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ABSTRACT

This paper is concerned with necessary and sufficient conditions on the presentation of a group in order for the group to be the fundamental group of a 3-manifold.

Chapter 0 is entirely devoted to background material and introduction.

Chapter I contains the development of the standard 2-complex associated with the presentation of a group as well as a proof of the isomorphism of the fundamental group of the complex and the original group.

Chapter II introduces the ideas of ordering a presentation of a group, the cycles of a presentation and the components of a presentation. Algorithms are presented for the computation of the number of cycles and the components.

In Chapter III the notion of a vertex manifold of an ordered presentation is described. Further the Euler characteristic of the vertex manifold is computed directly from the ordered presentation.

In Chapter IV the standard 2-complex of an ordered presentation is embedded in an oriented 3-dimensional CW-complex as a spine. This 3-dimensional complex is shown to have at most one non-manifold point. The neighborhood of this point is determined by the vertex manifold and is a ball if the vertex manifold is a sphere.

In Chapter V the orientable conditions are relaxed and the results are extended to include non-orientable manifolds.

The main result can be summarized as follows: Let G be a group with a presentation P and an ordering for P. Let $|R|$ be the number of appearances of all generators in P. Let η be the number of cycles, $|S|$ the number of generators and $|\gamma(P)|$ the number of components of P. Then the standard 2-complex associated with the presentation is the spine of a three-manifold if and only if

$\eta + 2|S| + 2 - 2|\gamma(P)| - |R| = 2$.

If this condition is satisfied, the construction of the 3-manifold is also presented.

CHAPTER O

INTRODUCTION

This paper is concerned with developing necessary and sufficient conditions for a group to be the fundamental group of a 3-manifold with boundary. These conditions will be criteria for a presentation which is chosen for the group. That is, conditions will be developed sufficient to guarantee that a presentation represents a group isomorphic to the fundamental group of a 3-manifold. And any fundamental group of a 3-manifold will have a presentation satisfying these conditions. The larger question must remain unanswered since Stallings in [1] shows it is impossible to find an algorithm to decide whether a finite presentation of a group defines a group isomorphic to the fundamental group of a 3-manifold.

In [2] L. Neuwirth attacks this problem from the point of view of conditions on the canonical 2-complex of a presentation. In this paper he restricts himself to closed orientable 3-manifolds.

A consecutive numbering system is employed for each chapter. That is, II.6 is the sixth numbered item in Chapter II regardless of whether it is a theorem or definition.

Let E^n be the set of all points in Euclidean n-space with distance from the origin less than or equal to one. Let U^n denote the interior of E^n, and S^{n-1} the boundary of E^n. We now define the notion of CW-complex as introduced by J. H. C. Whitehead in [3].

Definition 0.1. A Hausdorff space X is called a CW-complex if $X = \bigcup_{i=0}^{\infty} X^i$ where

1) $X^i \subset X^{i+1}$ for each $i \in \{0,1,2,\cdots\}$;
2) X^0 is a discrete space;
3) $X^n \setminus X^{n-1}$ is a collection of disjoint n-cells $\{e^n_\lambda\}$ and for each λ there is a continuous function such that f_λ maps U^n homeomorphically onto e^n_λ and $f_\lambda(S^{n-1}) \subset X^{n-1}$;
4) X has the weak topology;
5) X is closure finite, that is for each λ, $f_\lambda(S^{n-1})$ intersects a finite number of cells of dimension n-1 or less.

Definition 0.2. Suppose A is a set of points. By an abstract simplicial n-simplex, σ , in A is meant a subset $\{a_0, a_1, \cdots, a_n\}$ of $n+1$ distinct elements of A ; $a_0, a_1, \cdots, a_n$ are called vertices of σ . By a face of σ is meant any subset of σ . An abstract simplicial complex, K over A , is a set of abstract simplexes in A such that each simplex in K has all of its faces in K . Given $n+1$ linearly independent points $p_0, p_1, p_2, \cdots, p_n$ in an affine space, the geometric n-simplex spanned by $p_0, p_1, p_2, \cdots, p_n$ is defined as $\{\Sigma_{i=0}^{n} \lambda_i p_i \mid \Sigma_{i=0}^{n} \lambda_i = 1$ and $0 \leq \lambda_i \leq 1\}$. We denote the geometric n-simplex by $(p_0, p_1, \cdots, p_n)$. Suppose K is an abstract simplicial complex over $A = \{a_0, a_1, \cdots, a_m\}$. Let L be a real vector space having basis $\{b_0, b_1, \cdots, b_m\}$. Let α_i be the unit point on the vector b_i . By a geometric realization of the abstract simplicial complex K we mean the union of all geometric simplexes $(\alpha_{i(1)}, \alpha_{i(2)}, \cdots, \alpha_{i(j)})$ for which $\{a_{i(1)}, a_{i(2)}, \cdots, a_{i(j)}\}$ is an abstract simplex in A . The face of a geometric simplex corresponds to the abstract face. In practice the line between abstract and geometrical complexes blurs without incurring a loss of precision.

<u>Definition 0.3</u>. Let σ be the geometric simplex spanned by $p_0, p_1, \cdots, p_n$. The point $\Sigma_{i=0}^{n} \frac{1}{n+1} p_i$ is called the barycenter of σ . Each face of σ has a barycenter, since each face of σ is again a simplex. If a and b are

barycenters of simplices α and β respectively, then we write $a < b$ if α is a face of β. By the barycentric subdivision of a simplex σ we mean the complex formed by taking all simplexes whose vertices are barycenters of σ or its faces with the added condition that if a and b are vertices of the same simplex then $a < b$. By the barycentric subdivision of a geometrical complex we mean the union of the barycentric subdivisions of its component simplexes. The process can be iterated j times and the result is called the jth barycentric subdivision.

Definition 0.4. If K and L are geometrical simplicial complexes and f is a continuous function with domain K and range L, then f is called a simplicial map if the image of each vertex is a vertex and if each simplex in K is mapped onto a simplex in L in an affine manner. The map f will be called piecewise linear if there is a simplicial structure on K and L for which f is a simplicial map.

Definition 0.5. If K is a geometrical simplicial complex and L is a subcomplex of K, that is L is a complex and every simplex of L is also in K, then by a regular neighborhood of L in K is meant the subcomplex of K consisting of all simplexes of K having at least one face in L. This subcomplex is denoted by $R(L,K)$. The symbol

$R(L,K,j)$ represents the regular neighborhood of L in K when the simplicial structure on K and L is taken to be the jth barycentric subdivision.

<u>Definition 0.6</u>. Suppose f and g are functions from X to Y, then f and g are said to be homotopic if there exists a continuous function $H:X \times [0,1] \to Y$ such that $H(x,1) = f(x)$ and $H(x,0) = g(x)$ for each x in X. The function H is referred to as a homotopy.

A retract of a space X onto a subset A is a continuous function $r:X \to A$ where $r|A = id_A$. A deformation retract is a retract r that is homotopic to the identity on X and if H is the homotopy $H(a,t) = a$ for all a in A and t in $[0,1]$. A deformation retract r is called a strong deformation retract if the homotopy H has the additional property $H(x,t)$ is not in A, if x is not in A and t does not equal to 1.

<u>Definition 0.7</u>. Let K be a subset of some geometrical simplicial complex and v a point in the complex. By the cone over K with cone point v, denoted by $C(K,v)$, is meant $\{tv + (1-t)p | t \in [0,1], p \in K$ and if $p,q \in K$, $p \neq q$ then

CHAPTER I

PRELIMINARIES

There is a classical method of constructing a 2-dimensional CW-complex from a presentation of a group so that the fundamental group of the resulting complex is isomorphic to the original group. This method of construction will be presented in this chapter along with a proof of the fact that such an isomorphism is obtained.

By saying that P is a presentation of a group G we will mean that P is a triple (φ, S, R) ; where φ is a homomorphism from a free group F onto G , S is a set of generators for F , R is a subset of F with the property that the kernel of φ is the smallest normal subgroup of F containing R . Each member of R is called a relator, and each relator is expressed as a product of members of S or their inverses.

Let G be a group presented by $\varphi = (\varphi, S, R)$. A typical relator $r \in R$ can be expressed as

$g_{\alpha(1)}^{e(1)} g_{\alpha(2)}^{e(2)} \cdots g_{\alpha(|r|)}^{e(|r|)}$ where each $g_{\alpha(i)} \in S$, $e(i) = \pm 1$ where $g_{\alpha(i)}^{e(i)}$ is called an appearance and $|r|$ is the number of appearances in the relator r. For each $r \in R$, let D_r be a distinct copy of the planar disc with center at the origin and radius 1. Let $h_r:[0,|r|] \to \text{Bd } D_r$ be defined by $h_r(x) = e^{\frac{2\pi i x}{|r|}}$.

Lemma 1.1. For each $t \in \text{Bd } D_r$ there is a unique $k \in \{0,1,2,\cdots,|r|-1\}$ and a unique $\ell \in [0,1)$ such that $h_r(k+\ell) = t$.

Proof. The restriction of h_r to $[0,|r|)$ is one-to-one and onto the boundary of D_r. Also, each $x \in [0,|r|)$ can be written uniquely as $x = k+\ell$ when $k \in \{0,1,2,\cdots,|r|-1\}$ and $\ell \in [0,1)$. □

For each $g \in S$, let $C_g:[0,1] \to E^2$ be a simple closed curve such that if $h \in S$, $h \neq g$, $C_h((0,1)) \cap C_g((0,1)) = \emptyset$ and $C_h(0) = C_g(0)$. That is to say we take a loop for each generator so that the intersection of any two is a single point, say v. Let $B = \cup\{C_g([0,1]) \mid g \in S\}$. We are now ready to construct the CW-complex $K(P)$ associated with the presentation P.

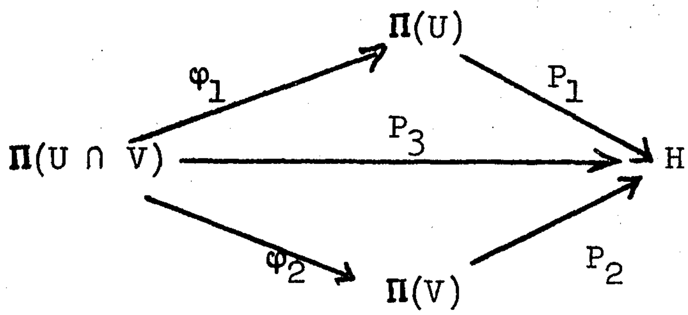
Definition 1.2. The CW-complex associated with the presentation $P = (\varphi,S,R)$ denoted by $K(P)$, is the complex with

B as a 1-skeleton and cells $\{D_r | r \in R\}$ attached to B in the following manner. Let $t \in \text{Bd } D_r$, then there exists a unique integer k and a unique $\ell \in [0,1)$ such that $h_r(k+\ell) = t$. We define the attaching map $A_r:\text{Bd } D_r \to B$ by

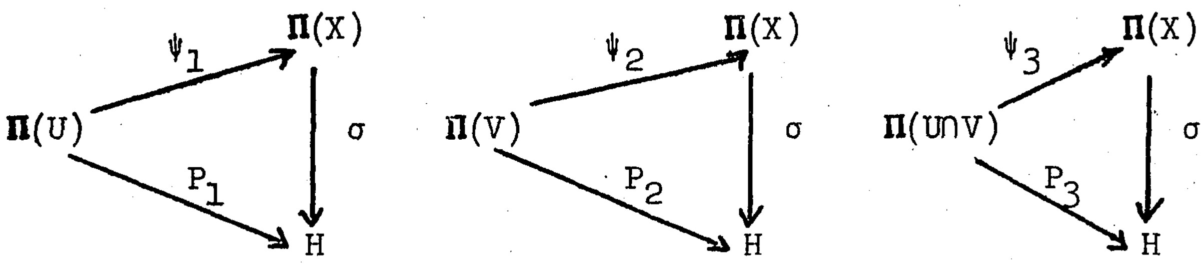
$$A_r(t) = \begin{cases} C_{g\alpha(k+1)}(\ell) & \text{if } e(k+1) = 1 \\ C_{g\alpha(k+1)}(1-\ell) & \text{if } e(k+1) = -1 . \end{cases}$$

The remainder of this chapter will be concerned with showing that $\Pi(K(P))$ is isomorphic to G. The major tools used in demonstrating this fact are the theorem of Seifert and Van Kampen, a generalization of this theorem and an associated lemma. Proofs of all three are presented by W. S. Massey in [4] and only the statements are included here.

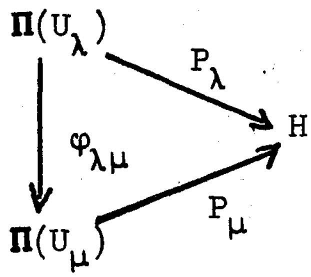
<u>Theorem 1.3</u>. (Seifert and Van Kampen [4]). Suppose U and V are arcwise connected open subsets of X such that $X = U \cup V$ and $U \cap V$ is non-empty and arcwise connected. Suppose all fundamental groups mentioned have base point $x_0 \in U \cap V$, and the homomorphisms φ_i and ψ_i, $i=1,2,3$ are induced by inclusion maps. Let H be any group and P_1, P_2 and P_3 be three homomorphisms such that the following diagram is commutative:



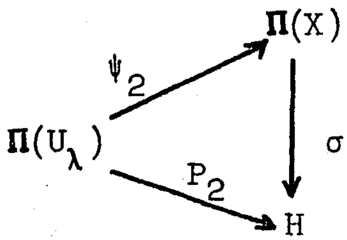
Then there exists a unique homomorphism σ such that the following three diagrams are commutative:



Theorem 1.4. (Massey [4]). Let X be an arcwise connected topological space and $x_0 \in X$. Let $\{U_\tau | \tau \in T\}$ be a covering of X by arcwise connected open sets such that for all $\tau \in T$, $x_0 \in U_\tau$. Assume that for any two indices $\tau', \tau'' \in T$ there exists an index $\tau \in T$ such that $U_{\tau'} \cap U_{\tau''} = U_\tau$. If $U_\lambda \subset U_\mu$, let $\varphi_{\lambda\mu}: \Pi(U_\lambda) \to \Pi(U_\mu)$ denote the homomorphism induced by inclusion. Let $\psi_\tau: \Pi(U_\tau) \to \Pi(X)$ be the homomorphism induced by inclusion. Let H be any group and let $P_\tau: \Pi(U_\tau) \to H$ be any collection of homomorphisms such that if $U_\lambda \subset U_\mu$, the following diagram commutes:



Then there exists a unique homomorphism $\sigma : \Pi(X) \to H$ such that for any $\tau \in T$ the following diagram is commutative:



Moreover this universal mapping condition characterizes $\Pi(X)$ up to a unique isomorphism.

The associated lemma can be stated briefly if we assume the notation and conditions of theorem 1.4.

<u>Lemma 1.5</u>. (Massey [4]). $\Pi(X)$ is generated by $\cup\{\psi_\tau(\Pi(U_\tau)) \mid \tau \in T\}$.

Unless otherwise indicated, all fundamental groups will have base point v , the vertex of $K(P)$. For the proof of the main theorem we will need a loop in $(\text{Int } D_r) \cup \{h_r(0)\}$ which is homotopic to h_r . Let $\overline{h}_r : [0, |r|] \to D_r$ be defined by

$$\overline{h}_r(x) = (\tfrac{1}{2} + \left| \frac{x - \frac{1}{2}|r|}{|r|} \right|) h_r(x) .$$

Lemma 1.6. $\overline{h}_r$ is homotopic to h_r in $D_r \setminus \{0\}$.

Proof. Define $H:[0,|r|] \times [0,1] \to D_r$ as follows;

$$H(x,t) = (1-t)h_r(x) + t\,\overline{h}_r(x) .$$

To complete the proof we show that if $x \in (0,|r|)$ and $t > 0$, then $0 < |H(x,t)| < 1$. This will guarantee that the homotopy takes place in $D\setminus\{0\}$, as well as to ascertain that $\operatorname{im} \overline{h}_r \subset \operatorname{Int} D_r \cup \{h_r(0)\}$.

$$|H(x,t)| = \left| (1-t)h_r(x) + t(\tfrac{1}{2} + \left| \frac{4 - \frac{1}{2}|r|}{|r|} \right|)h_r(x) \right| =$$

$$\left| h_r(x) \right| \left| (1-t) + t(\tfrac{1}{2} + \left| \frac{x - \frac{1}{2}|r|}{|r|} \right|) \right| =$$

$$\left| 1 - \tfrac{1}{2}t + t\left| \frac{x - \frac{1}{2}|r|}{|r|} \right| \right| .$$

If $x \in (0,|r|)$ and $t > 0$, then $0 < \left| \frac{4 - \frac{1}{2}|r|}{|r|} \right| < \frac{1}{2}$

and $\left| 1 - \frac{1}{2}t + t\left| \frac{x - \frac{1}{2}|r|}{|r|} \right| \right| < |1 - \frac{1}{2}t + (\frac{1}{2} - \epsilon)t| < 1$,

and $\left| 1 - \frac{1}{2}t + t\left| \frac{x - \frac{1}{2}|r|}{|r|} \right| \right| \geq |1 - \frac{1}{2}t| > 0$. This completes the proof. □

Additional notation is required in the proofs of the following theorems. The inclusion map from $(\mathrm{Int}\ D_r) \cup \{h_r(0)\}$ into $K(P)$ is denoted by ℓ_r . Let p_r be the image of the origin under ℓ_r . Let D_r' be defined as

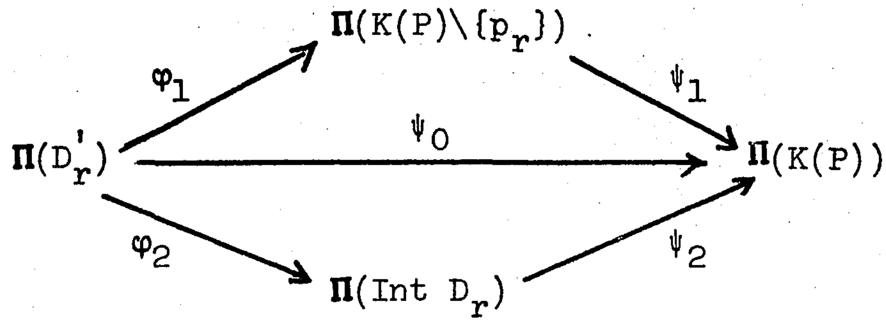
$$D_r' = (\mathrm{im}\ \ell_r \backslash \{p_r\}) \cup R(v,K(P),2) .$$

The fact that $R(v,K(P),2)$ is contractible to v is used in the following proof. Let $\alpha_r:[0,|r|] \to K(P)$ be defined as $\alpha_r(t) = A_r \circ h_r(t)$. We use the standard notation for the element of $\Pi(K(P))$ which contains the loop α_r , that is $[\alpha_r]$. We first consider the main theorem of the chapter in the case where the relating set R is finite.

<u>Theorem 1.7</u>. If the number of relators in P is finite, then $\Pi(K(P))$ is isomorphic to the group presented by P .

<u>Proof</u>. We will show that $\Pi(K(P))$ can be presented with generating set $\{[C_s] | s \in S\}$ and $\{[\alpha_r] | r \in R\}$ as the set of relators. The proof is by induction on the number of relators, and we begin the induction at zero. Suppose $R = \emptyset$. Then $\Pi(K(P)) = \Pi(B)$ which is generated by $\{[C_s] | s \in S\}$.

Suppose $R = \{r\}$. Consider the following commutative diagram induced by inclusion:

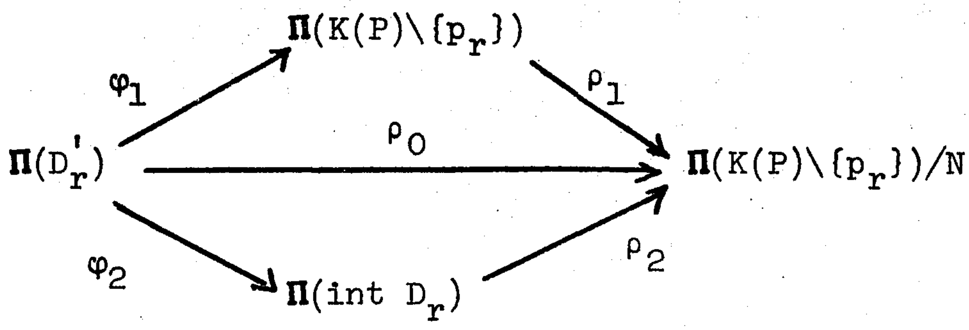


Now, $\Pi(K(P)\setminus\{p_r\})$ is a free group generated by $\{[C_s] | s \in S\}$, because B is a deformation retract of $K(P)\setminus\{p_r\}$ and $\Pi(B)$ is a free group generated by $\{[C_s] | s \in S\}$. To complete this step we must show ψ_1 is an epimorphism and $\ker \psi_1 = N$ where N is the smallest normal subgroup containing $\{[\alpha_r]\}$.

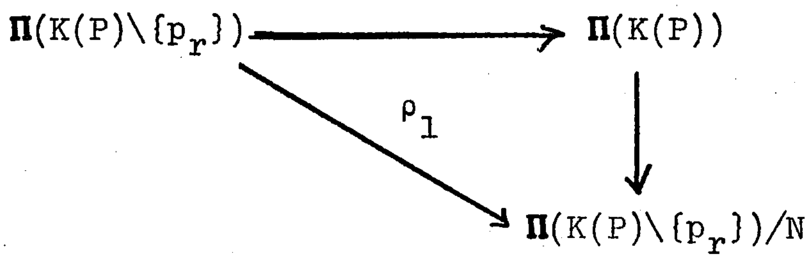
Applying Lemma 1.5 we have that $\Pi(K(P))$ is generated by $\operatorname{im} \psi_1 \cup \operatorname{im} \psi_2$. Since D_r is contractible, $\operatorname{im} \psi_2$ is the identity and $\Pi(K(P))$ is generated by $\operatorname{im} \psi_1$. Therefore ψ_1 is an epimorphism.

Next, we show that $[\alpha_r] \in \ker \psi_1$. By Lemma 1.6, $\bar{h}_r$ is homotopic to h_r . Therefore α_r , which equals $\ell_r \circ h_r$, is homotopic to $\ell_r \circ \bar{h}_r$. Therefore, $\varphi_1([\bar{h}_r]) = [\alpha_r]$. Using the commutativity of the diagram we have $\psi_1([\alpha_r]) = \psi_1(\varphi_1[\bar{h}_r]) = \psi_2(\varphi_2[\bar{h}_r]) = 1$. So $N \subset \ker \psi_1$. Next we show $\ker \psi_1 \subset N$.

Consider $\Pi(K(P)\setminus\{p_r\})/N$. We have the following commutative diagram in which φ_1 is the natural homomorphism and ρ_0 and φ_2 are trivial.

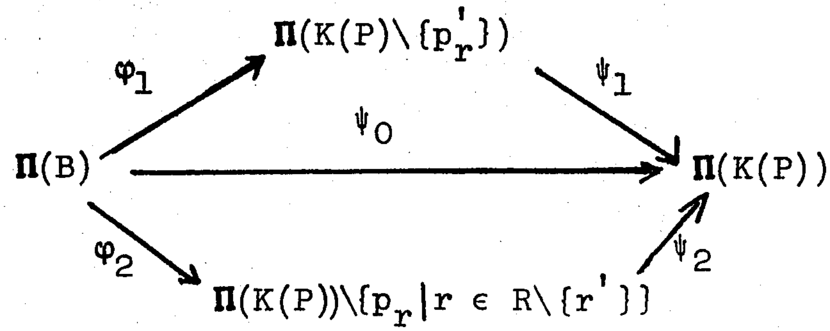


By the Seifert - Van Kampen Theorem, we have the following diagram which commutes;



We have $\ker \psi_1 \subset \ker \rho_1 = N$. And this concludes the case in which R has a single relator.

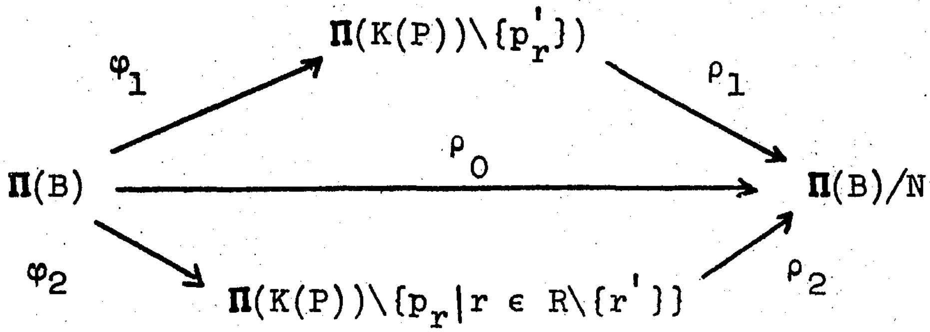
Suppose $R \neq \emptyset$. We make the following inductive assumptions. Choose $r' \in R$. Let $\psi : \Pi(B) \to \Pi(K(P) \setminus \mathrm{Int}\ D_{r'})$ be induced by inclusion, then assume ψ is an epimorphism and the kernel of ψ is the smallest normal subgroup of $\Pi(B)$ containing $\{[\alpha_r] \mid r \in R \setminus \{r'\}\}$. Inclusion maps induce the following commutative diagram;



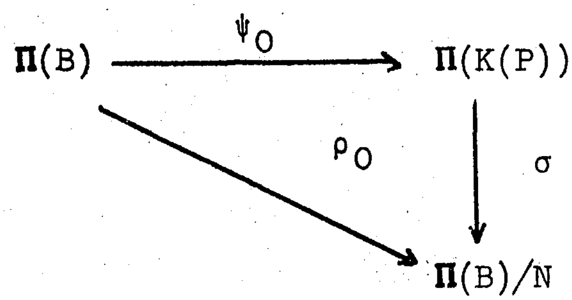
First note that $K(P)\setminus \text{Int } D'_{r'}$ is a deformation retract of $K(P)\setminus\{p_{r'}\}$. Therefore φ_1 is an epimorphism and $\ker \varphi_1$ is the smallest normal subgroup of $\Pi(B)$ containing $\{[\alpha_r] \mid r \in R\setminus\{r'\}\}$. Also $K(P)\setminus \cup\{\text{Int } D_r \mid r \in R\setminus\{r'\}\}$ is a deformation retract of $K(P)\setminus\{p_r \mid r \in R\setminus\{r'\}\}$, and we have φ_2 is an epimorphism and $\ker \varphi_2$ is the smallest normal subgroup of $\Pi(B)$ containing $\{[\alpha_{r'}]\}$.

Since $\Pi(K(P))$ is generated by $\text{im } \psi_1 \cup \text{im } \psi_2$ and φ_1 and φ_2 are epimorphisms, we have that $\text{im } \psi_0$ generates $\Pi(K(P))$ or equivalently, ψ_0 is an epimorphism. Let N be the smallest normal subgroup of $\Pi(B)$ containing $\{[\alpha_r] \mid r \in R\setminus\{r'\}\}$ then $N \subset \ker \varphi_1 \cup \ker \varphi_2$. Therefore $N \subset \ker \psi_0$. To complete the proof we must show $\ker \psi_0 \subset N$.

Consider the following diagram where ρ_0 is the natural homomorphism and ρ_i exists, $i = 1,2$, because $\ker \varphi_i \subset \ker \rho_0$;



Applying the Seifert - Van Kampen Theorem, we get the following commutative diagram:



Therefore $\ker \psi_0 \subset \ker \rho_0 = N$. This completes the proof of the theorem when R is finite.

We are now in position to prove the theorem in the generalized situation which can be stated as follows:

Theorem 1.8. The group presented by P is isomorphic to $\Pi(K(P))$.

Proof. We need to show that $\Pi(K(P))$ is the homomorphic image of $\Pi(B)$ where the kernel of the homomorphism is the smallest normal subgroup generated by $\{[\alpha_r] \mid r \in R\}$. Let $A = \{p_r \mid r \in R\}$. For each subset T of A such that $A\backslash T$

is finite, define $U_T = K(P)\backslash T$. Then $\{U_T | A\backslash T$ is finite$\}$ forms an open cover of $K(P)$ consisting of arcwise connected sets. Further if $A\backslash T$ and $A\backslash T'$ are finite, then so is $A\backslash T \cup T'$ and we have $U_T \cap U_{T'} = U_{T \cup T'}$. In other words this cover satisfies the hypothesis of the generalized Seifert - Van Kampen Theorem.

We note that U_T has for a deformation retract the complex $K(P')$ where P' is some presentation with a finite number of relators. Therefore $\Pi(U_T)$ is the homomorphic image of $\Pi(B)$, say $\varphi_T : \Pi(B) \to \Pi(U_T)$ and $\ker \varphi_T$ is the smallest normal subgroup of $\Pi(B)$ containing $\{[\alpha_r] | p_r \in A\backslash T\}$. If $U_T \subset U_{T'}$, inclusions induce the homomorphisms in the following commutative diagram:

$$\begin{array}{ccc} \Pi(U_T) & & \\ {\scriptstyle \varphi_{TT'}} \downarrow & \overset{\psi_T}{\searrow} & \\ & & \Pi(K(P)) \\ & \underset{\psi_{T'}}{\nearrow} & \\ \Pi(U_{T'}) & & \end{array}$$

Since B is a deformation retract of U_A we need only show $\psi_A : \Pi(U_A) \to \Pi(K(P))$ is an epimorphism and $\ker \psi_A$ is the smallest normal subgroup of $\Pi(U_A)$ containing $\{[\alpha_r] | r \in R\}$. We can conclude that ψ_A is an epimorphism since $\Pi(K(P))$ is generated by $\cup\{\operatorname{im} \psi_T | A\backslash T$ is finite$\}$ and $\psi_{A,T}$ is an epimorphism for each T. Further $\{[\alpha_a] | r \in R\} \subset \ker \psi_A$ since each $[\alpha_r]$ lies in $\ker \psi_T$ for some T. Finally, by applying Theorem 1.4 we see that N, the smallest normal subgroup of $\Pi(U_A)$ containing $\{[\alpha_r] | r \in R\}$, contains $\ker \psi_A$. This completes the proof. □

CHAPTER II

Three ideas are introduced and developed in this chapter; the ordering of a presentation of a group, the cycles of an ordered presentation and the components of a presentation. The development of these ideas include algorithms for counting both the cycles and the components. First, the ordering of a presentation will be defined.

We will be concerned with only finite presentations, that is a presentation $P = (\varphi, S, R)$ where both the generating set S and the set of relators R are finite. We will use the symbol $|S|$ to denote the number of generators. If $X \in S$ we will let $|X|$ be the number of appearances of X in all the relators, and $|R|$ be the number of appearances of all generators in R.

Definition 2.1. By an ordering of a generator X is meant a one-to-one function from $\{1,2,3,\cdots,|X|\}$ to the set of appearances of X.

Definition 2.2. By an ordering of a presentation

$P = (\varphi, S, R)$ is meant a collection of orderings containing one for each $X \in S$.

In practice we indicate the appearance of X which is the image of j by the subscript j. For example, consider the presentation $P = \{X, Y | XYX^{-1}Y^{-1}, XXY\}$, then one ordering for P can be represented as $\{X, Y | X_1Y_2X_4{}^{-1}Y_1^{-1},$ $X_2X_3Y_3\}$.

With each presentation we associate a set of points $A(P)$, consisting of four points for each appearance of each generator in S, that is $A(P)$ consists of $4|R|$ points. If the generator X has a j-th appearance, we name four of the points of $A(P)$ x_{2j-1}, x_{2j}, $\bar{x}_{2j-1}$ and $\bar{x}_{2j}$.

Suppose some member of the relators, say r, has in it the i-th appearance of X and the j-th appearance of Y. Then if the appearance of X is physically next to and preceding the appearance of Y, or, if the appearance of X is the last in r while the appearance of Y is the first, then we will say X_i is followed by Y_j. If the appearances have negative exponents then these will be included, and we could write for example X_i^{-1} is followed by Y_j^{-1}. We make the additional agreement that if an appearance of X is followed by an appearance of X they both have the same exponent. We can now define a relation T on $A(P)$ which will depend on the ordering chosen for P.

Definition 2.3. Let P be a presentation with an ordering. Let T be a subset of $A(P) \times A(P)$ defined as follows: If $r \in R$ and the i-th appearance of X is followed by the j-th appearance of Y in r then two pairs of $A(P) \times A(P)$ are selected for T according to the following scheme:

Table 2.4.

(1) if X_i is followed by Y_j then $(x_{2i},\overline{y}_{2j})$, $(x_{2i-1},\overline{y}_{2j-1}) \in T$.

(2) if X_i is followed by Y_j^{-1} then (x_{2i},y_{2j-1}), $(x_{2i},y_{2j}) \in T$

(3) if X_i^{-1} is followed by Y_j then $(\overline{x}_{2i},\overline{y}_{2j-1})$, $(\overline{x}_{2i-1},\overline{y}_{2j}) \in T$

(4) if X_i^{-1} is followed by Y_j^{-1} then $(\overline{x}_{2i},y_{2j})$, $(\overline{x}_{2i-1},y_{2j-1}) \in T$.

We now define a relation T' on $A(P)$ which does not depend on the ordering.

Definition 2.5. For each $X \in S$ and for each $i \in \{1,2,3,\cdots,|X|\}$ the pairs (x_2,x_3), $(\overline{x}_2,\overline{x}_3)$, (x_4,x_5), $(\overline{x}_4,\overline{x}_5),\cdots,(x_{2|X|},x_1)$ and $(\overline{x}_{2|X|},\overline{x}_1)$ are included in T' .

Definition 2.6. A member of T and a member of T' are said to form a link if one of the entries of the pair from T is identical to one of the entries of the pair from T' .

Two members, a and b, of $T \cup T'$ are said to be finitely linked if there is a subset $\{a_1, a_2, \cdots, a_n\}$ of $T \cup T'$ such that $a_1 = a$, $a_n = b$ and a_i and a_{i+1} form a link for each $i = 1, 2, 3, \cdots, n-1$.

By way of an example, if $(\overline{x}_2, \overline{x}_3)$ is in T' and $(y_2, \overline{x}_3)$ is in T, then they form a link. If (x_2, x_3) is in T' and (y_3, x_2) is in T, then these two form a link as well.

<u>Lemma 2.7</u>. Each member of T' forms a link with exactly two members of T.

<u>Proof</u>. It is clear from the definitions of T and T' that a member of $A(P)$ appears in exactly one member of T and exactly one member of T'. Therefore if (a,b) is a member of T' we know that a appears in exactly one pair of T and this pair forms a link with (a,b). Similarly the pair of T having b as an entry forms a link with (a,b). Further these two pairs in T must be distinct owing to the fact that two appearances of X, one following the other, cannot have different exponents. □

<u>Lemma 2.8</u>. Each member of T forms a link with exactly two members of T'.

<u>Proof</u>. If (a,b) is a member of T then there is a unique

member of T which contains a and a unique member of T which contains b. For the same reasons as in the proof of Lemma 2.7, these two pairs from T are necessarily distinct. □

<u>Lemma 2.9</u>. The relation of finitely linked is an equivalence relation on $T \cup T'$.

<u>Proof</u>. By Lemmas 2.7 and 2.8 each member of $T \cup T'$ forms a link with another member of $T \cup T'$ and therefore is finitely linked to itself by a chair consisting of three pairs.

The relation is immediately seen to be symmetric by revising the order of the chain. And transitivity follows by taking the union of two chains, where the last member of one is identical to the first of the other. □

<u>Definition 2.10</u>. By a cycle of an ordered presentation we will mean an equivalence class of the relation finitely linked on $T \cup T'$.

<u>Theorem 2.11</u>. Suppose Z is a cycle, then there is an ordering on the members of Z, say $Z = \{a_1, a_2, \cdots, a_n\}$ such that a_i and a_{i+1} form a link for each $i=1,2,\cdots,n-1$ and a_n and a_1 form a link.

<u>Proof</u>. The set $A(P)$ is finite, and so, each cycle of

$T \cup T'$ is finite. Let $a_1 \in T$ be any element of a cycle Z. By Lemma 2.8, a_1 forms a link with exactly two pairs in T'. Choose one of these pairs for a_2. Applying Lemma 2.7 we have that a_2 forms a link with exactly two pairs in T. One of these is a_1, and we name the other a_3. Iterated a finite number of times this process exhausts the members of Z. Further the last pair, a_n, must form a link with a_1. □

In any applications of the results of the latter chapters, it will be necessary to count the cycles of an ordered presentation. Therefore it will be advantageous to have a process of counting, which can be programmed. One approach is to adapt a technique used in Graph Theory to count the components of a graph. This technique is based on a matrix which we now define.

Let W be a simplicial complex with n vertices, $v_1, v_2, \cdots, v_n$. Let $D = (d_{i,j})$ be the n by n matrix defined by setting $d_{i,j} = 1$ if $\langle v_i, v_j \rangle$ is a simplex in W, otherwise set $d_{i,j} = 0$.

<u>Lemma 2.12</u>. If $D^k = (b_{i,j})$, then $b_{i,j}$ is the number of paths in W with initial point n_i and terminal point n_j which crosses $k-1$, not necessarily distinct, vertices.

<u>Proof</u>. The argument is by induction on k, and the case

when $k=1$ is just a rephrasing of the definition of D. Assume that the theorem is true for all values of $k < q$. Let $d_{i,j}$, $b_{i,j}$ and $C_{i,j}$ be the entries in the i-th row and the j-th column of D, D^q and D^{q-1} respectively. Then we have $b_{i,j} = \Sigma_{t=1}^{n} C_{i,t} d_{t,j}$. But $C_{i,t}$ is the number of paths from v_i to v_j crossing $q-1$ vertices. Since any must end in some simplex $\langle v_t, v_j \rangle$, that is for some t, and since we sum over all possible values of t, the proof is completed. □

<u>Lemma 2.13</u>. If v_i and v_j are vertices of a simplicial complex W, then v_i and v_j belong to the same component of W if and only if the entry in the i-th row and j-th column of $\Sigma_{k=1}^{n-k} D^k$ is non-zero.

<u>Proof</u>. The statement that v_i and v_j belong to the same component of W is equivalent to saying that there is a path in W from v_i to v_j crossing $n-2$ or fewer vertices. This in turn is equivalent to the fact that the i-th row and j-th column of some D^k, $0 < k \leq n-1$ is non-zero. Since all entries of D^k are positive, the proof is complete. □

The number of possible paths is of no consequence in these applications and a certain elegance is attained by suppressing this number.

<u>Definition 2.14</u>. Let W be a simplicial complex with

vertices $v_1, v_2, \cdots, v_n$. Let D be the matrix defined above. By the connectivity matrix of W we will mean a matrix obtained from $\Sigma_{k=1}^{n-1} D^k$ by replacing each non-zero entry with a 1 .

Theorem 2.15. Let W be a simplicial complex. The number of components of W equals the rank of the connectivity matrix.

Proof. If v_i and v_j are vertices in the same component of W the i-th row and the j-th row will have the same entries. Further if the i-th and j-th rows each have a 1 for their k-th entry then they are identical. Therefore not only all the rows associated with vertices of the same component identical, but by choosing a row from each component a set of independent rows is obtained. It follows that the rank of the matrix is the size of this set. □

Definition 2.16. We associate with the ordered presentation P the 1-dimensional abstract simplicial complex, called the cycle complex, having each member of $T \cup T'$ as vertices and containing a 1-simplex $\langle a, b \rangle$ if and only if a and b form a link.

Theorem 2.17. Let W be the cycle complex of an ordered presentation P . Then the number of cycles of P equals the row rank of the connectivity matrix of W .

Proof. The theorem follows immediately from the fact that there is a one-to-one correspondence between the cycles of P and the components of W . □

The final concept to be introduced in this chapter is that of components of a presentation.

Definition 2.18. Let P be a presentation. Let $\gamma(P)$ be the abstract simplicial complex, called the component complex of P , containing two vertices x and $\bar{x}$ for each generator X of P . The 1-simplexes of $\gamma(P)$ are characterized by the following:

Table 2.19.

(1) If X_i is followed by Y_j then $\langle x,\bar{y}\rangle \in \gamma(P)$.
(2) If X_i is followed by Y_j^{-1} then $\langle x,y\rangle \in \gamma(P)$.
(3) If X_i^{-1} is followed by Y_j then $\langle \bar{x},\bar{y}\rangle \in \gamma(P)$.
(4) If X_i^{-1} is followed by Y_j^{-1} then $\langle \bar{x},y\rangle \in \gamma(P)$.

The component complex $\gamma(P)$ is independent of the ordering on P . The number of components of $\gamma(P)$ will be of interest and is denoted by $|\gamma(P)|$.

Theorem 2.20. The rank of the connectivity matrix of $\gamma(P)$ equals $|\gamma(P)|$.

Proof. This fact follows immediately from Theorem 2.15.

This allows the number $|\gamma(P)|$ to be found by a a simple program for any application.

CHAPTER III

In this chapter we introduce the idea of the vertex manifold, V, of an ordered presentation. The complex V is then shown to be an orientable 2-manifold. We then show how to compute the Euler characteristic of V directly from an ordered presentation. In this construction we rely heavily on the notation, maps and techniques of the construction of $K(P)$ in Chapter I.

We begin with a group G which has a finite presentation $P = (\varphi, S, R)$. For each relator $r \in R$, let E_r be a copy of the unit ball in three spaces with center at the origin. By assuming that the planar disc D_r, used in the construction of $K(P)$, lies in the plane $Z = 0$, we have D_r regularly embedded in E_r, and we can use the attaching map A_r to attach E_r to the bouquet of loops B. Recalling that $Bd(D_r)$ was oriented and is an equatorial simple closed curve on the $Bd(E_r)$, we name the closure of each of the disc components of $Bd(E_r)\backslash Bd(D_r)$

by D_r' and D_r'' , the former reserved for the component on the right when proceeding in a positive direction around $Bd(D_r)$.

Definition 3.1. The CW-Complex associated with P , denoted by J(P) is the complex with B as a 1-skeleton and cells $\{E_r | r \in R\}$ attached to B by the attaching maps A_r .

It should be noted that not only is K(P) embedded in J(P) , but J(P) has K(P) as a strong deformation retract. We therefore have the following lemma.

Lemma 3.2. $\pi(J(P))$ is isomorphic to G .

Proof. $\pi(J(P))$ is isomorphic to $\pi(K(P))$ since K(P) is a deformation retract of J(P) , $\pi(K(P))$ is isomorphic to G by Theorem 1.8.

We are interested in when J(P) can be embedded in a 3-manifold, and this question leads us to consider the boundary of a regular neighborhood of the vertex v of J(P) . Let U = R(v,J(P),2) . By the boundary of U we will mean the union of the 2-cells in U which miss v . We denote this set by Bd(U) . The intersection of Bd(U) and the bouquet of loops B consists of $2|S|$ points, that is two for each $X \in S$. The first of these points encountered while proceeding from v in a positive direction around

C_x will be denoted by $\overline{x}$ while the second will be called x. Let $W_x = R(x, Bd(U), 4)$ and $W_{\overline{x}} = R(\overline{x}, Bd(U), x)$.

Two appearances in a presentation are said to form an adjacent pair if the first member of the pair is followed by the second member of the pair in some relator r. Here again it is understood that the last member of a relator is followed by the first. For example, if $X_1Y_2X_2^{-1}$ is a relator then it yields three adjacent pairs; (X_1,Y_2), (Y_2,X_2^{-1}) and (X_2^{-1},X_1). The $Bd(U)$ can be regarded as a collection of 2-cells each regularly embedded in some E_r. There is one such 2-cell for each adjacent pair. If (X_1,Y_2) is an adjacent pair, we name the associated 2-cell $D(X_1,Y_2)$. At first glance the natural association between an adjacent pair and a disc may not be clear but it is an intrinsic part of the construction. The disc $D(X_1,Y_2)$ intersects the bouquet in two points, one in C_x and one in C_y. Which of the points x, $\overline{x}$, y or $\overline{y}$ belong to $D(X_1,Y_2)$ is determined by the exponents of the two appearances. In this case $D(X_1,Y_2) \cap B = \{x,\overline{y}\}$. If the appearance of X had a negative exponent then $\overline{x}$ would be in the intersection. If the second appearance of the pair, Y, had an exponent of -1 then $\overline{y}$ would be replaced by y.

For each $X \in S$, W_x and $W_{\overline{x}}$ each consists of a collection of 2-cells. The 2-cells forming W_x all have the point x as their single common point, and the 2-cells of

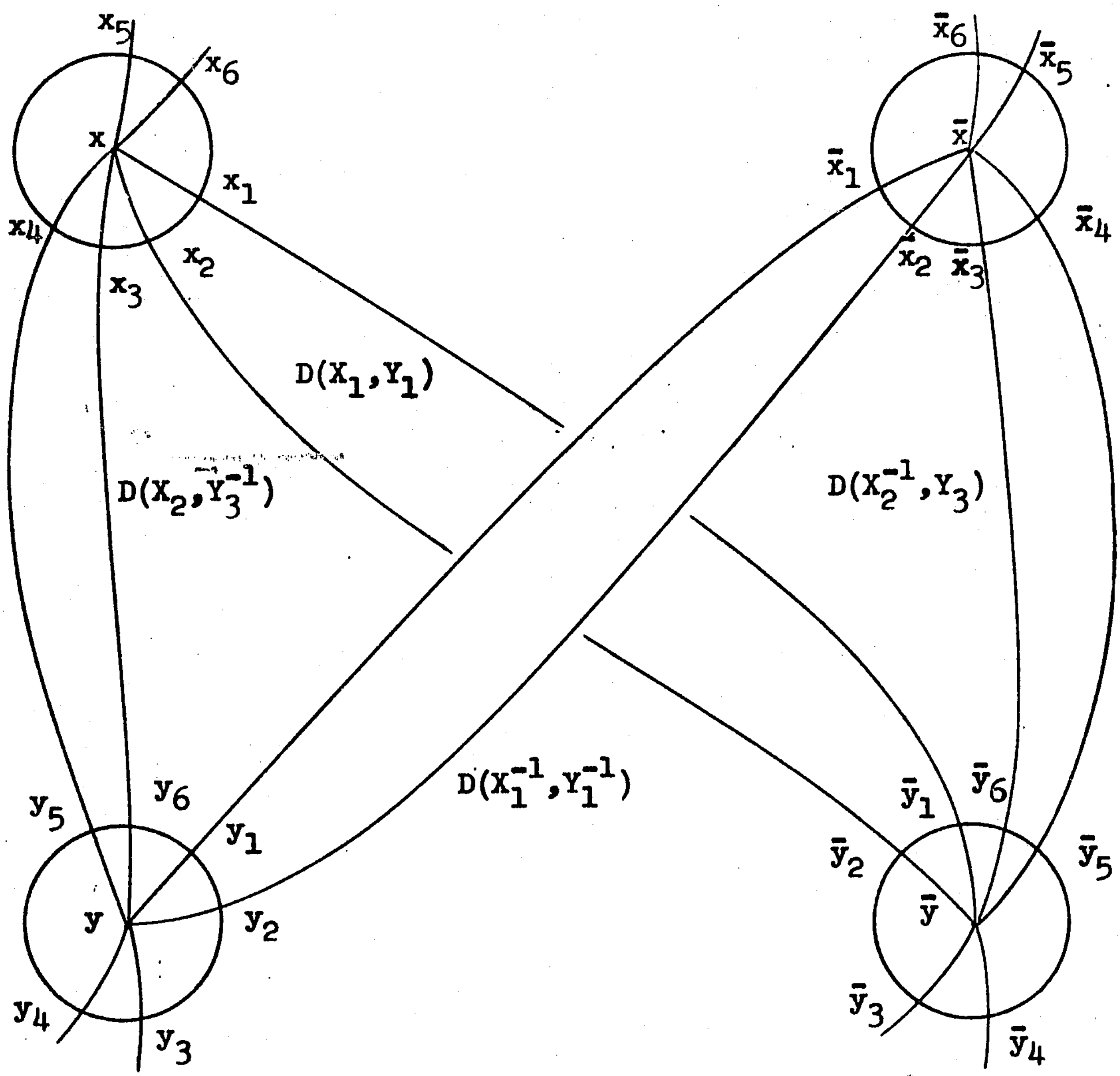
$W_{\overline{x}}$ all contain $\overline{x}$. For any two appearances s and t , the 2-cell D(s,t) intersects $\cup\{W_x \cup W_{\overline{x}} | X \in S\}$ in two 2-cells depending upon s and t . For example, $D(X_1,Y_2) \cap W_x$ and $D(X_1,Y_2) \cap W_{\overline{y}}$ are both 2-cells. Further, $Bd(D(X_1,Y_2)) \cap W_x$ is a 1-cell with x in the interior, and $Bd(D(X_1,Y_2)) \cap W_{\overline{y}}$ is a 1-cell with $\overline{y}$ in its interior. The naming of the endpoints of these two 1-cells is crucial to the construction of the vertex manifold. The naming scheme is based on the relation J of Chapter II , and is presented in the following table:

Table 3.3.

(1)	$D(X_i,Y_j)$	x_{2i-1}	x_{2i}	$\overline{y}_{2j}$	$\overline{y}_{2j-1}$
(2)	$D(X_i,Y_j^{-1})$	x_{2i-1}	x_{2i}	y_{2j-1}	y_{2j}
(3)	$D(X_i^{-1},Y_j)$	$\overline{x}_{2i-1}$	$\overline{x}_{2i}$	$\overline{y}_{2j-1}$	$\overline{y}_{2j}$
(4)	$D(X_i^{-1},Y_j^{-1})$	$\overline{x}_{2i-1}$	$\overline{x}_{2i}$	y_{2j}	y_{2j-1}

Line (1) means that in naming the points on the boundary of $D(X_i,Y_j)$, arbitrarily assign x_{2i} and x_{2i-1} to the end-points of the 1-cell $W_x \cap Bd(D(X_i,Y_j))$ and then proceeding around the boundary of $D(X_i,Y_j)$ beginning at x_{2i-1} passing through first x then x_{2i} the first endpoint of $W_{\overline{y}} \cap Bd(D(X_i,Y_j))$ is named $\overline{y}_{2j}$ and the second is called $\overline{y}_{2j-1}$.

Once the naming of these points is accomplished, the


x
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ȳ5
ȳ6
D(X1,Y1)
D(X2,Y3^-1)
D(X2^-1,Y3)
D(X1^-1,Y1^-1)

Figure 1

next step in the construction of the vertex manifold can be taken. For each generator X and each $j \in \{1,2,3,\cdots,|X|\}$ attach a disc $D(x,j)$ to W_x such that $D(x,j) \cap W_x$ is a 1-cell in the boundary of $D(x,j)$ and is the union of two 1-cells in the boundary of W_x, one having endpoints x_{2j-1} and x, and the second with endpoints x and $x_{2j \bmod (2|X|)}$. Similarly for each $j \in \{1,2,3,\cdots,|X|\}$ attach a disc $D(\overline{x},j)$ to $W_{\overline{x}}$ along a 1-cell in the boundary of $D(\overline{x},j)$ which is the union of two 1-cells in the boundary of $W_{\overline{x}}$ one with endpoints $\overline{x}_{2j-1}$ and $\overline{x}$ and the other having endpoints $\overline{x}$ and $\overline{x}_{2j \bmod (2|X|)}$. Let $\overline{W}_x = W_x \cup [\cup\{D(x,j) \mid j \in \{1,2,3,\cdots,|X|\}]$ and let $\overline{W}_{\overline{x}} = W_{\overline{x}} \cup [\cup\{D(x,j) \mid j \in 1,2,3,\cdots,|X|\}]$.

<u>Lemma 3.4</u>. $\overline{W}_x$ and $\overline{W}_{\overline{x}}$ are 2-cells.

<u>Proof</u>. Each 2-cell $D(x,j)$ is attached to $\overline{W}_x$ in a one to one fashion, and the only point of $\overline{W}_x$ which lies in more than one of the cells of $D(x,j)$ is the point x. It follows that $\overline{W}_x \setminus \{x\}$ is a collection of punctured discs. But for each $i \in \{0,1,2,\cdots,2|X|-2\}$, x_i and x_{i+1} are in the same component of $\overline{W}_x \setminus \{x\}$, since if i is even they lie in the same disc $D(x,i/2)$, and if i is odd they lie in the same component of $W_x \setminus \{x\}$. Therefore $\overline{W}_x \setminus \{x\}$ consists of a single component, that is $\overline{W}_x \setminus \{x\}$ is a single

punctured disc. This shows that $\overline{W}_x$ is a disc. A similar argument holds for $\overline{W}_{\overline{x}}$ which completes the proof. □

Let $V' = Bd(U) \cup \{\overline{W}_x \cup \overline{W}_{\overline{x}} | X \in S\}$.

<u>Lemma 3.5.</u> V' is a 2-manifold with boundary.

<u>Proof</u>. The closure of $Bd(U) \setminus \{\overline{W}_x \cup \overline{W}_{\overline{x}} | X \in S\}$ is a collection of disjoint 2-cells. Each of these 2-cells can be written as the closure of $D(s,t) \setminus \{\overline{W}_{\overline{x}} \cup \overline{W}_x | X \in S\}$ for some adjacent pair of appearances (s,t) . We denote this 2-cell by $D'(s,t)$. Then V' can be written as the union of cells of three types: $\overline{W}_x$, $\overline{W}_{\overline{x}}$ and $D'(s,t)$. Further if any two of these cells intersect they do so on a 1-cell contained in their boundary. It follows that V' is a 2-manifold with boundary. □

Since V is a 2-manifold with boundary, $Bd(V')$ is a collection of simple closed curves. Further $Q = \{y | y = x_j \text{ or } y = \overline{x}_j , x \in S \text{ and } j \in \{0,1,2,\cdots,2|x|-1\}$ is contained in $Bd(V')$; and $Bd(V') \setminus Q$ is a collection of open arcs.
The closure of each of these arcs is a 1-cell and will be referred to as $\langle s,t \rangle$ where s and t are the endpoints. For each $X \in S$ and $j \in \{1,2,3,\cdots,|X|\}$, $\langle x_{2j-1}, x_{2j \bmod (2|X|)} \rangle$ and $\langle \overline{x}_{2j-1}, \overline{x}_{2j \bmod (2|X|)} \rangle$ are

1-cells in $Bd(V')$. In fact there is a natural one-to-one correspondence between the pairs of L , defined in Chapter II , and these 1-cells.

The other half of the set of 1-cells comprising $Bd(V')$ lie in the boundary of the cells of the form $D'(s,t)$. Table 3.3 records the naming scheme for the points on the boundary of $D(s,t)$ and therefore on the boundary of $D'(s,t)$. The 1-cells from $Bd(V')$ lying in $D'(s,t)$ can be read from Table 3.3. If line (1) is used for $D'(s,t)$, then $\langle x_{2i},\overline{y}_{2j}\rangle$ and $\langle\overline{y}_{2j-1},x_{2j-1}\rangle$ are 1-cells in $Bd(V')$. The important idea is that there is a natural one-to-one correspondence between the pairs in the relation J of Chapter II and the 1-cells of $Bd(V')$ which lie in cells of the form $D'(s,t)$.

Lemma 3.6. For each cycle of an ordered presentation there is a simple closed curve in $Bd(V')$ such that, (s,t) is a member of the cycle if and only if $\langle s,t\rangle$ is contained in the simple closed curve.

Proof. Choose a cycle from the ordered presentation. For each pair (s,t) in the cycle there is a 1-cell $\langle s,t\rangle$ which is the closure of one of the components of

$$Bd(V')\backslash Q .$$

Further, if two members, (s,t) and (s',t') of a cycle form a link then the corresponding two 1-cells $<s,t>$ and $<s',t'>$ have a common endpoint. The converse of this statement is also true, and thus sufficient to prove the lemma. □

<u>Theorem 3.7</u>. The manifold V' is orientable.

<u>Proof</u>. We have previously represented V' as the union of cells of three types: $\overline{W}_X$, $\overline{W}_{\overline{X}}$ and $D'(s,t)$. The points $x_1, x_2, \cdots, x_{2|X|}$ all lie in the order of their subscripts around $Bd(\overline{W}_X)$, and the order will be used to orient $Bd(\overline{W}_X)$. The positive direction around the boundary will be in ascending order of subscripts. Similarly $Bd(\overline{W}_{\overline{X}})$ contains the points $\overline{x}_1, \overline{x}_2, \cdots, \overline{x}_{2|X|}$ in order, and the positive direction will be taken in the order of descending subscripts.

Each disc of type $D'(s,t)$ is contiguous to two disc of the remaining types, and each of these will induce an orientation on the boundary of $D'(s,t)$. We must show that these two orientations agree. Each of the four cases that are listed in Table 3.3 must be considered. Reference to Figure 1 will be helpful for the remainder of the proof.

The four points x_{2i-1}, x_{2i}, $\overline{y}_{2j}$, and $\overline{y}_{2j-1}$ lie in this order on the boundary $D'(X_i,Y_j)$ and therefore can be used to describe direction. Since x_{2i} precedes x_{2i-1} in the positive direction on $Bd(\overline{W}_X)$, x_{2i-1} will precede x_{2i}

in the positive direction around the boundary of $D'(X_i, Y_j)$, and therefore on this boundary $\overline{y}_{2j}$ precedes $\overline{y}_{2j-1}$. This dictates that in a positive direction about $Bd(W_{\overline{y}})$, $\overline{y}_{2j-1}$ precedes $\overline{y}_{2j}$ which agrees with the orientation established on $Bd(\overline{W}_y)$ previously.

The other three cases listed in Table 3.3 will be considered briefly. The orientation on $Bd(\overline{W}_x)$ imposes the positive direction to the boundary of $D'(X_i, Y_j^{-1})$ in the of the points $x_{2i-1}, x_{2i}, y_{2j-1}$ and y_{2j} . This agrees with the orientation which is imposed by the orientation on $Bd(\overline{W}_y)$. The boundary of $D'(X_i^{-1}, Y_j)$ intersects the two cells $W_{\overline{x}}$ and $W_{\overline{y}}$. Now $Bd(\overline{W}_{\overline{x}})$ is positive from $\overline{x}_{2i-1}$ to $\overline{x}_{2i}$, imparting a negative sense to the direction thru $\overline{x}_{2i-1}, \overline{x}_{2i}, \overline{y}_{2j-1}$ and $\overline{y}_{2j}$. This in turn says that the direction from $\overline{y}_{2j-1}$ thru $\overline{y}_{2j}$ on $Bd(W_{\overline{y}})$ is positive. On the boundary of $D'(X_i^{-1}, Y_j^{-1})$ consider the direction $\overline{x}_{2i-1}, \overline{x}_{2i}, y_{2j}, y_{2j-1}$. Both the orientation on $Bd(\overline{W}_{\overline{x}})$ and the orientation on $Bd(\overline{W}_y)$ induce a negative sense to this direction. This shows that the orientation placed on $\overline{W}_x$ and $\overline{W}_{\overline{x}}$ for the various generators X in S can be extended to all of V , which completes the proof. □

Before defining the vertex manifold the number of components of V must be considered. Note that the graph $\gamma(P)$, introduced by Definition 2.12, can be embedded in V' so that for each $X \in S$, the vertices x and $\overline{x}$ of the graph are identical to the vertices x and $\overline{x}$ of V, and if $\langle a,b\rangle$ is a 1-simplex in the graph, then $\langle a,b\rangle$ is regularly embedded in $D(a,b)$. It follows that V' contains $\gamma(P)$ as a strong deformation retract and the number of components of V' is the same as the number of components of $\gamma(P)$, that is $|\gamma(P)|$.

The vertex manifold can now be constructed. Each component has a boundary consisting of one or more simple closed curves. Choose one simple closed curve C_i, $i=1,2,3,\cdots,|\gamma(P)|$, from the boundary of each component. Let Q be a 2-sphere from which the interiors of $|\gamma(P)|$ disjoint 2-cells D_i, $i=1,2,3,\cdots,|\gamma(P)|$, have been removed. Choose an orientation for V' and an orientation for Q. A new complex is obtained from the disjoint union of Q and V' by identifying C_i homeomorphically to the $Bd(D_i)$ in a manner so that the positive direction about C_i agrees with the positive direction about $Bd(D_i)$. If $Bd(V')$ contains more simple closed curves than Q as boundary components, then attach a 2-cell to each of those remaining homeomorphically along the boundary of the 2-cell. The resulting complex is called V, the vertex manifold.

Theorem 3.8. The vertex manifold V is a connected, closed orientable 2-manifold.

Proof. The vertex manifold V was constructed from the disjoint union of several 2-manifolds with boundary V', Q and the collection of 2-cells. The intersection of any two of these 2-manifolds is empty or the union of disjoint simple closed curves lying in the boundary of each. Further if p is any point lying on one of these simple closed curves, then p belongs to exactly two of the manifolds. Therefore p has a neighborhood whose closure can be written as the union of two closed discs intersecting on a 1-cell and is therefore a closed disc containing p in the interior. It follows that V is a 2-manifold.

Clearly V is closed since if any boundary component of V remained after the identification of V' and Q, a 2-cell was attached to it. Since the positive direction of C_i and of $Bd(D_i)$ both agreed under the identification, it follows that V is orientable. To see that V is connected we note that Q is connected and that each component of V is attached to Q. This completes the proof. □

To determine the identity of V we compute $\chi(V)$, the Euler characteristic of V.

<u>Theorem 3.9</u>. Let $|R|$ be the number of appearances in R of all the generators, η the number of cycles, $|S|$ the number of generators and $|\Upsilon(P)|$ the number of components of V. Then $\chi(V)$ is given by

$$\chi(V) = \eta + 2|S| + 2 - 2|\gamma(P)| - |R| .$$

Proof. First we must choose a cell decomposition for V. Notice that $Bd(U)$ is embedded in V as a strong deformation retract of V' and $V\backslash Bd(U)$ is topologically equivalent to $V\backslash V'$. Now $Bd(U)$ consists of $2|S|$ vertices, $2|R|$ 1-cells and $|R|$ 2-cells. And $V\backslash Bd(U)$ consists of a sphere with $|\gamma(P)|$ holes, to which must be added $|\gamma(P)|-1$ 1-cells to produce a cell decomposition, and the $\eta - |\gamma(P)|$ cells which are the counterparts of the cells added to components of $Bd(V')$ remaining after the identification of V and Q. This is a cell structure of V with $|R| + 1 + \eta - |\gamma(P)|$ 2-cells, $2|R| + |\gamma(P)|-1$ 1-cells and $2|S|$ vertices. Therefore $\chi(V) = \eta + 2|S| + 2 - 2|\gamma(P)|-|R|$. □

The construction of V includes a number of options which causes speculation concerning the uniqueness of V. The following theorem shows that the options do not effect the net product.

Theorem 3.10. The vertex manifold V of an ordered presentation P is uniquely determined by P up to a homeomorphism.

Proof. Since V is a closed, orientable 2-manifold, then V is characterized by its Euler characteristic. Further the $\chi(V)$ is computable from the presentation. This completes the proof. □

CHAPTER IV

In this chapter the complex $J(P)$ is embedded in a three-dimensional CW-Complex M such that $\pi(M)$ is presented by P. It is shown that M contains at most one non-manifold point, and necessary and sufficient conditions are found on P for M to be a 3-manifold.

The regular neighborhood U of the vertex v in $J(P)$ can be expressed as $c(Bd(U),v)$, the cone over $Bd(U)$ with cone-point v. Let $i:Bd(U) \to V$ be the inclusion map. Then i can be extended to $i:c(Bd(U),v) \to c(V,v)$. Let $M' = J(P) \cup c(V,v)$, where $s \in J(P)$ is identified to $i(s) \in c(V,v)$.

Let $\overline{C}_x = cl(C_x \backslash U)$, that is $\overline{C}_x$ is a 1-cell contained in the generating loop C_x with endpoints in V and interior missing U.

<u>Lemma 4.1</u>. Each point in $M' \backslash (\{v\} \cup \{\overline{C}_x | X \in S\})$ is contained in a neighborhood whose closure is homeomorphic to a 3-ball.

Proof. Three types of points must be examined. Any point in $c(V,v)\backslash V \cup \{v\}$ is a manifold point since $c(V,v)\backslash V \cup \{v\}$ is topologically equivalent to $V \times (0,1)$. The second type is a point p such that $p \in M'\backslash\{v\} \cup \{\overline{C}_X | X \in S\}$ and such that $p \notin c(V,v)$. Then p lies in some E_r, either in the interior or on the boundary. If p is on the boundary of E_r there is a neighborhood about p which misses the equator and V, and therefore p is a manifold point. In other instance, with p in the interior, p is clearly a manifold point.

The third type point is one which lies in V, say q. If there is a neighborhood of q which lies entirely in $c(V,v)$, then there is a neighborhood of q whose closure is a 3-ball. Suppose each neighborhood of q intersects E_r. Then q is not a member of B, and we can choose a neighborhood of q, say O, such that $cl(O\backslash c(V,v))$ is a 3-ball, $O \cap c(V,v)$ is a 3-ball and the intersection of these two is a 2-cell in V. Therefore $cl(O)$ is a ball, which completes the proof. □

The next step is to attach 3-cells, in the form of wedges, along each $\overline{C}_X$ so that the points of $\overline{C}_X$ will no longer be non-manifold points. Let $\Gamma = c(V,v)\backslash V$, then $R(\overline{C}_X,M,4)\backslash\Gamma$ can be expressed as $(W_X \times [0,1]) \cup \overline{W}_X \cup \overline{W}_{\overline{X}}$, where $W_X \times \{0\} = W_X \subset \overline{W}_X$ and $W_X \times \{1\} = W_{\overline{X}} \subset \overline{W}_{\overline{X}}$. It has been established in Lemma 3.4 that $\overline{W}_X$ and $\overline{W}_{\overline{X}}$ are both 2-cells. Let D_X be a 2-cell, and let $h_X:\overline{W}_X \to D_X$ be a homeomorphism. Define $\overline{h}_X:(W_X \times [0,1]) \cup \overline{W}_X \cup \overline{W}_{\overline{X}} \to D_X \times [0,1]$ by $\overline{h}_X(s,t) = (h_X(s),t)$. Let $M = M' \cup \{D_X \times [0,1] | X \in S\}$ where $(s,t) \in W_X \times [0,1] \cup \overline{W}_X \cup \overline{W}_{\overline{X}}$ is identified to $\overline{h}_X(s,t)$.

<u>Lemma 4.2</u>. Each point of $M\setminus\{v\}$ has a neighborhood whose closure is a 3-ball.

<u>Proof</u>. We can express M as the union of two sets as follows: $M = Cl(M\setminus \cup \{D_X \times [0,1] | X \in S\} \cup (\cup\{D_X \times [0,1] | X \in S\})$. A point $p \in cl(M\setminus \cup \{D_X \times [0,1] | X \in S\}$, except v possibly, has a neighborhood whose closure is a 3-ball. The same is true for each of the points in $D_X \times [0,1]$ for each $X \in S$. The intersection of these two sets is a collection of 2-manifolds with boundary. There is a manifold for each generator X. If $X \in S$, $Cl(M\setminus \cup \{D_y \times [0,1] | Y \in S\} \cap D_X \times [0,1]$ is actually $\overline{W}_X \cup \overline{W}_{\overline{X}}$ with several 2-cells in the form of strips attaching $Bd(\overline{W}_X)$ to $Bd(\overline{W}_{\overline{X}})$. Therefore each point of the intersection has a neighborhood, say O, whose closure can be written as the union of two closed 3-balls, one in $cl(\cup\{D_y \times [0,1] | Y \in S\})$ and one in $Cl(M\setminus \cup \{D_y \times [0,1] | Y \in S\}$ whose intersection is a 2-cell. Therefore the closure of O is a 3-ball. □

<u>Lemma 4.3</u>. The complex M contains $K(P)$ as a deformation

retract.

Proof. Each $D_x \times [0,1]$ has $W_x \times [0,1] \cup \overline{W}_x \cup \overline{W}_{\overline{x}}$ as a deformation retract, and since $D_x \times [0,1] \cap M' = W_x \times [0,1] \cup \overline{W}_x \cup \overline{W}_{\overline{x}}$, we have that M' is a deformation retract of M. The regular neighborhood of v has been written as $c(V,v)$, and $V\backslash Bd(U)$ is a collection of open discs, one of which may be punctured several times. Therefore M', which can be written as $J(P) \cup c(V\backslash Bd(U),v)$ collapses to $J(P)$. Now $K(P)$ is a deformation retract of $J(P)$, and therefore $K(P)$ is a deformation retract of M which completes the proof. □

The following lemma follows immediately.

Lemma 4.4. $\pi(M)$ is isomorphic to the group presented by P.

Proof. In Chapter I we established that $\pi(K(P))$ is isomorphic to the group presented by P. Further, $K(P)$ is a deformation retract of M, and therefore $\pi(K(P))$ is isomorphic to $\pi(M)$. This completes the proof. □

By the boundary of M, denoted by $Bd(M)$, we will mean all points of $M\backslash\{v\}$ which do not have a neighborhood topologically equivalent to Euclidean 3-space. There are three sources for points in $Bd(M)$. The first is $V\backslash V'$.

The second is a bit more difficult to describe. Recall that D_r' and D_r'' are the two hemispheres of the boundary of the three cell E_r . Let $F_r' = Cl(D_r' \setminus \cup \{D_X \times [0,1] | X \in S\})$ and $F_r'' = Cl(D_r'' \setminus \cup \{D_X \times [0,1] | X \in S\})$. Then F_r' and F''_r are discs which lie in $Bd(M)$. Finally, $Cl(Bd(\overline{W}_X) \setminus Bd(W_X))$ is a collection of $|X|$ 1-cells with endpoints x_{2i} and $x_{(2i+1)mod(|X|)+1}$ for $i=1,2,3,\cdots,|X|$. Each such 1-cell will be identified as $\langle x_{2i}, x_{(2i+1)mod(|X|)+1}\rangle$ depending on its endpoints. Then the third source of boundary points are the 2-cells $\langle x_{2i}, x_{(2i+1)mod(|X|)+1}\rangle \times [0,1]$. For convenience we suppress $mod(|X|)+1$ and write simply $2i+1$, except for emphasis.

<u>Lemma 4.5</u>. The boundary of M is a closed 2-manifold.

<u>Proof</u>. There is only one potential non-manifold point in M , namely v . Therefore $Cl(M \setminus R(v,M,6))$ is a 3-manifold with boundary. Further each component of $Bd(M)$ is a component of the boundary of $Cl(M \setminus R(v,M,6))$. Since each component of the boundary of a 3-manifold is a closed 2-manifold, we have that $Bd(M)$ is a closed 2-manifold. □

We recall that the symbols $|R|$, η and $|\gamma(P)|$ respectively represent the total number of appearances in all of the relations of R , the number of cycles of the ordered presentation and the number of components of the graph

$\gamma(P)$. The symbol β is introduced to denote the number of relations in R .

<u>Lemma 4.6</u>. The Euler Characteristic of the boundary of M is given by $\chi(Bd(M)) = \eta - 2|\gamma(P)| - |R| + 2 + 2\beta$.

<u>Proof</u>. In the proof of Theorem 3.9 a cell decomposition of $V\backslash V'$ with $(\eta - |\gamma(P)| + 1)$ 2-cells in employed. We use the same one here. Also there are two 2-cells in $Bd(M)$ for each relator for a total of 2β . Finally there are $|R|$ 2-cells of the form $\langle x_{2i}, x_{2i+1}\rangle \times [0,1]$. This exhausts the 2-cells in $Bd(M)$, and therefore, one cell decomposition of $Bd(M)$ has a total of $(\eta - |\gamma(P)| + 1 + |R| + 2\beta)$ 2-cells.

Recalling that $Bd(V')$ is a collection of simple closed curves, we can exclude $Bd(V')$ from the computation. This excludes all of the vertices and all but $|\gamma(P)|-1$ 1-cells in Q and the $2|R|$ 1-cells of the form $\{x_{2i-1}\} \times [0,1]$ and $\{x_{2i}\} \times [0,1]$. Therefore $\chi(Bd(M)) =$ $\eta - |\gamma(P)| + |R| + 2\beta - (|\gamma(P)| - 1 + 2|R|) = \eta - 2|\gamma(P)| + 2 - |R| + 2\beta$. □

In order to identify $Bd(M)$, it is necessary to determine how many components it has. First, consider the closure of $Bd(M\backslash V)$, which is the union of discs of the form F_r' , F_r'' and $\langle x_{2i}, x_{(2i+1)}\rangle \times [0,1]$. The various cells of the types F_r' and F_r'' are pairwise disjoint. But each disc of the form $\langle x_{2i}, x_{(2i+1)}\rangle \times [0,1]$ is attached, along the 1-cells $\{x_j\} \times [0,1]$, to cells of the type F_r' or F_r''. This provides a connecting link between certain of these pairs. Which of these 2-cells are so connected is readable from the ordered presentation in the following manner. If X_i is an appearance in r and $X_{(i+1)\text{mod}(|X|)+1}$ is an appearance in s then F_r' and F_s'' will be connected by the disc $\langle x_{2i}, x_{(2i+1)\text{mod}(|X|)+1}\rangle \times [0,1]$. F_r' and F_s' will be similarly joined if X_i is an appearance in r and $X^{-1}_{(i+1)\text{mod}(|X|)+1}$ is an appearance in s. F_r'' and F_s'' will be joined by if X_i^{-1} is an appearance in r and $X_{(i+1)\text{mod}(|X|)+1}$ is an appearance in s. Finally, F_r'' and F_s' will be connected in this fashion whenever X_i^{-1} and $X^{-1}_{(i+1)\text{mod}(|X|)+1}$ are in r and s respectively.

Once again a matrix can be employed to determine the number of components of a complex. Let H be the 2β by 2β matrix with each row and column associated with one of the discs of the types F_r' or F_r''. Each entry is zero except where the row and column represent discs which are connected by some $\langle x_{2i}, x_{2i+1}\rangle \times [0,1]$, in which case the entry is one.

<u>Lemma 4.7</u>. The number of components of $Cl(Bd(M)\backslash V)$ is

the row rank of the adjusted matrix $\Sigma_{k=1}^{2\beta-1}H^k$.

Proof. This is an immediate consequence of Theorem 2.10. □

This does not determine the components of Bd(M) as yet, due to the facts that one component of Cl(Bd(M)\V) may not be attached to Q or may be attached to Q along several simple closed curves. That is Bd(M)\(Bd(M) ∩ V) consists of Q , a 2-sphere with $|\gamma(P)|$ holes, and a collection of 2-cells. Then of course each component of Cl(Bd(M)\V) which is attached to Q reduces the number of component of Bd(M) by one from the number of components of Cl(Bd(M)\V) . To compute the number of components of Bd(M) , it must be determined by inspection if the numbers of components of Cl(Bd(M)\V) is reduced by $\gamma(P)$ or fewer by attaching Q .

Since (Bd(M)) and the number of components of Bd(M) are both computable, all that is required to determine the identity of Bd(M) is to determine whether or not Bd(M) is orientable. First the orientability of M is considered.

Lemma 4.8. The complex M is orientable.

Proof. Theorem 3.8 includes the fact that V is orientable. Further an orientation on V induces an orientation on c(V,v) , M can be realized from c(V,v) by attaching |S| handles to c(V,v) , one for each generator, in an orientation preserving manner, and then attaching β cells so that each intersects $c(V,v)$ and the handles along an annulus. It follows that M is orientable. □

Lemma 4.9. The boundary of an orientable 3-manifold with boundary is orientable.

Proof. Let N be an orientable 3-manifold with boundary. If $Bd(N)$ is not orientable then $Bd(N)$ contains a copy of a Moibus band. Since N is orientable, $R(C,N,\sigma)$ a torus. But this is a contradiction since $R(C,Bd(N),\sigma) \subset Bd(R(C,N,\sigma))$ and the boundary of a torus cannot contain a Moebius band. □

Lemma 4.10. $Bd(M)$ is orientable.

Proof. Let $\Gamma' = R(v,M,\sigma)$. Then $Cl(M\backslash\Gamma')$ is an orientable 3-manifold. Further $Bd(M)$ is a component of $Bd(Cl(M\backslash\Gamma'))$. Lemma 4.9 implies that $Bd(Cl(M\backslash\Gamma'))$ is orientable, and therefore $Bd(M)$ being a component is orientable. □

At this point, beginning with an ordered presentation P , we can construct an orientable three dimensional cell complex M such that $\pi(M)$ is presented by P , and v is the only possible non-manifold point of M . Further, necessary and sufficient conditions for M to be a 3-manifold with boundary can now be given.

<u>Theorem 4.11</u>. M is a 3-manifold with boundary if and only if $\chi(V) = 2$, that is V is a sphere.

<u>Proof</u>. It is clear that if V is a 2-sphere then since $c(V,v)$ is a 3-ball, M is a manifold. On the other hand, if M is a manifold, then V is a manifold with spine v , that is U collapses to v . Then all regular neighborhood of v must be a ball, it follows that U is a ball and V is a 2-sphere. □

An interesting, but known, fact can be stated as a corollary. If Bd(M) is also a 2-sphere then be adding a ball to M a 3-manifold with empty boundary is obtained with fundamental group presented by P . From the equation $\chi(V) = \chi(Bd(M))$ the fact that $|S| = \beta$ can be inferred. Which says that the number of generators must equal the number of relators if a closed manifold is to be produced in this manner.

CHAPTER V

We now have necessary and sufficient conditions on an ordered presentation, P , for the complex, K(P) , to be the spine of an orientable 3-manifold with boundary. In this chapter we relax the orientability condition. The techniques and construction of Chapters II, III and IV are modified so that the resulting manifold may be non-orientable. We begin by introducing the notion of an oriented presentation.

Definition 5.1. By an orienting function for an ordered presentation, $P = (\varphi, S, R)$, we mean a function $\theta : S \to \{1, -1\}$. By an oriented ordered presentation P , we mean an ordered presentation P and an orienting function.

If P is an oriented ordered presentation then the CW-complex associated with P , K(P) , is not altered from that defined in Definition 1.2. Similarly the CW-complex J(P) as given in Definition 3.1 remains unchanged. In fact, the first change in the construction occurs in the naming scheme for four points on Bd(D(s,t)) were s and t are an adjacent pair. The naming scheme for the orientable case was presented in Table 3.3. We now list the changes which need to be made to introduce non-orientability.

Table 5.2.

(1')	$D(X_i, Y_j)$	x_{2i-1}	x_{2i}	$\overline{y}_{2j-1}$	$\overline{y}_{2j}$
(3')	$D(X_i^{-1}, Y_j)$	$\overline{x}_{2i-1}$	$\overline{x}_{2i}$	$\overline{y}_{2j}$	$\overline{y}_{2j-1}$
(4')	$D(X_i^{-1}, Y_j^{-1})$	$\overline{x}_{2i-1}$	$\overline{x}_{2i}$	y_{2j-1}	y_{2j}

Table 5.2 presents the adjustments to Table 3.3. There are corresponding adjustments to Table 2.4 which can be stated as follows:

Table 5.3.

(1) if X_i is followed by Y_j then $(x_{2i-1}, \overline{y}_{2j})$, $(x_{2i}, \overline{y}_{2j-1}) \in T$.

(3) if X_i^{-1} is followed by Y_j then $(\overline{x}_{2i-1}, \overline{y}_{2j-1})$, $(\overline{x}_{2i}, \overline{y}_{2j}) \in T$.

(4) if X_i^{-1} is followed by Y_j^{-1} then $(\overline{x}_{2i-1}, y_{2j})$, $(\overline{x}_{2i}, y_{2j-1}) \in T$.

These adjustments are applied depending on the value of the orientating function. If $\theta(X) = 1$ and $\theta(Y) = 1$ then Table 3.3 and Table 2.4 are used as they stand. If $\theta(X) = 1$ and $\theta(Y) = -1$, substitute 3' and 4' for 3 and

4. And if $\theta(X) = -1$ and $\theta(Y) = -1$ substitute $1'$ and $4'$ for 1 and 4. What we accomplish by making these substitutions is a reversing of the orientation on $Bd(\overline{W}_x)$ if $\theta(X) = -1$. That is, the discs $\overline{W}_x$ and $\overline{W}_{\overline{x}}$ are now constructed in the same fashion as Chapter III, with the result being that the order of ascending subscripts of the points $\overline{x}_i$ on $Bd(\overline{W}_{\overline{x}})$ and the points $\overline{y}_i$ on $Bd(\overline{W}_{\overline{y}})$ are in different directions if $\theta(X) \neq \theta(Y)$. We now investigate the result of these changes.

<u>Lemma 5.3</u>. Let P be an oriented ordered presentation and V' be the cell complex constructed as in Chapter III, using the appropriate substitutions from Table 5.2, then V' is orientable.

<u>Proof</u>. The boundaries of the regular neighborhoods of x and $\overline{x}$, for each $X \in S$ are oriented as follows: The points $x_1, x_2, x_3, \cdots, x_{|X|}$ appear in this order on $Bd(\overline{W}_x)$. We take as the positive direction the direction of increasing subscripts. The points $\overline{x}_1, \overline{x}_2, \overline{x}_3, \cdots, \overline{x}_{|X|}$ appear in this order on $Bd(\overline{W}_{\overline{x}})$. If $\theta(X) = 1$ the positive direction around $Bd(\overline{W}_{\overline{x}})$ is taken in the direction of descending subscripts. If $\theta(X) = -1$, the positive direction around $Bd(\overline{W}_{\overline{x}})$ is taken in the direction of ascending subscripts. Now each of the regular neighborhoods of the points $\{y | y = x$ or $y = \overline{x}, X \in S\}$ is oriented. Whenever one of these neighbor-

hoods is contiguous to a disc $D'(s,t)$ it induces an orientation on $D'(s,t)$. Then each disc $D'(s,t)$ has orientations induced upon it from two sources. We need to show that these two agree and each of the substitute naming schemes, $1'$, $3'$, and $4'$, must be examined with regard to this agreeing. If the manifold is constructed using Table 3.3, we have already established the theorem in Theorem 3.7.

<u>Case 1</u>. Suppose the four points to be named on $Bd(D'(X_i,Y_j))$ are named using $1'$ of Table 5.2. Proceeding around $Bd(D'(X_i,Y_j))$ we encounter, in order, the points x_{2i-1}, x_{2i}, $\overline{y}_{2j-1}$, and $\overline{y}_{2j}$. Since we are employing $1'$ we know $\theta(Y) = -1$. In which case $Bd(\overline{W}_{\overline{y}})$ is positive in the direction of ascending subscripts. Therefore the orientations induced on $Bd(D'(X_i,Y_j))$ agree.

<u>Case 2</u>. Line $3'$ is used from Table 5.2 to name the four points on $Bd(D'(X_i^{-1},Y_j))$. Proceeding around $Bd(D'(X_i^{-1},Y_j))$ the subscripts descend on the $\overline{x}'s$ and ascend on the $\overline{y}'s$. Therefore the orientations on $\overline{W}_{\overline{x}}$ and $\overline{W}_{\overline{y}}$ differ in the sense that one must be positive in the direction of ascending subscripts and the other negative. This is exactly what happens since $3'$ is substituted only when $\theta(X) \neq \theta(Y)$.

Case 3. Line 4' of Table 5.2 is used to name the four points on $Bd(D'(X_i^{-1}, Y_j^{-1}))$. Notice in one direction both the $\bar{x}'$s and y's are encountered in order of descending subscripts. But 4' is used only when $\theta(X) = -1$. And in this case both $Bd(\overline{W}_{\bar{x}})$ and $Bd(\overline{W}_y)$ are positive in the order of increasing subscripts. Therefore the two orientations induced on $Bd(D'(X_i^{-1}, Y_j^{-1}))$ agree. This completes Case 3 as well as the proof. □

The number of cycles of an ordered presentation plays a significant role in the construction of V in Chapter III. The number of cycles depended upon the ordering of the presentation. Now that the concept of an orientation function has been introduced, we need to investigate it's effect on the number of cycles and consequently on V . It is necessary when there is a choice of orders and orientations for P to be specific in our notation. Therefore if ξ is an ordering and θ is an orientation for P , we denote the number of cycles by $\eta(\xi,\theta)$, we denote the relation T defined in Table 2.4 by $T(\xi,\theta)$ and similarly V' and V by $V'(\xi,\theta)$ and $V(\xi,\theta)$. Further, recalling that ξ was a collection of functions, one for each generator, we denote by ξ_X the function of ξ associated with X . We let $\overline{\theta}$ be the orientation such that $\overline{\theta}(X) = 1$ for all $X \in S$. Employing this symbolism we can state the following lemma.

Lemma 5.4. Let P be a presentation with ordering ξ and orientation θ. Then there is an ordering ξ' such that $\eta(\xi,\theta) = \eta(\xi',\overline{\theta})$.

Proof. We define ξ' by defining ξ'_X for each $X \in S$ as follows:

$$\xi'_X = \begin{cases} |x| + 1 - \xi_X & \text{if } \theta(X) = -1 \\ \xi_X & \text{if } \theta(X) = 1 \end{cases}.$$

There is a one-to-one function γ from $T(\xi,\theta) \cup T'$ onto $T(\xi',\overline{\theta}) \cup T'$ defined as follows:

If $(x_i,y_j) \in T(\xi,\theta)$ let $\gamma((x_i,y_j)) = (x_i,y_j)$.

If $(x_i,\overline{y}_j) \in T(\xi,\theta)$ let

$$\gamma((x_i,\overline{y}_j)) = \begin{cases} (x_i,\overline{y}_{2|y|+1-j}) & \text{if } \theta(Y) = -1. \\ (x_i,\overline{y}_j) & \text{if } \theta(X) = 1. \end{cases}$$

If $(\overline{x}_i,y_j) \in T(\xi,\theta)$ let

$$\gamma(\overline{x}_i,y_j) = \begin{cases} (\overline{x}_{2|x|+1-i},y_j) & \text{if } \theta(X) = -1 \\ (\overline{x}_i,y_j) & \text{if } \theta(X) = 1 \end{cases}$$

If $(\overline{x}_i,\overline{y}_j) \in T(\xi,\theta)$ let

$$\gamma((\overline{x}_i,\overline{y}_j)) = \begin{cases} (\overline{x}_i,\overline{y}_j) & \text{if } \theta(X) = 1 \text{ and } \theta(Y) = 1 \\ (x_i,\overline{y}_{2|y|+1-j}) & \text{if } \theta(X) = 1 \text{ and } \theta(Y) = -1 \\ (\overline{x}_{2|x|+1-i},y_j) & \text{if } \theta(X) = -1 \text{ and } \theta(Y) = 1 \\ (\overline{x}_{2|x|+1-i},\overline{y}_{2|y|+1-j}) & \text{if } \theta(X) = -1 \text{ and } \theta(Y) = -1 . \end{cases}$$

If $(x_i,x_j) \in T'$ let

$\gamma((x_i,x_j)) = (x_i,x_j)$ and

$$\gamma((\overline{x}_i,\overline{x}_j)) = \begin{cases} (\overline{x}_i,\overline{x}_j) & \text{if } \theta(X) = 1 \\ (\overline{x}_{2|x|+1-j},\overline{x}_{2|x|+1-i}) & \text{if } \theta(X) = -1 . \end{cases}$$

Now a cycle of P when P is oriented and ordered by ξ and θ is subset of $T(\gamma,\theta) \cup T'$. We claim that the image under γ of a cycle in $T(\xi,\theta) \cup T'$ is a cycle in $T(\xi',\overline{\theta}) \cup T'$. To see this we need only take a pair from $T(\xi,\theta)$ and a pair from T' which form a link and show that their images under γ form a link.

A pair from $T(\gamma,\theta)$ and a pair from T' which form a link have a common entry. But from the definition of γ, if two pairs have a common entry images in $T(\gamma,\theta)$ and T' have a common entry, and therefore form a link. □

<u>Construction of the Vertex Manifold</u> $V(\xi,\theta)$. Let P be a presentation ordered and oriented by ξ and θ respectively. We have constructed the 2-manifold with boundary $V'(\xi,\theta)$ which has a boundary consisting of $\eta(\xi,\theta)$

simple closed curves. The number of components of $V(\xi,\theta)$, $\gamma(P)$, is independent of both the order and the orientation. The vertex manifold, $V(\xi,\theta)$, is constructed as in Chapter III. That is a simple closed curve is chosen from each of the components of the boundary of $V'(\xi,\theta)$, and a 2-sphere Q with the interiors of $|\gamma(P)|$ disjoint discs removed is attached to $V'(\xi,\theta)$ by identifying each of the selected simple closed curves to the boundary of one of the missing discs in Q . The manifold V is then completed by attaching a disc to each remaining boundary components of $V'(\xi,\theta) \cup Q$. Nothing in the new construction invalidates the proof of Theorem 3.8, and therefore $V(\xi,\theta)$ is a connected closed orientable 2-manifold.

<u>Theorem 5.5.</u> Let P be a presentation with order ξ and orientation θ . Then there is an orientation ξ' such that $V(\xi,\theta)$ is topologically equivalent to $V(\xi',\overline{\theta})$.

<u>Proof</u>. The closed 2-manifold $V(\xi,\theta)$ characterized by its Euler characteristic which is given by $\chi(V(\xi,\theta)) = \eta(\xi,\theta) + 2|S| + 2 - 2|\gamma(P)| - R$. And Lemma 5.4 says there is a ξ' such that $\eta(\xi,\theta) = \eta(\xi',\overline{\theta})$. Further, none of the other terms in the expression for X are dependent on θ . Therefore $\chi(V(\xi,\theta)) = \chi(\xi',\overline{\theta}))$ and we prove the theorem. □

<u>Construction of the near manifold</u> $M(Y, \ )$. For each $X \in S$, let F_X be a 2-cell and h_X be a homeomorphism from F_X onto $\overline{W}_X$. Let $T_{\overline{X}} = K(P) \cap \overline{W}_{\overline{X}}$ and $T_X = K(P) \cap \overline{W}_X$. Let $h_{\overline{X}}: \overline{W}_X \to \overline{W}_{\overline{X}}$ be a homeomorphism such that $h_{\overline{X}}(x_i) = \overline{x}_i$ and $h_{\overline{X}}(T_X) = T_{\overline{X}}$. Then attach the cell $F_X \times [0,1]$ to $C(V,v)$ by the attaching maps $h_X: F_X \times \{0\} \to \overline{W}_X$ defined by $h'_X(t,0) = h_X(t)$ and $h'_{\overline{X}}: F_X \times \{1\} \to \overline{W}_X$ defined by $h'_{\overline{X}}(t,1) = h_{\overline{X}}(t)$. Let $M' = c(V,v) \cup \{F_X | X \in S\}$. Then a regular neighborhood of the generating loops, $R(B,K(P),\alpha)$ is topologically equivalent to a subset of M' . This subset, which we call T , can be expressed as $T = c(K(P) \cap V,v) \cup \{T_X \times [0,1] | X \in S\}$.

<u>Lemma 5.6</u>. T is a strong deformation retract of M .

<u>Proof</u>. $C(V,v)$ collapses onto $C(V',v)$. Each handle, $F_X \times [0,1]$ collapses onto $T_X \times [0,1] \cup \overline{W}_X \cup \overline{W}_{\overline{X}}$. This leaves $C(V',v) \cup \{T_X \times [0,1] | X \in S\}$ which collapses to $C(K(P) \cap V,v) \cup \{T_X \times [0,1] | X \in S\}$ since V' collapses to $K(P) \cap V$.

Now $T \cap BdM'$ is a collection of simple closed curves, one such curve for each relator r , say C_r . Attach a 2-cell F_r to M' by identifying $Bd(F_r)$ to C_r . The resulting complex collapses to a 2-complex which is topologically equivalent to $K(P)$. Under what conditions can the discs F_r be fattened. The answer to this is if $R(C_r, BdM', 4)$ is a annulus.

Lemma 5.7. $R(C_r, BdM', 4)$ is an annulus if and only if the numbers of appearances of generators in R with orientation number -1 is even.

Proof. $R(C_r, BdM', 4)$ can be written as the union of discs of two types. The first type is what has been called D(s,t) where s and t form an adjacent pair of appearances in r . The remaining discs $cl(R(C_r, BdM', 4) \backslash V$ can each be expressed in the form $\langle x_i, x_{2i+1} \rangle \times [0,1]$ a subset of $T_x \times [0,1]$. In the construction, $\langle x_{2i}, x_{2i+1} \rangle \times \{1\}$ was identified to $\langle x_{2i}, x_{2i+1} \rangle$ with points of some subscripts being identified. But if (X) = -1 this introduced a half twist because of the way the points on Bd D(s,t) were named. Each time C_r crosses a handle of a generator with orientation number -1 a half twist is added.

Suppose M has the property that for each relator r , $R(C_r, BdM, 4)$ is an annulus. Then a 3-cell expressed as $H_r \times [0,1]$ can be attached to M' such that $Bd(H_r \times [0,1]) = R(C_r, BdM', 4)$. Let $M = M' \cup \{H_r \times [0,1] | r \in R\}$. □

Lemma 5.8. Each point of $M \backslash \{v\}$ has a neighborhood with closure topologically equivalent to a 3-cell.

<u>Proof</u>. Each point of $c(V,v)\setminus\{v\}$, $T_x \times [0,1]$ and $F_r \times [0,1]$ which does not lie in the discs $\overline{W}_x$, $\overline{W}_{\overline{x}}$ or the annulus $R(C_r, BdM', 4)$ certainly has a neighborhood of the required sort. Any point of M' lying in a disc $\overline{W}_x$ or $\overline{W}_{\overline{x}}$ has a neighborhood in $C(V,v)$ with closure of a 3-cell as well as a neighborhood in $T_r \times [0,1]$ with closure of a 3-cell and the intersection of these two neighborhoods is a 2-cell in V so that their union is a 3-ball. Therefore each point of $M\setminus\{v\}$ has a neighborhood in M with the closure topologically equivalent to a 3-cell.

If p is a member of $R(C_r, BdM', 4)$ then p has a neighborhood in $F_r \times [0,1]$ and a neighborhood in M' both closures equivalent to 3-cells with intersection equivalent to a 2-cell.

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