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Ming-shan Hsieh
*Louisiana State University and Agricultural & Mechanical College*

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# Quasi-normed Ideals of Operators on Banach Spaces

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

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by
Ming-Shan Hsieh
B.S., National Taiwan University, 1967
M.S., Louisiana State University, 1970
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**Approved:**

James R. Holmes
Major Professor and Chairman

James L. Trayham
Dean of the Graduate School

**EXAMINING COMMITTEE:**

Herron S. Collins

David P. Roelle

Bernard Madison

Richard M. Dudley

**Date of Examination:**

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ABSTRACT

The purpose of this dissertation is to develop further the theory and applications of quasi-normed ideals of bounded linear operators on Banach spaces. The theory was initiated by R. Schatten, A. Grothendieck and A. Pietsch. In Chapter I we present some basic results needed for this theory. Some questions of Pietsch [32] concerning the ideal of finitely approximable operators are answered.

In Chapter II ideals of nuclear type operators and their adjoints are studied. We show that the ideals of regular $(p,r,s)$-integral operators can be obtained by maximizing the ideals of $(p,r,s)$-nuclear operators.

In Chapter III the product and quotient of two quasi-normed ideals are introduced. It turns out that the conjugate ideal of a product ideal is a quotient ideal. This fact allows us to compute the adjoints of several specific ideals. We also provide some concrete examples and give a proof of the converse of a classical result of Grothendieck.

In Chapter IV we investigate some classes of operators with the aid of s-numbers on Banach spaces. In particular, an ideal of compact operators on Hilbert spaces is constructed to give a simple direct proof of the result of A. Brown, C. Pearcy and N. Salinas [1].

# CHAPTER 0

# INTRODUCTION

The theory of quasi-normed ideals on Banach spaces, as most abstract mathematical theories, began with the study of particular interesting examples. The classes of nuclear and absolutely summing operators were among the first examples of normed ideals. These classes of operators were investigated from the standpoint of tensor products by Schatten [39] and Grothendieck [13]. The appearance of Pietsch's book [28] indicated that those classes of operators could be studied without the notion of tensor product.

Since the appearance of [28], various special classes of operators between Banach spaces have been widely studied, generalized, and applied. In particular, in the paper [21] Lindenstrauss and Pełczyński demonstrated the power of the notion of p-absolutely summing operators in investigating the subspace structure of Banach spaces. For other results characterizing the structure of certain Banach spaces in terms of the behavior of various classes of operators, see

[12], [19], [41]. Recently, Pietsch, in his prebook [32], sketches the outline of ideal theory on Banach spaces. We follow [12] and [32] to further develop this theory and applications. We attempt to provide a unified approach.

In the remainder of this chapter, we provide the basic definitions, notations and some important theorems which will be used through this paper.

## §0.1 Fundamentals and Notations

All spaces we consider here are Banach spaces unless otherwise stated. For each Banach space $E$, we denote by $U_E$ the unit ball $\{x \in E: \|x\| \leq 1\}$ of $E$. A positive measure $\mu$ is a function, defined on a $\sigma$-algebra $\Sigma$ of some set $\Omega$, whose range is in $[0,\infty]$ and which is countably additive. In particular, $\mu$ is called a probability measure if $\mu(\Omega) = 1$.

If $\Omega$ is a topological Hausdorff space, then a finite Borel regular measure $\mu$ on $\Omega$ is defined as a positive measure on the $\sigma$-algebra of the Borel sets contained in $\Omega$ such that $\mu(B) = \sup\{\mu(K) | K \subseteq B, K \text{ compact}\}$ for all Borel sets $B \subset \Omega$.

If $A$ is a compact Hausdorff space, then $C(A)$ is the Banach space of all scalar valued continuous functions on $A$ equipped with the supremum norm. A continuous linear functional $\emptyset$ on $C(A)$ is called a Radon measure on $A$. The

Riesz representation theorem establishes a one-to-one correspondence between positive Radon measures and finite Borel regular measures on $A$ by the integral $\langle f,\phi\rangle = \int_A f(a)d\mu(a)$ .

For each positive measure $\mu$ , we denote by $L_p(\mu) = L_p(\Omega,\Sigma,\mu)$ , $1 \leq p \leq \infty$ , the Banach space of equivalence classes of measurable functions on the measure space $(\Omega,\Sigma,\mu)$ whose pth powers are integrable (resp. are essentially bounded if $p = \infty$ ) and with norm

$$\|f\| = \begin{cases} \int |f(x)^p d\mu & \text{if } 1 \leq p < \infty \\ \text{ess. sup } |f(x)| & \text{if } p = \infty . \end{cases}$$

If $(\Gamma,\Sigma,\mu)$ is the discrete measure space on a set $\Gamma$ with $\mu(\{x\}) = 1$ for every $x \in \Gamma$ , we denote $L_p(\mu)$ by $\ell_p(\Gamma)$ i.e. the Banach space of all scalar-valued functions $f$ on the set $\Gamma$ for which

$$\|f\| = \begin{cases} (\sum_\alpha |f(\alpha)|^p)^{1/p} < \infty & \text{if } p < \infty \\ \sup_\alpha |f(\alpha)| < \infty & \text{if } p = \infty . \end{cases}$$

If $\Gamma$ is the set of all positive integers, we also denote $\ell_p(\Gamma)$ by $\ell_p$ , while $\ell_p^n$ will denote $\ell_p(\Gamma)$ with $\Gamma = \{1,2,\cdots,n\}$ , $n < \infty$ .

The subspace of $\ell_\infty(\Gamma)$ consisting of those functions which vanish at infinity is denoted by $c_o(\Gamma)$ (resp. $c_o$ if $\Gamma$ is the set of positive integers).

Furthermore, if $\{E_\alpha\}_{\alpha\in\Gamma}$ is an indexed family of Banach spaces, we denote by $(\bigoplus_{\alpha\in\Gamma} E_\alpha)_p$ , $1 \leq p < \infty$ (resp. by $(\bigoplus_{\alpha\in\Gamma} E_\alpha)_\infty$) the Banach space consisting of all functions $x = \{x_\alpha\}_{\alpha\in\Gamma}$ with $x_\alpha \in E_\alpha$ for all $\alpha$ and $\sum_{\alpha\in\Gamma} \|x_\alpha\|_{E_\alpha}^p < \infty$ (resp. $\sup_{\alpha\in\Gamma} \|x_\alpha\|_{E_\alpha} < \infty$) under the obvious norm.

By operator, or map, we will mean a bounded linear operator. The collection $\mathcal{L}(E,F)$ of all operators from $E$ into $F$ is a Banach space equipped with the norm $\|T\| = \sup_{x\in U_E} \|Tx\|$ .

For each Banach space $E$ , we denote by $I_E$ the identity map of $E$ . A projection $P$ is an element of $\mathcal{L}(E,E)$ such that $P^2 = P$ .

A subspace is a closed linear subset. If $M$ is a subspace of $E$ , $J_M^E$ denotes the injection map from $M$ into $E$ . For each subspace $N$ of $E$ , $Q_N^E$ denotes the canonical map from $E$ onto the quotient space $E/N$ . By $E'$ we mean the space of all continuous linear functionals on $E$ with $\|f\| = \sup_{x\in U_E} |\langle x,f\rangle|$ for $f \in E'$ .

For $T \in \mathcal{L}(E,F)$ , the dual operator $T'$ of $T$ is the element of $\mathcal{L}(F',E')$ defined by $\langle T'b,x\rangle = \langle b,Tx\rangle$ for all $b \in F'$ , $x \in E$ .

For each $x \in E$ , let $J_E x : a \to \langle x,a\rangle$ , then $J_E$ is an operator for $E$ into the bidual space $E''$ and

$\|J_E x\| = \|x\|$ . $J_E$ is called the canonical injection. The space $E$ is called reflexive if $J_E$ is onto.

An operator $T \in \mathcal{L}(E,F)$ is called finite, written $T \in \mathfrak{F}(E,F)$ , if $T$ has finite dimensional range. An operator $T \in \mathfrak{F}(E,F)$ if and only if there are elements $a_1, a_2, \cdots, a_n \in E'$ and elements $y_1, y_2, \cdots, y_n \in F$ such that $Tx = \sum_{i=1}^{n} \langle x, a_i \rangle y_i$ for $x \in E$ . We usually write

$$T = \sum_{i=1}^{n} a_i \otimes y_i \ .$$

For each finite operator $T = \sum_{i=1}^{n} a_i \otimes x_i \in \mathfrak{F}(E,E)$ , the trace of $T$ is defined by $Tr(T) = \sum_{i=1}^{n} \langle x_i, a_i \rangle$ . It is easy to see that $Tr(T)$ is well defined and $Tr(T) = Tr(T') = Tr(J_E T)$ for $T \in \mathfrak{F}(E,E)$ . Moreover, if $T \in \mathfrak{F}(E,F)$ and $S \in \mathcal{L}(F,E)$ , then $Tr(ST) = Tr(TS)$ .

An operator $T \in \mathcal{L}(E,F)$ is finitely approximable. denoted by $T \in C(E,F)$ , if $T$ belongs to the closure of $\mathfrak{F}(E,F)$ under the operator norm $\|\cdot\|$ . An operator $T \in \mathcal{L}(E,F)$ is compact, respectively, weakly compact if $T(U_E)$ is relatively compact, respectively, relatively weakly compact. We use $K(E,F)$, $[W(E,F)]$ for the set of all compact [weakly compact] operators from $E$ to $F$ . An operator $T \in \mathcal{L}(E,F)$ is completely continuous if for each weakly convergent sequence $(x_i) \in E$ , $(Tx_i)$ is norm convergent. The class of completely continuous operators is denoted by

$V(E,F)$ .

## § 0.2 Some Geometric Properties of Banach Spaces

A Banach space $E$ has the approximation property (a.p., in short) if for each compact set $A \subset E$ there is an operator $S \in \mathfrak{F}(E,E)$ such that $\|(I_E - S)(x)\| \leq 1$ for $x \in A$ .

If $S$ can be chosen so that $\|S\| \leq 1$ , then $E$ is said to have the metric approximation property (m.a.p., in short). It is very easy to prove that $E$ has a.p. if and only if $K(F,E) = C(F,E)$ for every $F$ . For numerous equivalent formations of the approximation property for a Banach space, see [13].

The following remarkable result was demonstrated by Enflo [6]. It also provided a negative solution of the famous basis problem in Banach spaces.

0.2.1 Theorem. There is a Banach space which does not have the approximation property.

A Banach space $F$ has the extension property if for each $T_o \in \mathcal{L}(M,F)$ , $M$ a subspace of $E$ , there is an operator $T \in \mathcal{L}(E,F)$ with $T_o = TJ_M^E$ and $\|T_o\| = \|T\|$ for any Banach space $E$ .

0.2.2 Theorem. For each set $\Gamma$ , $\ell_\infty(\Gamma)$ has the extension property.

0.2.3 Theorem. For each Banach space $E$, $J_E^\infty: x \to \langle x, a \rangle$ is an injection map from $E$ into the Banach space $\ell_\infty(U_E, )$ with $\|J_E^\infty\| = 1$.

A Banach space $E$ is said to have the lifting property if to each operator $T_o \in \mathcal{L}(E, F/N)$ and each number $\epsilon > 0$, there is an operator $T \in \mathcal{L}(E,F)$ with $T_o = Q_N^F T$ and $\|T\| \leq (1+\epsilon)\|T_o\|$ where $N$ is a subspace of an arbitrary Banach space $F$.

0.2.4 Theorem. For each set $\Gamma$, the Banach space $\ell_1(\Gamma)$ has the lifting property.

0.2.5 Theorem. For each Banach space $E$, the map $Q_E^1: (\lambda_x) \to \sum_{x \in U_E} \lambda_x x$ is a surjection from $\ell_1(U_E)$ onto $E$ with $\|Q_E^1\| = 1$.

Finally, we also need the "Principle of Local Reflexivity" established by Lindenstrauss and Rosenthal [22] (see also [18]).

The Principle of Local Reflexivity

Let $E$ be a Banach space (regarded as a subspace of $E''$). Let $U$ and $V$ be finite dimensional subspace of $E''$ and $E'$, respectively, and let $\epsilon > 0$. Then there exists a one-to-one operator $T: U \to E$ with $T(x) = x$ for all $x \in E \cap U$, $f(Te) = e(f)$ for all $e \in U$, $f \in V$ and

$\|T\|\|T^{-1}\| < 1 + \epsilon$ .

## § 0.3 Certain Classes of Operators

In this section we give four fundamental classes of operators which were studied in detail by Grothendieck [13] and Pietsch [28].

0.3.1 Definition. An operator $T \in \mathcal{L}(E,F)$ is called nuclear if there are elements $a_n \in E'$ and elements $y_n \in F$ with $\sum_{n=1}^{\infty} \|a_n\|\|y_n\| < +\infty$ such that $Tx = \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n$ for $x \in E$ . For each nuclear operator $T$ , we set $\nu(T) = \inf\{ \sum_{n=1}^{\infty} \|a_n\|\|y_n\| < \infty\}$ where the infimum is taken over all possible representations of $T$ . We denote by $N(E,F)$ the set of all nuclear operators from $E$ into $F$ . The space $N(E,F)$ is a Banach space under the norm $\nu$ .

0.3.2 Definition. An operator $T \in \mathcal{L}(E,F)$ is called integral if there is a number $\rho \geq 0$ such that $|\mathrm{Tr}(TA)| \leq \rho\|A\|$ for $A \in \mathfrak{F}(F,E)$ .

We set $i(T) = \inf \rho$ . The collection $I(E,F)$ of all integral operators forms a Banach space under the norm $i$ .

0.3.3 Definition. An operator $T \in \mathcal{L}(E,F)$ is called absolutely summing if there is a number $\rho \geq 0$ $\sum_{i=1}^{n} \|Tx_i\| \leq \rho \sup\{ \sum_{i=1}^{n} |\langle x_i, a \rangle| : \|a\| \leq 1\}$ for all finite sets $\{x_1, x_2, \cdots, x_n\} \subset E$ .

We set $\pi(T) = \inf \rho$ . The collection $\mathbf{\Pi}(E,F)$ of all absolutely summing operators from $E$ into $F$ forms a Banach space under the norm $\pi$ .

0.3.4 Definition. An operator $T \in \mathcal{L}(E,F)$ is called quasi-nuclear if there is a sequence of linear functionals $a_n \in E'$ with $\sum_{n=1}^{\infty} \|a_n\| < +\infty$ such that $\|Tx\| \leq \sum_{n=1}^{\infty} |\langle x, a_n \rangle|$ for $x \in E$ .

We set $\nu^Q(T) = \inf\{ \sum_{n=1}^{\infty} \|a_n\| \}$ where the infimum is taken over all sequences $\{a_n\}$ with the above property. The collection $N^Q(E,F)$ of all quasi-nuclear operators forms a Banach space under the norm $\nu^Q$ .

0.3.5 Theorem. The product $ST$ of two absolutely summing operators $T \in \mathbf{\Pi}(E,F)$ and $S \in \mathbf{\Pi}(F,G)$ is nuclear and $\nu(ST) \leq \pi(S)\pi(T)$ .

For other basic and important properties of $N(E,F)$, $I(E,F)$, $\mathbf{\Pi}(E,F)$ and $N^Q(E,F)$ , see [27] and [28].

# CHAPTER I

## QUASI-NORMED IDEALS OF OPERATORS

In this chapter we present some results concerning the theory of quasi-normed ideals of operators on Banach spaces, essentially continuing a study initiated by R. Schatten, A Grothendieck and A. Pietsch. For the sake of completeness we include the basic definitions and some useful known results from the recent works [31], [12] and, in particular, the pre-book of Pietsch [32].

The ideal theory in the ring $\mathcal{L}(H)$ of bounded linear operators on an infinite dimensional separable Hilbert space $H$ was first studied by J. W. Calkin [2]. It was shown that any two-sided ideal $A(H)$ in the ring $\mathcal{L}(H)$ is sandwiched, by set inclusion, between the maximal ideal $K(H)$ of all compact operators and the smallest ideal $\mathfrak{F}(H)$ of all finite rank operators. The most important ideals in the ring $\mathcal{L}(H)$ are the ideals $S_p(H)$ , $(0 < p < \infty)$ , introduced by Von Neumann and R. Schatten [37]. The ideal $S_p(H)$ consists of all compact operators with $\text{Trace}[(T^*T)^{p/2}] < \infty$ .

Ideals of operators on Hilbert spaces have been extensively studied, see e.g. [9], [11] and [38].

For an ideal theory on arbitrary Banach spaces, it is useful to consider the class of all operators between Banach spaces. Following Pietsch [31], we let $\mathcal{L}$ denote the class of all bounded linear operators between arbitrary Banach spaces and $\mathcal{L}(E,F)$ the set of all such operators between specific Banach spaces $E$ and $F$. We say that a subclass $A$ of $\mathcal{L}$ is an ideal of operators if for each component $A(E,F) = A \cap \mathcal{L}(E,F)$, one has the following properties:

i) If $a \in E'$ and $y \in F$, then $a \otimes y \in A(E,F)$.

ii) $A(E,F)$ is a linear subspace of $\mathcal{L}(E,F)$.

iii) If $S \in \mathcal{L}(G,E)$, $T \in A(E,F)$ and $R \in \mathcal{L}(F,H)$ then $RTS \in A(G,H)$.

It is easy to see that the class $\mathfrak{F}$ of all finite operators forms the smallest ideal.

Remark. We could also consider ideals of operators defined only on a certain class of Banach spaces, for example, the class of all finite dimensional Banach spaces. In case this class consists of one single Banach space $E$, we obtain the usual ideal concept in the ring $\mathcal{L}(E)$.

Several important ideals which have been studied during the past five decades are $K$: compact operators;

W: weakly compact operators, V: completely continuous operators (see [5]), $S_s$: strictly singular operators and $S_c$: strictly consingular operators (see [26]). The ideals W, V, $S_s$ and $S_c$ all contain K and $V(H) = S_s(H) = S_c(H) = K(H)$ for each Hilbert space H .

## § 1.1 Quasi-normed Ideals of Operators

The study of the ideal $S_p(H)$ , and, in particular, the isolation of the trace class $S_1(H)$ and Hilbert-Schmidt class $S_2(H)$ of operators on Hilbert space was the main achievement in the monograph [38], [39] by R. Schatten. Grothendieck in his remarkable memoir [13] extended the theory of tensor products of Banach spaces (and of locally convex spaces). In this framework, Grothendieck introduced the notions of nuclear and "semi-intégrale à droite" operators. Later, A. Pietsch, using ideas of Grothendieck but without using notions from tensor products, introduced and studied classes of operators (called p-nuclear and p-absolutely summing operators) in his series of papers [27], [28], [30]. For another study of those classes of operators treated by the theory of tensor product, we refer to the papers [36], [37] by Saphar.

For the study of certain classes of operators, Pietsch introduced the notion of ideal norm, which was similar to the $\otimes$-norm of Grothendieck and Schatten.

1.1.1 Definition. A function $\alpha$ from an ideal $A$ to the non-negative real number is called an ideal quasi-norm if one has the following properties:

i) If $a \in E'$ and $y \in F$, then $\alpha(a \otimes y) = \|a\|\|y\|$.

ii) There is a $\rho \geq 1$ for which

$\alpha(S+T) \leq \rho[\alpha(S) + \alpha(T)]$ for $S,T \in A(E,F)$.

iii) If $S \in \mathcal{L}(G,E)$, $T \in A(E,F)$ and $R \in \mathcal{L}(F,H)$ then $RTS \in A(G,H)$ and $\alpha(RTS) \leq \|R\|\alpha(T)\|S\|$.

If it is possible to take $\rho = 1$ in ii) we obtain an ideal norm.

If the condition ii) is replaced by

ii') There is $p$ with $0 < p \leq 1$ for which

$\alpha(S+T)^p \leq \alpha(S)^p + \alpha(T)^p$ for $S,T \in A(E,F)$,

$\alpha$ is called a p-ideal norm on $A$.

It is easy to show that any p-ideal norm is an ideal quasi-norm.

An ideal $A$ together with an ideal quasi-norm [ideal norm] $\alpha$ is called a quasi-normed ideal [normed ideal] and denoted by $[A,\alpha]$.

It follows from the definition that all components $A(E,F)$ of a quasi-normed ideal $[A,\alpha]$ are quasi-normed spaces equipped with the quasi-norm $\alpha$. We say a quasi-normed ideal $[A,\alpha]$ is complete if each component $A(E,F)$ is a complete topological vector space with topology induced

by the quasi-norm $\alpha$ . A complete normed ideal is simply called a Banach ideal.

It is easy to see that $[\mathfrak{F},\|\ \|]$ is a normed ideal. Also, $[C,\|\ \|]$, $[K,\|\ \|]$, $[W,\|\ \|]$ and $[\mathcal{L},\|\ \|]$ are Banach ideals. If we denote by $N$ , $I$, $\Pi$ and $N^Q$ the class of all nuclear, integral, absolutely summing and quasi-nuclear operators respectively, then $[N,\nu]$, $[I,i]$, $[\Pi,\pi]$ and $[N^Q,\nu^Q]$ are Banach ideals.

We now give some elementary properties of a quasi-norm $\alpha$ .

1.1.2 Lemma. If $\alpha$ is a quasi-norm on an ideal $A$ , then $\|T\| \leq \alpha(T)$ for all $T \in A$ .

Proof. Let $T \in \mathcal{L}(E,F)$ and $S = g' \otimes x \in \mathcal{L}(G,E)$ with $g' \in G'$ , $\|g'\| = 1$ and $\|x\| = 1$ . Then $\|S\| = \|x\| = 1$ . Furthermore, $TS = g' \otimes (Tx)$ , hence
$\|Tx\| = \|Tx\|\|g'\| = \|g' \otimes Tx\| = \|TS\| = \alpha(TS) \leq \alpha(T)\|S\| = \alpha(T)$ .

1.1.3 Definition. Two ideal quasi-norms $\alpha_1$ and $\alpha_2$ defined on an ideal $A$ are equivalent if there is a $\rho \geq 0$ such that $\alpha_1(T) \leq \rho\alpha_2(T)$ and $\alpha_2(T) \leq \rho\alpha_1(T)$ for all $T \in A$ .

We now give a generalization of a result by R. Schatten [38].

1.1.4 Theorem. Let $A$ be an ideal. Then any two quasi-norms with respect to which $A$ is a complete quasi-normed ideal, are equivalent.

Proof. Let $[A,\alpha_1]$ and $[A,\alpha_2]$ be two complete quasi-normed ideals. We define the mapping $\psi:A(E,F) \to A(E,F)$ by $\psi(T) = T$ for all $T \in A(E,F)$. Let $\{T_n\}$ converge to $T$ in $[A(E,F),\alpha_1]$ and $\{\psi(T_n)\}$ converge to $S$ in $[A(E,F),\alpha_2]$, then from Lemma 1.1.2, $\lim\|T_n-T\| \leq \lim \alpha_1(T_n-T) = 0$ and $\lim\|T_n-S\| = \lim\|\psi(T_n)-S\| \leq \lim \alpha_2(T_n-S) = 0$. Hence $T = S$, which means $\psi(T) = S$ and $\psi$ is closed. If $\alpha_1$ and $\alpha_2$ are not equivalent, there exist Banach spaces $E_n$, $F_n$ and operators $T_n:E_n \to F_n$ with $\alpha_1(T_n) > n$ and $\alpha_2(T_n) = \frac{1}{n^2}$. Let $E = (\oplus E_n)_\infty$ and $F=(\oplus F_n)_\infty$, and let $T = \sum_{n=1}^{\infty} J_n T_n Q_n$ where $Q_n:E \to E_n$ denotes the canonical projection and $J_n:F_n \hookrightarrow F$ denotes the canonical injection. Also let $P_n:F \to F_n$ be the canonical projection and $I_n:E_n \hookrightarrow E$ be the canonical injection. Then $\alpha_2(T) = \alpha_2(\sum_{n=1}^{\infty} J_n T_n Q_n)$ $\leq \sum_{n=1}^{\infty} \alpha_2(J_n T_n Q_n) \leq \rho \sum_{n=1}^{\infty} \|J_n\|\alpha_2(T_n)\|Q_n\| = \rho \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$ and $T \in A_2(E,F)$. But, $T_n = P_n T I_n$ and so $\alpha_1(T) = \|P_n\|\alpha_1(T)\|I_n\|$ $\geq \alpha_1(P_n T I_n) = \alpha_1(T_n) > n$. That is, $\alpha_1(T) = \infty$. This contradicts the closed graph theorem (on $A(E,F)$).

1.1.5 Theorem. If $[A,\alpha]$ is a Banach ideal, then $N \subseteq A$. Also, $\|T\| \leq \alpha(T) \leq \nu(T)$ for any operator $T$.

Proof. If $T \in N(E,F)$, then for each $\epsilon > 0$ there is a representation $T = \sum_{n=1}^{\infty} a_n \otimes y_n$ such that $\sum_{n=1}^{\infty} \|a_n\|\|y_n\| < \nu(T) + \epsilon$. Let $T_m = \sum_{n=1}^{m} a_n \otimes y_n$ and let $m > p$. Then

$$\alpha(T_m - T_p) = \alpha\Big(\sum_{n=p+1}^{m} a_n \otimes y_n\Big) \leq \sum_{n=p+1}^{m} \alpha(a_n \otimes y_n) = \sum_{n=p+1}^{m} \|a_n\|\|y_n\| ,$$

that is, $(T_m)$ is $\alpha$-Cauchy. Since $[A,\alpha]$ is a Banach ideal, there is an operator $S \in A(E,F)$ such that $\alpha(S-T_m) \to 0$. Hence $\|S-T_m\| \to 0$ by Lemma 1.1.2. But $\|T-T_m\| \to 0$ since $\| \| \leq \nu$. Thus $S = T$. This proves that $T \in A(E,F)$ and

$$\alpha(T) \leq \sum_{n=1}^{\infty} \alpha(a_n \otimes y_n) = \sum_{n=1}^{\infty} \|a_n\|\|y_n\| < \nu(T) + \epsilon .$$

If $T \notin N(E,F)$, $\nu(T) = +\infty$ and there is nothing to prove.

To investigate the relationship between the ideal norm (quasi-norm) and $\otimes$-norm ($\otimes$-quasi-norm) of Grothendieck, we need the following terms used by Pietsch in [32].

1.1.6 Definition. An ideal quasi-norm is said to be elementary if it is defined only on the class of all operators between finite dimensional Banach spaces, called elementary operators. An ideal quasi-norm $\alpha$ on an ideal $A$ is called an extension of the elementary norm $\emptyset$, if the restriction of $\alpha$ to elementary operators agrees with $\emptyset$.

Remark. Since for two arbitrary finite dimensional Banach

spaces $E$ and $F$, $\mathcal{L}(E,F)$ can be identified with the algebraic tensor product $E' \otimes F$, there is a one-to-one correspondence between elementary quasi-norms [norms] and $\otimes$-quasi-norms [$\otimes$-norms]. Moreover, any ideal quasi-norm can be considered as the extension of some $\otimes$-quasi-norm.

1.1.7 Definition. Let $\alpha_1$ and $\alpha_2$ be extensions of the same elementary quasi-norm $\phi$ on an ideal $A$. We say $\alpha_1$ is coarser than $\alpha_2$ or $\alpha_2$ is finer than $\alpha_1$ if $\alpha_1(T) \geq \alpha_2(T)$ for all $T \in A$.

1.1.8 Theorem [32]. For each elementary quasi-norm $\phi$, there exists an ideal quasi-norm $\phi^+$ (respectively, $\phi^-$) on the ideal $\mathfrak{F}$ of finite operators so that $\phi^+$ (respectively, $\phi^-$) is the coarsest (the finest) extension of $\phi$.

Proof. Every $T \in \mathfrak{F}(E,F)$ can be written in the form $T = BSA$ with $X$, $Y$ finite dimensional Banach spaces and $A \in \mathcal{L}(E,X)$, $B \in \mathcal{L}(Y,F)$ and $S \in \mathcal{L}(X,Y)$. We set $\phi^+(T) = \inf\{\|B\|\phi(S)\|A\|\}$, where the infimum is taken over all such representations of $T$. It is clear that $\phi^+$ is an ideal quasi-norm with $\phi^+ \geq \phi$ and for every extending ideal norm $\alpha$ of $\phi$ on $\mathfrak{F}$, we have $\phi^+ \geq \alpha$. On the other hand, for each $T \in \mathfrak{F}(E,F)$ and finite dimensional spaces $X$ and $Y$, $A \in \mathcal{L}(X,E)$, $\|A\| \leq 1$, $B \in \mathcal{L}(F,Y)$, $\|B\| \leq 1$, we set $\phi^-(T) = \sup\{\phi(BTA)\}$, where the supremum is taken over all finite dimensional spaces

$X$, $Y$ and operators $A$, $B$. Then $\phi^-$ is an ideal quasi-norm on $\mathfrak{F}$ and $\phi^- \leq \phi$. For each extension $\alpha$ of $\phi$ on $\mathfrak{F}$, we have $\phi^- \leq \alpha$.

1.1.9 Definition. A quasi-norm $\alpha$ on the ideal $\mathfrak{F}$ is said to be upper semi-continuous, respectively, lower semi-continuous if $\alpha^+ = \alpha$, respectively, $\alpha^- = \alpha$.

It is easy to see that $\| \ \|$ and $\pi$ are both upper and lower semi-continuous. The integral norm $i$ is lower, but not upper semi-continuous. The nuclear norm $\nu$ is neither upper nor lower semi-continuous.

Remark. The upper semi-continuity of a quasi-norm $\alpha$ will play a crucial role in differentiating between the conjugation and adjoint operations in Section 1.2.

Finally, we end this section with some more results of Pietsch [32] which will be used to classify the quasi-normed ideals.

1.1.10 Definition. Two quasi-normed ideals $[A,\alpha]$ and $[B,\beta]$ are elementary, respectively, finitely equivalent if the ideal quasi-norms $\alpha$ and $\beta$ agree on all elementary, respectively, finite operators.

The collection of all quasi-normed ideals can be partially ordered by the relation $[A,\alpha] \subset [B,\beta]$ if and only

if $A \subset B$ and $\alpha(T) \geq \beta(T)$ for all $T \in A$.

1.1.11 Definition. A complete quasi-normed ideal $[M,m]$ is called minimal, repsectively, maximal if for each quasi-normed ideal $[A,\alpha]$ which is elementary equivalent to $[M,m]$, we have $[M,m] \subset [A,\alpha]$, respectively, $[A,\alpha] \subset [M,m]$.

Pietsch has observed [32] that an application of Zorn's lemma yields the following result.

1.1.12 Theorem. For each quasi-normed ideal $[A,\alpha]$, there exists a minimal quasi-normed ideal $[A^{min},\alpha^{min}]$ and a maximal ideal $[A^{max},\alpha^{max}]$ which are both elementary equivalent to $[A,\alpha]$.

1.1.13 Theorem. $[A^{min},\alpha^{min}] = [\overline{\mathfrak{F}},\overline{\alpha}^{+}]$ where $\overline{\mathfrak{F}}(E,F)$ is the completion of $\mathfrak{F}(E,F)$ with respect to the quasi-norm $\alpha^{+}$.

1.14 Corollary. Every minimal quasi-normed ideal is contained (as a point set) in the normed ideal $[K,\|\ \|]$.

1.1.15 Theorem. $T \in [A^{max}(E,F),\alpha^{max}]$ if and only if there is $\rho \geq 0$ such that $\alpha(VTU) \leq \rho\|V\|\|U\|$ for arbitrary finite dimensional Banach spaces $X$ and $Y$ and all operators $U \in \mathcal{L}(X,E)$ and $V \in \mathcal{L}(F,Y)$. In this case, $\alpha^{max} = \inf \rho$.

Proof. Let $\alpha^{max}(T) = \sup_{\|V\|=\|U\|=1} \alpha(VTU)$. It is easy to show that the collection of all $T:E \to F$ such that $\alpha^{max}(T) < +\infty$

forms a complete quasi-normed ideal with ideal quasi-norm $\alpha^{max}$ . Furthermore, for any quasi-normed ideal $[B,\beta]$ which is elementary equivalent to $\alpha$ , we have

$$\alpha^{max}(T) = \sup_{\|V\|=\|U\|=1} \alpha(VTU) = \sup_{\|V\|=\|U\|=1} \beta(VTU) \leq \beta(T) .$$

§ 1.2 Associated Quasi-normed Ideals of $[A,\alpha]$

For any quasi-normed ideal of operators $[A,\alpha]$ , we can associate the following normed ideals.

i) The conjugate ideal $[A^{\Delta},\alpha^{\Delta}]$: $A^{\Delta}(E,F)$ is the class of all operators $T \in \mathcal{L}(E,F)$ for which there is a $\rho \geq 0$ such that for any $U \in \mathfrak{F}(F,E)$, $|\text{Trace } UT| \leq \rho\alpha(U)$ . Here $\alpha^{\Delta}(T) = \inf \rho$ .

ii) The adjoint ideal $[A^{*},\alpha^{*}]$: $A^{*}(E,F)$ is the class of all operators $T \in \mathcal{L}(E,F)$ for which there is a $\rho \geq 0$ such that for all finite dimensional Banach spaces $X$, $Y$ and for all operators $V \in \mathcal{L}(X,E)$ , $U \in \mathcal{L}(Y,X)$ and $W \in \mathcal{L}(F,Y)$ $|\text{Trace } WTVU| \leq \rho\|W\|\|V\|\alpha(U)$ . Here $\alpha^{*}(T) = \inf \rho$ .

It is easy to show that $[A^{\Delta},\alpha^{\Delta}]$ and $[A^{*},\alpha^{*}]$ both are Banach ideals. Thus, the operations $\Delta$ and $*$ essentially depend on the quasi-norm $\alpha$ restricted to the ideal $\mathfrak{F}$ .

1.2.1 Theorem. Two quasi-normed ideals of operators are elementary, respectively, finitely equivalent if and only if they have the same adjoint, respectively conjugate ideal.

The proof follows from the definitions.

1.2.2 Theorem. If $\alpha$ is an upper semi-continuous quasi-norm on the ideal $\mathfrak{I}$, then $[A^{\Delta},\alpha^{\Delta}] = A^{*},\alpha^{*}]$.

Proof. That $[A^{\Delta},\alpha^{\Delta}] \subset [A^{*},\alpha^{*}]$ is immediate from the definition. To see the other inclusion, let $T \in A^{*}(E,F)$ and let $U \in \mathfrak{I}(F,E)$. Since $\alpha$ is upper semi-continuous, there are finite dimensional spaces $X$ and $Y$, operators $V \in \mathcal{L}(F,X)$, $U_{o} \in \mathcal{L}(X,Y)$, $W \in \mathcal{L}(Y,E)$ such that $\|V\|\alpha(U_{o})\|W\| \leq \alpha(U) + \epsilon$. Hence $|\mathrm{Trace}(UT)| = |\mathrm{Trace}(WU_{o}VT)| \leq$ $\leq \alpha^{*}(T)\|V\|\alpha(U_{o})\|W\| \leq \alpha^{*}(T)(\alpha(U)+\epsilon)$, therefore, $T \in A^{\Delta}(E,F)$ and $\alpha^{\Delta}(T) \leq \alpha^{*}(T)$.

1.2.3 Corollary. For any ideal quasi-norm $\alpha$,

$$(\alpha^{+})^{\Delta} = \alpha^{*}.$$

We have been unable to prove the converse of 1.2.2.

In general, however, we have the following result of Pietsch [31] (see also [12]).

1.2.4 Theorem. Let $T \in \mathcal{L}(E,F)$. If $E$ and $F$ both have m.a.p. then $\alpha^{*}(T) = \alpha^{\Delta}(T)$ for any ideal quasi-norm $\alpha$.

1.2.5 Corollary. For any ideal quasi-norm $\alpha$, $\alpha^{\Delta *} = \alpha^{**}$.

1.2.6 Corollary. For any upper semi-continuous quasi-norm $\alpha$, $\alpha^{*\Delta} = \alpha^{\Delta\Delta}$.

1.2.7 Theorem [31]. If $[A,\alpha]$ is a quasi-normed ideal, then $[A,\alpha] \subset [A^{**},\alpha^{**}]$ .

Proof. Consider the diagram $Y \xrightarrow{W} E \xrightarrow{T} F \xrightarrow{V} X \xrightarrow{U} Y$ with X, Y finite dimensional, we have $|\mathrm{Trace}(VTWU)| \leq \alpha^{\Delta}(U)\alpha(VTW) = \alpha^{*}(U)\alpha(VTW) \leq \alpha(T)\|V\|\alpha^{*}(U)\|W\|$ . Hence $\alpha^{**}(T) \leq \alpha(T)$ .

A normed ideal $[A,\alpha]$ is perfect if $[A,\alpha] = [A^{**},\alpha^{**}]$ . The following result is contained in [31].

1.2.8 Theorem. For each quasi-normed ideal $[A,\alpha]$ , the following are equivalent.

i) There is a quasi-normed ideal $[B,\alpha]$ such that $[A,\alpha] = [B^{*},\alpha^{*}]$ .

ii) $[A,\alpha]$ is perfect.

iii) $[A,\alpha]$ is maximal.

1.2.9 Corollary. $\alpha$ is lower semi-continuous if and only if $\alpha^{**} = \alpha$ .

1.2.10 Theorem [12]. Let $T \in \mathcal{L}(E,F)$ . If E or F has m.a.p., then $\alpha^{\Delta\Delta}(T) \leq \alpha(T)$ , with equality when $\alpha$ is perfect.

To any quasi-normed ideal $[A,\alpha]$ , it is natural to consider the following ideal, called the dual ideal $[A',\alpha']$: An operator $T \in A'(E,F)$ if and only if $T' \in A(F',E')$ . Here $\alpha'(T) = \alpha(T')$ . We say that a quasi-normed ideal

$[A,\alpha]$ is <u>symmetric</u> if $A \subset A'$. If $[A,\alpha] = [A',\alpha']$ then $[A,\alpha]$ is said to be <u>fully symmetric</u>. It is well-known that $[N,\nu]$ and $[I,i]$ are symmetric, while $[\Pi,\pi]$ is not symmetric. Examples of fully symmetric normed ideals are $[\mathfrak{F},\|\ \|]$, $[K,\|\ \|]$, $[W,\|\ \|]$ and $[\mathfrak{L},\|\ \|]$. Pietsch asked in [32] if the ideal $[C,\|\ \|]$ of finitely approximable operators is fully symmetric. The answer is affirmative and is an easy application of the principle of local reflexivity.

1.2.11 <u>Theorem</u>. The ideal $[C,\|\ \|]$ is fully symmetric.

<u>Proof</u>. It is clear that $C \subset C'$. To see the other direction, let $T \in C'(E,F)$. By definition, $T' \in C(F',E')$ and thus $T$ is compact. Let $\epsilon_n \to 0$ and choose $y_1^n,\cdots,y_{k_n}^n \in F$ such that if $x \in E, \|x\| \leq 1$, then there are $y_i^n$ with $\|y_i^n - Tx\| < \epsilon_n$. Let $\{L_n\}$ be a sequence of finite rank operators such that $\|L_n - T'\| \to 0$ and let $Z_n = [y_1^n,\cdots,y_{k_n}^n, L_n'(E)] \subset F''$. Then, by the principle of local reflexivity, there exists $V_n : Z_n \to F$ such that $\|V_n\| \leq 1 + \epsilon_n$ and $V_n y_i^n = y_i^n$. We claim that $\|V_n L_n' - T\| \to 0$. Indeed, let $x \in E$ and $\|x\| \leq 1$, choose $y_i^n$ such that $\|Tx - y_i^n\| \leq \epsilon_n$, then

$$\begin{aligned}\|V_n L_n' x - Tx\| &\leq \|V_n L_n' x - y_i^n\| + \\ &+ \|y_i^n - Tx\| \leq \|V_n L_n' x - V_n y_i^n\| + \epsilon_n \\ &\leq (1 + \epsilon_n)\|L_n' x - y_i^n\| + \epsilon_n\end{aligned}$$

$$\leq (1 + \epsilon_n)(\|L_n'x - Tx\| + \|Tx - y_i^n\|) + \epsilon_n$$

$$\leq (1 + \epsilon_n)(\|L_n'|_E - T|_E\| + \epsilon_n) + \epsilon_n$$

$$\leq (1 + \epsilon_n)(\|L_n' - T''\| + \epsilon_n) + \epsilon_n \ .$$

Thus $T$ is approximable by finite operators.

We also include a non-trivial result of Pietsch [31].

1.2.12 Theorem. For any quasi-norm $\alpha$, $\alpha^{*\prime} = \alpha'^{*}$.

Finally, we state a duality theorem which was proved by U. Schwarz [40].

1.2.13 Theorem. If $E'$ and $F$ have m.a.p., then each $S \in A^*(F,E'')$ defines an $\alpha$-bounded linear functional of norm $\alpha^*(S)$ by the formula $\langle T,S\rangle = \text{Trace } ST$. Thus, $[A^*(F,F''),\alpha^*]$ and $[A^{min}(E,F),\alpha]'$ are isometrically isomorphic.

## §1.3 Injective and Projective Quasi-normed Ideals

We next define four new quasi-normed ideals from a given quasi-normed ideal $[A,\alpha]$; these formations arise from the theory of tensor product of Banach spaces.

i) Right injective envelope of $[A,\alpha]$, denoted $[A\backslash,\alpha\backslash]$: $T \in A\backslash(E,F)$ if and only if $J_F^\infty \circ T \in A(E,\ell^\infty(U_{F'}))$ and $\alpha\backslash(T) = \alpha(J_F^\infty \circ T)$ where $J_F^\infty$ is the canonical injection of $F$ into $\ell^\infty(U_{F'})$.

ii) Left injective envelope of $[A,\alpha]$ , denoted $[/A,/\alpha]$: $T \in /A(E,F)$ if and only if $T \circ Q_E^1 \in A(\ell^1(U_E),F)$ and $/\alpha(T) = \alpha(T \circ Q_E^1)$ , where $Q_E^1$ is the canonical surjection of $\ell_1(U_E)$ onto $E$ .

iii) Right projective envelope of $[A,\alpha]$ , denoted $[A/,\alpha/]$: $T \in A/(E,F)$ if and only if there is $S \in A(E,\ell^1(U_F))$ with $T = Q_F^1 \circ S$ and $\alpha/(T) = \alpha(S)$ .

iv) Left projective envelope of $[A,\alpha]$ , denoted $[\backslash A,\backslash\alpha]$: $T \in \backslash A(E,F)$ if and only if there is $S \in A(\ell^\infty(U_{E'}),F)$ with $T = S \circ J_E^\infty$ and $\backslash\alpha(T) = \alpha(S)$ .

Remark. The right injective envelope, respectively, left injective envelope, of $[A,\alpha]$ was called the injective hull, respectively, the projective hull by A. Pietsch.

1.3.1 Definition. A quasi-normed ideal $[A,\alpha]$ is right injective, respectively, left injective, right projective, respectively, left projective, if $[A,\alpha] = [A\backslash,\alpha\backslash]$ , respectively, $[A,\alpha] = [/A,/\alpha]$, $[A,\alpha] = [A/,\alpha/]$ , respectively, $[A,\alpha] = [\backslash A,\backslash\alpha]$ .

The following theorem is immediate from the definitions.

1.3.2 Theorem. i) $[(/A)',(/\alpha)'] = [(A')\backslash,(\alpha')\backslash]$

ii) $[(A\backslash)',(\alpha\backslash)'] = [/(A'),/(\alpha')]$

iii) $[(/A)^*,(/\alpha)^*] = [(A^*)/,(\alpha^*)/]$

iv) $[(A\backslash)^*,(\alpha\backslash)^*] = [\backslash(A^*),\backslash(\alpha^*)]$ .

It is easy to verify that the ideals $\mathfrak{F}$ and $K$ are both right injective and left injective. Pietsch asked in [32] if the ideal $[C,\|\ \|]$ is either right or left injective. The answer to both questions is negative and is an easy application of Enflo's result [6] that $C \neq K$.

1.3.3 Theorem. The ideal $[C,\|\ \|]$ is not right injective.

Proof. Let $T \in \mathcal{L}(E,F)$. Since $\ell^\infty(U_{F'})$ has a.p., $J_F^\infty \circ T \in C(E,\ell^\infty(U_{F'}))$ if and only if $J_F^\infty \circ T \in K(E,\ell^\infty(U_{F'}))$ if and only if $T \in K(E,F)$. Thus if $C$ is right injective, then $C = K$ which contradicts Enflo's result.

1.3.4 Theorem. The ideal $[C,\|\ \|]$ is not left injective.

Proof. Let $T \in \mathcal{L}(E,F)$ and $Q$ a canonical surjection. From the full symmetry of $C$ (Theorem 1.2.11) we have $TQ \in C(\ell^1(U_E),F)$ if and only if $(TQ)' \in C(F',L_\infty(\mu))$. Since $L_\infty$-spaces have a.p., the following are equivalent:

i) $(TQ)' \in K(F',L_\infty(\mu))$; ii) $T' \in K(F',E')$,

iii) $T \in K(E,F)$. Indeed, we need only show i) implies ii): since $Q'$ is an isometry and $Q'T'(U_{F'})$ is relatively compact, for each sequence $(x_n) \in Q'T'(U_{F'})$, there are subsequences $(x_{n_m})$ and $x_o$ such that $x_{n_m} \to x_o$ and $x_{n_m} = Q'T'(y_m)$. This implies $x_o \in Q'\overline{R(T')}$, hence $Q'^{-1}(x_{n_m}) = T'(y_m)$ tends to $Q'^{-1}(x_o) \in \overline{R(T')}$, so $T'(y_m)$

clusters and $T' \in K$ . Thus $C$ is not right injective since $C \neq K$ .

Finally, for each quasi-normed ideal $[A,\alpha]$ , we associate another new quasi-normed ideal $[A^R,\alpha^R]$ , called the regular envelope of $[A,\alpha]$ , as follows: $T \in A^R(E,F)$ if and only if $J_F T \in A(E,F'')$ and $\alpha^R(T) = \alpha(J_F T)$ , where $J_F : F \to F''$ is the natural injection. A quasi-normed ideal $[A,\alpha]$ is regular if and only if $[A^R,\alpha^R] = [A,\alpha]$ . It is known that $[\mathfrak{F},\|\ \|]$, $[K,\|\ \|]$, $[I,i]$, $[\Pi,\pi]$ and $[\mathcal{L},\|\ \|]$ are regular. The regularity of $[C,\|\ \|]$ was asked by Pietsch in [32].

1.3.5 Theorem. The ideal $[C,\|\ \|]$ is regular.

Proof. If $J_F S \in C(E,F'')$ , then $(J_F S)'' \in C(E'',F'''')$ . Since there is a norm one projection $P$ from $F''''$ onto $F''$ with $S'' = PJ''_F S''$ , it follows that $S'' \in C(E'',F'')$. Thus $S' \in C(F',E')$ and $S \in C(E,F)$ since $C$ is fully symmetric.

Analyzing the proof of the above theorem, we can see that any fully symmetric quasi-normed ideal of operators is regular.

1.3.6 Theorem. A quasi-normed ideal of operators is fully symmetric if and only if it is regular and symmetric.

Proof. The necessity part is easy, so we only show the sufficiency part. Let $T \in A(E,F)$ . From the symmetry of $[A,\alpha]$ , we have $\alpha(T') = \alpha'(T) \leq \alpha(T)$ and $\alpha(T'') = \alpha'(T') \leq$ $\leq \alpha(T')$ . From the regularity, $\alpha(T) = \alpha(J_F T) = \alpha(T'' J_E) \leq$ $\leq \alpha(T'')$ . Hence $\alpha(T) \leq \alpha(T'') \leq \alpha(T') = \alpha'(T) \leq \alpha(T)$ , and $A = A'$ .

## § 1.4 Normed Sequence Ideals

The theory of solid, symmetric sequence spaces has been studied for a long time. These spaces play an important role in the study of ideals of operators on infinite dimensional separable Hilbert spaces [2], [8], [11].

Let $\ell_\infty$ be the set of all bounded sequences $(x_i)$ . It is easy to see that $\ell_\infty$ is a commutative ring with norm $\|(x_i)\|_\infty = \sup_i |x_i|$ .

1.4.1 Definition. A pair $[a,\mu]$ is called a normed sequence ideal if $a \subset \ell_\infty$ and $\mu$ a non-negative function satisfy the following:

i) $(1,0,0,\cdots) \in a$ and $\mu(1,0,0,\cdots) = 1$ .

ii) If $(x_i)$, $(y_i) \in a$ , then $(x_i + y_i) \in a$ and $\mu(x_i + y_i) \leq \mu(x_i) + \mu(y_i)$ .

iii) For $(x_i) \in \ell_\infty$ and $(y_i) \in a$ , $(x_i\, y_i) \in a$ and $\mu(x_i y_i) \leq \|(x_i)\|_\infty \mu(y_i)$ .

iv) For each permutation $\sigma$ of the positive integers, if $(x_i) \in a$ , then $(x_{\sigma(i)}) \in a$ and $\mu(x_i) = \mu(x_{\sigma(i)})$ .

We note that a normed sequence ideal has been called a symmetric sequence space in the literature.

The most important normed sequence ideals are $[\ell_p, \| \|_p]$, $(1 \leq p \leq \infty)$ and $[c_o, \| \|_\infty]$. The space $c$ of all convergent sequences is not a normed sequence ideal.

Following the standard concepts of ring theory, we naturally consider the quotient of a normed sequence ideal $[\mathcal{A}, \mu]$ by another normed sequence ideal $[\mathcal{B}, \nu]$.

1.4.2 Definition. Let $[\mathcal{A}, \mu]$ and $[\mathcal{B}, \nu]$ be normed sequence ideals, let $\mathcal{A}/\mathcal{B} = \{\xi \in \ell^\infty \mid \xi\mathcal{A} \subseteq \mathcal{B}\}$ where $\xi\mathcal{A} = \{(\xi_i x_i) \mid (x_i) \in \mathcal{A}\}$ and the function $\mu/\nu(\xi) = \sup_{\substack{\nu(x)\leq 1 \\ x\in\mathcal{B}}} \mu(\xi x)$, then $[\mathcal{A}/\mathcal{B}, \mu/\nu]$ is a normed sequence ideal, called the normed quotient of $[\mathcal{A}, \mu]$ by $[\mathcal{B}, \nu]$.

In particular, if $[\mathcal{B}, \nu] = [\ell_1, \| \|_1]$, we denote $\mathcal{A}^x = \ell_1/\mathcal{A}$ and $\mu^x = \| \|_1/\mu$, and $[\mathcal{A}^x, \mu^x]$ is called the adjoint sequence ideal of $[\mathcal{A}, \mu]$ or the Kothe dual of $[\mathcal{A}, \mu]$.

We also consider another useful normed sequence ideal. Let $\ell_{t:p} = \{(x_i) \in \ell_\infty \mid \sum_{n=1}^{\infty} (n^{1/t-1/p} |\hat{x}_n|)^p < \infty$, where $(|\hat{x}_n|)$ is the non-increasing rearrangement of $(|x_n|)\}$ and let $\|(x_i)\|_{t:p} = [\sum_{n=1}^{\infty} (n^{1/t-1/p} |\hat{x}_n|)^p]^{1/p}$. Then, $[\ell_{t:p}, \| \|_{t:p}]$ is a Banach space, called a Lorentz sequence space. We note

that $\ell_{p:p} = \ell_p$ and

$$\ell_{t:p} \subseteq \ell_p \quad \text{if} \quad 1 \leq t \leq p \leq \infty$$

$$t_{t:p} \supseteq \ell_p \quad \text{if} \quad 1 \leq p \leq t \leq \infty \ .$$

The following result, which is well-known in the theory of interpolation, is needed in Chapter II. We give a direct proof.

1.4.3 Theorem. For $1 \leq p, t < \infty$, $(\ell_{t:p})' = \ell_{t':p'}$ where $1/p + 1/p' = 1$; $1/t + 1/t' = 1$ .

Proof. If $(x_i) \in \ell_{t':p'}$, then $\sum_{n=1}^{\infty} |x_n y_n|$

$$= \sum_{n=1}^{\infty} (n^{1/t' - 1/p'} |x_n|)(n^{1/t-1/p} |y_n|)$$

$$\leq [\sum_{n=1}^{\infty} (n^{1/t'-1/p'} |x_n|)^{p'}]^{1/p'} [\sum_{n=1}^{\infty} (n^{1/t-1/p} |y_n|)^{p}]^{1/p} \quad \text{for}$$

all $(y_i) \in \ell_{t:p}$ . Hence $(x_i) \in \ell'_{t:p}$ . Conversely, let $(x_i) \in \ell'_{t:p}$ . Then there is $\rho \geq 0$ such that

$$\sum_{n=1}^{\infty} |x_n y_n| \leq \rho \max_{\sigma} [\sum_{n=1}^{\infty} (n^{1/t-1/p} |y_{\sigma(n)}|)^{p}]^{1/p} \quad \text{for} \quad (y_i) \in \ell_{t:p} \ .$$

If we take $y_n = (n^{1/t'-1/p'})^{p'} |x_n|^{p'-2} x_n$ then

$$\sum_{n=1}^{\infty} (n^{1/t'-1/p'} |x_n|)^{p'} \leq \rho [\sum_{1}^{\infty} (n^{(1/t'-1/p')(p'-1)} |x_n|^{p'-1})^{p}]^{1/p} \ .$$

Thus, we have $[\sum_{n=1}^{\infty} (n^{1/t'-1/p'} |x_n|)^{p'}]^{1/p'} \leq \rho$ and

$(x_i) \in \ell_{t':p'}$ .

Remark. It is easy to show that $(\ell_{\infty:p})' \neq \ell_{1:p'}$ .

Finally, we need the following notation: If $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are normed sequence ideals and $(x_i y_i) \in \mathcal{C}$ for $(x_i) \in \mathcal{A}$, $(y_i) \in \mathcal{B}$ we write $\mathcal{A} \circ \mathcal{B} \subset \mathcal{C}$ . For example, $\ell_p \circ \ell_{p'} \subset \ell_1$ if $1/p + 1/p' = 1$ . More generally, $\ell_p \circ \ell_s \subset \ell_r$ if $1/p + 1/s = 1/r$ .

## CHAPTER II

## IDEALS OF NUCLEAR TYPE OPERATORS AND THEIR ADJOINTS

Since the remarkable class of nuclear operators was introduced in the Memoir of Grothendieck [13], various operators $T \in \mathcal{L}(E,F)$ with representation $\sum_{i=1}^{\infty} \lambda_i a_i \otimes y_i$ have been studied by considering certain summability properties for the sequences $(a_i) \subset E'$, $(y_i) \subset F$ and the scalar sequence $(\lambda_i)$. Our aim in this chapter is to provide a unified construction so that the individual known classes of operators can be clearly singled out.

In order to start the theory in as general a setting as possible, we follow Pietsch [32] and consider three sequence ideals $\theta$, $\mathcal{R}$, $\mathcal{S}$ with $\theta \circ \mathcal{R} \circ \mathcal{S} \subset \ell_1$, and call an operator $T \in \mathcal{L}(E,F)$ $(\theta,\mathcal{R},\mathcal{S})$-nuclear if there are sequences $(a_i) \subset E', (y_i) \subset F$ and scalar sequence $(\lambda_i)$ with $(\lambda_i) \in \theta$, $(\langle x,a_i\rangle) \in \mathcal{R}$ and $(\langle y_i,b\rangle) \in \mathcal{S}$ such that $T(x) = \sum_{i=1}^{\infty} \lambda_i \langle x,a_i\rangle y_i$ for $x \in E$. It is easy to see the class $N_{\theta,\mathcal{R},\mathcal{S}}$ of $(\theta,\mathcal{R},\mathcal{S})$-nuclear operators forms an ideal.

In the following we first investigate the class of $(\mathcal{Q},\mathcal{R},\mathcal{S})$-nuclear operators when $\mathcal{Q} = \ell_p$ or $\ell_{t:p}$ $(0 < t \leq p \leq \infty)$, $\mathcal{R} = \ell_r$, $\mathcal{S} = \ell_s$ $(0 < r, s \leq \infty$, and when $r,s = \infty$, we consider $c_o$ instead of $\ell_\infty)$.

## §2.1 (t:p,r,s)-Nuclear Operators

2.1.1 Definition. An operator $T \in \mathcal{L}(E,F)$ is called (p,r,s)-nuclear, respectively, (t:p,r,s)-nuclear $(0 < p,r,s,t \leq \infty$ and $1/p + 1/r + 1/s \geq 1)$ if there are scalar sequence $(\lambda_i) \in \ell_p$, respectively, $\ell_{t:p}$ and elements $a_i \in E'$ and $y \in F$ with

$$\epsilon_r(a_i) = \sup_{\|x\|\leq 1} \left( \sum_{i=1}^{\infty} |\langle x,a_i\rangle|^r \right)^{1/r} < \infty \quad \text{and}$$

$$\epsilon_s(y_i) = \sup_{\|b\|\leq 1} \left( \sum_{i=1}^{\infty} |\langle y_i,b\rangle|^s \right)^{1/s} < \infty \quad \text{such that } T \text{ can be}$$

represented as $Tx = \sum_{i=1}^{\infty} \lambda_i \langle x,a_i\rangle y_i$. We denote by $N_{p,r,s}$, respectively, $[N_{t:p,r,s}]$ the class of all (p,r,s)-nuclear, [(t:p,r,s)-nuclear] operators and set $\nu_{p,r,s}(T) = \inf\{\|(\lambda_i)\|_p \epsilon_r(a_i)\epsilon_s(y_i)\}[\nu_{t:p,r,s}(T)=\inf\{\|(\lambda_i)\|_{t:p}\epsilon_r(a_i)\epsilon_s(y_i)\}]$, where the infimum is taken over all representations of $T$ as above.

Remark. If $t = p$, then $N_{p,r,s} = N_{t:p,r,s}$

$t > p$, then $N_{p,r,s} \subset N_{t:p,r,s}$

$t < p$, then $N_{p,r,s} \supset N_{t:p,r,s}$.

Thus, our (t:p,r,s)-nuclear operators include the

(p,r,s)-nuclear operators of Pietsch [32].

The following theorem is immediate from the definition.

2.1.2 Theorem. If $1 \le r,s \le \infty$, then an operator $T \in N_{t:p,r,s}(E,F)$ has a factorization $E \xrightarrow{A} \ell_r \xrightarrow{D_\lambda} \ell_{s'} \xrightarrow{B} F$, where $A \in \mathcal{L}(E,\ell_r)$, $B \in \mathcal{L}(\ell_{s'},F)$ and $D_\lambda(\xi) = (\lambda_i \xi_i)$ with $(\lambda_i) \in \ell_{t:p}$. In this case $\nu_{t:p,r,s}(T) = \inf\|A\|\|D_\lambda\|\|B\|$, where the infimum is taken over all such representations of T.

2.1.3 Theorem. If $1/p + 1/r + 1/s \ge 1$, $t \le p$, then $[N_{t:p,r,s}, \nu_{t:p,r,s}]$ is a complete quasi-normed ideal.

Proof. Let $1/p + 1/r + 1/s = 1/q \ge 1$, i.e., $q \le 1$. We show that $[\nu_{t:p,r,s}(T_1+T_2)]^q \le [\nu_{t:p,r,s}(T_1)]^q + [\nu_{t:p,r,s}(T_2)]^q$. Indeed, for $\epsilon > 0$, there are sequences $(\lambda_i^{[k]}) \in \ell_{t:p}$, $(a_i^{[k]}) \subset E'$ and $(y_i^{[k]}) \subset F'$ such that $T_k = \sum_{i=1}^{\infty} \lambda_i^{[k]} a_i^{[k]} \otimes y_i^{[k]}$ with $\|(\lambda_i^{[k]})\|_{t:p} \le [\nu_{t:p,r,s}(T_k) + \epsilon]^{q/p}$, $\epsilon_r(a_i^{[k]}) \le [\nu_{t:p,r,s}(T_k) + \epsilon]^{q/r}$ and $\epsilon_s(y_i^{[k]}) \le [\nu_{t:p,r,s}(T_k) + \epsilon]^{q/s}$ $(k=1,2)$. Then

$$[\nu_{t:p}(T_1+T_2)]^q \le [\|(\lambda_i^{[1]}\|_{t:p}\epsilon_r(a_i^{[1]})\epsilon_s(y_i^{[1]})$$

$$+ \|(\lambda_i^{[2]})\|_{t:p}\epsilon_r(a_i^{[2]})\epsilon_s(y_i^{[2]})]^q \le \{[\nu_{t:p,r,s}(T_1)+\epsilon]^{q/p+q/r+q/s}$$

$$+ [\nu_{t:p,r,s}(T_2) + \epsilon]^{q/p+q/r+q/s}\}^q$$

$$= \{[\nu_{t:p,r,s}(T_1)+\epsilon] + [\nu_{t:p,r,s}(T_2) + \epsilon]\}^q$$

$$\leq [\nu_{t:p,r,s}(T_1) + \epsilon]^q + [\nu_{t:p,r,s}(T_2) + \epsilon]^q .$$

2.1.4 Corollary. If $1/p + 1/r + 1/s = 1$, $t \leq p$, then $[N_{t:p,r,s}, \nu_{t:p,r,s}]$ is a Banach ideal.

Remark. The diagonal operator $D_\lambda : \ell_r \to \ell_s$, for $1 \leq r,s \leq \infty$ and $t \leq p$ with $D_\lambda(\xi_i) = \lambda_i \xi_i$ and $(\lambda_i) \in \ell_{t:p}$ is the prototype of $(t:p,r,s)$-nuclear operator with $\nu_{t:p,r,s}(D_\lambda) = \|(\lambda_i)\|_{t:p}$. We will repeatedly use this remark in the consideration of adjoint ideals.

2.1.5 Definition. For $1 \leq p \leq \infty$, an operator $T \in N_{p,\infty,p'}(E,F)$ is called a p-nuclear operator. This concept has been studied by Pietsch [27] and Saphar [36]. We denote $N_{p,\infty,p'}$ by $N_p$.

2.1.6 Definition. For $0 < p \leq 1$, $T \in N_{p,\infty,\infty}(E,F)$ is called the pth order Fredholm operator.

These operators have been studied by Grothendieck ([13], Chapter II).

2.1.7 Theorem. An operator $T \in N_{p,\infty,\infty}(E,F)$ if and only if $T$ has a representation $T = \sum_{i=1}^{\infty} a_i \otimes y_i$ with $(\|a_i\|) \in \ell_r$, $(\|y_i\|) \in \ell_s$ and $1/r + 1/s = 1/p$ and $\nu_{p,\infty,\infty}(T) = \inf(\sum_1^\infty \|a_i\|^r)(\sum_1^\infty \|y_i\|^s)^{1/s}$.

The proof is easy so we omit it.

Remark. The space $N_{p,\infty,\infty}(E,F)$ $(0 < p \leq 1)$ is exactly equal to $E' \overset{\wedge}{\otimes}_{s_{r,s}} F$ if $E'$ or $F$ has a.p. Here, the quasi-$\otimes$-norm $s_{r,s}$ is defined by $s_{r,s}(u) = \inf(\sum_1^n \|a_i\|^r)^{1/r} (\sum_1^n \|y_i\|^s)^{1/s}$, where the infimum is taken over all the representations of $u = \sum_{i=1}^n a_i \otimes y_i$.

2.1.8 Theorem. An operator $T \in N_{\infty,p,p'}(E,F)$, $1/p + 1/p' = 1$ if and only if $T$ factors through $\ell_p$.

In this case, we identify $N_{\infty,p,p'}$ by $\mathfrak{I}_p$, a class of operators studied by Pietsch [29].

2.1.9 Theorem. The ideal $[N_{t:p,r,s}, \nu_{t:p,r,s}]$ is minimal, $(1 < t \leq p < \infty$ and $1/p + 1/r + 1/s \geq 1)$.

Proof. It is easy to see that $[N_{t:p,r,s}(E,F), \nu_{t:p,r,s}] = [\overline{\mathfrak{F}}(E,F), \nu^+_{t:p,r,s}]$ where $\nu^+_{t:p,r,s}(T) = \inf \sum_{i=1}^n \|(\lambda_i)\|_{t:p} \epsilon_r(a_i)\epsilon_s(y_i)$ is the finest extension of the elementary quasi-norm $\nu_{t:p,r,s}$ on the ideal $\mathfrak{F}$ of finite operators.

2.1.10 Theorem. The ideal $[N_{\infty,p,p'}, \nu_{\infty,p,p'}]$ is not minimal.

Proof. It is easy to see that $N_{\infty,p,p'}$ is not contained in the ideal $K$ of compact operators. For example, the identity operator on $\ell_p$ is in $N_{\infty,p,p'}$ but not in $K$.

We note that the minimal ideal which is elementary equivalent to $N_{\infty,p,p'}$ is the class of operators which factor compactly through $\ell_p$, denoted by $C_p$, see Johnson [17].

Remark. Figiel [7] modified Johnson's results and gave necessary and sufficient conditions on a Banach space $Z$ so that every approximable operator by operators of finite rank admits a compact factorization through $Z$ and every compact operator admits a factorization through a subspace of $Z$.

## § 2.2 The Maximalization of $[N_{p,r,s}, \nu_{p,r,s}]$

To find the maximal ideal $[N^{max}_{p,r,s}, \nu^{max}_{p,r,s}]$ we define a new class of operators.

2.2.1 Definition. An operator $T \in \mathcal{L}(E,F)$ is $(p,r,s)$-integral $(1/p + 1/r + 1/s = 1,\ 1 \leq p,r,s \leq \infty)$ if there is a measure space $(\Omega,\Sigma,\mu)$ such that the following diagram is commutative

(*)
$$\begin{array}{ccc} E & \xrightarrow{\ T\ } & F \\ {\scriptstyle A}\downarrow & & \uparrow{\scriptstyle B} \\ L_r(\mu) & \xrightarrow[M_\varphi]{} & L_{s'}(\mu) \end{array}$$

where $A$, $B$ are bounded and $M_\varphi$ is the operator given by the multiplication of a function $\varphi \in L_p(\mu)$. We denote

by $I_{p,r,s}$ the class of all $(p,r,s)$-integral operators and set $i_{p,r,s}(T) = \inf \|A\|\|M_\varphi\|\|B\|$ where the infimum is taken over all measure spaces $(\Omega, \Sigma, \mu)$ and operators $A$, $B$ and $M_\varphi$.

2.2.2 Theorem. $[I_{p,r,s}, i_{p,r,s}]$ is a Banach ideal.

The proof follows in the same fashion as proposition 1 of Kwapien [19].

2.2.3 Theorem. In the case $s' < r$, the factorization (*) is equivalent to the following

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle A_1}\downarrow & & \uparrow{\scriptstyle B_1} \\ L_r(\nu) & \xrightarrow[J]{} & L_{s'}(\nu) \end{array}$$

where $\nu$ is a probability measure, $J$ the canonical injection and $A_1$ and $B_1$ are bounded.

Proof. Let $d\nu = \frac{|\varphi|^p}{\|\varphi\|^p}\, d\mu$ and define $A_1 : E \to L_r(\nu)$ by $A_1(x) = \operatorname{sgn}(\varphi)|\varphi|^{-p/r}A(x)$ and $B_1 : L_{s'}(\nu) \to F$ by $B_1(g) = B(|\varphi|^{p/s'}g)$. It is easy to see $B_1 J A_1 = T$ and $\|A_1\| \le \|A\|\|\varphi\|^{-p/r}$, $\|B_1\| \le \|B\|\|\varphi\|^{-p/s'}$. Hence $\|A_1\|\|B_1\| \le \|A\|\|B\|\|\varphi\|$.

Remark. $I_{p,\infty,p'}$ is the class of p-integral operators, studied by A. Pietsch [27]. In this case, we denote $I_{p,\infty,p'}$ by $I_p$ .

2.2.4 Theorem. i) $N_{p,r,s}(E,F) \subset I_{p,r,s}(E,F)$ and $\nu_{p,r,s} \geq i_{p,r,s}$ .

ii) If $E'$ or $F$ has m.a.p. and $T \in N_{p,r,s}(E,F)$ , then $\nu_{p,r,s}(T) = i_{p,r,s}(T)$ .

iii) $[N_{p,r,s},\nu_{p,r,s}]$ and $[I_{p,r,s},i_{p,r,s}]$ are elementary equivalent.

Proof. i) and iii) are easy. The proof of ii) follows from the argument of Theorem 36 in Persson and Pietsch [27].

2.2.5 Definition. An operator $T \in \mathcal{L}(E,F)$ is called regular (p,r,s)-integral if and only if $J_F T \in I_{p,r,s}(E,F'')$ . We write $T \in I^R_{p,r,s}(E,F)$ and define $i^R_{p,r,s}(T) = i_{p,r,s}(J_F T)$ .

We note that $[I^R_1,i^R_1] = [I,i]$ .

Question. Find p,r,s so that $[I^R_{p,r,s},i^R_{p,r,s}] = [I_{p,r,s},i_{p,r,s}]$.

Recently, Johnson and Figiel have observed that $[I_1,i_1] \neq [I^R_1,i^R_1]$ .

From Theorem 2.2.4 and the fact that $I^R_{p,r,s} \supset I_{p,r,s}$ it is conceivable that $I^R_{p,r,s} = N^{max}_{p,r,s}$ . To give an affirmative answer of this conjecture of Pietsch [32], we need the concept of ultraproducts of Banach spaces,

developed by Dacunha - Castelle and Krivine [4].

Let $(E_i)_{i\in I}$ be a family of Banach spaces over an arbitrary index set $I$. The collection of all bounded families $(x_i)_{i\in I}$, $x_i \in E_i$, forms a linear space, denoted by $\ell_\infty(E_i)_{i\in I}$.

If $\mathcal{U}$ is an ultrafilter on $I$ tending to infinity, then the limit $\lambda_{\mathcal{U}}(x_i) = \lim_{\mathcal{U}} \|x_i\|$ always exists and $\ell_\infty(E_i)$ is a semi-normed linear space under the semi-norm $\lambda_{\mathcal{U}}$. We denote by $(E_i)_{\mathcal{U}}$ the quotient space $\ell_\infty(E_i)_{i\in I}/C_{\mathcal{U}}(E_i)$, where $C_{\mathcal{U}}(E_i) = \{(x_i) \in \ell_\infty(E_i) : \lambda_{\mathcal{U}}(x_i) = 0\}$ and $(E_i)_{\mathcal{U}}$ is called <u>the ultraproduct of the Banach spaces $(E_i)_{i\in I}$ with respect to the ultrafilter $\mathcal{U}$</u>.

If $(F_i)_{i\in I}$ is another family of Banach spaces and $(T_i)_{i\in I}$ is a uniformly bounded family of operators $T_i \in \mathcal{L}(E_i, F_i)$, then an operator $(T_i)_{\mathcal{U}}$ from $(E_i)_{\mathcal{U}}$ into $(F_i)_{\mathcal{U}}$ can be defined in a natural way by $(T_i)_{\mathcal{U}}(x_i)_{\mathcal{U}} = (T_i x_i)_{\mathcal{U}}$ and $(T_i)_{\mathcal{U}}$ is called <u>the ultraproduct of the operators $(T_i)_{i\in I}$ with respect to the ultrafilter $\mathcal{U}$</u>.

If $E, F$ are Banach spaces, we denote by $\mathcal{M}$ the family of all finite dimensional subspaces $M$ of $E$ and $\mathcal{N}$ the family of all finite codimensional subspaces of $F$. If we take the set $I$ as the collection of all pairs $i = (M,N)$, $M \in \mathcal{M}$, $N \in \mathcal{N}$, then there exists an ultrafilter $\mathcal{U}$ on $I$ such that $\mathcal{U}$ contains the sections $\mathcal{S}(i_o) = \{i \in I : i = (M,N), M \supset M_o, N \subset N_o\}$ for each

$i_o = (M_o, N_o) \in I$ .

If we set $E_i = M$ , $F_i = F/N$ and $T_i = Q_N^F T J_M^E$ for $i = (M,N) \in I$ , we have the following lemma of Pietsch [33].

2.2.6 Lemma. If $T \in \mathcal{L}(E,F)$ , then there exists operators $J \in \mathcal{L}(E,(E_i)_{\mathcal{U}})$ and $Q \in \mathcal{L}((F_i)_{\mathcal{U}},F'')$ with $\|J\| \leq 1$ and $\|Q\| \leq 1$ such that $J_F T = Q(T_i)_{\mathcal{U}} J$ .

In order to show that $[I^R_{p,r,s}, i^R_{p,r,s}]$ is maximal we also need the following lemma of [20].

2.2.7 Lemma. Suppose for each $i \in I$ , $T_i \in \mathcal{L}(E,F)$ is such that $i_{p,r,s}(T_i) \leq 1$ , then $i_{p,r,s}((T_i)_{\mathcal{U}}) \leq 1$ .

Proof. By assumption, we have

$T_i : E_i \xrightarrow{A_i} L_r(\Omega_i,\mu_i) \xrightarrow{M_{f_i}} L_{s'}(\Omega_i,\mu_i) \xrightarrow{B_i} F_i$ with $\|A_i\| \leq 1 + \epsilon$ , $\|B_i\| \leq 1 + \epsilon$ and $M_{f_i}$ is the operator given by multiplication of a positive function $f_i$ with $\|f_i\|_{L_p} \leq 1$ $(1/p = 1/s' - 1/r)$ . By passing to the ultraproduct, we have $(E_i)_{\mathcal{U}} \xrightarrow{A} (L_r(\Omega_i,\mu_i))_{\mathcal{U}} \xrightarrow{M} (L_{s'}(\Omega_i,\mu_i))_{\mathcal{U}} \xrightarrow{B} (F_i)_{\mathcal{U}}$ with $\|A\| \leq 1 + \epsilon$ , $\|B\| \leq 1 + \epsilon$ and $\|M\| \leq 1$ . According to Dacunha - Castelle and Krivine [4], $(L_r(\Omega_i,\mu_i))_{\mathcal{U}}$ and $(L_{s'}(\Omega_i,\mu_i))_{\mathcal{U}}$ are identified respectively with the spaces $L_r(\Omega_1,\mu_1)$ and $L_{s'}(\Omega_2,\mu_2)$ .

In the case $r = s'$ , the proof is attained, since $(T_i)_{\mathcal{U}}$ is factored through an $L_r$-space.

In the case $s' < r$, we have to show $i_{p,r,s}(M) \leq 1$. In fact, $(L_r(\Omega_i,\mu_i))_{\mathcal{U}}$ and $(L_{s'}(\Omega_i,\mu_i))_{\mathcal{U}}$ are lattices with the order defined by $f \leq g$ if and only if there are representatives $(f_i)$ and $(g_i)$ of $f$ and $g$ such that $f_i \leq g_i$ for $i \in I$. It follows that $M = (M_{f_i})_{\mathcal{U}}$ is a positive operator from $L^r(\Omega_1,\mu_1)$ into $L^{s'}(\Omega_2,\mu_2)$ with $\|M\| \leq 1$. Since $M$ admits a factorization $L_r(\Omega_1,\mu_1) \overset{V}{\to} L_r(\Omega_2,\mu_2) \overset{M_f}{\to} L_{s'}(\Omega_2,\mu_2)$ with $\|V\| \leq \|M\| \leq 1$ and $\|f\|_{L_p} \leq 1$, $i_{p,r,s}(M) \leq 1$. The case $s' < r$ follows in a similar fashion.

2.2.8 Theorem. $[I^R_{p,r,s}, i^R_{p,r,s}] = [N^{\max}_{p,r,s}, \nu^{\max}_{p,r,s}]$.

Proof. Let $T \in \mathcal{L}(E,F)$ be such that for finite dimensional subspaces $X$ and $Y$ and $U \in \mathcal{L}(X,E)$, $V \in \mathcal{L}(F,Y)$, $i^R_{p,r,s}(VTU) \leq 1$ we claim that $i^R_{p,r,s}(T) \leq 1$.

For each $i = (M,N)$, the operator $T_i = J_M T Q_N$ from $M$ into $F/N$ has $i_{p,r,s}(T_i) \leq 1$. By Lemma 2.2.7, we can obtain the operator $(T_i)_{\mathcal{U}}$ from $(E_i)_{\mathcal{U}}$ into $(F_i)_{\mathcal{U}}$, where $E_i = M_i$, $F_i = F/N$ with $i_{p,r,s}(T_i)_{\mathcal{U}} \leq 1$.

From Lemma 2.2.6 there are norm $\leq 1$ operators $J: E \to (E_i)_{\mathcal{U}}$ and $Q: (F_i)_{\mathcal{U}} \to F''$ such that $J_F \circ T = Q \circ (T_i)_{\mathcal{U}} \circ J$. Hence we have $i^R_{p,r,s}(T) = i_{p,r,s}(J_F \circ T) \leq 1$.

Remark. We note that $I^R_{\infty,p,p'} = \Gamma_p$ the class of operators which factor through $L_p(\mu)$ for some measure space $(\Omega,\Sigma,\mu)$ [19].

## § 2.3 Adjoint Ideals of (t:p,r,s)-Nuclear Operators

In this section a concrete representation of the adjoint ideals $[N^*_{p,r,s}, \nu^*_{p,r,s}]$ and respectively, $[N^*_{t:p,r,s}, \nu^*_{t:p,r,s}]$ will be given.

We first suppose $T \in N^*_{p,r,s}(E,F)$ where $1/p + 1/r + 1/s \geq 1$ , $p \geq 1$ . Consider the diagram

$$\ell^n_{s'} \xrightarrow{V} E \xrightarrow{T} F \xrightarrow{U} \ell^n_r \xrightarrow{D_\lambda} \ell^n_{s'}$$

where $V' : a \to (\langle a, x_i \rangle)_{i=1}^n$ , $D_\lambda : (\xi_i)_{i=1}^n \to (\lambda_i \xi_i)_{i=1}^n$ and $U : y \to (\langle y, b_i \rangle)_{i=1}^n$ . Then we can write $TVD_\lambda U = \sum_{i=1}^n \lambda_i b_i \otimes Tx_i$

and $|\mathrm{Trace}(UTVD_\lambda)| = |\mathrm{Trace}(TVD_\lambda U)|$

$= |\sum_{i=1}^n \lambda_i \langle Tx_i, b_i \rangle| \leq \nu^*_{p,r,s}(T) \nu_{p,r,s}(D_\lambda) \|U\| \|V\|$

$= \nu^*_{p,r,s} \|(\lambda_i)\|_p \|U\| \|V\|$ . Hence,

$$\left(\sum_{i=1}^n |\langle Tx_i, b_i \rangle|^{p'}\right)^{1/p'} \leq \nu^*_{p,r,s}(T) \epsilon_s(x_i)\, \epsilon_r(b_i) .$$

This leads us to the following definition of Pietsch [32].

2.3.1 Definition. An operator $T \in \mathcal{L}(E,F)$ is called $(p,r,s)$-absolutely summing; $0 < p,r,s \leq \infty$ , $1/r + 1/s \geq 1/p$ , if there is a constant $\rho \geq 0$ such that for each finite sequence $(x_i)_{i=1}^n \subset E$ and $(b_i)_{i=1}^n \subset F'$ , we have $\left(\sum_{i=1}^n |\langle Tx_i, b_i \rangle|^p\right)^{1/p} \leq \rho\, \epsilon_r(x_i) \epsilon_s(b_i)$ . We denote by $\Pi_{p,r,s}$ the class of all $(p,r,s)$-absolutely summing operators and

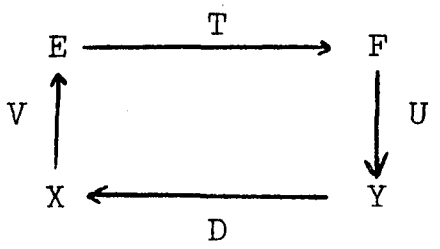
set $\pi_{p,r,s}(T) = \inf \rho$ , $\rho$ satisfying the above. The class of $(t:p,r,s)$-absolutely summing operators are defined similarly by considering $(\langle Tx_i, b_i\rangle) \in \ell_{t:p}$ .

2.3.2 Theorem. For $1 \leq p,r,s \leq \infty$

i) $[\Pi_{p,r,s}, \pi_{p,r,s}]$ is a Banach ideal

ii) $[N^*_{p,r,s}, \nu^*_{p,r,s}] = [\Pi_{p',s,r}, \pi_{p',s,r}]$ .

Proof. i) is easy. To prove ii), we have seen $N^*_{p,r,s} \subset \Pi_{p',s,r}$ and $\pi_{p',s,r}(T) \leq \nu^*_{p,r,s}(T)$ . To see the other direction, let $T \in \Pi_{p',s,r}(E,F)$ and consider the following diagram




with X, Y finite dimensional, U, V bounded and $D = \sum_{i=1}^{n} \lambda_i b_i \otimes x_i$ such that $\|(\lambda_i)\|_{p'} \epsilon_s(x_i)\epsilon_r(b_i) \leq \nu_{p,r,s}(U) + \epsilon$ . Then $|\mathrm{Trace}(UATB)| = |\sum_{i=1}^{n} \lambda_i \langle ATBx_i, b_i\rangle|$

$\leq \|(\lambda_i)\|_p \pi_{p',s,r}(T)\|A\|\|B\|\epsilon_s(x_i)\epsilon_r(b_i)$

$\leq \pi_{p',s,r}(T)\|A\|(\nu_{p,r,s}(U)+\epsilon)\|B\|$ . Hence $T \in N^*_{p,r,s}$ and $\nu^*_{p,r,s}(T) \leq \pi_{p',s,r}(T)$ .

2.3.3 Theorem [32]. i) $T \in \Pi_{p,r,\infty}(E,F)$ if and only if

there is $\rho \geq 0$ such that

$$\{\sum_{i=1}^{n} \|Tx_i\|^p\}^{1/p} \leq \rho \sup_{\|a\|\leq 1} \{\sum_{i=1}^{n} |\langle x_i, a\rangle|^r\}^{1/r} \quad \text{for}$$

$x_1, x_2, \cdots, x_n \in E$ .

ii) $T \in \Pi_{p,\infty,s}(E,F)$ if and only if there is $\rho \geq 0$ such that $\{\sum_{1}^{n} \|T' b_i\|^p\}^{1/p} \leq \rho \sup_{\|y\|\leq 1} \{\sum_{i=1}^{n} |y, b_i\rangle|^s\}^{1/s}$ for $b_1, b_2, \cdots, b_n \in F'$ .

The proof is easy by using the Hahn-Banach theorem.

Remark. i) $\Pi_{p,r,\infty}$ is the class of $(p,r)$-absolutely summing operators in the sense of Mitiagin and Pełczyński [24], and $\Pi_{p,p,\infty}$ is the class of p-absolutely summing operators denoted by $\Pi_p$ and studied by Pietsch [30].

ii) $\Pi_{p,\infty,p}$ and $\Pi_{1,p,p'}$ are the class of p'-strongly absolutely summing operators, respectively, the class of Cohen p-nuclear operators [3]. We write $\Pi_{p,\infty,p} = D_{p'}$ and $\Pi_{1,p,p'} = J_p$ .

Since $I_{p,r,s}$ and $N_{p,r,s}$ are elementary equivalent, we have

2.3.4 Theorem. $[I^*_{p,r,s}, i^*_{p,r,s}] = [\Pi_{p',s,r}, \pi_{p',s,r}]$ . In particular $[I^*_p, i^*_p] = [\Pi_{p'}, \pi_{p'}]$ .

From the maximality of $[I^R_{p,r,s}, i^R_{p,r,s}]$ , we have

2.3.5 Theorem. $[\Pi^*_{p,r,s}, \pi^*_{p,r,s}] = [I^R_{p',s,r}, i^R_{p',s,r}]$ . In particular $[\Pi^*_p, \pi^*_p] = [I^R_{p'}, i^R_{p'}]$ .

2.3.6 Theorem. $[\Pi^\Delta_{p,r,s}, \pi^\Delta_{p,r,s}] = [I^R_{p',s,r}, i^R_{p',s,r}]$ .

Similarly, we can state

2.3.7 Theorem. $[N^*_{t:p,r,s}, \nu^*_{t:p,r,s}] = [\Pi_{t':p',s,r}, \pi_{t':p',s,r}]$ .

We note that the class $\Pi_{t:p,r,\infty}$ has been studied by K. Miyazaki [25].

Finally, we summarize in a table the Banach ideals $N_{p,r,s}$ and their associated ideals ($1 \leq p,r,s < \infty$ and $1/p + 1/r + 1/s = 1$) .

| a | b | c | $N_{a,b,c}$ | $N^*_{a,b,c}$ | $N^{**}_{a,b,c}$ |
|---|---|---|---|---|---|
| p | r | s | $N_{p,r,s}$ | $\Pi_{p',s,r}$ | $I^R_{p,r,s}$ |
| $\infty$ | p | p' | $F_p$ | $J_{p'}$ | $\Gamma_p$ |
| p | $\infty$ | p' | $N_p$ | $\Pi_{p'}$ | $I^R_p$ |
| p' | p | $\infty$ | $(N_{p'})'$ | $D_{p'}$ | $(I^R_{p'})'$ |

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# CHAPTER III

# PRODUCT AND QUOTIENT IDEALS

In this chapter, we introduce the product and quotient of two quasi-normed ideals $[A,\alpha]$ and $[B,\beta]$. It turns out that the conjugate ideal of a product ideal is a quotient ideal. We also provide some examples and give a proof of the converse of a classical result of Grothendieck.

## § 3.1 Product Ideals

Let $[A,\alpha]$, $[B,\beta]$ be two given quasi-normed ideals. We denote by $B\circ A$ the class of all operators $T \in \mathcal{L}(E,F)$ such that there is a Banach space $G$ and operators $U \in A(E,G)$, $V \in B(G,F)$ such that $T = V\circ U$. We set $(\beta\circ\alpha)(T) = \inf \beta(V)\alpha(U)$.

3.1.1 Theorem. $[B\circ A,\beta\circ\alpha]$ is a quasi-normed ideal.

Proof. If $a \in E'$ and $y \in F$, then $a \otimes y$ factors through an one-dimensional Banach space $G$ and $(\beta\circ\alpha)(a\otimes y) \leq \alpha(1\otimes y)\beta(a\otimes 1) = \|a\|\|y\|$. On the other hand,

$\|a\|\|y\| = \|a\otimes y\| \leq \|1\otimes y\|\|a\otimes 1\| \leq \alpha(1\otimes y)\beta(a\otimes 1)$ . Hence $(\beta\circ\alpha)(a\otimes y) = \|a\|\|y\|$ .

Let $T_1$, $T_2 \in B\circ A(E,F)$ and suppose $T_i$ has an $(A,B)$-factorization $(U_i,V_i)$ through the space $G_i$ , $(i=1,2)$ . Write $G = (G_1\oplus_\infty G_2)_\infty$ and let $P_i$ be the projection of $G$ onto $G_i$ , then $(U_1+ U_2, V_1P_1 + V_2P_2)$ is an $(A,B)$-factorization of $T_1+ T_2$ through $G$ .

To see that $\beta\circ\alpha$ is a quasi-norm, we let $\epsilon > 0$ , then there is an $(A,B)$-factorization $(U_i,V_i)$ of $T_i$ through $G_i$ $(i=1,2)$ such that $\beta(V_i)\alpha(U_i) \leq (\beta\circ\alpha)(T_i) + \epsilon/4$ . Then $(\beta\circ\alpha)(T_1+ T_2) \leq \beta(V_1P_1+V_2P_2)\alpha(U_1+U_2) \leq 2\beta(V_1P_1+V_2P_2)$ $\max(\alpha(U_1),\alpha(U_2))$ .

If we take $\alpha(U_1) = \alpha(U_2)$ , then

$(\beta\circ\alpha)(T_1+T_2) \leq 2[\beta(V_1) + \beta(V_2)]\alpha(U_1)$

$= 2[\beta(V_1)\alpha(U_1) + \beta(V_2)\alpha(U_2)] \leq 2[(\beta\circ\alpha)(T_1) + (\beta\circ\alpha)(T_2) + \epsilon/2]$

$= 2[(\beta\circ\alpha)(T_1) + (\beta\circ\alpha)(T_2)] + \epsilon$ .

It is easy to show that if $S \in \mathcal{L}(G,E)$ , $T \in B^\circ A(E,F)$ and $R \in \mathcal{L}(F,H)$ , then $RTS \in B^\circ A(G,H)$ and $(\beta\circ\alpha)(RTS) \leq \|R\|(\beta\circ\alpha)(T)\|S\|$ .

Remark. In general, we are not able to show that $\beta\circ\alpha$ is a norm even if both $\alpha$ and $\beta$ are ideal norms.

We now give an example of a product ideal. Using the same techniques as in the proof of Proposition 2 in [19] by Kwapien, we obtain the following result.

3.1.2 Theorem. $[\Pi_{p,r,s}, \pi_{p,r,s}] = [\Pi'_s \circ \Pi_r, \pi'_s \circ \pi_r]$ where $1/p = 1/r + 1/s \leq 1$ and $1 \leq r, s < \infty$ .

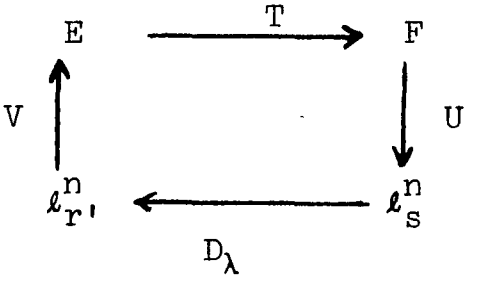
To prove our theorem, we need a lemma.

3.1.3 Lemma. If $1/p = 1/r + 1/s \leq 1$ , $1 \leq r, s < \infty$ and $a,b$ are positive real numbers, then

$$\frac{(ab)^p}{p} = \inf_{0<t<\infty} \left(\frac{t^r a^r}{r} + \frac{b^s}{st^s}\right) .$$

The proof is easy, so we omit it.

Proof of Theorem. Let $T \in \Pi_{p,r,s}(E,F)$ . We consider the following diagram




with $U: y \to (\langle y, b_i \rangle)_{i \leq n}$ , $(b_i)_{i \leq n} \subset F'$ ; $V': b \to (\langle x_i, a \rangle)_{i \leq n}$, $(x_i)_{i \leq n} \subset E$ and $D_\lambda(\xi_i) = (\lambda_i \xi_i)_{i \leq n}$ , $(\lambda_i) \in \ell^n_{p'}$ . Then $|\mathrm{Trace}(D_\lambda UTV)| = |\sum_{i=1}^n \lambda_i \langle Tx_i, b_i \rangle| \leq \pi_{p,r,s}(T)\|A\|\|B\|\nu_{p',s,r}(D_\lambda)$ $= \pi_{p,r,s}(T)\, \epsilon_s(b_i)\, \epsilon_r(x_i) \|(\lambda_i)\|_{p'}$ . Taking the supremum over $(\lambda_i)$ with $\|(\lambda_i)\|_{p'} \leq 1$ , we have $(\sum_{i=1}^n |\langle Tx_i, b_i \rangle|^p)^{1/p} \leq \pi_{p,r,s}(T) \epsilon_r(x_i) \epsilon_s(b_i)$ .

By Lemma 3.13, we have

$$\sum_{i=1}^{n} |\langle Tx_i, b_i\rangle|^p \leq [\pi_{p,r,s}(T)]^p\{(\tfrac{p}{r})[\epsilon_r(x_i)]^r + (\tfrac{p}{s})[\epsilon_s(b_i)]^s\} .$$

Let $K_1 \subset E'$ and $K_2 \subset F''$ be the weak-star closure of the extreme points of the closed unit balls and set $K = K_1 \times K_2$. For $(a,b) \in E \times F'$, let $f_{(x,b)} \in C(K)$ be given by

$$f_{(x,b)}(a,y'') = (\tfrac{p}{r})|\langle x,a\rangle|^r + (\tfrac{p}{s})|\langle b,y''\rangle|^s .$$

Let $A$ be the closed convex hull of $\{f_{(x,b)} \in C(K) : |\langle Tx,b\rangle| = 1\}$ and $B = \{f \in C(K): f(t) < [\pi_{p,r,s}(T)]^{-p}\}$. Then $A$ and $B$ are disjoint convex sets with $B$ open; in fact, if $f_{(x_i,b_i)}$ satisfies $|\langle Tx_i,b_i\rangle| = 1$, $i=1,2,\cdots,n$, $0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^{n} \lambda_i = 1$, then

$$1 = \sum_{i=1}^{n} \lambda_i |\langle Tx_i,b_i\rangle|^p = \sum_{i=1}^{n} |\langle T(\lambda_i^{1/r} x_i), (\lambda_i^{1/s} b_i)\rangle|^p$$

$$\leq [\pi_{p,r,s}(T)]^p\{(\tfrac{p}{r})[\epsilon_r(\lambda_i^{1/r} x_i)]^r + (\tfrac{p}{s})[\epsilon_s(\lambda_i^{1/s} b_i)]^s\}$$

$$\leq [\pi_{p,r,s}(T)]^p \sup_{(a,y'')\in K} [\sum_{i=1}^{n} \lambda_i f_{(x_i,b_i)}](a,y'') .$$

By the separation theorem for convex sets and the Riesz theorem, there is a probability measure $\sigma$ on $K$ such that $[\pi_{p,r,s}(T)]^{-p} \leq \sigma(f)$ for $f \in A$. This gives

$$\pi_{p,r,s}^{-p}(T) \leq (\tfrac{p}{r})\int_K |\langle x,a\rangle|^r \sigma(da,d\nu) + (\tfrac{p}{s})\int_K |\langle b,\nu\rangle|^s \sigma(da,d\nu)$$

whenever $|\langle Tx,b\rangle| = 1$ . Replacing x,b by tx and $t^{-1}b$ respectively and applying Lemma 3.13, we obtain

$$\pi_{p,r,s}^{-p}(T) \leq \left(\int_K |\langle x,a\rangle|^r d\sigma\right)^{p/r} \left(\int_K |\langle b,\nu\rangle|^s d\sigma\right)^{p/s}$$

whenever $|\langle Tx,b\rangle| = 1$ , which in turn implies

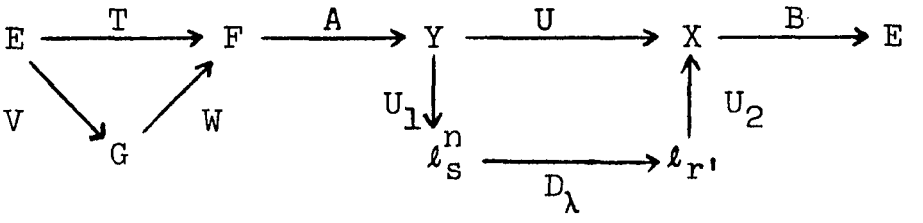
$$(*) \quad |\langle Tx,b\rangle| \leq \pi_{p,r,s}(T) \left(\int_K |\langle x,a\rangle|^r d\sigma\right)^{1/r} \left(\int_K |\langle b,\nu\rangle|^s d\sigma\right)^{1/s}$$

for all $x \in E$, $b \in F'$ .

Now let $V \in \mathcal{L}(E,L_r(K,\sigma))$ be defined by $V(x)(a) = \langle x,a\rangle$ on $K_1$ and $V(x)(y'') = 0$ on $K_2$ . Similarly, let $W_o \in \mathcal{L}(F',L_s(K,\sigma))$ be defined by $W_o(b)(y'') = \langle b,y''\rangle$ on $K_2$ and $W_o(b)(a) = 0$ on $K_1$ . Let $G = \overline{V(E)}$ in $L_r(K,\sigma)$ and $H = \overline{W_o(F')}$ in $L_s(K,\sigma)$ . By Pietsch's theorem (see [30]) $V \in \Pi_r(E,G)$ , $W_o \in \Pi_s(F',H)$ and $\pi_r(V) \leq 1$ , $\pi_s(W_o) \leq 1$ . Since $\|Vx\| = \left(\int_K |\langle x,a\rangle|^r d\sigma(a)\right)^{1/r}$, and $\|W_o b\| = \left(\int_K |\langle b,y''\rangle|^s d\sigma(y'')\right)^{1/s}$ , (*) implies $|\langle Tx,b\rangle| \leq \pi_{p,r,s}(T)\|Vx\|\|W_o b\|$ , that is, there exists an operator $Z:G \to H'$ such that $J_F T = W_o' Z V$ and $\|Z\| \leq \pi_{p,r,s}(T)$ where $J_F:F \to F''$ is the natural injection. Indeed, Z is defined by $(W_o')^{-1} J_F T V^{-1}$ . Let $W = W_o' Z$ , then $J_F T = WV$ and $\pi_s'(W) \leq \|Z\|\pi_s'(W_o') \leq \|Z\| \leq \pi_{p,r,s}(T)$ . Hence $(\pi_s' \circ \pi_r)(T) \leq \pi_{p,r,s}(T)$ .

On the other hand, assume $T = WV$ , where $V \in \Pi_r(E,G)$

and $W' \in \Pi_{s'}(F',G')$ . Let X and Y be finite dimensional Banach spaces, let $A \in \mathcal{L}(F,Y)$ , $U \in N_{p',s,r}(Y,X)$ and $B \in \mathcal{L}(X,E)$ . We consider the diagram




where $U = U_2 D_\lambda U_1$ , $U_1 \in \mathcal{L}(Y,\ell_s^n)$ , $U_2 \in \mathcal{L}(\ell_{r'},X)$ and $D_\lambda(\xi_i) = (\lambda_i \xi_i)_{i \le n}$ , $(\lambda_i) \in \ell_{p'}$ so that $\|U_2\|\|D_\lambda\|\|U_1\| \le \nu_{p',s,r}(U) + \epsilon$ .

We have to prove that $|\text{Trace } UATB| \le \pi_r(V)\pi_{s'}(W')\nu_{p',s,r}(U)\|A\|\|B\|$ . But $VBU_2 \in \Pi_s(\ell_{r'}^n,G)$ and $(U_1AW)' \in \Pi_s(\ell_{s'}^n,G')$ . Also $\Pi_r^* = I_{r'}^R$ . Thus we have, $|\text{Trace } UATB| = |\text{Trace } D_\lambda U_1 AWVBU_2|$

$\le i_{r'}^R(D_\lambda U_1 AW)\pi_r(VBU_2) \le \pi_{r'}(W'A'U_1'D_\lambda')\pi_r(VBU_2)$

$\le \|(\lambda_i)\|_{p'}\pi_s(W'A'U_1')\pi_r(VBU_2) \le \pi_r(V)\pi_s(W')(\nu_{p',s,r}(U)+\epsilon)\|A\|\|B\|$ .

3.1.4 Theorem. i) $[B \circ \backslash A, \beta \circ \backslash \alpha] = [\backslash(B \circ A), \backslash(\beta \circ \alpha)]$

ii) $[\backslash B \circ A, \backslash \beta \circ \alpha] = [(B \circ A)/, (\beta \circ \alpha)/]$

iii) $[\backslash B \circ \backslash A, \backslash \beta \circ \backslash \alpha] = [\backslash(B \circ A)/, \backslash(\beta \circ \alpha)/]$ .

Proof. i) By definition if $T \in (B \circ \backslash A)(E,F)$ , we can write $T: E \xrightarrow{V} G \xrightarrow{W} F$ with $V \in \backslash A(E,G)$ and $W \in B(G,F)$ . However,

we can factor $V$ as $E \xrightarrow{J_E^\infty} \ell^\infty(U_{E'}) \xrightarrow{W} G$ with $W \in A(\ell^\infty(U_{E'}),G)$. Thus $T \in \backslash(B\circ A)(E,F)$ and $(\beta\circ\backslash\alpha)(T) \geq \backslash(\beta\circ\alpha)(T)$. Conversely if $T \in \backslash(B\circ A)(E,F)$, by the definition of left projective envelope, $T$ can be factored $E \xrightarrow[J_E^\infty]{} \ell_\infty(U_{E'}) \xrightarrow[S]{} F$ with $S \in B\circ A\,(\ell_\infty(U_{E'}),F)$ and $S = V\circ U$, where $U \in A(\ell_\infty(U_{E'}),G)$ and $V \in B(G,F)$. Hence $U\circ J_E^\infty \in \backslash A(E,G)$, $T \in (B\circ\backslash A)(E,F)$ and $\backslash(\beta\circ\alpha)(T) \geq (\beta\circ\backslash\alpha)(T)$.

Statements ii) and iii) follow in a similar manner.

3.1.5 Corollary. i) $[(\Pi_s)'\circ I_r^R, (\pi_s)'\circ i_r^R] = [\backslash\Pi_{p,r,s}, \backslash\pi_{p,r,s}]$

ii) $[(I_s^R)'\circ\Pi_r, (i_s^R)'\circ\pi_r] = [\Pi_{p,r,s/}, \pi_{p,r,s/}]$

iii) $[(I_s^R)'\circ(I_r^R), (i_s^R)'\circ i_s^R] = [\backslash\Pi_{p,r,s/}, \backslash\pi_{p,r,s/}]$.

§ 3.2 Quotient Ideals

Let $[A,\alpha]$ and $[B,\beta]$ be two quasi-normed ideals. We denote by $B/A(E,F)$ the class of all operators $T \in \mathcal{L}(E,F)$ such that for each Banach space $G$ and $U \in A(F,G)$, $U\circ T \in B(E,G)$. We also set $\beta/\alpha(T) = \sup_{\alpha(U)_G \leq 1} \beta(U\circ T)$.

3.2.1 Theorem. If $[A,\alpha]$ and $[B,\beta]$ are two normed ideals, then $[B/A,\beta/\alpha]$ is a normed ideal.

Proof. i) If $a \in E'$, $y \in F$, then $U\circ(a\otimes y) = a\otimes U(y) \in B(E,G)$ and $a\otimes y \in B/A(E,F)$. Moreover,

$\beta/\alpha(a\otimes y) = \sup_{\alpha(U)\le 1} \beta(a\otimes U(y)) = \sup_{\alpha(U)\le 1} \|a\|\|U(y)\| \le \|a\|\|y\|$ .

If we take $U = b\otimes z$ with $b \in U_F'$ and $z \in U_G$, then

$\beta/\alpha(a\otimes y) \ge \beta((b\otimes z)\circ(a\otimes y)) = \beta[a\otimes b(y)z] = |b(y)|\beta(a\otimes z)$

$= |b(y)|\|a\|\|z\| = |b(y)|\|a\|$ . Hence $\beta/\alpha(a\otimes y) \ge \|y\|\|a\|$ .

ii) If $T,S \in B/A(E,F)$, then $U\circ(T+S) = U\circ T + U\circ S \in B(E,G)$ for $U \in A(F,G)$ . Hence $T+S \in B/A(E,F)$ and

$\beta/\alpha(S+T) = \sup_{\alpha(U)\le 1} \beta[U\circ(S+T)] \le \sup_{\alpha(U)\le 1} [\beta(U\circ S)+\beta(U\circ T)]$

$\le \sup_{\alpha(U)\le 1} \beta(U\circ S) + \sup_{\alpha(U)\le 1} \beta(U\circ T) = \beta/\alpha(S) + \beta/\alpha(T)$ .

iii) If $S \in \mathcal{L}(F,H)$ and $T \in B/A(E,F)$, then $V\circ S \in A(F,G)$ for $V \in A(H,G)$ and $V\circ(S\circ T) = (V\circ S)\circ T \in B(E,G)$ . Hence $S\circ T \in B/A(E,H)$ and $\beta/\alpha(S\circ T) = \sup_{\alpha(V)\le 1} \beta(V\circ(S\circ T))$

$= \sup_{\alpha(V)\le 1} \|S\|\beta(V\circ(\frac{S}{\|S\|}\circ T)) \le \sup_{\alpha(U)\le 1} \|S\|\beta(U\circ T) = \|S\|\beta/\alpha(T)$ ,

since $\alpha(V\circ\frac{S}{\|S\|}) \le \alpha(V) \le 1$ .

iv) If $W \in \mathcal{L}(H,E)$, $T \in B/A(E,F)$, then $U\circ T \in B(E,G)$ for $U \in A(F,G)$ and $(U\circ T)\circ W = U\circ(T\circ W) \in B(H,G)$ for $U \in A(F,G)$ . Hence $T\circ W \in B/A(H,F)$ and $\beta/\alpha(T\circ W)$

$= \sup_{\alpha(U)\le 1} \beta(U\circ(T\circ W)) \le \sup_{\alpha(U)\le 1} \beta(U\circ T)\|W\| = \beta/\alpha(T)\|W\|$ for $W \in \mathcal{L}(H,E)$ .

3.2.2 Corollary. If $[B,\alpha]$ is complete, so is $[B/A,\beta/\alpha]$ .

Remark. i) If $[A,\alpha] \subseteq [B,\beta]$, then $B/A(E,F) = \mathcal{L}(E,F)$

ii) $[B,\beta] \subset [B/A,\beta/\alpha]$ for all $[A,\alpha]$ .

A classical result of Grothendieck shows that the composition of two 2-absolutely summing operators is nuclear. We show the converse is also true.

3.2.2 Lemma. $I/A \subset A^*$ for any normed ideal $[A,\alpha]$ .

Proof. If $T \in I/A(E,F)$ , then $S \circ T \in I(E,G)$ for all $S \in A(F,G)$ . We claim that there is a constant $c \geq 0$ such that $i(ST) \leq c\alpha(S)$ for any finite dimensional space $G$ and operator $S \in A(F,G)$ . Indeed, suppose for each $n$, there are finite dimensional spaces $G_n$ and $S_n \in A(F,G_n)$ with $i(S_nT) > n$ , and $\alpha(S_n) = \frac{1}{n^2}$ . Let $S = \sum_{n=1}^{\infty} J_n S_n$ , where $J_n : G_n \to (\bigoplus_{n=1}^{\infty} G_n)_\infty$, then $\alpha(S) \leq \sum_{n=1}^{\infty} \alpha(J_n S_n) \leq \sum_{n=1}^{\infty} \alpha(S_n) < +\infty$ and $S \in A(F,G)$ . But $S_nT = P_nST$ , where $P_n : (\bigoplus_{n=1}^{\infty} G_n)_\infty \to G_n$ a projection. We thus have $i(ST) \geq i(S_nT) > n \to \infty$ , hence $ST \notin I(E,G)$ and this gives a contradiction.

Now let $X$, $Y$ be finite dimensional, $U \in \mathcal{L}(X,E)$ , $V \in \mathcal{L}(F,Y)$ and $S \in A(Y,X)$ . Then

$|\mathrm{Trace}(USVT)| \leq \|U\| \|SVT\|^{\Delta} = \|U\| i(SVT) \leq c\|U\| \alpha(SV) \leq$
$\leq c\|U\| \|V\| \alpha(S)$ , so that $\alpha^*(T) \leq c$ and $T \in A^*$ .

3.2.3 Theorem. $N_1/\Pi_2 = \Pi_2$ .

Proof. From Grothendieck's result, we have $\Pi_2 \subset N_1/\Pi_2$ .

Following Lemma 3.2.2, we have $N_1/\Pi_2 \subset \Pi_2^* = I_2 = \Pi_2$ .

We mention that $\Pi_2 \circ \Pi_2 \neq N_1$ . Indeed if $T \in \Pi_2 \circ \Pi_2$ , then $T$ is fully nuclear in the sense of [41] and there are nuclear operators which are not fully nuclear.

3.2.4 Theorem. i) $\Pi_p = N_1/N_{p'} = I_1/I_{p'}$

ii) $I_p = N_1/N_{p'}^Q = I_1/\Pi_{p'}$ $(1/p + 1/p' = 1)$ .

Proof. i) From the multiplication theorem of Pietsch [27], we have $\Pi_p \subset N_1/N_{p'}$ and $\Pi_p \subset I_1/I_{p'}$ . Also, by Lemma 3.2.2, it follows that $N_1/N_{p'} \subset I_1/N_{p'} \subset N_{p'}^* = \Pi_p$ and $I_1/I_{p'} \subset I_{p'}^* = \Pi_p$ . ii) Similarly, $I_p \subset N_1/N_{p'}^Q \subset I_1/N_{p'}^Q$, $\subset N_{p'}^{Q*} = I_p$ and $I_p \subset I_1/\Pi_{p'} \subset \Pi_{p'}^* = I_p$ .

Call $[B/A, \beta/\alpha]$ the right quotient of $[B,\beta]$ by $[A,\alpha]$ . We now define the following analogous class of operators.

Let $[A,\alpha]$ and $[B,\beta]$ be two given quasi-normed ideals. We denote $A\backslash B$ the class of all operators $T$ such that for each $U \in A(G,E)$ the composition $T \circ U \in B(G,F)$ and set $\alpha\backslash\beta(T) = \sup_{\alpha(U)_G \leq 1} \beta(T \circ U)$ . $[A\backslash B, \alpha\backslash\beta]$ is a quasi-normed ideal, and is called the left quotient of $[B,\beta]$ by $[A,\alpha]$ .

3.2.5 Theorem. i) $[(B/A)', (\beta/\alpha)'] = [A'\backslash B', \alpha'\backslash\beta']$

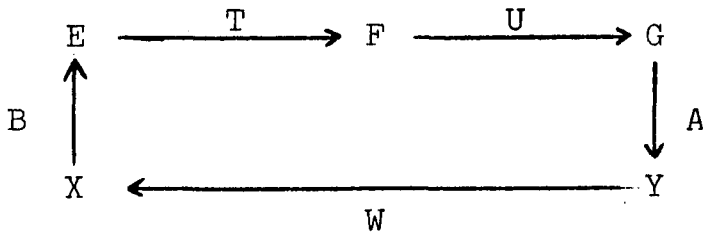
ii) $[(A\backslash B)', (\alpha\backslash\beta)'] = [B'/A', \beta'/\alpha']$ .

The proof is easy, so we omit it.

Now we are able to construct a right quotient normed ideal which includes a case considered by Kwapien [19].

3.2.6 Theorem. $[I^R_{p,r,s}, i^R_{p,r,s}] = [(I^R_{r'})'/\Pi_s\ ,\ (i^R_{r'})'/\Pi_s]$ $(1/p + 1/r + 1/s = 1)$ .

Proof. Let $T \in I^R_{p,r,s}(E,F)$ and consider the diagram




with $U \in \Pi_s(F,G)$ , $W' \in \Pi_r(X',Y')$ and X, Y are finite dimensional. From Theorem 3.1.2, $WAU \in \Pi_{p',s,r}(F,X)$ and

$|\mathrm{Trace}(WAUTB)| \le \pi_{p',s,r}(WAU)\pi^*_{p',s,r}(TB)$

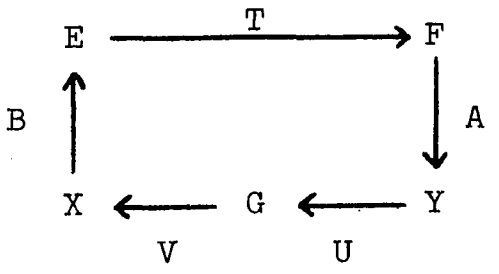
$\le \pi_s(U)\pi_r(W')\|A\| i^R_{p,r,s}(T)\|B\| = \pi_s(U) i^R_{p,r,s}(T)\|A\|\|B\|\pi_r(W')$ .

Thus $(UT)' \in \pi^*_r(G',E') = I^R_{r'}(G',E')$ and $i^R_{r'}((UT)')$

$\le \pi_s(U) i^R_{p,r,s}(T)$ . Thus $(i^R_{r'})'/\pi_s(T) \le i^R_{p,r,s}(T)$ .

On the other hand, let $T \in (I^R_{r'})'/\Pi_s(E,F)$ and for $U \in \pi_s(F,G)$ , we have $i^R_{r'}((UT)') \le \pi_s(U)(i^R_{r'})'/\pi_s(T)$ .

Considering the diagram




with $U \in \Pi_s(Y,G)$ and $V' \in \Pi_r(X',G')$, we have $|\text{Trace } VUATB| = |\text{Trace } B'(UAT)'V'|$. But $(UAT)' \in I^R_{r'}(G',E')$ and $V' \in \pi_r(X',G')$. Therefore $|\text{Trace } VUATB|$ $\leq \pi_r((BV)'i^R_{r'}((UAT)')) \leq \|B\|\pi_r(V')\pi_s(U)(i^R_r)'/\pi_s(T)\|A\|$.

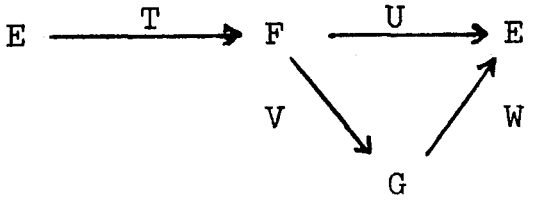
Hence $i^R_{p,r,s}(T) \leq (i^R_{r'})'/\pi_s(T)$.

We next establish a duality theorem that show the relationship between product and quotient normed ideals.

3.2.7 Theorem. For normed ideals $[A,\alpha]$ and $[B,\beta]$, $[(B\circ A)^\Delta, (\beta\circ\alpha)^\Delta] = [B^\Delta/A, \beta^\Delta/\alpha]$.

Proof. Let $T \in (B\circ A)^\Delta(E,F)$ and consider the diagram $E \xrightarrow{T} F \xrightarrow{U} G \xrightarrow{V} E$ with $U \in A(F,G)$ and $V \in \mathfrak{F}(G,E)$, then $|\text{Trace } VUT| \leq (\beta\circ\alpha)^\Delta(T)\beta\circ\alpha(VU) \leq (\beta\circ\alpha)^\Delta(T)\beta(V)\alpha(U)$ $= [(\beta\circ\alpha)^\Delta(T)\alpha(U)]\beta(V)$. Hence $U\circ T \in B^\Delta(E,G)$ for $U \in A(F,G)$ and $\beta^\Delta(U\circ T) \leq (\beta\circ\alpha)^\Delta(T)\alpha(U)$. Thus $T \in B^\Delta/A(E,F)$ and $\beta^\Delta/\alpha(T) \leq (\beta\circ\alpha)^\Delta(T)$.

On the other hand, let $T \in B^\Delta/A(E,F)$ and consider the diagram




with $U \in \mathfrak{F}(F,E)$ . Then as indicated in the diagram, there exists a Banach space $G$ and operators $V \in A(F,G)$ and $W \in B(G,E)$ with $U = WV$ and $(\beta\circ\alpha)(U) + \epsilon \geq \alpha(V)\beta(W)$ .

Since $V \in A(F,G)$ , $V\circ T \in B^{\Delta}(E,G)$ , and so $|\text{Trace } UT| = |\text{Trace } WVT| \leq \beta^{\Delta}(VT)\beta(W) \leq \beta^{\Delta}/\alpha(T)\alpha(V)\beta(W) \leq \beta^{\Delta}/\alpha(T)[(\beta\circ\alpha)(U) + \epsilon]$ . Hence $T \in (B\circ A)^{\Delta}(E,F)$ and $(\beta\circ\alpha)^{\Delta}(T) \leq (\beta^{\Delta}/\alpha)(T)$ .

Remark. We are not able to prove the above theorem for the adjoint operation $*$ in general. However, we have the following corollary.

3.2.8 Corollary. If both $E$ and $F$ have m.a.p., then $(B\circ A)^{*}(E,F) = B^{*}/A(E,F)$ and $(\beta\circ\alpha)^{*}(T) = \beta^{*}/\alpha(T)$ for all $T$ .

3.2.9 Theorem. If both $E$ and $F$ have m.a.p. and $\beta$ is a perfect ideal norm, then $[(B/A)^{*},(\beta/\alpha)^{*}] = [B^{*}\circ A,\beta^{*}\circ\alpha]$ .

Proof. Since $[B,\beta]$ is a perfect normed ideal $[B^{*}\circ A,\beta^{*}\circ\alpha] = [B^{***}\circ A,\beta^{***}\circ\alpha] = [(B^{**}/A)^{*},(\beta^{**}/\alpha)^{*}] = [(B/A)^{*},(\beta/\alpha)^{*}]$ .

3.2.10 Corollary. If $\beta$ is perfect and both $E$ and $F$ have m.a.p., then $(B \circ A)(E,F)$ is a Banach space under the norm $\beta \circ \alpha$ .

## § 3.4 Sum and Intersection of ideals

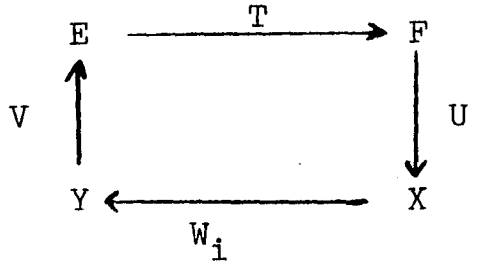
Let $[A,\alpha]$ and $[B,\beta]$ be two given quasi-normed ideals. We can form $A \cap B$ and $A + B$ by defining the components $(A \cap B)(E,F) = A(E,F) \cap B(E,F)$ and $(A+B)(E,F) = [A(E,F), B(E,F)]$ , the linear span of $A(E,F)$ and $B(E,F)$ in $\mathcal{L}(E,F)$ .

If we set $(\alpha \cap \beta)(T) = \max\{\alpha(T), \beta(T)\}$ and $(\alpha+\beta)(T) = \inf\{\alpha(R)+\beta(S) : T = R+S,\ R \in A,\ S \in B\}$ , then it is easy to show that $[A \cap B, \alpha \cap \beta]$ and $[A+B, \alpha+\beta]$ both are quasi-normed ideals.

If $A$, $B$ are Banach ideal, then so are $[A \cap B, \alpha \cap \beta]$ and $[A+B, \alpha+\beta]$ .

3.4.1 Theorem. $[(A+B)^*, (\alpha+\beta)^*] = [A^* \cap B^*, \alpha^* \cap \beta^*]$ .

Proof. If $T \in A^* \cap B^*(E,F)$ , we consider the diagram



with X, Y finite dimensional spaces, then

$|\text{Trace } WUTV| = |\text{Trace } (W_1 + W_2)UTV|$

$\leq |\text{Trace } W_1UTV| + |\text{Trace } W_2UTV| \leq \alpha^*(T)\|U\|\|V\|\alpha(W_1)$

$+ \beta^*(T)\|U\|\|V\|\beta(W_2) \leq \max(\alpha^*(T),\beta^*(T))\|U\|\|V\|[\alpha(W_1)+\beta(W_2)]$ .

Hence $(\alpha+\beta)^*(T) \leq \max(\alpha^*(T),\beta^*(T)) = (\alpha^* \cap \beta^*)(T)$ and

$T \in (A+B)^*(E,F)$ .

On the other hand, let $T \in (A+B)^*(E,F)$ . Then,

$|\text{Trace } WUTV| \leq (\alpha+\beta)^*(T)\|U\|\|V\|(\alpha+\beta)(W) \leq (\alpha+\beta)^*(T)\|U\|\|V\|\alpha(W)$ .

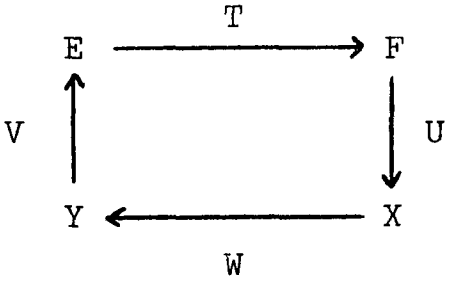
Similarly, $|\text{Trace } WUTV| \leq (\alpha+\beta)^*(T)\|U\|\|V\|\beta(W)$ . Thus,

$\alpha^*(T) \leq (\alpha+\beta)^*(T)$ and $\beta^*(T) \leq (\alpha+\beta)^*(T)$ . Hence

$\max(\alpha^*(T),\beta^*(T)) \leq (\alpha+\beta)^*(T)$ and $T \in (A^* \cap B^*)(E,F)$ .

3.4.2 Theorem. $[(A \cap B)^*, (\alpha \cap \beta)^*] \supset [A^*+B^*, \alpha^*+\beta^*]$ .

Proof. If $T \in (A^*+B^*)(E,F)$ , we consider the diagram




with U, V bounded and X, Y finite dimensional. Then

$|\text{Trace}(WUTV)| \leq |\text{Trace } WUT_1V| + |\text{Trace } WUT_2V|$

$\leq \alpha^*(T_1)\alpha(W)\|U\|\|V\| + \beta^*(T_2)\beta(W)\|U\|\|V\|$

$\leq (\alpha^*(T_1) + \beta^*(T_2))\|U\|\|V\|\max(\alpha(W),\beta(W))$

$= [\alpha^*(T_1) + \beta^*(T_2)]\|U\|\|V\|(\alpha\cap\beta)(W)$ . Hence $T \in (A\cap B)^*(E,F)$ and $(\alpha\cap\beta)^*(T) \leq (\alpha^*+\beta^*)(T)$ .

Remark. We are not able to show that $[(A\cap B)^*,(\alpha\cap\beta)^*]$ $= [A^*+B^*,\alpha^*+\beta^*]$ . However, if $[A,\alpha]$ and $[B,\beta]$ are both perfect, we have $[A^*+B^*,\alpha^*+\beta^*] = [(A^*+B^*)^{**},(\alpha^*+\beta^*)^{**}]$ $= [(A^{**}\cap B^{**})^*,(\alpha^{**}\cap\beta^{**})^*] = [(A\cap B)^*,(\alpha\cap\beta)^*]$ .

Example. For any ideals A, B it is clear that $A \subset A+B$ , $B \subset A+B$ . In general, A+B can be distinct from either A or B , e.g. $\pi_p + K \neq K$ or $\pi_p$ . Indeed, let $T:\ell_2 \to \ell_2$ be defined by $Te_n = \frac{1}{\sqrt{n}}\, e_n$ and let $i = \ell_1 \to \ell_2$ be the natural inclusion. Then $i+T \notin K$ or $T \notin \pi_p$ .

# CHAPTER IV

## QUASI-NORMED IDEALS OF OPERATORS DETERMINED BY s-NUMBERS

The nth s-number of a compact operator $T$ on an infinite dimensional Hilbert space $H$ is defined as the nth eigenvalue (ordered according to magnitude) of the operator $|T| = (T^*T)^{1/2}$. It is well known that the s-numbers play an important role in the theory of operators on Hilbert space. Indeed, s-numbers have been used to construct (symmetric) normed ideals in the ring $\mathcal{L}(H,H)$. This work has been compiled in the book of Gohberg and Krein [11].

For operators between Banach spaces, several definitions of sequences of numbers which coincide with s-numbers in the case of Hilbert space have appeared in the literature. Some of these have been collected in the paper [24] of Mitiagin and Pełczyński. Recently, Pietsch collected some important properties of s-numbers in Hilbert spaces and gave an axiomatic definition of s-numbers, see [34].

## § 4.1 Axiomatic definition of s-numbers

Following Pietsch [34], we say that a map $s:T \to (s_n(T))$ from the ideal $\mathcal{L}$ into the sequence of non-negative numbers is an s-number function provided that the following conditions are satisfied $(n=1,2,\cdots)$

i) $\|T\| = s_1(T) \geq s_2(T) \geq s_3(T) \geq \cdots \geq 0$ for $T \in \mathcal{L}$.

ii) $s_n(S+T) \leq s_n(S) + \|T\|$ for $S,T \in \mathcal{L}(E,F)$

iii) $s_n(RTS) \leq \|R\| s_n(T) \|S\|$ for $S \in \mathcal{L}(G,E)$, $T \in \mathcal{L}(E,F)$ and $R \in \mathcal{L}(F,H)$

iv) If $\dim(T) < n$, then $s_n(T) = 0$

v) If $\dim(T) \geq n$, then $s_n(I_E) = 1$.

The number $s_n(T)$ is said to be the nth s-number of the operator $T$.

We note that, from iii), v), the converse of statement iv) is also valid.

Pietsch showed that s-numbers of operators in Hilbert space $H$ are determined uniquely by the above axioms.

An s-number function $s$ is called additive if the following improvement of condition ii) is made

ii') $s_{m+n-1}(S+T) \leq s_m(S) + s_n(T)$ for $S,T \in \mathcal{L}(E,F)$ $(m,n=1,2,\cdots)$.

We will see that the additivity of an s-number function is required to construct a quasi-normed ideal.

Some Special s-number Functions

Let $T \in \mathcal{L}(E,F)$, we define the nth approximation number

$$a_n(T) = \inf\{\|T-A\| : A \in \mathcal{L}(E,F),\ \dim(A) < n\}$$

the nth Kolmogorov number

$$d_n(T) = \inf_{\dim N<n} \sup_{x\in U_E} \|Tx\|_{F/N} = \inf_{\dim N<n} \{\|Q_N^F T\|\}$$

the nth Gelfand number

$$c_n(T) = \inf_{\operatorname{codim} M<n} \sup_{x\in U_E\cap M} \|Tx\| = \inf_{\operatorname{codim} M<n} \{\|TJ_M^E\|\}.$$

It has been shown (see [34]) that a, d, c are additive s-number functions. It is unknown whether all s-numbers are additive.

§ 4.1 Ideals of Operators and Ideals of Sequences

Given an operator ideal A and two sequence ideals $\mathcal{P}$ and $\mathcal{Q}$, we denote $\ell_{(A,\mathcal{P},\mathcal{Q})}$ the class of sequences $(\alpha_n) \in \ell_\infty$ for which the diagonal operator $D_\lambda : \mathcal{P} \to \mathcal{Q}$ with $D_\lambda(\xi_n) = (\alpha_n\xi_n)$ belongs to $A(\mathcal{P},\mathcal{Q})$. It is easy to see that $\ell_{(A,\mathcal{P},\mathcal{Q})}$ forms a sequence ideal.

In [42], Tong (also see Holub [15]) showed that $\ell_{(N,\ell_p,\ell_q)} = \ell_h$, where

$$h = \begin{cases} 1 & \text{if } 1 \le q \le p \le \infty \\ \frac{pq}{pq+p-q} & \text{if } 1 \le p < q < \infty \\ \frac{1}{p-1} & \text{if } q = \infty \text{ and } 1 < p < \infty \\ \infty & \text{if } q = \infty \text{ and } p = 1 . \end{cases}$$

Also Garling [10] computed the sequence ideals $\ell(\Pi_r, \ell_p, \ell_q)$ and $\ell(N_r, \ell_p, \ell_q)$ .

We note that the correspondence $(A, \theta, 2) \to \ell_{(A,\theta,2)}$ is not one-to-one in general.

On the other hand, for each s-number function $s$ and each sequence ideal $a$ , we can associate an ideal of operators by

$$A_a^s = \{T \in L = (s_n(T)) \in a\} .$$

We set $\alpha_a^s(T) = \|(s_n(T))\|_a$ .

4.2.1 Theorem. For any additive s-number function $s$ and a sequence ideal $a$ , $[A_a^s, \alpha_a^s]$ is a complete quasi-normed ideal.

The proof is routine, so we omit it.

For the approximation number function $a$ , we use $\ell_p(E,F)$ for $A^a_{\ell_p}(E,F)$ .

The following result is immediate from the definition of the Kolmogorov and Gelfand numbers.

4.2.2 Theorem. i) $\ell_p\backslash(E,F) = A^c_{\ell_p}(E,F)$

ii) $/\ell_p(E,F) = A^d_{\ell_p}(E,F)$ .

4.2.3 Theorem. [9] The mapping $a \to A^s_a(H,H)$ establishes a one-to-one correspondence between the sequence ideals and ideals of operators on Hilbert spaces.

It is well-known and easy to prove that every ideal $A(H,H)$ is contained in $K(H,H)$ . But, given an operator $T \in K(H,H)$ , can we assert that there exists an ideal $A$ other than $K$ so that $T \in A(H,H)$ ? This question has been affirmatively answered by Brown, Pearcy and Salinas [1]. The proof uses some ideas of Von Neumann-Calkin [2] and is rather complicated and indirect.

We present an alternative direct proof. To this end, we need the following result.

4.2.4 Lemma. If $\{\beta_n\}$ is an increasing sequence with $\{\frac{1}{\beta_n}\} \in c_o \backslash \bigcup_{p>0} \ell_p$ , then there is a sequence $\{\alpha_n\}$ with $\alpha_n \neq 0$ such that $\sum_{n=1}^{\infty} \alpha_n < +\infty$ , $\sum_{n=1}^{\infty} \alpha_n\beta_n = +\infty$ and $\{\alpha_n\beta_n\}$ is monotone decreasing to $0$ .

Proof. Without loss of generality, we assume $\beta_n > 1$ for all $n$ . Choose $N_1$ and $t_1$ so that $\beta_{N_1} > 2$ and $\sum_{n=1}^{N_1} (\frac{1}{\beta_n})^{t_1} < \frac{1}{2}$ . Then, choose $r_1 > t_1$ and $M_1 > N_1$ such that

$\frac{1}{2}(\frac{1}{2}) < \sum_{n=N_1}^{M_1} (\frac{1}{\beta_n})^{r_1} < 2(\frac{1}{2})$ . Again, we choose $N_2 > M_1$ and $t_2 > r_1$ such that $\beta_{N_2} > 2$ and $\sum_{n=M_1}^{N_2} (\frac{1}{\beta_n})^{t_2} < \frac{1}{4}$ .

Choose $r_2 > t_2$ and $M_2 > N_2$ such that $\frac{1}{2}(\frac{1}{3}) < \sum_{n=N_2}^{M_2} (\frac{1}{\beta_n})^{r_2} < 2(\frac{1}{3})$ . In general, we choose $M_m > N_m > M_{m-1}$ and $r_m > t_m > r_{m-1}$ so that

$\beta_{N_m} > m$ , $\sum_{n=M_{m-1}}^{N_m} (\frac{1}{\beta_n})^{t_m} < \frac{1}{2^m}$ and $\frac{1}{2}(\frac{1}{m+1}) < \sum_{n=N_m}^{M_m} (\frac{1}{\beta_n})^{r_m} < 2(\frac{1}{m+1})$ .

Let

$$k_j = \begin{cases} t_i & \text{if } M_{i-1} \le j \le N_i \\ r_i & \text{if } N_i < j < M_i \end{cases} .$$

Then $\sum_{j=1}^{\infty} (\frac{1}{\beta_j})^{k_j} \ge \sum_{m=1}^{\infty} [\sum_{n=N_m}^{M_m} (\frac{1}{\beta_n})^{r_m}] \ge \frac{1}{2} \sum_{m=1}^{\infty} \frac{1}{m+1} = \infty$ and

$$\sum_{j=1}^{\infty} (\frac{1}{\beta_j})^{k_j+1} \le \sum_{m=1}^{\infty} [\sum_{M_{m-1}}^{N_m} (\frac{1}{\beta_n})^{t_m+1}] + \sum_{m=1}^{\infty} [\sum_{N_m}^{M_m} (\frac{1}{\beta_n})^{r_m+1}]$$

$$\le \sum_{m=1}^{\infty} \frac{1}{\beta_{N_m}} \cdot \frac{1}{2^m} + \sum_{m=1}^{\infty} \frac{1}{\beta_{M_m}} \cdot \frac{2}{m+1} \le \sum_{m=1}^{\infty} \frac{1}{m2^m} + \sum_{m=1}^{\infty} \frac{2}{m(m+1)} < +\infty .$$

Thus, letting $\alpha_j = (\frac{1}{\beta_j})^{k_j+1}$ we have $\sum_{j=1}^{\infty} \alpha_j = \sum_{j=1}^{\infty} (\frac{1}{\beta_j})^{k_j+1} < +\infty$, $\alpha_n \beta_n = (\frac{1}{\beta_j})^{k_j}$ , which is monotone decreasing to 0 , and

$$\sum_{j=1}^{\infty} \alpha_j \beta_j = \sum_{j=1}^{\infty} \left(\frac{1}{\beta_j}\right)^{k_j} = +\infty \ .$$

4.2.5 Theorem. If $T \in K(H,H)$, then there exists an ideal $A \neq K$ such that $T \in A(H,H)$.

Proof. Without loss of generality we may assume $T \in K(H,H) \setminus \bigcup_{p>0} S_p(H,H)$. Let $\beta_n = \frac{1}{s_n(T)}$, then $\frac{1}{\beta_n} \in c_0 \setminus \bigcup_{p>0} \ell_p$.

By Lemma 4.2.4, there is a sequence $\{\alpha_n\} \in \ell_1$ such that $\{\alpha_n\beta_n\}$ is monotone decreasing to 0 and $\sum_{n=1}^{\infty} \alpha_n\beta_n = +\infty$.

Let $\Lambda(T) = \{S \in \mathcal{L}(H,H): \sum_{n=0}^{\infty} s_n(S)\alpha_n\beta_n < +\infty\}$. Since $\sum_{n=1}^{\infty} s_n(T)\alpha_n\beta_n = \sum_{n=1}^{\infty} \alpha_n < +\infty$, $T \in \Lambda(T)$.

If $\dim S < +\infty$, then $s_n(S) = 0$ for large enough $n$ and so $\Lambda(T)$ contains the finite rank operators. Also, if $R,S \in \Lambda(T)$, then from the additivity of $s_n$, $\alpha_{2n}\beta_{2n}s_{2n}(R+S) \leq \alpha_n\beta_n s_n(R) + \alpha_n\beta_n s_n(S)$ and $\alpha_{2n+1}\beta_{2n+1}s_{2n+1}(R+S) \leq \alpha_{2n}\beta_{2n}s_{2n}(R+S)$. Thus $\Lambda(T)$ is a linear space. Also it is easy to see that $\Lambda(T)$ is closed under left and right composition by bounded linear operators. So $\Lambda(T)$ is an ideal.

But, since $\sum_{n=1}^{\infty} \alpha_n\beta_n = \infty$, there is a sequence $\{\gamma_n\}$ decreasing to zero such that $\sum_{n=1}^{\infty} \gamma_n\alpha_n\beta_n = +\infty$. If we write $T = \sum_{n=1}^{\infty} s_n(T) f_n \otimes y_n$, where $(f_n)$ and $(y_n)$ are suitable orthonormal sets in $H$, and let $T_0 = \sum_{n=1}^{\infty} \gamma_n f_n \otimes y_n$, then

$T_o \in K\backslash\Lambda(T)$ .

Using the recent results [16] and [35] it is easy to generalize the above construction to include many other classes of Banach spaces. We recall the following definition [16]: Two Banach spaces $E$ and $F$ form a Bernstein pair if for any positive, decreasing null sequence $(b_n)$ there is a $T \in K(E,F)$ such that

$$(*) \qquad 0 < \inf_n \frac{\alpha_n(T)}{b_n} \leq \sup_n \frac{\alpha_n(T)}{b_n} < +\infty$$

where $\alpha_n(T)$ is the nth approximation number of $T$ .

4.2.6 Theorem. There exists a complete quasi-normed ideal $\Lambda(T)$ such that $\Lambda(T)(E,F) \neq K(E,F)$ whenever $\langle E,F\rangle$ is a Bernstein pair.

Proof. By the result of Pietsch [35], if $K(E,F) = \ell_p(E,F)$ for some $p > 0$ , then $\min(\dim E, \dim F) < +\infty$ . Thus we may assume that $(\alpha_n(T)) \in c_o \backslash \bigcup_{p>0} \ell_p$ . Let $\beta_n = \frac{1}{\alpha_n(T)}$ and let $\alpha_n$ and $\gamma_n$ be as in the proof of Theorem 4.2.5. Let $\Lambda(T) = \{S \in K: \sum_{n=1}^{\infty} \alpha_n(S)\alpha_n\beta_n < +\infty\}$ . It is clear from the proof of 4.2.5 that $\Lambda(T)$ is a quasi-normed ideal with quasi-norm $\rho(S) = \sum_{n=1}^{\infty} \alpha_n(S)\alpha_n\beta_n$ . The completeness of $\Lambda(T)$ is easily established.

If $\langle E,F\rangle$ is a Bernstein pair, then by (*) there is

an $S \in K(E,F)$ such that $\inf_n \frac{\alpha_n(S)}{\gamma_n} > 0$ and thus, $S \notin \Lambda(T)(E,F)$ .

We remark that all classical Banach spaces form Bernstein pairs. In particular $\langle L_p(\mu), L_q(\nu)\rangle$ is a Bernstein pair for all $1 \leq p, q \leq \infty$ and measures $\mu$ , $\nu$ [16].

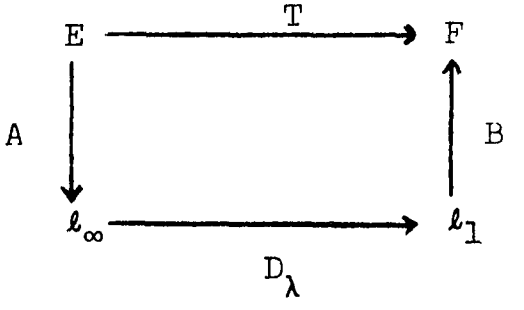
## § 4.3 A Special Quasi-normed Ideal

In [23], Marcus showed some relationship between the ideals $A^a_{\ell_p}$, $A^d_{\ell_p}$, $A^a_{\ell_{p:\infty}}$ and $N_{p,\infty,\infty}$ $(0 < p \leq 1)$ . To see some other interesting results, we need a lemma first given in [16].

4.3.1 Lemma. If $D_\lambda : \ell_\infty \to \ell_p$ is a diagonal operator determined by a sequence $(\lambda_i)_\infty$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\lambda_n \to 0$ , then $a_n(D_\lambda) = (\sum_{i=n} |\lambda_i|^p)^{1/p}$ .

4.3.2 Theorem. $N_{1/1+p:1,\infty,\infty}(G,F) \subset \ell_{1/p:1}(E,F)$ $(1 \leq p \leq \infty)$ for any Banach spaces $E$ and $F$ .

Proof. If $T \in N_{1/1+p:1,\infty,\infty}$ , then $T$ has factorization



with $D_\lambda \sim (\lambda_n) \in \ell_{1/1+p:1}$ and $\lambda_n \to 0$ that is, $\sum_{n=1}^{\infty} n^p |\lambda_n| < \infty$ . From Lemma 4.3.2, we have $a_n(D_\lambda) = \sum_{i=n}^{\infty} |\lambda_i|$ , therefore, $\sum_{n=1}^{\infty} n^{p-1} a_n(D_\lambda) = \sum_{n=1}^{\infty} (n^{p-1} \sum_{i=n}^{\infty} |\lambda_i|) \leq \sum_{n=1}^{\infty} n^p |\lambda_n| < \infty$.

Hence $D_\lambda \in A^a_{\ell_{1/p:1}}(\ell_\infty, \ell_1)$ and

$T = BD_\lambda A \in A^a_{\ell_{1/p:1}}(E,F) = \ell_{1/p:1}(E,F)$ .

It is well-known that $\ell_1(E,F) \subset N_{1,\infty,\infty}(E,F)$ (see [28]). From the above theorem, we have

4.3.3 Corollary. $N_{1/2:1,\infty,\infty}(E,F) \subset \ell_1(E,F)$ for Banach spaces E, F .

Finally, we conclude by giving a general result.

4.3.4 Theorem. $N_{p,r,r'}(E,F) \subset \ell_p(E,F)$ for $1 \leq r \leq \infty$ .

Proof. If $T \in N_{p,r,r'}$ , then T has factorization through a diagonal operator $D_\lambda \sim (\lambda_i)$ from $\ell_r$ to $\ell_r$ with $(\lambda_i) \in \ell_p$ . But $a_n(D_\lambda) = \lambda_n$ for $n=1,2,\cdots$ . Therefore $D_\lambda \in \ell_p(\ell_r, \ell_r)$ and $T \in \ell_p(E,F)$ .

BIBLIOGRAPHY


1. Brown, A., Pearcy, C. and N. Salinas, "Ideals of compact operators on Hilbert spaces," Michigan Math. J. 18(1971), 373-384.
2. Calkin, J. W., "Two-sided ideals and congruences in the ring of bounded operators in Hilbert spaces," Ann. of Math. 42(2)(1941), 839-873.
3. Cohen, J. S., "Absolutely p-summing, p-nuclear operators and their conjugates," Math Ann. 201(1973), 177-200.
4. Dacunha-Castelle, D. and J. L. Krivine, "Application des ultraproduits a l'etude des espaces et des algebres de Banach," Studia Math. 41(1972), 315-334.
5. Dunford, N., and J. Schwarz, Linear operators, I and II, Interscience, New York, 1958-1963.
6. Enflo, P., "A counterexample to the approximation property," Acta Math. 130(3-4)(1973), 309-315.
7. Figiel, T., "Factorization of compact operators and applications to the approximation problem," Studia Math. 45(1973), 191-210.

8. Garling, D. J. H., "On symmetric sequence spaces," Proc. London Math. Soc. 16(3)(1966), 85-106.

9. Garling, D. J. H., "On ideals of operators in Hilbert spaces," Proc. London Math. Soc. 17(3)(1967), 115-138.

10. Garling, D. J. H., "Lattice bounding, radonifying and absolutely summing mappings," (to appear).

11. Gohberg, I. C. and M. G. Kreĭn, Introduction to the Theory of Linear Nonselfadjoint Operators, A.M.S., Trans. of Math. Monographs, (18)(1968).

12. Gordon, Y., Lewis D. R. and J. R. Retherford, "Banach ideals of operators with applications," Journal of Funct. Anal. 14(1)(1973), 85-129.

13. Grothendieck, A., "Produits tensoriels topologiques et espaces nucleaires," Memoirs A.M.S. 16(1955).

14. Grothendieck, A., "Résumé de la théorie métrique des produits tensoriels topologiques," Bol. Soc. Matem. Sao Paulo 8(1956), 1-79.

15. Holub, J. R., "Diagonal nuclear maps in sequence spaces," Math. Ann. 191(1971), 326-332.

16. Hutton, C. V., Morrell, J. S. and J. R. Retherford, "Diagonal operators, approximation numbers and Kolmogoroff diameters," (to appear in Bull. Amer. Math. Soc.).

17. Johnson, W., "Factoring compact operators," Israel J. Math. 9(1971), 337-345.

18. Johnson, W., Rosenthal, H. P., and M. Zippin, "On bases, finite dimensional decompositions and weaker structures in Banach spaces," Israel J. Math. 9(1971), 488-506.

19. Kwapień, S., "On operators factorizable through $L_p$-spaces," Bull. Soc. Math. de France 31-32 (1972), 215-225.

20 Lapreste, J. P., "Operateurs se factorisant par un espace $L_p$ d'apres S. Kwapień," Seminaire Maurey-Schwartz (1972-1973).

21. Lindenstrauss J. and A. Pełczyński,"Absolutely summing operators in $\ell_p$-spaces and their applications," Studia Math. 9(1968), 275-326.

22. Lindenstrauss J. and H. P. Rosenthal, "The $\ell_p$-spaces," Israel J. Math. 7(1969), 325-349.

23. Marcus, A. S., "Some criteria for the completeness of a system of root vectors of a linear operator in a Banach space," Amer. Math. Soc. Transl. 85(2) (1969), 51-91.

24. Mitiagin B. S. and A. Pełczyński, "Nuclear operators and approximative dimension, Int. Math. Congress (1966), 366-375.

25. Miyazaki, K. I., "(p,q;r)-absolutely summing operators," J. Math. Soc. Japan, 24(2)(1972), 341-354.

26. Pełczyński A., "On strictly singular and strictly consingular operators, Parts I, II,"Bull. Acad. Polon. Sér. Math. Astronom. Phys. 13(1965), 31-41.

27. Persson, A. and A. Pietsch, "p-nukleare und p-integrale Abbildungen in Banachräumen," Studia Math. 33(1969), 19-62.

28. Pietsch, A., Nuclear Locally Convex Spaces, Springer-Verlag, Berlin-Heidelberg-New York, 1972. (Transl. of Nukleare Lokalkonvexe Räume, Berlin-Akademie-Verlag, 1965.)

29. Pietsch, A., "$l_p$-factorizable operators in Banach spaces, Acta Sci. Math. (Szeged) 31 (1970), 117-123.

30. Pietsch, A., "Absolut p-summierende Abbildungen in normierten Räumen," Studia Math. 28(1967), 333-353.

31. Pietsch, A., "Adjungierte normierten Operatorenideale," Math. Nach. 48(1971), 189-211.

32. Pietsch, A., Theorie der Operatorenideale, (Zusammenfassung), Jena, 1972.

33. Pietsch, A., "Ultraprodukte von Operatoren in Banachräumen," (to appear).

34. Pietsch, A.,"s-number of operators in Banach spaces," (to appear).

35. Pietsch, A., "The ideals $\ell_p(E,F)$ , $0 < p < \infty$ , are small," (to appear).

36. Saphar, P., "Produits tensoriels d'espaces de Banach et classes d'applications linéaires," Studia Math. 38(1970), 71-100.

37. Saphar, P., "Applications p-décomposantes et p-absolument sommantes," Israel J. Math. 11(1973), 164-179.

38. Schatten, R., Norm Ideals of Completely Continuous Operators, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960.

39. Schatten, R., A Theory of Cross Spaces, Princeton Univ. Press, 1950.

40. Schwarz, U.,"Adjungierte Operatorenideale," Math. Nach. 57(1973), 293-308.

41. Stegall, C. P. and J. R. Retherford, "Fully nuclear and completely nuclear operators with applications to $\mathcal{L}_1$ and $\mathcal{L}_\infty$-spaces," Trans. Amer. Math. Soc. 163(1972), 457-492.

42. Tong, A., "Diagonal nuclear operators on $l_p$-spaces," Trans. Amer. Math. Soc. 143(1969), 235-247.

# VITA

Ming-Shan Hsieh was born in Tainan, Taiwan on June 3, 1945. He received a B.S. degree in mathematics from National Taiwan University in June, 1967. He then entered Louisiana State University in September, 1968, where he received his M.S. degree in August, 1970, and married Ming-Yuan Grace Chang in May, 1972. At present he is a candidate for the degree of Doctor of Philosophy in the Department of Mathematics at Louisiana State Uuniversity, Baton Rouge.