On Reductive Operators.

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ABSTRACT

Reductive operators are the principal concern of this dissertation. The most important problem concerning reductive operators is that of determining whether or not every reductive operator is normal. In Chapter I we show that a reductive operator \( A \) is normal if \( f(A) \) is normal for some analytic function \( f \) on \( \{z: |z| \leq \|A\|\} \). It is also shown that reductive quasinormal operators are normal. Concerning the spectrum of a reductive operator \( A \), we prove that \( \sigma(A) = \pi(A) \), the approximate point spectrum of \( A \). An example is given to show that the set of reductive operators on an infinite-dimensional Hilbert space is not closed in the strong operator topology.

Chapter II is devoted to the study of inverses of reductive operators. We show that if \( A \) is reductive and if \( 0 \) is in the unbounded component of the resolvent set \( \rho(A) \), then \( A^{-1} \) is reductive. This result is used to demonstrate that the bilateral shift is a direct sum of
two reductive operators. We also show that a reductive operator $A$ is normal if and only if $A \circ A$ is reductive, thus obtaining a new equivalence to the invariant subspace conjecture.

In Chapter III we investigate powers (positive and negative) of reductive operators. With various restrictions on the spectrum of an operator $A$, we show that $A$ is reductive if $A^n$ is reductive and that $A^n$ is reductive if $A$ is reductive.

Chapter IV is chiefly concerned with quasinilpotent operators and their invariant subspaces. This departs only slightly from the topic of reductive operators, because any transitive operator is reductive. We show that a quasinilpotent operator $A$ has a proper invariant subspace if and only if, for some non-zero vectors $x$ and $y$, the analytic function $\lambda \mapsto ((\lambda - A)^{-1}x, y)$ on $\mathbb{C} \setminus \{0\}$ has a pole at $\lambda = 0$. 
INTRODUCTION

This section introduces most of the notation and conventions used in the remainder of the text. It also mentions most of the background knowledge which will be required of the reader.

The symbols $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}$ and $\mathbb{C}$ will denote the sets of integers, positive integers, real numbers and complex numbers respectively. If $\lambda \in \mathbb{C}$, then $\overline{\lambda}$ will denote the conjugate of $\lambda$.

A Hilbert space $H$ will always be a complex Hilbert space. The inner product of two vectors $x$ and $y$ in $H$ will be denoted by $(x,y)$ and $\|x\|$ will denote the norm of the vector $x$. A unit vector is a vector $x$ such that $\|x\| = 1$.

By an operator $A$ on a Hilbert space $H$, we will mean a continuous linear transformation of $H$ into itself. The adjoint of an operator $A$ on $H$ will be denoted by $A^*$, so $(Ax,y) = (x,A^*y)$ for all vectors $x$ and $y$ in $H$. 
The symbol $I$ represents the identity operator on $H$, that is $Ix = x$ for all vectors $x \in H$. For convenience we will use $(\lambda - A)$ to represent the operator $(\lambda \cdot I - A)$. If $A$ is an operator on $H$ and if there exists an operator $B$ on $H$ such that $BA = AB = I$, then we say that $A$ is invertible and that $B$ is the inverse of $A$, written $B = A^{-1}$. The symbol $\|A\|$ will denote the operator norm of the operator $A$ on $H$, that is, $\|A\| = \text{sup} \{\|Ax\| : x \in H, \|x\| = 1\}$. The symbol $\mathcal{B}(H)$ designates the set of all operators on $H$.

We will assume that the reader is familiar with the uniform, strong operator and weak operator topologies on $\mathcal{B}(H)$. The uniform topology on $\mathcal{B}(H)$ is the topology generated by all sets of the form $N(A; \epsilon) = \{B \in \mathcal{B}(H) : \|B - A\| < \epsilon\}$ where $\epsilon > 0$. The strong operator topology on $\mathcal{B}(H)$ is the topology generated by all sets of the form

$$N(A; x_1, \ldots, x_n; \epsilon) = \{B \in \mathcal{B}(H) : \|(B - A)x_i\| < \epsilon, \ i = 1, 2, \ldots, n\}$$

where $\epsilon > 0$, $n \in \mathbb{Z}^+$ and $x_1, \ldots, x_n$ are arbitrary vectors in $H$. The weak operator topology on $\mathcal{B}(H)$ is the topology generated by all sets of the form

$$N(A; x_1, \ldots, x_n; y_1, \ldots, y_n; \epsilon) = \{B \in \mathcal{B}(H) : |(B - A)x_i, y_i| < \epsilon, \ i = 1, 2, \ldots, n\}$$

where $\epsilon > 0$, $n \in \mathbb{Z}^+$ and
The strong operator topology is stronger than the weak operator topology and weaker than the uniform topology. Further discussion on these can be found in [5], [6], [7], [12] or [16].

A linear manifold \( \mathcal{M} \) of \( H \) is a subset of \( H \) which is closed under the operations of vector addition and scalar multiplication. A subspace of \( H \) is a closed linear manifold of \( H \). If \( \mathcal{X} \) is a subset of \( H \), \( \mathcal{V}(x:x \in \mathcal{X}) \) will denote the linear manifold generated by \( \mathcal{X} \) and \( \mathcal{V}(x:x \in \mathcal{X}) \) will denote the subspace generated by \( \mathcal{X} \). The symbol \( \overline{\mathcal{X}} \) will denote the closure of the set \( \mathcal{X} \). Also, for any subset \( \mathcal{X} \) of \( H \), \( \mathcal{X}^\perp \) will denote the orthogonal complement of \( \mathcal{X} \), that is, \( \mathcal{X}^\perp = \{ y \in H : (x,y) = 0 \text{ for all } x \in \mathcal{X} \} \). \( \mathcal{X}^\perp \) is always a subspace of \( H \).

The subspaces \( \{0\} \) and \( H \) are called the trivial subspaces of \( H \) and all others are called proper subspaces. A subspace \( \mathcal{M} \) of \( H \) is invariant for an operator \( A \) if \( \{Ax:x \in \mathcal{M}\} \subseteq \mathcal{M} \). We say that \( \mathcal{M} \) reduces \( A \) if \( \mathcal{M} \) is invariant for both \( A \) and \( A^* \) or, equivalently, if both \( \mathcal{M} \) and \( \mathcal{M}^\perp \) are invariant for \( A \). The restriction of an operator \( A \) to an invariant subspace \( \mathcal{M} \) is denoted by \( A|_\mathcal{M} \).

An operator \( A \) is Hermitian if \( A = A^* \), normal if
\[ AA^* = A^*A, \] isometric if \[ \|Ax\| = \|x\| \] for all \( x \in H \), and unitary if \( AA^* = A^*A = I \). If \( A \) is defined and isometric on a dense subspace of \( H \) and if the range of \( A \) is dense in \( H \), then \( A \) has a unique extension to a unitary operator on \( H \). A projection is an operator \( A \) which satisfies \( A = A^* = A^2 \). \( A \) is said to be quasi-normal if \( A \) commutes with \( A^*A \), and reductive if every invariant subspace for \( A \) reduces \( A \). A transitive operator is an operator which has no proper invariant subspaces. Clearly every operator on a one-dimensional Hilbert space \( H \) is transitive - and these are the only known examples. Operators which are not transitive are called intransitive. A compact operator is an operator which maps any bounded sequence of vectors to a set with a convergent subsequence. Every operator on a finite-dimensional Hilbert space is compact.

A normal part of an operator \( A \) is the restriction of \( A \) to a reducing subspace on which it is normal. An operator is said to be normal-free if it has no normal part.

We let \( \mathcal{N}(A) = \{ x \in H : Ax = 0 \} \), the null-space of \( A \), and we let \( \mathcal{R}(A) = \{ Ax : x \in H \} \), the range of \( A \). The relationships \( \mathcal{N}(A) = \mathcal{R}(A^*)^\perp \) and \( \mathcal{R}(A) = \mathcal{N}(A^*)^\perp \) are valid for any operator \( A \) on \( H \).
The spectrum of $A$, $\sigma(A)$, is $\{\lambda \in \mathbb{C} : (\lambda - A) \text{ is not invertible}\}$. The resolvent of $A$, $\rho(A)$, is the complement of $\sigma(A)$. The operator-valued function $\lambda \mapsto (\lambda - A)^{-1}$ defined on $\rho(A)$ is called the resolvent function of the operator $A$. For any pair of vectors $x$ and $y$ in $H$, the function $\lambda \mapsto ((\lambda - A)^{-1}x, y)$ is an analytic function on $\rho(A)$, hence we say the resolvent function is analytic on $\rho(A)$.

Following Dunford and Schwartz ([7], p. 580) we define

$$\sigma_p(A) = \{\lambda \in \sigma(A) : N(\lambda - A) \neq \{0\}\},$$

$$\sigma_r(A) = \{\lambda \in \sigma(A) : N(\lambda - A) = \{0\} \text{ and } \overline{R(\lambda - A)} \neq H\},$$

and

$$\sigma_c(A) = \{\lambda \in \sigma(A) : N(\lambda - A) = \{0\} \text{ and } \overline{R(\lambda - A)} = H\}.$$

The sets $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_c(A)$ are called, respectively, the point spectrum, residual spectrum and continuous spectrum of $A$. Clearly $\sigma_p(A)$, $\sigma_r(A)$ and $\sigma_c(A)$ are disjoint sets whose union is $\sigma(A)$. Also $\sigma_r(A) \subset \sigma_p(A^*)$ because $R(A)^\perp = N(A^*)$. An eigenvalue of $A$ is a number $\lambda \in \sigma_p(A)$, that is a number $\lambda$ such that $(\lambda - A)x = 0$ for some unit vector $x \in H$, and this vector is called an eigenvector of $A$ corresponding to the eigenvalue $\lambda$. A number $\lambda \in \sigma(A)$ is called an approximate eigenvalue of $A$ if there exists a sequence $\{x_n\}$ of vectors in $H$ such that $\|x_n\| = 1$ for each $n$ and $\|(\lambda - A)x_n\| \to 0$. 
The set of all approximate eigenvalues of $A$ will be denoted by $\pi(A)$ and will be called the approximate point spectrum of $A$.

An operator $A$ is said to be nilpotent if $A^n = 0$ for some $n \in \mathbb{Z}^+$, and quasinilpotent if $\sigma(A) = \{0\}$.

A vector $x \in H$ is called a cyclic vector for an operator $A$ on $H$ if $\overline{\{A^n x : n \in \mathbb{Z}^+ \text{ or } n=0\}} = H$. The commutant of an operator $A$ on $H$ is $\{A\}' = \{B \in B(H) : BA = AB \text{ and } BA^* = A^*B\}$. B. Fuglede [9] has shown that if $A$ is normal and $BA = AB$, then also $BA^* = A^*B$. Thus for a normal operator $A$ on $H$ we have $\{A\}' = \{B \in B(H) : BA = AB\}$. A subalgebra $\mathfrak{A}$ of $B(H)$ is called a $*$-subalgebra of $B(H)$ (or simply a $*$-algebra) if $A \in \mathfrak{A}$ implies $A^* \in \mathfrak{A}$. An element $A$ of a $*$-subalgebra $\mathfrak{A}$ of $B(H)$ is called a positive element of $\mathfrak{A}$ if $A = B^*B$ for some $B \in \mathfrak{A}$. For any operator $A$, $\{A\}'$ is a $*$-algebra and is closed in the weak operator topology.

A continuous linear functional on an algebra $\mathfrak{A}$ is a continuous linear mapping from $\mathfrak{A}$ into $C$. A positive continuous linear functional $f$ on a $*$-algebra $\mathfrak{A}$ satisfies the additional condition that $f(A) \geq 0$ whenever $A$ is a positive element of $\mathfrak{A}$. If $\mathfrak{A}$ is a commutative uniformly closed $*$-subalgebra of $B(H)$ containing $I$ and
If \( \mathcal{J} \) is a maximal ideal in \( \mathcal{A} \), then the factor ring \( \mathcal{A}/\mathcal{J} \) is isometrically isomorphic to \( C \). Thus each maximal ideal \( \mathcal{J} \) determines a continuous linear functional on \( \mathcal{A} \). The space \( M \) of all maximal ideals of \( \mathcal{A} \) is a closed subset of the unit ball of the dual of \( \mathcal{A} \) and is thus a compact Hausdorff space in the relative weak topology. Moreover, \( \mathcal{A} \) is isometrically isomorphic to the algebra \( C(M) \) of all continuous complex-valued functions on \( M \). The isometric isomorphism from \( \mathcal{A} \) to \( C(M) \) is called the Gelfand transform and the symbol \( \hat{A}(m) \) denotes the image of an operator \( A \in \mathcal{A} \). Proofs of the facts stated above and further information on this topic can be found in Naimark [16] or Dunford and Schwartz [7], Vol. II.

Another non-trivial result we will assume is the spectral theorem for normal operators. A good exposition of this may be found in Halmos [10]. The theorem states that every normal operator \( A \) on \( H \) has a representation of the form \( A = \int \int \lambda dE(\lambda) \) where, for each Borel set \( \Delta \subset C \), \( E(\Delta) \) is a projection in \( B(H) \), \( E(C) = I \), \( E(\bigcup_{n=1}^{\infty} \Delta_n) = \sum_{n=1}^{\infty} E(\Delta_n) \) whenever \( \{\Delta_n\}_{n=1}^{\infty} \) is a disjoint sequence of Borel subsets of \( C \), and \( E(\Delta_1)E(\Delta_2) = 0 \) whenever \( \Delta_1 \cap \Delta_2 = \emptyset \). Moreover, if \( BA = AB \) then
E(Δ)B = BE(Δ) for every Borel subset Δ of C. In particular, each projection E(Δ) commutes with A itself. If the only values of E(Δ) are I and 0 then A is a multiple of I. The family of projections \{E(Δ)\} is called the spectral family of the normal operator A.

If f is a complex-valued function which is defined and analytic on a neighborhood U of σ(A) for some operator A ∈ B(H) we will simply say that f is analytic on σ(A). The set F_A will be the set of all functions analytic on σ(A) (where the neighborhood U depends on the function). If f ∈ F_A we can find a rectifiable Jordan curve Γ in U which contains σ(A) in its interior and define an operator f(A) in B(H) by the formula

\[ f(A) = \frac{1}{2\pi i} \int_{\Gamma} f(\lambda)(\lambda - A)^{-1} d\lambda \]

where dλ denotes Lebesgue measure on the complex plane. (Note that Γ is a positive distance from σ(A) since Γ and σ(A) are disjoint compact sets.) Also, the operator f(A) so defined is independent of the particular curve Γ chosen because the resolvent function \( \lambda \to (\lambda - A)^{-1} \) is analytic on ρ(A) and the Cauchy integral theorem is valid. The correspondence from F_A into B(H) defined above has the following properties:
(1) if $f(\lambda) = 1$ then $f(A) = I$; (2) if $f(\lambda) = \lambda$ then $f(A) = A$; (3) if $f(\lambda) = a_1 f_1(\lambda) + a_2 f_2(\lambda)$ where $f_1, f_2 \in F_A$, $a_1, a_2 \in \mathbb{C}$, then $f \in F_A$ and $f(A) = a_1 f_1(A) + a_2 f_2(A)$; (4) if $f(\lambda) = f_1(\lambda)f_2(\lambda)$ where $f_1, f_2 \in F_A$, then $f \in F_A$ and $f(A) = f_1(A)f_2(A)$; and (5) if the sequence $\{f_n(\lambda)\}$ converges uniformly to $f(\lambda)$ in some neighborhood $\mathcal{U}$ of $\sigma(A)$ and $f_n \in F_A$ for each $n$, then $f \in F_A$ and the sequence $\{f_n(A)\}$ converges in the uniform topology to $f(A)$. In particular, if $f$ is a polynomial in $\lambda$, then $f(A)$ is the corresponding polynomial in $A$. This procedure of constructing analytic functions of an operator will be referred to as the operational calculus. Some good sources of information on the operational calculus are Dunford and Schwartz [7], Lorch [15] and Naimark [16].

Finally, the reader may find it helpful to have at least a vague idea of the concept of a direct integral. For this the reader is referred to Naimark [16], Dixmier [5], or Pedersen [17].
CHAPTER I
THE REDUCTIVE OPERATOR CONJECTURE

Reductive operators, as mentioned in the introduction, are operators for which every invariant subspace is a reducing subspace. The name 'reductive operator' is of recent origin and was suggested by Paul Halmos. In earlier literature these operators have traveled under the names of 'completely normal operators', as in Dyer and Porcelli [8]; 'operators with property (P)' as in Wermer [23]; or simply as 'operators for which every invariant subspace is reducing', as in Sarason [20].

The first proposition below is very elementary and its proof will be omitted.

Proposition 1.1. The following are equivalent for $A \in \mathcal{B}(H)$:
(a) $A$ is reductive.
(b) $A^*$ is reductive.
(c) If a subspace $\mathcal{M}$ is invariant for $A$, so is $\mathcal{M}^\perp$.
(d) A subspace \( \mathcal{M} \) is invariant for \( A \) iff it is invariant for both of the Hermitian parts of \( A \).

(e) \( \lambda - A \) is reductive for some \( \lambda \in \mathbb{C} \).

(f) \( \lambda - A \) is reductive for all \( \lambda \in \mathbb{C} \).

(g) \( \lambda A \) is reductive for some non-zero \( \lambda \in \mathbb{C} \).

(h) \( \lambda A \) is reductive for all \( \lambda \in \mathbb{C} \).

In particular, we note that the set of reductive operators is closed under the operations of translation, scalar multiplication and the taking of adjoints.

**Proposition 1.2.** If \( A \in B(\mathcal{H}) \) and the subspace \( \mathfrak{m} \) reduces \( A \), then \( A^*|_{\mathfrak{m}} = (A|_{\mathfrak{m}})^* \).

**Proof.** If \( x, y \in \mathfrak{m} \), then \( (A^*|_{\mathfrak{m}}x, y) = (A^*x, y) = (x, Ay) = (x, A|_{\mathfrak{m}}y) \). Since \( x \) and \( y \) are arbitrary in \( \mathfrak{m} \), we have \( (A|_{\mathfrak{m}})^* = A^*|_{\mathfrak{m}} \).

**Proposition 1.3.** If \( A \) is reductive, then \( \sigma_r(A) = \emptyset \).

**Proof.** Suppose \( \lambda \in \sigma_r(A) \). Then \( \mathfrak{N}(\lambda - A) = \{0\} \) and \( \mathfrak{R}(\lambda - A) \) is not dense in \( \mathcal{H} \). Hence \( \{0\} \neq \mathfrak{R}(\lambda - A) \perp = \mathfrak{N}(\lambda - A)^* \) and so there is a vector \( x \neq 0 \) in \( \mathfrak{N}(\lambda - A)^* \).

It follows that \( A^*x = \overline{\lambda}x \) and \( \mathfrak{m} = \overline{\mathfrak{V}}\{x\} \) is invariant for \( A^* \). By Proposition 1.1(b), \( A^* \) is reductive so \( \mathfrak{m} \) is also invariant for \( A \). Also, \( (A|_{\mathfrak{m}})^* = A^*|_{\mathfrak{m}} = \overline{\lambda} \cdot I|_{\mathfrak{m}} = (\lambda \cdot I|_{\mathfrak{m}})^* \) so \( A = \lambda \cdot I \) on \( \mathfrak{m} \). Thus we have \( \lambda \in \sigma_p(A) \),
a contradiction because \( \sigma_r(A) \) and \( \sigma_p(A) \) are disjoint. We thus conclude that \( \sigma_r(A) = \emptyset \).

Hermitian operators are clearly reductive because \( A = A^* \). Normal operators need not be reductive as is shown by the following example:

**Example 1.4.** Let \( H \) be a separable infinite-dimensional Hilbert space and let \( \{e_n: n \in \mathbb{Z}\} \) be an orthonormal basis for \( H \). Let \( \mathcal{U} \) be the operator on \( H \) defined by \( \mathcal{U}e_n = e_{n+1} \) for \( n \in \mathbb{Z} \). The adjoint \( \mathcal{U}^* \) of \( \mathcal{U} \) is the operator defined by \( \mathcal{U}^*e_n = e_{n-1} \) for \( n \in \mathbb{Z} \), and \( \mathcal{U} \) is called a bilateral shift on \( H \). We have \( \mathcal{U}\mathcal{U}^* = \mathcal{U}^*\mathcal{U} = I \) so \( \mathcal{U} \) is normal (unitary, in fact). However, \( \mathcal{U} \) is not reductive because every subspace of the form \( \overline{\mathcal{V}\{e_k,e_{k+1},\cdots\}} \) is invariant for \( \mathcal{U} \) but not for \( \mathcal{U}^* \).

We saw that our example above was, in fact, a unitary operator. It is interesting to note that this is, in a sense, the only example of a unitary operator which is not reductive. John Wermer has shown in [23] that a unitary operator is reductive iff it does not have a reducing subspace on which it acts like a bilateral shift.

The statement that every reductive operator is normal is an open question, which we shall refer to as the reductive operator conjecture. T. Andô has shown in [1] that
a compact operator is reductive iff it is normal.

Another open question, more widely known than the reductive operator conjecture, is the statement that every operator on a Hilbert space of dimension greater than 1 has a proper invariant subspace. This we shall refer to as the invariant subspace conjecture.

In the early 1930's J. von Neumann proved (but did not publish) the result that every compact operator on a Hilbert space of dimension greater than 1 has a proper invariant subspace. This was later extended to arbitrary Banach spaces by N. Aronszajn and K. T. Smith [2]. More recently it has been shown that polynomially compact operators have proper invariant subspaces (cf. Bernstein and Robinson [3], Halmos [10]). An easy generalization of this is the fact that every analytically compact operator has a proper invariant subspace. Some other results in this direction can be found in Chernoff and Feldman [4].

John Dyer and Pasquale Porcelli have proven the following remarkable result [8]:

**Theorem 1.5.** The following two statements are equivalent:

(a) The reductive operator conjecture is true.

(b) The invariant subspace conjecture is true.

It is easy to show that (a) \(\Rightarrow\) (b). Suppose that
(b) is false. Then we have a transitive operator $A$ on a Hilbert space $H$ of dimension greater than 1.

Clearly $A$ is reductive because its only invariant subspaces are $\{0\}$ and $H$. However, $A$ is not normal because normal operators are not transitive (by the spectral theorem for normal operators, for example). Hence (a) is false.

The proof of the converse is more difficult and requires some knowledge of direct integral theory. Some good references on this subject are Pedersen [17] and Dixmier [5]. Before indicating the proof that $(b) \Rightarrow (a)$ in Theorem 1.5, we will pause to develop some necessary machinery. First we note that, in attacking the reductive operator conjecture, it suffices to consider only reductive operators $A$ with a cyclic vector $\xi_0$. This simplification is possible because $\overline{\{A^n\xi_0 : n \in \mathbb{Z}^+ \cup \{0\}\}}$ reduces $A$ and $A$ is normal iff its restriction to each such subspace is normal. Also, by Ando's result, we may assume that $\overline{\{A^n\xi_0 : n \in \mathbb{Z}^+ \cup \{0\}\}}$ is infinite-dimensional. Thus we assume for the remainder of this proof that $H$ is a separable infinite-dimensional Hilbert space and that $A$ is a reductive operator on $H$ with a cyclic vector $\xi_0$ such that $\|\xi_0\| = 1$. Let $E$ be a maximal commutative $*$-subalgebra of $\{A\}'$. Let $E_{\xi_0} = \overline{\{B\xi_0 : B \in E\}}$, a subspace
If $\mathbb{E}_0 \neq H$, we can find a vector $\xi_1 \in (\mathbb{E}_0')^\perp$ such that $\|\xi_1\| = 1$. It is easy to show that the subspaces $\mathbb{E}_0$ and $\mathbb{E}_1$ are orthogonal. If $\mathbb{E}_0 \oplus \mathbb{E}_1 \neq H$, we can choose a vector $\xi_2 \in (\mathbb{E}_0 \oplus \mathbb{E}_1)^\perp$ and continue in this manner. Since $H$ is separable we can thus write $H = \bigoplus_{i=0}^{\infty} \mathbb{E}_i$. We will define $H' = \{ \sum_{i=0}^{n} B_i \xi_i : B_i \in \mathbb{E}, n \in \mathbb{Z}^+ \cup \{0\} \}$, a linear manifold which is dense in $H$. For each vector $\xi_i$ we can define a continuous positive linear functional $f_i$ on the algebra $E$ by $f_i(B) = (B \xi_i, \xi_i)$. Since the algebra $E$ is isometrically isomorphic to $C(M)$, the algebra of continuous functions on the maximal ideal space $M$ of $E$, each functional $f_i$ may be considered as a functional on $C(M)$. As a result, each functional $f_i$ induces a regular Borel measure $\mu_i$ on $M$ and we have $f_i(B) = \int_{M} \hat{B}(m) d\mu_i(m)$ for each $B \in E$, where $\hat{B}(m)$ denotes the Gelfand transform of $B$. Because $\xi_0$ is a cyclic vector for $A$, it can be shown that the support of the measure $\mu_0$ is $M$ and that each $\mu_i$ is absolutely continuous with respect to $\mu_0$. Also, the vectors $\xi_i$, $i=1,2,\ldots$, can be chosen so that $d\mu_i = x_{M_1} d\mu_0$ where $x_{M_1}$ is the characteristic function of a $\mu_0$-measurable set.
$M_1$ contained in $M$. Next we define a linear map $\mathcal{U}_1'$ from $\overline{\mathcal{E}_1}$ into $C(M_1)$ by $\mathcal{U}_1'(\mathcal{B}_1) = \hat{\mathcal{B}}(m)|_{M_1}$. The map $\mathcal{U}_1'$ is densely defined on $\overline{\mathcal{E}_1}$ with range dense in $L^2(M_1, \mu_1)$ and for $\mathcal{B} \in E$ we have $\|\mathcal{U}_1'\mathcal{B}_1\|^2 = (\mathcal{U}_1'\mathcal{B}_1, \mathcal{U}_1'\mathcal{B}_1) = (\hat{\mathcal{B}}(m)|_{M_1}, \hat{\mathcal{B}}(m)|_{M_1}) = \int_{M_1} \hat{\mathcal{B}}^*(m)\hat{\mathcal{B}}(m)\,d\mu_1(m) = \int_{M_1} \mathcal{B}^*(m)\mathcal{B}(m)\,d\mu_1(m) = (\mathcal{B}_1, \mathcal{B}_1) = \|\mathcal{B}_1\|^2$. Thus each map $\mathcal{U}_1'$ has a unique extension to a unitary operator $\mathcal{U}_1$ which maps $\overline{\mathcal{E}_1}$ onto $L^2(M_1, \mu_1)$. Using the symbol $\sim$ to mean 'is unitarily equivalent to', we have $H \sim \bigoplus_{i=0}^\infty L^2(M_1, \mu_1)$ via the unitary operator $\mathcal{U} = \bigoplus_{i=0}^\infty \mathcal{U}_1$. Let $H_0 = \overline{\{\xi_i : i \in \mathbb{Z}^+ \cup \{0\}\}}$ and for each $m \in M$, let $H(m) = \overline{\{\xi_i : m \in M_1\}}$, a subspace of the separable Hilbert space $H_0$. For a vector $\mathbf{\eta} \in H'$ (defined previously) of the form $\mathbf{\eta} = \sum_{i=0}^n \mathbf{B}_i \xi_i$ where $\mathbf{B}_i \in E$, we can define a vector-valued function $\mathbf{\eta}(m)$ on $M$ by the equation $\mathbf{\eta}(m) = \sum_{i=0}^n \mathbf{B}_i(m)\hat{\mathbf{B}}(m)\xi_i$. Then $\mathbf{\eta}(m) \in H(m)$ for each $m \in M$ and furthermore

$$
\|\mathbf{\eta}\|^2 = \sum_{i=0}^n \|\mathbf{B}_i \xi_i\|^2 = \sum_{i=0}^n \|\mathcal{U}_1 \mathbf{B}_i \xi_i\|^2 = \sum_{i=0}^n \int_{M_1} |\hat{\mathbf{B}}(m)|^2\,d\mu_1(m) = \sum_{i=0}^n \int_{M} |\hat{\mathbf{B}}(m)|^2\chi_{M_1}(m)\,d\mu(m) = \sum_{i=0}^n \int_{M} |\hat{\mathbf{B}}(m)|^2\chi_{M_1}(m)\,d\mu(m) = \int_{M} \|\mathbf{\eta}(m)\|^2\,d\mu(m) .
$$

Thus the map $V' : \mathbf{\eta} \rightarrow \{\mathbf{\eta}(m)\}_m \in M$ maps a
dense subset of \( H \) onto a dense subset of \( \int_M H(m) d\mu(m) \) in an isometric fashion and so

\[
(1.1) \quad H \cong \int_M H(m) d\mu(m)
\]

via the unitary map \( V \) which extends \( V' \) and is defined on all of \( H \). For vectors \( \eta \in H' \), the vector-valued function \( \{\eta(m)\}_{m \in M} \) is defined everywhere on \( M \). This is not the case for all vectors in \( H \). If \( \eta \in H \) is arbitrary, we can find a sequence \( \{\eta_n\} \) of vectors in \( H' \) converging to \( \eta \) such that the sequence of functions \( \{\eta_n(m)\}_{m \in M} \) converges pointwise a.e. \( (\mu) \) on \( M \). This allows us to define \( \eta(m) \) a.e. \( (\mu) \) on \( M \) and this is the best we can do.

Our next task is to represent an operator \( B \) on \( H \) as an operator on the direct integral of the Hilbert spaces \( \{H(m)\}_{m \in M} \). In general, we cannot represent all operators in \( B(H) \) in such a fashion, but we are able to represent every operator in \( E' \) in this way. In particular, the reductive operators \( A \) and \( A^* \) have such a representation.

Let \( \xi_1 \in H \) be as before and let \( \eta \in H' \). Let \( B \in E' \). Then \( \eta(m) \) is defined everywhere on \( M \) and \( (B\xi_1)(m) \) is defined a.e. \( (\mu) \) on \( M \). We let \( N_1 = \{m \in M: (B\xi_1)(m) \text{ is not defined}\} \) and \( N = \bigcup_{i=0}^{\infty} N_i \).
For \( m \in M \setminus N \) we define \( f(\eta(m)) = (\eta(m), (B_{\xi_i})'(m)) \). Then \( f \) is a linear functional on the dense linear manifold \( H' \cap H(m) \) of \( H(m) \) and \( |f(\eta(m))| \leq \|\eta(m)\| \cdot \|(B_{\xi_i})'(m)\| \), so \( f \) is bounded. As a result of this, \( f \) extends uniquely to a bounded linear functional \( F \) defined on all of \( H(m) \) with \( \|F\| = \|f\| \leq \|(B_{\xi_i})'(m)\| \). By the Riesz representation theorem, there is a unique vector \( \xi_i(m) \) in \( H(m) \) such that \( F(\xi(m)) = (\xi(m), \xi_i(m)) \) for all \( \xi(m) \in H(m) \). We will define an operator \( B(m) \) on \( H(m) \) by letting \( B(m)\xi_i(m) = \xi_i(m) \) for \( i = 0, 1, \ldots \), and extending linearly. (Note that the \( \{\xi_i(m)\}_{i=0}^{\infty} \) contain a basis for \( H(m) \).) For \( m \in N \) we will define \( B(m) \) to be the zero operator on \( H(m) \). The operators \( B(m) \) so defined can be shown to have the following properties: (a) \( B(m)\xi(m) = (B_{\xi_i})'(m) \) a.e. \( \mu \) for \( \xi \in H \); (b) \( B(m)\eta(m) = (B_{\eta})'(m) \) for all \( m \in M, \eta \in H' \); (c) \( \|B(m)\| \leq \|B\| \) for all \( m \in M \). We write

\[(1.2) \quad B = B = \int_M B(m)d\mu(m) \text{ where} \]

\[
\int_M B(m)d\mu(m)\int_M \xi(m)d\mu(m) = \int_M B(m)\xi(m)d\mu(m) = \\
= \int_M (B_{\xi_i})'(m)d\mu(m) = (B_{\xi_i})' \cdot 
\]

If \( B_1 \) and \( B_2 \) are any two operators in \( E' \) we
can show that \((B_1 + B_2)(m) = B_1(m) + B_2(m)\) a.e. \((\mu)\),
\(B_1B_2(m) = B_1(m)B_2(m)\) a.e. \((\mu)\), and that \((B_1^*)^* = (B_1(m))^*\) a.e. \((\mu)\). As a result we have

\[ B_1 + B_2 = \int_M [B_1(m) + B_2(m)] d\mu(m) \]  
\[ B_1 \cdot B_2 = \int_M B_1(m) \cdot B_2(m) d\mu(m) \quad \text{and} \]
\[ B_1^* = \int_M [B_1(m)]^* d\mu(m) . \]

We now return to the proof of Theorem 1.5. Let \(A\) be a reductive operator on the separable infinite-dimensional Hilbert space \(H\) and let \(\xi_0\) be a cyclic vector for \(A\) with \(\|\xi_0\| = 1\). According to the above construction, we have a representation.

\[ A = \int_M A(m) d\mu(m) \]

where for each \(m\), \(A(m)\) is an operator on the Hilbert space \(H(m)\) as defined above. Dyer and Porcelli showed that the 'coordinate operators' \(A(m)\) are transitive a.e. \((\mu)\). If we assume that the invariant subspace conjecture is true, then each operator \(A(m)\) must be an operator on a 1-dimensional Hilbert space \(H(m)\). Due to the fact that operators on a 1-dimensional Hilbert space commute, we have
\[ \Lambda \Lambda^* = \int M \Lambda(m) A^*(m) d\mu(m) \]
\[ = \int M A^*(m) A(m) d\mu(m) \]
\[ = A^* A \]

and so \( \Lambda \) is normal. This shows that \((b) \Rightarrow (a)\) and completes the proof of Theorem 1.5.

Some progress towards a solution of the reductive operator conjecture is given by the following result:

**Proposition 1.6.** If \( \Lambda \) is reductive and if \( f(\Lambda) \) is normal for some non-constant analytic function \( f \) on \( \{z : |z| \leq \|\Lambda\|\} \), then \( \Lambda \) is normal.

**Proof.** Suppose that \( f \) is as described above and \( f(z) = \sum_{p=0}^{\infty} a_p z^p \) is the power series expansion of \( f \) about the point \( z = 0 \). Then \( \sum_{p=0}^{\infty} a_p \Lambda^p \) converges in the uniform topology and so \( f(\Lambda) = \sum_{p=0}^{\infty} a_p \Lambda^p \). According to (1.6) we have \( \Lambda = \int M \Lambda(m) d\mu(m) \) where the operators \( \Lambda(m) \) are transitive a.e. \((\mu)\). If we let \( f_n(\Lambda) = \sum_{p=0}^{n} a_p \Lambda^p \), then
\[ f_n(\Lambda)(m) = f_n(\Lambda(m)) \quad \text{a.e.} \quad (\mu) \quad \text{and} \quad f_n(\Lambda) = \int M f_n(\Lambda)(m) d\mu(m) = \int_M f_n(\Lambda(m)) d\mu(m) \]. Hence for almost all \( m \) we have
f_n(A(m)) = f_n(A(m)) for all n and \( \{f_n(A(m))\} \) converges in the uniform topology to \( f(A(m)) \). (Note that \( f(A(m)) \) is defined because \( ||A(m)|| \leq ||A|| \).) It follows that \( f(A)(m) = f(A(m)) \) a.e. \( (\mu) \) and that

\[
f(A) = \int_M f(A(m)) \, d\mu(m).
\]

Since \( f(A) \) is normal, \( f(A(m)) \) is normal a.e. \( (\mu) \). If \( f(A(m)) \) is normal, then the spectral family of \( f(A(m)) \) commutes with \( A(m) \) because \( A(m) \) commutes with \( f(A(m)) \). As a result, \( f(A(m)) \) must be a multiple of the identity operator \( I_m \) on \( H(m) \) whenever \( f(A(m)) \) is normal, say \( f(A(m)) = \alpha_m \cdot I_m \). If we let \( g(z) = f(z) - \alpha_m \), then \( g(A(m)) = 0 \) so \( A(m) \) is analytically compact. This implies that \( A(m) \) must be an operator on a 1-dimensional Hilbert space \( H(m) \), because \( A(m) \) is transitive. This happens a.e. \( (\mu) \) on \( M \) and it follows that \( A \) is normal.

As an application of Proposition 1.6 we obtain the following result:

**Corollary 1.7.** If \( A \) is reductive and if \( A^n \) commutes with \( (A^*)^m \) for some \( m, n \in \mathbb{Z}^+ \), then \( A \) is normal.

**Proof.** If \( A^n \) commutes with \( (A^*)^m \), then \( A^{nm} \) is normal.

Our next result states that every operator in \( B(H) \) has a decomposition into normal and normal-free parts.
Proposition 1.8. Every operator $A$ in $B(H)$ can be expressed as the direct sum of a normal operator and a normal-free operator. This decomposition is unique.

Proof. The set of all reducing subspaces for $A$ on which $A$ is normal is partially ordered by inclusion, and Zorn's lemma gives us a maximal such subspace $H_1$. Hence $A|_{H_1}$ is normal and $A|_{H_1^\perp}$ is normal-free and we have the desired decomposition. The uniqueness is clear.

Alternatively, we can define the subspace $H_1$ by $H_1 = \{W(A,A^*)(AA^*-A^*A)x: x \in H, W(A,A^*)$ is a word in $A$ and $A^*\}$. It is easy to show that $H_1$ reduces $A$, $A|_{H_1}$ is normal, and if $K$ reduces $A$ and $A|_K$ is normal, then $K \subset H_1$.

We can now restrict our attention to reductive normal-free operators in studying the reductive operator problem.

Proposition 1.9. If $A$ is reductive normal-free on $H$, then $\sigma(A) = \sigma_c(A)$, that is, $A$ has no eigenvalues.

Proof. If $\lambda \in \sigma_p(A)$, then there is a non-zero vector $x \in H$ such that $(\lambda - A)x = 0$. This means $\overline{V}(x)$ reduces $A$ and the restriction of $A$ to this one-dimensional subspace is certainly normal, a contradiction. Hence $\sigma_p(A) = \emptyset$. We saw earlier that $\sigma_r(A) = \emptyset$ for any reductive
operator $A$ so $\sigma(A) = \sigma_c(A)$.

**Proposition 1.10.** If $A$ is reductive normal-free on $H$ and if $\mathcal{M}$ is an invariant subspace for $A$, then $[Ax : x \in \mathcal{M}]$ is dense in $\mathcal{M}$.

**Proof.** Since $A$ is reductive, $\mathcal{M}$ reduces $A$ and $A|_{\mathcal{M}}$ is again reductive normal-free. We may, therefore, without loss of generality, assume that $\mathcal{M} = H$. Now, either $A$ is invertible in which case $A$ is onto, or else $0 \in \sigma_c(A)$ which again means $A$ has dense range.

Our next result is an extension of Proposition 1.9.

**Proposition 1.11.** If $p$ is any non-constant polynomial and if $A$ is reductive normal-free on $H$, then $\sigma(p(A)) = \sigma_c(p(A))$.

**Proof.** It suffices to show that neither $[p(A)]$ nor $[p(A)]^*$ can have an eigenvalue. Suppose $\lambda$ is an eigenvalue of $p(A)$. Factor the polynomial $p(A) - \lambda$ into linear factors $\prod_{i=1}^{n} (A - \alpha_i)$ (assuming without loss of generality that $p$ is monic). If $\lambda$ is a eigenvalue of $p(A)$ there is a vector $x \neq 0$ in $H$ such that $\sum_{i=1}^{n} (A - \alpha_i)x = 0$. As a result one of the $\alpha_i$'s must be an eigenvalue of $A$, a contradiction of Proposition 1.9.
Thus $p(A)$ has no eigenvalues. Since $A^*$ is also reductive normal-free, we can show in a similar manner that $[p(A)]^*$ has no eigenvalues and the result follows.

Recall that an operator $A$ is said to be quasinormal if $A$ commutes with $A^*A$.

**Proposition 1.12.** If $A$ is quasinormal and reductive on $H$, then $A$ is normal.

**Proof.** It suffices to show that $A$ has no normal-free part. We will assume that $A$ is normal-free, (because the normal-free part would also be quasinormal and reductive), and arrive at a contradiction. Since $A$ is quasinormal, we have $0 = A(A^*A) - (A^*A)A = (A^*A - A^*A)A$. The operator $A$ has dense range by Proposition 1.10, so $AA^*$ and $A^*A$ agree on a dense subset of $H$. This means $AA^* = A^*A$, which cannot be because $A$ is normal-free.

The full power of a reductive operator is not used in proving the above result. As a matter of fact the following statement is true:

**Proposition 1.13.** An operator $A$ on $H$ is normal iff $A$ is quasinormal and $\eta(A^*)$ reduces $A$.

**Proof.** If $A$ is normal it is certainly quasinormal.
Also \( \eta(A^*) \) is clearly invariant for \( A^* \). If \( x \in \eta(A^*) \), then we need to show that \( Ax \in \eta(A^*) \). But for such an \( x \) we have \( A^*(Ax) = A(A^*x) = 0 \), hence \( \eta(A^*) \) reduces \( A \).

Conversely suppose that \( A \) is quasinormal and that \( \eta(A^*) \) reduces \( A \). Then \( A|\eta(A^*) \) is the zero operator which is clearly normal, and we only need to consider \( A|\eta(A^*) \). On \( \eta(A^*) \), \( A \) is still quasinormal and has dense range, so the proof of Proposition 1.12 shows that \( A|\eta(A^*) \) is normal. Thus \( A \) is normal.

It is well-known that an operator \( A \) in \( B(H) \) is invertible iff it satisfies the following two conditions:

\[
\begin{align*}
(1.7) & \quad \mathcal{R}(A) \text{ is dense in } H \text{ and} \\
(1.8) & \quad \|Ax\| \geq c\|x\| \text{ for some } c > 0 \text{ for all } x \in H.
\end{align*}
\]

**Proposition 1.14.** If \( A \) is reductive on \( H \), then \( A \) is invertible iff (1.8) holds.

**Proof.** It suffices to show that (1.8) implies (1.7) for reductive operators. Suppose (1.8) holds and that \( \xi \in \mathcal{R}(A)^{\perp} \). Then \( \xi \) is orthogonal to \( AA^*\xi \) and so \( (\xi, AA^*\xi) = 0 \). Thus \( (A^*\xi, A^*\xi) = 0 \) which implies \( A^*\xi = 0 \). Now, \( \overline{V(\xi)} \) reduces \( A \) so we also have \( A\xi = 0 \). Since (1.8) holds, we have \( 0 = \|A\xi\| \geq c\|\xi\| \) and so \( \xi = 0 \). It follows that \( A \) has dense range, i.e., (1.7) holds.
We might remark here that Proposition 1.14 is also true if 'reductive' is replaced by 'normal'.

Proposition 1.15. If $A$ is reductive on $H$ and $\lambda \in \sigma(A)$, then $\lambda$ is an approximate eigenvalue for $A$, (that is, $\sigma(A) = \pi(A)$).

Proof. The number $\lambda$ is in $\sigma(A)$ iff (1.8) fails for the operator $\lambda - A$. Thus for each $n$ we can find a unit vector $x_n$ such that $\|(\lambda - A)x_n\| < \frac{1}{n}$. This means that $\lambda$ is an approximate eigenvalue for $A$.

Proposition 1.16. The set of reductive operators on $H$ is not closed in the strong operator topology on $B(H)$.

Proof. Let $H$ be a separable infinite-dimensional Hilbert space and let $\{e_i : i \in \mathbb{Z}\}$ be an orthonormal basis for $H$. Define a sequence $\{u_n\}$ of operators on $H$ by

$$u_n(e_i) = \begin{cases} e_{i+1} & \text{if } -n \leq i < n \\ e_{-n} & \text{if } i = n \\ e_i & \text{if } |i| > n \end{cases}$$

For each $n$, $u_n u_n^* = u_n^* u_n = I$, and the sequence $\{u_n\}$ converges to the bilateral shift in the strong operator topology. For each $n$, $u_n^{2n+1} = I = u_n u_n^*$ so $u_n^* = u_n^{2n}$. As a result, every invariant subspace for $u_n$ is
invariant for $\gamma_n^*$ and $\gamma_n$ is reductive. The bilateral shift, as we saw earlier, is not reductive.

It is natural to ask what the SOT (strong operator topology) closure of the set of reductive operators is. It is known that the SOT closure of the set of normal operators is the set of subnormal operators (operators with normal extensions, cf. [12]). If the reductive operator conjecture is true, the SOT closure of the set of reductive operators would necessarily be contained in the set of subnormal operators. If one can show that the set of reductive operators is SOT dense in $B(H)$, this would show the reductive operator conjecture to be false, because $B(H)$ contains operators which are not subnormal.
CHAPTER II
INVERSES OF REDUCTIVE OPERATORS

R. G. Douglas has asked whether or not the inverse of an invertible intransitive operator is intransitive (cf. [12]). This question is clearly equivalent to the question of whether or not the inverse of an invertible transitive operator is transitive. A seemingly more general question is indicated by the following:

Proposition 2.1. If the inverse of each invertible reductive operator is reductive, then the inverse of each invertible transitive operator is transitive.

Proof. Suppose $A$ is transitive and $A^{-1}$ exists. Then $A$ is reductive so $A^{-1}$ is reductive by our hypothesis. Any invariant subspace for $A^{-1}$ reduces $A^{-1}$, and hence reduces $A$ as well. Since $A$ has no proper reducing subspaces, $A^{-1}$ has no proper invariant subspaces and $A^{-1}$ is transitive.
The converse of Proposition 2.1 has not been settled. According to (1.6), we can express a reductive operator $A$ as a direct integral of a family $\{A(m)\}_{m \in M}$ of transitive operators. If $A$ is invertible, then the "coordinate operators" $A(m)$ are invertible for almost all $m \in M$. If the inverse of a transitive operator is transitive, then $A^{-1}$, being the direct integral of the family $\{A^{-1}(m)\}_{m \in M}$, is a direct integral of a family of transitive operators. The problem is that we do not know that a direct integral of a family of transitive operators is reductive. Indeed, we do not even know when the direct sum of two transitive operators is reductive. (However, we shall soon see that if transitive operators exist on Hilbert spaces of dimension greater than 1, it is not true in general.) The following results suggest that this problem is non-trivial.

**Proposition 2.2** (Sarason). If $A$ is a normal operator on $H$, then the weakly closed algebra generated by $A$ is a $*$-algebra iff $A$ is reductive.

**Proof.** The reader is referred to [20].

**Proposition 2.3.** If $A$ is reductive and normal on $H$, then $A \oplus A$ is reductive and normal on $H \oplus H$. 
Proof. Let \( W(A) \) denote the weakly-closed algebra generated by \( A \). Let \( T \in W(A \otimes A) \) and let \( x_1 \otimes x_2 \) and \( y_1 \otimes y_2 \) be unit vectors in \( H \otimes H \). Let \( \epsilon > 0 \) be arbitrary. There exists a polynomial \( p \) such that

\[
|((T - p(A \otimes A))y_1 \otimes y_2, x_1 \otimes x_2)| < \frac{\epsilon}{3}.
\]

Then

\[
|((T^* - \overline{p}(A^* \otimes A^*))x_1 \otimes x_2, y_1 \otimes y_2)| < \frac{\epsilon}{3}
\]

also, where \( \overline{p} \) denotes the polynomial whose coefficients are the conjugates of the coefficients of \( p \). By Proposition 2.2, \( W(A) \) is a \(*\)-algebra, hence \( W(A) \) is strongly closed (cf. [5], p. 43, [6], p. 333, [16], p. 448). Thus, there exists a polynomial \( q \) such that

\[
||(p(A^*) - q(A))x_1|| < \frac{\epsilon}{3}
\]

for \( i = 1, 2 \). Hence

\[
|((T^* - q(A \otimes A))x_1 \otimes x_2, y_1 \otimes y_2)| \leq
\]

\[
|((T^* - \overline{p}(A^* \otimes A^*))x_1 \otimes x_2, y_1 \otimes y_2)|
\]

\[
+ |((\overline{p}(A^*) - q(A))x_1, y_1)|
\]

\[
+ |((\overline{p}(A^*) - q(A))x_2, y_2)| < \frac{\epsilon}{3} + \frac{\epsilon}{3}||y_1|| + \frac{\epsilon}{3}||y_2|| < \epsilon.
\]

The same calculations work for any finite sequence \( \{x_1^{(n)} \otimes x_2^{(n)}, y_1^{(n)} \otimes y_2^{(n)}\}_{n=1}^N \) and so \( T^* \in W(A \otimes A) \). The operator \( A \otimes A \) is clearly normal, (because \( A \) is), and \( W(A \otimes A) \) is a \(*\)-algebra, so by Proposition 2.2 \( A \otimes A \) is reductive.

Proposition 2.4. If \( A \) is any operator on \( H \), and if \( A \otimes A \) is reductive, then \( A \) is normal and reductive.

Proof. If \( A \otimes A \) is reductive, then \( A \) must be reductive, because it is the restriction of \( A \otimes A \) to the
reducing subspace $H \oplus \{0\}$. Consider the subspace of $H \oplus H$ defined by $\mathcal{M} = \{x \odot Ax : x \in H\}$. This subspace is invariant for $A \odot A$, so by our hypothesis it is also invariant for $A^* \odot A^*$. Let $x \in H$ be arbitrary. Then $(A^* \odot A^*)(x \odot Ax) = A^*x \odot A^*Ax \in \mathcal{M}$ so by the definition of $\mathcal{M}$ we have $A^*Ax = AA^*x$. Hence $A$ is normal.

**Corollary 2.5.** The following statements are equivalent:

(a) If $A$ is reductive, then $A \odot A$ is reductive.

(b) If $A$ is reductive, then $A$ is normal (i.e., the reductive operator conjecture is true).

**Proof.** Suppose (a) is true and $A$ is a reductive operator on $H$. Then $A \odot A$ is reductive on $H \oplus H$, and by Proposition 2.4 $A$ is normal. Conversely, suppose (b) is true and $A$ is a reductive operator on $H$. Then $A$ is both reductive and normal, so $A \odot A$ is reductive on $H \oplus H$ by Proposition 2.3.

Getting back to the problem of the inverse of an invertible reductive operator, let us first recall the resolvent function $\lambda \rightarrow (\lambda - A)^{-1}$ for $\lambda \in \rho(A)$. For any two vectors $x, y \in H$, the mapping $\lambda \rightarrow ((\lambda - A)^{-1}x, y)$ is an analytic function on the open set $\rho(A)$.

**Proposition 2.6.** If the subspace $\mathcal{M}$ of $H$ is invariant for the operator $A$ on $H$ and if $\lambda$ is in the unbounded
component of $\rho(A)$, then $m$ is invariant for $(\lambda A)^{-1}$.

**Proof.** For $|\lambda| > \|A\|$ we have $(\lambda A)^{-1} = \sum_{p=0}^{\infty} \lambda^{-p}A^p$.

It follows that $(\lambda A)^{-1}$ is invariant on $m$ whenever $|\lambda| > \|A\|$. Choose vectors $x \in m$ and $y \in m^\perp$. Then the analytic function $\lambda \mapsto ((\lambda A)^{-1}x, y)$ is zero for $|\lambda| > \|A\|$, hence it is zero everywhere on the unbounded component of $\rho(A)$. Since $x \in m$ and $y \in m^\perp$ are arbitrary, the result follows.

**Proposition 2.7.** If $0$ is in the unbounded component of $\rho(A)$, then $A$ and $A^{-1}$ have exactly the same invariant subspaces.

**Proof.** Since $0$ is in the unbounded component of $\rho(A)$, we can find a curve $\Gamma$ which contains $\sigma(A)$ in its interior such that $0$ is exterior to $\Gamma$, and such that $\Gamma$ lies entirely in the unbounded component of $\rho(A)$. The function $\lambda \mapsto \lambda^{-1}$ is analytic on $\Gamma$ and its interior so, according to the operational calculus, we have

$$A^{-1} = \frac{1}{2\pi i} \oint_{\Gamma} \lambda^{-1}(\lambda A)^{-1}d\lambda.$$ 

If $m$ is an invariant subspace for $A$, it is also invariant for $(\lambda A)^{-1}$ for each $\lambda \in \Gamma$. As a result $m$ is invariant for $A^{-1}$. Since $0$ is also in the unbounded component of $\rho(A^{-1})$, the rest of the proposition follows by symmetry.
Corollary 2.8. If 0 is in the unbounded component of $\rho(A)$, then
(a) $A^{-1}$ is transitive if $A$ is transitive.
(b) $A^{-1}$ is intransitive if $A$ is intransitive.
(c) $A^{-1}$ is reductive if $A$ is reductive.

Proof. Statements (a) and (b) are immediate by Proposition 2.7. Suppose $A$ is reductive and $\mathcal{M}$ is invariant for $A^{-1}$. Then $\mathcal{M}$ is invariant for $A$ by Proposition 2.7. Hence $\mathcal{M}$ reduces $A$, and so $\mathcal{M}$ reduces $A^{-1}$. This means that $A^{-1}$ is reductive.

Corollary 2.9. If $A$ is reductive, transitive or intransitive, respectively, there exists a translation $(\alpha+A)$ of $A$ such that $(\alpha+A)^{-1}$ exists and is reductive, transitive or intransitive, respectively.

Proof. We may assume $A \neq 0$. Choose $\alpha = 2\|A\| > 0$. Then $0$ is in the unbounded component of $\rho(\alpha+A)$, so we apply Proposition 2.8 to the operator $\alpha+A$.

The next result gives a sufficient (but not necessary) condition on $\sigma(A)$ to insure that $0$ is in the unbounded component of $\rho(A)$.

Proposition 2.10. If for some real number $t$,
$\sigma(A) \cap e^{it} \cdot \sigma(A) = \emptyset$, then $0$ is in the unbounded
Proof. Choose $\delta > 0$ so that $\text{dist}(\sigma(A), e^{it}\sigma(A)) > 3\delta$, and cover $\sigma(A)$ with a finite collection of balls with radius $\delta$ and centers in $\sigma(A)$. Call this open cover $\mathcal{U}$. It follows that $\mathcal{U} \cap e^{it}\mathcal{U} = \emptyset$ by our choice of $\delta$. Each component of the boundary of $\mathcal{U}$ is a Peano space. If no component of $\mathcal{U}$ separates 0 from $\infty$, there is a path from 0 to $\infty$ missing $\mathcal{U}$, and hence it also misses $\sigma(A)$. Thus we may suppose that some component $\mathcal{U}_0$ of $\mathcal{U}$ separates 0 from $\infty$. Let $C_\infty$ denote the unbounded component of $C \backslash \mathcal{U}_0$, and let $B = \text{bdry}(\mathcal{U}_0) \cap C_\infty$. Then $B$ is a Peano space and $B$ separates 0 from $\infty$ (cf. [14], Theorem 3.3). Let $\lambda_0$ be a point on $B$ with minimal modulus and let $\mu_0$ be a point on $B$ with maximal modulus. Then $\lambda_0 \not\in e^{it}B$, so $\lambda_0$ is path-connected to 0 in $C \setminus e^{it}B$ because components of $C \setminus e^{it}B$ are path-connected. Likewise, $\mu_0 \not\in e^{it}B$, so $\mu_0$ is path-connected to $\infty$. Now, $e^{it}B$ separates 0 from $\infty$ because $B$ does, and $B$ is path-connected so $B$ contains a path from $\mu_0$ to $\lambda_0$. Then $B$ must intersect $e^{it}B$, because every path from $\lambda_0$ to $\mu_0$ intersects $e^{it}B$. This means $\mathcal{U} \cap e^{it}\mathcal{U} \neq \emptyset$, a contradiction. Hence no component of $\mathcal{U}$ separates 0 from $\infty$, so $\sigma(A)$ does not separate 0 from $\infty$. 
Note that the truth of the invariant subspace conjecture will imply that the inverse of an invertible transitive operator is transitive, (because all transitive operators are then operators on one-dimensional Hilbert spaces). It was (and still is) hoped that the truth of the reductive operator conjecture would imply that the inverse of an invertible reductive operator is reductive. At the time of this writing, a proof of the above implication has not been achieved. However, the following information may be enlightening:

Suppose that \( A \) is a normal reductive operator and that \( A^{-1} \) exists. We let \( \Delta_1 = \{ z \in \mathbb{C} : \text{re}(z) < 0 \} \) and \( \Delta_2 = \{ z \in \mathbb{C} : \text{re}(z) \geq 0 \} \). Then \( \Delta_1 \) and \( \Delta_2 \) are two disjoint Borel subsets of \( \mathbb{C} \), and we have two corresponding projections \( E_1 = E(\Delta_1) \) and \( E_2 = E(\Delta_2) \) in the spectral family of \( A \). Furthermore, \( E_1E_2 = 0 = E_2E_1 \) because \( \Delta_1 \) and \( \Delta_2 \) are disjoint, and \( E_1 + E_2 = I \) because \( \Delta_1 \cup \Delta_2 = \mathbb{C} \). We let \( H_1 = \{ x \in \mathcal{H} : E_1x = x \} \) and \( A_i = A_j|_{H_j} \) for \( i=1,2 \). Then \( A = A_1 \oplus A_2 \) on the Hilbert space \( \mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H} \). Note that each of the operators \( A_1 \) and \( A_2 \) is normal and reductive (because \( \mathcal{H}_1 \) reduces \( A_1 \)), and that \( \sigma(A_1) \subset \overline{\mathbb{A}_1} \cap \sigma(A) \) so that the hypotheses of Corollary 2.8 are satisfied. As a result \( A_1^{-1} \) and \( A_2^{-1} \) are reductive operators on the Hilbert spaces
$U_1$ and $U_2$ respectively. This means $A^{-1} = A_1^{-1} \oplus A_2^{-1}$ is the direct sum of two reductive normal operators. We know that $A_1, A_2, A_1^{-1}, A_2^{-1}$ and $A_1 \oplus A_2$ are normal and reductive. Can we prove that $A_1^{-1} \oplus A_2^{-1}$ is reductive? The fact that $A_1 \oplus A_2$ is reductive must not be neglected because the statement is false otherwise. (In other words, it is false that the direct sum of any two reductive operators is reductive.) For example, let $U$ be a bilateral shift and decompose $U$ in the manner described above. We then have $U = U_1 \oplus U_2$ and $U_1^{-1} = U_1^*$ for $i=1,2$. Since every invariant subspace for $U_i$ is invariant for $U_i^{-1} = U_i^*$, $i=1,2$, we have the result that the non-reductive operator $U$ is the direct sum of two reductive operators. Thus we have:

Proposition 2.11. The direct sum of two reductive operators need not be reductive.

The same example can be used to demonstrate the following:

Proposition 2.12. The sum of two commuting reductive operators need not be reductive.

Proof. Let $U$ be a bilateral shift and let $\Lambda_1, \Lambda_2$ be as before. Then $U = U_1 + U_2$ where $U_1 = U E_1$ for
Clearly $\mathcal{U}_1 \mathcal{U}_2 = 0 = \mathcal{U}_2 \mathcal{U}_1$ so we only need to show that $\mathcal{U}_1$ and $\mathcal{U}_2$ are reductive. Now

$\mathcal{U}_1 = \mathcal{U}_1|_{H_1} \oplus 0$ and $\mathcal{U}_2 = 0 \oplus \mathcal{U}_2|_{H_2}$ where $H_1, H_2$ are as in the preceding example and $W(\mathcal{U}_1) = W(\mathcal{U}_1|_{H_1}) \oplus \{0\}$,

$W(\mathcal{U}_2) = \{0\} \oplus W(\mathcal{U}_2|_{H_2})$. By Proposition 2.2, $W(\mathcal{U}_1)$ and $W(\mathcal{U}_2)$ are $*$-algebras, so $\mathcal{U}_1$ and $\mathcal{U}_2$ are reductive as desired.
CHAPTER III
POWERS OF REDUCTIVE OPERATORS

In this chapter we will investigate powers of operators. In particular we will be concerned with finding sufficient conditions on an operator $A$ on $H$ that subspaces invariant for $A^n$ ($n \neq 0$) are invariant for $A$.

We shall restrict our attention to the case where $0$ is in the unbounded component of $\rho(A)$. This implies that there is a path $S$ from $0$ to $\infty$ which misses $\sigma(A)$. We may, without loss of generality, assume that $S$ is simple and that $S$ coincides with the negative real axis in some neighborhood of $\infty$. For such a path $S$, we will define $L_S(\lambda)$ to be the branch of the logarithm function on $C \setminus S$ which is real for large positive values of $\lambda$. The function $L_S(\lambda)$ is then analytic on $\sigma(A)$, so we can choose a path $\Gamma$ which encloses $\sigma(A)$ and which does not intersect $S$ and define the operator
(3.1) \[ L_S(A) = \frac{1}{2\pi i} \int \frac{L_S(\lambda)(\lambda - A)^{-1} d\lambda}{\Gamma} \]

by the operational calculus.

**Proposition 3.1.** If \( L_S(A) \) is the operator defined above, then \( A = \exp(L_S(A)) \).

**Proof.** The function \( \exp L_S(\lambda) \) is analytic on \( \sigma(A) \), so
\[
\exp(L_S(A)) = \frac{1}{2\pi i} \int_{\Gamma} \exp(L_S(\lambda))(\lambda - A)^{-1} d\lambda
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma} \lambda \cdot (\lambda - A)^{-1} d\lambda
\]
\[
= A.
\]

Thus it is possible to recover the operator \( A \) from the operator \( L_S(A) \).

**Proposition 3.2.** If a path \( S \) from \( 0 \) to \( \infty \) misses \( \sigma(A^n) \), \( n \neq 0 \), then \( A = \exp \left( \frac{1}{n} L_S(A^n) \right) \).

**Proof.** The function \( \exp \left( \frac{1}{n} \cdot L_S(\lambda^n) \right) \) is analytic on \( \sigma(A) \), so
\[
\exp \left( \frac{1}{n} L_S(A^n) \right) = \frac{1}{2\pi i} \int_{\Gamma} \exp \left( \frac{1}{n} L_S(\lambda^n) \right)(\lambda - A)^{-1} d\lambda
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma} \exp \left( L_S(\lambda) \right)(\lambda - A)^{-1} d\lambda
\]
Proposition 3.3. If $S$ is a path from 0 to $\infty$ missing $\sigma(A)$, then $A$ and $L_s(A)$ have exactly the same invariant subspaces.

Proof. If $m$ is invariant for $A$, then $(\lambda - A)^{-1}$ is invariant on $m$ for all $\lambda \in \Gamma$ because $\Gamma$ lies in the unbounded component of $\rho(A)$. Thus $L_s(A)$ is invariant on $m$.

Conversely, if $m$ is invariant for $L_s(A)$, it is also invariant for $\exp(L_s(A)) = A$.

Corollary 3.4. If $S$ is a path from 0 to $\infty$ missing $\sigma(A)$, then

(a) $A$ is transitive iff $L_s(A)$ is transitive
(b) $A$ is reductive iff $L_s(A)$ is reductive.

Proof. Part (a) is immediate by Proposition 3.3. Suppose $A$ is reductive and $m$ is invariant for $L_s(A)$. Then $m$ is invariant for $A$, so $m$ reduces $A$. The projection $P$ which maps $H$ onto $m$ commutes with $A$, hence $P$ commutes with $(\lambda - A)^{-1}$ for each $\lambda \in \Gamma$, hence $P$ commutes with $L_s(A)$. This means that $m$ reduces $A$. 

\[
= \frac{1}{2\pi i} \int_{\Gamma} \lambda \cdot (\lambda - A)^{-1} d\lambda 
= A.
\]
Conversely, if $L_S(A)$ is reductive and $\mathcal{M}$ is invariant for $A$, then $\mathcal{M}$ reduces $L_S(A)$ by Proposition 3.3. This means that $P$ commutes with $L_S(A)$, hence $P$ commutes with $\exp(L_S(A)) = A$ and $\mathcal{M}$ reduces $A$.

**Proposition 3.5.** If $n \neq 0$ and if $S$ is a path from 0 to $\infty$ missing $\sigma(A^n)$, then every invariant subspace for $A^n$ is invariant for $A$.

**Proof.** By Proposition 3.2 we have $A = \exp\left(\frac{1}{n} L_S(A^n)\right)$. If $\mathcal{M}$ is invariant for $A^n$, it is also invariant for $L_S(A^n)$ by Proposition 3.3. Thus it is also invariant for $\exp\left(\frac{1}{n} \cdot L_S(A)\right) = A$.

If $n > 0$, the converse of Proposition 3.5 is clearly true. If $n < 0$ and if $S'$ is a path from 0 to $\infty$ missing $\sigma(A)$, then every invariant subspace for $A$ is invariant for $A^{-1}$ by Proposition 2.7, and so is invariant for $A^n$. Thus we have:

**Corollary 3.6.** (a) If $n < 0$ and if $S$ and $S'$ are paths from 0 to $\infty$ missing $\sigma(A)$ and $\sigma(A^n)$ respectively, then $A$ and $A^n$ have exactly the same invariant subspaces.

(b) If $n > 0$ and if $S$ is a path from 0 to $\infty$ mis-
sing $\sigma(A^n)$, then $A$ and $A^n$ have exactly the same invariant subspaces.

**Proposition 3.7.** If $n \neq 0$, if $S$ is a path from 0 to $\infty$ missing $\sigma(A^n)$, and if $A$ is reductive, then $A^n$ is reductive.

**Proof.** If $A^n$ is invariant on $\mathfrak{m}$, then so is $A$ by Proposition 3.5. Thus the projection $P$ onto $\mathfrak{m}$ commutes with $A$, hence it commutes with $A^n$. It follows that $A^n$ is reductive.

**Proposition 3.8.** (a) If $n > 0$, if $S$ is a path from 0 to $\infty$ missing $\sigma(A^n)$, and if $A^n$ is reductive, then $A$ is reductive.

(b) If $n < 0$, if $S$ and $S'$ are paths from 0 to $\infty$ missing $\sigma(A)$ and $\sigma(A^n)$ respectively, and if $A^n$ is reductive, then $A$ is reductive.

**Proof of (a).** Suppose $\mathfrak{m}$ is invariant for $A$. Then $\mathfrak{m}$ is also invariant for $A^n$, so by our hypothesis, $\mathfrak{m}$ is invariant for $(A^*)^n = (A^n)^*$. The path $\overline{S} = \{\overline{\lambda}: \lambda \in S\}$ misses $\sigma((A^*)^n)$, so $\mathfrak{m}$ is invariant for $A^*$ by Proposition 3.5. This proves that $A$ is reductive.

**Proof of (b).** Suppose $\mathfrak{m}$ is invariant for $A$. By
Corollary 3.6 (a) $\mathcal{M}$ is also invariant for $A^n$. Since $A^n$ is reductive, $\mathcal{M}$ is invariant for $(A^*)^n$, and again $S$ is a path from $0$ to $\infty$ missing $\sigma((A^*)^n)$, so $\mathcal{M}$ is invariant for $A^*$ by Proposition 3.5. Thus in this case we also have shown that $A$ is reductive.

One consequence of Proposition 3.5 is this: If $S$ is a path from $0$ to $\infty$ missing $\sigma(A^n)$, $n \neq 0$, and if $A$ is transitive, then $A^n$ is transitive. Our next result will show, in particular, that no non-zero power of a transitive operator can have a finite-dimensional invariant subspace.

Proposition 3.9. If $A$ is a reductive normal-free operator on $H$ and $n \neq 0$, then any proper invariant subspace for $A^n$ must be infinite-dimensional.

Proof. We will suppose that $A^n$ has a finite-dimensional invariant subspace $\mathcal{M}$ and show that we must have $\mathcal{M} = \{0\}$. First we note that $A^k(\mathcal{M})$, the image of $\mathcal{M}$ under $A^k$, is finite-dimensional for every $k \in \mathbb{Z}$. If $n > 0$ we will let $\mathcal{M}_n = \mathcal{M} + A(\mathcal{M}) + \cdots + A^{n-1}(\mathcal{M})$, the linear manifold generated by the manifolds $\{A^k(\mathcal{M}): 0 \leq k \leq n-1\}$. Notice that $\mathcal{M}_n$ is finite-dimensional and hence $\mathcal{M}_n$ is closed (i.e., $\mathcal{M}_n$ is a subspace). Also $\mathcal{M}_n$ is invariant for $A$ because $A(A^{n-1}(\mathcal{M})) = A^n(\mathcal{M}) \subset \mathcal{M}$. Because $A$ is reductive, \[ \mathcal{M}_n \subset \mathcal{M}_n \] and so $\mathcal{M}_n = \mathcal{M}$.
the restriction of $A$ to $\mathcal{H}_n$ is reductive. However, any reductive operator on a finite-dimensional space must be normal (cf. Ando [1]). Thus we must have $\mathcal{H}_n = \{0\}$ and $\mathcal{H} = \{0\}$, because $A$ is normal-free.

In the case where $n < 0$, we only need to note that $A^n(\mathcal{H}) \subset \mathcal{H}$ implies $\mathcal{H} \subset A^{-n}(\mathcal{H})$, and since $A^{-n}$ cannot increase dimension we have $\mathcal{H} = A^{-n}(\mathcal{H})$. Thus $\mathcal{H}$ is invariant for $A^{-n}$ and $-n > 0$. Now the proof for positive powers applies and we conclude that $\mathcal{H} = \{0\}$. 
CHAPTER IV
RESOLVENT TECHNIQUES

This chapter grew out of an attempt to find invariant subspaces for quasinilpotent operators.

Recall that the resolvent function $\lambda \mapsto (\lambda - A)^{-1}$ of an operator $A$ on $H$ is an analytic function defined on the open set $\rho(A)$. This means that, for any two vectors $x$ and $y$ in $H$, the function $\lambda \mapsto ((\lambda - A)^{-1}x, y)$ is analytic on $\rho(A)$. For $|\lambda| > \|A\|$ the equation

$$(\lambda - A)^{-1} = \sum_{p=0}^{\infty} \lambda^{-p-1}A^p$$

is valid, where the series converges in the uniform topology on $B(H)$. This gives the Laurent expansion

$$(4.1) \quad ((\lambda - A)^{-1}x, y) = \sum_{p=0}^{\infty} (A^p x, y)\lambda^{-p-1}, \quad |\lambda| > \|A\|.$$  

By 'resolvent techniques', we will mean growth conditions on the resolvent function. These have been investigated by several others previously. The reader is referred to Putnam [19] and Stampfli [21] and to their
bibliographies for further information. An operator will be called a \((G_n)\) operator if the inequality

\[(4.2) \quad \|\{(\lambda - A)^{-1}\} \leq K/(\text{dist} [\lambda, \sigma(A)])^n\]

is satisfied for some positive integers \(K\) and \(n\) for all \(\lambda \in \rho(A)\). If this inequality is satisfied on a set \(U \cap \rho(A)\), where \(U\) is a neighborhood of \(\sigma(A)\), we will say that \(A\) is locally \((G_n)\). If \(\lambda_0 \in \sigma(A)\), and if there exists a neighborhood \(U\) of \(\lambda_0\) such that (4.2) holds in \(U \cap \rho(A)\), we will say that \(A\) is \((G_n)\) at \(\lambda_0\). Thus a locally \((G_n)\) operator is \((G_n)\) at each point in its spectrum, and conversely, if an operator is \((G_n)\) at each point in its spectrum it is locally \((G_n)\).

If \(\lambda_0 \in \sigma(A)\), and if \(\{\lambda_k\}_{k=1}^\infty\) is a sequence in \(\rho(A)\) converging to \(\lambda_0\) such that inequality (4.2) holds for each \(\lambda_k\), we will say that \(A\) is sequentially \((G_n)\) at \(\lambda_0\). A sequentially \((G_n)\) operator is an operator which is sequentially \((G_n)\) at each point in its spectrum. Clearly, \(A\) is sequentially \((G_n)\) at \(\lambda_0\) if it is locally \((G_n)\) at \(\lambda_0\).

A weaker growth condition on an operator \(A\) is given by the inequality

\[(4.3) \quad |((\lambda - A)^{-1}x,y)| \leq K/[\text{dist} (\lambda, \sigma(A))]^n ,\]
where \( x, y \in \mathbb{H} \) and \( k, n \) are positive integers. An operator which satisfies (4.3) on \( \rho(A) \) will be called a \((G_n, x, y)\) operator. We can define the notions of locally \((G_n, x, y)\), \((G_n, x, y)\) at \( \lambda_0 \in \sigma(A) \), sequentially \((G_n, x, y)\) at \( \lambda_0 \in \sigma(A) \), and sequentially \((G_n, x, y)\) in a manner analogous to the case for \((G_n)\) operators above.

It has been shown that every hyponormal operator \( A \) (that is, \( A^*A - AA^* \) is positive) is a \((G_1)\) operator (cf. [18], p. 58). A \((G_1)\) operator is locally \((G_n)\) for \( n > 1 \); simply let \( \mathcal{U} = \{ z \in \rho(A) : \text{dist}(z, \sigma(A)) < 1 \} \). Thus we have many examples of operators which satisfy the above restrictions. It is easy to show that any of the above restrictions are invariant under translations and multiplication by non-zero scalars, so if \( A \) satisfies one of the above restrictions so do \( \alpha + A \) and \( \beta A \) for \( \beta \neq 0 \).

An important and still unsolved problem is the question of whether or not every quasinilpotent operator has a proper invariant subspace. In what follows, we will impose a growth condition of the type (4.2) on a quasinilpotent operator. The result obtained shows that this is a very strong restriction on a quasinilpotent operator. It forces the operator to be nilpotent.

Note that for the case of a quasinilpotent operator
A we have \( \text{dist} [\lambda, \sigma(A)] = |\lambda| \).

**Proposition 4.1.** If \( A \) is quasinilpotent, and if there exists a neighborhood \( \mathcal{U} \) of 0 and an analytic function \( f(\lambda) \) on \( \mathcal{U} \) such that \( \| f(\lambda)(\lambda-A)^{-1} \| \leq K \) for some constant \( K \) and for all \( 0 \neq \lambda \in \mathcal{U} \), then \( A \) is nilpotent. Conversely, if \( A \) is nilpotent of order \( N \), then \( \|\lambda^N(\lambda-A)^{-1}\| \) is bounded in a neighborhood of \( 0 \).

**Proof.** Suppose \( f(\lambda) \) is analytic and \( \| f(\lambda)(\lambda-A)^{-1} \| \leq K \) in some neighborhood \( \mathcal{U} \) of 0. This forces \( f(0) = 0 \), because \( \| (\lambda-A)^{-1} \| \rightarrow \infty \) as \( \lambda \rightarrow 0 \). Let \( f(\lambda) = \lambda^N \cdot g(\lambda) \), where \( g(0) \neq 0 \). We may assume that \( g(\lambda) \) is bounded away from 0 on \( \mathcal{U} \) so that \( \|\lambda^N(\lambda-A)^{-1}\| \) is bounded on \( \mathcal{U} \) by \( K/\inf \{ |g(\lambda)| : \lambda \in \mathcal{U} \} \). Now, for any two vectors \( x \) and \( y \) in \( H \), the function \((\lambda^N(\lambda-A)^{-1}x,y)\) is bounded in the neighborhood \( \mathcal{U} \), so it has a removable discontinuity at 0. It follows, by the Cauchy integral theorem, that \( (A^Nx,y) = \frac{1}{2\pi i} \int_{\Gamma} (\lambda^N \cdot (\lambda-A)^{-1}x,y) d\lambda = 0 \)

where \( \Gamma \) is any curve in \( \mathcal{U} \) containing 0 in its interior. Since \( x \) and \( y \) are arbitrary, we have \( A^N = 0 \).

Conversely, suppose \( A^N = 0 \). Then for \( \lambda \neq 0 \), we have \( (\lambda-A)^{-1} = \sum_{p=0}^{N-1} \lambda^{-p} A^p \) and...
\[ \|\lambda^N(\lambda-A)^{-1}\| = \| \sum_{p=0}^{N-1} A^p \lambda^{N-p-1} \| \leq \sum_{p=0}^{N-1} \| A^p \| \text{ for } |\lambda| < 1. \]

Thus we see that a quasinilpotent operator \( A \) is nilpotent iff it is locally \((G_n)\) for some \( n \). For sequentially \((G_n)\) operators our results are not as impressive.

**Proposition 4.2.** If \( A \) is sequentially \((G_n)\) and nilpotent of order \( m \), then \( m \leq n \).

**Proof.** We have a sequence \( \{\lambda_j\}_{j=1}^\infty \) converging to 0 such that \( \|(\lambda_j-A)^{-1}\| \leq K/|\lambda_j|^n \) for \( j=1,2,... \). Since \( A^m = 0 \) we have \( (\lambda_j-A)^{-1} = \sum_{p=0}^{m-1} \lambda_j^{p-1} A^p \), so

\[ \| \lambda_j^{m-1} \sum_{p=0}^{m-1} \lambda_j^{p-1} A^p \| = \| \sum_{p=0}^{m-1} \lambda_j^{n-p-1} A^p \| \leq K. \]

If \( m > n \), we have \( \| \lambda_j^{n-m} A^{m-1} \| - \| \sum_{p=0}^{m-2} \lambda_j^{n-p-1} A^p \| \leq K \), but this is impossible because

\[ \| \lambda_j^{n-m} A^{m-1} \| - \| \sum_{p=0}^{m-2} \lambda_j^{n-p-1} A^p \| \geq |\lambda_j^{n-m}| \cdot \| A^{m-1} \| - \sum_{p=0}^{m-2} |\lambda_j^{n-p-1}| \cdot \| A^p \|, \]

and the right hand side of the above inequality becomes infinite as \( \lambda_j \to 0 \). Hence \( m \leq n \) as claimed.

This means that a sequentially \((G_n)\) nilpotent operator is locally \((G_n)\). In particular, a sequentially
(G₁) nilpotent operator must be the zero operator. A similar result by B. Wadhwa [22] states that any sequentially \((G₁)\) algebraic operator is normal.

Proposition 4.3. If \(A\) is quasinilpotent and sequentially \((Gₙ)\), that is, if \(A\) is quasinilpotent, \(λ_k → 0\) and \(\|λ_k^n(λ_k-A)^{-1}\|\) is bounded, then \(\|A^n(λ_k-A)^{-1}\|\) is bounded. In particular, \(0\) is an approximate eigenvalue for \(A\).

Proof. Since \(A\) is quasinilpotent we have \((λ_k-A)^{-1} = \sum_{p=0}^{∞} λ_k^{-p-1}A^p\). Let \(K\) be a bound for \(\|λ_k^n(λ_k-A)^{-1}\|\).

Then \(K ≥ \|λ_k^n(\sum_{p=0}^{∞} λ_k^{-p}A^p)\| = \|\sum_{p=0}^{∞} λ_k^nλ_k^{-p}A^p\| ≥ \|\sum_{p=0}^{∞} λ_k^nλ_k^{-p}A^p\| - \|\sum_{p=0}^{∞} λ_k^nλ_k^{-p}A^p\|\). There is no loss of generality in assuming that \(|λ_k| < 1\) for each \(k\). Then \(\|\sum_{p=0}^{n-1} λ_k^nλ_k^{-p}A^p\| ≤ \sum_{p=0}^{n-1} \|A^p\|\). If we let \(K' = K + \sum_{p=0}^{∞} \|A^p\|\), we have \(\|\sum_{p=0}^{∞} λ_k^nλ_k^{-p}A^p\| = \|\sum_{p=0}^{∞} λ_k^nλ_k^{-p}A^{n+p}\| = \|A^n(\sum_{p=0}^{∞} λ_k^nλ_k^{-p}A^p)\| = \|A^n(λ_k-A)^{-1}\| ≤ K'\) as desired. Next we will show that \(0 \in π(A^n)\). Since \(\|(λ_k-A)^{-1}\| → ∞\) as \(k → ∞\) we may assume \(\|(λ_k-A)^{-1}\| > k\). Then we can find a unit vector \(x_k\) such that \(\|(λ_k-A)^{-1}x_k\| > k\). Let
\[ y_k = \frac{(\lambda_k - A)^{-1}x_k}{\| (\lambda_k - A)^{-1}x_k \|} \]. It follows that

\[ \| A^n y_k \| = \| \frac{A^n(\lambda_k - A)^{-1}x_k}{\| (\lambda_k - A)^{-1}x_k \|} \| \leq \frac{k}{k} \rightarrow 0 \text{ as } k \rightarrow \infty \text{ and } 0 \in \pi(A^n) \]

as claimed. Now by Problem 59 of [12], \( \pi(A^n) = [\pi(A)]^n = [0] \) so \( \pi(A) = [0] \) and the proof is complete.

A growth condition more closely related to the problem of finding invariant subspaces for quasinilpotent operators is given by the inequality (4.3). We first prove the following result for an arbitrary operator in \( B(H) \).

**Proposition 4.4.** If \( A \in B(H) \), \( x \) and \( y \) are non-zero vectors in \( H \), and the function \( \lambda \rightarrow ((\lambda - A)^{-1}x,y) \) on \( \rho(A) \) has a meromorphic extension to the entire complex plane, then \( A \) has a proper invariant subspace.

**Proof.** Suppose \( f(\lambda) = ((\lambda - A)^{-1}x,y) \) has a meromorphic extension \( F(\lambda) \) to all of \( C \). Let \( \alpha_1, \cdots, \alpha_n \) be the distinct poles of \( F \), and let \( Y_i \) be the order of the pole at \( \alpha_i \) for \( i = 1, \cdots, n \). Let \( G(\lambda) = \prod_{i=1}^{n} (\lambda - \alpha_i)^{Y_i} F(\lambda) \).

Then \( G(\lambda) \) is analytic on all of \( C \). For any positive integer \( m \), and any curve \( C \) in \( \rho(A) \) enclosing \( \sigma(A) \), we have
\[ (A^m \prod_{i=1}^{\infty} (A - \alpha_i)^{-1} x, y) = \frac{1}{2\pi i} \int_\Gamma (\lambda^m \prod_{i=1}^{\infty} (\lambda - \alpha_i)^{-1}] (\lambda - A)^{-1} x, y) \, d\lambda \]
\[ = \frac{1}{2\pi i} \int_\Gamma \lambda^m G(\lambda) \, d\lambda = 0. \]

Since \( y \neq 0 \), the vector \( \prod_{i=1}^{\infty} (A - \alpha_i)^{-1} x \) is not cyclic for \( A \). If this vector is non-zero, \( A \) has a proper invariant subspace, namely \( \bigvee \{ A^m \prod_{i=1}^{\infty} (A - \alpha_i)^{-1} x : m \in \mathbb{Z}^+ \cup \{0\} \} \).

If \( \prod_{i=1}^{\infty} (A \alpha_i)^{-1} x = 0 \) then one of the numbers \( \alpha_i \) is an eigenvalue for \( A \) and again \( A \) has a proper invariant subspace.

The next result is a converse to Proposition 4.4.

**Proposition 4.5.** If \( A \in \mathcal{B}(H) \) has a proper invariant subspace, then there exist non-zero vectors \( x, y \in H \) such that the restriction of the function \( f(\lambda) = ((\lambda - A)^{-1} x, y) \) to the unbounded component of \( \rho(A) \) has a meromorphic extension to all of \( C \).

**Proof.** Suppose \( \mathcal{M} \) is a proper invariant subspace for \( A \). Choose non-zero vectors \( x \in \mathcal{M} \) and \( y \in \mathcal{M}^\perp \). By Proposition 2.6, \( ((\lambda - A)^{-1} x, y) = 0 \) for all \( \lambda \) in the unbounded component of \( \rho(A) \). Thus \( F(\lambda) = 0 \) for all \( \lambda \in C \) is a meromorphic extension to all of \( C \).

Thus, in the case where \( \sigma(A) \) is a finite set of
points, $A$ has a proper invariant subspace iff for some vectors $x, y \neq 0$ the function $\lambda \to ((\lambda-A)^{-1}x, y)$ defined on $\rho(A)$ has a pole (possibly of order 0) at each point in $\sigma(A)$. For a quasinilpotent operator $A$, we have

**Corollary 4.6.** If $A$ is quasinilpotent, then $A$ has a proper invariant subspace iff $|((\lambda-A)^{-1}x, y)| \leq K/|\lambda|^n$ for some constants $K$ and $n$ and for $\lambda$ in some neighborhood $\mathcal{U}$ of 0, that is, $A$ is locally $(G_n, x, y)$ for some non-zero vectors $x, y \in H$ and some $n$.

Thus we see that a transitive operator on a separable infinite-dimensional Hilbert space (if one exists) must be badly behaved. That is to say, for any pair of non-zero vectors $x, y \in H$, the function $\lambda \to ((\lambda-A)^{-1}x, y)$ must have an essential singularity at $\lambda = 0$.

In light of the bad behavior of transitive quasinilpotent operators as witnessed above, it is clear that a transitive quasinilpotent operator on a separable infinite-dimensional Hilbert space must be sequentially $(G_n, x, y)$ for any integer $n \geq 0$, and for any vectors $x, y \in H$. This is true because the function $\lambda \to ((\lambda-A)^{-1}x, y)$ has values arbitrarily close to 0 in any neighborhood of 0. Thus the notion of sequentially
Proposition 4.7. If the dimension of $H$ is greater than 1, and if $A$ is a quasinilpotent operator on $H$, then $A$ is sequentially $(G_0, x, y)$ for some non-zero vectors $x$ and $y$ in $H$.

Proof. If $A$ has a proper invariant subspace, we choose non-zero vectors $x \in \mathbb{M}$ and $y \in \mathbb{M}^\perp$. Then for any sequence $\{\lambda_k\}$ converging to zero we have $((\lambda_k - A)^{-1}x, y) = 0$. If, on the other hand, $A$ has no proper invariant subspaces we choose $x$ and $y$ arbitrarily. Since the function $\lambda \mapsto ((\lambda - A)^{-1}x, y)$ has an essential singularity at 0, we can choose a sequence $\{\lambda_k\}$ converging to zero such that $|((\lambda_k - A)^{-1}x, y)| < 1$ for each $k$. 

$(G_n, x, y)$ operators will not aid the search for invariant subspaces for quasinilpotent operators. In fact, we have the following result:
BIBLIOGRAPHY


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