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Pseudo-Interiors of Hyperspaces.

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FOREWORD

The title of this dissertation refers more to the latter two chapters than to the work as a whole. Chapter I gives an exposition of the topology of Z-sets and capsets in Q , such as developed by Anderson in [2], [3] and [4]. However, the proofs and organization are rather different, and various simplifications have been made. Most of what is new in this chapter has been included in the Master's Thesis of the author [15].

An alternative treatment on Z-sets can be found in Chapters I and II of T. A. Chapman's Notes on Hilbert Cube Manifolds (unpublished).

Chapters II and III consist entirely of new material.

My first acquaintance with Infinite-Dimensional Topology was through a course taught in "Texas-style" by Professor R. D. Anderson during his stay in Amsterdam in 1970-1971. I feel very much indebted to him for this most inspiring introduction to his field. Several proofs in Chapter I resulted from work I did for this course.

The material for Chapters II and III was developed during my stay in 1972-1974 at LSU, under partial support of NSF grant GP 34635X . I received much help and encouragement - in the form of discussions, suggestions, comments and readings of various versions of the manuscript - from Professors R. D. Anderson, D. W. Curtis and R. M. Schori.

Finally, I wish to thank the typist, Monica Loftin, for the excellent job she has done.

TABLE OF CONTENTS

CHAPTER		PAGE
	FOREWORD	ii
	ABSTRACT	v
	INTRODUCTION	1
I	HILBERT CUBE TOPOLOGY	
	Preliminaries	6
	The Homeomorphism Extension Theorem for Compact Subsets of s	9
	The Homeomorphism Extension for Z-sets	17
	Capsets	26
	Alternative Definitions of Z-sets	36
II	PSEUDO-INTERIORS FOR 2^Q AND $C(Q)$ AND RELATED RESULTS	42
III	PSEUDO-INTERIORS FOR 2^I	53
	BIBLIOGRAPHY	66
	VITA	69

ABSTRACT

The Hilbert cube Q is the countable infinite product of intervals I^∞ - where $I = [-1,1]$, topologized by the product topology and furnished with a suitable metric. Points of Q are denoted by $x = (x_i)_i$, with $x_i \in I$. We consider the following subsets of Q:

1. The pseudo-boundary $B(Q) = \{x \mid \text{for some } i, |x_i| = 1\}$.
2. Its complement $s = (-1,1)^\infty$ which is called the pseudo-interior of Q. It is shown by R. D. Anderson that s is homeomorphic to $(\cong)$ ℓ_2.
3. The closed subsets K of Q such that for each $\epsilon > 0$ there exists a map $f:Q \to Q-K$ with $d(f,id_Q) < \epsilon$. These are called Z-sets.

In Chapter I, certain well-known theorems about these subsets are proved. We mention especially:

1. The Homeomorphism Extension Theorem: any homeomorphism between two Z-sets in Q can be extended to an autohomeomorphism of Q.
2. The non-empty Z-sets are exactly those closed subsets of

Q which can be mapped by an autohomeomorphism of Q onto a set which projects onto a point in infinitely many coordinates.

3. A topological characterization of the pseudo-boundary.

Let 2^X be the space of all non-empty compact subsets of a metric space X and let C(X) be the space of non-empty compact connected subsets, both with the Hausdorff metric d_H which is defined by $d_H(A,B) = \inf\{\epsilon \mid A \subset U_\epsilon(B)$ and $B \subset U_\epsilon(A)\}$. D. W. Curtis and R. M. Schori showed that $2^X \cong Q$ for X a non-degenerate Peano continuum, and that $C(X) \cong Q$ for X a non-degenerate Peano continuum without free arcs. In particular, it follows that $2^X \cong C(X) \cong Q$ if $X \cong Q$ or X is a compact connected Q-manifold. Also, we have $2^I \cong Q$, which was proved earlier by Schori and West.

In Chapter II, it is shown that the collection of non-empty Z-sets in Q (or in a compact connected Q-manifold M) is a topological pseudo-interior for 2^Q (or 2^M). As a corollary one obtains that $2^{\ell_2} \cong \ell_2$, and that $2^M \cong \ell_2$ for M a connected ℓ_2-manifold. Corresponding results are obtained for the collection of non-empty Z-sets in C(Q) or C(M), and also it follows that $C(\ell_2) \cong \ell_2$ and that $C(M) \cong \ell_2$ for M a connected ℓ_2-manifold.

In Chapter III, it is shown that the collection of topological Cantor sets in the interval I and the collection of non-empty zero-dimensional subsets of I are topological pseudo-interiors for 2^I.

INTRODUCTION

By the Hilbertcube Q we mean the countable infinite product of intervals I^∞ or $[-1,1]^\infty$ with the product topology. If $x = (x_i)_{i\geq 1}$ and $y = (y_i)_{i\geq 1}$ are two points of Q, then their distance $d(x,y)$ is defined as $\Sigma_{i\geq 1} 2^{-i} \cdot |x_i - y_i|$. By the pseudo-interior of Q we mean the subset $s = (-1,1)^\infty$; its complement $Q-s$ is called the pseudo-boundary $B(Q)$ or BQ of Q. It is easily seen that s is a dense G_δ in Q. Anderson proved in [1] that s is homeomorphic to the Hilbertspace ℓ_2. The complement $B(Q)$ is also dense in Q, so the pseudo-boundary of Q is only a restricted infinite-dimensional analogue of the boundary of a finite-dimensional n-cell.

We consider two classes of subsets of Q, viz. Z-sets and capsets. Both concepts have played an important role in I-D topology and especially Z-sets are the focus of continued interest. K is a Z-set in Q if for every ϵ there exists a map $f: Q \to Q-K$ such that $d(f,id) < \epsilon$ (where id denotes the identity-mapping; sometimes we shall

write id_X instead of id for the identity-mapping on X).
The following facts are easy to verify:

1) The property of being a Z-set in Q is topologically invariant, i.e., invariant under autohomeomorphisms of Q.
2) A closed subset of a Z-set is a Z-set.
3) A finite or closed countable union of Z-sets is a Z-set.
4) Examples of Z-sets are compact subsets of s and closed subsets of Q which project onto a point in infinitely many coordinates. For there exist maps f of Q into itself whose image $f(Q)$ is disjoint from such a set, and such that f leaves the lower-numbered coordinates unchanged.

The definition of Z-set can be generalized for a larger class of spaces, in particular for Q-manifolds and (manifolds of) infinite-dimensional topological vector spaces. In Geoghegan-Summerhill [11] a version of Z-sets for Euclidean spaces is introduced. See also the remarks after Lemma II.1. Corollary I.9 and Theorem I.10 state the two most important facts about Z-sets: that (1) any Z-set in Q can be mapped by an autohomeomorphism of Q onto a subset which projects onto a point in infinitely many coordinates, and that (2) any homeomorphism between two Z-sets can be extended to an autohomeomorphism of Q.

A <u>capset</u> is a subset of Q which is equivalent to

B(Q) under an autohomeomorphism of Q . In Chapter I more practical characterizations will be given. It is a key observation that there exist capsets which are entirely contained in s (Proposition I.7). Other useful facts are that for any capset M and any Z-set K , M-K is a capset (Corollary I.13) and that the union of a capset with a countable number of Z-sets is again a capset (Proposition I.15). One use of capsets is to show that certain spaces are homeomorphic to ℓ_2 by exhibiting an embedding into Q with a capset as remainder. This principle will be applied in Corollaries II.3 and II.5.

In Chapter I the most important facts about Z-sets and capsets in Q will be proved. Our treatment is rather different from previous ones (most theorems from Chapter I have appeared originally in either of Anderson's papers [1] - [4]). The several stages in the proof of the Homeomorphism Extension Theorem for compact subsets of s are entirely standard. However, the autohomeomorphism of Q which maps B(Q) into s (Proposition I.7) is obtained by a direct geometrical construction, and requires very little preliminary work. This causes changes in the entire organization.

At the end of the chapter several alternative definitions of Z-set, equivalent for Q and s or ℓ_2 , will be given (Theorem I.18).

For X a metric space, the hyperspace 2^X of X is the collection of non-empty compact subsets of X (for non-compact X, in other treatments 2^X is sometimes understood to be the collection of all non-empty closed subsets), with metric $d_H(A,B) = \inf \{\epsilon \mid A \subset U_\epsilon(B)$ and $B \subset U_\epsilon(A)\}$, where $U_\epsilon(Y)$ denotes the ϵ-neighborhood of the set Y. If d and d' induce the same topology on X then d_H and d'_H induce the same topology on 2^X. This metric is called the Hausdorff metric. By $C(X)$ we denote the subspace of 2^X consisting of all non-empty subcontinua of X. We have the following:

Theorem A (Curtis-Schori [9]). *2^X is homeomorphic to the Hilbert cube iff X is a non-degenerate Peano-continuum.*

Theorem B (Curtis-Schori [9]). *$C(X)$ is homeomorphic to Q iff X is a non-degenerate Peano-continuum without free arcs* (i.e., not having a topological open interval as an open subset).

Remark 1. In [16], Schori and West proved Theorem A for the case $X = I$. This result is used in the proof of the general case.

Remark 2. It is easily seen that the hyperspace of non-empty subcontinua of an interval, $C(I)$, is homeomorphic to a two-cell.

In Chapter II, we identify pseudo-interiors for 2^X and $C(X)$, where X is a Hilbertcube (Theorem II.2) or a compact connected Hilbertcube manifold (Theorem II.4); viz. the collection of non-empty Z-sets in X for 2^X and the collection of non-empty connected Z-sets for $C(X)$. The proofs rest on the aforementioned results of Curtis and Schori, and for the case where X is a manifold also on the Triangulation Theorem for Q-manifolds [8]. As a corollary we obtain that, for $X \cong \ell_2$ or X an ℓ_2-manifold, both 2^X and $C(X)$ are homeomorphic to ℓ_2 (Corollary II.3 and II.5).

In Chapter III, we prove that both the collection of topological Cantor sets in I and the collection of non-empty zero-dimensional closed subsets of I form pseudo-interiors for 2^I (Theorem III.4). Here we use Schori-West [16].

Unfortunately, the author has been unable to generalize the above results, even for finite graphs instead of I. It might be worth mentioning that the proofs of Chapter II have very little in common with those of Chapter III.

CHAPTER I

HILBERTCUBE TOPOLOGY

Preliminaries. For each $n > 0$, we can write $Q = I^n \times Q_{n+1}$, where $Q_{n+1} = \Pi_{i \geq n+1} I_i$. By $p_n : Q \to I_n$ we mean the projection onto the n^{th} coordinate; by $p_n' : Q \to I^n$ the projection onto the first n coordinates. Note the difference between I^n and I_n. For any non-empty subset C of N, we write $Q_C = \Pi_{n \in C} I_n$, and $s_C = \Pi_{n \in C} \overset{\circ}{I}_n$, where $\overset{\circ}{I}_n$ is the combinatorial interior of I_n, and $p_C : Q \to Q_C$ the projection onto Q_C. The end-faces W_n^+ and W_n^- are the sets $\{x \in Q | x_n = 1\}$ and $\{x \in Q | x_n = -1\}$ respectively. Trivially $B(Q)$ is the union of all endfaces of Q. We call a subset K of Q deficient in the n^{th} coordinate if $p_n(K)$ is a point. We call K infinitely deficient if K is deficient in infinitely many coordinates.

Homeomorphisms are always understood to be onto. We write sometimes "$X \cong Y$" instead of "X is homeomorphic to Y", and "$(X,X') \cong (Y,Y')$" instead of "there exists a homeomorphism $h:X \to Y$ such that $h(X') = Y'$ ". The distance between two maps or homeomorphisms f and $g:X \to Y$, where Y is compact metric, is defined as $d(f,g) = \sup_{x \in X} d(f(x),g(x))$. If f is a homeomorphism, then obviously $d(f,id) = d(f^{-1},id)$ and $d(g,h) = d(gf,hf)$. For $f:X \to X$, we sometimes say that f is small (or ϵ-small) instead of "$d(f,id_X)$ is small (or less than ϵ)". The space of auto-homeomorphisms of a topological space X is denoted by $H(X)$.

One convenient property of the Hilbertcube is stated in the following

<u>Proposition I.1</u> (Mapping Replacement Theorem). <u>Let</u> $f:X \to Q$ <u>be a map from a separable metric space into the Hilbertcube. Then for each</u> $\epsilon > 0$, <u>there exists an embedding</u> $f':X \to Q$ <u>such that</u> $f'(X)$ <u>is an infinitely deficient subset of</u> s <u>and</u> $d(f',f) < \epsilon$.

<u>Proof</u>. It is well-known that every separable metric space can be embedded in s. So let $g:X \to s$ be any embedding. Define, for $\delta \in (0,1)$ and M any integer and $x \in X$, $f_{M,\delta}(x) = (\delta \cdot p_1 f(x), \cdots, \delta \cdot p_M f(x), p_1 g(x), 0, p_2 g(x), 0, \cdots)$. Then $f_{M,\delta}$ is an embedding because g is, and $f_{M,\delta}$ is ϵ-close to f if M is sufficiently large and δ sufficiently close to 1, because in that case f and $f_{M,\delta}$

"almost" coincide in the most significant coordinates.

For several constructions in this chapter, we obtain a homeomorphism with certain properties as a limit of inductively constructed homeomorphisms. We can ensure convergence _to a homeomorphism_ if at each stage the next homeomorphism can be chosen arbitrarily close to the identity. More formally, let $(f_i)_i$ be a sequence of maps $f_i : X \to X$ such that the sequence $f_1, f_2 \circ f_1, f_3 \circ f_2 \circ f_1, \cdots$ has a continuous limit; then the limit is denoted $L\Pi_i f_i$ and is called the _infinite left product_ of the sequence $(f_i)_i$. We have the following theorem (due to Fort [10] in a slightly different form):

Theorem I.2 (The Convergence Criterion). _Let X be a compact metric space and let $(h_i : X \to X)_i$ be a sequence of autohomeomorphisms. Then $L\Pi_i h_i$ is a homeomorphism if for any i_ (1) $d(h_{i+1}, id) < 2^{-i}$ _and_
(2) $d(h_{i+1}, id) < 3^{-i} \cdot \inf\{d(h_i \circ \cdots \circ h_1(x), h_i \circ \cdots \circ h_1(y)) \mid d(x,y) \geq 1/i\}$.

Proof. Convergence to a continuous limit is ensured by the Cauchy condition $d(h_{i+1}, id) < 2^{-i}$, which is equivalent to $d(h_{i+1} \circ \cdots \circ h_1, h_i \circ \cdots \circ h_1) < 2^{-i}$. Because X is compact, the only other thing we have to show is that $L\Pi_i h_i$ is one-to-one. This follows from (2) since points which are at least $1/i$ apart are prevented from being mapped onto the same point in the limit by the size-restrictions on $h_{i+1}, h_{i+2}, \cdots$.

<u>The Homeomorphism Extension Theorem for Compact Subsets of</u> s .
Eventually we want to prove the Homeomorphism Extension Theorem for Z-sets (Theorem I.10). For this we will need the concept of <u>basic core set</u> (<u>bcs</u>), to be defined later. It will be seen that any two basic core sets are equivalent under an autohomeomorphism $h \in H(Q)$.

The proof of the general Homeomorphism Extension Theorem is broken up in the following steps:

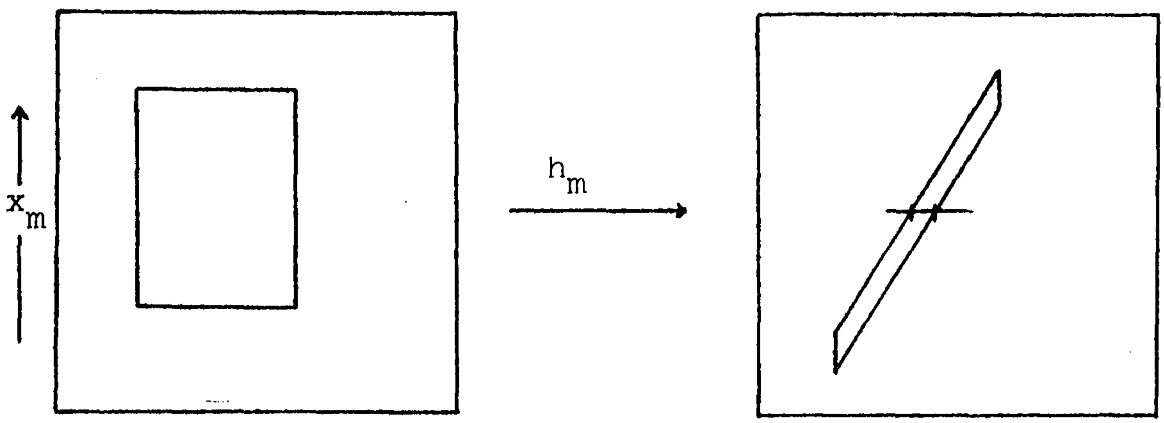
1) A Homeomorphism Extension Theorem for compact subsets of s (Proposition I.5).
2) It is shown that (Q,BQ) is homeomorphic to (Q,M) , where M is a basic core set (Proposition I.7).
3) It is shown that for a bcs M and any Z-set K , $(Q,M) \cong (Q,M-K)$, and as a consequence that $(Q,BQ) \cong (Q,BQ-K)$ (Proposition I.8). (In fact, a weaker version of I.8 would suffice, but later on we will need the stronger statement.)

Combining the above results one can easily obtain the general version of the Homeomorphism Extension Theorem.

<u>Lemma I.3</u> (Anderson [1]). <u>Let</u> K <u>be a compact subset of</u>

s , N *a positive integer and* ϵ *a positive real number. Then there exists a homeomorphism* $\varphi : Q \to Q$ *such that* (1) *for some* $n > N$, $p_n(\varphi(K))$ *is a single point in* $(-1,1)$, (2) $d(\varphi, id) < \epsilon$ *and* (3) *for any endface* $W = \{x | x_i = 1\}$ *or* $\{x | x_i = -1\}$, $\varphi(W) = W$ *and therefore* $\varphi(BQ) = BQ$.

Proof. The set K is contained in a cube $K' = \Pi_n [a_n, b_n] \subset s$. Let $n > N$ be so large that for any $x, y \in Q$, if $p'_{n-1}(x) = p'_{n-1}(y)$ then $d(x,y) < \epsilon$. First we find a homeomorphism h such that any line in the direction of the n^{th} coordinate intersects $h(K')$ in at most one point. Let, for any $m > n$, h_m be a PL autohomeomorphism of the 2-cell $I_n \times I_m$ as indicated in Figure I.1 below, which deforms



x_n Fig. I.1

$[a_n,b_n] \times [a_m,b_m]$ into a slanted figure such that horizontal intersections with it have diameter $\leq 2^{-m}$, and which leaves the x_n-coordinate unchanged. Define $h(x) = (x_1,\ldots,x_n,y_{n+1},y_{n+2},\ldots)$, where $(x_n,y_m) = h_m(x_n,x_m)$. Obviously h is a homeomorphism, and, for any $m > n$, by the definition of h_m, the intersection of $h(K')$ with any interval in the x_n-direction has diameter at most 2^{-m}. Hence the intersection is a point.

For the second and last step we construct a $g \in H(Q)$ which maps $h(K')$ into the hyperplane $p_n^{-1}(0)$. On any interval in the x_n-direction $L_x = \{y \mid m \neq n \Rightarrow y_m = x_m\}$, g will act as $f_q:[-1,1] \to [-1,1]$, where f_q is a PL homeomorphism which maps $[-1,q]$ linearly onto $[-1,0]$ and $[q,1]$ linearly onto $[0,1]$, and where $q \in (-1,1)$ will be specified later. We write $Q = I^{n-1} \times Q_{n+1} \times I_n$, and $x = (\hat{x},x_n)$, where $\hat{x} \in I^{n-1} \times Q_{n+1}$. Let $\hat{K}$ be the projection of $h(K')$ on $I^{n-1} \times Q_{n+1}$. We define $F':\hat{K} \to (-1,1)$ by $F'(\hat{x}) = y$, where $y \in (-1,1)$ is the unique point such that $(\hat{x},y) \in h(K')$. By Tietze's lemma, F' can be extended to $F:I^{n-1} \times Q_{n+1} \to (-1,1)$. Define g by $g(\hat{x},x_n) = (\hat{x},f_{F(\hat{x})}(x_n))$ (see Figure I.2). Then g is one-to-one onto because g leaves intervals L_x invariant and is one-to-one onto on each L_x. Furthermore

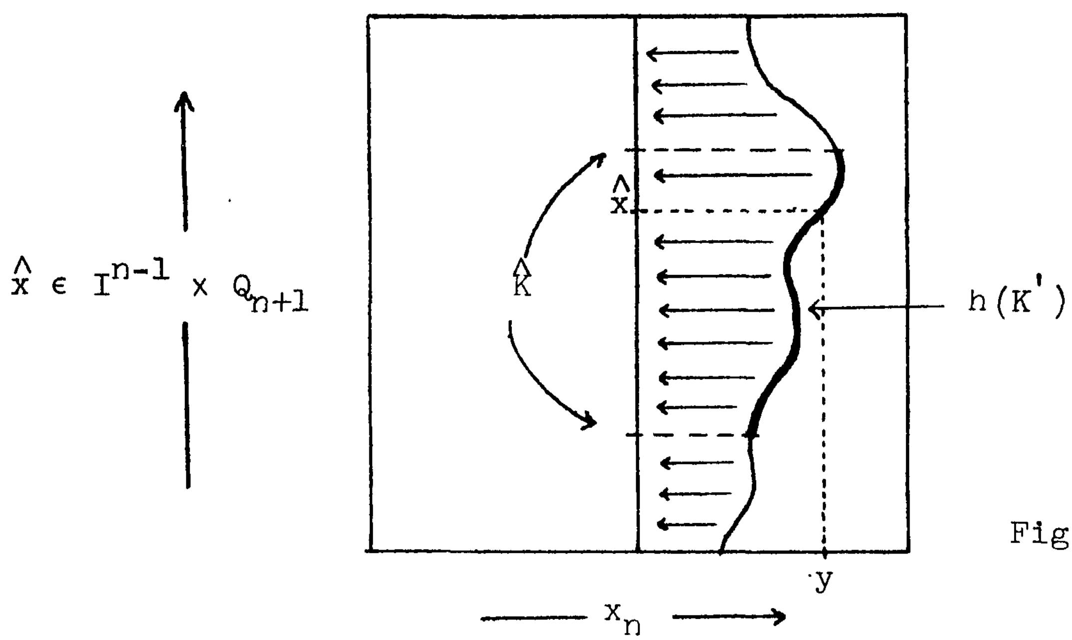


Fig. I.2

$g(h(K'))$ is a subset of the hyperplane $\{x_n = 0\}$ and $g \circ h$ is ϵ-close to id_Q because it does not alter the first $n-1$ coordinates of any point. Finally, from the construction it follows that $x_i = \pm 1$ iff $p_i \circ g \circ h(x) = \pm 1$. Therefore $\varphi = g \circ h$ is the desired homeomorphism.

Corollary I.4. Let K be a compact subset of s and ϵ a positive number. Then there exists a homeomorphism $f: Q \to Q$ such that $f(BQ) = BQ$ and $f(K)$ is infinitely deficient.

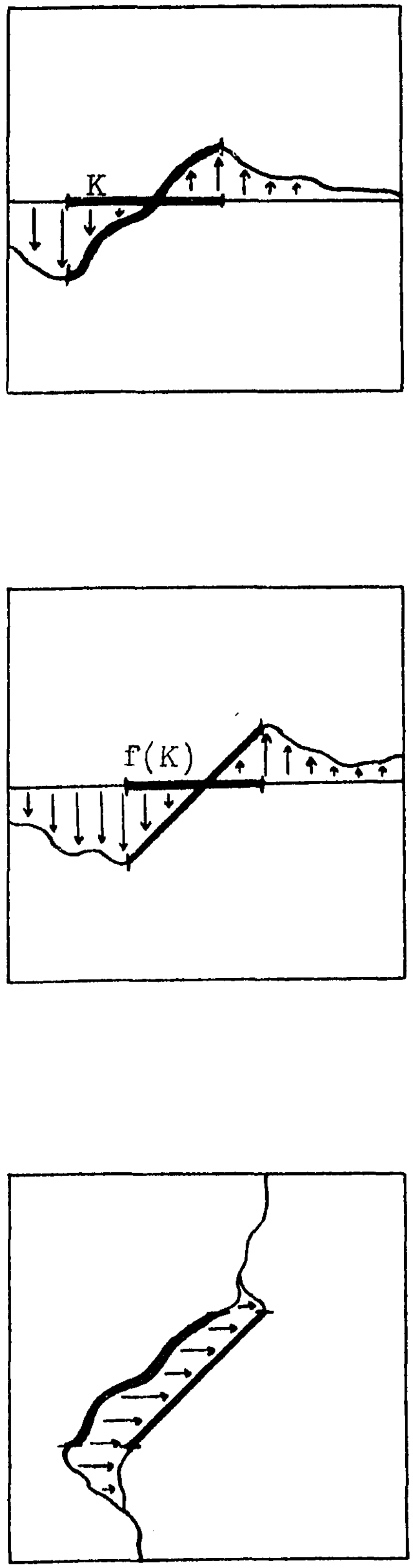
Proof. We can write $Q = \Pi_i Q_{C_i}$ where $C_i \cap C_j = \emptyset$ and $\cup_i C_i = N$. We can apply Lemma I.3 to each of the copies Q_{C_i} of Q and the compact subsets $P_{C_i}(K)$ of s_{C_i} and obtain a homeomorphism $f_i: Q_{C_i} \to Q_{C_i}$ such that $f_i(P_{C_i}(K))$ is deficient in some coordinate $n_i \in C_i$. Then the homeomorphism $f: Q \to Q$ defined by $p_{C_i} \circ f(x) = f_i(x_{C_i})$ maps K

onto a set of infinite deficiency. Moreover f is ϵ-close to the identity mapping if the maps f_{C_i} are sufficiently close to the identity.

Proposition I.5. Let $f:K \to f(K)$ be a homeomorphism between two compact subsets of s such that $d(f, id_K) < \epsilon$. Then f can be extended to an autohomeomorphism f' of Q which maps $B(Q)$ onto $B(Q)$ such that $d(f', id) < \epsilon$.

Proof. The idea of the proof is basically due to Klee [14], and modified by Barit [6] as to satisfy the smallness condition. Let $d(f, id_K) = \epsilon_1 < \epsilon$ and let $\delta = (\epsilon - \epsilon_1)/5$. Since there exist autohomeomorphisms of Q which map $K \cup f(K)$ onto a subset of s of infinite deficiency and which are arbitrarily close to the identity, we may assume that $K \cup f(K) \subset Q^C \times \{0\}$, where C is such that $d(p_C, id) < \delta$. For convenience of notation however, we shall write $Q \times Q$ instead of $Q^C \times Q^{N-C}$ but keep in mind that the second copy of Q has a small diameter. We write $p_I(x,y) = x$ and $p_{II}(x,y) = y$. Instead of $p_I(K)$ and $p_I(f(K))$ we write K and $f(K)$. The extension f will be a composition $h_2^{-1}h_3h_1$. (See Figure I.3.)

For the construction of h_1, apply Tietze's theorem to each of the functions $p_i \circ f$ to obtain a function $f^*: Q \to s$ which is an extension of $f:K \to f(K)$. Let, for any point $x \in (-1,1)$, $\varphi_x:[-1,1] \to [-1,1]$ be the PL map



h_1

h_2

h_3

Fig. I.3

which maps $[-1,0]$ linearly onto $[-1,x]$ and $[0,1]$ linearly onto $[x,1]$. Let, for $x \in s$, $F_x:Q \to Q$ be defined by $p_i F_x(y) = \varphi_{x_i}(y_i)$. Now define $h_1:Q \times Q \to Q \times Q$ by $h_1(x,y) = (x,F_{f^*(x)}(y))$. Then on each set $\{x\} \times Q$, h_1 is equal to $F_{f^*(x)}$ and maps $(x,0)$ onto $(x,f^*(x))$. In particular K is mapped onto the "graph" of f. Furthermore $d(h_1,id_{Q \times Q}) < \delta$. For h_2 we use a similar construction, except that f is replaced by id_K and f^* by an extension of id_K to a map id^* from Q into s.

We want h_3 to map the graph of f (not of f^*) onto the graph of $id_{f(K)}$. We use a modification of the trick for h_1 and h_2: let, for $x,y \in (-1,1)$, $\varphi_{x,y}:[-1,1] \to [-1,1]$ be the PL map which maps $[-1,x]$ linearly onto $[-1,y]$ and $[x,1]$ linearly onto $[y,1]$. Let, for x and y in s, $F_{x,y}$ be defined by $p_i F_{x,y}(z) = \varphi_{x_i,y_i}(z_i)$ for $z \in Q$. Let $id^{**}:Q \to s$ be an extension of $id_{f(K)}$, and $f^{**}:Q \to s$ an extension of f^{-1}. We cannot define $h_3(x,y) = (F_{f^{**}(y),id^{**}(y)}(x),y)$ since we are not sure whether f^{**} and id^{**} are sufficiently close, i.e., $\epsilon_1 + \delta$-close together. In other words, if $d(f^{**},id^{**})$ gets large then so does h_3. But, using a Urysohn function which is 1 on $f(K)$ and 0 on $\{y \mid d(f^{**}(y),id^{**}(y)) > \epsilon_1 + \delta\}$, we can replace f^{**} and

id^{**} by f^+ and id^+ which coincide with f^{**} and id^{**} respectively on K, and both of which coincide with the average of f^{**} and id^{**} for those x for which they would have been too far apart (i.e., more than $\epsilon_1 + \delta$) . We define $h_3(x,y) = (F_{f^+(y), id^+(y)}(x), y)$. Then $h_2^{-1} h_3 h_1 : Q \times Q \to Q \times Q$ is the desired extension of f .

<u>Corollary 1.6.</u> a) <u>In Proposition I.5, we can furthermore require that f' is the identity outside $U_\epsilon(K)$</u> .

b) <u>In addition, if $K \cup f(K)$ projects onto a point in all but finitely many coordinates, then for some n , f' can be constructed as the product of an autohomeomorphism of I^n and $id_{Q_{n+1}}$</u> .

<u>Proof</u>. a) Let $\delta = (\epsilon - \epsilon_1)/8$ instead of $(\epsilon - \epsilon_1)/5$. We can accomplish a) by, in the construction of h_1 and h_2 , replacing f^* and id^* by functions which coincide with f^* and id^* on K and $f(K)$ respectively, and which are zero outside a δ-neighborhood of K and $f(K)$ respectively, using suitable Urysohn functions. In the construction of h_3 , $\varphi_{x,y}$ has to be replaced by $\varphi'_{x,y}$ which, if $x \leq y$, is the identity outside $[x-\delta, y+\delta]$, maps $[x-\delta, x]$ onto $[x-\delta, y]$ and $[x, y+\delta]$ onto $[y, y+\delta]$. The case $x \geq y$ is treated analogously. Furthermore, the

Urysohn function which regulates the "averaging out" of f^{**} and id^{**} has to be zero if either $f^{**}(x)$ or $id^{**}(x)$ is more than δ away from $f(K)$. b) Left to the reader.

<u>The Homeomorphism Extension Theorem for</u> Z-<u>sets</u>. To prove the general Homeomorphism Extension Theorem, we use another copy of $B(Q)$. A <u>basic core set</u> (<u>bcs</u>) <u>structured on the core</u> $\Pi_i[a_i,b_i]$, where $-1 < a_i < b_i < 1$, is the set $\{x \in s \mid \text{for all but finitely many } i,\ x_i \in [a_i,b_i]\}$. Notice that one obtains the same bcs from two cores which differ in only finitely many a_i and b_i . It is easy to prove, using a coordinatewise defined homeomorphism, that any two basic core sets are homeomorphic under an autohomeomorphism of Q . From the definition it follows that any basic core set is invariant under a homeomorphism h with $h(B(Q)) = B(Q)$ such that h changes at most finitely many coordinates of any point. It is easily seen that basic core sets are σ-compact, e.g., the basic core set structured on the core $[-\frac{1}{2},\frac{1}{2}]^\infty$ can be written as $\cup_n \Pi_{i \le n}[-1+\frac{1}{n}, 1-\frac{1}{n}] \times \Pi_{i>n}[-\frac{1}{2},\frac{1}{2}]$. (The set $\{x \in s \mid \text{for all but finitely many } i,\ x_i = 0\}$, which is the countable union of finite-dimensional compacta, might be considered as a "basic core set structured on a degenerate core"; such sets, f-d capsets, have played an important role in Infinite-Dimensional topology, but we will not concern ourselves with them).

Below we will construct an autohomeomorphism $h = \Pi_i h_i$

of Q which maps $B(Q)$ onto a basic core set. Convergence to a homeomorphism will be ensured by the convergence criterion. The proof involves two ideas:

1) The terms of the sequence $(h_i \circ h_{i-1} \circ \cdots \circ h_1)_i$ map any given endface $W_i^{\pm}$ into higher and higher indexed endfaces and away from lower indexed endfaces, in such a way that in the limit the endface is mapped disjoint from all of BQ. More explicitly, there is an increasing sequence $(n_i)_i$ such that for any i, $\cup_{j\leq n_i}(W_j^+ \cup W_j^-)$ is mapped successively into $W^+_{n_i+1}, W^+_{n_{i+1}+1}, W^+_{n_{i+2}+1}, \cdots$ by $h_i, h_{i+1}\circ h_i, h_{i+2}\circ h_{i+1}\circ h_i, \cdots$. We shall write $h^{(i)}$ for $L\Pi_{j\geq i} h_j$.

2) By imposing some side-conditions on the h_i (conditions 2) - 4) below), we can accomplish that for any i there exists an infinite product $\Pi_j[a_j, b_j] \subset s$ such that h_i maps $W^+_{n_{i-1}+1}$ onto $\Pi_{j\leq n_i}[a_j, b_j] \times \{1\} \times Q_{n_i+2}$, $h_{i+1}\circ h_i$ maps $W^+_{n_{i-1}+1}$ onto $\Pi_{j\leq n_{i+1}}[a_j, b_j] \times \{1\} \times Q_{n_{i+1}+2}, \cdots$ and therefore that the limit $h^{(i)} = L\Pi_{j\geq i} h_j$ maps $W^+_{n_{i-1}+1}$ onto $\Pi_{j\geq 1}[a_j, b_j]$. Up to finitely many j, $(a_j)_j$ and $(b_j)_j$ will not depend on the choice of i; to be specific, if $i < i'$ then from $j = i'$ on, we will get the same a_j and b_j. It will be seen that $\cup_i h^{(i)}(W^+_{n_{i-1}+1})$ is a basic

core set, where any of the sets $\Pi_j[a_j,b_j]$ referred to above can be considered as the core. Adding to the above the observation that $h^{(i)}(W^+_{n_{i-1}+1})$ is contained in $h(h_1^{-1}\circ \cdots \circ h_{i-1}^{-1})(BQ) = h(BQ)$, it is not hard to prove that $h(BQ) = \cup_i h^{(i)}(W^+_{n_{i-1}+1})$, and therefore a basic core set.

Proposition I.7. For every ϵ there exists an autohomeomorphism $h \in H(Q)$ such that $h(BQ)$ is a basic core set and $d(h,id_Q) < \epsilon$.

Proof. Let, for the finite-dimensional cube I^i, the faces $\{x|x_j = \pm 1\}$ be denoted by $F^{\pm}_{i,j}$. Let, for each pair (i,n) such that $i \geq n \geq 1$, $h^*_{i,n}$ be a PL-autohomeomorphism of I^{i+1} such that the following conditions are satisfied:

1) $d(h^*_{i,n},id) < 2^{-i+2}$
2) $h^*_{i,n}$ maps $\cup_{j\leq i}(F^+_{i+1,j} \cup F^-_{i+1,j})$ into $\{x \in F^+_{i+1,i+1} |$ for all $k \leq i$, $|x_k| \leq 1-2^{-i}\}$.
3) $h^*_{i,n}(F^+_{i+1,n})$ is a product of intervals (the $i+1^{st}$ interval degenerate) and
4) on $\{x \in F^+_{i+1,n} |$ for all $k \neq n,i+1$, $|x_k| \leq 1-2^{-n+1}\}$, $h^*_{i,n}$ is linear and changes only the n^{th} and $i+1^{st}$ coordinate.

Figure I.4 is a diagram for $h^*_{1,1}$ and $h^*_{2,2}$ (the size

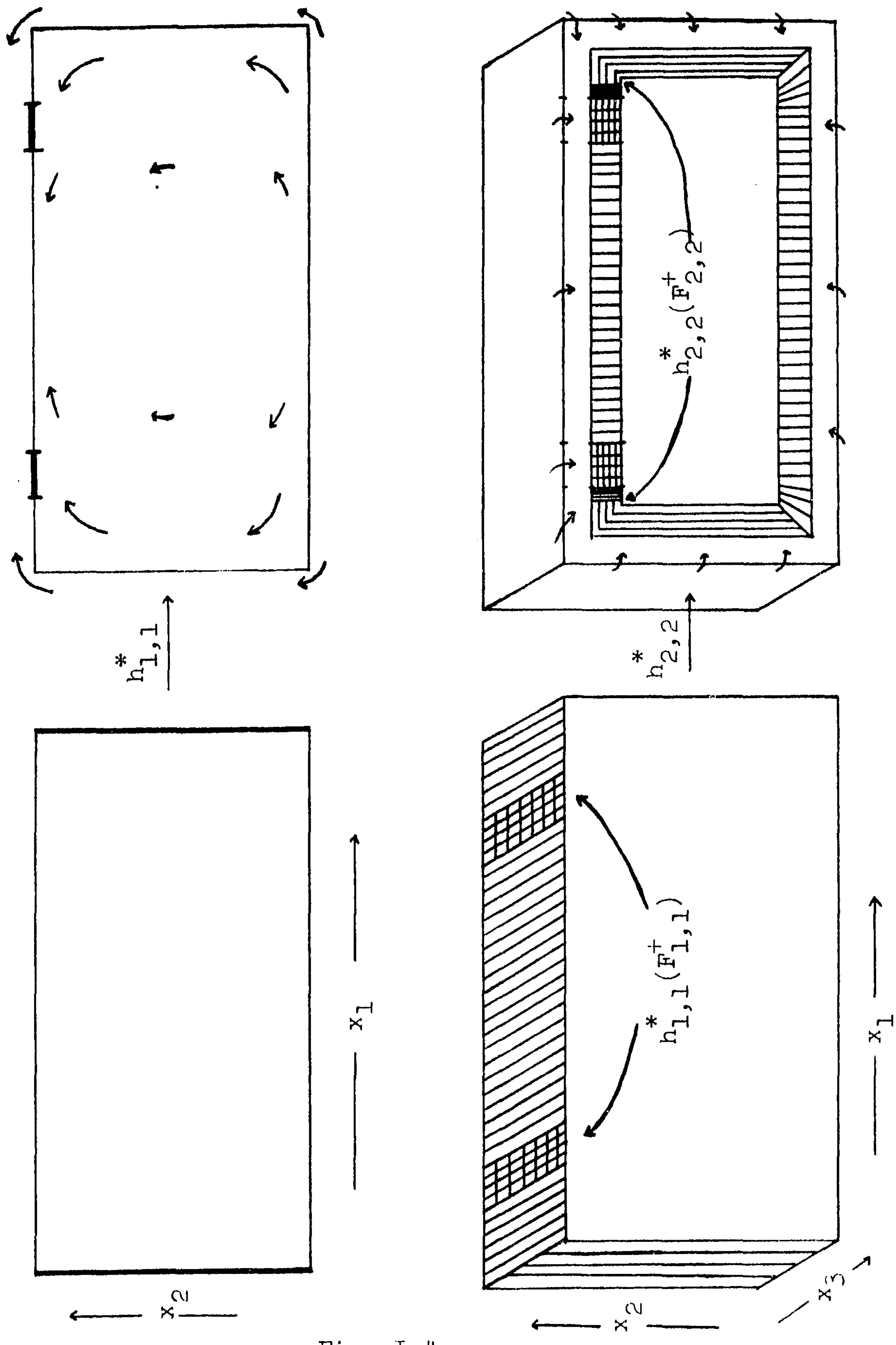


Fig. I.4

of the different coordinates as shown, reflects their relative importance for the metric).

Let $h_{i,n} = h^{*}_{i,n} \times id_{Q_{i+2}}$. Notice that for each i and n , $h_{i,n}(BQ) = BQ$. Select, using the Convergence Criterion, an increasing sequence $(n_i)_i$ such that $n_0+1 = n_1$ and the infinite left product $h = L\Pi_{i\geq 1} h_{n_i, n_{i-1}+1}$ is a homeomorphism. Write $h_i = h_{n_i, n_{i-1}+1}$. Regarding $h(W^+_{n_0+1})$, observe that for $x \in W^+_{n_0+1}$, by 4) $p_i \circ h(x) = p_i \circ h_j \circ \cdots \circ h_1(x) \in (-1,1)$ for the smallest j such that $n_j \geq i$. This is because x is mapped consecutively in $\{x \in W^+_{n_1+1} \mid \text{ for all } k \leq n_1, |x_k| \leq 1-2^{-n_1}\}$, in $\{x \in W^+_{n_2+1} \mid \text{ for all } k \leq n_2, |x_k| \leq 1-2^{-n_2}\}$ etc. Thus $h(W^+_{n_0+1}) \subset s$. By 3) and the above (see also the remarks preceding the proposition), $h(W^+_{n_0+1})$ is a product of closed subintervals $[a_{j,1}, b_{j,1}]$ of $(-1,1)$. By similar arguments $h^{(i)}(W^+_{n_{i-1}+1})$ is a product of closed subintervals $[a_{j,i}, b_{j,i}]$ of $(-1,1)$. Moreover for all $j \geq n_i$, $a_{j,i} = a_{j,1}$ and $b_{j,i} = b_{j,1}$, as can be seen by comparing the sets $W^+_{n_{i-1}+1}$ and $h_{i-1} \circ \cdots \circ h_1(W^+_{n_0+1})$ and their images under $h_i, h_{i+1} \circ h_i$ etc.

The set $\cup_i h^{(i)}(W^+_{n_{i-1}+1})$ is easily seen to be a basic core set, structured on the core $h(W^+_{n_0+1}) = \Pi_j [a_{j,1}, b_{j,1}]$.

We show that this set equals $h(BQ)$: write $h(BQ) = h(\cup_k W_k^+ \cup W_k^-)$. For any k there is an i such that $h_{i-1}\circ \cdots \circ h_1(W_k^+ \cup W_k^-) \subset W^+_{n_{i-1}+1}$. Therefore $h(BQ) \subset \cup_i h(h_{i-1}\circ \cdots \circ h_1)^{-1}(W^+_{n_{i-1}+1})$. Conversely, $W^+_{n_{i-1}+1} \subset BQ = (h_{i-1}\circ\cdots\circ h_1)(BQ)$, and thus, for each i, $h^{(i)}W^+_{n_{i-1}+1} \subset h(BQ)$.

The proof is concluded by the observation that h can be made arbitrarily small by choosing n_1 large enough.

<u>Proposition I.8</u>. a) <u>For any basic core set</u> M , <u>any</u> Z-<u>set</u> K <u>and any</u> $\epsilon > 0$, <u>there exists an</u> $h \in H(Q)$ <u>such that</u> $d(h,id) < \epsilon$ <u>and</u> $h(M-K) = M$ <u>and</u> h <u>is the identity outside an</u> ϵ-<u>neighborhood of</u> K .

b) <u>For any</u> Z-<u>set</u> K <u>and any</u> $\epsilon > 0$, <u>there exists an</u> $h \in H(Q)$ <u>such that</u> $d(h,id) < \epsilon$ <u>and</u> $h(BQ - K) = B(Q)$ <u>and</u> h <u>is the identity outside an</u> ϵ-<u>neighborhood of</u> K .

<u>Proof</u>. Obviously b) is a consequence of a) and Proposition I.7. We prove a) in five steps. Let a <u>standard</u> n-<u>cell</u> in Q be any set $\Pi_{i\leq n}[a_i,b_i] \times \{(0,0,\cdots)\}$, where $-1 < a_i < b_i < 1$.

<u>Step 1</u>. <u>Moving</u> K <u>off a standard</u> n-<u>cell</u>. Let $C_n = \Pi_{i\leq n}[a_i,b_i] \times \{(0,0,\cdots)\}$. For any $\delta > 0$, there is

an $f \in H(Q)$ such that (1) $d(f,id) < \delta$, (2) $f(K) \cap C_n = \emptyset$, (3) $f(B(Q)) = B(Q)$, (4) f changes only finitely many coordinates, and therefore $f(M) = M$. For, since K is a Z-set, there exists a map $\varphi: Q \to Q-K$ with $d(\varphi, id) < \delta/2$. Let $\eta = \min(\delta/2, d(K, \varphi(C_n)))$. Approximate $\varphi | C_n$ by an embedding φ' such that $d(\varphi, \varphi') < \eta$ and $\varphi'(C_n)$ is a compact subset of s which projects onto 0 in all but finitely many coordinates. Then $\varphi'(C_n) \cap K = \emptyset$. By Corollary I.6, $\varphi': C_n \to \varphi'(C_n)$ can be extended to $f' \in H(Q)$ with $d(f', id) < \delta$ and $f'(B(Q)) = B(Q)$ and which changes only finitely many coordinates, and which therefore maps M onto M . Then $f = f'^{-1}$ is the desired homeomorphism.

Step 2. Moving K off a given infinite-dimensional cube in s . Now let $C = \Pi_n [a_n, b_n]$ be any infinite-dimensional subcube of s . Then we shall show that for any δ , there exists a homeomorphism $f: Q \to Q$ satisfying (1) - (4) from step 1 with C instead of C_n . For let N be so large that $p_N'(x) = p_N'(y)$ implies $d(x,y) < \delta/2$. Let $C_N = \Pi_{i \leq N} [a_i, b_i] \times \{(0,0,\cdots)\}$. Applying step 1, find $g \in H(Q)$ satisfying (1) - (4) from step 1 for C_N and $\delta/2$. Then $g(K)$ is disjoint from an open neighborhood of C_N , and in particular disjoint from a set $\Pi_{i \leq N} [a_i, b_i] \times \Pi_{N < i \leq M} [a_i', b_i'] \times Q_{M+1}$. Let h be a map, affecting only the $N+1^{th}$ until the M^{th} coordinate, which maps $\Pi_{i\leq N}[a_i,b_i] \times \Pi_{N<i\leq M}[a_i',b_i'] \times Q_{M+1}$ onto $\Pi_{i\leq M}[a_i,b_i] \times Q_{M+1}$. Then $hg(K)$ is disjoint from C , hg is δ-close to the identity, maps $B(Q)$ onto $B(Q)$ and changes only finitely many coordinates and therefore maps M onto M . Therefore $f = hg$ is as desired.

Step 3. $f = id$ outside a small neighborhood of C . According to Corollary I.6 we may suppose that g is the identity outside a small neighborhood of C_N . The same can be accomplished for h by making use of Urysohn functions as in the proof of Proposition I.5: Let $r:I^N \to I$ be 0 outside a small neighborhood of $\Pi_{i\leq N}[a_i,b_i]$ and 1 on $\Pi_{i\leq N}[a_i,b_i]$. Let $\psi:I^{M-N} \to I^{M-N}$ be a PL homeomorphism which maps $\Pi_{N<i\leq M}[a_i',b_i']$ onto $\Pi_{N<i\leq M}[a_i,b_i]$. Write $x = (x_I,x_{II},x_{III})$ where $x_I = (x_1,\cdots,x_N)$, $x_{II} = (x_{N+1},\cdots,x_M)$ and $x_{III} = (x_{M+1},x_{M+2},\cdots)$. Define
$h'(x) = (x_I, r(x_I)\cdot\psi(x_{II}) + (1-r(x_I))\cdot x_{II}, x_{III})$. If N is chosen sufficiently large, then $f = h'\circ g$ is the identity outside an arbitrarily small neighborhood of C .

Step 4. $f = id$ outside a small neighborhood of K . It is also possible to find an $f \in H(Q)$, satisfying (1) - (4) of step 1 for C and any $\delta > 0$, such that f is the identity outside an arbitrarily small neighborhood of $K \cap C$: Let $\{C^{(1)},\cdots,C^{(n)}\}$ be some "canonical" decomposition of C into small closed subcubes. Suppose $C^{(1)},\cdots,C^{(k)}$ are the subcubes that intersect K . Construct f_1 , such that

$f_1(K) \cap C^{(1)} = \emptyset$, say $d(f_1(K), C^{(1)}) = \delta_1$. Construct f_2 such that $d(f_2, id) < \delta_1/2$ and $f_2 f_1(K) \cap C^{(2)} = \emptyset$. Then we still have $f_2 f_1(K) \cap C^{(1)} = \emptyset$. Working our way through $C^{(3)} - C^{(k)}$ we get the desired homeomorphism.

Step 5. Moving K off countably many cubes in s. Let $M = \cup_i M_i$ be any basic core set, where $(M_i)_i$ is an increasing sequence of geometrical cubes. Let $h_1 \in H(Q)$ have the properties (1) - (4) of step 1 for $\epsilon/2$ and M_1, and such that h_1 is the identity outside $U_\epsilon(K)$. Let $\delta_2 < \min(\epsilon/4, \frac{1}{2} \cdot d(h_1(K), M_1))$ be small enough with regard to the Convergence Criterion. Let $h_2 \in H(Q)$ satisfy (1) - (4) of step 1 for δ_2 and M_2, and be equal to the identity outside $U_\epsilon(K) \cap h_1(U_{\epsilon/2}(K))$ (which is a neighborhood of $h_1(K)$). Then $d(h_2 h_1(K), M_1) > 1/2 \cdot d(h_1(K), M_1)$. For the inductive step, we let $\delta_n < \min(\epsilon \cdot 2^{-n}, \delta_1 \cdot 2^{-n+1}, \cdots, \delta_{n-1} \cdot 2^{-1})$ and sufficiently small for the convergence criterion, and we let h_n satisfy (1) - (4) from step 1 for $(h_{n-1} \cdot \cdots \cdot h_1)(K)$ and M_n and δ_n, and we let h_n be the identity outside $U_\epsilon(K) \cap h_{n-1} \circ \cdots \circ h_1(U_{\epsilon \cdot 2^{-n+1}}(K))$. Then $h = L\Pi_i h_i$ has distance less than ϵ to id_Q and is the identity outside $U_\epsilon(K)$. For any point x not in K, h is equal to some finite composition $h_n \circ \cdots \circ h_1$ which changes only finitely many coordinates of x. Therefore h maps $M-K$ into M and maps $Q - (M \cup K)$ into $Q-M$. Finally, $h(K)$ has positive distance to every set M_i , and therefore $h(K) \cap M = \emptyset$. But then $h(M-K) = M$, and we have proved a).

Corollary I.9. *A closed subset K of Q is a Z-set in Q iff there is an $h \in H(Q)$ which maps K onto a set of infinite deficiency.*

Proof. By Proposition I.8 b), K can be mapped into s , and by Corollary I.4, the image of K can be made infinitely deficient subsequently.

Theorem I.10. *Let $f:K \to f(K)$ be a homeomorphism between two Z-sets in Q . Then there exists an f' in $H(Q)$ which is an extension of f . Moreover, if $d(f,id_K) = \epsilon_1 < \epsilon$ then f' can be chosen in such a way that $d(f',id_Q) < \epsilon$ and f' is the identity outside an ϵ-neighborhood of K .*

Proof. Let $\delta = (\epsilon-\epsilon_1)/6$. Let $g \in H(Q)$ map $B(Q)-(K \cup f(K))$ onto $B(Q)$ and be δ-close to the identity. Then $d(gfg^{-1},id_K) < \epsilon_1 + 2\delta$. Let h be an autohomeomorphism of Q which extends $gfg^{-1}:g(K) \to gf(K)$ such that $d(h,id_Q) < \epsilon_1 + 3\delta$ and h is the identity outside $U_{\epsilon_1+3\delta}(K)$ (which set contains $g(U_{\epsilon_1+\delta}(K))$) . Let $f' = g^{-1}hg$, then f' has the required properties.

Capsets. In [3], R. D. Anderson introduced the concept of capset (in Theorem I.12 below, it will be shown that $B(Q)$

and any basic core set are capsets. For the closely related but more general concept of $(G,\mathcal{K})$-skeletoid, introduced at about the same time, see Bessaga-Pełczyński [6]. A subset M of Q is a <u>capset</u> (for Q) if M can be written as a countable increasing union $\cup_i M_i$ of Z-sets such that for each $\epsilon > 0$, $n > 0$ and for each Z-set K in Q there exists an $m \geq n$ and an $h \in H(Q)$ such that $h(K) \subset M_m$, $d(h,id) < \epsilon$ and $h|M_n = id_{M_n}$. Obviously the concept of capset is topologically invariant. We remark in passing that the property of being a capset, as well as that of being a Z-set, can be defined for subsets of s or ℓ_2, and that many of the theorems about Z-sets and capsets remain valid. There exists a finite-dimensional analogue, viz. f-d capsets (see Anderson [3]). These were already briefly touched upon at the discussion of basic core sets.

<u>Theorem I.11</u>. <u>Suppose</u> M <u>and</u> N <u>are two capsets in</u> Q. <u>Then for each</u> $\epsilon > 0$ <u>there exists an</u> $h \in H(Q)$ <u>such that</u> $h(M) = N$ <u>and</u> $d(h,id_Q) < \epsilon$.

<u>Proof</u>. Let the decompositions $M = \cup_i M_i$ and $N = \cup_i N_i$ satisfy the conditions in the definition of capset. We construct h as a composition $\cdots g_2^{-1} \circ h_2 \circ g_1^{-1} \circ h_1$. Without further mentioning, it is understood that at each stage the next homeomorphism is constructed in accordance with the con-

vergence criterion.

Applying the definition of capset for N, we can find $h_1 \in H(Q)$ such that for some n_1, $h_1(M_1) \subset N_{n_1}$. Since $h_1(M)$ is a capset we can find $g_1 \in H(Q)$ such that for some $m_1, g_1(N_{n_1}) \subset h_1(M_{m_1})$ or equivalently $g_1^{-1}h_1(M_{m_1}) \supset N_{n_1}$, and such that moreover $g_1|h_1(M_1) = g_1^{-1}|h_1(M_1) = \text{id}$. Then, since $h_1(M_1) \subset N_{n_1}$, also $g_1^{-1}h_1(M_1) \subset N_{n_1}$. Construct h_2 such that for some $n_2, h_2 \circ g_1^{-1} \circ h_1(M_{m_1}) \subset N_{n_2}$ and $h_2|N_{n_1} = \text{id}$. Then again $h_2 \circ g_1^{-1} \circ h_1(M_{m_1}) \supset N_{n_1}$ and $h_2 \circ g_1^{-1} \circ h_1(M_1) \subset N_{n_1}$. Continuing with the inductive construction of maps $g_2, h_3, g_3, \cdots \in H(Q)$ which create and preserve appropriate inclusion-relations, we obtain a sequence of which the infinite left product $h = L\Pi_i g_i^{-1} \circ h_i$ is a homeomorphism with on the one hand $h|M_{m_i} = h_i \circ g_{i-1}^{-1} \circ \cdots \circ h_1|M_{m_i}$, and therefore $h(M_{m_i}) \subset N_{n_{i+1}} \subset N$, and on the other hand $(L\Pi_{j>i} g_j^{-1} \circ h_j)|N_{n_i} = \text{id}|N_{n_i}$, or in other words $h^{-1}|N_{n_i} = (g_i^{-1} \circ h_i \circ \cdots \circ h_1)^{-1}|N_{n_i}$, and therefore $h(M) \supset h(M_{m_i}) \supset N_{n_i}$. Together these show that $h(M) = N$.

<u>Theorem I.12</u>. a) <u>Any basic core set is a capset.</u>

b) <u>The pseudo-boundary is a capset.</u>

<u>Proof</u>. By Proposition I.7, b) follows from a). For the

proof of a) we use the following notion: If X is a countably infinite product $\Pi_i[a_i,b_i]$, where for each i , $a_i < b_i$, then we call $\Pi_i(a_i,b_i)$ the <u>pseudo-interior</u> $PsI(X)$ of X . The proof is divided into two sublemmas:

<u>Sublemma 1. If N is a countable union of geometric subcubes N_i of s such that for each i , $N_i \subset PsI(N_{i+1})$ and such that $\cup_i N_i$ is dense in Q , then N is a capset.</u>

<u>Proof</u>. Obviously N is a countable union of Z-sets. Notice that each N_i can be written as $\Pi_j[a_{i,j},b_{i,j}]$, where for each j , $(a_{i,j})_i$ strictly decreases to -1 and $(b_{i,j})_i$ strictly increases to $+1$. Let a Z-set K , $\epsilon > 0$ and a positive integer i be given. For each $k > i$ there is a coordinatewise defined homeomorphism $f_k : Q \xrightarrow{onto} N_k$ which leaves N_i pointwise fixed. For sufficiently high k , $d(f_k, id_Q) < \epsilon$. By applying the Homeomorphism Extension Theorem to $f_k|K \cup N_i$ we obtain the autohomeomorphism of Q that proves the capset property for N . (This is the only place in the proof of Theorem I.12 where we need the Homeomorphism Extension Theorem.)

For the proof of the theorem, it clearly suffices to prove the following sublemma.

<u>Sublemma 2. For any bcs M there exists an $h \in H(Q)$ such</u>

that $h(M)$ is a union of cubes as described in Sublemma 1.

Proof. Since any core can be translated coordinatewise to any other core, it is trivial that for any two basic core sets M and M', $(Q,M) \cong (Q,M')$. Let M be the particular bcs with core $[-\frac{1}{2},\frac{1}{2}]^\infty$. Let
$M_i = \Pi_{j\leq i}[-1+1/i,1-1/i] \times \Pi_{j>i} I_j$; then $M = \cup_i M_i$. We will construct h as an infinite left product $L\Pi_{i\geq 2} h_i$, which will converge by the Convergence Criterion, and where each h_i maps M_i into $PsI(M_i)$, while satisfying certain side-conditions.

Step 1. We construct h_2 such that $h_2(M_2)$ is a geometrical cube $\Pi_j[a_j,b_j]$ in $PsI(M_2)$ and such that for points not in M_2 only finitely many coordinates are changed. For such an h_2, $h_2(M) = M$, as the reader easily checks for himself. Let $U_i = [-\frac{1}{2} - 2^{-i}, \frac{1}{2} + 2^{-i}]^i \times Q_{i+1}$. Then $(U_i)_i$ is a neighborhood basis for M_2. We construct h_2 as an infinite left product $L\Pi_{i\geq 2} f_i$, where f_i is the product of an autohomeomorphism of I^i and the identity on Q_{i+1}, and where f_i is the identity outside $f_{i-1}\circ\cdots\circ f_2(U_i)$. We use the Convergence Criterion to make the left product converge.

Let f_2 be the product of the identity on Q_3 and a PL map on I^2, which maps (see Fig. I.5) $[-\frac{1}{2},\frac{1}{2}]^2$

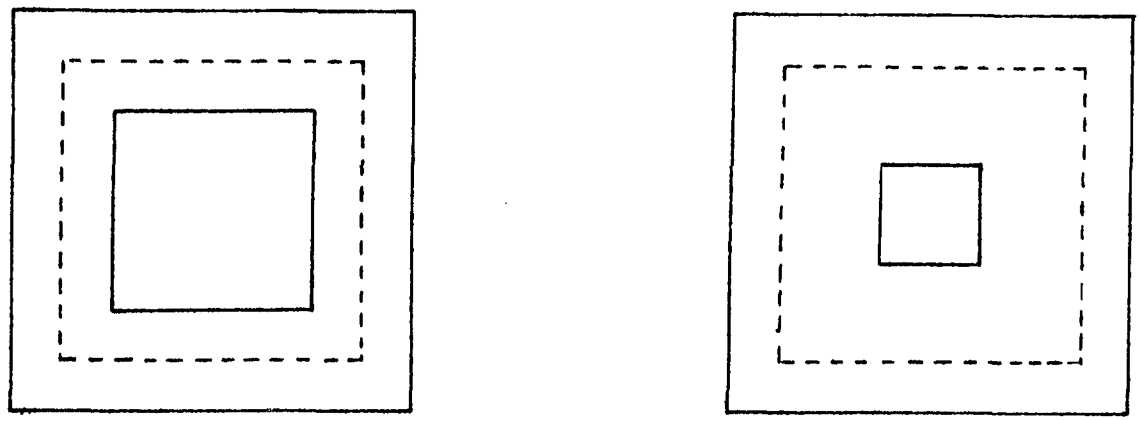


Fig. I.5

onto a square $[-\frac{1}{2}+\epsilon_2, \frac{1}{2}-\epsilon_2]^2$ in its own interior and is the identity outside $[-\frac{3}{4}, \frac{3}{4}]^2$ (the first two coordinates of U_2) . Let f_3 be the product of the identity on Q_4 and a PL map on I^3 which shrinks $[-\frac{1}{2}+\epsilon_2, \frac{1}{2}-\epsilon_2]^2 \times [-\frac{1}{2}, \frac{1}{2}]$ to a set $[-\frac{1}{2}+\epsilon_2, \frac{1}{2}-\epsilon_2]^2 \times [-\frac{1}{2}+\epsilon_3, \frac{1}{2}-\epsilon_3]$, in such a way that f_3 is the identity outside $f_2(U_3)$, and is small enough for the convergence criterion (see Fig. I.6). Inductively we construct the remaining f_i in a similar manner. The left product $L\Pi_i f_i$ is the desired homeomorphism h_2 . Notice that we may assume that for each $x \in Q$ and each integer i , $x_i \leq p_i h_2(x) \leq 0$ or $0 \leq p_i h_2(x) \leq x_i$.

<u>Second and Inductive Step</u>. By similar constructions, obtain

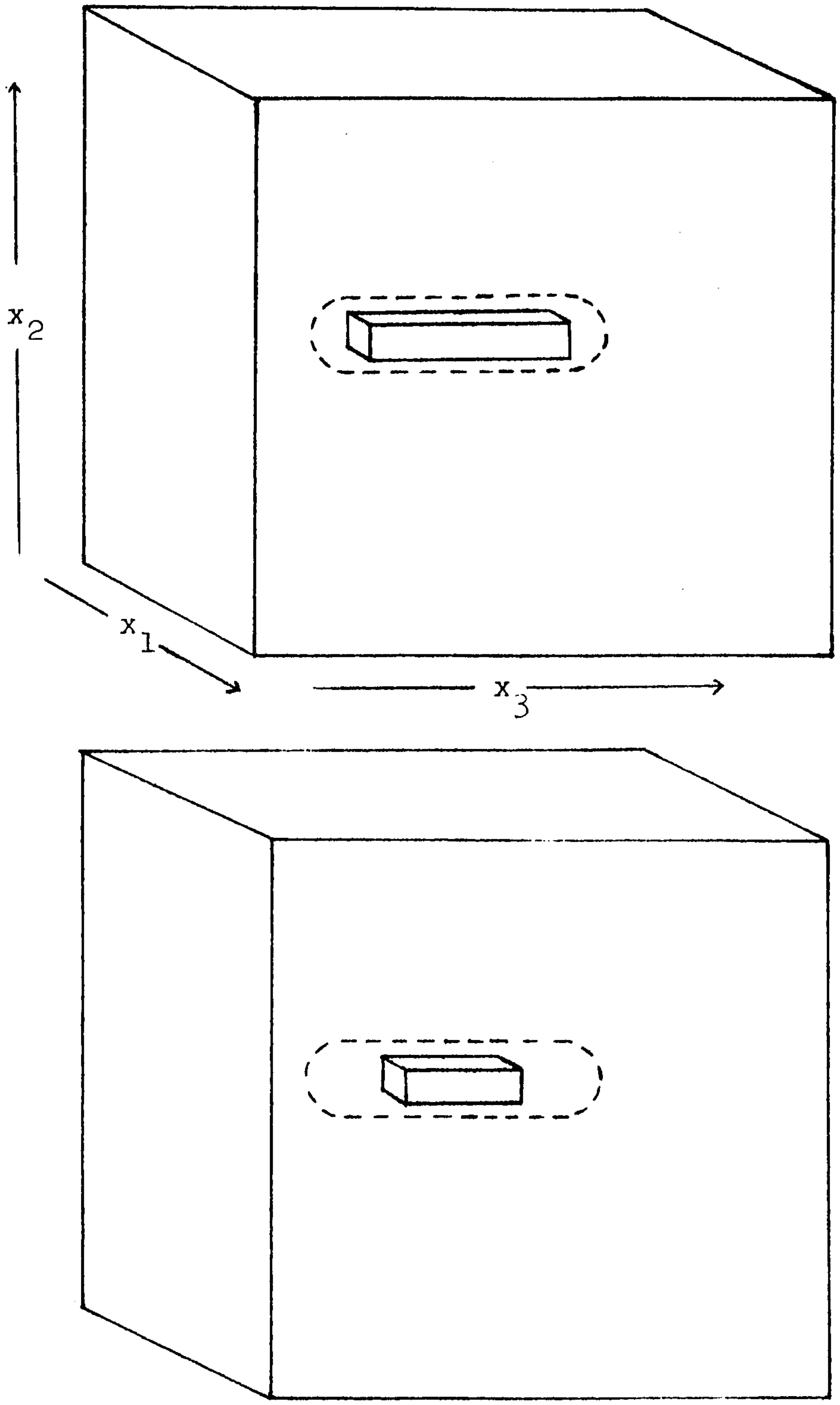


Fig. 1.6

a sufficiently small $h_3 \in H(Q)$ such that $h_2(M_2) \subset PsI(h_3(M_3)) \subset h_3(M_3) \subset PsI(M_3)$. It is geometrically obvious that we can require that $h_3|h_2(M_2) = id$, and that for each x and i , $x_i \leq p_i h_3(x) \leq 0$ or $0 \leq p_i h_3(x) \leq x_i$. Then $h_3 h_2(M_2) = h_2(M_2)$ is contained in $PsI(h_3 h_2(h_2^{-1}(M_3))) = h_3(M_3)$, and $h_3(M) = M$.

Inductively construct sufficiently small h_i such that $h_{i-1}(M_{i-1}) \subset PsI(h_i(M_i)) \subset h_i(M_i) \subset PsI(M_i)$ and such that $h_i|h_{i-1}(M_{i-1}) = id$ and for each x and k , $x_k \leq p_k h_i(x) \leq 0$ or $0 \leq p_k h_i(x) \leq x_k$, and $h_i(M) = M$.

Because of the condition $|p_k h_i(x)| \leq |x_k|$, we have for each i , $h_{i-1}^{-1} \circ \cdots \circ h_2^{-1}(M_i) \supset M_i$. Let $h = L\Pi_i h_i$. Then $h(M) = \cup_i h(M_i) \subset \cup_i h(h_{i-1}^{-1} \circ \cdots \circ h_2^{-1}(M_2)) = \cup_i h_i(M_i)$ and $\cup_i h_i(M_i) = \cup_i h(h_{i-1}^{-1} \circ \cdots \circ h_2^{-1}(M_i)) \subset h(M)$, and therefore $h(M) = \cup_i h_i(M_i)$. It is easily seen that $\cup_i h_i(M_i)$ is a union of geometrical cubes as described in Sublemma 1.

<u>Corollary I.13</u>. <u>For each capset</u> M , <u>each</u> $\epsilon > 0$ <u>and each Z-set</u> K <u>there exists</u> $h \in H(Q)$ <u>such that</u> $d(h, id_Q) < \epsilon$, $h(M-K) = M$ <u>and</u> $h = id$ <u>outside</u> $U_\epsilon(K)$.

<u>Proof</u>.

By the topological equivalence of all capsets and by Theorem I.12, M is equivalent to a bcs under some $h \in H(Q)$; since Proposition I.8 proves the corollary for basic core sets, this completes the proof.

Proposition I.14. *Let* $f: K \to f(K)$ *be a homeomorphism between two Z-sets in* Q *such that* $f(K) \cap BQ = f(K \cap BQ)$ *and* $d(f, id_K) < \epsilon$; *then there is an* $f' \in H(Q)$ *such that* f' *extends* f , $f'(BQ) = BQ$ *and* $d(f', id_Q) < \epsilon$.

Proof. Let $L \subset Q$ be any Z-set. Using the Homeomorphism Extension Theorem, first find $g_1 \in H(Q)$ such that $g_1(L) \subset s$; next find $g_2 \in H(Q)$ such that $g_2 g_1(L) \cap g_1(L) = \emptyset$ and $d(g_1^{-1} g_2 g_1, id_Q) < \epsilon/2$. Then $g_1^{-1} g_2 g_1(L) \cap L = \emptyset$. Let $\delta = d(g_1^{-1} g_2 g_1(L), L)$. Since both BQ and $g_1^{-1} g_2 g_1(BQ)$ are capsets, there exists a $g_3 \in H(Q)$ such that $d(g_3, id) < \min(\epsilon/2, \delta)$ and $g_3 g_1^{-1} g_2 g_1(BQ) = BQ$. Then $d(g_3 g_1^{-1} g_2 g_1(L), L) > 0$ and $d(g_3 g_1^{-1} g_2 g_1, id_Q) < \epsilon$.

Now consider $K \cup f(K)$ as a Z-set L as above. Let $d(f, id_K) = \epsilon_1 < \epsilon$, and $\delta = \epsilon - \epsilon_1$. There exists a $\delta/4$-small $\varphi \in H(Q)$ such that $\varphi(K \cup f(K)) \cap (K \cup f(K)) = \emptyset$ and in particular $K \cap \varphi f(K) = \emptyset$, and such that $\varphi(BQ) = BQ$. Define $f^+: K \cup \varphi f(K) \to K \cup \varphi f(K)$ by $f^+|K = \varphi \circ f$ and $f^+|\varphi \circ f(K) = (\varphi \circ f)^{-1}$. Then $d(f^+, id_{K \cup \varphi f(K)}) < \epsilon_1 + \delta/4$. Let, by Proposition I.8, h be an autohomeomorphism of Q such that $h(BQ - (K \cup \tilde{f}(K))) = BQ$ and $d(h, id) < \delta/4$. Using Proposition I.5, let $g \in H(Q)$ be such that $d(g, id_Q) < \delta/4 + \epsilon_1$, $g|h(K \cup f^+(K)) = hf^+h^{-1}|h(K \cup f^+(K))$ and $g(B(Q)) = B(Q)$. Then $h^{-1} \circ g \circ h$ is an autohomeomorphism of Q extending f^+ such that $d(h^{-1}gh, id) < \epsilon_1 + 3\delta/4$ and

$h^{-1}gh(BQ) = BQ$. We check the last statement:

$$\begin{aligned} h^{-1}gh(BQ) &= h^{-1}gh([BQ - (K \cup f^+(K))] \cup [BQ \cap (K \cup f^+(K))] \\ &= h^{-1}g(BQ) \cup f^+(BQ \cap (K \cup f^+(K))) \\ &= h^{-1}(BQ) \cup (BQ \cap (K \cup f^+(K))) \\ &= BQ . \end{aligned}$$

Then $\varphi^{-1} \circ (h^{-1} \circ g \circ h)$ is the desired homeomorphism.

Corollary I.13 states that for any capset M and any Z-set K , $M-K$ is again a capset. Complementary to this we have the following useful proposition.

__Proposition I.15__. __The union of a capset and a__ σ-Z-__set__ (countable union of Z-sets) __is again a capset__.

__Proof__. Let $M = \cup_i M_i$ be a capset, with $\{M_i\}_i$ having the properties listed in the definition of capset. Let $K = \cup_i K_i$ be a countable increasing union of Z-sets.

We show that $M \cup K$ is a capset by constructing a homeomorphism $H:Q \to Q$ such that $H(M \cup K) = M$. This homeomorphism will be an infinite left product $\mathbf{L\Pi}_i G_i$ such that for some increasing sequence $(n_i)_i$, (1) G_i is the identity on M_{n_i} , (2) $G_{i-1} \circ \cdots \circ G_1$ embeds K_{i-1} in M_{n_i} (thus G_i is also the identity on $G_{i-1} \circ \cdots \circ G_1(K_{i-1})$) , and (3) $G_i(M \cup \cup_{j=1}^{i} K_j) = M$.

Let $n_1 = 1$. For the construction of G_1 , choose, for sufficiently large n_2, an embedding $\varphi: K_1 \cup M_{n_1} \to M_{n_2}$ which is the identity on M_{n_1}. Notice that $\varphi(K_1 \cup M_{n_1})$ is a Z-set because it is contained in a Z-set. By Corollary I.13, let $f \in H(Q)$ be such that $f(M - (K_1 \cup M_{n_1})) = M$ and let $g \in H(Q)$ be such that $g(M \cup \varphi(K_1 \cup M_{n_1})) = M$. Let $h \in H(Q)$ be such that $h(M) = M$ and $hf(K_1 \cup M_{n_1})) = g\varphi(K_1 \cup M_{n_1})$; then $G_1 = g^{-1}hf$ maps K_1 into M_{n_2} and $M \cup K_1$ onto M, leaving M_{n_1} pointwise fixed. Observe that all the above homeomorphisms can be constructed arbitrarily small.

The inductive step is similar. The reader can verify for himself that, if appropriate size restrictions hold, the infinite left product H of a sequence of such homeomorphisms G_i is an autohomeomorphism of Q mapping $M \cup \bigcup_i K_i$ onto M.

The following two characterizations of capsets will be needed in Chapters II and III. The first characterization (I.17) is known in the folklore, the second (I.18) is especially designed by the author for the proofs in Chapter II.

<u>Corollary I.16</u>. <u>Suppose</u> M <u>is a countable union of compact subsets of</u> Q <u>such that</u>

1) <u>For every</u> $\epsilon > 0$ <u>there exists a map</u> $h: Q \to Q-M$ <u>such</u>

that $d(h,id) < \epsilon$

2) M contains a set $\cup_i M_i$ such that for each i, $M_i \cong Q$ and M_i is a Z-set in M_{i+1}

3) For each $\epsilon > 0$, there exists an i and a map $h:Q \to M_i$ such that $d(h,id_Q) < \epsilon$.

Then M is a capset for Q.

Proof. From 1) it follows that M is a countable union of Z-sets and that every compact subset of M is a Z-set. We show that $\cup_i M_i$ is a capset. Let ϵ, j and a Z-set K be given. By 3) there exist $i > j$ and $h:Q \to M_i$ such that $d(h,id_Q) < \epsilon/4$. By the Mapping Replacement Theorem there exists an embedding $g:Q \to M_i$ which maps Q onto a Z-set in M_i such that $d(h,g) < \epsilon/4$. Then $d(g,id_Q) < \epsilon/2$. By the Homeomorphism Extension Theorem for M_i, there exists a homeomorphism $f:M_i \to M_i$ which extends $g^{-1}|g(K \cap M_j)$ and such that $d(f,id) < \epsilon/2$. Then $f \circ g:K \to M_i$ is an embedding of K into M which is the identity on $K \cap M_j$ and such that $d(f \circ g, id) < \epsilon$. Extending $f \circ g$ to an ϵ-small $F \in H(Q)$, we see that $\cup_i M_i$ is a capset, and therefore M as well.

A map $F = (F_t)_t : X \times I \to X$ is an isotopy if for each $t \in I$, $F_t = F(\cdot,t):X \to X$ is an embedding. Below I is replaced by $[1,\infty]$.

Corollary I.17. Suppose M is a σ-compact subset of Q such that

1) For every ϵ there exists a map $h:Q \to Q-M$ such that $d(h,id) < \epsilon$.

2) There exists an isotopy $F = (F_t)_t : Q \times [1,\infty] \to Q$ such that $F_\infty = id_Q$ and $F|Q \times [1,\infty]$ is a 1-1 map into M .

Then M is a capset for Q .

Proof. Define $M_i = F([-1 + 1/i, 1 - 1/i]^\infty \times [1,i])$ and $h_i:Q \to M_i$ by $h_i(x) = F_i((1-1/i)\cdot x)$. Since $\lim_{i\to\infty} d(id_Q,h_i) = 0$ and M_i is a Z-set in M_{i+1} , the conditions of Corollary I.16 are satisfied.

Alternative definitions of Z-sets. Nowadays, the definition of Z-set presented here is the one most commonly used in Infinite-Dimensional Topology. This definition is closely related to (ii) below, which is due to Toruńczyk [17]. (v) below is the original definition of Anderson [3]. Definition (vi) is used in Chapman's "Notes on Hilbert Cube Manifolds."

Theorem I.18. For a closed subset K of Q the following are equivalent:

(i) K is a Z-set.

(ii) For every $n > 0$, the set $\{f \in Q^{I^n} | f(I^n) \cap K = \emptyset\}$ is dense in Q^{I^n} (here X^Y denotes the space of all maps from Y to X , topologized by the compact-open topology).

(iii) There exists an $h \in H(Q)$ such that $h(K)$ has infinite deficiency.

(iv) For every $\epsilon > 0$ there exists an $h \in H(Q)$ such that $h(K)$ has infinite deficiency and $d(h, id_Q) < \epsilon$.

(v) For every non-empty homotopically trivial open subset O of Q, the set $O-K$ is again non-empty and homotopically trivial.

(vi) For any open subset O of Q and any open cover $\mathcal{O}$ of O, there exists a map $f: O \to O-K$ such that f is limited by $\mathcal{O}$ (i.e., for any $x \in O$, $\{x, f(x)\}$ is contained in some element of $\mathcal{O}$).

Proof. We show

a) (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (v)

b) (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i)

c) (i) $\Leftrightarrow$ (iv)

a) (i) $\Rightarrow$ (ii): Suppose K is a Z-set. Let $f: I^n \to Q$ and $\epsilon > 0$ be given. Let $g: Q \to Q-K$ be such that $d(g, id_Q) < \epsilon$, then $d(f, g \circ f) < \epsilon$ and $g \circ f(I^n) \subset Q-K$.

(ii) $\Rightarrow$ (i): Suppose K satisfies (ii). Let n be sufficiently large that $p_n'(x) = p_n'(y)$ implies $d(x,y) < \epsilon/2$. Let $e: I^n \to Q$ be the natural embedding $e(x_1, \cdots, x_n) = (x_1, \cdots, x_n, 0, 0, \cdots)$. Let $f: I^n \to Q-K$ such that $d(f,e) < \epsilon/2$. Then $d(f \circ p_n', id_Q) < \epsilon$ and $(f \circ p_n')(Q) \subset Q-K$.

(ii) $\Rightarrow$ (v): Let $O \subset Q$ be open, non-empty and homotopically trivial. Since K is nowhere dense, $O-K$ is non-empty. Since K is an ANR, we only have to show that all homotopy groups of $O-K$ are trivial. Let n and $f:\partial I^{n-1} \to O-K$ be given. Extend f to $f':I^n \to O$, using homotopic triviality of O. Because K satisfies (ii), there exists a map $g:I^n \to Q-K$ such that $d(g,f') < \epsilon$, where $\epsilon < d(f'(I^n),Q-O)$ and $\epsilon < d(f(\partial I^{n-1}),K)$. Then $f(I^n) \subset O-K$, and if a map $F:\partial I^{n-1} \times I \to Q$ is defined by linear interpolation between f and $g|\partial I^{n-1}$ then the image of F lies entirely within $O-K$. Since I^n is homeomorphic to the union of itself and a cylinder $\partial I^n \times I$ attached to its boundary, an extension $f^+:I^n \to O-K$ of f can be constructed from F and g.

(v) $\Rightarrow$ (ii): Let n and $f:I^n \to Q$ be given and let $\epsilon > 0$. For sufficiently small δ, $d(x,x') < \delta$ implies $d(f(x),f(x')) < \epsilon \cdot 2^{-n-1} = \epsilon_1$ for any two points x and x' of I^n. Subdivide I^n in equal subcubes $\mathcal{J}_1,\cdots,\mathcal{J}_k$ of diameter less than δ and let T_i be the i-skeleton of this cellular subdivision. By an induction on the skeleta one constructs $f_i:T_i \to Q-K$, where $d(f|T_0,f_0) < \epsilon \cdot 2^{-n-1}$ and such that for any i-cell D of C_i, the diameter of $f_i(D)$ is less than $\epsilon \cdot 2^{-n+i-1}$. This can be done by applying (v) to an open convex set containing the image of the (combinatorial) boundary of D. The details are left to the reader. This completes (a).

(b): (i) $\Rightarrow$ (iv) follows from the Homeomorphism Extension Theorem and the Mapping Replacement Theorem, and

(iv) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) are trivial.

(c): (i) $\Rightarrow$ (vi) can be shown by embedding K in an endface by an autohomeomorphism of Q and applying a simple geometric argument. (vi) $\Rightarrow$ (i) is trivial.

This proves the theorem.

CHAPTER II

PSEUDO-INTERIORS FOR 2^Q AND RELATED RESULTS

First we show (Theorem II.2) that the collection of (connected) Z-sets in Q forms a pseudo-interior for 2^Q $(C(Q))$ by verifying the conditions of Lemma I.17. Thus we rely heavily on the facts that $2^Q \cong Q$ and $C(Q) \cong Q$ [9]. As a corollary, we show that $2^{\ell_2} \cong \ell_2$ (Corollary II.3). Next these results are generalized to the manifold case (Theorem II.4 and Corollary II.5).

Notation. By X^Y we mean the space of all continuous mappings from Y into X endowed with the compact-open topology.

Lemma II.1. a) *The collection of Z-sets in Q is a G_δ in 2^Q.*

b) *The collection of connected Z-sets in Q is a G_δ in $C(Q)$.*

Proof. a) Let $\mathcal{Z}_i = \{K \in 2^Q \mid \exists g \in Q^Q : g(Q) \cap K = \emptyset$ and

$d(g,id_Q) < 1/i\}$. Obviously $\mathcal{Z}_i$ is an open subset of 2^Q and $\mathcal{Z} = \cap_i \mathcal{Z}_i$ is exactly the collection of Z-sets in Q .
b): This is a direct consequence of a).

Remark. Lemma II.1 has a finite-dimensional analogue. In [11], Geoghegan and Summerhill give generalizations to Euclidean n-space E^n for many infinite-dimensional notions and results. In [11], Section 3, they define what they call Z_m-sets and strong Z_m-sets in E^n for $0 \leq m \leq n-2$. For $(n,m) \neq (3,0),(4,1)$ or $(4,2)$, the Z_m-sets and strong Z_m-sets coincide. A third possible definition is: " K is a Z_m^*-set if for all $i \leq m+1$, the maps from I^i into $E^n \backslash K$ lie dense in $(E^n)^{I^i}$ " . This definition is easily seen to imply the definition of Z_m-set given in [11] and to be implied by the definition of strong Z_m-set. The collection of Z_m^*-sets can be written as a countable intersection of open sets: let, for all $i \leq m+1$, $\{f_k^i\}_k$ be a countable dense subset of $(E^n)^{I^i}$. Let

$\mathcal{Z}_{i,k} = \{K \in 2^{E^n} | \exists g \in (E^n)^{I^i} : g(I^i) \cap K = \emptyset$ and $d(g,f_k^i) < 1/k\}$. Then $\bigcap_{\substack{i \leq m+1 \\ k=1,2,\cdots}} \mathcal{Z}_{i,k}$ is exactly the collection of Z_m^*-sets. Moreover, this set is dense in 2^{E^n} since the collection of finite subsets of E^n is a subcollection of it. If $m \leq n-3$, its intersection with $C(E^n)$ is also dense in $C(E^n)$ since the collection of compact connected one-dimensional rectilinear polyhedra is a subcollection and is dense in $C(E^n)$.

<u>Theorem II.2.</u> a) <u>The collection $\mathcal{Z}$ of Z-sets in Q is a pseudo-interior for 2^Q</u> .

b) <u>The collection $\mathcal{Z}_C$ of connected Z-sets in Q is a pseudo-interior for $C(Q)$</u> .

<u>Proof</u>. Note that Lemma I.17 is stated in terms of the pseudo-boundary and Theorem II.2 in terms of the pseudo-interior. The maps h and $(F_t)_t$ which are asked for in the lemma will map connected sets onto connected sets, so that they prove a) and b) simultaneously.

As remarked in the Introduction, every compact subset of s is a Z-set in Q . Therefore the map $h:Q \to s$, defined by $h(x) = (1-\epsilon)\cdot x = ((1-\epsilon)\cdot x_1,(1-\epsilon)\cdot x_2,\cdots)$ induces a map $2^h:2^Q \to \mathcal{Z}$ as asked for in 1) of Lemma I.17.

We shall construct F_t so that for $K \in 2^Q$ and $t < \infty$ the set $F_t(K)$ will be the union of two intersecting sets, one of which carries all information about K and the other of which is not a Z-set. First we consider the case that t is an integer. We define a sequence of maps $(f_i)_i:Q \to Q$ by

$$f_i(x) = (1 - \frac{1}{i}) \cdot (x_1,\cdots,x_{2i},0,x_{2i+1},0,x_{2i+2},\cdots) .$$

Obviously $f_i(Q)$ is contained in s and projects onto 0

in all odd coordinates $\geq 2i+1$. We define another auxiliary operator $T_{j,c}:2^Q \to 2^Q$, where $j \geq 1$ and $c \in [0,2]$:

$$T_{j,c}(K) = \{(x_1,\cdots,x_{j-1},x_j+y,x_{j+1},\cdots) \mid |y| \leq c \text{ and } |x_j+y| \leq 1 \text{ and } (x_i)_i \in K\} .$$

As c varies from 0 to 2 , $T_{j,c}(K)$ is transformed continuously from K into a set which occupies the whole interval in the j^{th} direction. We have:

$$T_{j,0}(K) = K \text{ and } T_{j,2}(p_j'^{-1}(p_j'(K))) = p_{j-1}'^{-1}(p_{j-1}'(K)) .$$

If $p_j(K) = \{0\}$ then $c = 2$ can be replaced by $c = 1$ in the above formula. Now we set:

$$F_i(K) = T_{2i+3,\frac{1}{2}}(f_i(K)) \cup p_{2i+3}'^{-1}(p_{2i+3}'(f_i(K))) .$$

For every K this is a non-Z-set since the second term contains a subset of the form $p_j'^{-1}(x_1,\cdots,x_j)$ with $-1 < x_i < 1$ for $i = 1,\cdots,j$. Furthermore, $p_{2i+3}^{-1}(\frac{1}{2}) \cap F_i(K) = p_{2i+3}^{-1}(\frac{1}{2}) \cap T_{2i+3,\frac{1}{2}}(f_i(K))$ is a translation of $f_i(K)$ in the direction of the $2i+3^{rd}$ coordinate, and therefore the first term contains all information about K , and F_i is one-to-one.

Before we describe f_t for arbitrary t , we restrict ourselves to $k = i + \frac{n-1}{n}$ where $i \geq 1$ and $n \geq 1$:

$$f_i(x) =$$

$$\left(1 - \frac{1}{i}\right)\cdot(x_1, \cdots, x_{2i}, 0, x_{2i+1}, 0, x_{2i+2}, 0, x_{2i+3}, 0, x_{2i+4}, 0, \cdots)$$

$$f_{i+\frac{1}{2}}(x) =$$

$$\left(1 - \frac{1}{i+\frac{1}{2}}\right)\cdot(x_1, \cdots, x_{2i}, x_{2i+1}, 0, 0, x_{2i+2}, 0, x_{2i+3}, 0, x_{2i+4}, 0, \cdots)$$

$$f_{i+\frac{2}{3}}(x) =$$

$$\left(1 - \frac{1}{i+\frac{2}{3}}\right)\cdot(x_1, \cdots, x_{2i}, x_{2i+1}, x_{2i+2}, 0, 0, 0, x_{2i+3}, 0, x_{2i+4}, 0, \cdots)$$

$$f_{i+\frac{3}{4}}(x) =$$

$$\left(1 - \frac{1}{i+\frac{3}{4}}\right)\cdot(x_1, \cdots, x_{2i}, x_{2i+1}, x_{2i+2}, 0, x_{2i+3}, 0, 0, 0, x_{2i+4}, 0, \cdots)$$

$$f_{i+\frac{4}{5}}(x) =$$

$$\left(1 - \frac{1}{1+\frac{4}{5}}\right)\cdot(x_1, \cdots, x_{2i}, x_{2i+1}, x_{2i+2}, 0, x_{2i+3}, 0, x_{2i+4}, 0, 0, 0, \cdots)$$

. . . .

$$f_{i+1}(x) =$$

$$\left(1 - \frac{1}{i+1}\right)\cdot(x_1, \cdots, x_{2i}, x_{2i+1}, x_{2i+2}, 0, x_{2i+3}, 0, x_{2i+4}, 0, x_{2i+5}, \cdots) .$$

For $t \in \left(i + \frac{n-1}{n}, i + \frac{n}{n+1}\right)$, f_t is defined by linear interpolation between $f_{i+\frac{n-1}{n}}$ and $f_{i+\frac{n}{n+1}}$. This way $f_t(Q)$ projects onto 0 in all odd coordinates $\geq 2i+3$ if $t \leq i+1$.

For $i \geq 1$ and $u \in [0,1]$ we define

$$F_{i+u}(K) = T_{2i+3,\frac{1}{2}(1-u)} \circ T_{2i+5,\frac{1}{2}u}(f_{i+u}(K))$$

$$\cup\ T_{2i+4,2-2u} \circ T_{2i+5,1-u}(p'^{-1}_{2i+5}(p'_{2i+5}(f_{i+u}(K)))) .$$

Note that this is consistent with the previous definition of $F_i(K)$. We check:

1) $(F_t)_t$ is continuous. For finite t this follows from the continuity of the operators $T_{j,c}, 2^{f_t}$ and $p'^{-1}_j \circ p'_j$; for $t \to \infty$ it is easily seen that $F_t(K) \to K$.

2) For every K, $F_t(K)$ is a non-Z-set if t is finite, for it contains a subset of the form $p'^{-1}_j(x_1,\cdots,x_j)$ with $-1 < x_i < 1$ for $i=1,\cdots,j$.

3) $F = (F_t)_t$ is one-to-one on $2^Q \times [0,\infty)$: for the determination of t from $F_t(K)$, note that $t \in (i,i+1]$ iff $p_j(F_t(K)) = [-1,1]$ for all $j > 2i+5$ and for no odd $j \leq 2i+5$. Once it is determined that $t \in (i,i+1]$, then on that interval t is in one-to-one correspondence with $p_{2i+3} \circ F_t(K) = [-(1-(t-i))/2,(1-(t-i))/2]$ (recalling that $p_{2i+3} \circ f_t(x) = 0$ for $x \in Q$ and $t \leq i+1$).
Finally, for $t = i+u$ and $u \in (0,1]$,

$F_t(K) \cap p_{2i+3}^{-1}((1-u)/2) \cap p_{2i+5}^{-1}(u/2)$ is a copy of K in a canonical way. Note that this set does not intersect the second term

$$T_{2i+4,2-2u} \circ T_{2i+5,1-u}(p'^{-1}_{2i+5} p'_{2i+5}(f_{i+5}(K)))$$

since the latter set projects onto 0 in the $2i+3$rd coordinate, and if $u = 1$ also in the $2i+5$th coordinate.

4) If K is connected, then $F_t(K)$ is connected since $F_t(K)$ is the union of two connected sets which intersect in $f_t(K)$.

The following corollary answers a question posed by R. M. Schori:

<u>Corollary II.3</u>. <u>Both the collection of compact subsets of ℓ_2 and the collection of compact connected subsets of ℓ_2 are homeomorphic to ℓ_2</u>.

<u>Proof</u>. According to [1], ℓ_2 is homeomorphic to $s = (-1,1)^\infty$. Thus it is sufficient to show that the collection $\mathcal{L}$ (or $\mathcal{L}_C$) of closed (connected) subsets of Q which are contained in s forms a pseudo-interior for 2^Q (or $C(Q)$). Since this collection is a subset of $\mathcal{J}$ ($\mathcal{J}_C$) we only have to verify condition 1) of Lemma I.17 and to show that $\mathcal{J}$ and $\mathcal{J}_C$ are G_δ's. But the map 2^h from the proof of Theorem II.2 actually maps 2^Q and

$C(Q)$ in $\mathcal{L}$ and $\mathcal{L}_C$ respectively, showing 1) of Lemma I.17. Finally, we can write $\mathcal{L}$ ($\mathcal{L}_C$) as a G_δ by $\cap_i \{K \subset Q \mid K$ is closed (and connected) and $p_i(K) \subset (-1,1)\}$. This completes the proof of the corollary.

We have similar results about hyperspaces of Hilbert cube manifolds. A separable metric space M is a <u>Hilbert cube manifold</u> or Q-<u>manifold</u> if M is locally homeomorphic to Q . In [8], Chapman proved that every Q-manifold M is triangulable, i.e., $M \cong |P| \times Q$, where P is a countable locally finite complex. If M is compact, then P can be chosen finite and even such that $|P|$ is a combinatorial manifold with boundary. We denote points of $|P| \times Q$ by (q,x) or $(q,(x_i)_i)$ and define the projection maps $p_i(q,x) = x_i$ and $p_P(q,x) = q$. For a given triangulation $M = |P| \times Q$, a closed subset $K \subset M$ is called i-<u>deficient</u> if $p_i(K)$ is a point, and <u>infinitely deficient</u> if K is i-deficient for infinitely many i . A closed subset K of a compact Q-manifold M is a Z-set if for every ϵ there is a map $f:M \to M-K$ such that $d(f,id_M) < \epsilon$. Only a restricted version of the Homeomorphism Extension Theorem holds, since homotopy conditions have to be met.

<u>Theorem II.4</u>. <u>If</u> M <u>is a compact connected</u> Q-<u>manifold, then</u>

a) <u>the collection</u> $\mathcal{Z}^M$ <u>of</u> Z-<u>sets in</u> M <u>is a pseudo-interior for</u> 2^M .

b) <u>the collection</u> $\mathcal{Z}_C^M$ <u>of connected</u> Z-<u>sets in</u> M <u>is a pseudo-interior for</u> C(M) .

<u>Proof</u>. As observed above, by [8], we may write $M = |P| \times Q$, where $|P|$ is a compact finite-dimensional manifold with boundary. Again we apply Lemma I.17, where the M from the lemma is $2^M - \mathcal{Z}^M$ or $C(M) - \mathcal{Z}_C^M$ respectively. As before one can prove that $\mathcal{Z}^M$ and $\mathcal{Z}_C^M$ are G_δ-sets in 2^M and C(M) respectively. Condition 1) of the lemma is proved by the map 2^h, where $h(p,x) = (p,(1-\epsilon)\cdot x)$.

Let $H: |P| \times [1,\infty] \to |P|$ be an isotopy such that $H_\infty = \mathrm{id}$ and $H_t(|P|) \subset |P| - |\partial P|$ for finite t (remember that we assume that $|P|$ is a compact manifold with boundary). Consider the map $F: 2^Q \times [1,\infty] \to 2^Q$ defined in the proof of Theorem II.2. Define, for $q \in |P|$ and $K \subset Q$, $G_t(\{q\} \times K) = \{H_t(q)\} \times F_t(K)$. If L is a subset of $|P| \times Q$, then L can be written as a union $\bigcup_{q \in p_P(L)} \{q\} \times L_q$. Now define $G_t(L) = \bigcup_{q \in p_P(L)} G_t(\{q\} \times L_q)$ Then $G = (G_t)_t$ satisfies 2) of Lemma I.17. We need only show that $G_t(L)$ is a closed set.

From the definition of $F_t(K)$ one readily sees that $F_t(K) = \bigcup_{x \in K} F_t(\{x\})$. Therefore we can write $G_t(L) =$
$= \bigcup_{(q,x) \in L} \{H_t(q)\} \times F_t(\{x\})$. Let $(r_i,y_i)_i$ be a sequence in $G_t(L)$ converging to (r,y). We have to show that

$(r,y) \in G_t(L)$. Let $r_i = H_t(q_i)$ and $y_i \in F_t(\{x_i\})$, where $(q_i,x_i) \in L$. There is a subsequence (q_{i_k},x_{i_k}) converging to some point $(q,x) \in L$. Then $H_t(q) = \lim_k r_{i_k} = r$, and by continuity of F_t we have that $y \in F_t(\{x\})$. Therefore $(r,y) \in G_t(L)$.

Corollary II.5. *For any connected ℓ_2-manifold M , both the collection 2^M of compact subsets of M and the collection C(M) of connected compact subsets of M are homeomorphic to ℓ_2 .*

Proof. According to [8] we can triangulate $M = |P| \times \ell_2$, where P is a locally finite simplicial complex. Of course, now we cannot assume that $|P|$ is a manifold with boundary.

Let K be a compact (connected) subset of M , then K has a closed neighborhood $|P'| \times \ell_2$, where P' is a finite (connected) subcomplex of P . The collection $\mathcal{O}^{P'}$ ($\mathcal{O}_C^{P'}$) of compact (connected) subsets of M which are contained in the topological interior of $|P'| \times \ell_2$ is an open neighborhood of K . Its closure in 2^M (C(M)) , the set $\{K \subset M | K$ is compact (and connected) and $K \subset |P'| \times \ell_2\}$, is a pseudo-interior for $2^{|P'| \times Q}$ (for $C(|P'| \times Q)$) if we identify ℓ_2 with $(-1,1)^\infty \subset Q$. This is proved by an argument similar to that in the proof of Corollary II.3.

Therefore $\mathcal{O}^{P'}$ $(\mathcal{O}_C^{P'})$ is an open subset of a copy of ℓ_2, showing that 2^M $(C(M))$ is an ℓ_2-manifold.

Next we show that 2^M $(C(M))$ is homotopically trivial. By [12], this will prove that 2^M $(C(M))$ is homeomorphic to ℓ_2. Let a map $f:\partial I^n \to 2^M$ (or $f:\partial I^n \to C(M)$) be given. Then $Y' = \bigcup_{y\in\partial I^n} f(y)$ is a compact union of compact sets, and therefore a compact subset of M. Choose a finite connected subcomplex P' of P and a compact convex subset D of ℓ_2 such that $Y' \subset |P'| \times D$. Then $f(\partial I^n) \subset 2^{|P'| \times D}$ $(f(\partial I^n) \subset C(|P'| \times D))$. Moreover, $2^{|P'| \times D}$ and $C(|P'| \times D)$ are contractible: define, for $K \in 2^{|P'| \times D}$ $(K \in C(|P'| \times D))$ and for $t \in [0,T]$ where T is sufficiently large, $H(K,t)$ to be the closed t-neighborhood of K in some fixed convex metric for $|P'| \times D$. Then H is a contraction of $2^{|P'| \times D}$ (or $C(|P'| \times D)$). Therefore f can be extended to $\bar{f}:I^n \to 2^{|P'| \times D} \subset 2^M$ (to $\bar{f}:I^n \to C(|P'| \times D) \subset C(M)$).

CHAPTER III

PSEUDO-INTERIORS FOR 2^I

In this chapter we show that both the collection of zero-dimensional subsets of I and the collection $\mathcal{C}$ of Cantor sets in I are pseudo-interiors for 2^I. We use Lemma I.16. It seems reasonable that similar statements are true for the hyperspace of more general spaces, but the author has been unable to prove a comparable statement even for the hyperspace of a finite graph.

In this chapter $I = [0,1]$.

Lemma III.1. a) The collection $\mathcal{O}$ of zero-dimensional closed subsets of a compact metric space X is a G_δ in 2^X.

b) The collection $\mathcal{C}$ of Cantor sets in X is a G_δ in 2^X.

Proof. a) The collection $\mathcal{O}_n = \{A \subset X \mid A$ is closed and all components of A have diameter less than $1/n\}$ is an open subset of 2^X. For let $(A_i)_i \to A$, where $A_i \notin \mathcal{O}_n$

for all i . We show that $A \notin \mathcal{O}$. For every i there is a component K_i of A_i with diameter at least $1/n$. The sequence $(K_i)_i$ has a subsequence $(K_{i_k})_k$ which converges to a set K which is closed, connected and has diameter at least $1/n$ and is a subset of A . Therefore $A \notin \mathcal{O}_n$. b) We write $\mathcal{C}_n = \{A \subset X \,|\, A$ is closed and for all $x \in A$, there is a $y \neq x$ in A such that $d(x,y) < 1/n\}$. Since Cantor sets are exactly the compact metric spaces which are zero-dimensional and have no isolated points, it follows that $\mathcal{C} = \mathcal{O} \cap \bigcap_n \mathcal{C}_n$. We show that $\mathcal{C}_n$ is an open subset of 2^X : let $(A_i)_i \to A$, where $A_i \notin \mathcal{C}_n$ for all i . There is a sequence $(q_i)_i$ such that $U_{1/n}(q_i) \cap A_i = \{q_i\}$. This sequence has a limit point q and it is easily seen that $U_{1/n}(q) \cap A = \{q\}$. Therefore $A \notin \mathcal{C}_n$.

<u>Main Lemma III.2</u>. <u>There exist arbitrarily small maps $h: 2^I \to \mathcal{C}$</u> .

<u>Proof</u>. The map h will be defined as a composition

$$2^I \xrightarrow{\;f\;} \mathrm{FSI} \xrightarrow{\;h_N^+\;} \mathrm{FSC} \xrightarrow{\;g\;} \mathcal{C}, \qquad h = g \circ h_N^+ \circ f$$

where FSI (<u>F</u>inite <u>S</u>equences of <u>I</u>ntervals) is a collection of finite sequences of intervals, to be defined later.

and FSC (Finite Sequences of Cantor sets)

is a collection of finite sequences of topological Cantor sets, which will also be defined later on. The map f will be discontinuous, but g, h_N^+ and $g \circ h_N^+ \circ f$ are continuous. In the subsequent discussion we assume a fixed $\epsilon < \frac{1}{2}$, and N is the largest integer such that $N \cdot \epsilon \leq 1$. The map $h = g \circ h_N^+ \circ f$ will have distance less than 3ϵ to the identity.

Step 1. The set FSI. Let FSI_n be the set of all sequences of n terms $\langle [a_1,b_1],\cdots,[a_n,b_n] \rangle$ such that

i) $0 \leq a_1$ and $b_n \leq 1$

ii) $a_{i+1} \geq b_i$, i.e., the intervals do not overlap

iii) $b_i - a_i \geq 2n \cdot \epsilon^2$ if $1 < i < n$

iv) $b_i - a_i \geq n \cdot \epsilon^2$ if $i=1,n$.

The metric on FSI_n is

$\rho_n(\langle [a_1,b_1],\cdots,[a_n,b_n] \rangle, \langle [a_1',b_1'],\cdots,[a_n',b_n'] \rangle)$

$= \max_i \max(|a_i'-a_i|, |b_i'-b_i|)$. Define $FSI = \cup_{n=1}^N FSI_n$, where N is defined as above. Note that for $n > N$, $FSI_n = \emptyset$ since for any element X of FSI_n, the sum of the lengths of the intervals of X is at least $(n-1) \cdot 2n \cdot \epsilon^2 > (n-1) \cdot 2\epsilon > 2-2\epsilon > 1$ since $\epsilon < \frac{1}{2}$, whereas X is a collection of non-overlapping subintervals of $[0,1]$. We choose the following metric on FSI :

$\rho(X,Y) = \rho_n(X,Y)$ if $\{X,Y\} \subset FSI$, i.e., if both X and Y consist of n intervals, and $\rho(X,Y) = 1$ if for no n, $\{X,Y\} \subset FSI_n$, i.e., if X and Y have a different number of terms.

Step 2. The function $f:2^I \to FSI$. Let $A \in 2^I$; then $U_\epsilon(A)$, the open ϵ-neighborhood of A, is a finite union of disjoint subintervals of I, open relative to I. Let $f(A) = \langle [a_1,b_1],\cdots,[a_n,b_n] \rangle$, where the intervals $[a_i,b_i]$ are the closures of the components of $U_\epsilon(A)$, arranged in increasing order; e.g., if $U_\epsilon(A) = (a_1,b_1) \cup (b_1,b_2)$ then $f(A) = \langle [a_1,b_1],[b_1,b_2] \rangle$, and not $\langle [a_1,b_2] \rangle$. This assignment is not continuous: Let $A_\delta = \{0, 2\epsilon + \delta\}$. If $\delta \geq 0$, then $f(A_\delta) = \langle [0,\epsilon],[\epsilon+\delta, 3\epsilon+\delta] \rangle$ but if $\delta < 0$ then $f(A_\delta) = \langle [0, 3\epsilon+\delta] \rangle$. But apart from this phenomenon f is continuous in the following sense: Let $\delta < \epsilon$ and suppose for some $A,B \in 2^I$, $d_H(A,B) < \delta$, where d_H denotes the Hausdorff distance (see the Introduction). Then each gap of $U_\epsilon(A \cup B)$ (including a gap consisting of one point) corresponds to, i.e., is contained in, a gap of $U_\epsilon(A)$, since for $\delta < \epsilon$ it cannot lie left or right from $U_\epsilon(A)$. Conversely, each gap in $U_\epsilon(A)$ which has length $\geq 2\delta$ corresponds to, i.e., contains, a gap of $U_\epsilon(A \cup B)$. Let $f_B(A)$ be a function from 2^I to FSI which is obtained from $f(A)$ by replacing each gap in $U_\epsilon(A)$ which has no counterpart in $U_\epsilon(A\cup B)$ by a degenerate gap (see Fig. III.1);

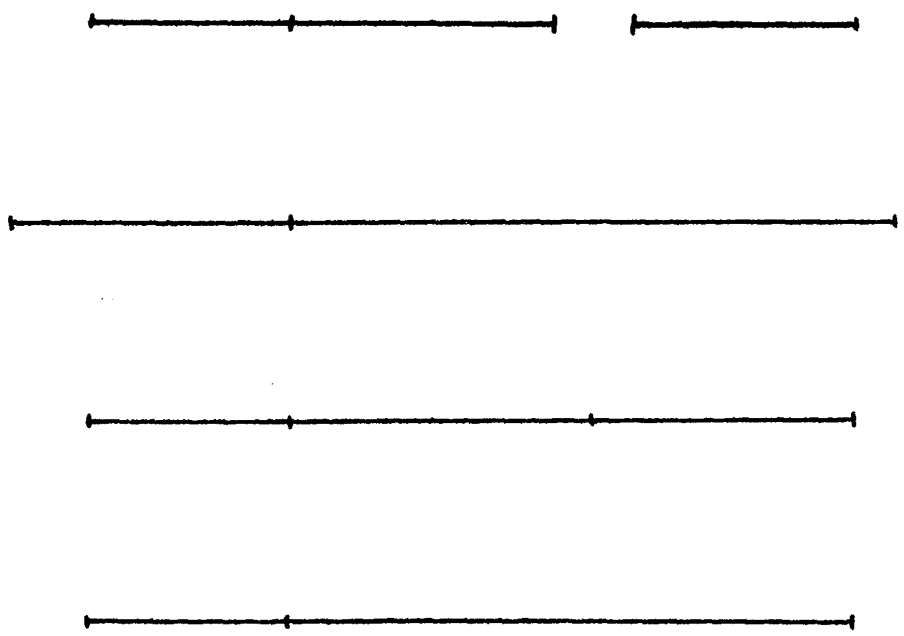


Fig. III.1

e.g., if $f(A) = <[a_1,b_1],[a_2,b_2]>$ with $a_2-b_1 < 2\delta$ and if $U_\epsilon(A\cup B) = (a_1',b_2')$ with $0 \leq b_2' - b_2 < \delta$ and

$0 \leq a_1 - a_1' < \delta$, then let $f_B(A) = <[a_1,\frac{b_1+a_2}{2}],[\frac{b_1+a_2}{2},b_2]>$. Let $f^B(A)$ eliminate the degenerate gaps thus obtained (but not the other degenerate gaps); e.g., in the above example $f^B(A) = <[a_1,b_2]>$. Then for $d_H(A,B) < \delta$ we have $d(f^B(A), f^A(B)) < \delta$ and also $d(f(A), f_B(A)) < \delta$ and $d(f(B), f_A(B)) < \delta$. These notations will be used in the proof of the continuity of $g \circ h_N^+ \circ f$.

Step 3. The set FSC. Let C be a topological Cantor set such that $C \subset I$ and $\{0,1\} \subset C$ and $d_H(C,I) < \epsilon$. Let $C(a,b)$ be the image of C under the linear map which maps 0 onto a and 1 onto b . For $[a,b] \subset [0,1]$ we also have $d_H(C(a,b),[a,b]) < \epsilon$. We define FSC_n to be the collection of all sequences of n terms $\langle C(a_1,b_1), \cdots, C(a_n,b_n) \rangle$ such that

i) $0 \leq a_1 \leq \cdots \leq a_n \leq 1$

ii) $0 \leq b_1 \leq \cdots \leq b_n \leq 1$

iii) $a_i < b_i$ for $1 \leq i \leq n$.

Thus the sets $C(a_i,b_i)$ may overlap. Define $FSC = \cup_{n=1}^N FSC_n$. The metric of FSC is somewhat analogous to that on FSI: If $X = \langle C(a_1,b_1), \cdots, C(a_n,b_n) \rangle$ and $Y = \langle C(a_1',b_1'), \cdots, C(a_n',b_n') \rangle$ then $\rho(X,Y) = \max_i d_H(C(a_i,b_i), C(a_i',b_i'))$ and if for no n , $\{X,Y\} \subset FSC_n$ then $\rho(X,Y) = 1$.

Step 4. The map $g: FSC \to \mathcal{C}$. We simply let $g(X)$ be the union of the terms of X . Obviously g is continuous. Notice that by the characterization of Cantor sets given in the proof of Lemma III.1, $g(X)$ is indeed a Cantor set.

Step 5. Construction of h_N^+. From the remark at Step 2 it is easily seen that the function

$\varphi:\langle[a_1,b_1],\cdots,[a_n,b_n]\rangle \to \langle C(a_1,b_1),\cdots,C(a_n,b_n)\rangle$ does not yield a continuous composition $g\circ\varphi\circ f$. Instead, we construct by induction a map $h_n:FSI_n \to FSC_n$ and set $h_n^+ = \cup_{i=0}^n h_i$ (i.e., h_n^+ is the function which assigns $h_i(X)$ to X if $X \in FSI_i$ and $i \leq n$). The following induction hypotheses should be satisfied:

i) If $X = \langle[a_1,b_1],\cdots,[a_n,b_n]\rangle$, then $h_n(X) = \langle C(a_1',b_1'),\cdots,C(a_n',b_n')\rangle$, where $a_1' = a_1$ and $b_n' = b_n$.

ii) Additivity at "large" gaps. If X can be broken up into Y and Z where $Y = \langle[a_1,b_1],\cdots,[a_i,b_i]\rangle$ and $Z = \langle[a_{i+1},b_{i+1}],\cdots,[a_n,b_n]\rangle$ and $a_{i+1} - b_i \geq 2\epsilon^2$ then $h_n(X) = \langle C(a_1,b_1'),\cdots,C(a_n',b_n)\rangle$, where $h_{n-1}^+(Y) = \langle C(a_1,b_1'),\cdots,C(a_i',b_i')\rangle$ and $h_{n-1}^+(Z) = \langle C(a_{i+1}',b_{i+1}'),\cdots,C(a_n',b_n)\rangle$. In particular, by i) $b_i' = b_i$ and $a_{i+1}' = a_{i+1}$.

iii) If $X = \langle[a_1,b_1],\cdots,[a_i,b_i],[b_i,b_{i+1}],\cdots,[a_n,b_n]\rangle$, that is, if $a_{i+1} = b_i$, and if $Y = \langle[a_1,b_1],\cdots,[a_i,b_{i+1}],\cdots,[a_n,b_n]\rangle$, and if, moreover, $h_{n-1}(Y) = \langle C(a_1,b_1'),\cdots,C(a_i',b_{i+1}'),\cdots,C(a_n',b_n)\rangle$, then $h_n(X) = \langle C(a_1,b_1'),\cdots,C(a_i',b_{i+1}'),C(a_i',b_{i+1}'),C(a_{i+2}',b_{i+2}'),\cdots,C(a_n',b_n)\rangle$, i.e., $a_i' = a_{i+1}'$

and $b_i' = b_{i+1}'$ and $gh_n(X) = gh_{n-1}(Y)$.

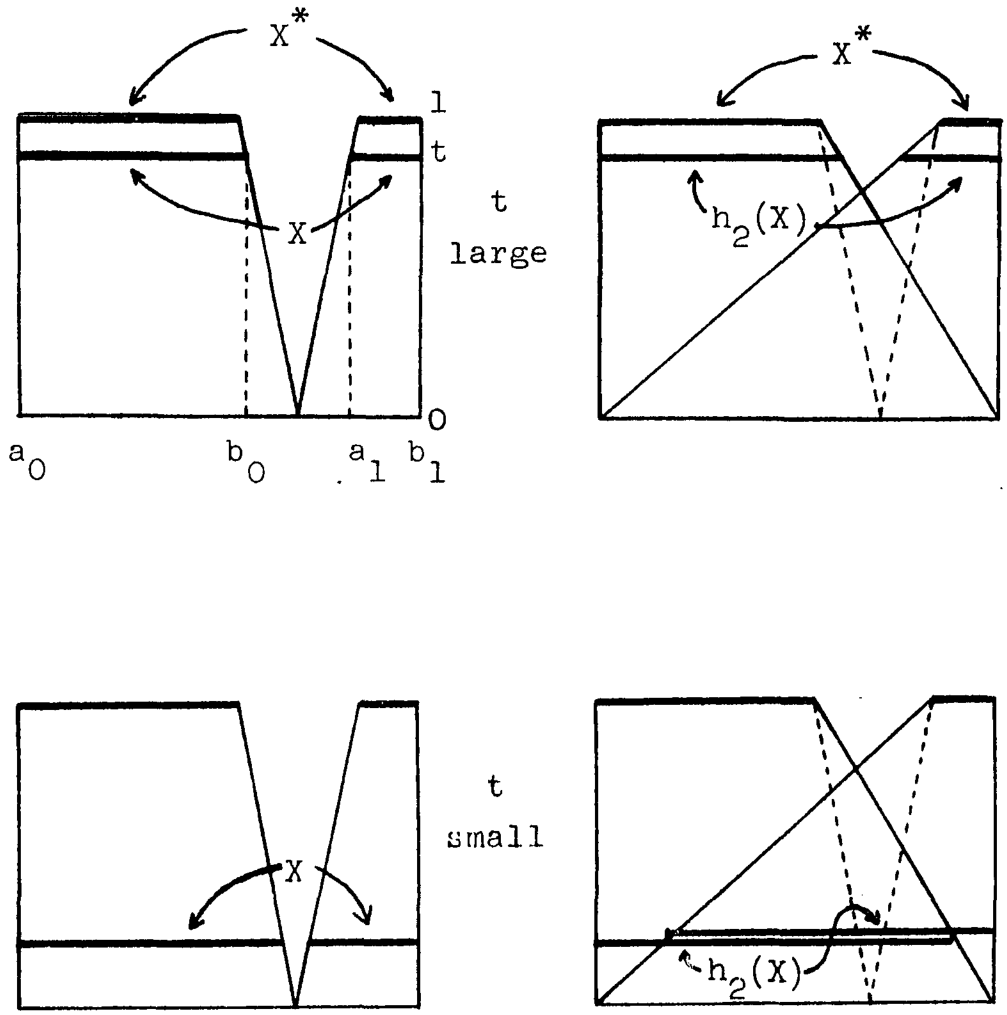
These induction hypotheses, and especially iii), will be seen to insure continuity of $g \circ h_N^+ \circ f$. We give now the inductive construction of $h_n : FSI_n \to FSC_n$.

<u>n = 1</u> : set $h_1(\langle[a_1,b_1]\rangle) = \langle C(a_1,b_1)\rangle$, in accordance with i) .

<u>n = 2</u> : let $X = \langle[a_1,b_1],[a_2,b_2]\rangle$ with both $b_1 - a_1 \geq \epsilon^2$ and $b_2 - a_2 \geq \epsilon^2$ and with $a_2 - b_1 \geq 0$. If $a_2 = b_1$ then according to iii) we have $h_2(X) = \langle C(a_1,b_2),C(a_1,b_2)\rangle$. If $a_2 - b_1 \geq 2\epsilon^2$, then according to ii), we have $h_2(X) = \langle C(a_1,b_1),C(a_2,b_2)\rangle$. If $a_2 - b_1 = t \cdot 2\epsilon^2$ with $0 < t < 1$, then b_1' and a_2' are constructed as in Figure III.2 (the pictures show what happens if t is large (upper pictures), and what happens if t is small, (lower pictures)).

In formulas: let $X^* = \langle[a_1,b_1^*],[a_2^*,b_2]\rangle$ be the result of enlarging the gap (b_1,a_2) symmetrically from its midpoint by a factor $1/t$. Thus $a_2^* - b_1^* = 2\epsilon^2$. We put $h_2(X) = \langle C(a_1,t \cdot b_1^* + (1-t) \cdot b_2),C(t \cdot a_2^* + (1-t) \cdot a_1,b_2)\rangle$. Note that this is consistent with the case $a_2 = b_1$ and $a_2 - b_1 \geq 2\epsilon^2$ as treated above.

<u>n + 1</u> : Suppose h_n^+ is already defined. Let $X = \langle[a_1,b_1],\cdots,[a_{n+1},b_{n+1}]\rangle \in FSI_{n+1}$. If for all i , $a_{i+1} - b_i = 0$, i.e., if all gaps are degenerate, then by repeated application of iii) we find that for all i ,



Note that above and below we have different X but the same X^*.

Fig. III.2

$C(a_1', b_1') = C(a_1, b_{n+1})$. If $\max_i (a_{i+1} - b_i) \geq 2\epsilon^2$, then $h_{n+1}(X)$ is determined by ii). If for several i , $a_{i+1} - b_i \geq 2\epsilon^2$ then it is easily seen, using ii) for h_n^+ , that $h_{n+1}(X)$ is independent of the choice of the gap at which X is broken up into Y and Z . So let us assume that the length of the largest gap $\max_i (a_{i+1} - b_i) = 2t \cdot \epsilon^2$ with $0 < t < 1$. Let X^* be the result of widening each gap symmetrically from its midpoint by a factor $1/t$, so that the largest gap of X^* has width $2\epsilon^2$. Now break up X^* into Y and Z , where the gap in between Y and Z has width $2\epsilon^2$. The reader may check that Y and Z are elements of $FSI_1 \cup \cdots \cup FSI_n$, in particular that they consist of intervals of sufficient length, noting that since Y and Z have less terms than X , they are allowed to consist of smaller intervals. Therefore $h_n^+(Y)$ and $h_n^+(Z)$ are defined. Let $h_n^+(Y) = \langle C(a_1, b_1^*), \cdots, C(a_i^*, b_i^*) \rangle$ and $h_n^+(Z) = \langle C(a_{i+1}^*, b_{i+1}^*), \cdots, C(a_{n+1}^*, b_{n+1}) \rangle$. The construction of $h_{n+1}(X)$ from $h_n^+(Y)$ and $h_n^+(Z)$ is shown in Figure III.3.

In formulas: $h_{n+1}(X) = \langle C(a_1, t \cdot b_1^* + (1-t) \cdot b_{n+1}),$ $C(t \cdot a_2^* + (1-t) \cdot a_1, t \cdot b_2^* + (1-t) \cdot b_{n+1}), \cdots, C(t \cdot a_{n+1}^* +$ $(1-t) \cdot a_1, b_{n+1}) \rangle$. Thus each Cantor set is stretched somewhat toward $C(a_1, b_{n+1})$: only a little if t is close to 1 and almost all the way if t is close to 0 .

It is an easy exercise to check the induction hypo-

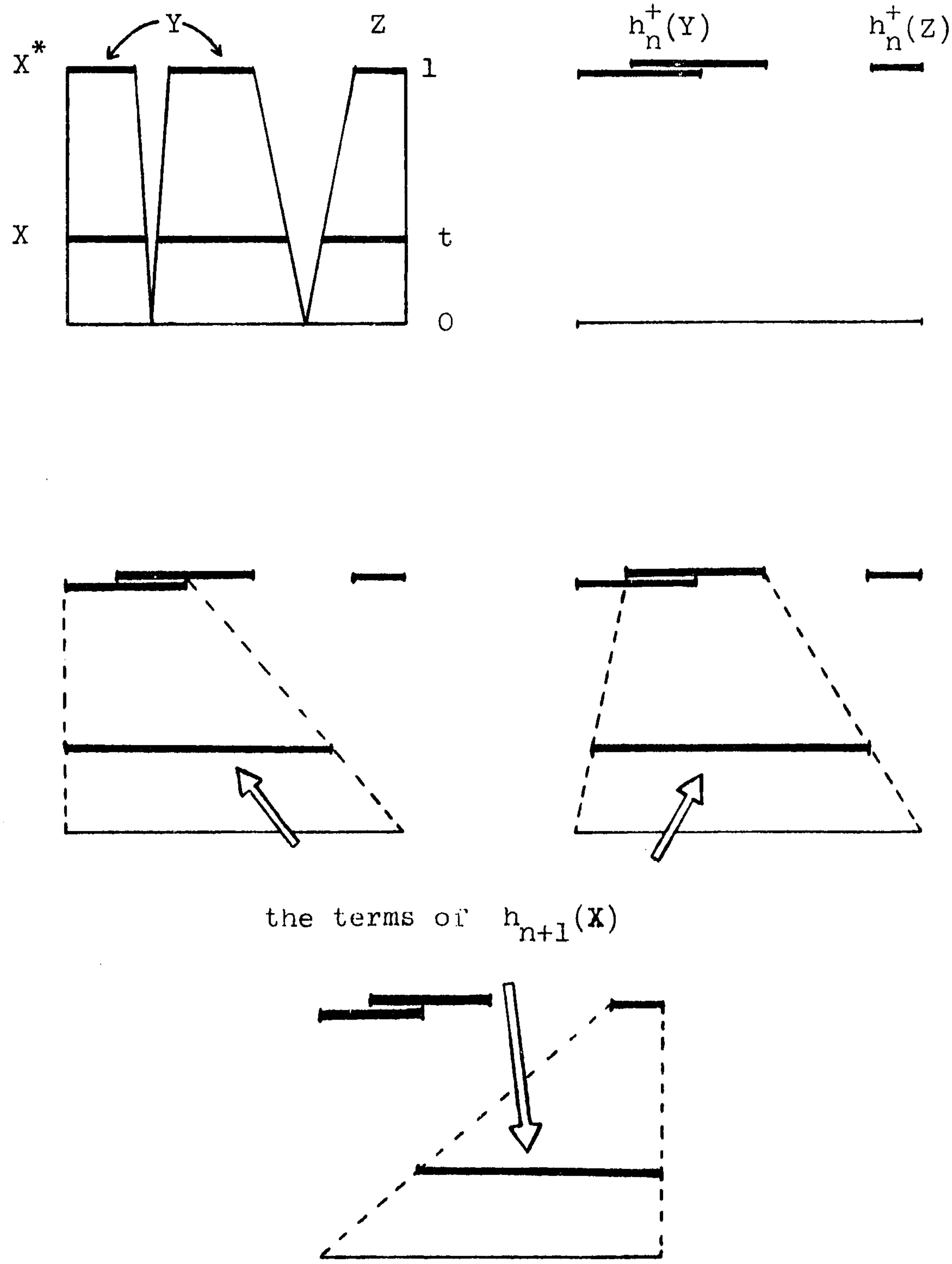


Fig. III.3

theses and to prove that $d_H(A, g\circ h_n^+\circ f(A)) < 3\epsilon$. To show continuity, we refer to the functions f_B and f^B , defined at Step 2. From the remarks there and the continuity of g and h_N^+ and the fact that $g\circ h_N^+\circ f_B(A) = g\circ h_N^+\circ f^B(A)$ for any two $A,B \in 2^I$, we easily see that $g\circ h_N^+\circ f$ is continuous.

Let $I^* = \{\{t\}|t \in I\} \subset 2^I$. Then I^* is a Z-set in 2^I , since the map $f:2^I \to 2^I$ defined by $f(K) = Cl(U_\epsilon(K))$ is an ϵ-small map from 2^I into $2^I - I^*$. Moreover, $I^* \cap \mathcal{C} = \emptyset$. Therefore the inclusion of I^* in Lemma III.3 is harmless according to Corollary I.13.

<u>Lemma III.3</u>. <u>The set</u> $(2^I - \mathcal{O}) \cup I^*$ <u>contains a family of copies of</u> Q <u>as asked for in Lemma</u> I.16 <u>sub</u> 2).

<u>Proof</u>. For $K \subset I$, let $[a_K, b_K]$ be the smallest closed interval containing K . Define $M_\epsilon \subset 2^I$ by

$M_\epsilon = \{K \subset I | K$ is closed and $[a_K + (1-\epsilon)\cdot(b_K - a_K), b_K] \subset K\}$.

Let $\tilde{K}_\epsilon$ be the image of K under a linear map which maps a_K onto a_K and b_K onto $a_K + (1-\epsilon)\cdot(b_K-a_K)$. In formulas: $\tilde{K}_\epsilon = \{a_K + (1-\epsilon)\cdot(t-a_K) | t \in K\}$. Let $h_\epsilon(K) = \tilde{K}_\epsilon \cup [a_K + (1-\epsilon)\cdot(b_K - a_K), b_K]$. Then h_ϵ is a homeomorphism of 2^I onto M_ϵ with distance $\leq \epsilon$ to the identity. Since Lemma III.2 and the remark on I^* show that every closed subset of $(2^I - \mathcal{O}) \cup I^*$ is a Z-set, it follows that M_ϵ is a Z-set in 2^I. Because for $\delta < \epsilon$, $h_\delta^{-1}(M_\epsilon) = M_{(\epsilon-\delta)/(1-\delta)}$ is a Z-set in 2^I by the same token, we see that M_ϵ is a Z-set in M_δ. Therefore the family $\{M_{1/i}\}_i$ satisfies 2) of Lemma I.16, both for $M = (2^I - \mathcal{O}) \cup I^*$ and for $M = 2^I - \mathcal{C}$.

Combining Lemmas III.2 and III.3, we obtain the main theorem of this chapter:

Theorem III.4. *Both the collection of topological Cantor sets and the collection of zero-dimensional subsets in* I *are pseudo-interiors for* 2^I.

Finally, we mention the following conjecture:

Conjecture (R. M. Schori). *The collection of finite subsets of* I *is an fd capset for* 2^I.

BIBLIOGRAPHY

[1] R. D. Anderson, "Hilbert space is homeomorphic to the countable infinite product of lines," Bull. Amer. Math. Soc. 72(1966), 515-519.

[2] R. D. Anderson, "Topological Properties of the Hilbert Cube and the Infinite Product of Open Intervals," Trans. Amer. Math. Soc. 126(1967), 200-216.

[3] R. D. Anderson, "On topological infinite deficiency," Mich. Math. J. 14(1967), 365-383.

[4] R. D. Anderson, "On sigma-compact subsets of infinite-dimensional spaces," Trans. Amer. Math. Soc. (to appear).

[5] R. D. Anderson, T. A. Chapman, "Extending homeomorphisms to Hilbert cube manifolds," Pacific J. of Math. 38(1971), 281-293.

[6] W. Barit, "Small extensions of small homeomorphisms," Notices Amer. Math. Soc. 16(1969), 295.

[7] C. Bessaga, A. Pełczyński, "Estimated extension theorem, homogeneous collections and skeletons and their

application to topological classification of linear metric spaces," Fund. Math. 69(1970), 153-190.

[8] T. A. Chapman, "All Hilbert cube manifolds are triangulable," preprint.

[9] D. W. Curtis, R. M. Schori, "Hyperspaces of Peano continua are Hilbert cubes," preprint.

[10] M. K. Fort, Jr., "Homogeneity of Infinite Products of Manifolds with Boundary," Pacific J. Math. 12(1962), 879-884.

[11] Ross Geoghegan, R. Richard Summerhill, "Pseudo-boundaries and pseudo-interiors in Euclidean spaces and topological manifolds," Trans. Amer. Math. Soc. (to appear).

[12] D. W. Henderson, R. Schori, "Topological classification of infinite-dimensional manifolds by homotopy type," Bull. Amer. Math. Soc. 76(1969), 121-124.

[13] D. W. Henderson, J. E. West, "Triangulated infinite-dimensional manifolds," Bull. Amer. Math. Soc. 76(1970), 655-660.

[14] V. L. Klee, Jr., "Some Topological Properties of Convex Sets," Trans. Amer. Math. Soc. 78(1955), 30-45.

[15] N. Kroonenberg, "Some Simplified Proofs in Infinite-Dimensional Topology," Master's Thesis, Mathematisch Instituut, Universiteit van Amsterdam, 1972.

[16] R. Schori, J. E. West, "2^I is homeomorphic to the Hilbert cube," Bull. Amer. Math. Soc. 78(1972), 402-406.

[17] H. Toruńczyk, "Remarks on Anderson's paper 'On topological infinite deficiency,' " Fund. Math. 66(1970), 393-401.

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