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Integral cohomology of the Siegel modular variety of degree two and level three

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INTEGRAL COHOMOLOGY OF THE SIEGEL MODULAR VARIETY OF DEGREE TWO AND LEVEL THREE

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Abstract

In this thesis work Deligne's spectral sequence $E_r^{p,q}$ with integer coefficients for the embedding of the Siegel modular variety of degree two and level three, $\mathcal{A}_2(3)$, into its Igusa compactification, $\mathcal{A}_2(3)^*$ is investigated. It is shown that $E_3 = E_\infty$ and this information is applied to compute the cohomology groups of $\mathcal{A}_2(3)$ over the integers.

Introduction

For a positive integer n the Siegel upper-half space of degree n, $\mathfrak{S}_n$, is defined to be the space

$$\mathfrak{S}_n = \{\tau \in M_n(\mathbb{C}) |\ {}^t\tau = \tau,\ \operatorname{Im}\tau > 0\}.$$

The symplectic group $\mathrm{Sp}(2n, \mathbb{R})$ acts on $\mathfrak{S}_n$ in the following way:

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} : \tau \mapsto (A\tau + B)(C\tau + D)^{-1}$$

where $\gamma \in \mathrm{Sp}(2n, \mathbb{R})$ and $\tau \in \mathfrak{S}_n$.

A Siegel modular variety $\mathcal{A}_n(\Gamma)$ of degree n is defined as the quotient space $\Gamma\backslash\mathfrak{S}_n$ of the Siegel upper-half space by the action of an arithmetic subgroup Γ of symplectic group $\mathrm{Sp}(2n, \mathbb{Q})$. Of particular interest are those Siegel modular varieties for which the arithmetic group Γ is a principal congruence subgroup $\Gamma_n(m)$, in which case the corresponding variety is called the *Siegel modular variety of degree n and level m*, denoted by $\mathcal{A}_n(m)$. These varieties are important from various perspectives:

1. They naturally occur as the moduli space of principally polarized abelian varieties with level structures.

2. Automorphic forms for the group $\mathrm{Sp}(2n, \mathbb{R})$ and its metaplectic covering typically appear as sections of vector bundles over these spaces.

3. One has

$$H^*(\Gamma\backslash\mathfrak{S}_n, \mathbb{Q}) \cong H^*(\Gamma, \mathbb{Q}).$$

The isomorphism still holds over the integers if the group Γ is torsion-free. Therefore this is a way to compute the cohomology of certain arithmetic subgroups of $\mathrm{Sp}(2n, \mathbb{Q})$.

Although there is a substantial amount of information on the Siegel modular varieties, these spaces are still poorly understood. In fact we know the rational (co)homology of only a few of them. The known cases are the following:

- **degree 1:** These are better known as modular curves (the symplectic group of degree 1 is just the special linear group). The topological properties of these curves is studied in the nineteenth century and the arithmetic of them is a current topic of research.

- **degree 2, levels 1 and 2:** Results are due to R. Lee and S. Weintraub ([10],[11]). They also computed the Hodge numbers for the Igusa compactification of $\mathcal{A}_2(4)$ in [13].

- **degree 2, levels 3 and 4:** J. Hoffman and S. Weintraub computed the rational cohomology of $\mathcal{A}_2(3)$ in [7] and $\mathcal{A}_2(4)$ in [6].

- **degree 3, level 1:** The rational cohomology is computed by R. Hain in [5].

In this work, the main result is the determination of the integral cohomology of $\mathcal{A}_2(3)$. Much of the work in this direction is done in [7] and [8]. The latter paper contains information about integral cohomology of $\mathcal{A}_2(3)$, and in fact the only cases left open by this work is the determination of torsion parts in $H^3(\mathcal{A}_2(3), \mathbb{Z})$ and $H^4(\mathcal{A}_2(3), \mathbb{Z})$. The types of the torsion parts of these groups are also given in [8]. According to this the torsion part of $H^3(\mathcal{A}_2(3), \mathbb{Z})$ may have elements of order divisible by 2 or 3 and the torsion of $H^4(\mathcal{A}_2(3), \mathbb{Z})$ may have elements of order

3 only. Our computations agrees also with the existing results on other integral cohomology groups. We can summarize our main result in this work as follows:

$$H^q(\mathcal{A}_2(3),\mathbb{Z}) = \begin{cases} \mathbb{Z} & q=0; \\ \mathbb{Z}^{21}\oplus\mathbb{Z}/2\oplus(\mathbb{Z}/3)^{10} & q=2; \\ \mathbb{Z}^{139}\oplus(\mathbb{Z}/2)^{15}\oplus(\mathbb{Z}/3)^{35} & q=3; \\ \mathbb{Z}^{81} & q=4; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The method we use to determine the integral cohomology groups is specific to the case $\mathcal{A}_2(3)$. In this special case, Deligne's integral spectral sequence

$$E_2^{p,q} = H^p(D^{[q]},\mathbb{Z}) \;\Rightarrow\; H^{p+q}(\mathcal{A}_2(3),\mathbb{Z})$$

degenerates at E_3, where $D^{[0]} = \mathcal{A}_2(3)^*$ is the Igusa compactification of $A_2(3)$ and $D^{[q]}$ is the disjoint union of q by q intersections of the components of the boundary $\mathcal{A}_2(3)^* - \mathcal{A}_2(3)$. Once we know this fact the computations of cohomology groups of $\mathcal{A}_2(3)$ are fairly easy, because the differentials of this spectral sequence, which are Gysin homomorphisms, can be easily implemented, as matrices of intersection numbers of cycle classes, in a software. One has to find, of course, a set of generators for $H^*(\mathcal{A}_2(3)^*,\mathbb{Z})$, in terms of dual cycles, and has to know the intersection numbers of certain cycle classes to do this. Fortunately this tedious task is carried out by J. Hoffman and S. Weintraub [7]: the cohomology groups $H^p(\mathcal{A}_2(3)^*,\mathbb{Z})$ are free of ranks $1,0,61,0,61,0$ and 1 for $p=0,1,\ldots,6$ ([7, theorem 1.1]) and one can choose a generator sets for H^2 and H^4 using cycle classes of components of the boundary and Humbert surfaces. This is another factor making the computations easier because the incidence geometry of these subvarieties is explained by a combinatorial topology called Tits building with scaffolding.

The article [7] sets as a background for this thesis work and much of the information is taken from it. In the next chapter we go over briefly the basic definitions about Siegel modular varieties and basic facts about $\mathcal{A}_2(3)$. Second chapter includes the proof of the degeneracy of the Deligne's integral spectral sequence and the main result.

Chapter 1

The Siegel Modular Variety of Degree Two and Level Three

The background material for this chapter is [7], [8] [9] and [14]. All of the information related to the Siegel modular variety of degree 2 and level 3 is coming from first two.

1.1 Siegel Modular Varieties

The set of n by n complex symmetric matrices with the positive-definite imaginary part, is called the *Siegel upper-half space* of degree n denoted by $\mathfrak{S}_n$ which is the n-dimensional version of the upper-half plane $\mathfrak{S}_1$:

$$\mathfrak{S}_n = \{\tau \in M_n(\mathbb{C}) |\ {}^t\tau = \tau,\ \operatorname{Im}\tau > 0\}.$$

The real symplectic group of degree n, $\mathrm{Sp}(2n, \mathbb{R})$ consists of all real square matrices X of dimension $2n$ satisfying

$${}^tX \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} X = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

where I_n is the identity matrix of dimension n. This means that if

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

then the square-blocks, A, B, C and D must satisfy the following relations:

$${}^tAC = {}^tCA, \quad {}^tBD = {}^tDB \quad \text{and} \quad {}^tAD - {}^tCB = I_n$$

The group $\mathrm{Sp}(2n, \mathbb{R})$ acts on Siegel upper-half space $\mathfrak{S}_n$ (see [14]), the action is given by

$$X \cdot \tau = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \tau = (A\tau + B)(C\tau + D)^{-1} \tag{1.1}$$

for τ in $\mathfrak{S}_n$ and for $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\mathrm{Sp}(2n, \mathbb{R})$ where A, B, C and D are the n by n portions of X. Actually the more is true:

Proposition 1.1. *1. The action of* $\mathrm{Sp}(n, \mathbb{R})$ *on* $\mathfrak{S}_n$ *is transitive*

2. The group $\mathbf{Aut}(\mathfrak{S}_n)$ *of biholomorphic automorphisms of* $\mathfrak{S}_n$ *is isomorphic to* $\mathrm{Sp}(2n, \mathbb{R})/\pm 1$.

The proof can be found in [14]. Observe that in dimension one we have a more familiar situation namely the action of $\mathrm{SL}(2, \mathbb{R}) = \mathrm{Sp}(2, \mathbb{R})$ on the upper-half plane.

It can be easily proved that the isotropy group

$$\mathbf{Iso}(\sqrt{-1}I_n) := \{M \in \mathrm{Sp}(2n, \mathbb{R}) : M \cdot (\sqrt{-1}I_n) = \sqrt{-1}I_n\},$$

which is a maximal compact subgroup of $\mathrm{Sp}(2n, \mathbb{R})$, is isomorphic to the unitary group

$$U(n) := \{X \in \mathrm{GL}(n, \mathbb{R}) : X\ {}^tX = I_n\}.$$

Therefore $\mathfrak{S}_n$ is a homogeneous space

$$\mathfrak{S}_n \cong \mathrm{Sp}(2n, \mathbb{R})/U(g).$$

Next we consider the quotient space $\Gamma \backslash \mathfrak{S}_n$ of $\mathfrak{S}_n$ by arithmetic subgroups of $\mathrm{Sp}(2n, \mathbb{Q})$. These quotient spaces are, in general, called *Siegel Modular Varieties.* They actually are quasi-projective varieties by a theorem of W. Baily and A. Borel [1].

Definition 1.2. A subgroup Γ of $\mathrm{Sp}(2n, \mathbb{R})$ is called an *arithmetic subgroup* if

(i) Γ is contained in $\mathrm{Sp}(2n, \mathbb{Q})$.

(ii) for a rational faithful representation $\rho : \mathrm{Sp}(2n, \mathbb{Q}) \to \mathrm{GL}(m, \mathbb{Q})$ the image $\rho(\Gamma)$ is commensurable with $\rho(\mathrm{Sp}(2n, \mathbb{Z})$.

Commensurable, here, means that $\rho(\Gamma) \cap \rho(\mathrm{Sp}(2n, \mathbb{Z}))$ has finite index in both $\rho(\Gamma)$ and $\rho(\mathrm{Sp}(2n, \mathbb{Z}))$.

The action of any arithmetic subgroup Γ of $\mathrm{Sp}(2n, \mathbb{Q})$ on $\mathfrak{S}_n$ is properly discontinuous. This means that for all $\tau \in \mathfrak{S}_n$ there exists a neighborhood U of τ in $\mathfrak{S}_n$ such that $\{M \in \Gamma : \ M \cdot U \cap U \neq \emptyset\}$ is a finite set. This is a direct consequence of the following more general fact:

Lemma 1.3. *Let G be a topological group and K be a compact subgroup. Any discrete subgroup Γ of G acts on G/K (with quotient topology) properly discontinuously.*

By a theorem of Cartan [3] we have the following corollary.

Corollary 1.4. *For any arithmetic subgroup Γ of $\mathrm{Sp}(2n, \mathbb{R})$ $\Gamma\backslash\mathfrak{S}_n$ admits a canonical structure of a normal analytic space with the following universal property: a map $f : \Gamma\backslash\mathfrak{S}_n \to X$ into an analytic space X is holomorphic if and only if the composition $f \circ p : \mathfrak{S}_n \to X$ is holomorphic, where p is the projection map.*

If an arithmetic subgroup Γ acts without fixed points, the quotient space turns out to be smooth. Not all arithmetic subgroups of $\mathrm{Sp}(2n, \mathbb{R})$ act fixed point freely on $\mathfrak{S}_n$. However the fact that an arithmetic subgroup Γ has a subgroup Γ' of finite index implies that $\Gamma\backslash\mathfrak{S}_n$ can only have finite quotient singularities.

The *principal congruence subgroups*,

$$\Gamma_n(m) = \{X \in \mathrm{Sp}(2n, \mathbb{Z}) |\ X \equiv I_{2n} \mod m\},$$

act without fixed point for $m \geq 3$ hence the resulting Siegel modular variety, *of degree n and level m*, is smooth as a complex manifold for $m \geq 3$. Although the action of $\Gamma_2(2)$ is not without fixed points the Siegel modular variety of degree 2 and level 2 is still smooth.

The notation $\mathcal{A}_n(m)$ is used to denote the Siegel modular variety of degree n and level m. Our main object of study in this work is $\mathcal{A}_2(3)$ and it is birationally isomorphic to the Burkhardt quadric: the subvariety of the projective space $\mathbb{P}^4\mathbb{C}$ defined by

$$J_4 = Y_0^4 - Y_0(Y_1^3 + Y_2^3 + Y_3^3 + Y_4^3) + 3Y_1Y_2Y_3Y_4. \tag{1.2}$$

The Siegel modular variety of degree n and level m is the moduli spaces of n dimensional complex abelian varieties with level structures.

1.2 Compactification of Siegel Modular Varieties

There are several types of compactifications of Siegel modular varieties. Let Γ be an arithmetic subgroup of $\mathrm{Sp}(2n, \mathbb{R})$.

Satake Compactification $(\boldsymbol{\Gamma\backslash\mathfrak{S}_n})^{sa}$: Satake compactification is the oldest compactification of Siegel modular varieties, constructed by I. Satake. The idea is a generalization of compactification of modular curves, which are also Siegel modular varieties of degree 1. It is proven by Baily and Borel that, with a suitable topology, this is a projective variety. However, unlike the 1-dimensional case it is no longer non-singular and the serious nature of the singularities restricts the usefulness of it by algebraic means.

Borel-Serre Compactification $(\boldsymbol{\Gamma\backslash\mathfrak{S}_n})^{bs}$ Borel-Serre compactification of a Siegel modular variety is a manifold with corners which is obtained by a process called *blowing up the Tits building.* Using this Borel and Serre give a formula for the virtual cohomological dimensions:

$$c = \mathrm{vcd}(\Gamma) = \dim_{\mathbb{R}}(\mathfrak{S}_n) - \mathrm{rank}(\mathrm{Sp}_{2n}) = n^2.$$

as well as the duality theorem, [2, theorem 11.5.1],

$$H^i(\Gamma, \mathbb{Z}) \simeq H_{c-i}(\Gamma, I)$$

where $I = H^c(\Gamma, \mathbb{Z}[\Gamma])$ is the dualizing module of Γ which is isomorphic to $\mathbb{Z}$.

Toroidal Compactification $(\boldsymbol{\Gamma\backslash\mathfrak{S}_n})^*$ First constructed by Igusa on Siegel modular varieties of degree 2 and generalized by Mumford and his coworkers to locally symmetric domains. Toroidal compactification generally depends on a choice of a fan, but in the case of degree 2 Siegel modular varieties there is essentially a unique choice and referred also as *Igusa compactification*. When $\Gamma(m)$ is torsion-free, this is a smooth, projective variety and the boundary

$$\partial\mathcal{A}_2(m) = \mathcal{A}_2(m)^* - \mathcal{A}_2(m)$$

is a divisor with normal crossings. The toroidal compactification $\mathcal{A}_2(m)^*$ may have finite quotient singularities due to the existence of torsion in $\Gamma(m) = \Gamma_2(m) \subset \mathrm{Sp}(4, \mathbb{Z})$. We know that $\Gamma(m)$ is torsion-free if $m \geq 3$ and $\Gamma(2)$ has torsion. However, $\mathcal{A}_2(2)^*$ is still smooth. The next section includes a geometric description of $A_2(3)^*$.

1.3 Important Subvarieties of $\mathcal{A}_2(3)^*$

In cohomology point of view, there are important subvarieties of the Igusa compactification $\mathcal{A}_2(3)^*$. These are the components of the boundary $\partial\mathcal{A}_2(3)^*$, which is a divisor with normal crossings and the Humbert surfaces. Indeed the articles [7] and [8] show that if we know enough about these subvarieties we can determine the (co)homology groups of both $\mathcal{A}_2(3)$ and $\mathcal{A}_2(3)^*$.

Each boundary component M of $\mathcal{A}_2(3)$ is an elliptic modular surface over the modular curve $\Gamma_1(3)\backslash\mathfrak{S}_1$, i.e. there is a surjection $\pi : \mathcal{A}_2(3) \to \Gamma_1(3)\backslash\mathfrak{S}_1$ such that each fiber $\pi^{-1}(p)$ over a smooth point p of $\Gamma_1(3)\backslash\mathfrak{S}_1$ is an elliptic curve. The modular curve $\Gamma_1(3)\backslash\mathfrak{S}_1$ has four cusps and the fibers corresponding to those are triangles of $\mathbb{P}^1$ formed by intersection of M with other boundary components (figure 1.1).

A Humbert surface is a subvariety of $\mathcal{A}_2(m)$ which is the image in $\mathcal{A}_2(m)$ of a subvariety of $\mathfrak{S}_2$ defined by an equation of the form

$$az_1 + a_2z_2 + cz_3 + d(z_2^2 - z_1z_3) + e = 0,$$

where a, b, c, d and e are integers and $\begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix}$. The number $\Delta = b^2 - 4(ac+de)$ is the discriminant of the Humbert surface. From now on we call Humbert surfaces of discriminant 1 simply Humbert surfaces.

There are 40 boundary components, each of which is an elliptic modular surface of level 3, and 45 Humbert surfaces, each of which is isomorphic to $\mathcal{A}_1(3)^* \times \mathcal{A}_1(3)^* \cong \mathbb{P}^1 \times \mathbb{P}^1$.

The intersection configuration of these subvarieties is explained by the finite geometry of $\mathbb{P}^3(\mathbb{F}_3)$ together with the standard symplectic form

$$([x_1, x_2, x_3, x_4], [y_1, y_2, y_3, y_4]) \mapsto x_3y_1 + x_2y_4 - x_3y_1 - x_4y_2.$$

where x_i and y_j are the Plücker coordinates of two points x and y of $\mathbb{P}^3(\mathbb{F}_3)$. Note that whether the symplectic product of two points is zero is well-defined. We say that two points x and y are *isotropic* to each other (or one is isotropic to the other) if their symplectic product is zero and *anisotropic* if not.

Now we briefly describe the correspondence between this geometry and the incidence relations of the special subvarieties. We first look into the space $\mathbb{P}^3(\mathbb{F}_3)$: there are 40 points and 130 lines, each of which containing 4 points. For two points l_1 and l_2 in $\mathbb{P}^3(\mathbb{F}_3)$, the line passing through l_1 and l_2 is of the form $l_1 \wedge l_2 = \{l_1, l_2, l_1 + l_2, l_1 - l_2\}$. There are 40 isotropic lines i.e. lines whose points are pairwise isotropic, and 90 anisotropic ones. For each anisotropic line δ there is a unique anisotropic line $\delta^\perp$ such that points of δ are isotropic to those of $\delta^\perp$. We will call them *anisotropic pairs*. The correspondence is as follows:

- The points l of the projective space $\mathbb{P}^3(\mathbb{F}_3)$ index the boundary components $D(l)$ of $\mathcal{A}_2(3)^*$ and the Humbert surfaces $H(\Delta)$ are indexed by sets $\Delta = \{\delta, \delta^\perp\}$ (this Δ should not be confused with the discriminant) consisting of anisotropic pairs.

- Two boundary components $D(l_1)$ and $D(l_2)$ intersect in a subvariety isomorphic to complex projective line $\mathbb{P}^1$ if and only if the points l_1 and l_2 of $\mathbb{P}^3(\mathbb{F}_3)$ are isotropic i.e. their symplectic product is zero. The intersection of components $D(l_i)$, $i = 1, \dots q$ will be denoted by $D(l_1, \dots, l_q)$.

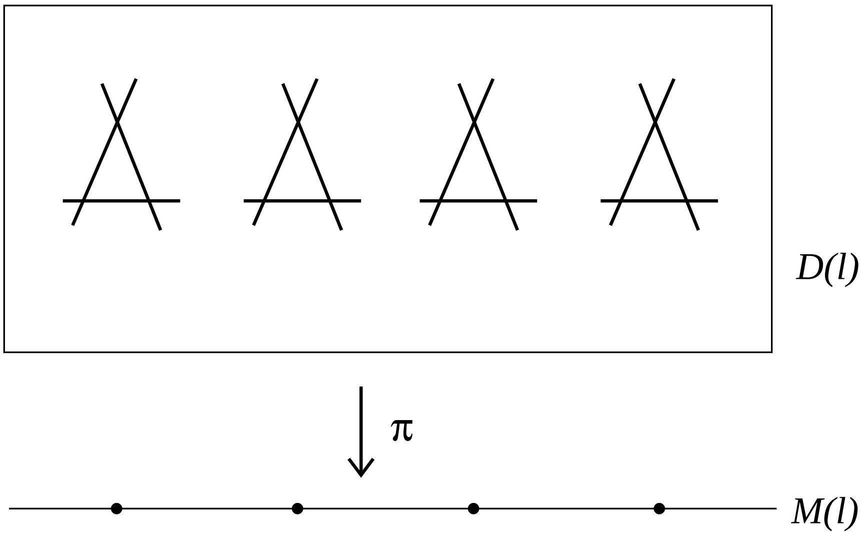

FIGURE 1.1. A boundary component

The figure 1.1 shows the incidence relations mentioned above on a boundary component $D(l)$. Each line is the intersection of $D(l)$ with another boundary component whose index l' is isotropic to l. As we mentioned earlier, each boundary component $D(l)$ is an elliptic surface over a modular curve $M(l)$, i.e. there is a map $\pi : D(l) \to M(l)$ whose generic fibres are elliptic curves and the fibres over the cusps of $M(l)$ are the triangles in $D(l)$.

- The intersection of any four of $D(l)$'s is empty. Therefore the boundary components corresponding to points l_1, l_2, l_3 and l_4 of an isotropic line h form a tetrahedron $C(h)$ with $\mathbb{P}^1$ edges (figure 1.2).

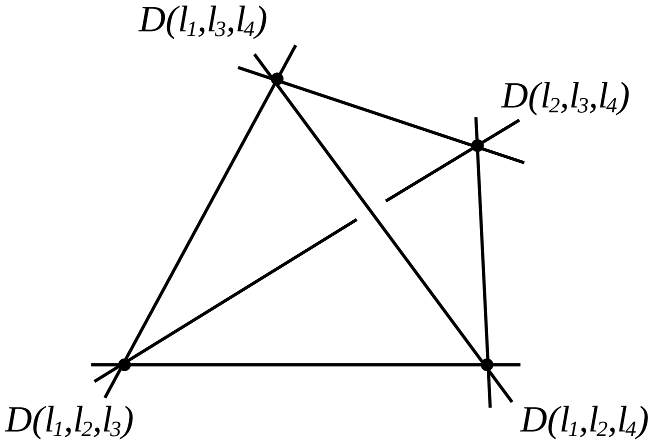

FIGURE 1.2. $C(h)$

- A Humbert surface $H(\Delta)$, $\Delta = \{\delta, \delta^{\perp}\}$, meets a boundary component $D(l)$, $l \in \mathbb{P}^3(\mathbb{F}_3)$ if and only if $l \in \delta$ or $l \in \delta^{\perp}$ and in this case we denote the intersection $D(l) \cap H(\Delta)$ by $S(l, \Delta)$ (figures 1.3 and 1.4).

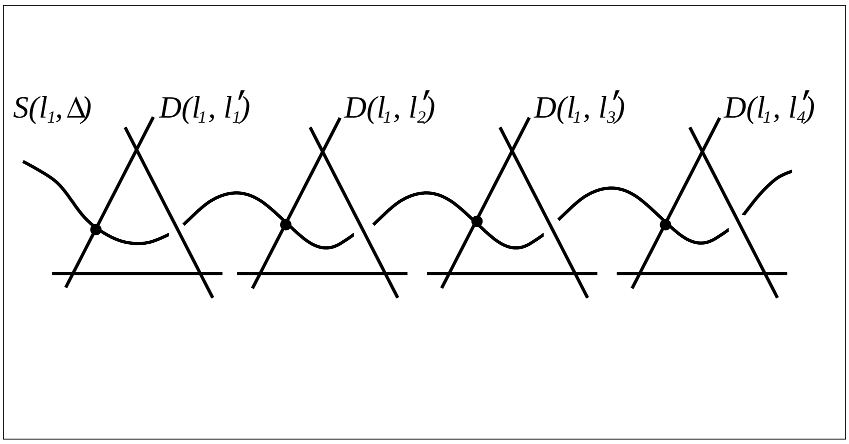

FIGURE 1.3. $S(l_1, \Delta)$ in $D(l_1)$

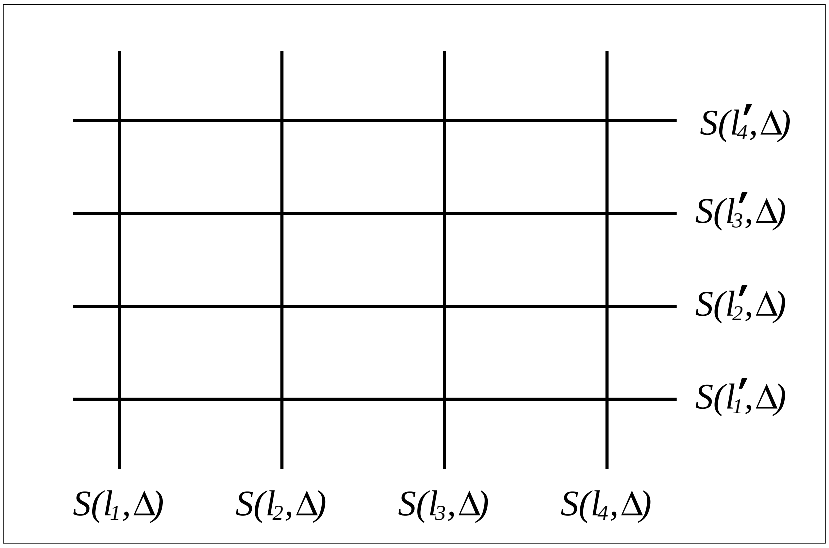

FIGURE 1.4. $H(\Delta)$, $\delta = \{l_1, l_2, l_3, l_4\}$ and $\delta^\perp = \{l'_1, l'_2, l'_3, l'_4\}$

- For each $l \in \mathbb{P}^3(\mathbb{F}_3)$ there are 12 other points of $\mathbb{P}^3(\mathbb{F}_3)$ which are isotropic to l and 27, anisotropic to l. Hence there are 4 isotropic lines and 9 anisotropic lines containing l. So there are 9 $S(l,\Delta)$'s contained in $D(l)$.

Some of the results of [7] which are used in order to make explicit computation of the Deligne's spectral sequence are the following:

Theorem 1.5. *1. The cohomology groups $H^j(\mathcal{A}_2(3)^*, \mathbb{Z})$ of are given by*

$$H^j(\mathcal{A}_2(3)^*, \mathbb{Z}) = \begin{cases} \mathbb{Z} & j = 0, 6, \\ \mathbb{Z}^{61} & j = 2, 4, \\ 0 & \textit{otherwise} \end{cases}$$

2. The 85 classes $[H(\Delta)]$ and $[D(l)]$ generate $H_4(\mathcal{A}_2(3)^*, \mathbb{Z})$.

3. The 130 classes $\{h_1(\Delta), h_2(\Delta)\}$ and $d(l)$ generate $H_2(\mathcal{A}_2(3)^*, \mathbb{Z})$ where $\{h_1(\Delta), h_2(\Delta)\} = \{S(l,\Delta), S(l',\Delta)\}$ and

$$[d(l)] = [D(l,l')] + \sum_i [S(l,\Delta_i)]$$

The intersection numbers of some important cycle classes are given in the following theorem which is a restatement of lemmas 3.5-3.10 of [7].

Theorem 1.6.

$$D(l_1)\cdot D(l_1,l_2) = \begin{cases} 1 & \text{if } l, l_1, l_2 \text{are pairwise isotropic} \\ -2 & \text{if } l = l_1 \text{ or } l = l_2 \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

$$H(\Delta)\cdot D(l_1,l_2) = \begin{cases} 1 & \text{if } l_1 \in \delta \text{ and } l_2 \in \delta^\perp \text{ or vice-versa} \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

$$D(l')\cdot S(l,\Delta) = \begin{cases} 1 & \text{if } l \in \delta \text{ and } l' \in \delta^\perp \text{ or vice-versa} \\ 0 & \text{otherwise} \end{cases} \tag{1.5}$$

$$H(\Delta')\cdot S(l,\Delta) = \begin{cases} -1 & \text{if } \Delta = \Delta' \\ 0 & \text{otherwise} \end{cases} \tag{1.6}$$

$$\tag{1.7}$$

1.3.1 The Cohomology of Boundary Components

We denote by $D^{[q]}$ the disjoint union of q by q intersections of boundary components. Let $D(l)$ be a boundary component. By [8, corollary 2.2] we know that

$$H_p(D(l),\mathbb{Z}) = H^p(D(l),\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } p = 0,4, \\ \mathbb{Z}^{10} & \text{if } p = 2, \\ 0 & \text{otherwise} \end{cases} \tag{1.8}$$

The second homology group $H_2(D(l),\mathbb{Z})$ is generated freely by 9 cycle classes $[S(l,\Delta)]$ and $[d(l)] = [D(l,l')] + \sum_i [S(l,\Delta_i)]$.

Each $D(l,l')$ is isomorphic to $\mathbb{P}^1$ unless it is empty and $H^j(\mathbb{P}^1,\mathbb{Z}) = \mathbb{Z}$ for $j = 0$ or 2 and 0 for other values of j. Therefore $H^j(D^{[2]},\mathbb{Z})$ for $j = 0$ or 2 is free of rank equal to the number of isotropic pairs (l,l') of distinct points in $\mathbb{P}^3(\mathbb{F}_3)$. There are

240 of them so we have

$$H^j(D^{[2]},\mathbb{Z}) = \begin{cases} \mathbb{Z}^{240} & j = 0, 2, \\ 0 & \text{otherwise} \end{cases}$$

Similarly since there are 160 pairwise isotropic triples (l, l', l'') of points of $\mathbb{P}^3(\mathbb{F}_3)$, $D^{[3]}$ is the union of 160 distinct points and hence $H^0(\mathbb{D}^{[3]},\mathbb{Z}) = \mathbb{Z}^{160}$.

Chapter 2
The Integral Cohomology of $\mathcal{A}_2(3)$

2.1 The Rational Cohomology of $\mathcal{A}_2(3)$

The rational cohomology groups of the Siegel modular variety $\mathcal{A}_2(3)$ is computed by J. Hoffman and S. Weintraub. The ranks of the cohomology groups are (as found in [7, p.4])

$$\mathrm{rank} H^i(\mathcal{A}_2(3), \mathbb{Q}) = \begin{cases} 1 & i = 0, \\ 21 & i = 2, \\ 139 & i = 3, \\ 81 & i = 4, \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

which are also the ranks of the cohomology of the principal congruence subgroup $\Gamma_2(3)$. The same authors have also obtained the following partial result related to the cohomology of $\mathcal{A}_2(3)$ with integer coefficients ([8, p.35]):

$$\begin{aligned} H^0(\mathcal{A}_2(3), \mathbb{Z}) &= \mathbb{Z} \\ H^2(\mathcal{A}_2(3), \mathbb{Z}) &= \mathbb{Z}^{21} \oplus \mathbb{Z}/2 \oplus (\mathbb{Z}/3)^{10} \\ H^3(\mathcal{A}_2(3), \mathbb{Z}[1/6]) &= \mathbb{Z}[1/6]^{139} \\ H^4(\mathcal{A}_2(3), \mathbb{Z}[1/3]) &= \mathbb{Z}[1/3]^{81} \end{aligned} \tag{2.2}$$

This means that the integral cohomology of $\mathcal{A}_2(3)$ may have $\mathbb{Z}/3$ torsion at dimension 4 and $\mathbb{Z}/2$ or $\mathbb{Z}/3$ torsion at dimension 3. We will see in the next section that our computations of the integral cohomology agree with these results.

Now we briefly explain how the ranks (2.1) are computed in [7]. The first step is to compute the ranks of $H^i(\mathcal{A}_2(3)^*, \mathbb{Q})$, by finding the zeta function of a variety B

obtained by resolving the singularities of the Burkhardt's quadric (1.2): for every prime power q congruent to 1 modulo 3 the zeta function of B regarded as a scheme over $\mathbb{F}_q$ is

$$Z(B/\mathbb{F}_q, u) = \frac{1}{(1-u)(1-qu)^{61}(1-q^2u)^{61}(1-q^3u)} \tag{2.3}$$

By comparison theorems in étale cohomology this gives the integral cohomology groups of $\mathcal{A}_2(3)^*$, (i.e. they all are free of ranks 1, 0, 61, 0, 61, 0, 1, for dimensions $1, \ldots, 6$. The computation of the zeta function is a very tedious task carried out in [7].

To compute the rational cohomology of $\mathcal{A}_2(3) \subset \mathcal{A}_2(3)^*$, Deligne's spectral sequence is used:

$$E_2^{p,q} = H^p(D^{[q]}, \mathbb{Q}) \;\Rightarrow\; H^*(\mathcal{A}_2(3), \mathbb{Q}) \tag{2.4}$$

where $D^{[0]} = \mathcal{A}_2(3)^*$ and $D^{[q]}$ is the disjoint union of the intersections of q boundary components for $q \geq 1$. The computation using the Deligne's spectral sequence involves determining the set of generators for all cohomology groups of $\mathcal{A}_2(3)^*$ by means of cycles of $\mathcal{A}_2(3)$ and the computation of intersection numbers of these cycles, which is a work done in [7].

Deligne's spectral sequence (2.4) is the Leray's spectral sequence for an inclusion $X \to \overline{X}$ where X is a smooth variety embedded in a smooth complete variety $\overline{X}$ as a Zariski open dense subset and $\partial X = \overline{X} - X$ is a divisor with normal crossings and it degenerates at E_3 modulo torsion. We will show, in the next chapter that, for our special case, the Deligne's integral spectral sequence

$$E_2^{p,q} = H^p(D^{[q]}, \mathbb{Z}) \;\Rightarrow\; H^*(\mathcal{A}_2(3), \mathbb{Z})$$

degenerates at E_3 too, and this establishes our main result.

2.2 Deligne's Spectral Sequence over Integers

The following theorem establishes the main idea of computations of the integral cohomology of $\mathcal{A}_2(3)$. As in the last section we denote by $D^{[q]}$ the disjoint union of intersections of q of the boundary components of $\mathcal{A}_2(3)^*$.

Theorem 2.1. *The Deligne's spectral sequence with integer sequence*

$$E_2^{p,q} = H^p(D^{[q]}, \mathbb{Z}) \ \Rightarrow H^{p+q}(\mathcal{A}_2(3), \mathbb{Z}) \tag{2.5}$$

degenerates at level three, i.e. $E_3 = E_\infty$.

All $E_2^{p,q}$ of the spectral sequence (2.5) are free and in the following chart all the nontrivial maps ranks and (p,q) coordinates are placed.

$$\begin{tikzcd}
\begin{matrix}(0,3)\\ \mathbf{160}\end{matrix} \arrow[drr] & & & & & & \\
\begin{matrix}(0,2)\\ \mathbf{240}\end{matrix} \arrow[drr] & \begin{matrix}(1,2)\\ \mathbf{0}\end{matrix} & \begin{matrix}(2,2)\\ \mathbf{240}\end{matrix} \arrow[drr] & & & & \\
\begin{matrix}(0,1)\\ \mathbf{40}\end{matrix} \arrow[drr] & \begin{matrix}(1,1)\\ \mathbf{0}\end{matrix} & \begin{matrix}(2,1)\\ \mathbf{400}\end{matrix} \arrow[drr] & \begin{matrix}(3,1)\\ \mathbf{0}\end{matrix} & \begin{matrix}(4,1)\\ \mathbf{40}\end{matrix} \arrow[drr] & & \\
\begin{matrix}(0,0)\\ \mathbf{1}\end{matrix} & \begin{matrix}(1,0)\\ \mathbf{0}\end{matrix} & \begin{matrix}(2,0)\\ \mathbf{61}\end{matrix} & \begin{matrix}(3,0)\\ \mathbf{0}\end{matrix} & \begin{matrix}(4,0)\\ \mathbf{61}\end{matrix} & \begin{matrix}(5,0)\\ \mathbf{0}\end{matrix} & \begin{matrix}(6,0)\\ \mathbf{1}\end{matrix}
\end{tikzcd}$$

The differentials of the spectral sequence

$$d : E_2^{p,q} = H^p(D^{[q]}, \mathbb{Z}) \to E_2^{p+2,q-1} = H^{p+2}(D^{[q-1]}, \mathbb{Z})$$

is a direct sum $\sum (-1)^i d_{i*}$ of Gysin homomorphisms d_{i*} associated to inclusions

$$d_i : D(l_1, \ldots, l_q) \to D(l_1, \ldots, \hat{l}_i, \ldots, l_q).$$

The Gysin homomorphisms are easy to understand: if x is a cohomology class corresponding to an algebraic cycle of codimension r then $d_i(x)$ is the class of the same cycle of codimension $r+1$ on $D(l_1, \ldots, \hat{l}_i, \ldots, l_q)$. As it is obvious from the definition of the differentials, we have to fix an order "$<$" once and for all for the points of $\mathbb{P}^3(\mathbb{F}_3)$

From the chart above, we see that there are only three complexes to be considered. The first one consists of only one nontrivial map

$$d : E_2^{0,1} \to E_2^{2,0}$$

which, by the remarks following the proof of [7, theorem 4.6], is injective and the image can be completed to a basis of $E_2^{2,0} = H^2(\mathcal{A}_2(3)^*, \mathbb{Z})$. The other complexes are labeled $S^\bullet$ and $T^\bullet$ in [7, p.34]. Here we keep the same notation are write $d_S^\bullet$ and $d_T^\bullet$ for the differentials:

$$S^\bullet: \ 0 \to E_2^{0,2} \xrightarrow{d_S^1} E_2^{2,1} \xrightarrow{d_S^2} E_2^{4,0} \to 0 \tag{2.6}$$

$$T^\bullet: \ 0 \to E_2^{0,3} \xrightarrow{d_T^1} E_2^{2,2} \xrightarrow{d_T^2} E_2^{4,1} \xrightarrow{d_T^3} E_2^{6,0} \to 0. \tag{2.7}$$

We have canonical free bases for $E_2^{p,q}$ for $(p,q) \neq (4,0)$ or $(2,0)$. Since all $E_2^{p,q}$ are free, by universal coefficient theorem

$$E_2^{p,q} = H^p(D^{[q]}, \mathbb{Z}) \simeq \mathrm{Hom}(H_p(D^{[q]}, \mathbb{Z}), \mathbb{Z}).$$

Via this isomorphism all of the maps above can be interpreted as matrices of intersection of cycle classes. Therefore the matrix representation of the maps d_S^1, d_T^1, d_T^2 and d_T^3 can be constructed easily: the entries are the intersection numbers of the cycles indexing rows and columns and the following is how we label rows and columns.

- $\boldsymbol{d_S^1}$: the columns are indexed by $D(l, l')$, (l, l') run over all isotropic pairs, $l < l'$ and the rows are separated into 40 groups of 10. These groups are indexed by $l \in \mathbb{P}^3(\mathbb{F}_3)$. The first one of rows corresponding to an l is labeled $d(l)$ and the rest 9 are indexed by 9 cycles $S(l, \Delta)$ where if $\Delta = \{\delta, \delta^\perp\}$ then $l \in \delta \cup \delta^\perp$.

$\boldsymbol{d_T^1}$: the columns are indexed by $D(l_1, l_2, l_3)$ where l_1, l_2 and l_3 are pairwise isotropic, $l_1 < l_2 < l_3$, the rows are indexed by $D(l, l')$, (l, l') runs over all isotropic pairs, $l < l'$.

$\boldsymbol{d_T^2}$: the columns are indexed by $D(l, l')$, (l, l') runs over all isotropic pairs, $l < l'$ and rows are indexed by $D(l)$'s.

$\boldsymbol{d_T^3}$: a row matrix with 1 in each entry.

The construction of d_S^2 is not different from the others, the only difference is that we don't have a canonical basis for $E_2^{4,0} = H^4(\mathcal{A}_2(3)^*, \mathbb{Z})$. This causes no trouble at all, we construct the matrix representation by indexing the columns with as the rows of the representation matrix of d_S^2 and the rows with 40 $D(l)$'s and 45 $H(\Delta)$'s.

Remark 2.2. Note that we now represent d_S^2 by a 85×400 matrix whose rank is 61. Moreover the Smith normal form of this matrix has 61 1's on the diagonal showing that d_S^2 is surjective.

Now we will prove theorem 2.1 by investigating each piece of the spectral sequence (2.5).

Proof of Theorem 2.1. Since we know that the $H^p(\mathcal{A}_2(3), \mathbb{Z}) = 0$ for $p \geq 5$ we actually need to verify

$$E_3^{p,q} = E_\infty^{p,q} \tag{2.8}$$

only when $p + q \leq 4$, therefore there are only 15 cases to check. However, to see that (2.8) holds for the rest of the spectral sequence is elementary. It is a consequence of biregularity of the spectral sequence (2.5) that $E_3^{p,q} = E_\infty^{p,q}$ for $(p, q) = (0,0), (1,0), (0,1), (1,1), (2,0)$ and $(2,1)$. On the other hand $E_2^{p,q} = 0$

for $(p,q) = (1,*), (3,0), (3,1)$ hence (2.8) holds for these values of p and q. The remaining part that needs to be checked is when $(p,q) = (0,2), (0,3), (0,4), (2,2)$ and $(4,0)$. Since $E_3^{3,0} = H^3(D^{[0]}, \mathbb{Z}) = 0$, we have

$$\begin{tikzcd}
0 \arrow[r] & E_3^{0,2} \arrow[r] \arrow[d, equal] & E_3^{3,0} = 0 \\
0 \arrow[r] & E_4^{0,2} \arrow[r] & 0
\end{tikzcd}$$

so $E_3^{0,2} = E_\infty^{0,2}$. Similarly we have

$$\begin{tikzcd}
0 \arrow[r] & E_3^{2,2} \arrow[r] \arrow[d, equal] & E_3^{5,0} = \text{a subquotient of } E_2^{5,0} = 0 \\
0 \arrow[r] & E_4^{2,2} \arrow[r] & 0
\end{tikzcd}$$

therefore $E_3^{2,2} = E_\infty^{2,2}$. To prove $E_3^{4,0} = 0$ (therefore $E_3^{4,0} = E_\infty^{4,0}$) we use the fact that the map

$$\begin{tikzcd}
E_2^{2,1} \arrow[r, "d_S^2"] \arrow[d, equal] & E_2^{4,0} \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{Z}^{400} & \mathbb{Z}^{61} &
\end{tikzcd} \tag{2.9}$$

is surjective. This is seen by computing the smith normal form of the matrix representation of d_S^2 using `Maple` or any other software with matrix computation capabilities (see the Remark 2.2 above). Because of this we have, by the following diagram,

$$\begin{tikzcd}
0 \arrow[r] & E_3^{0,3} \arrow[r] \arrow[d, equal] & E_3^{3,1} = 0 \\
0 \arrow[r] & E_4^{0,3} \arrow[r] \arrow[d, equal] & E_4^{4,0} = 0 \\
0 \arrow[r] & E_5^{0,3} \arrow[r] & 0
\end{tikzcd}$$

$E_3^{0,3} = E_\infty^{0,3}$. It remains to prove that $E_3^{0,4} = E_\infty^{0,4}$. Since $E_4^{4,1}$ is a subquotient of $E_3^{4,1} = 0$ it is 0 itself. Similarly $E_5^{5,0} = 0$ because $E_2^{5,0} = 0$. So from the diagram

$$\begin{tikzcd}
0 \arrow[r] & E_3^{0,4} \arrow[r] \arrow[d, equal] & E_3^{3,2} = 0 \\
0 \arrow[r] & E_4^{0,4} \arrow[r] \arrow[d, equal] & E_4^{4,1} = 0 \\
0 \arrow[r] & E_5^{0,4} \arrow[r] \arrow[d, equal] & E_5^{5,0} \\
0 \arrow[r] & E_6^{0,4} \arrow[r] & 0
\end{tikzcd}$$

we see that $E_3^{0,4} = E_\infty^{0,4}$ and this finishes the proof of the theorem. $\square$

2.3 The Main Result

In this section we simply write H^p instead of $H^p(\mathcal{A}_2(3), \mathbb{Z})$.

To determine the cohomology groups of $\mathcal{A}_2(3)$ with integer coefficients, we compute the groups $E_3^{p,q}$. They give the Gr^W_{p+2q} of, so called, *weight filtration* W of H^{p+q}. More precisely we have

$$E_3^{p,q} = E_\infty^{p,q} = \mathrm{Gr}^W_{p+2q} H^{p+q}.$$

The fact that the weight filtration is increasing is important, because of this fact we can completely determine the torsion parts.

Now suppose we have a sequence of maps

$$\mathbb{Z}^r \xrightarrow{P} \mathbb{Z}^s \xrightarrow{Q} \mathbb{Z}^t \tag{2.10}$$

given by means of matrices P and Q such that $QP = 0$. To find the homology group $H = \ker Q / \mathrm{Im}\, P$, we find the smith normal form of Q, i.e. we find matrices $U \in \mathrm{GL}(s, \mathbb{Z})$ and $V \in \mathrm{GL}(t, \mathbb{Z})$ such that the matrix VQU has all nonzero entries

$e_1, e_2, \ldots, e_u$ on the diagonal and the entries satisfy $e_1|e_2|\cdots|e_u$.

$$\begin{tikzcd}
\mathbb{Z}^r \arrow[r, "P"] \arrow[dr, "U^{-1}P"'] & \mathbb{Z}^s \arrow[r, "Q"] & \mathbb{Z}^t \arrow[d, "V"] \\
& \mathbb{Z}^s \arrow[u, "U"] \arrow[r, "VQU"] & \mathbb{Z}^t
\end{tikzcd}$$

The homology of sequences

$$\mathbb{Z}^r \xrightarrow{P} \mathbb{Z}^s \xrightarrow{Q} \mathbb{Z}^t \quad \text{and} \quad \mathbb{Z}^r \xrightarrow{U^{-1}P} \mathbb{Z}^s \xrightarrow{VQU} \mathbb{Z}^t$$

at the middle are isomorphic and we find the of the kernel: $\ker Q \simeq \ker(VQU) = \mathbb{Z}^{s-u}$. So it remains to find the cokernel of the map

$$U^{-1}P : \mathbb{Z}^r \to \ker(VQU) = \mathbb{Z}^{s-u}$$

which can be done by computing the Smith normal form of $U^{-1}P$: if the nonzero entries of the Smith normal form of this matrix are $d_1, \ldots, d_v$ with $d_1|d_2|\cdots|d_v$, which appear as diagonal entries, then $v \leq u$ and

$$\mathrm{Coker}(U^{-1}P : \mathbb{Z}^r \to \mathbb{Z}^{s-u}) = \mathbb{Z}^{u-v} \oplus \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_v.$$

It is clear that we can ignore U because it is invertible and therefore P and U^{-1} have the same Smith normal form.

To compute E_3 components of the spectral sequence we carry out the same computations in `Maple` for the matrix representations of $d_S^\bullet$ and $d_T^\bullet$. We obtain the following:

$$\mathrm{Gr}_i^W H^2 \simeq \begin{cases} \mathbb{Z}^{21} \oplus \mathbb{Z}/2 \oplus (\mathbb{Z}/3)^{10} & i = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Gr}_i^W H^3 \simeq \begin{cases} \mathbb{Z}^{99} \oplus (\mathbb{Z}/3)^{20} \oplus (\mathbb{Z}/6)^{15} & i = 4 \\ \mathbb{Z}^{40} & i = 6 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{Gr}_i^W H^4 \simeq \begin{cases} \mathbb{Z}^{81} & i = 6 \\ 0 & \text{otherwise} \end{cases}$$

At this point we can see the ranks of and sizes of the torsion parts of the cohomology groups. One elementary but very important detail needs to be explained here. Since the groups Gr_i^W are merely the successive quotients of the weight filtration, one cannot tell, in general, the cohomology groups explicitly from a data like the one above. In our case, on the other hand, we are able to do it: since the weight filtration of cohomology groups are biregular and $W(H^2)$ and $W(H^4)$ are just one-step filtrations, we have

$$W^i(H^2) = \begin{cases} \mathbb{Z}^{21} \oplus \mathbb{Z}/2 \oplus (\mathbb{Z}/3)^{10} & i \geq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$W^i(H^4) = \begin{cases} \mathbb{Z}^{81} & i \geq 6 \\ 0 & \text{otherwise} \end{cases}$$

this means that $H^2 = W^2(H^2)$ and $H^4 = W^6(H^4)$.

For the weight filtration H^3 we have the following situation:

$$W^5 \simeq W^4 \simeq \mathbb{Z}^{99} \oplus (\mathbb{Z}/2)^{15} \oplus (\mathbb{Z}/3)^{35} \text{ and } W^6/W^5 \simeq \mathbb{Z}^{40}.$$

Hence $H^3 = W^6(H^3)$ can be written as an extension of groups

$$0 \to \mathbb{Z}^{99} \oplus (\mathbb{Z}/2)^{15} \oplus (\mathbb{Z}/3)^{35} \longrightarrow H^3 \longrightarrow \mathbb{Z}^{40} \to 0. \tag{2.11}$$

Since $\mathbb{Z}^{40}$ is a free, the sequence (2.11) splits. This means that

$$H^3 \simeq \mathbb{Z}^{99} \oplus (\mathbb{Z}/2)^{15} \oplus (\mathbb{Z}/3)^{35} \oplus \mathbb{Z}^{40}.$$

In summary we get the integral cohomology groups of $\mathcal{A}_2(3)$ explicitly,

$$H^q(\mathcal{A}_2(3), \mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0; \\ \mathbb{Z}^{21} \oplus \mathbb{Z}/2 \oplus (\mathbb{Z}/3)^{10} & q = 2; \\ \mathbb{Z}^{139} \oplus (\mathbb{Z}/2)^{15} \oplus (\mathbb{Z}/3)^{35} & q = 3; \\ \mathbb{Z}^{81} & q = 4; \\ 0 & \text{otherwise.} \end{cases}$$

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Appendix
The Maple Code

The following is the `Maple` code of the computations. The code is supposed to run with `Maple V` release 4 compiler. The little change in the code will enable it to run with more recent versions of `Maple`. In the code, the grammatical rules of programming are not strictly followed although certain conventions are adopted, e.g. the names of processes start with an underscore character.

```
with(linalg):                                # necessary to use linear algebra package
```

The (Plücker) Coordinates of Points in $\mathbb{P}^3\mathbb{F}_3$

```
L := array(1..40,1..4):                      # the coordinate matrix to be filled:.
x1 := 0:                                     #   each row is coordinates of a point.
x2 := 0:                                     #   this matrix provides an order amongst the points
x3 := 0:
x4 := 0:
for i from 1 to 40 do
   if x4 <> 1 then
      x4 := x4 + 1;
   elif x3 <> 1 then
      x4 := -1;
      x3 := x3 + 1;
   elif x2 <> 1 then
      x4 := -1;
      x3 := -1;
      x2 := x2 + 1;
   else
      x4 := -1;
      x3 := -1;
      x2 := -1;
      x1 := x1 + 1;
   fi;
   L[i,1] := x1;
   L[i,2] := x2;
   L[i,3] := x3;
   L[i,4] := x4;
od:

_Lplus := proc(i,j)
   local k;
   k := 1;
   while (k <= 40) do
      if ( L[i,1] + L[j,1] - L[k,1] mod 3 = 0 and
      L[i,2] + L[j,2] - L[k,2] mod 3 = 0 and
      L[i,3] + L[j,3] - L[k,3] mod 3 = 0 and
      L[i,4] + L[j,4] - L[k,4] mod 3 = 0 ) or
      ( -L[i,1] - L[j,1] - L[k,1] mod 3 = 0 and
```

```
      -L[i,2] - L[j,2] - L[k,2] mod 3 = 0 and
      -L[i,3] - L[j,3] - L[k,3] mod 3 = 0 and
      -L[i,4] - L[j,4] - L[k,4] mod 3 = 0 ) then
         RETURN(k);
         k := 41;
      else
         k := k + 1;
      fi;
   od;
end:

_Lminus := proc(i,j)
   local k;
   k := 1;
   while (k <= 40) do
      if ( L[i,1] - L[j,1] - L[k,1] mod 3 = 0 and
      L[i,2] - L[j,2] - L[k,2] mod 3 = 0 and
      L[i,3] - L[j,3] - L[k,3] mod 3 = 0 and
      L[i,4] - L[j,4] - L[k,4] mod 3 = 0 ) or
      ( -L[i,1] + L[j,1] - L[k,1] mod 3 = 0 and
      -L[i,2] + L[j,2] - L[k,2] mod 3 = 0 and
      -L[i,3] + L[j,3] - L[k,3] mod 3 = 0 and
      -L[i,4] + L[j,4] - L[k,4] mod 3 = 0 ) then
         RETURN(k);
         k := 41;
      else
         k := k + 1;
      fi;
   od;
end:

_Sprod := proc(i,j)
   RETURN( L[i,3]*L[j,1] + L[i,4]*L[j,2]
   - L[i,1]*L[j,3] - L[i,2]*L[j,4] mod 3);
end:
```

Lines in $\mathbb{P}^3\mathbb{F}_3$

```
V := array(1..130,1..4):
r := 1:
for i from 1 to 37 do
   for j from i+1 to 38 do
      if _Lplus(i,j) > j and _Lminus(i,j) > j then
         V[r,1] := i;
         V[r,2] := j;
         if _Lplus(i,j) < _Lminus(i,j) then
            V[r,3] := _Lplus(i,j);
            V[r,4] := _Lminus(i,j);
         else
            V[r,3] := _Lminus(i,j);
            V[r,4] := _Lplus(i,j);
         fi;
         r := r + 1;
      fi;
   od:
od:

_IsBelongTo := proc(point_num, line_num)
   if V[line_num,1] = point_num or
   V[line_num,2] = point_num or
   V[line_num,3] = point_num or
   V[line_num,4] = point_num then
      RETURN(1);
```

```
   else
      RETURN(0);
   fi;
end:

Isotropic := array(1..40):
r := 1:
for i from 1 to 130 do
   if _Sprod(V[i,1],V[i,2]) = 0 then
      Isotropic[r] := i;
      r := r + 1;
   fi;
od:

Unisotropic := array(1..90):
UnisotropicPairs := array(1..45,1..2):
r := 1:
s := 1:
for i from 1 to 130 do
   if i <> Isotropic[s] then
      Unisotropic[r] := i;
      r := r + 1;
   else
      if s < 40 then
         s := s + 1;
      fi;
   fi;
od;

r := 1:
for i from 1 to 89 do
   for j from i+1 to 90 do
      if ( _Sprod(V[Unisotropic[i],1],V[Unisotropic[j],1]) = 0 and
      _Sprod(V[Unisotropic[i],1],V[Unisotropic[j],2]) = 0 and
      _Sprod(V[Unisotropic[i],2],V[Unisotropic[j],1]) = 0 and
      _Sprod(V[Unisotropic[i],2],V[Unisotropic[j],2]) = 0)then
         UnisotropicPairs[r,1] := Unisotropic[i];
         UnisotropicPairs[r,2] := Unisotropic[j];
         r := r + 1;
      fi;
   od;
od;
```

The Sequence U

```
dU:=array(1..130,1..40):
for i from 1 to 45 do
   for j from 1 to 40 do
      if _IsBelongTo(j,UnisotropicPairs[i,1]) = 1 then
         dU[2*i-1,j] := 0;
         dU[2*i,j] := 1;
      elif _IsBelongTo(j,UnisotropicPairs[i,2]) = 1 then
         dU[2*i-1,j] := 1;
         dU[2*i,j] := 0;
      else
         dU[2*i-1,j] := 0;
         dU[2*i,j] := 0;
      fi;
   od:
od:
for i from 1 to 40 do
   for j from 1 to 40 do
      if i = j then
```

```
         dU[90+i,j] := -2;
      elif _Sprod(i,j) = 0 then
         dU[90+i,j] := 1;
      else
         dU[90+i,j] := 0;
      fi;
   od;
od:

idU := ismith(dU):

diagonal := 1:                                    # to count the entries of the diagonal
count := 0:
for i from 1 to 40 do
   if diagonal = idU[i,i] then
      count := count + 1;
   else
      printf(`the number of %d on the diagonal is %d\ n`,
      diag,count);
      diagonal := idU[i,i];
      count := 1;
   fi;
od;
printf(`the number of %d on the diagonal is %d\ n`,
diagonal,count);
```

output:

```
the number of 1 on the diagonal is 30
the number of 3 on the diagonal is 9
the number of 6 on the diagonal is 1
```

The Sequence S

```
dS1_col := array(1..240,1..2):
r := 1:
for i from 1 to 39 do
   for j from i+1 to 40 do
      if _Sprod(i,j) = 0 then
         dS1_col[r,1] := i;
         dS1_col[r,2] := j;
         r := r + 1;
      fi;
   od;
od:

dS1_row := array(1..360,1..2):
r := 1:
for i from 1 to 45 do
   for j from 1 to 4 do
      dS1_row[r,1] := V[UnisotropicPairs[i,1],j];
      dS1_row[r,2] := i;
      r := r + 1;
   od:
   for j from 1 to 4 do
      dS1_row[r,1] := V[UnisotropicPairs[i,2],j];
      dS1_row[r,2] := i;
      r := r + 1;
   od:
od:

dS1 := array(1..400,1..240):
for i from 1 to 360 do
```

```
   for j from 1 to 240 do
      if dS1_row[i,1] = dS1_col[j,1] and
      (_IsBelongTo(dS1_col[j,2], UnisotropicPairs[dS1_row[i,2],1]) = 1 or
      _IsBelongTo(dS1_col[j,2], UnisotropicPairs[dS1_row[i,2],2]) = 1) then
         dS1[i,j] := 1;
      elif dS1_row[i,1] = dS1_col[j,2] and
      (_IsBelongTo(dS1_col[j,1], UnisotropicPairs[dS1_row[i,2],1]) = 1 or
      _IsBelongTo(dS1_col[j,1], UnisotropicPairs[dS1_row[i,2],2]) = 1) then
         dS1[i,j] := -1;
      else
         dS1[i,j] := 0;
      fi;
   od:
od:

for i from 361 to 400 do
   for j from 1 to 240 do
      if i - 360 = dS1_col[j,1] then
         dS1[i,j] := 1;
      elif i - 360 = dS1_col[j,2] then
         dS1[i,j] := -1;
      else
         dS1[i,j] := 0;
      fi;
   od:
od:

idS1 := ismith(dS1):

diagonal := 1:                                    # to count the entries of the diagonal
count := 0:
for i from 1 to 240 do
   if diagonal = idS1[i,i] then
      count := count + 1;
   else
      printf(`the number of %d on the diagonal is %d\ n`,
      diagonal,count);
      diagonal := idS1[i,i];
      count := 1;
   fi;
od;
printf(`the number of %d on the diagonal is %d\ n`,
diag,count);
```

output:

```
the number of 1 on the diagonal is 205
the number of 3 on the diagonal is 20
the number of 6 on the diagonal is 15
```

```
dS2 := array(1..85,1..400):
dS2_col := dS1_row:
for i from 1 to 40 do
   for j from 1 to 360 do
      if _Sprod(i,dS2_col[j,1]) = 0 and
      i <> dS2_col[j,1] and
      (_IsBelongTo(i,UnisotropicPairs[dS2_col[j,2],1]) = 1 or
      _IsBelongTo(i,UnisotropicPairs[dS2_col[j,2],2]) = 1) then
         dS2[i,j] := 1;
      else
         dS2[i,j] := 0;
      fi;
   od;
```

```
od:
for i from 41 to 85 do
   for j from 1 to 360 do
      if i - 40 = dS2_col[j,2] then
         dS2[i,j] := -1;
      else
         dS2[i,j] := 0;
      fi;
   od;
od:
for i from 1 to 40 do
   for j from 361 to 400 do
      if i = j - 360 then
         dS2[i,j] := -2;
      elif _Sprod(i,j - 360) = 0 then
         dS2[i,j] := 1;
      else
         dS2[i,j] := 0;
      fi;
   od;
od:
for i from 41 to 85 do
   for j from 361 to 400 do
      dS2[i,j] := 0;
   od:
od:

idS2 := ismith(dS2):

diagonal := 1:                                    # to count the entries of the diagonal
count := 0: for i from 1 to 85 do
   if diagonal = idS2[i,i] then
      count := count + 1;
   else
      printf(`the number of %d on the diagonal is %d\ n`,
      diagonal,count);
      diagonal := idS2[i,i];
      count := 1;
   fi;
od;
printf(`the number of %d on the diagonal is %d\ n`,
diagonal,count);
```

output

```
the number of 1 on the diagonal is 61
the number of 0 on the diagonal is 24
```

The sequence *T*

```
dT1_col := array(1..160,1..3):
r := 1:
for i from 1 to 38 do
   for j from i + 1 to 39 do
      for k from j + 1 to 40 do
         if _Sprod(i,j) = 0 and
         (_Lplus(i,j) = k or
         _Lminus(i,j) = k) then
            dT1_col[r,1] := i;
            dT1_col[r,2] := j;
            dT1_col[r,3] := k;
            r := r + 1;
         fi;
```

```
      od;
   od;
od:
dT1_row := dS1_col:
dT1 := array(1..240,1..160):
for i from 1 to 240 do
   for j from 1 to 160 do
      if (dT1_row[i,1] = dT1_col[j,1] and
      dT1_row[i,2] = dT1_col[j,2]) or
      (dT1_row[i,1] = dT1_col[j,2] and
      dT1_row[i,2] = dT1_col[j,3]) then
         dT1[i,j] := -1;
      elif (dT1_row[i,1] = dT1_col[j,1] and
      dT1_row[i,2] = dT1_col[j,3]) then
         dT1[i,j] := 1;
      else
         dT1[i,j] := 0;
      fi;
   od:
od:

idT1 := ismith(dT1):

diagonal := 1:                                    # to count the entries of the diagonal
count := 0:
for i from 1 to 160 do
   if diagonal = idT1[i,i] then
      count := count + 1;
   else
      printf(‘the number of %d on the diagonal is %d\ n‘,
      diagonal,count);
      diagonal := idT1[i,i];
      count := 1;
   fi;
od;
printf(‘the number of %d on the diagonal is %d\ n‘,
diagonal,count);
```

output

```
the number of 1 on the diagonal is 120
the number of 0 on the diagonal is 40
dT2 := array(1..40,1..240):
dT2_col := dT1_row:
for i from 1 to 40 do
   for j from 1 to 240 do
      if i = dT2_col[j,1] then
         dT2[i,j] := -1;
      elif i = dT2_col[j,2] then
         dT2[i,j] := 1;
      else
         dT2[i,j] := 0;
      fi;
   od;
od:

idT2 := ismith(dT2):

diagonal := 1:                                    # to count the entries of the diagonal
count := 0:
for i from 1 to 40 do
   if diagonal = idT2[i,i] then
      count := count + 1;
   else
```

```
      printf(‘the number of %d on the diagonal is %d\ n‘,
      diagonal,count);
      diagonal := idT2[i,i];
      count := 1;
   fi;
od;
printf(‘the number of %d on the diagonal is %d\ n‘,
diagonal,count);
```

output

```
the number of 1 on the diagonal is 39
the number of 0 on the diagonal is 1
```

Vita

Mustafa Arslan was born April 5, 1973, in Alaca Turkey. He finished his undergraduate studies in mathematics at Middle East Technical University in June 1996. After the graduation he worked three years as a research assistant at the same institution specializing in complex analysis. In August 1999 he started his studies in Louisiana State University Mathematics Department. During his years in LSU he studied the Siegel modular varieties under the supervision of Prof. Jerome W. Hoffman. He is currently a candidate for doctoral degree in mathematics to be awarded in May 2006.