $t_n < \lambda_n$. Letting $x_n = [u(t_n, \lambda_n)v(t_n)]$, and
\[ y_n = \lambda_n^{-1}[u(t_n, \lambda_n)v(t_n) - v(t_n)] \]
we obtain that
\[ y_n \in A(t_n) x_n \]
and that
\[ \|y_n\| \leq (1-\lambda_n^2)\|A(t_n)v(t_n)\| \]
Thus, it follows from Lemma 2.7 that $\|y_n\|$ is bounded.

Because $X$ is uniformly convex, $y_{n_k} \to y$ for some sub-
sequence $\{k_n\}$ of $\{k_n\}$ and some $y \in X$. Using Lemma 2.7
and the fact that $v_n(t) \to u(t)$ if and only if $u_n(t) \to u(t)$,
it is not hard to show that $x_n \to u(t)$. Thus by Condition $\mathcal{M}$
$u(t), y) \in A(t), i.e., u(t) \in D.$
Lemma 3.6: Let $X^*$ be uniformly convex. Let 
\[ A(t) : 0 \leq t \leq T \] be a family of $u$-dissipative, multi-
valued operators with common domain $D$ satisfying 
condition $P$. Let $A(t)x \in B(t)x$ for $x \in D$, where 
\[ B(t) : 0 \leq t \leq T \] is also $u$-dissipative. Suppose that 
\[ A(t) \in D_p \text{ for each } t \in [0,T] \text{.} \] Suppose that for each 
\[ x \in D_p \text{, there is a ball } B(x,r) \text{ and a number } \lambda \geq 0 \text{ such that} \]
\[ (x,r) \cap D_p \subset R(I-\lambda A(t)) \]
for $0 \leq \lambda \leq \lambda_0$ and $0 \leq t \leq T$. Suppose that \( 0 \leq s \leq b \leq T \), and that there is a function \( u : [s,b] \rightarrow X \) satisfying:
\( u(t) \in D_p \text{ for } t \in [s, b] \), \( u \) is differentiable a.e. on \( (s,b) \), and \( u'(t) \in k(t)A(t)u(t) \) a.e. on \( (s,b) \). Then \( u(t) \in D \text{ a.e.} \) on \( (s,b) \), and \( u'(t) \in A(t)u(t) \) a.e. on \( (s,b') \).

Proof: Choose \( t \in (s,b') \) so that \( x \equiv u(t) \in D_p \), \( u \) is differentiable at \( t \), and \( u'(t) \in k(t)u(t) \). Let \( \lambda_0 \) and \( r \) be chosen such that 
\[ B(x,r) \cap D_p \subset R(I-\lambda A(t)) \]
for $0 \leq \lambda \leq \lambda_0$ and $0 \leq t \leq T$. Note that for \( t' \) close to 
\( t \), \( u(t') \in B(x,r) \), since \( u \) is continuous at \( t \). Choose 
an increasing sequence \( \{t_n\} \) so that \( t_n \rightarrow t \) and let 
\[ \lambda_n = t - t_n. \] Then
u(t_n) \in P(x,r) \cap D_B \subset R(1-\lambda_n A(t_n))

for large n. So \((1-\lambda_n A(t_n))^{-1}u(t_n) = x_n \in D\), and
\[u(t_n) = x_n - \lambda_n y_n\]
for some \(y_n \in A(t_n)x_n \subset A(t_n)x_n\). Since \(u'(t) \in B(t)u(t)\) and \(B(t)\) is \(\omega\)-dissipative, we have that
\[
\langle u'(t) - y_n, F(x-x_n) \rangle \leq \omega\|x-x_n\|^2
\]
by Propositions 1.6 and 1.7. Now let
\[
\phi(s) = u'(t) - \frac{u(s) - u(t)}{s-t}.
\]
Then
\[
\phi(t_n) = u'(t) - \frac{u(t_n) - u(t)}{t_n - t}
= u'(t) - \frac{x_n - \lambda_n y_n - x}{-\lambda_n}
= u'(t) - \frac{x-x_n}{\lambda_n} - y_n.
\]
Thus,
\[
\phi(t_n) + \frac{x-x_n}{\lambda_n} = u'(t) - y_n,
\]
and
\[
\langle \phi(t_n) + \frac{x-x_n}{\lambda_n}, F(x-x_n) \rangle \leq \omega\|x-x_n\|^2.
\]
So
\[
\langle \phi(t_n), F(x-x_n) \rangle \leq \frac{\lambda_n^{\omega-1}}{\lambda_n} \|x-x_n\|^2,
\]
or
\[
\frac{1-\lambda_n^\omega}{\lambda_n} \|x-x_n\|^2 \leq \langle \phi(t_n), -F(x-x_n) \rangle \leq \|\phi(t_n)\| \cdot \|x-x_n\|,
\]
or
\[ \|x_n - x\| \leq \frac{\lambda_n^r}{1 - \lambda_n^r} \|\phi(t_n)\|. \]

Therefore, \( x_n \to x \). Now
\[ \|u'(t) - y_n\| \leq 3\|\phi(t_n)\| \]
for large \( n \), so that \( y_n \to u'(t) \). Hence Condition \( R \) shows that \( u'(t) \in A(t)u(t) \) and the proof is complete.

**Lemma 3.7:** Let \( X \) be a reflexive Banach space and \( v(t) \) be a sequence in \( L_p(a,b; X) \), \( p > 1 \), such that \( \|v_n(t)\| \) is bounded for almost all \( t \in (a,b) \). Let \( V(t) \) denote the set of weak cluster points of \( \{v_n(t)\} \). If \( v_n \) converges weakly to \( u \) in \( L_p(a,b; X) \) then \( u(t) \) belongs to the closed convex hull of \( V(t) \) a.e. on \( (a,b) \).

**Proof:** See Kato's paper, [14].

**Proof of Theorem 3.3:** The proof of the uniqueness of solution is standard and will not be given here.

Define \( B(t) \), an extension of \( A(t) \) for \( t \in [0,T] \), as follows:

\[ D(B(t)) = D \text{ for } 0 \leq t \leq T, \text{ and} \]
\[ B(t)x = \text{closed convex hull of } A(t)x. \]

\( B(t) \) is \( \omega \)-dissipative for each \( t \in [0,T] \). Let \( 0 < b' < b \), and \( x \in D \). Then by Theorem 2.4, the sequence \( \{u_n\} \) of functions defined in Theorem 3.2 converges uniformly to
\((s,t)x\) on \([s,b']\).

Let \(u(t) = u(s,t)x\). Then by Lemma 3.4, \(u(t) \in C\) for \(t \in [s,b']\). Also note that since \(u\) is Lipschitz continuous and \(X^*\) is uniformly convex, \(u'(t)\) exists almost everywhere on \((s,s+b')\).

Let \(\{v_n\}\) be the sequence of functions defined in Lemma 3.5. We show that \(v_n' \to u'\) in \(L_p(s,s+b';X)\) for \(1 < p < \infty\). Since \(u_n\) converges uniformly to \(u\), it follows that \(v_n\) also converges uniformly to \(u\) on \([s,s+b']\). Thus

\[
\int_s^{s+b'} v_n'(t)f(t)dt \to -\int_s^{s+b'} u(t)f(t)dt,
\]

for \(f \in C'_o(s,s+b';\mathbb{R})\) where \(C'_o(a,b;\mathbb{R})\) denotes the continuously differentiable real-valued functions which vanish outside of \((a,b)\). Note also that

\[
\int_s^{s+b'} v_n'(t)f(t)dt \to \int_s^{s+b'} u'(t)f(t)dt
\]

for \(f \in C'_o(s,s+b';\mathbb{R})\), so that

\[
g(\int_s^{s+b'} v_n'(t)f(t)dt) \to g(\int_s^{s+b'} u'(t)f(t)dt)
\]

for \(f \in C'_o(s,s+b';\mathbb{R})\) and \(g \in X^*\). Since the Lipschitz norm of \(v_n'\) is bounded, it follows that some subsequence \(\{v_{n_k}'\}\) of \(\{v_n'\}\) converges to some element \(w\) belonging to \(L_p\), for \(1 < p < \infty\). Thus,
for \( f \in C^1_0(s,s+b';K) \) and \( g \in X^* \), so it follows that 
\[ w = u' \] and \( v_n' \to u' \) in \( L_p \).

Next observe that from Condition \( \mathcal{M} \) it follows that 
the set of weak cluster points of \( \{v'_n(t)\} \), denoted by \( V(t) \), 
is contained in \( A(t)u(t) \). Thus, from Lemma 3.7 we have 
that

\[ u'(t) \in V(t) \subseteq \text{closed convex hull of } A(t)u(t) \]
\[ = B(t)u(t). \]

Now since \( \{A(t) : 0 \leq t \leq T\} \) and \( \{B(t) : 0 \leq t \leq T\} \)
satisfy the hypothesis of Lemma 3.6, we obtain that 
\( u'(t) \in A(t)u(t) \) a.e. on \( (s,s+b') \).

Next we show that \( u \) is a solution on \([s,T]\). Let 
\( u \) be a solution on \([s,T_1]\), where \( T_1 \) is maximal. If 
\( T_1 \neq T \), choose \( t_n \to T \). Then \( u(t_n) \to u_0 \in X \), because 
\( u \) is Lipschitz continuous. Since \( \{A(t) : 0 \leq t \leq T\} \)
satisfies Condition \( \mathcal{M} \), we have that \( u_0 \in D \). Hence we may 
consider the problem

\[ u'(t) \in A(t)u(t), \ u(T_1) = u_0. \]

It will have a solution \( v(t) \) on \([T_1,T_0)\).
Letting $f(t) = \begin{cases} u(t) & \text{on } [s, T_1) \\ v(t) & \text{on } [T_1, T_{u_0}) \end{cases}$,
we extend the original solution, contradicting the maximality of $T_1$.

This concludes the proof of Theorem 3.3.
CHAPTER IV

A LOCAL ABSTRACT CAUCHY PROBLEM

In [1], Brezis and Pazy show that if $X$ is uniformly convex, and $A$ is a dissipative, demi-closed operator which satisfies Condition 1, then the initial value problem

$$u'(t) \in Au(t), \quad u(0) = x$$

has a unique global solution. The techniques of Brezis and Pazy may also be used in solving a local abstract Cauchy problem of a similar nature. We confine our attention to multi-valued operators which satisfy the Condition $\mathcal{J}$ below.

Definition 4.1: Let $\{A(t) : 0 \leq t \leq T\}$ be a family of multi-valued operators with common domain $D$. $\{A(t)\}$ is said to satisfy Condition $\mathcal{J}$ if there is a non-decreasing sequence of sets $\{D_n\}$ so that $D = \bigcup_{n=1}^{\infty} D_n$, and
Multi-valued operators satisfying similar conditions are studied in [3].

In the pages which follow, the problem

\[ u'(t) \in A_n(t)u(t), \quad u(0) = x \in D_1 \]

will be denoted by \( ACP_n \), and the problem

\[ u'(t) \in A(t)u(t), \quad u(0) = x \]

will be denoted by \( ACP \).

We now consider the problem of finding a solution of \( ACP \) with \( u(0) = x \in D_1 \), where \( \{A(t)\} \) satisfies Condition \( \mathcal{J} \). It is easy to show that if \( x \in D_1 \), and \( ACP_n \) has a solution on \([0,b_n)\) and \( b_n \to b \), then \( ACP \) has a solution on \([0,b)\).

We prove this in Theorem 4.6. With only the Condition \( \mathcal{J} \), this solution does not have to be unique, as the following example illustrates.

**Example 4.2:** Define \( A : \mathbb{R} \times [0,1] \setminus (0,1) \times (0,1) \to \mathbb{R} \times \mathbb{R} \), by

\[
A(x,y) = \begin{cases}
(0,0), & -\infty < x \leq 0, \quad 0 \leq y \leq 1 \\
(x,0), & 0 \leq x \leq 1, \quad y = 1 \\
(1,0), & 1 \leq x < \infty, \quad 0 \leq y \leq 1 \\
(\sqrt{x},0), & 0 \leq x \leq 1, \quad y = 0.
\end{cases}
\]

Let \( D_n = D(A) \setminus (0,n^{-2}) \times (0) \). Then \( A(t) = A \) satisfies
Condition \( J \). In particular, each \( A_n \) is \( n \)-dissipative, closed, and satisfies condition \( J \). Thus it follows from Theorem 3.3 that ACP with \( x = (0,0) \) has a unique solution. This solution is, of course, \( u_n(t, (0,0)) \). However, ACP with \( x = (0,0) \) has \( u(t) = (t^2/4, 0) \) for \( 0 \leq t \leq 2 \) as a solution, together with \( u(t) = (0,0) \). We therefore make the following definition.

**Definition 4.3:** Let \( \{A(t) : 0 \leq t \leq T\} \) be a family of multi-valued operators with common domain \( D \), which satisfies condition \( J \), with \( D = \bigcup_{n=1}^{\infty} D_n \). We say that \( u \) is a solution of ACP with respect to \( \{D_n\} \) on \([0,b)\), if for each \( t_o \in (0,b) \), \( u \) is a solution of ACP on \([0,t_o)\) for some \( n = n(t_o) \).

In Example 4.2, \( u(t) = (t^2/4, 0) \) is not a solution of ACP with respect to \( \{D_n\} \), while \( u(t) = (0,0) \) is such a solution.

**Theorem 4.4:** Let \( \{A(t) : 0 \leq t \leq T\} \) be a family of multi-valued operators with common domain \( D \). If \( \{A(t)\} \) satisfies Condition \( J \) with \( D = \bigcup_{n=1}^{\infty} D_n \), then ACP with \( x \in D \) has at most one solution with respect to \( \{D_n\} \) on the interval \([0,b)\).

**Proof:** Suppose that \( u \) and \( v \) are solutions of ACP with respect to \( \{D_n\} \) on \([0,b)\). Choose \( t_o \in (0,b) \). Then
$u(t) \in D_n$ and $u'(t) \in A_n(t)u(t)$ for some $n$ and almost all $t \in (0,b)$. Also, $v(t) \in D_m$ and $v'(t) \in A_m(t)v(t)$ for some $m$ and almost all $t \in [0,b)$. Suppose that $m < n$. Then by Proposition 1.8,

$$\frac{d}{dt}\|u(t) - v(t)\| = \langle u'(t) - v'(t), f(t) \rangle$$

$$\leq w_n \|u(t) - v(t)\|^2$$

where we have used that

$$v'(t) \in A_m(t)v(t) \subset A_n(t)v(t),$$

and that,

$$f(t) \in F(u(t) - v(t)).$$

Hence, if $t \in [0,t_0)$ then

$$\|u(t) - v(t)\| \leq w_n \int_0^t \|u(s) - v(s)\| ds,$$

and it follows that $u(t) = v(t)$. Since $t_0$ was arbitrary, $u(t) = v(t)$ for all $t \in [0,b)$.

**Remark:** An interesting question whose answer is still unknown to the author is: If $u$ is a solution of ACP with respect to $\{D_n\}$ and $v$ is a solution of ACP with respect to $\{E_n\}$, where $D = \bigcup_{n=1}^{\infty} D_n$ and $E = \bigcup_{n=1}^{\infty} E_n$, does $u = v$?
Theorem 4.1: Let \( A(t) : 0 \leq t \leq T \) be a family of multi-valued operators with common domain \( D \), which also satisfies Condition \( J \). If \( u_n \) is a solution of \( ACP_n \) on \( [0,b_n) \), and if \( 0 < b_1 < b_2 < \ldots \), then \( u_n = u_{n+1}|_{(0,b_n)} \).

**Proof:** Let \( t \in (0,b_n) \), and \( f(t) \in F(u_n(t) - u_{n+1}(t)) \).

By Proposition 1.8, we have that

\[
\|u_n(t) - u_{n+1}(t)\| \frac{d}{dt}\|u_n(t) - u_{n+1}(t)\| = \langle u_n'(t) - u_{n+1}'(t), f(t) \rangle \\
\leq w_n\|u_n(t) - u_{n+1}(t)\|^2.
\]

Hence, as before, \( u_n(t) = u_{n+1}(t) \) for \( t \in (0,b_n) \).

Theorem 4.6: Let \( A(t) : 0 \leq t \leq T \) be a family of multi-valued operators with common domain \( D \), which satisfies Condition \( J \). If \( u_n \) is a solution of \( ACP_n \) on \( [0,b_n) \), with \( \{b_n\} \) increasing and \( b_n \to b \), then

\[
u(t) = \begin{cases} u_1(t) & \text{on } [0,b_1) \\
u_{n+1}(t) & \text{on } [b_n,b_{n+1}) \quad n=1,2,\ldots\end{cases}
\]

is a solution of \( ACP \) with respect to \( \{D_n\} \) on \( [0,b) \).

**Proof:** The function \( u \) is well-defined by Theorem 4.5. Choose \( t \in [0,b) \), then either \( 0 \leq t < b_1 \), or \( b_{n-1} \leq t < b_n \) for some \( n, n=2,3,\ldots \). In either case we have that
Thus, \( u \) is a solution of \( ACF_n \) on \([0,t)\) and the result follows.

**Theorem 4.7:** Let \( X^* \) be uniformly convex. Let 
\((A(t) : 0 \leq t \leq T)\) be a family of multi-valued operators with common domain \( D \). Suppose that \( \{A(t)\} \) is \( \omega \)-dissipative and satisfies Conditions \( S, \mathcal{K}, \) and \( \mathcal{C} \). Let \( \mathcal{A}(t) \) denote the smallest extension of \( A(t) \) for which \( \{\mathcal{A}(t) : 0 \leq t \leq T\} \) satisfies Condition \( \mathcal{M} \). Let \( \mathcal{A}(t)x \) denote the closed convex hull of \( \mathcal{A}(t)x \) for \( x \in D(\mathcal{A}(t)) \). Then there is a number \( b > 0 \) and a unique function \( u : [0,b) \rightarrow X \) such that 
\[ u'(t) \in \mathcal{A}(t)u(t) \text{ a.e. on } [0,b), \text{ and } u(0) = x \in D. \]

**Proof:** Let \( b \) be as in Lemma 2.6. Let \( u_n : [0,b) \rightarrow X \), be as in Theorem 3.2. Let \( v_n : [0,b) \rightarrow X \), be as in Lemma 3.5. Then by Condition \( \mathcal{M} \), we have as in the proof of Lemma 3.5, that if \( t \in [0,b) \) and \( u_n(t) \rightarrow u(t) \), then 
\[ u(t) \in D(\mathcal{A}(t)) \subset D(\mathcal{A}(t)). \]

Now as in the proof of Theorem 3.3, it follows that \( u'(t) \) exists a.e. on \((0,b)\), and \( u'_n \rightarrow u' \) in \( L^p(0,b;X) \) for \( 1 < p < \infty \). Again, as in the proof of Theorem 3.3, it follows that the set of weak cluster points of \( \{v'_n(t)\} \), denoted by \( V(t) \), is contained in \( \mathcal{A}(t)u(t) \). Finally, using Lemma 3.7, we conclude that
for almost all $t \in (0,b)$. This concludes the proof of Theorem 4.7.

**Theorem 4.9:** Let $X^*$ be uniformly convex. Let 

$\{A(t) : 0 \leq t \leq T\}$ be a family of multi-valued operators with common domain $D$. Suppose that $\{A(t)\}$ satisfies Condition $\mathcal{J}$ with $D = \bigcup_{n=1}^{\infty} D_n$, and each $\{A_n(t) : 0 \leq t \leq T\}$ satisfies Conditions $\mathcal{J}$, $\mathcal{K}$, and $\mathcal{C}$, and that

$$\mathcal{A}_n(t) \subseteq A_{n+1}(t)$$

(see Theorem 4.7) for each positive integer $n$ and each $t \in [0,T]$. Then there is a number $b > 0$ and a unique function $u: [0,b) \rightarrow X$ so that $u$ is a solution of ACP with respect to $\{D_n\}$ on $[0,b)$.

**Proof:** The family $\{A_n(t) : 0 \leq t \leq T\}$ satisfies Conditions $\mathcal{J}$, $\mathcal{K}$, and $\mathcal{C}$, so by Theorem 4.7 there exists a number $b_n > 0$ and a function $u_n: [0,b_n) \rightarrow X$ such that $u_n(t) \in \mathcal{A}_n(t)u(t)$ a.e. on $[0,b_n)$ for $n=1,2,3,\ldots$. Thus, $u_n(t) \in A_{n+1}(t)u(t)$ a.e. on $[0,b_n)$. If the sequence $\{b_n\}$ is bounded, we let $b = \sup_b b_n$; otherwise, let $b = +\infty$. 

In either instance, if $b_k = b$ for some $k$, we define
A \rightarrow X \text{ by } u(t) = u_k(t). \text{ If } b_n \neq b \text{ for all } n, \text{ then we select an increasing subsequence } \{b_{n_k}\} \text{ of } \{b_n\} \text{ so that } b_{n_k} \rightarrow b \text{ and define } u:[0,b) \rightarrow X \text{ as in Theorem 4.6, and the result follows from that theorem.}

The uniqueness follows from Theorem 4.5.

We now consider alternative approaches to solving the problem

\[ u'(t) \in A(t)u(t), \quad u(0) = x, \]

where \( \{A(t)\} \) satisfies Condition \( J. \) In order to facilitate our discussion we consider the specific case in which \( X = \mathbb{R}, \) \( A(t)x = Ax = x^2 \) for \( x \in \{x \in \mathbb{R} : x \geq 0\} = D(A), \) and \( D_n = [0,n). \) Letting \( b_n = (n-x)/nx \) for \( x \in D_n, \) we obtain \( u_n(t) = x/(1-tx) \) as solutions of \( \text{ACP}_n \) on \([0,b_n).\)

Theorem 3.1 of \([1]\) does not apply to \( \text{ACP}_n \) because \( \tilde{A}_n \) is not closed, and it does not apply directly to \( u' \in \tilde{A}_n u, \) or \( u' \in \tilde{A}_n u \) because the operators \( \tilde{A}_n \) and \( \tilde{A}_n^* \) do not satisfy Condition 1 of \([1].\)

In Theorem 4.8 we are able to solve this problem by using the methods (but not the results) of \([1]\) because \( \tilde{A}_n \subseteq A_{n+1}. \)

It should be noted that Theorem 3.1 of \([1]\) can be applied in a different way, at least in this example.
\begin{align*}
A_n^* x &= \begin{cases}
A_n x & \text{if } 0 \leq x < n \\
[0,n^2] & \text{if } x = n
\end{cases}
\end{align*}

Then $A_n^*$ is closed and satisfies Condition I. Furthermore, the solution $v_n$ of $v' \in A_n^* v$ agrees with $u_n$ on $[0,b_n]$ and is constant thereafter.

It is not clear that this method of finding closed extensions of $A_n$ which satisfy Condition I can be generalized. This process may sometimes be possible, at least in Hilbert spaces, by using methods like those in [2]. However, even if such extensions can be found, it is not clear how a solution of $ACP$ can be constructed from the solutions $v_n$ of $v' \in A_n^* v$. This is because we would not know in general that $v_n'(t) \in A v_n(t)$, or even that $v_n(t) \in D$, for small positive $t$. It can happen, for example, that $A_n v$ is properly contained in $A_n^* v$ for $v \in D_n$, and $D_n$ is not open in $D(A_n^*)$. See Example 2 in [7].

We now show that a family which satisfies Condition $J$ generates a local evolution system in the sense defined below.

**Definition 4.9:** Let $\{A(t) : 0 \leq t \leq T\}$ be a family of multi-valued operators with common domain $D$. $\{A(t)\}$ is said to generate the local evolution system $U(t,s)$ if
\[ u(t,s)x = \prod_{i=1}^{n} A_i(x) \text{ for each } x \text{ in a dense subset of } D(U(t,s)). \]

**Definition 4.10:** Let \( \{A(t) : 0 \leq t \leq T\} \) be a family of multi-valued operators with common domain \( D \). Let \( \{D_n\} \) be a non-decreasing sequence of sets so that \( D = \bigcup_{n=1}^{\infty} D_n \).

Let \( A_n(t) = A(t) \big|_{D_n} \). A local evolution system \( \{U(t,s)\} \) on \( D \) is said to be **exponentially generated by** \( \{A(t)\} \) via \( \{D_n\} \) if

1. \( A_n(t) \) generates a local evolution system \( U_n(t,s) \) on \( D_n \) for each \( n \),
2. \( D(t,s) = \bigcup_{n=1}^{\infty} D_n(t,s) \), where \( D_n(t,s) = D(U_n(t,s)) \), and
3. \( U(t,s) \supset U_n(t,s) \) for \( n=1,2,\ldots \), and \( 0 \leq s \leq t \leq T \).

**Example 4.11:** It should be noted here that even though \( \{U(t,s)\} \) is exponentially generated by \( \{A(t)\} \) via some sequence of sets \( \{D_n\} \) that \( \{U(t,s)\} \) does not have to be given by an exponential formula in terms of the family \( \{A(t) : 0 \leq t \leq T\} \). For let \( \{A(t) : 0 \leq t \leq T\} \) be given by \( A(t)x = 2tx^2 \) for \( x \in (-\infty, +\infty) \), \( D_k = (-\infty, k) \), and \( A_k(t) = A(t) \big|_{D_k} \). Each \( A_k(t) \) is \( L_k \)-dissipative and satisfies conditions \( \mathcal{D}, \mathcal{K}, \) and \( \mathcal{C} \). However, \( A(t) \) is not one-to-one, so

\[ (1 - \frac{t-s}{n} A(s+i(t-s)))^{-1} \text{ for } i=1,2,\ldots,n \]
is not a function, and an exponential formula would be meaningless.

**Theorem 4.1.1**: Let \{A(t) : 0 \leq t \leq T\} be a family of multi-valued operators with common domain \(D\), which satisfies Condition \(\mathcal{J}\), with \(D = \bigcup_{n=1}^{\infty} D_n\). Suppose that each \(A_n(t) : 0 \leq t \leq T\) satisfies Conditions \(\mathcal{J}, \mathcal{K}\), and \(\mathcal{C}\). Then there is a local evolution system \(\{U(t,s)\}\) on \(D\), which is exponentially generated by \{A(t)\} via \(D_1\).

Furthermore, if \(\{V(t,s)\}\) is exponentially generated by \(A(t)\) via \(D_n\), then \(U(t,s) = V(t,s)\) on their common domains.

**Proof**: Since \(\{A_n(t)\}\) satisfies Conditions \(\mathcal{J}, \mathcal{K}\), and \(\mathcal{C}\), we have that \(\{A_n(t)\}\) generates a local evolution system \(\{U_n(t,s)\}\) on \(D_n\) for each \(n\), by Theorem 3.5. Let \(J_k(t,\lambda)\) denote the inverse of \((I - \lambda A_k(t))\). Then for each \(x\) in a dense subset of \(D_k \cap \bigcap_{m=1}^{\infty} D_m(t,s)\) it follows that

\[
U_k(t,s)x = \prod_{s=1}^{t} J_k(t,s)x.
\]

By using the fact that

\[
J_m(s + \frac{t-s}{n}, \frac{t-s}{n})x = J_m(s + \frac{t-s}{n}, \frac{t-s}{n})x
\]

for \(x \in D_k \cap D_m\) and for large \(n, i=1,2,\ldots,n\), it is not hard to show that

\[
U_k(t,s)x = U_m(t,s)x \text{ for } x \in D_k(t,s) \cap D_m(t,s).
\]

Now letting \(D(t,s) = \bigcup_{k=1}^{\infty} D_k(t,s)\) and \(U(t,s) : D(t,s) \to \overline{D}\)
be defined by \( U(t,s)x = U_k(t,s)x \) for \( x \in U_k(t,s) \), we will show that \( U(t,s) \) is a local evolution system on \( D \).

The previous remark shows that \( U(t,s) \) is well-defined. We now check the four properties of Definition 2.3.

(i) \( U = \bigcup_{k=1}^{\infty} \bigcup_{k=1}^{\infty} D_k(t,s) \)

(ii) \( D(t,r) \subseteq D(s,r) \) for \( 0 \leq r \leq s \leq t \leq T \) because \( D_k(t,r) \subseteq D_k(s,r) \) for \( k=1,2,3,\ldots \).

(iii) \( U(t,t)x = x \) for \( x \in D(t,t) \) because \( U_k(t,t)x = x \) for each \( k \), and \( D(t,t) = \bigcup_{k=1}^{\infty} D_k(t,t) \).

(iv) Let \( x \in D(t,r) \). Then \( x \in D_k(t,r) \) for some \( k \), and

\[
U(s,r)x = U_k(s,r)x \in D_k(t,s) \subseteq \bigcup_{k=1}^{\infty} D_k(t,s) = D(t,s).
\]

Also,

\[
U(t,s)U(s,r)x = U_k(t,s)U_k(s,r)x = U_k(t,r)x = U(t,r)x.
\]

The proof of the last statement of the theorem is clear.
BIBLIOGRAPHY


APPENDIX

The propositions found in the Appendix were used freely in the text and are collected here for easy reference. In order that the Appendix be self-contained we restate the definition below.

**Definition:** Let \( \{a_k\}, \{b_k\} \) be sequences of non-negative real numbers so that \( a_k + b_k = 1, k=1,2,\ldots \). Let \( r, s, \) and \( t \) be non-negative integers. For \( r \leq s \) and \( 0 \leq t \leq s-r+1 \) let

\[
A(r,s,t) = \{ (x_r, x_{r+1}, \ldots, x_s) \in \mathbb{R}^{s-r+1} : \text{exactly } t \text{ of the components are } 1 \text{ and the remaining components are } 0 \).
\]

Let \( f: A(r,s,t) \to \mathbb{R} \) be given by

\[
f(x_r, x_{r+1}, \ldots, x_s) = \prod_{i=1}^{s} \eta_i \text{ where } \left\{ \begin{array}{ll}
\eta_i = a_i & \text{if } x_i = 1 \\
\eta_i = b_i & \text{if } x_i = 0
\end{array} \right.
\]

Finally, define \( [a,b]_t = \sum_{y \in A(r,s,t)} f(y) \).
For notational convenience if \( s < r \) and \( t \geq 0 \) define

\[
[a,b]_t^s = 1.
\]

**Remark:** Notice that \([a,b]_t^s \) is a sum consisting of \( \binom{s-r+1}{r} \) terms, each term being a product of \( t \) \( a \)'s and \( s-r+1-t \) \( b \)'s, where \( \binom{p}{q} \) denotes the binomial coefficient.

A property of \([a,b]_t^s \) which we will make frequent use of is that

\[
[a,b]_t^s \sum_{j=0}^{s} \binom{s+j}{r} a^{s+1} b^j = [a,b]_t^{s+1}.
\]

**A.1:** If \( 0 < a_i < 1 \) and \( a_i + b_i = 1 \) for \( i=1,2,\ldots,n, \)

then \( \sum_{j=0}^{n} [a,b]_j^j = 1. \)

**Proof:** We will use induction. The result is clear for \( n = 1. \) Suppose the result is true for the positive integer \( n. \)

\[
\sum_{j=0}^{n+1} [a,b]_j^j = \sum_{j=0}^{n} [a,b]_j^j + [a,b]_{n+1}^n
\]

\[
= [a,b]_0^n + \sum_{j=1}^{n} [a,b]_{j-1}^a_{n+1} + [a,b]_j^b_{n+1}
\]

\[
+ [a,b]_{n+1}^n
\]
\[
\begin{align*}
&= \sum_{j=1}^{n} [a, b]^0_{a, b}^n + \sum_{j=1}^{n} [a, b]^1_{a, b}^n + \sum_{j=1}^{n} [a, b]^n_{a, b}^n \\
&+ \sum_{j=1}^{n} [a, b]^1_{a, b}^n + \sum_{j=1}^{n} [a, b]^2_{a, b}^n + \cdots + \sum_{j=1}^{n} [a, b]^{n-1}_{a, b}^n + [a, b]^n_{a, b}^n \\
&= \sum_{j=1}^{n} [a, b]^0 b_{a, b}^n + [a, b]^1 b_{a, b}^n + \cdots + [a, b]^{n-1} b_{a, b}^n + [a, b]^n b_{a, b}^n \\
&= \sum_{j=1}^{n} [a, b] j (b_{a, b}^n + a_{a, b}^n) \\
&= \sum_{j=1}^{n} [a, b] j \\
&= 1.
\end{align*}
\]

**Proof:** If \( n = 1 \), then

\[
\sum_{j=1}^{1} [a, b] j = [a, b]_1 = a_1.
\]

Suppose the result is true for the positive integer \( n \), then

\[
\begin{align*}
\sum_{j=1}^{n+1} [a, b] j \\
&= \sum_{j=1}^{n} [a, b] j + (n+1) [a, b]_{n+1} \\
&= \sum_{j=1}^{n} [a, b] j + [a, b]_{n+1} + \sum_{j=1}^{n} [a, b]_{n+1} \\
&= \sum_{j=1}^{n} [a, b]_{j-1} a_{a, b}^n + [a, b]_{j} b_{a, b}^n + [a, b]_{n+1} a_{a, b}^n + [a, b]_{n+1} b_{a, b}^n
\end{align*}
\]
\[ \sum_{j=1}^{n} j^{n} \left[ a, b \right]_{j}^{b_{n+1}} = \sum_{j=1}^{n} j^{n} \left[ a, b \right]_{j}^{a_{n+1}} \]

\[ \sum_{j=1}^{n} j^{n} \left[ a, b \right]_{j}^{b_{n+1}} = \sum_{j=0}^{n} \left[ a, b \right]_{j}^{a_{n+1}} \]

\[ \sum_{j=1}^{n} j^{n} \left[ a, b \right]_{j}^{a_{n+1}} \]

\[ \sum_{j=1}^{n} j^{n} \left[ a, b \right]_{j}^{(b_{n+1} + a_{n+1})} + \sum_{j=0}^{n} \left[ a, b \right]_{j}^{a_{n+1}} \]

\[ \sum_{j=1}^{n} a_{j} + a_{n+1} \]

\[ \sum_{j=1}^{n+1} a_{j} \]

A.3: If \( 0 < a_{i} \leq 1 \) and \( a_{i} + b_{i} = 1 \) for \( i = 1, 2, \ldots, n \),

then

\[ \sum_{j=1}^{n} j^{n} \left[ a, b \right]_{j} = 2 \sum_{i=1}^{n} a_{i} a_{j} + \sum_{i=1}^{n} a_{i}. \]

\[ \sum_{i<j<n} \]

Proof: For \( n = 1 \) the result is clear. Suppose the result is true for the positive integer \( n \). Then,

\[ \sum_{j=1}^{n+1} j^{n+1} \left[ a, b \right]_{j} \]

\[ = \sum_{j=1}^{n} j^{n} \left[ a, b \right]_{j}^{a_{n+1}} + \sum_{j=1}^{n} j^{n} \left[ a, b \right]_{j}^{b_{n+1}} \]

\[ + (n+1)^{2} \sum_{j=1}^{n} \left[ a, b \right]_{j}^{a_{n+1}} \]
\[ a_{i+1} = \text{if } 0 < a_i < 1, \text{ and } a_i + b_i = 1 \text{ for } i=1,2,\ldots,n, \]

then

\[ \sum_{j=1}^{\infty} [a,b]_j = \left( \sum_{i=1}^{n} a_i \right)^2 - \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} a_i. \]

**Proof:**

\[ \sum_{j=1}^{\infty} [a,b]_j = 2 \sum_{i=1}^{n} a_i a_j + \sum_{i=1}^{n} a_i \]

\[ = \left( \sum_{i=1}^{n} a_i \right)^2 - \sum_{i=1}^{n} a_i^2 + \sum_{i=1}^{n} a_i. \]

\[ A.5: \text{ Assume that} \]

(i) \( 0 < a_i < 1, \) \( a_i + b_i = 1 \) for \( i=1,2,\ldots,n, \)

(ii) \( e_{n,m} \geq 0 \) for \( m,n \geq 1, \) and

(iii) \( d_{k,j} \leq a_j d_{k-1,j-1} + b_j d_{k,j-1} + e_{k,j}. \)
Then
\[ d_{1,n} \leq b_1 \cdots b_n d_{1,0} + \sum_{j=1}^{n} \left[ a_{n-j+1} \right] b_0 d_{0,n-j} + \sum_{i=1}^{n} \left[ a_{i+1} \right] b_0 e_{i+1,1} \text{ for } n \geq 1, 2, \ldots. \]

**Proof:** Suppose \( n = 1 \), then
\[ d_{1,1} \leq a_1 d_{0,0} + b_1 d_{1,0} + e_{1,1} \]

\[ = b_1 d_{1,0} + \sum_{j=1}^{1} a_{2-j} b_0 d_{0,1-j} + \sum_{i=1}^{1} a_{i+1} b_0 e_{i+1,1}. \]

Assume the result is true for the positive integer \( n \).

\[ d_{1,n+1} \leq a_{n+1} d_{0,n} + b_{n+1} d_{1,n} + e_{1,n+1} \]

\[ \leq a_{n+1} d_{0,n} + b_{n+1} \left[ b_1 \cdots b_n d_{1,0} + \sum_{j=1}^{n} a_{n-j+1} \left[ a_{n-j+1} \right] b_0 d_{0,n-j} + \sum_{i=1}^{n} a_{i+1} b_0 e_{i+1,1} \right] + e_{1,n+1} \]

\[ = b_1 b_2 \cdots b_{n+1} d_{1,0} + \sum_{j=1}^{n+1} a_{n+2-j} \left[ a_{n+2-j} \right] b_0 d_{0,n+1-j} + \sum_{i=1}^{n+1} a_{i+1} b_0 e_{i+1,1} + e_{1,n+1} \]

\[ = b_1 b_2 \cdots b_{n+1} d_{1,0} + \sum_{j=1}^{n+1} a_{n+2-j} \left[ a_{n+2-j} \right] b_0 d_{0,n+1-j} + \sum_{i=1}^{n+1} a_{i+1} b_0 e_{i+1,1}. \]
A.6: Let the assumptions of A.5 be satisfied. Then
\[ d_{m,k} \leq \sum_{j=0}^{k} [a,b]^j d_{m-j,0} + \sum_{l=0}^{k-1} \sum_{i=1}^{k-l} [a,b]^l e_{m-l,i}. \]

Proof:

Let \( k \in \mathbb{N} \): if \( m+n \leq k \), \( m \neq 0 \), \( n \neq 0 \) and \( m \geq n \), then
\[ d_{m,n} \leq \sum_{j=0}^{n} [a,b]^j d_{m-j,0} + \sum_{l=0}^{n-1} \sum_{i=1}^{n-l} [a,b]^l e_{m-l,i}. \]

We show that \( S = \{2,3,4,\ldots\} \). It is routine to check that \( 2 \in S \). We suppose that \( k \in S \) and show that \( k+1 \in S \). Assume that \( m+n \leq k+1 \), \( m \neq 0 \), \( n \neq 0 \), and \( m \geq n \). Because \( 2 \in S \) the desired inequality is true for \( m-1 \) and \( n-1 \). Hence we assume that \( m \neq 1 \) and that \( n \neq 1 \) below. From the recursion relation (A.5, (iii)) we have that
\[ d_{m,n} \leq a_n d_{m-1,n-1} + b_n d_{m,n-1} + e_{m,n}. \]

Because

(i) \( (m-1) + (n-1) = k + 1 - 2 = k - 1 < k \),
(ii) \( m - 1 \neq 0 \), \( n - 1 \neq 0 \), and
(iii) \( m - 1 \geq n - 1 \),

we have from the induction hypothesis that
\[ d_{m-1,n-1} \leq \sum_{j=0}^{n-1} [a,b]^j d_{m-1-j,0} + \sum_{l=0}^{n-2} \sum_{i=1}^{n-1-l} [a,b]^l e_{m-1-l,i}. \]
Also since

(i) \( m \cdot (n-1) \cdot k + 1 - 1 = k \),

(ii) \( m > 0 \), \( n - 1 \neq 0 \), and

(iii) \( m = n \neq n - 1 \),

we have that

\[
d_{m,n-1} \leq \sum_{j=0}^{n-1} \sum_{l=0}^{n-1-l} (a,b) j! d_{m-j,0} m_l e_{m-l,i} \]

Thus,

\[
d_{m,n} = a_n d_{m-1,n-1} + b_n d_{m,n-1} + e_{m,n}
\]

\[
\leq a_n \sum_{j=0}^{n-1} \sum_{l=0}^{n-1-l} (a,b) j! d_{m-1-j,0} m_l e_{m-l,i} \]

\[
+ b_n \sum_{j=0}^{n-1} \sum_{l=0}^{n-1-l} (a,b) j! d_{m-j,0} m_l e_{m-l,i} + e_{m,n}
\]

\[
\leq b_n \sum_{l=1}^{n-1} (a,b) 0! d_{m-0,0} m_l e_{m-0,i} \]

\[
+ a_n \sum_{l=1}^{n-1} \sum_{j=1}^{n-1-l} (a,b) j! d_{m-j,0} m_l e_{m-j,i} + e_{m,n}
\]

\[
\leq b_n \sum_{l=1}^{n-1} (a,b) 0! d_{m-0,0} m_l e_{m-0,i} \]

\[
+ a_n \sum_{l=1}^{n-1} \sum_{j=1}^{n-1-l} (a,b) j! d_{m-j,0} m_l e_{m-j,i} + e_{m,n}
\]

\[
\leq b_n \sum_{l=1}^{n-1} (a,b) 0! d_{m-0,0} m_l e_{m-0,i} \]

\[
+ a_n \sum_{l=1}^{n-1} \sum_{j=1}^{n-1-l} (a,b) j! d_{m-j,0} m_l e_{m-j,i} + e_{m,n}
\]

\[
\leq b_n \sum_{l=1}^{n-1} (a,b) 0! d_{m-0,0} m_l e_{m-0,i} \]

\[
+ a_n \sum_{l=1}^{n-1} \sum_{j=1}^{n-1-l} (a,b) j! d_{m-j,0} m_l e_{m-j,i} + e_{m,n}
\]

\[
\leq b_n \sum_{l=1}^{n-1} (a,b) 0! d_{m-0,0} m_l e_{m-0,i} \]

\[
+ a_n \sum_{l=1}^{n-1} \sum_{j=1}^{n-1-l} (a,b) j! d_{m-j,0} m_l e_{m-j,i} + e_{m,n}
\]
\[
\begin{align*}
    &n \quad n \quad n-1 \\
    &\prod_{a,b}^n d_{m,n} + \sum_{j=1}^n \sum_{m-j=0}^n [a,b]_j d_{m-j,0} + [a,b]_n d_{m-n,0} \\
    &+ \sum_{j=1}^n \sum_{m-j=0}^n [a,b]_j d_{m-j,0} + \sum_{n-1}^n \sum_{\ell=0}^{n-\ell} [a,b]_\ell d_{m-\ell, \ell} + \sum_{n-1}^n \sum_{\ell=0}^{n-\ell} [a,b]_\ell d_{m-\ell, \ell} \text{ for } 1 \leq m \leq n.
\end{align*}
\]

\textbf{A.7:} Let the assumptions of A.5 be satisfied. Then

\[
    d_{m,n} \leq \sum_{j=0}^m \sum_{a,b}^n [a,b]_j d_{m-j,0} + \sum_{j=m-n+1}^n \sum_{a,b}^n [a,b]_{m-j+1} \text{ for } 1 \leq m \leq n.
\]

\textbf{Proof:} Let \( S \) denote the set of positive integers \( k \) such that if \( m+n \leq k, m \neq 0, \) and \( m \leq n, \) then

\[
    d_{m,n} \leq \sum_{j=0}^m \sum_{a,b}^n [a,b]_j d_{m-j,0} + \sum_{j=m-n+1}^n \sum_{a,b}^n [a,b]_{m-j+1} \text{ for } 1 \leq m \leq n.
\]
\[ d_{m,n} = \sum_{j=0}^{n} \sum_{i=1}^{m} [a,b]_j d_{m-j,0} + \sum_{j=1}^{n} \sum_{i=1}^{n-j+1} [a,b]_{m-1} d_{0,n-j} \]

\[ = \sum_{k=0}^{m-1} \sum_{i=1}^{n-k} [a,b]_k e_{m-k,i} \]

To show that \( k = 1, 2, 3, 4, \ldots \). It is routine to check that \( k = 1 \). We suppose that \( k \in S \) and show that \( k+1 \in S \). Let \( m \geq k+1 \), \( m \neq 0 \), and \( m \leq n \). If \( m = 1 \) then \( d_{1,n} \) satisfies the desired inequality by A.6. If \( m = n \) then \( d_{m,n} \) satisfies the desired inequality from A.6. We therefore assume that \( m \neq 1 \) and \( m \neq n \). From the recursion relation we have that

\[ d_{m,n} = a d_{m-1,n-1} + b d_{m,n-1} + e_{m,n} \]

We want to use the induction hypothesis on the terms \( d_{m-1,n-1} \) and \( d_{m,n-1} \), and so we check that both \( m-1,n-1 \) and \( m,n-1 \) satisfy the desired relations.

(i) \( (m-1) + (n-1) \leq k + 1 - 2 = k - 1 < k \),

(ii) \( m - 1 \neq 0 \), and

(iii) \( m - 1 \leq n - 1 \).

Thus,

\[ d_{m-1,n-1} \leq \sum_{j=0}^{n-1} [a,b]_j d_{m-1-j,0} \]

\[ + \sum_{j=1}^{n} \sum_{i=1}^{n-j+1} [a,b]_{m-2} d_{0,n-j} \]

\[ + \sum_{k=0}^{m-2} \sum_{i=1}^{n-k} [a,b]_k e_{m-k,i} \]

\[ + \sum_{k=1}^{m-1} \sum_{i=1}^{n-k} [a,b]_k e_{m-k,i} \]

\[ + \sum_{k=0}^{m-1} \sum_{i=1}^{n-k} [a,b]_k e_{m-k,i} \]
Also,

(i) \( m + (n-1) \leq k \),

(ii) \( m \neq 0 \), and

(iii) \( m \leq n-1 \) since \( m < n \).

Thus,

\[
d_{m,n-1} \leq \sum_{j=0}^{n-1} [a,b]_{j} d_{m-j,0} + \sum_{j=m}^{n-1} a_{n-j} [a,b]_{j} d_{m-1,d_{0,n-1-j}} \]

\[
\quad + \sum_{k=0}^{m-1} \sum_{i=1}^{n-1} [a,b]_{i} e_{k-m-k,i}.
\]

Therefore,

\[
d_{m,n} = a_{n} d_{m-1,n-1} + b_{n} d_{m,n-1} + e_{m,n}
\]

\[
\quad = a_{n} \sum_{j=0}^{n-1} [a,b]_{j} d_{m-1-j,0} + \sum_{j=m}^{n-1} a_{n-j} [a,b]_{j} d_{m-1,d_{0,n-1-j}} \]

\[
\quad + \sum_{k=0}^{m-2} \sum_{i=1}^{n-1} [a,b]_{i} e_{k-m-k,i}.
\]

Therefore,

\[
d_{m,n} = a_{n} \sum_{j=0}^{n-1} [a,b]_{j} d_{m-j,0} + \sum_{j=m}^{n-1} a_{n-j} [a,b]_{j} d_{m-1,d_{0,n-1-j}} \]

\[
\quad + \sum_{k=0}^{m-2} \sum_{i=1}^{n-1} [a,b]_{i} e_{k-m-k,i}.
\]

\[
\quad - b_{n} \sum_{j=0}^{n-1} [a,b]_{j} d_{m-1,d_{0,n-1-j}}.
\]
\[ a_{n} = a_{n-m+1} \] for \( n = m-2 \) to \( n-m \).

A.8: If \( 0 < a_i \leq 1 \) and \( a_i + b_i = 1 \) for \( i = 1, 2, \ldots, n \), then

\[ \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} a_{j} a_{k+1} \] for \( n = 2, 3, \ldots \).
Proof: If $n = 2$, then

\[
\sum_{k=1}^{n-1} \frac{a_j a_{k+1}}{a} \bigg/ [a, b]_0 = a_1 a_2 [a, b]_0 = a_1 a_2.
\]

and

\[
\sum_{j=1}^{n} (j-1)[a, b]_j = [a, b]_n - a_1 a_2.
\]

Thus the result is true for $n = 2$. Suppose the result is true for the positive integer $n$. Then

\[
\sum_{k=1}^{n} \sum_{j=1}^{n} a_j a_{k+1} [a, b]_0
\]

\[
= \sum_{k=1}^{n-1} \sum_{j=1}^{n} a_j a_{k+1} [a, b]_0 + \sum_{j=1}^{n} a_j a_{n+1}
\]

\[
= \sum_{j=2}^{n} \sum_{1}^{n} (j-1)[a, b]_j b_{n+1} + \sum_{j=2}^{n} \sum_{1}^{n} (j-1)[a, b]_j a_{n+1}
\]

\[
= \sum_{j=2}^{n} \sum_{1}^{n} (j-1)[a, b]_j b_{n+1} + [a, b]_j a_{n+1}
\]

\[
= \sum_{j=2}^{n} \sum_{1}^{n} (j-1)[a, b]_j a_{n+1}
\]

\[
= \sum_{j=2}^{n} (j-1) [a, b]_j.
\]

This proves the result for $n+1$. 

A.2: If $0 < a_i \leq 1$ and $a_i + b_i = 1$ for $i = 1, 2, \ldots, n$, then

$$
\Sigma_{k=1}^{m+1} \sum_{j=1}^{k+2} a_j a_{k+1} [a, b]_{m-1} = \sum_{j=m+1}^{n} (j-m) [a, b]_j
$$

for $m = 1, 2, 3, \ldots, n$.

Proof:

Let $L = \{ \ell \in \mathbb{N} : \text{if } m + n \leq \ell, m < n, \text{ and } m \neq \emptyset \}$, then

$$
\sum_{k=1}^{n-m} \sum_{j=1}^{n} a_j a_{k+1} [a, b]_{m-1} = \sum_{j=m+1}^{n} (j-m) [a, b]_j.
$$
we will show that $S = \{3, 4, 5, \ldots\}$. It is easy to show that $S$. We suppose that $k \in S$ and show that $k+1 \in S$.

Assume that $m - 1, m < n$, and $m > 0$. From [4] and [5], respectively, we have that, if $m - 1$ or if $m = l$, then the above equality holds. Hence, assuming that $m - 1$ and $m < l$, it is not hard to see that

$$
\sum_{k=1}^{l-m} \sum_{j=1}^{l-m} a_j a_{k+1} [a, b]_{m-1} = \sum_{j=m+1}^{l} a_j [a, b]_l,
$$

and that,

$$
\sum_{k=1}^{l-m-(m-1)} \sum_{j=1}^{l-m} a_j a_{k+1} [a, b]_{m-2} = \sum_{j=m+1}^{l} a_j [a, b]_l.
$$

Also, note that

$$
\sum_{k=1}^{l-m-(m-1)} \sum_{j=1}^{l-m} a_j a_{k+1} [a, b]_{m-2} = \sum_{j=1}^{l-m} a_j [a, b]_m - \sum_{j=1}^{l-m} a_j [a, b]_{m-2}.
$$

Now we show that $l+1 \in S$.

$$
\sum_{k=1}^{l+1-m} \sum_{j=1}^{l+1-m} a_j a_{k+1} [a, b]_{m-1} = \sum_{k=1}^{l+1-m} \sum_{j=1}^{l+1-m} a_j a_{k+1} [a, b]_{m-1} + \sum_{k=1}^{l+1-m} \sum_{j=1}^{l+1-m} a_j a_{k+1} [a, b]_{m-1}.
$$
\[ \sum_{l-m}^{l+1-m} \sum_{j=1}^{\ell + 1-2m} \left( \left[ \left[ a, b \right] \right] \right)_{m-2}^{a} \ell + 1-m \]

\[ \sum_{k=1}^{\ell - 2m} k \quad \ell - m \]

\[ \sum_{k=1}^{\ell - 2m} k \quad \ell - m \]

\[ \sum_{k=1}^{\ell - 2m} k \quad \ell - m \]

\[ \sum_{j=1}^{\ell + 1-2m} \left( \left[ \left[ a, b \right] \right] \right)_{m-1}^{b} \ell + 1-m \]

\[ \sum_{j=m+1}^{l-m} \left( j-m \right) \left[ \left[ a, b \right] \right]_{j}^{b} \ell + 1-m \]

\[ \sum_{j=m+1}^{l-m} \left( j-m \right) \left[ \left[ a, b \right] \right]_{j}^{b} \ell + 1-m \]

\[ \sum_{j=1}^{\ell - 2m+1} \left( \left[ \left[ a, b \right] \right] \right)_{m-2}^{a} \ell + 1-m \]

\[ \sum_{j=1}^{\ell - 2m+1} \left( \left[ \left[ a, b \right] \right] \right)_{m-2}^{a} \ell + 1-m \]

\[ \sum_{j=m}^{\ell - m} \left( \left( j-(m-1) \right) \right) \left[ \left[ a, b \right] \right]_{j}^{a} \ell + 1-m \]

\[ \sum_{j=1}^{\ell - m} \left( \left[ \left[ a, b \right] \right] \right)_{j}^{a} \ell + 1-m \]

\[ \sum_{j=m+1}^{l-m} \left( j-m \right) \left[ \left[ a, b \right] \right]_{j}^{b} \ell + 1-m \]

\[ \sum_{j=m+1}^{l-m} \left( j-m \right) \left[ \left[ a, b \right] \right]_{j}^{b} \ell + 1-m \]
If $0 \leq a_i \leq 1$ and $a_i + b_i \leq 1$ for $i = 1, \ldots, n$, and $a_1 + a_2 + \ldots + a_n = m$, then

$$
\sum_{j=m+1}^{\ell-m} \sum_{j=m}^{n} \binom{j-m}{a,b} = \sum_{j=0}^{m} \binom{m-j}{a,b}.
$$

Proof:

$$
\sum_{j=0}^{m} \sum_{j=m+1}^{n} \binom{j-m}{a,b} = \sum_{j=m+1}^{n} \binom{j-m}{a,b} + \sum_{j=0}^{m} \binom{j-m}{a,b}.
$$

$$
= \sum_{j=0}^{n} \binom{j-m}{a,b}.
$$

$$
= \sum_{j=0}^{n} \binom{j}{a,b} - m \sum_{j=0}^{n} \binom{a,b}{j}.
$$

$$
= \sum_{j=1}^{n} a_j - m.
$$

$$
= 0.
$$
A.12: If $0 < a_i \leq 1$ and $a_1 + b_1 = 1$ for $i = 1, 2, \ldots, n,$ and $a_1 + a_2 + \ldots + a_n = m,$ then

$$
\sum_{j=0}^{m} \sum_{l=0}^{n} (m-j) [a_i, b_l] = \sum_{j=m}^{n} \sum_{k=1}^{l} a_k a_{n-j+1} n-j+2 [a_i, b_l]^{m-1}.
$$

Proof:

Let $s_1 = \sum_{j=0}^{m} \sum_{l=0}^{n} (m-j) [a_i, b_l]$, 
$s_2 = \sum_{j=m}^{n} \sum_{k=1}^{l} a_k a_{n-j+1} n-j+2 [a_i, b_l]^{m-1}$, and 
$s_3 = \sum_{k=1}^{n-m} \sum_{j=1}^{k+2} a_k a_{j+k+1} [a_i, b_l]^{m-1}$, and 
$s_4 = \sum_{j=m+1}^{n} \sum_{l=0}^{n} (j-m) [a_i, b_l]$. 

From A.11, $s_4 = s_1$, and from A.10, $s_3 = s_4$. Changing the index of summation in $s_2$ implies that $s_2 = s_3$. Thus $s_1 = s_4 = s_3 = s_2$ and the equality is established.
VITA

Alban Joseph Roques was born in Paulina, Louisiana on February 3, 1941. He was graduated from Lutcher High School in 1959. Three months later, he entered Nicholls State University in Thibodaux, Louisiana and received his B. S. from Nicholls in 1963. He entered graduate school at L. S. U. in September, 1963, and received his M. S. from there in 1965. After graduation he accepted a position with Chrysler Corporation, Space Division as a computer programmer. Here he met and married Nancy Louise Adams. During the academic year 1968-1969, he served as Instructor at Southeastern Louisiana University in Hammond, Louisiana. In June, 1969 he became a graduate teaching assistant at L. S. U. and continued his graduate studies in Mathematics. In the academic year 1971-1972, he acted as an Instructor at L. S. U. In June, 1972, he again became a graduate teaching assistant at L. S. U., where he is currently a candidate for the degree of Doctor of Philosophy in Mathematics.
EXAMINATION AND THESIS REPORT

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Major Field: Mathematics

Title of Thesis: Local Evolution Systems in General Banach Spaces

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