1973

Approximation-Numbers of Bounded Linear-Operators.

Charlene Victoria Hutton
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The Department of Mathematics

by

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ABSTRACT

This paper is a study of operators of type $\ell^p, \ell^p(E,F)$, and related topics. In Chapter I we compute exactly or obtain asymptotic bounds for the approximation numbers of diagonal operators on the $\ell^p$-spaces, essentially completing a study initiated by Pietsch. As a corollary we are able to compute the Kolmogoroff diameters of certain compacta in the $\ell^p$-spaces. Using these calculations we are able to prove the $\ell^p$-analog of the Grothendieck-Tong theorem for diagonal nuclear operators.

In Chapter II we obtain some results on the space $\ell^p(E,F), 0<p\leq 1$, which parallel Grothendieck's development of the strongly $p$-summable operators $L^{(p)}(E,F)$. We show that the spaces $\ell^p(E,F)$ and $L^{(p)}(E,F)$ are, in general, distinct for $0<p\leq 1$ and prove that $T\in \ell^p(E,F)$ if and only if $T'\in \ell^p(F',E'), 0<p\leq \infty$.

In Chapter III we introduce a new class of operators $\mathfrak{F}_p(E,F)$ on Banach spaces determined by various growth conditions on the approximation numbers or by certain representations as tensor products. Also, motivated by some results of Retherford and Stegall, we introduce the class of operators $\mathcal{F}_R^p(E,F)$ and study the relationships between the classes $\ell^p(E,F), L^{(p)}(E,F), \mathfrak{F}_p(E,F), \mathcal{F}_R^p(E,F)$, and $F_p(E,F)$ (this latter class of operators introduced by Marcus).

In Chapter IV we give an application to interpolation theory, extending the results of Oloff on interpolation between the spaces $\mathcal{L}(\mathcal{H}, \mathcal{H}), \mathcal{H}$ a separable Hilbert space, to spaces of diagonals of type $\ell^p$ on the $\ell^q$-spaces using the K- and L-methods of Peetre.
INTRODUCTION

In his paper [28] Pietsch introduced a class of operators on Banach spaces which he called operators of type $\ell^p$, $0 < p \leq \infty$. This dissertation is a study of the operators of type $\ell^p$, their relationship to various other classes of operators on Banach spaces, and their relationship to geometric properties of the spaces on which they are defined.

We begin by giving basic definitions and establish the notation which will be used throughout this paper. We will also state several results which will be useful in later chapters.

Throughout this paper all spaces are Banach spaces, unless otherwise stated. We will denote the Banach space of all bounded, linear, operators from $E$ to $F$ by $\mathcal{L}(E,F)$ with $\|T\| = \sup_{x \in E} \|Tx\|$. By operator, or map, we will always mean a bounded, linear operator. A projection $P$ is an element of $\mathcal{L}(E,E)$ such that $P^2 = P$. A closed subspace $E_0$ of $E$ is said to be complemented in $E$ if there is a projection $P \in \mathcal{L}(E,E)$ with $P(E) = E_0$. If $T \in \mathcal{L}(E,F)$ then $T|_{E_0}$ denotes the restriction of $T$ to $E_0$. For $(x_i)_{i=1}^n \subset E$, $[x_i:1 \leq i \leq n]$ denotes the subspace of $E$ spanned by $(x_i)_{i=1}^n$.

By $E'$ we mean the Banach space of all continuous linear functionals on $E$ with $\|f\| = \sup_{x \in E} |<x,f>|$ for $f \in E'$. If $f \in E'$ and $y \in F$ then by $f \otimes y$ we mean the rank one operator from $E$ to $F$ defined by
\[ f \circ y(x) = \langle x, f \rangle y \text{ for every } x \in E. \] For \( T \in \mathcal{L}(E,F) \) the adjoint operator \( T^* \) of \( T \) is the element of \( \mathcal{L}(F',E') \) defined by \( \langle T^*y', x \rangle = \langle y', Tx \rangle \) for all \( y' \in F', x \in E \).

By \( L^p \), \( 0 < p < \infty \), we mean the set of all sequences of scalars \( \xi = (\xi_i)_{i=1}^{\infty} \) with \( \sum_{i=1}^{\infty} |\xi_i|^p \) finite for \( 0 < p < \infty \), and \( \sup |\xi_i| \) finite if \( p = \infty \). For \( \xi \in L^p \), \( \|\xi\|_p = \left( \sum_{i=1}^{\infty} |\xi_i|^p \right)^{1/p} \) if \( 1 \leq p < \infty \) and \( \|\xi\|_\infty = \sup |\xi_i| \). By \( C_0 \) we mean the subspace of \( L^\infty \) consisting of all null sequences. By \( L^p(n) \), \( 0 < p < \infty \), we mean the space of all \( n \)-tuples given the norm \( \|\cdot\|_p \). We will often identify \( L^p(n) \) with \( \{(\xi_i)_{i=1}^{\infty} \in L^p : \xi_i = 0 \text{ for all } i \geq n+1\} \). We will always denote the \( i \)-th unit vector of \( L^p \) by \( e_i \). For \( 1 \leq p < \infty \) we will denote the unit ball of \( L^p \) by \( U^p \); that is, \( U^p = \{\xi \in L^p : \|\xi\|_p \leq 1\} \), and the unit ball of \( E \) by \( U^E \).

**Definition 0.1** For \( T \in \mathcal{L}(E,F) \) the \( k \)-th approximation number of \( T \), \( \alpha_k(T) \), is defined by

\[
\alpha_k(T) = \inf \|T - A\|
\]

the infimum being taken over all \( A \in \mathcal{L}(E,F) \) of rank at most \( k \), \( k = 0, 1, 2, \ldots \).

We note that the approximation numbers of an operator have the following properties:

i) \( \alpha_0(T) = \|T\| \)

ii) \( \alpha_k(T) \geq \alpha_{k+1}(T) \) for all \( k \)

iii) \( \alpha_k(S + T) \leq \alpha_j(S) + \alpha_n(T), \ j+n=k \)

iv) \( \alpha_k(ST) \leq \alpha_j(S) \alpha_n(T), \ j+n=k \)

v) \( \alpha_k(\lambda T) = |\lambda| \alpha_k(T) \) for all \( k \) and scalars \( \lambda \)

vi) \( |\alpha_k(S) - \alpha_k(T)| \leq \|S - T\| \) for all \( k \)
Definition 0.2 An operator $T \in \mathcal{L}(E,F)$ is said to be of type $\mathcal{A}^p$
from $E$ to $F$ if $(\alpha_k(T))^\infty_{k=0} \in \ell^p$, $0 < p < \infty$, or $(\alpha_k(T))^\infty_{k=0} \in C_0$ if $p = \infty$.

These operators have been studied extensively on Hilbert spaces (Krein-Gohberg [4], Schatten [31]) and it is well known (see for example, [27]) that for compact operators on a Hilbert space,
$\alpha_k(T) = \lambda_{k+1}(T)$, where $(\lambda_k(T))^\infty_{k=1}$ is the sequence of eigenvalues of $\sqrt{T^*T}$ arranged in decreasing order and repeated according to multiplicity.

If $p > 0$ Markus and Macaev [18] have shown that
\[
\sum_{n=1}^\infty |\lambda_n(T)|^p \leq K(p) \sum_{n=1}^\infty \alpha_{n-1}(T)^p \lambda_n(1 + \|T\| \alpha_{n-1}(T)^{-1})
\]
for every compact $T \in \mathcal{L}(E,E)$.

The relationship between approximation numbers and eigenvalues of an operator on a Banach space has been studied by Markus in [17]. An operator $T \in \mathcal{L}(E,E)$ is said to be an \$\mathcal{K}\$-operator if its spectrum is real and its resolvent satisfies $\|(T - \lambda I)^{-1}\| \leq C |\text{Im } \lambda|^{-1}$, $\text{Im } \lambda \neq 0$, $C$ independent of $\lambda$. It is well known that $T \in \mathcal{L}(\mathcal{K},\mathcal{K})$ is an \$\mathcal{K}\$-operator with $C = 1$ if and only if $T$ is self-adjoint.

The following theorem of Markus [17] gives the relationship between the eigenvalues of a compact \$\mathcal{K}\$-operator on $E$ and its approximation numbers.

**Theorem**: If $T \in \mathcal{L}(E,E)$ is a compact \$\mathcal{K}\$-operator then $\delta_n(T) \leq \alpha_n(T)^p$
$\leq \frac{2}{2C} |\lambda_n(T)| \leq 8C(C+1)\delta_n(T)$ where $C$ is the \$\mathcal{K}\$-constant. Thus if $p > 0$ the convergence of any one of the series $\sum_{n=1}^\infty |\lambda_n(T)|^p$, $\sum_{n=1}^\infty \alpha_n(T)^p$, $\sum_{n=1}^\infty \delta_n(T)^p$ implies the convergence of the other two.

Here $\delta_n(T)$ is the $n^{th}$ Kolmogoroff diameter of $T$, which will be defined below.
For $0 < p < \infty$ let $\rho_p(T) = \left( \sum_{k=0}^{\infty} \alpha_k(T)^p \right)^{1/p}$ and $\rho_\infty(T) = ||T||$. It follows from (0.1) that for $p \geq 1$, $\rho_p$ satisfies:

1) $\rho_p(T) = 0$ if and only if $T = 0$

2) $\rho_p(\lambda T) = |\lambda| \rho_p(t)$ for all scalars $\lambda$

3) $\rho_p(S + T) \leq K(p) (\rho_p(S) + \rho_p(T))$ where $K(p)$ is a constant depending only on $p$.

Following Pietsch we will denote by $\ell^p(E,F)$ the set of all $T \in \mathcal{L}(E,F)$ such that $\rho_p(T) < +\infty$. Pietsch [27], [28] has shown that $\ell^p(E,F)$ equipped with the topology generated by $\rho_p$ is a complete, metrizable, topological vector space which is, in general, not locally convex. It is clear from the definitions that the finite rank operators are $\rho_p$-dense in $\ell^p(E,F)$.

For $0 < p \leq 1$ we have the following representation theorem for operators of type $\ell^p$ due to Pietsch [27].

**Theorem 0.3** For $0 < p \leq 1$ and $T \in \ell^p(E,F)$ there exist sequences

$$(f_i)_{i=1}^\infty \subseteq U_E, (y_i)_{i=1}^\infty \subseteq U_F,$$

and scalars $(\lambda_i)_{i=1}^\infty \in \ell^p$ with $|\lambda_i| \geq |\lambda_{i+1}|$ for all $i$ such that $T = \sum_{i=1}^\infty \lambda_i f_i \otimes y_i$ and

$$\left( \sum_{i=1}^\infty |\lambda_i|^p \right)^{1/p} \leq C(p) \rho_p(T)$$

where $C(p)$ is a constant depending only on $p$.

**Definition 0.4** If $B$ is a bounded set in $E$ then the $n$th Kolmogoroff diameter of $B$, $\delta_n(B)$, is defined by

$$\delta_n(B) = \inf \inf \{ \delta > 0 : B \subseteq \delta U_n + E_n \}$$

where $E_n$ is an $n$-dimensional subspace of $E$. For $T \in \mathcal{L}(E,F)$, the $n$th Kolmogoroff diameter of $T$, $\delta_n(T)$, is defined by $\delta_n(T) = \delta_n(T(U_{E_n})$.
These $\delta$-numbers of $\mathcal{T}(E,F)$ were first defined and studied by Kolmogoroff [12] and later by Bessaga, Mitiajin, Pelczynski, Pietsch, Rolewicz and others in connection with Gelfand's problem [3, p.6] of characterization of Grothendieck's nuclear spaces [5] by degree of approximation by finite-dimensional spaces. The characteristic $\delta_n(T)$ is the best non-linear approximation to $T$ and $\alpha_n(T)$, the best linear approximation. It is well known (see for example [27]) that $T$ is compact if and only if $\delta_n(T) \rightarrow 0$ and $T$ is the norm limit of finite rank operators if and only if $\alpha_n(T) \rightarrow 0$. We will need the following results of Pietsch [27].

**Theorem 0.5** For $\mathcal{T}(E,F)$ we have $\delta_n(T) \leq \alpha_n(T) \leq (n+1)\delta_n(T)$ for all $n$.

**Theorem 0.6** Let $P\subseteq\mathcal{L}(E,E)$ be a projection of norm one onto an $(n+1)$-dimensional subspace of $E$, $B$ a bounded set in $E$, and $\delta > 0$. If $U_E \cap P(E) \subseteq \delta^{-1}B$ then $\delta_n(B) \geq \delta$.

Using the result of Kadec-John [11], [9] that there is a projection of $E$ onto each of its $n$-dimensional subspaces of norm at most $\sqrt{n}$ we show in Chapter I that $\delta_n(T) \leq \alpha_n(T) \leq (\sqrt{n} + 1)\delta_n(T)$ for all $n$. We do not know the best possible value, $f(n)$, such that $\alpha_n(T) \leq f(n) \delta_n(T)$ for all $n$ and $\mathcal{T}(E,F)$. It follows from Enflo's counterexample to the approximation problem [2] that there are spaces $E$ and $F$ and $\mathcal{T}(E,F)$ such that $\left(\delta_n(T)\right)_{n=0}^{\infty} \in C_0$ but $\left(\alpha_n(T)\right)_{n=1}^{\infty} \notin C_0$. Thus $f(n)$ is, in general, not a constant.

In order to calculate Kolmogoroff diameters, the following theorem of Krein, Krosnoselski, and Millman [14] will be useful.

**Theorem 0.7** If $E_{n+1}$ is an $(n+1)$-dimensional subspace of $E$ then $\delta_k(U_E \cap E_{n+1}) = 1$ for $k \leq n$. 


We next recall the definitions of various classes of operators which we will need later on.

**Definition 0.8** An operator $T$ from $E$ to $F$ is said to be absolutely $p$-summing, $p \geq 1$, if there is a constant $M > 0$ such that for all finite sequences $(x_i^n) \subseteq E$, $n \geq 1$, we have

$$\left( \sum_{i=1}^{n} \| Tx_i \|^{p} \right)^{1/p} \leq M \sup_{f \in U_E} \left( \sum_{i=1}^{n} | \langle x_i, f \rangle |^{p} \right)^{1/p}.$$

Absolutely 1-summing operators will simply be called absolutely-summing, and $\Pi_p(E,F)$ will denote the collection of absolutely $p$-summing operators.

**Definition 0.9** An operator $T \in \mathcal{L}(E,F)$ is said to be $p$-nuclear, $p \geq 1$, if there exist sequences $(f_i^\infty) \subseteq E'$, $(y_i^\infty) \subseteq F$ such that $T = \sum_{i=1}^{\infty} f_i \otimes y_i$ and $\sum_{i=1}^{\infty} \| f_i \|^{p} < +\infty$, $\sup_{g \in U_F} \left( \sum_{i=1}^{\infty} | \langle y_i, g \rangle |^{q} \right)^{1/q} < +\infty$, $1/p + 1/q = 1$.

The 1-nuclear operators coincide with the nuclear operators introduced by Grothendieck [5], and will be referred to as nuclear.

We will use $N(E,F)$ and $N_p(E,F)$ to denote the nuclear and $p$-nuclear operators from $E$ to $F$ respectively. For $T \in N(E,F)$ let $\nu(T) = \inf_{i=1}^{\infty} \| f_i \| \| y_i \| : T = \sum_{i=1}^{\infty} f_i \otimes y_i \}.

**Remarks 0.10** If $T \in N(E,F)$ then $T$ factors through a nuclear diagonal $\delta : C_0 \to \ell^1$. Indeed, choose a representation of $T$, $T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i$ where $(\lambda_i^\infty) \in \ell^1$, $(\| f_i \|) \in C_0$, and $(y_i^\infty) \subseteq U_F$ (this is possible by [5]). Then $T$ has the factorization.
where \( Ax = (\langle x, f_i \rangle)_{i=1}^{\infty} \), \( A = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i \), and \( B(\xi_i)_{i=1}^{\infty} = \sum_{i=1}^{\infty} \xi_i y_i \).

Since \((\lambda_i)_{i=1}^{\infty} \in l^1\), \( A \) is nuclear.

**Definition 0.11**

If \( H_i \) are Hilbert spaces, \( i=1,2 \) then \( T \in \mathcal{L}(H_1, H_2) \) is said to be **Hilbert-Schmidt** if for every complete orthonormal set \((\delta_i)_{i=1}^{\infty} \subset H_1\), \((f_i)_{i=1}^{\infty} \subset H_2\) we have that \( \sum_{i=1}^{\infty} |\langle T\delta_i, f_i \rangle|^2 \) is finite. We will denote the Hilbert-Schmidt operators from \( H_1 \) to \( H_2 \) by \( \mathcal{H}S(H_1, H_2) \).

Pietsch [27] has shown that \( \mathcal{P}_2(H_1, H_2) = \ell^2(H_1, H_2) = \mathcal{H}S(H_1, H_2) \).

Closely related to the nuclear operators are the fully nuclear operators introduced by Retherford-Stegall [29]. An operator \( T \in \mathcal{L}(E, F) \) is said to be **fully nuclear** if its astriction \( T_a : E \rightarrow \overline{T(E)} \) is nuclear. It is immediate from the definitions that every fully-nuclear operator is nuclear. In [29] Retherford-Stegall show that in any two infinite dimensional spaces \( E \) and \( F \) there are infinite dimensional subspaces \( E_o \) of \( E \), \( F_o \) of \( F \), and \( T \in \mathcal{L}(E_o, F_o) \) such that \( T \) is nuclear but not fully nuclear. We will denote the fully nuclear operators from \( E \) to \( F \) by \( \mathcal{H}N(E, F) \).

We will need the following theorem of Retherford-Stegall [29].

**Theorem 0.12**

If \( \mathcal{L}(E, F) = \mathcal{H}N(E, F) \) then one of \( E \), \( F \) is finite dimensional.
We point out that the composition of two absolutely 2-summing operators is fully nuclear. Indeed, if \( T \in \pi_2(E,F) \), \( S \in \pi_2(F,G) \) then \( S \) and \( T \) have the factorizations:

![Diagram](attachment:diagram.png)

When \( K, K_1 \) are compact, Hausdorff spaces and \( J, J_1 \) natural injections \([25]\). It follows from \([5], [25]\), that \( J_1A_1BJ \) is nuclear, hence \( J_1A_1BJA \) is nuclear. Let \( X = B_1^{-1}(ST(E)) \) and let \( P:L^2(\mu_1) \to X \) be a projection. Then \( PJ_1A_1BJA:E \to X \) is nuclear, hence \( B_1PJ_1A_1BJA:E \to ST(E) \) is nuclear, but \( B_1PJ_1A_1BJA = (ST)_a \).

We next define the \( L_p \) - spaces, \( 1 \leq p \leq \infty \), introduced by Lindenstrauss-Pelczynski \([15]\). These spaces are a proper generalization of the \( L_p(S,\Sigma,\mu) \) spaces and are defined as follows:

**Definition 0.13** Let \( 1 \leq p \leq \infty \). A space \( E \) is a \( L_p,\mu \) - space if for each finite-dimensional subspace \( X \) of \( E \) there exists a subspace \( Y \) of \( E \) such that \( X \subseteq Y \) and \( d(Y, L^p(n)) \leq \lambda, n = \dim Y \).

By \( d(E,F) \) we mean the Banach-Mazur distance, \( d(E,F) = \inf \|T\| \|T^{-1}\| \) where \( T \) is an isomorphism from \( E \) onto \( F \).

We will say that \( E \) is a \( L_p \)-space if it is a \( L_p,\mu \)-space for some \( \lambda \) \([15]\).

We will also need the "Principle of Local Reflexivity" established by Lindenstrauss and Rosenthal \([16]\) (see also Johnson, Rosenthal, and Zippin \([30]\)).

**Principle of Local Reflexivity:** Let \( E \) be regarded as a subspace of \( E'' \) and let \( X \) be a finite-dimensional subspace of \( E'' \). There exists a finite dimensional subspace \( Y \) of \( E \) and a one-to-one
operator \( \varphi: E'' \to Y \) such that \( \| \varphi \| \| \varphi^{-1} \| \leq 1 + \epsilon \) and \( \varphi(x) = x \) for every \( x \in X \cap E \).
CHAPTER I
DIAGONAL OPERATORS ON THE $\ell^q$-SPACES

In this chapter we compute exactly or give asymptotic estimates for the approximation numbers and Kolmogoroff diameters of diagonal operators on the $\ell^q$-spaces, essentially completing a study initiated by Pietsch [28]. As a consequence, we are able to give examples of operators which are of type $\ell^p$ but not absolutely p-summing or p-nuclear, $p \geq 1$. The existence of nuclear operators which are not of type $\ell^1$ was shown by Pietsch [28]. We are also able to compute the Kolmogoroff diameters of certain compacta in the $\ell^q$-spaces, obtaining as a corollary some results of Macaev (see [17]).

Definition 1.1 By a diagonal operator $T: \ell^p \rightarrow \ell^q$ we mean an operator corresponding to multiplication by a sequence of scalars $(\lambda_i)_{i=1}^{\infty}$, real or complex. We will denote this by writing $T \sim (\lambda_i)_{i=1}^{\infty}$.

In computations which follow we will assume that $|\lambda_i| \geq |\lambda_{i+1}|$ for all $i$. We do not lose any generality in assuming this, as will be made more precise later.

We begin by computing the approximation numbers for a very special diagonal, namely the natural injection $I: \ell^p \rightarrow \ell^q$ where $1 \leq p \leq q \leq \infty$.

Proposition 1.2 If $I: \ell^1 \rightarrow \ell^\infty$ is the injection operator then $\alpha_k(I) = 1/2$ for each $k$. 
Proof:

To show that $\alpha_k(I) \leq 1/2$ for all $k$ it suffices to show that $\alpha_1(I) \leq 1/2$. Let $e_0$ denote the element $(1/2, 1/2, \ldots)$ of $\ell^\infty$ and define a rank 1 operator $A: \ell^1 \to \ell^\infty$ by $A(\xi_i) = \sum_{i=1}^{\infty} \xi_i e_0$ for $(\xi_i) \in \ell^1$. Then $\|I-A\| = \sup_i \| (I-A)e_i \| = \sup_i \| e_i - e_0 \| = 1/2$, hence $\alpha_1(I) \leq 1/2$.

Now suppose that $\alpha_k(I) < 1/2$ for some $k$. Let $\epsilon > 0$ be such that $\alpha_k(I) < 1/2 - \epsilon$ and choose an operator $A: \ell^1 \to \ell^\infty$ of rank at most $k$ such that $\|I-A\| \leq \alpha_k(I) + \epsilon/2 < 1/2 - \epsilon/2$. Then

(1.2.1) $\sup_i \| (I-A)e_i \| < 1/2 - \epsilon/2$.

If $A e_i = (a_{ij})^\infty_{j=1} \in \ell^\infty$ then $A e_i \in B_A$ for each $i$ where

$B_A = \{ \xi \in \ell^1 : \| \xi \|_\infty \leq \| A \| \}$. Since $A$ is finite rank, $B_A$ is relatively compact. By (1.2.1) we have for each $i$

$$|1 - a_{ii}| < 1/2 - \epsilon/2$$

and

$$|a_{ij}| < 1/2 - \epsilon/2 \text{ for every } j, j \neq i.$$ If $i \neq n$ then

$$\| A e_i - A e_n \| = \sup_j |a_{ij} - a_{nj}|$$

$$\geq |a_{ii} - a_{ni}|$$

$$\geq(1/2 + \epsilon/2-[1/2-\epsilon/2])$$

$$= \epsilon$$

Thus the sequence $(A e_i)_{i=1}^{\infty} \subset B_A$ can have no convergent subsequence, contradicting the fact that $B_A$ is relatively compact. Therefore,
\( \alpha_k(T) \geq 1/2 \) for each \( k \).

**Corollary** (to the proof) 1.3 If \( U_1, U_\infty \) denotes the unit ball of \( \ell^1, \ell^\infty \) respectively, then \( \delta_k(U_1, U_\infty) = 1/2 \) for all \( k \).

**Proposition 1.4** If \( 1 < p \leq \infty \) and \( I: \ell^p \to \ell^\infty \) is the natural injection then \( \alpha_k(I) = 1 \) for all \( k \).

**Proof:**

The case \( p = \infty \) was proved by Pietsch [27] and is a special case of (1.20), so suppose that \( p < \infty \). Clearly \( \alpha_k(I) \leq 1 \) for all \( k \). If \( \alpha_k(I) < 1 \) for some \( k \) choose \( \epsilon > 0 \) and \( A \in \mathcal{L}(\ell^p, \ell^\infty) \) of rank at most \( k \) such that \( \|IA\| < 1 - \epsilon/2 \). Since \( A \) is finite rank, \( A \) has a representation \( A = \sum_{i=1}^{k} f_i \otimes y_i \) where \( f_i \in \ell^p \) (\( \frac{1}{p} + \frac{1}{p'} = 1 \)) and \( y_i \in \ell^\infty \) for each \( i \). By the choice of \( A \) we have

\[
\sup_j \| (I-A)e_j \|_\infty \leq \| I-A \| < 1-\epsilon/2.
\]

If \( y_i = (y_{ij})^\infty_{j=1} \) then we have

\[
|1 - \sum_{i=1}^{k} \langle e_j, f_i \rangle y_{ij}| < 1-\epsilon/2 \quad \text{for all } j
\]

hence

\[
(1.4.1) \quad \sum_{i=1}^{k} |\langle e_j, f_i \rangle| > \epsilon/2M
\]

for all \( j \) where \( M = \max \|y_i\|_\infty \). Since \( f_i \in \ell^p \) there exists an index \( j_0 \) such that

\[
(\sum_{j=j_0}^{\infty} |\langle e_j, f_i \rangle|^{p'})^{1/p'} < \epsilon/2kM
\]

for \( 1 \leq i \leq k \). In particular, \(|\langle e_{j_0}, f_i \rangle| < \epsilon/2kM \) for all \( i, 1 \leq i \leq k \). It now follows from (1.4.1) that
This contradiction shows that \( \alpha_k(I) \geq 1 \) for all \( k \).

**Proposition 1.5** For \( 1 < p \leq \infty \) we have \( \delta_n(U_p, U_\infty) = 2^{-1/p} \) for all \( n \).

**Proof:**

In order to calculate \( \delta_n(U_p, U_\infty) = \delta_n(I) \) where \( I: \mathbb{E}^p \rightarrow \mathbb{E}^\infty \) is natural injection, we will use the equivalent formulation

\[
\delta_n(I) = \inf \{ \sup_{\|\xi\|_p \leq 1} d(\xi, F_n)_n \mid F_n \text{ is an } n\text{-dimensional subspace of } \mathbb{E}^\infty \}.
\]

Letting \( F_0 \) denote the 1-dimensional subspace of \( \mathbb{E}^\infty \) generated by \( e_0 = (1,1,1,1,\ldots) \) it is not difficult to see that

\[
\sup_{\|\xi\|_p \leq 1} d(\xi, F_0) = 2^{-1/p}.
\]

Thus \( \delta_1(I) \), hence \( \delta_n(I) \), is at most \( 2^{-1/p} \) for all \( n \). That \( \delta_n(I) = 2^{-1/p} \) for all \( n \) follows from the fact that

\[
\inf \{ \sup_{\|\xi\|_p \leq 1} d(\xi, F_n)_n \mid F_n \text{ is an } n\text{-dimensional subspace of } \mathbb{E}^\infty \} = \inf \{ \sup_{\|\xi\|_p \leq 1} d(\xi, G_n)_n \mid G_n \text{ is an } n\text{-dimensional subspace of } \mathbb{E}^\infty \text{ which is equi-distant from } (e_i)_\infty \} = \sup_{\|\xi\|_p \leq 1} d(\xi, F_0) = 2^{-1/p}.
\]

**Proposition 1.6** Let \( 1 \leq p \leq q < \infty \) and \( I: \mathbb{E}^p \rightarrow \mathbb{E}^q \) is natural injection then \( \alpha_k(I) = 1 \) for all \( k \).

**Proof:**

If \( 1 < p \leq q < \infty \) let \( I_n: \mathbb{E}^n \rightarrow \mathbb{E}^\infty \) denote the injection operator, \( n < \infty \). Since \( I_p = I \mid I_q \) we have from (1.4)

\[
1 = \alpha_k(I_p) = \alpha_k(I_q) \leq \|I_q\|\alpha_k(I) = \alpha_k(I)
\]

for each \( k \). That \( \alpha_k(I) \leq 1 \) for each \( k \) is clear.
If $p = 1$ and $\alpha_k(I) < 1$ for some $k$, choose $\epsilon > 0$ and

$\lambda \in \ell^1, \ell^q)$ of rank at most $k$ such that $\|I-A\| < 1 - \epsilon/2$. Let

$B_A = \{ \xi \in \ell^1 : \|\xi\| \leq \|A\| \}$. Since $A$ is finite rank $B_A$ is relatively
compact and $A e_i = (a_{i,j})_{j=1}^\infty \in B_A$ for each $i$. Let $(y_i)_{i=1}^m \subseteq B_A$ be an $\epsilon/10$ net for $B_A$. If $y_i = (y_{i,j})_{j=1}^\infty$ then there is an index

$j_0$ such that $(\sum_{j=1}^\infty |y_{i,j}|^q)^{1/q} < \epsilon/10$ for each $i$, $1 \leq i \leq m$. Also for

each $i$ there exists an index $q(i)$ such that

$\left( \sum_{j=1}^\infty |a_{i,j} - y_{q(i),j}|^q \right)^{1/q} < \epsilon/10$.

In particular,

$\left( \sum_{j=1}^\infty |a_{i,j} - y_{q(i),j}|^q \right)^{1/q} < \epsilon/10$.

Thus

$\left( \sum_{j=1}^\infty |a_{i,j}|^q \right)^{1/q} \leq \left( \sum_{j=1}^\infty |a_{i,j} - y_{q(i),j}|^q \right)^{1/q} + \left( \sum_{j=1}^\infty |y_{q(i),j}|^q \right)^{1/q}$

$< \epsilon/10 + \epsilon/10 = \epsilon/5$

for each $i$. Hence $|a_{i,j}| < \epsilon/5$ for each $i$ and each $j \geq j_0$.

Since

$1 - \epsilon/2 > \sup_i \|(I-A)e_i\| = (|1-a_{i,i}|^q + \sum_{j=1, j \neq i}^\infty |a_{i,j}|^q)^{1/q}$

we have $|1-a_{i,i}| < 1 - \epsilon/2$ for all $i$. If $i = j_0$ we have $|a_{j_0,j_0}| < \epsilon/5$

hence

$1 - \epsilon/5 < 1 - a_{j_0,j_0} < 1 - \epsilon/2$.

Therefore we must have $\alpha_k(I) \geq 1$ for all $k$. That $\alpha_k(I) \leq 1$ is
clear.
Corollary (to the proof) 1.7 If $1 \leq p \leq q < \infty$ then

$$\delta_n(U_p, U_q) = 1 \quad \text{for each } n.$$

We now turn our attention to computing the approximation numbers of arbitrary diagonal operators on the $\mathcal{L}^q$-spaces. As with the injection operator, we will have to consider different cases depending on the relation between $p$ and $q$. We will first consider the case of a diagonal from $\mathcal{L}^p$ to $\mathcal{L}^\infty$, $1 \leq p < \infty$. In [28] Pietsch showed that there is a constant $C > 0$ such that

$$C \sum_{i=k+1}^{\infty} |\lambda_i| \leq \alpha_k(T) \leq \sum_{i=k+1}^{\infty} |\lambda_i| \quad \text{for all } k \text{ and diagonals } T: \mathcal{L}^\infty \rightarrow \mathcal{L}^1.$$

We are able to show that for such $T$, $\alpha_k(T) = \sum_{i=k+1}^{\infty} |\lambda_i|$. First we need two lemmas.

Lemma 1.8 If $P$ be an n-dimensional polytope in Euclidean n-space then its boundary is the union of its (n-1)-dimensional faces. (1.8) is well-known. For a proof see [6].

Lemma 1.9 Let $P$ be as in (1.8) and let $V$ be a k-dimensional manifold, $1 \leq k \leq n$. If $P \cap V \neq \emptyset$ then there is an (n-k)-dimensional face $F$ of $P$ such that $P \cap V \neq \emptyset$.

Proof:

The result is clear for $n=1$ so suppose the Lemma holds for $n-1$. By (1.8) $P \cap V \neq \emptyset$ implies that there is an (n-1)-dimensional face $F_1$ of $P$ such that $V \cap F_1 \neq \emptyset$. Let $V_1$ denote the affine manifold spanned by $F_1$. Then $V \cap V_1$ is an affine manifold of $V_1$ and the dimension of $V \cap V_1 \geq k-1$. Clearly we may assume that $k > 1$. Let $V_2$ be an affine manifold of $V \cap V_1$ of dimension $k-1$ which intersects $F_1$. Now $F_1$ is an (n-1)-dimensional polytope so by the induction hypothesis there is an (n-1)-(k-1)=(n-k)-dimensional
face $F$ of $F_1$ such that $F \cap V_2 \neq \emptyset$. But $F$ is also an $(n-k)$-dimensional face of $P$ and $F \cap V \neq \emptyset$.

**Theorem 1.10** Let $1 \leq p < \infty$ and $T: \ell^p \to \ell^\infty$ be a diagonal, $T \sim (\lambda_i)^\infty_{i=1}$. Then $\delta_k(T) = \alpha_k(T) = (\sum_{i=k+1}^\infty |\lambda_i|^p)^{1/p}$ for all $k$.

**Proof:**

Clearly $\alpha_k(T) \leq (\sum_{i=k+1}^\infty |\lambda_i|^p)^{1/p}$ for each $k$. The prove the opposite inequality let $A: \ell^p \to \ell^p$ be an arbitrary operator of rank $k$, say $A = \sum f_j \otimes y_j$ where $f_j \in (\ell^p)'$ and $y_j = (y_{jn})^\infty_{n=1} \in \ell^p$ for each $j$. Then $A\xi = (\sum <\xi, f_j>y_{jn})^\infty_{j=1} \in \ell^p$ for each $\xi \in \ell^\infty$ and

$$||T-A|| = \sup_{||\xi||_\infty=1} (\sum_{n=1}^\infty |\lambda_n|^p)^{1/p} \leq (\sum_{i=k+1}^\infty |\lambda_i|^p)^{1/p}.$$ 

where $\xi = (\xi_n)^\infty_{n=1} \in \ell^\infty$. We will show that for any $m>k$ there exists $\xi \in \ell^\infty, ||\xi||_\infty \leq 1$, such that

$$\left( \sum_{i=1}^m |\lambda_i|^p \right)^{1/p} \geq (\sum_{i=k+1}^\infty |\lambda_i|^p)^{1/p}.$$ 

Indeed, let $m>k$ and let $V = \{\xi \in \ell^\infty: <\xi, f_j> = 0 \text{ for all } j, 1 \leq j \leq k\}$. Then $V$ is a subspace of $\ell^\infty(m)$ of dimension $\geq m-k$ which intersects the unit ball of $\ell^\infty(m)$. By (1.9) $V$ intersects a $k$-dimensional face of the unit ball in $\ell^\infty(m)$. Therefore there exists $\xi \in \ell^\infty(m)$ and indices $i_1, \ldots, i_{m-k}, 1 \leq i \leq m$ for each $j$, such that $|\xi_{i_j}| = 1$ for $1 \leq j \leq m-k$. For this $\xi = (\xi_i)^m_{i=1}$ let $\xi^m = (\xi_{i_j})^\infty_{i=1} \in \ell^\infty$ where $\xi^m_{i_j} = \xi_{i_j}$ for $1 \leq i \leq m$ and $\xi^m_{i_j} = 0$ for $i > m$. Then

$$\left( \sum_{n=1}^m |\lambda_n|^p \right)^{1/p} \geq (\sum_{i=k+1}^\infty |\lambda_i|^p)^{1/p}.$$ 

Indeed, let $m=k$ and let $V = \{\xi \in \ell^\infty(m): <\xi, f_j> = 0 \text{ for all } j, 1 \leq j \leq k\}$. Then $V$ is a subspace of $\ell^\infty(m)$ of dimension $= m-k$ which intersects the unit ball of $\ell^\infty(m)$. By (1.9) $V$ intersects a $k$-dimensional face of the unit ball in $\ell^\infty(m)$. Therefore there exists $\xi \in \ell^\infty(m)$ and indices $i_1, \ldots, i_{m-k}, 1 \leq i \leq m$ for each $j$, such that $|\xi_{i_j}| = 1$ for $1 \leq j \leq m-k$. For this $\xi = (\xi_i)^m_{i=1}$ let $\xi^m = (\xi_{i_j})^\infty_{i=1} \in \ell^\infty$ where $\xi^m_{i_j} = \xi_{i_j}$ for $1 \leq i \leq m$ and $\xi^m_{i_j} = 0$ for $i > m$. Then

$$\left( \sum_{i=k+1}^\infty |\lambda_i|^p \right)^{1/p} \geq (\sum_{i=k+1}^\infty |\lambda_i|^p)^{1/p}.$$
Therefore $||T-A|| \geq \sup \{|| (T-A)^m ||_p : m \}$

Since \( A \) was an arbitrary rank \( k \) operator we have

\[
\alpha_k(T) \geq (\sum_{i=k+1}^{\infty} |\lambda_i|^{1/p})^{1/p}.
\]

To see that $\delta_k(T) \geq (\sum_{i=k+1}^{\infty} |\lambda_i|^{1/p})^{1/p}$, hence equal to $\alpha_k(T)$, apply (0.7) and proceed as above.

Remark: If we do not assume that $|\lambda_i| \geq |\lambda_{i+1}|$ for all \( i \) then a slight modification of the argument given in (1.10) gives the equation

\[
\alpha_k(T) = \inf_{\sigma \in \Sigma(k)} \left( \sum_{i \in \sigma} |\lambda_i|^{1/p} \right)^{1/p}
\]

where $\Sigma(k)$ denotes the collection of all subsets $\sigma$ of the positive integers which contain at most $k$ elements. Thus by replacing the computation of $\alpha_k(T)$ in the following propositions by the obvious modification of (*), we may assume that the sequence $\langle \lambda_i \rangle_{i=1}^{\infty}$ is actually non-increasing.

We have the following Corollary to (1.10) which was attributed to Macaev by Marcus [14].

Corollary 1.11 Let $\beta_{n} = \sum_{n=1}^{\infty} \epsilon \beta_{n}^{1} = \beta_{n+1} \geq 0$ and let

\[
K = \{(a_{n})_{n=1}^{\infty} \in \ell^{1} : |a_{n}| \leq \beta_{n} \text{ for all } n\}.
\]

Then

\[
\delta_{n}(K) = \sum_{k=n+1}^{\infty} \beta_{n}.
\]

By replacing polytopes by normal hulls of certain elements in normal Kothe sequence spaces, and using techniques similar to those above P. Johnson [10] has been able to generalize (1.10), proving the following Theorem.
Theorem 1.12 If $1 < q < p < \infty$ and $T: \ell^p \to \ell^q$ is a diagonal, $T \sim (\lambda_i^{\infty})_{i=1}^{\infty}$, then

$q_k(T) = \left( \sum_{i=k+1}^{\infty} |\lambda_i|^p \right)^{1/q}$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{n}$.

In order to calculate the approximation numbers of a diagonal from $\ell^p$ to $\ell^q$, $1 \leq p \leq q \leq \infty$, we first prove the following lemma.

Lemma 1.13 For fixed $n$ let $(y_i)_{i=1}^{n} \subseteq \ell^\infty$ be such that

$\|e_i - y_i\|_\infty < 1/2$ for all $i$, $1 \leq i \leq n$. Then at least $n-1$ of the $y_i$'s are linearly independent.

Proof:

If $\mathcal{U}_i = \{g \in \ell^\infty : \|y_i - g\|_\infty < 1/2\}$ then since $\|e_i - y_i\|_\infty < 1/2$ for all $i$ we have $\mathcal{U}_i \cap \mathcal{U}_j = \emptyset$ if $i \neq j$. Also, no more than 2 of the vectors $y_{k-1}, y_k, y_{k+1}$ can belong to any one coordinate plane. Thus the convex hull of $(y_i)_{i=1}^{n}$ is a non-degenerate simplex having $n$ vertices and dimension at least $n-1$. The desired result now follows.

Proposition 1.14 If $T: \ell^1 \to \ell^\infty$ is a diagonal, $T \sim (\lambda_i^{\infty})_{i=1}^{\infty}$ then

$\delta_k(T) \leq \frac{1}{2} |\lambda_{k+2}|$ for all $k$.

Proof:

Suppose that $\delta_k(T) < \frac{1}{2} |\lambda_{k+2}|$ for some $k$. Then since

$\delta_k(T) = \inf \{ \|\Pi_F T\|_F \text{ is a } k\text{-dimensional subspace of } \ell^\infty \}$

and $\Pi_F: \ell^\infty \to \ell^\infty/F$ the natural quotient map) there is a $k$-dimensional subspace $F$ of $\ell^\infty$ such that $\sup d(\lambda_i e_i, F) = \|\Pi_F T\|_F < \frac{1}{2} |\lambda_{k+2}|$.

Choose $(x_i)_{i=1}^{\infty} \subseteq F$ such that $\|\lambda_i e_i - x_i\|_\infty < \frac{1}{2} |\lambda_{k+2}|$ for each
and let $y_i = \lambda_i^{-1}x_i \in F$. Then

$$(*) \quad \|e_i - y_i\|_\infty < \frac{1}{2} \left\| \frac{\lambda_{k+2}}{\lambda_i} \right\|$$

for each $i$. In particular, $(*)$ holds for $1 \leq i \leq k+2$. Since $|\lambda_{k+2}/\lambda_i| < 1$ for all $i$, $1 \leq i \leq k+2$, the vectors $(y_i)_{i=1}^{k+2}$ satisfy the hypothesis of (1.13), hence at least $k+1$ of the $y_i$'s are linearly independent. But $y_i \in F$ for all $i$ and $F$ is $k$-dimensional. Thus $\delta_k(T) \leq \frac{1}{2} |\lambda_{k+2}|$ for all $k$.

**Proposition 1.15** If $1 \leq p \leq \infty$ and $T: \ell^p \to \ell^q$ is a diagonal, $T \sim (\lambda_i)_{i=1}^\infty$, then $\delta_k(T) \leq \frac{1}{2} |\lambda_{k+2}|$.

**Proof:**

If $\delta_k(T) < \frac{1}{2} |\lambda_{k+2}|$ for some $k$, then as in (1.14) there is a $k$-dimensional subspace $F$ of $\ell^q$, vectors $(y_i)_{i=1}^{k+2} \subset F$ with

$$\|e_i - y_i\|_\infty < \frac{1}{2}$$

for all $i$, $1 \leq i \leq k+2$. The result now follows from (1.13).

**Theorem 1.16** Let $1 \leq p \leq q \leq \infty$ and $T: \ell^p \to \ell^q$ be a diagonal, $T \sim (\lambda_i)_{i=1}^\infty$. Then

$$\frac{1}{2} |\lambda_{k+2}| \leq \alpha_k(T) \leq |\lambda_{k+1}|.$$

**Proof:**

Clearly $\alpha_k(T) \leq |\lambda_{k+1}|$. To see that $\alpha_k(T) \leq \frac{1}{2} |\lambda_{k+2}|$ let $\tilde{T} = I_2T I_1$ where $I_1: \ell^1 \to \ell^p$, $I_2: \ell^q \to \ell^\infty$ are natural
injections. Then \( \tilde{T} : l \to l'' \) is a diagonal, \( T \sim (\lambda_i)_{i=1}^{\infty} \). By (1.15), (1.14) we have

\[
\frac{1}{2} \left| \lambda_{k+2} \right| \leq \delta_k(T) \leq \alpha_k(T) = \left| \lambda_{k+1} \right| \leq \| I_2 T I_1 \| \leq \| I_2 \| \alpha_k(T) \| I_1 \|
\]

In the special case \( p=q \) it is well known [19], [20] that for a diagonal \( T: l^p \to l^p \), \( \alpha_k(T) = \delta_k(T) = |\lambda_{k+1}| \), where \( T \sim (\lambda_i)_{i=1}^{\infty} \).

The proof which we give here uses Kolmogoroff diameters to calculate the approximation numbers. In [27] Pietsch proved that for \( T \in \mathcal{L}(E,F) \)

\( \delta_k(T) \leq \alpha_k(T) \leq (k+1) \delta_k(T) \) for each \( k \). Using a result of Kadec-John [11], [9] we are able to replace \( k+1 \) by \( \sqrt{k} + 1 \).

**Theorem 1.17** If \( T \in \mathcal{L}(E,F) \) then for each \( k \)

\( \delta_k(T) \leq \alpha_k(T) \leq (\sqrt{k} + 1) \delta_k(T) \).

**Proof:**

If \( T(U) \subseteq \delta U_F + F_n \) where \( F_n \) is an \( n \)-dimensional subspace of \( F \) choose a projection \( P : F \to F_n \) such that \( \|P\| \leq \sqrt{n} \) (this is possible by [7], [9]). Define \( A : E \to F \) by \( A = PT \). Then \( A \) is rank \( n \) and for \( x \in U_E \) there are elements \( v \in \delta U_F \) and \( y \in F_n \) such that \( Tx = \delta V + y \). Thus

\[
\| (T-A)x \| = \| (T-PT)x \| \\
= \| \delta V + y - (\delta PV + y) \| \\
= \| \delta (V-PV) \| \\
\leq \delta (1 + \sqrt{n}).
\]

Therefore, \( \alpha_n(T) \leq \| T-A \| \leq \delta (1 + \sqrt{n}) \). Since \( \delta \) was arbitrary, we have the desired inequality.

It is known that \( T \in \mathcal{L}(E,F) \) is nuclear if \( \delta_n(T) = 0 (n^{-3/2} - \epsilon) \) for some \( \epsilon > 0 \) [20]. Using (1.17) we are able to show that if \( \delta_n(T) = 0 (n^{-3} - \epsilon) \) then \( T \) is of type \( \ell^1 \), hence nuclear. This answers a question raised by Mitiajin in [19].
Corollary 1.18 Let $T \in \mathcal{L}(E,F)$, if $\delta_n(T) = 0(n^{-3/2-\epsilon})$ for some $\epsilon > 0$, then $T \in \ell^1(E,F)$.

Proof:

By (1.17) we need only show that $\sum_{n=1}^{\infty} n^{1/2} \delta_n(T)$ is finite. But $n^{1/2} \delta_n = \frac{1}{n^{1+\epsilon}} n^{3/2+\epsilon} \delta_n$; by hypothesis $(n^{3/2+\epsilon} \delta_n)_{n=1}^{\infty}$ is bounded and since $\epsilon > 0$, $\sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}}$ converges.

The next proposition is well known. See for example, Pietsch [27].

Proposition 1.19 Let $(\lambda_n)_{n=1}^{\infty}$ be a decreasing sequence of positive numbers, $1 \leq p \leq \infty$, and let $B = \{(\xi_i)_{i=1}^{\infty} \in \ell^p : (\sum_{i=1}^{\infty} \lambda_i^{-p} |\xi_i|^p)^{1/p} \leq 1\}$. Then $\delta_n(B) = \lambda_{n+1}$.

Proof:

Let $\mathcal{E}(n)$ denote the subspace of $\mathcal{E}$ generated by $e_1, \ldots, e_n$. Then if $\xi = (\xi_i)_{i=1}^{\infty} \in B$, $\sum_{i=1}^{n} \xi_i e_i \in \mathcal{E}(n)$ and

$$\|\xi - \sum_{i=1}^{n} \xi_i e_i\|_p = (\sum_{i=n+1}^{\infty} |\xi_i|^p)^{1/p}$$

$$= (\lambda_{n+1}^{-p} \sum_{i=n+1}^{\infty} \lambda_i^{-p} |\xi_i|^p)^{1/p}$$

$$\leq \lambda_{n+1}^{-p} \left(\sum_{i=n+1}^{\infty} |\xi_i|^p\right)^{1/p}$$

$$\leq \lambda_{n+1}^{-p}.$$ 

Thus $B \subseteq \lambda_{n+1} U_p + \mathcal{E}(n)$ and so $\delta_n(B) \leq \lambda_{n+1}$.

Given $n$, let $P_n : \mathcal{E} \to \mathcal{E}(n+1)$ be the canonical projection. Then if $\xi \in \mathcal{E}(n)$ and $\|\xi\|_p \leq 1$
Thus $\xi \lambda^{-1} B$. By (0.6) we have $\delta_n(B) \geq \lambda_{n+1}$.

**Theorem 1.20** If $T: \ell^p \to \ell^p$, $1 \leq p \leq \infty$, is a diagonal, $T \sim (\lambda_i)_{i=1}^\infty$, then $\alpha_k(T) = \delta_k(T) = |\lambda_{k+1}|$.

**Proof:**

We will first consider the case $1 \leq p < \infty$. It follows from (1.16) and (1.17) that $\delta_k(T) \leq \alpha_k(T) = |\lambda_{k+1}|$. To see that $\delta_k(T) \geq |\lambda_{k+1}|$ we will show that $T(U_p) \supseteq B$, where $B$ is as in (1.19). It will then follow that $\delta_k(T) \geq \delta_k(B) = |\lambda_{k+1}|$. If $\xi = (\xi_i)_{i=1}^\infty \in B$ then $(\lambda_i^{-1} \xi_i)_{i=1}^\infty \in U_p$ hence $\xi = T(\lambda_i^{-1} \xi_i)_{i=1}^\infty \in T(U_p)$. Thus $B \subseteq T(U_p)$.

If $p=\infty$ then by (1.16) $\alpha_k(T) \leq |\lambda_{k+1}|$ for all $k$. If $A: \ell^\infty \to \ell^\infty$ is of rank at most $k$, choose $\xi = (\xi_i)_{i=1}^\infty \in \ell^\infty$ such that $\|\xi\|_\infty = 1$, $\xi_i = 0$ for all $i > k+1$, and $A\xi = 0$. Then $\|T-A\| \geq \|(T-A)\xi\|_\infty \geq |\lambda_{k+1}|$. Since $A$ was arbitrary, $\alpha_k(T) \geq |\lambda_{k+1}|$ for all $k$.

For a diagonal $T: \ell^p \to \ell^q$ with $1 \leq p < q < \infty$, $T \sim (\lambda_i)_{i=1}^\infty$ there may be better bounds on $\alpha_k(T)$ than those found in (1.16).

In general, however, $\alpha_k(T) \neq |\lambda_{k+1}|$. In the case $p=1$ and $q=2$ we
are able to calculate $\alpha_k(T) = \delta_k(T)$ if $(\lambda_i)_{i=1}^\infty$ is a null sequence. To do this we need two lemmas, the first of which is combinatorial in nature.

**Lemma 1.21** For a fixed $k$ and $n \geq k+1$ if $(\lambda_i)_{i=1}^{n+1}$ are scalars satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > \lambda_{n+1} = 0$ then there is a unique integer $i$, $k+1 \leq i \leq n$, such that

$$\lambda_1 \geq \left( \frac{\frac{i-k}{\sum_{j=1}^{n} \lambda_j^{-2}}} \right)^{1/2} > \lambda_{i+1}.$$

**Proof:**

Suppose that there is no such integer $i$, $k+1 \leq i \leq n$. Then since

$$\frac{n-k}{\sum_{j=1}^{n} \lambda_j^{-2}} > 0 = \lambda_{n+1}$$

we must have

$$\frac{n-1}{\sum_{j=1}^{n-1} \lambda_j^{-2}} < n-k$$

or

$$1 + \frac{n-1}{\sum_{j=1}^{n-1} \lambda_j^{-2}} < n-k$$

that is

$$\lambda_n^2 < \frac{(n-1)-k}{\sum_{j=1}^{n-1} \lambda_j^{-2}}.$$

Since the Lemma does not hold, we must have
\[ \lambda_{n-1}^2 < \frac{(n-1)-k}{\sum_{j=1}^{n-1} \lambda_j^2} \]

hence

\[ 1 + \lambda_{n-1}^2 \sum_{j=1}^{n-2} \lambda_j^2 < (n-1)-k \]

or

\[ \lambda_{n-1}^2 < \frac{(n-2)-k}{\sum_{j=1}^{n-2} \lambda_j^2} \]

Again, since the Lemma does not hold, we have

\[ \lambda_{n-2}^2 < \frac{(n-2)-k}{\sum_{j=1}^{n-2} \lambda_j^2} \]

Continuing in this manner we will eventually have

\[ \lambda_{k+1}^2 < \frac{(k+1)-k}{\sum_{j=1}^{k+1} \lambda_j^2} \]

or

\[ 1 < 1 + \lambda_{k+1}^2 \sum_{j=1}^{k} \lambda_j^{-2} < 1 \]

This contradiction shows that there exist such \( i, k+1 \leq i \leq n \).

Now suppose the Lemma holds for another integer \( s, k+1 \leq s \leq n \), with say \( s < i \). Then

\[ \lambda_i^2 > \left( \sum_{j=1}^{i-k} \lambda_j^{-2} \right)^{1/2} > \lambda_{i+1}^2 \geq \lambda_{i+2}^2 \geq \ldots \geq \lambda_s^2 \geq \left( \sum_{j=1}^{s-k} \lambda_j^{-2} \right)^{1/2} > \lambda_{s+1}^2 \]
\[
\begin{align*}
\text{(1.1)} \quad \sum_{i=1}^{s} \lambda^{-2}_j &> \sum_{j=1}^{s-k} \lambda^{-2}_j \\
\text{hence} \quad \frac{1}{s-k} \sum_{i=1}^{s} \lambda^{-2}_j &> \frac{s}{s-k} \sum_{j=1}^{s} \lambda^{-2}_j \\
\text{hence} \quad \sum_{j=1}^{s} \lambda^{-2}_j &\leq \frac{s-i}{s-k} \sum_{j=1}^{s} \lambda^{-2}_j \\
\text{that is} \quad \sum_{j=1}^{s} \lambda^{-2}_j &\leq \frac{s-i}{s-k} \sum_{j=1}^{s} \lambda^{-2}_j \\
\text{thus} \quad \sum_{j=1}^{i} \lambda^{-2}_j &\leq \frac{k-i}{s-k} \sum_{j=1}^{s} \lambda^{-2}_j.
\end{align*}
\]

But then we would have
\[
\sum_{j=1}^{i} \lambda^{-2}_j \leq \frac{s-k}{s-k} \sum_{j=1}^{s} \lambda^{-2}_j.
\]

from (1.1). Therefore \(s=i\).

\textbf{Lemma 1.22} Let \(T: \ell^1(n) \to \ell^2(n)\) be defined by \(T e_i = \lambda_i e_i\), with \((\lambda_i)^n\) as in (1.21). Then \(\alpha_k(T) = \inf \{ \max_{i \leq n} \lambda_i (1 - \sum_{j=1}^{k} x_j) \}^{1/2} : x \in \ell^2(n)\) with \((x_m, x_n) = \delta_{mn}\) for \(1 \leq m,n \leq k\).

Here \((\cdot, \cdot)\) denotes the usual inner product on \(\ell^2(n)\).

\textbf{Proof:}

Let \(Q_k\) denote the right-hand side of the above equation, \(\epsilon > 0\), and \(A \in \mathcal{L}(\ell^1(n), \ell^2(n))\) be of rank \(k \leq n\). If \(H = A(\ell^1(n))\), we can assume that \(H\{x_i : i \leq k, (x_i, x_j) = \delta_{ij}\}\). Then
\[ \|T_n - A\| = \max_{1 \leq n} \| (T_n - A)e_i \|_2 = \max_{1 \leq n} \| \lambda_i e_i - Ae_i \|_2. \]

If \( x_i = (x_{ij})^n \) then \( d(\lambda_i e_i, H) = \lambda_i (1 - \sum_{j=1}^k x_{ij}^2)^{1/2} \) for each \( i \).

Thus \( \|T_n - A\| \geq \max_{1 \leq n} \lambda_i (1 - \sum_{j=1}^k x_{ij}^2)^{1/2} \) since \( \| \lambda_i e_i - Ae_i \|_2 \geq d(\lambda_i e_i, H). \) Hence \( \alpha_k(T) \geq \sigma_k \) for each \( k \leq n \).

Now let \( (x_i)^k \subseteq \ell^2(n) \) be such that \( (x_i, x_j) = \delta_{ij}, \ 1 \leq i, j \leq k, \)
where \( x_i = (x_{ij})^n \). Let \( H = [x_i : 1 \leq k] \) and let \( P: \ell^2(n) \to H \) be the projection \( P e_i = \sum_{j=1}^k (e_i, x_j)x_j \). Then

\[ d(\lambda_i e_i, H) = \| \lambda_i e_i - Pe_i \|_2 = \lambda_i (1 - \sum_{j=1}^k x_{ij}^2)^{1/2} \]

for each \( i, 1 \leq i \leq n \). Let \( I: \ell^1(n) \to \ell^2(n) \) be the natural injection and define \( S: \ell^2(n) \to \ell^2(n) \) by \( S e_i = \lambda_i e_i, 1 \leq i \leq n \).

Then \( \lambda_i e_i, H \) is of rank \( k \) and

\[ \|T_n - PSI\| = \max_{1 \leq n} \| (T_n - PSI)e_j \|_2 \]
\[ = \max_{1 \leq n} \| \lambda_j e_j - Pe_j \|_2 \]
\[ = \max_{1 \leq n} \lambda_j (1 - \sum_{m=1}^k x_{mj}^2)^{1/2}. \]

Therefore \( \alpha_k(T_n) \leq \max_{1 \leq n} \lambda_j (1 - \sum_{m=1}^k x_{mj}^2)^{1/2}. \) Since \( (x_i)^k \) was an arbitrary orthonormal sequence in \( \ell^2(n) \), it follows that \( \alpha_k(T) \leq \sigma_k \).

Observe that
\[ a_k = \inf \{ \max_{i=1}^{n} \lambda_i (1 - A_i)^{1/2} : A_i \in [0,1], 1 \leq i \leq n \} \]

\[ = \inf \{ \max_{i=1}^{n} \lambda_i B_i^{1/2} : B_i \in [0,1] \text{ and } \sum_{i=1}^{n} B_i = n-k \} \]

\[ = \inf \{ \max_{k \leq i \leq n} \lambda_i B_i^{1/2}, \ldots, \lambda_r B_r^{1/2}, \lambda_{r+1} \} : B_i \in [0,1] \text{ and } \sum_{i=1}^{r} B_i = r-k \}. \]

**Theorem 1.23** For fixed integers \( k, n, n \geq k+1 \) there is a unique integer \( r = r(k,n) \), \( k+1 \leq r \leq n \), such that \( a_k(T_n) = \left( \frac{r-k}{\sum_{j=1}^{r} \lambda_j^{-2}} \right)^{1/2} \),

where \( (\lambda_j)_{j=1}^{n} \) is an in (1.21) and \( T_n \) as in (1.22).

**Proof:**

Choose \( r = r(k,n) \) as in Lemma 1.21. If \( r \leq s \leq n \) then

\[ \max_{i=s} \{ \lambda_i B_i^{1/2}, \lambda_{s+1} \} = \max_{i=r} \{ \lambda_i B_i^{1/2}, \lambda_{r+1} \} \]

\[ = \max_{i \geq r} \{ \lambda_i B_i^{1/2} \} \]

\[ = \left( \frac{r-k}{\sum_{j=1}^{r} \lambda_j^{-2}} \right)^{1/2} \]

where \( B_i \in [0,1] \) and \( \sum_{i=1}^{s} B_i = s-k \). If \( k \leq s \leq r \) then for \( B_i \in [0,1] \) and

\[ \sum_{i=1}^{s} B_i = s-k \] we have

\[ \max_{i=s} \{ \lambda_i B_i^{1/2}, \lambda_{s+1} \} = \lambda_{s+1} \geq \lambda_r \]
Since \( \max_{1 \leq i \leq r} \lambda_i C_i^{1/2} = \left( \frac{r-k}{\sum_{j=1}^{r} \lambda_j^{-2}} \right)^{1/2} \) where \( C_i \in [0,1] \) and \( \sum_{i=1}^{r} C_i = r-k \),

we have, by the choice of \( r \),

\[
\lambda_{s+1} \geq \lambda_r \geq \max_{1 \leq i \leq r} \{ \lambda_i C_i^{1/2}, \lambda_{r+1} \} = \left( \frac{r-k}{\sum_{j=1}^{r} \lambda_j^{-2}} \right)^{1/2}.
\]

Therefore

\[
\max_{1 \leq i \leq s} \lambda_i B_i^{1/2}, \lambda_{s+1} \geq \left( \frac{r-k}{\sum_{j=1}^{r} \lambda_j^{-2}} \right)^{1/2},
\]

thus

\[
\alpha_k = \inf_{k \leq s \leq n} \{ \max_{1 \leq i \leq s} \lambda_i B_i^{1/2}, \lambda_{s+1} : B_i \in [0,1], \sum_{i=1}^{s} B_i = s-k \}
\]

\[
= \left( \frac{r-k}{\sum_{j=1}^{r} \lambda_j^{-2}} \right)^{1/2}, \text{ where } r \text{ is the unique integer, } k+1 \leq r \leq n, \text{ satisfying (1.21)}.
\]

**Theorem 1.24** If \( T : \ell^1 \rightarrow \ell^2 \) is a diagonal, \( T \sim (\lambda_i)_i \), where \((\lambda_i)_i \) is null sequence, then

\[
\alpha_k(T) = \lim_{n \to \infty} \left( \frac{r_{n-k}}{\sum_{j=1}^{r_{n} \lambda_j^{-2}} \right)^{1/2},
\]

where, for each \( n \geq k+1 \), \( r_n \) is the unique integer, \( k+1 \leq r_n \leq n \), satisfying (1.21).

**Proof:**

Since \((\lambda_i)_i \) is a null sequence \( ||T-T_n|| \to 0 \) where \( T_n = TP_n \) and \( P_n : \ell^1 \to \ell^1(n) \) is the natural projection. The Theorem now follows from (1.23).
Example 1.25 Let $T: \ell^1 \to \ell^2$ be the diagonal, $T \sim (i^{-1/2})_{i=1}^{\infty}$, and let $T_n$ be as in (1.24). For $n \geq k+1$, $r(k,n)$ will be the unique integer obtained in (1.23).

<table>
<thead>
<tr>
<th>$k=1$</th>
<th>$n$</th>
<th>$r(1,n)$</th>
<th>$\alpha_1(T_n)$</th>
</tr>
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<tbody>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>$1/\sqrt{3}$</td>
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<tr>
<td>3</td>
<td>3</td>
<td>3</td>
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</tr>
<tr>
<td>4</td>
<td>3</td>
<td>3</td>
<td>$1/\sqrt{3}$</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>3</td>
<td>$1/\sqrt{3}$</td>
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<tr>
<td>and $\alpha_1(T) = 1/3$.</td>
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</table>

<table>
<thead>
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<td>3</td>
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</tr>
<tr>
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<td>4</td>
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<td>$1/\sqrt{5}$</td>
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<td>6</td>
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<td>$1/\sqrt{5}$</td>
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<td>7</td>
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<tr>
<td>and $\alpha_2(T) = 1/\sqrt{5}$.</td>
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For $k = 3$

<table>
<thead>
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<th>$r(3,n)$</th>
<th>$\alpha_3(T_n)$</th>
</tr>
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<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>$1/\sqrt{10}$</td>
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<tr>
<td>5</td>
<td>5</td>
<td>5</td>
<td>$\sqrt{2}/\sqrt{15}$</td>
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<td>6</td>
<td>6</td>
<td>6</td>
<td>$1/\sqrt{7}$</td>
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<td>8</td>
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<td>$1/\sqrt{7}$</td>
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<td>9</td>
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</table>
and $\alpha_3(T) = 1/7$.

In general we have $\alpha_k(T) = 1/\sqrt{2k+1}$.

We next prove a result about the Kolmogoroff diameters of an arbitrary operator $T : \ell^1 \to F$ where $F$ is any Banach space.

**Theorem 1.26** If $T \in \mathcal{L}(\ell^1, F)$ then $\alpha_k(T) = \delta_k(T)$ for all $k$.

**Proof:**

If $T(U_k) \subseteq \delta U_F + F_k$ where $F_k$ is a subspace of $F$ of dimension $k$ then there are elements $x_i \in U_F$ and $y_i \in F_k$ such that $Te_i = \delta x_i + y_i$ for each $i$. Define $A : \ell^1 \to F$ by $Ae_i = y_i$ and extend linearly to all of $\ell^1$. Since $\sup_i \|y_i\| \leq \|T\| + \delta$ we have $\|A\| \leq \|T\| + \delta$. Also, $A(\ell^1) \subseteq F_k$ so $A$ is of rank at most $k$.

Therefore

$$\alpha_k(T) \leq \|T - A\| = \sup_i \|(T - A)e_i\| = \sup_i \|\delta x_i\| \leq \delta.$$ 

Since $\delta$ was arbitrary we have $\alpha_k(T) \leq \delta_k(T)$. Equality now follows from (1.17).

If $B = \{(\lambda, \xi) : \|\xi\|_p \leq 1\}$ and $B_1 = \{(\lambda, \xi) : \|\xi\|_1 \leq 1\}$, where $(\lambda_i)_{i=1}^\infty$ is a non-increasing sequence of positive scalars, then from (1.20) and (1.26) it follows that $\delta_k(B) = \delta_k(B_1) = \lambda_{k+1}$ for all $k$.

We point out that (1.26) obviously extends to $\ell^1(\Gamma)$, $\Gamma$ arbitrary. Less obvious is the fact that if $E$ is a separable $\ell^1, \lambda$-space then there exists $C > 0$, $C$ depending only on $\lambda$, such that $\delta_k(T) \leq \alpha_k(T) \leq C \delta_k(T)$ for all $k$ and $T \in \mathcal{L}(E, F)$, $F$ any Banach space.

Indeed, Johnson, Rosenthal, and Zippin [30] have shown that for such $E$ there exist finite dimensional subspaces $E_n \subseteq E$, projections $P_n : E \to E_n$ with $\|P_n\| \leq C(\lambda)$ for all $n$ such that $d(E_n, \ell^1(\lambda_n)) \leq \lambda_n E_n \subseteq E_{n+1}$.
and \( E = \bigcup_{n=1}^{\infty} E_n \). Moreover \( P_m P_n = P_m \) for \( m \geq n \). The result now follows from the above construction.

We now summarize the main results of this chapter.

**Theorem 1.27** If \( T: \ell^p \to \ell^q \) is a diagonal operator, \( T \sim (\lambda_i^1)_{i=1}^\infty \), then

1. If \( 1 \leq p = q \leq \infty \) we have \( \delta_k(T) = \alpha_k(T) = |\lambda_{k+1}| \).
2. If \( 1 \leq q < p = \infty \) we have \( \delta_k(T) = \alpha_k(T) = \left( \sum_{i=k+1}^{\infty} |\lambda_i|^q \right)^{1/q} \).
3. If \( 1 < q < p < \infty \) we have \( \alpha_k(T) = \left( \sum_{i=k+1}^{\infty} |\lambda_i|^s \right)^{1/s}, \frac{1}{p} + \frac{1}{s} = \frac{1}{q} \).
4. If \( 1 \leq p < q \leq \infty \) we have \( 1/2 \left| \lambda_{k+2} \right| \leq \alpha_k(T) \leq |\lambda_{k+1}| \).

**Corollary 1.28** If \( T: \ell^p \to \ell^q \) is as in (1.27) then \( T \in \ell^F(\ell^p,\ell^q) \) if and only if

1. \( (\lambda_i^1)_{i=1}^\infty \subseteq \ell^F \) if \( 1 \leq p = q \leq \infty \)
2. \( (\mu_i^1)_{i=1}^\infty \subseteq \ell^F \) if \( 1 \leq q < p \leq \infty \) where
\[
\mu_i = \left( \sum_{j=i+1}^{\infty} |\lambda_j|^s \right)^{1/s}, \frac{1}{p} + \frac{1}{s} = \frac{1}{q}, \text{ (if } p = \infty \text{ take } s = q). \]

This characterization of diagonals of type \( \ell^p \) on the \( \ell^q \)-spaces is closely related to the Grothendieck-Tong [5], [32] results on nuclear diagonals on the \( \ell^q \)-spaces, and to the results of Markus on \( \mathcal{K} \)-operators [17].

In [32] Tong (see also Holub [8]) showed that a diagonal \( T: \ell^q \to \ell^s, 1 \leq q, s \leq \infty, T \sim (\lambda_i^1)_{i=1}^\infty \), is nuclear if and only if \( (\lambda_i^1)_{i=1}^\infty \subseteq \ell^h \) and, moreover, \( \nu(T) = \| (\lambda_i^1)_{i=1}^\infty \|_h \) where
Thus we see that in most cases the nuclear and type $\ell^1$ diagonals on the $\ell^q$-spaces are distinct. For example, in the case $1 \leq q < s < \infty$, we see that a diagonal $T \sim (\lambda_i^q)_{i=1}^{\infty}$ from $\ell^q$ to $\ell^s$ is nuclear if $(\lambda_i^q)_{i=1}^{\infty} \in \ell^h$, $h = \frac{qs}{qs+q-s} > 1$, whereas to be of type $\ell^1$, we must have $(\lambda_i^q)_{i=1}^{\infty} \in \ell^1$. If $T: \ell^q \to \ell^1$ we see that $T$ is nuclear if $(\lambda_i^q)_{i=1}^{\infty} \in \ell^1$ but we require $\sum_{i=1}^{\infty} \lambda_i$ to be finite in order for $T$ to be of type $\ell^1$.

1.29 Remarks:

1) Pietsch has shown that the composition of three absolutely-summing operators is of type $\ell^1$ [27]. We give an example which shows that this is the best result possible. Define $T: \ell^1 \to \ell^2$ by $T \sim \left(n^{-1/2}[\ln(n+1)]^{-1}\right)_{n=1}^{\infty}$. We will show that $T$ is the composition of two absolutely-summing operators, hence fully nuclear, but $T$ is not of type $\ell^1$. Let $I: \ell^1 \to \ell^2$ be injection operator and define a diagonal $S: \ell^2 \to \ell^2$ $S \sim \left(n^{-1/2}[\ln(n+1)]^{-1}\right)_{n=1}^{\infty}$. 
Then $T = SI$. It is well known that $I$ is absolutely-summing [5]; it is clear that $S$ is Hilbert-Schmidt, hence absolutely-summing [5], [27]. $T$ is not of type $\ell^1$ however, since by (1.16),

$$\alpha_n(T) \geq 1/2 \, n^{-1/2} \left[ \sum_{n=1}^{\infty} (n(n+1))^{-1} \right]$$

and

$$(n^{-1/2} \left[ \sum_{n=1}^{\infty} (n(n+1))^{-1} \right])_{n=1}^{\infty} \notin \ell^1.$$

ii) It follows from (1.28) and the results of Tong-Grothendieck [32], [5], that the nuclear and type $\ell^1$ diagonals from $\ell^p$ to $\ell^p$ coincide, $1 \leq p \leq \infty$. However we are able to give an example of a nuclear operator on $\ell^1$ which is not of type $\ell^1$. Indeed, let $T: \ell^1 \to \ell^2$ be a nuclear operator which is not of type $\ell^1$ (take $T$ as in (i), for example). By (0.10) $T$ admits a factorization

\[ T = DA, \]

where $D$ is nuclear. But then $DA: \ell^1 \to \ell^1$ is nuclear but not of type $\ell^1$ (otherwise $T$ would also be of type $\ell^1$).

iii) If $T \in \mathcal{B}(E,F)$, $p > 1$, we cannot, in general say very much about $T$ other than that it is compact. Indeed, for fixed $p > 2$ choose $(\lambda_i)_{i=1}^{\infty} \in \ell^p$, $|\lambda_i| \geq |\lambda_{i+1}|$, but $(\lambda_i)_{i=1}^{\infty} \notin \ell^2$. Construct the diagonal $T: \ell^2 \to \ell^2$, $T \sim (\lambda_i)_{i=1}^{\infty}$. Then $T$ is of type $\ell^p$ by (1.28) and yet $T$ is not absolutely q-summing for any $q$. If $T$ were absolutely q-summing for some $q$ then $T$ would be Hilbert-Schmidt [27] hence of type $\ell^2[27]$. But by (1.28) $T$ is not of type $\ell^2$. 

\[ \begin{array}{ccc}
\ell^1 & \xrightarrow{T} & \ell^2 \\
A & \downarrow & B \\
C & \xleftarrow{D} & \ell^1
\end{array} \]
Since p-summing operators need not be compact (e.g., the natural injection $I: l^1 \to l^2$ is absolutely-summing [5], [15], but certainly not compact), there are many operators which are p-summing, $p \geq 1$, but not of type $l^q$ for any $q$. Thus, in general, there is no relation between $\pi_p(E,F)$ and $\ell^p(E,F)$, $p \geq 1$.

iv) Using (1.28) it is also easy to construct operators on $l^2$ which are of type $\ell^p$, $p > 2$, but not q-nuclear for any $q > 2$. Indeed let $T$ be as in (iii). Then $T$ is of type $\ell^p$ but not q-nuclear for any $q$ (otherwise $T$ would be absolutely q-summing [24], hence Hilbert-Schmidt [24], [27]).

In general, the only relationship holding between the classes $N_\pi(E,F)$, $\ell^p(E,F)$, and $\pi_p(E,F)$, $p \geq 1$, are

(a) $\ell^1(E,F) \subseteq N(E,F)$ with equality if $E,F$ are Hilbert spaces [27], [28].

(b) $\ell^2(l^2, l^2) = NS(l^2, l^2) = \pi_p(l^2, l^2) \supset N_\pi(l^2, l^2) \supset N_\pi(l^2, l^2) [15], [24], [28].
CHAPTER II
L^p(E,F) AND SETS OF TYPE L^p

In the second chapter of his memoir [5], Grothendieck introduced a class of operators on Banach spaces which he called strongly p-summable ("de puissance p. ème sommable") for 0 < p ≤ 1. Namely \( T \in \mathcal{L}(E,F) \) is said to be strongly p-summable, 0 < p ≤ 1, if it has a representation \( T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i \), where \( (\lambda_i)_{i=1}^{\infty} \in c_0^p, (f_i)_{i=1}^{\infty} \in c_0 \), \( (y_i)_{i=1}^{\infty} \subset F \), and \( \|f_i\|, \|y_i\| \leq 1 \). Following Grothendieck, we will let \( L^p(E,F) \) denote the complete, metrizable, topological vector space of all strongly p-summable operators from \( E \) to \( F \) equipped with the topology generated by \( S_p(T) = \inf \{ \sum_{i=1}^{\infty} |\lambda_i|^p : T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i \} \). Observe that \( L^1(E,F) = N(E,F) \) and \( S_1 = \nu \).

It follows from (0.3) that for 0 < p ≤ 1, \( L^p(E,F) \subset L^p(E,F) \) and \( S_p(T) \leq C(p) \rho_p(T) \), where \( C(p) \) is a constant depending only on p. Thus the Fredholm theory developed by Grothendieck for strongly p-summable operators carries over to operators of type \( L^p \), 0 < p ≤ 1.

The classes \( L^p(E,F) \) and \( L^p(E,F) \) are, in general, distinct. In (1.29, (i)) we gave an example of a nuclear operator which was not of type \( L^1 \). For 0 < p < 1, examples of strongly p-summable operators which are not of type \( L^p \) can be readily constructed. Indeed, let \( T: \ell^\infty \to \ell^1 \) be the diagonal corresponding to \( (\lambda_n)_{n=1}^{\infty} \), where \( \lambda_n \) is defined as follows: let \( \beta > 1 \) be such that \( \beta p > 1 \) and let \( \beta_n = [n+1]^{(p-1)/p} \left[ \ell n(n+1) \right]^{-\beta} \). Let \( \lambda_n = \beta_n^{-\beta} n+1 \). As we shall
see in (3.22) $T \in \mathcal{L}^p(\ell^\infty, \ell^1)$ but $T$ is not of type $\ell^q$ for any $q < p(1-p)^{-1}$. Since $p < p(1-p)^{-1}$, $T$ is not of type $\ell^p$.

Grothendieck also introduced a collection of subsets of a Banach space which he called sets of magnitude $p$, $0 < p \leq 1$. Namely, a bounded set $A$ of $E$ is said to be of magnitude $p$ in $E$ if the canonical map $E_{\Gamma(A)} \to E$ is strongly $p$-summable. Here $\Gamma(A)$ denotes the convex, circled hull of $A$ and $E_{\Gamma(A)} = \bigcup_{n=1}^{\infty} n\Gamma(A)$ is normed by the Minkowski functional of $\Gamma(A)$. The relationship between strongly $p$-summable operators and sets of magnitude $p$ is shown by the following theorem of Grothendieck [5§2].

**Theorem:** Let $T \in \mathcal{L}(E,F)$ and $0 < p \leq 1$. Then

i) If $T$ takes $U_E$ to a subset of magnitude $p$ in $F$ then $T \in \mathcal{L}^p(E,F)$.

ii) If $T \in \mathcal{L}^p(E,F)$ and is one-to-one, or the kernel of $T$, $\ker(T)$, is complemented in $E$, then $T$ takes $U_E$ to a subset of magnitude $p$ in $F$.

iii) If $T \in \mathcal{L}^p(E,F)$ and $0 < p \leq 2/3$ then $T$ takes $U_E$ to a subset of magnitude $q$ in $F$, $1/q = 1/p - 1/2$.

Motivated by this result we make the following definition.

**Definition 2.1** A bounded set $A \subseteq E$ is said to be a set of type $\ell^p$, $0 < p \leq 1$, if the canonical map $E_{\Gamma(A)} \to E$ is of type $\ell^p$.

The results which we next present are direct analogs of Grothendieck's results on strongly $p$-summable operators.

**Proposition 2.2** Let $0 < p \leq 1$ and $T \in \mathcal{L}(E,F)$. If $T$ takes $U_E$ to a set of type $\ell^p$ in $F$ then $T \in \mathcal{L}^p(E,F)$.

**Proof:**

Let $W = T(U_E)$ and consider the commutative diagram
where $\tilde{T}$ is the operator "defined" by $T$ and $J$ is the canonical map.

By hypothesis, $W$ is a set of type $\mathcal{E}^p$ in $F$ hence $J \in \mathcal{E}^p(F_W, F)$, but then $T = JT$ is of type $\mathcal{E}^p$.

**Proposition 2.3** If $0 < p \leq 1$ and $T \in \mathcal{E}^p(E, F)$ is one-to-one then $T$ takes $U_E$ to a set of type $\mathcal{E}^p$ in $F$.

**Proof:**

Let $W = T(U_E)$ and

$$
\begin{array}{c}
\begin{tikzpicture}
  \node (E) at (0,0) {$E$};
  \node (F) at (2,0) {$F$};
  \node (J) at (2,-2) {$J$};
  \node (TW) at (1,-2) {$F_W$};
  \node (T) at (1,0) {$T$};
  \draw [->] (E) to node [midway] {$T$} (F);
  \draw [->] (T) to node [midway, left] {$\tilde{T}$} (TW);
  \draw [->] (TW) to node [midway, right] {$J$} (J);
\end{tikzpicture}
\end{array}
$$

be as in (2.2). We must show that $J \in \mathcal{E}^p(F_W, F)$. To this end, choose $A : E \to F$ of rank at most $k$ such that $\|T-A\| < \alpha_k(T) + \epsilon$. Define $\tilde{A} : F_W \to F$ by $\tilde{A} = AT^{-1}$. $\tilde{A}$ is well defined since both $A$ and $T^{-1}$ are continuous ($T^{-1} : F_W \to E$). Also, $\tilde{A}$ is of rank at most $k$. Now

$$\alpha_k(J) \leq \|J-\tilde{A}\| = \sup_{y \in W} \|J - \tilde{A}y\|_F = \sup_{x \in U_E} \|JT_x - \tilde{A}T_x\|_F = \sup_{x \in U_F} \|T_x - Ax\|_F$$

Thus $\alpha_k(J) \leq \alpha_k(T)$ for each $k$, hence $J \in \mathcal{E}^p(F_W, F)$.

**Proposition 2.4** Let $0 < p \leq 1$ and $T \in \mathcal{E}^p(E, F)$ with $\ker(T)$ complemented in $E$. Then $T$ takes $U_E$ to a set of type $\mathcal{E}^p$ in $F$.

**Proof:**

Since $\ker(T)$ is complemented we can write $E = \ker(T) \oplus X$. Let $P : E \to X$ denote the projection of $E$ onto $X$ and let $T_X = T|_X$. Since $T$ is of type $\mathcal{E}^p$ so is $T_X$ (indeed, $\alpha_k(T_X) \leq \alpha_k(T)$) and $T_X$ is one-to-one. By (2.3) $T_X$ takes the unit ball in $X$ to a set
of type $\mathcal{E}^p$ in $F$. Since $U_X = U_E \cap X$ and $T_R P(U_E) \subseteq \|p\|_R (U_X)$ it follows that $T_R P(U_E)$ is a set of type $\mathcal{E}^p$ in $F$ (clearly, a subset of a set of type $\mathcal{E}^p$ in $F$ is also of type $\mathcal{E}^p$ in $F$). But $T_R P(U_E) = T(U_E)$.

**Proposition 2.5** Let $0 < p \leq 1$. If $T \in \mathcal{E}^p(E,F)$ then $T$ takes $U_E$ to a set of type $\mathcal{E}^r$ in $F$, $1/2 = 1/p - 1$.

**Proof:**

Since $\mathcal{E}^p(E,F) \subseteq L^{(p)}(E,F)$, $0 < p \leq 1$, there exist sequences $(f_i)_{i=1}^\infty \subseteq U_E$, $(y_i)_{i=1}^\infty \subseteq U_F$, and $(\lambda_i)_{i=1}^p \in \mathcal{E}^p$, $|\lambda_i| \geq |\lambda_{i+1}|$ such that $T = \sum_{i=1}^\infty \lambda_i f_i \otimes y_i$. Therefore $T$ has the factorization

\[
\begin{array}{c}
E \\
\Downarrow A \\
\Downarrow D \\
\Downarrow B \\
\Downarrow 2 \\
\Downarrow F \\
E 
\end{array}
\]

where $Ax = (\langle \alpha, \lambda_i^{p/2} f_i \rangle)_{i=1}^\infty$, $D(\xi_i)_{i=1}^\infty = (\lambda_i^{1-p} \xi_i)_{i=1}^\infty$, and $B(\xi_i)_{i=1}^\infty = \sum_{i=1}^\infty \lambda_i^{p/2} \xi_i y_i$. By (1.20) $\alpha_k^r(D) = |\lambda_{k+1}|^{1-p}$, hence $D$ is of type $\mathcal{E}^r$, $1/r = 1/p - 1$. But then $BD \in \mathcal{E}^r(2^2,F)$ and Ker $(BD)$ is complemented in $2^2$ and so by (2.4) $BD$ takes $U_2$ to a set of type $\mathcal{E}^r$ in $F$. Since $A(U_E) \subseteq (E | \lambda_i |^{p})_{i=1}^\infty U_2$ we have $T(U_E) = BDA(U_E) \subseteq (E | \lambda_i |^{p})_{i=1}^\infty BD(U_2)$, thus $T(U_E)$ is a set of type $\mathcal{E}^r$ in $F$.

**Theorem 2.6** If $T \in \mathcal{E}(E,F)$ is the norm limit of finite rank operators then $\alpha_k^r(T) = \alpha_k^r(T^*) = \alpha_k^r(T^*)$ for all $k$.

**Proof:**

Let $J:F \to F''$ be the natural embedding and $S:E'' \to F$ finite rank. Let $\beta_k(S) = \inf\{||S - A||: A:E'' \to F \text{ is of rank at most } k\}$. Choose
$A_k:E'' \to F''$ such that $\|JS-A_k\|_F < \alpha_k(JS) + \epsilon$. Let $G = [JS(E'') \cup A_k(E'')]$. By the Principle of Local Reflexivity [16] there exists a one-to-one operator $\varphi:G \to F$ such that $\|\varphi\| = 1$, $\|\varphi^{-1}\| \leq 1 + \epsilon$, and $\varphi|_{G \cap J(F)}$ is the identity. Consider $\varphi A_k:E'' \to F$ if $\|x''\| \leq 1$, we have

$$\|(S-\varphi A_k)x''\|_F = \|(\varphi J S - \varphi A_k)x''\|_F$$

$$\leq \|JSx'' - A_kx''\|_F''$$

hence

$$\|S - \varphi A_k\| \leq \|JS - A_k\|_F'' < \alpha_k(JS) + \epsilon$$

Thus

$$\beta_k(S) \leq \alpha_k(JS) \leq \|J\| \beta_k(S) = \beta_k(S).$$

Therefore for finite rank $S:E'' \to F$, $\beta_k(S) = \alpha_k(JS)$.

Since $T$ is the norm limit of finite rank operators, $T^{**}:E^{**}E'' \to J(F)$ and there exists a sequence $(S_n) \subset \mathcal{L}(E'', J(F))$ of finite rank operators such that $\lim_{n \to \infty} \|S_n - T^{**}\| = 0$. Choose $N$ such that $\|T^{**} - S_n\| < \epsilon/2$ for $n \geq N$. Since $|\beta_k(J^{-1}T^{**}) - \beta_k(J^{-1}S_n)|$ $\leq \|T^{**} - S_n\|$ we have

$$\beta_k(J^{-1}T^{**}) < \beta_k(J^{-1}S_n) + \epsilon/2$$

for all $k$ and $n \geq N$. From the above we obtain

$$\beta_1(T^{**}) \leq \beta_k(J^{-1}T^{**}) < \beta_k(J^{-1}S_n) + \epsilon/2$$

$$= \alpha_k(S_n) + \epsilon/2.$$ 

But $|\alpha_k(T^{**}) - \alpha_k(S_n)| \leq \|T^{**} - S_n\| < \epsilon/2$, that is $\alpha_k(S_n) + \epsilon/2 < \alpha_k(T^{**}) + \epsilon$. Thus $\beta_k(T^{**}) < \alpha_k(T^{**}) + \epsilon$. 

Since $\epsilon > 0$ was arbitrary we have the desired inequality. Clearly, we always have $\alpha_k(T^{**}) \leq \beta_k(T^{**})$ since $\beta_k(T^{**}) = \inf\{||T^{**} - A||: A:E'' \to J(F) \text{ is of rank at most } k\}$ and $\alpha_k(T^{**}) = \inf\{||T^{**} - A||: A:E'' \to F \text{ is of rank at most } k\}$. Therefore,

$$\alpha_k(T) \leq \beta_k(T^{**}) = \alpha_k(T^{**}) \leq \alpha_k(T) \leq \alpha_k(T),$$

which gives the desired equalities.

**Corollary 2.7** An operator $T \in \mathcal{L}^p(E, F)$, $1 \leq p \leq \infty$, if and only if $T^{*} \in \mathcal{L}^p(F', E')$ if and only if $T^{**} \in \mathcal{L}^p(E'', F'')$.

**Theorem 2.8** Let $0 < p \leq 1$, $T \in \mathcal{L}^p(E, F)$, and $N$ a closed subspace of $E$ contained in $\text{Ker}(T)$. Then $\widetilde{T}: E/N \to F$ is of type $\ell^r, 1/r = 1/p - 1$. If $M$ is a closed subspace of $F$ containing $T(E)$ then $\widetilde{T}: E \to M$ is of type $\ell^r, 1/r = 1/p - 1$. (Here $\widetilde{T}$ is the canonical map defined by $T$).

**Proof:**

First consider the operator $\widetilde{T}: E/N \to F$ defined by $T$. By (2.5) $T$ takes $U_E$ to a set of type $\ell^r$ in $F$, $1/r = 1/p - 1$. If $U_N$ denotes the unit ball in $E/N$ then $U_N = U_E + N$. Thus $\widetilde{T}(U_N) = T(U_E)$ and it follows from (2.2) that $\widetilde{T}$ is of type $\ell^r$.

Now suppose that $M$ is a closed subspace of $F$ containing $T(E)$. Let $\widetilde{T}: E \to M$ be the operator defined by $T$. By (2.7) $T^{*} \in \mathcal{L}^p(F', E')$ and $M^0 \subset \text{Ker}(T^{*})$. By the above the operator $\widetilde{T}': F'/M^0 \to E'$ defined by $T^{*}$ is of type $\ell^r$, $1/r = 1/p - 1$. Again by (2.7) $(\widetilde{T}')^*: E'' \to M^{oo}$ is of type $\ell^r$, but $(\widetilde{T}')^* = \widetilde{T}^{**}$. The fact that $\widetilde{T}$ is of type $\ell^r$ now follows from (2.7).

**Corollary 2.9** If $M$ is a closed subspace of $F$, $A$, a subset of $M$ which is a set of type $\ell^p$ in $F$, $0 < p \leq 1$, then $A$ is a set of type $\ell^r$ in $M$, $1/r = 1/p - 1$. 


Pietsch has remarked that if \( \mathcal{L}(E,F) = \ell^p(E,F) \), \( 0 < p \leq 1/2 \), then one of \( E,F \) is finite dimensional. Using (0.12) we have been able to extend this result to the case \( 0 < p \leq 2/3 \).

**Theorem 2.10** If \( 0 < p \leq 2/3 \) and \( T \in \mathcal{L}^p(E,F) \) then \( T \) is fully nuclear.

**Proof:**

If \( T \in \mathcal{L}^p(E,F) \) then \( T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i \) where \( (\lambda_i)_{i=1}^{\infty} \in \ell^p \) with \( |\lambda_1| \geq |\lambda_{i+1}| \) and \( (f_i)_{i=1}^{\infty} \subseteq U_E \), \( (y_i)_{i=1}^{\infty} \subseteq U_F \). Thus \( T \) has the following factorization:

\[
\begin{array}{cccc}
E & \xrightarrow{T} & F \\
\downarrow{U} & & \uparrow{V} \\
\ell^\infty & \xrightarrow{\mathcal{B}} & \ell^2 \\
\end{array}
\]

where \( Ux = (\langle x, f_i \rangle)_{i=1}^{\infty} \) \( \mathcal{B}(\xi_i)_{i=1}^{\infty} = (\lambda_1^{2/3} \xi_i)_{i=1}^{\infty} \), and \( V(\xi_i)_{i=1}^{\infty} = \sum_{i=1}^{\infty} \lambda_i^{1/3} \xi_i y_i \). Now \( \mathcal{B} = \sum_{i=1}^{\infty} \lambda_i^{2/3} e_i \otimes e_i \) hence \( \mathcal{B} \) is nuclear. Let \( H = V^{-1}(T(E)) \). Then \( H \) is a closed subspace of \( \ell^2 \), hence complemented. Let \( P: \ell^2 \to H \) denote the projection of \( \ell^2 \) onto \( H \).

Now \( P\mathcal{B}U:E \to H \) is nuclear (since \( \mathcal{B} \) is nuclear), hence \( V P\mathcal{B}U:E \to T(E) \) is nuclear. But \( V P\mathcal{B}U = T_a \) and so \( T \) is fully nuclear.

**Corollary 2.11** If \( 0 < p \leq 2/3 \) and \( \mathcal{L}(E,F) = \ell^{2/3}(E,F) \) then one of \( E,F \) is finite dimensional.

**Proof:**

It follows from (2.10) that every operator from \( E \) to \( F \) is fully nuclear. The result 2.11 now follows from (0.12).
We next obtain some results concerning geometric properties of sets of type $\mathcal{L}^p$, $0 < p \leq 1$.

**Proposition 2.12** Let $0 < p \leq 1$ and $A$ a set of type $\mathcal{L}^p$ in $E$.

Then $A$ is contained in the closed, circled, convex hull of a sequence $(\lambda_i x_i)^{\infty}_{i=1}$ where $(x_i)^{\infty}_{i=1}$ is bounded in $E$ and $(\lambda_i)^{\infty}_{i=1} \in \mathcal{L}^r$, $1/r = 1/p - 1$.

**Proof:**

Suppose that $A$ is closed, circled, and convex. Then since $E \rightarrow E$ is of type $\mathcal{L}^p$ there exist bounded sequences $(f_i)^{\infty}_{i=1} \subseteq (E_A)'$, $(y_i)^{\infty}_{i=1} \subseteq E$, and a sequence $(\mu_i)^{\infty}_{i=1} \in \mathcal{L}^p$ such that

$$x = \sum_{i} \mu_i <x, f_i>y_i \text{ for each } x \in E_A.$$  

We may assume that $\sum_{i} |\mu_i|^p \leq 1$ and $|<x, f_i>| \leq 1$ for each $i$ and $x \in A$. Then if $x \in A$ we have $x = \sum_{i} \beta_i z_i$ where $\beta_i = \mu_i^{1-p} <x, f_i>$ and $z_i = \mu_i^{1-p} y_i$. Since $\sum_{i} |\beta_i| \leq 1$, $x$ is in the closed, circled, convex hull of the sequence $(z_i)^{\infty}_{i=1}$. To finish the proof take $\lambda_i = \mu_i^{1-p}$ and $x_i = y_i$.

**Proposition 2.13** If $(x_i)^{\infty}_{i=1}$ is a bounded sequence in $E$ and $(\lambda_i)^{\infty}_{i=1} \in \mathcal{L}^p$, $0 < p \leq 1$, then the closed, circled, convex hull of $(\lambda_i x_i)^{\infty}_{i=1}$ is a set of type $\mathcal{L}^r$ in $E$, $1/r = 1/p - 1$.

**Proof:**

Consider $\ell^1 \rightarrow \ell^1 \rightarrow E$ where $A(\xi_i)^{\infty}_{i=1} = (\lambda_i \xi_i)^{\infty}_{i=1}$ and $B(\xi_i)^{\infty}_{i=1} = \sum_{i=1}^{\infty} \xi_i x_i$. Since $\alpha_k(A) \leq |\lambda_k|$, $A$, hence $BA$, is of type $\mathcal{L}^p$. By (2.5) $BA(U_1)$ is a set of type $\mathcal{L}^r$ in $E$, $1/r = 1/p - 1$.

But $BA(U_1)$ is precisely the closed, circled, convex hull of $(\lambda_i x_i)^{\infty}_{i=1}$. 
CHAPTER III

THE SPACES \(L^p(E,F)\), \(J_p(E,F)\), AND \(J^p(E,F)\)

The somewhat curious fact that an operator may have a certain property (e.g. nuclearity) and yet its restriction fails to have this property has been pointed out by Grothendieck [512] and by Retherford-Stegall [29]. Motivated by these results we are led to ask if the restriction of an operator of type \(L^p\) is again of type \(L^p\). In the case \(0 < p \leq 1\), Theorem 2.8 suggests that the answer to this question is no. In what follows we study the relationship between the approximation numbers of an operator and those of its restriction.

Definition 3.1 For \(T \in \mathcal{L}(E,F)\) let \(\beta_k(T) = \inf \|T - A\|\), the inf being taken over all operators \(A : E \to \overline{T(E)}\) of rank at most \(k\). Then \(\mathcal{J}^p(E,F) = \{T \in \mathcal{L}(E,F) : (\beta_k(T))_{k=0}^\infty \subseteq L^p\}\).

It follows from the definitions that \(\alpha_k(T) \leq \beta_k(T)\) for all \(k\). Of course if the range of \(T\) is dense in \(F\) then we have equality.

In light of the above definition, (2.8) can be stated: If \(T \in \mathcal{J}^p(E,F)\) and \(0 < p \leq 1\) then \(T \in \mathcal{L}^q(E,F)\) where \(1/q = 1/p - 1\).

In (3.3) we give a simple example of an operator for which \(\alpha_k \neq \beta_k\) for any \(k\). But first we prove the following Proposition.

Proposition 3.2 If \(I : L^p \to L^\infty\) is the natural injection, \(1 \leq p \leq \infty\), then \(\beta_k(I) = 1\) for all \(k\).

Proof:

If \(p = \infty\) the result follows from (1.20), so consider the case \(1 \leq p < \infty\). Clearly \(\beta_k(I) \leq 1\) for all \(k\). If \(\beta_k(I) < 1\) for some \(k\) choose \(\epsilon > 0\) and \(A : L^p \to \overline{I(L^p)}\) (closure taken in \(L^\infty\)) of rank at
most $k$ such that $||I-A|| < 1-\epsilon$. If $B = \{x \in A^P: ||x||_\infty \leq ||A||\}$ then $B$ is relatively compact since $A$ is finite rank. Let $(y_i)_{i=1}^n \subset B$ be an $\epsilon/4$-chain for $B$. Observe that $\exists \in I(\ell^P)$ implies that $\exists \in C_0$, thus there exists an index $j_0$ such that if $y_i = (y_{ij})_{j=1}^\infty$ then $|y_{ij_0}| < \epsilon/4$ for all $i$, $1 \leq i \leq n$. Now $Ae_i \in B$ for each $i$ and therefore for each $i$ there exists an index $p(i)$, $1 \leq p(i) \leq n$, such that $||Ae_i - y_{p(i)}||_\infty \leq \epsilon/4$. If $Ae_i = (a_{ij})_{j=1}^\infty$ for each $i$ then $|a_{ij} - y_{p(i)}| \leq \epsilon/4$ for all $j$. Since $||I-A|| < 1-\epsilon$ we have

$$1-\epsilon > \sup_i ||(I-A)e_i||_\infty = \sup_{i,j \neq 1} \{|1-a_{ii}|, |a_{ij}|\}.$$ 

Thus $|1-a_{ii}| < 1-\epsilon$ for all $i$. Letting $i = j_0$ we have $|a_{j_0 j_0} - y_{p(j_0)}| \leq \epsilon/4$ hence $|a_{j_0 j_0}| < \epsilon/2$. But then

$$1-\epsilon/2 < 1-a_{j_0 j_0} < 1-\epsilon.$$ 

This contradiction shows that $\beta_k(I) \leq 1$ for all $k$.

**Corollary 3.3** If $I: \ell^1 \rightarrow \ell^\infty$ is the natural injection then $\alpha_k(I) < \beta_k(I)$ for all $k$.

**Corollary 3.4** If $1 \leq p \leq q \leq \infty$ and $I: \ell^P \rightarrow \ell^q$ is the natural injection then $\beta_k(I) = 1$ for all $k$.

**Proposition 3.5** If $1 \leq p \leq \infty$ and $T: \ell^P \rightarrow \ell^\infty$ is a diagonal, $T \sim (\lambda_i)_{i=1}^\infty$ where $\lim_{i \rightarrow \infty} \lambda_i = 0$ then $\alpha_k(T) = \beta_k(T)$ for all $k$. 


Proof:

Let \( P_n : \ell^p \rightarrow \ell^p(n) \) be the natural projection and let
\( T_n : \ell^p \rightarrow \ell^\infty(n) \) be defined by \( T_n = TP_n \). Since \( (\lambda_i) \sim C \lim_{n \to \infty} \|T - T_n\| = 0 \),
hence \( \alpha_k(T_n) \to \alpha_k(T) \) uniformly in \( k \). Since \( T_n(\ell^p) = \ell^\infty(n) \),
\( \alpha_k(T_n) = \beta_k(T_n) \) for all \( k \) and \( n \). To finish the proof it suffices
to show that \( \beta_k(T_n) \to \beta_k(T) \) for each \( k \). Since \( T_n = TP_n \) we have
\[
\beta_k(T_n) = \alpha_k(T_n) = \alpha_k(TP_n) \leq \alpha_k(T) \leq \beta_k(T)
\]
for each \( k \) and \( n \). For fixed \( k \) choose \( n > k \) and let \( A : \ell^p \rightarrow \ell^\infty(n) \)
be an operator of rank \( \leq k \) such that \( \|T_n - A\| < \beta_k(T_n) + 2^{-n} \). Then
\[
\beta_k(T) \leq \|T - A\| \leq \|T - T_n\| + \|T_n - A\| < \|T - T_n\| + \beta_k(T_n) + 2^{-n}.
\]
But then
\[
\beta_k(T_n) \leq \beta_k(T) \leq \|T - T_n\| + \beta_k(T_n) + 2^{-n}
\]
hence
\[
\lim_{n \to \infty} \beta_k(T_n) \leq \beta_k(T) \leq \lim_{n \to \infty} \beta_k(T_n)
\]
that is,
\[
\lim_{n \to \infty} \beta_k(T_n) = \beta_k(T) \text{ for each } k.
\]

Proposition 3.6 If \( T \in \ell_2^p(E,F) \) and \( \overline{T(E)} \) is complemented in \( F \) then
\( T \in \ell_2^p(E,F) \). In particular, if \( F \) is a \( \ell_2 \)-space then \( \ell^p(E,F) = \ell_2^p(E,F) \) for every \( E \).

We next introduce the spaces of \( p \)-factorable operators from
\( E \) to \( F \), \( p \geq 1 \), and study the relationship between these and various
other classes of operators on Banach spaces.
In Chapter I we saw that for a diagonal $T: \ell^\infty \to \ell^1$, 

$T \sim (\lambda_i)^\infty$, to be of type $\ell^1$ it is necessary and sufficient that 

$\sum_{i=1}^{\infty} |\lambda_i|$ converge. It is easy to see that if $T\in \mathcal{L}(E,F)$ has a 

representation $T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i$ where $(f_i)^\infty \subset U_E$, $(y_i)^\infty \subset U_F$, and 

$\sum_{i=1}^{\infty} |\lambda_i|$ is finite then $T$ factors through a diagonal from $\ell^\infty$ to $\ell^1$ 

which is of type $\ell^1$. Indeed, we have 

\[ \begin{array}{ccc} 
E & \xrightarrow{T} & F \\
\downarrow A & & \downarrow B \\
\ell^\infty & \xrightarrow{\beta} & \ell^1 
\end{array} \]

where $Ax = (\langle x, f_i \rangle)^\infty$, $\beta \sim (\lambda_i)^\infty$, and $B(\xi_i)^\infty = \sum_{i=1}^{\infty} \xi_i y_i$. Since 

$\alpha_k(B) \leq \sum_{i=k+1}^{\infty} |\lambda_i|$ we have 

$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \alpha_k(B) \leq \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |\lambda_i| = \sum_{i=1}^{\infty} |\lambda_i|$, 

hence $\beta$ is of type $\ell^1$.

We next show that this representation actually characterizes 

the operators which factor through type $\ell^1$ diagonals from $\ell^\infty$ to $\ell^1$.

**Definition 3.7** We will say that $T\in \mathcal{L}(E,F)$ is $\ell^1$-factorable if $T$ 

factors through a diagonal $B: \ell^\infty \to \ell^1$ of type $\ell^1$.

**Proposition 3.8** $T\in \mathcal{L}(E,F)$ is $\ell^1$-factorable if and only if $T$ has 

a representation $T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i$ where $(f_i)^\infty \subset U_E$, $(y_i)^\infty \subset U_F$, 

and $\sum_{i=1}^{\infty} |\lambda_i|$ converges.
proof:

We have already observed the necessity, so suppose that $T$ has such a factorization. Then $T = BA$ where

\[
\begin{array}{c}
E \\
\downarrow B \\
\uparrow A \\
F
\end{array}
\]

with $D \sim (\lambda_i)_{i=1}^\infty$. We may assume that $|\lambda_i| \leq |\lambda_{i+1}|$ for all $i$ by a permutation of the indices if necessary. Since $B \in l^1(\ell^\infty, \ell^1)$ it follows from (1.28) that $\sum_{i=1}^\infty |\lambda_i|$ is finite. Define $f_i \in E'$ by $f_i = e_i \circ \frac{A}{||A||}$ and let $y_i = \frac{Be_i}{||B||}$. If $\mu_i = \lambda_i ||A|| ||B||$ then for $x \in E$ we have

\[
\sum_{i=1}^\infty \mu_i <x,f_i>y_i = \sum_{i=1}^\infty \lambda_i <e_i, Ax> Be_i
\]

\[
= B (\sum_{i=1}^\infty \lambda_i <e_i, Ax> e_i)
\]

\[
= BD (e_i, Ax)^\infty_{i=1}
\]

\[
= BSAx = Tx.
\]

hence $T = \sum_{i=1}^\infty \mu_i f_i \otimes y_i$ is the desired representation of $T$.

Clearly $\ell^1$-factorable operators are of type $\ell^1$. In order to see that there are spaces $E, F$ and operators $T \in \mathcal{L}(E, F)$ which are of type $\ell^1$ but not $\ell^1$-factorable we need the following theorem of H. Weyl [34]. (See Grothendieck [5§2]).
Theorem 3.9 Let \( \mathcal{H} \) be a Hilbert space and \( 0 < p \leq 1 \). If \( T \in L^p(\mathcal{H},\mathcal{H}) \) and if \( (z_i)_{i=1}^{\infty} \) is the sequence of eigenvalues of \( T \), arranged in order of decreasing modulus and repeated according to multiplicity, then
\[
\sum_{i=1}^{\infty} |z_i|^p \leq S_p(T)
\]

Let \( T: \ell^2 \to \ell^2 \) be the diagonal \( T \sim (i^{-3/2})_{i=1}^{\infty} \). That \( T \) is of type \( \ell^1 \) follows from (1.28). If \( T \) were \( \ell^1 \)-factorable then by (3.8) there exist sequences \( (f_i)_{i=1}^{\infty} \subset \ell^2 \), \( (y_i)_{i=1}^{\infty} \subset \ell^2 \) and scalars \( (\mu_i)_{i=1}^{\infty} \) with \( \sum_{i=1}^{\infty} i|\mu_i| \) finite such that \( T = \sum_{i=1}^{\infty} \mu_i f_i \otimes y_i \). But then
\[
\sum_{i=1}^{\infty} |\mu_i|^{2/3} = \sum_{i=1}^{\infty} (i|\mu_i|)^{2/3} i^{-2/3} \leq (\sum_{i=1}^{\infty} i|\mu_i|)^{2/3} (\sum_{i=1}^{\infty} i^{-2})^{1/3} < +\infty
\]
hence \( T \) is strongly \( 2/3 \)-summable, that is \( S_{2/3}(T) < +\infty \). By Theorem 3.9, since the sequence of eigenvalues of \( T \) is precisely \( (i^{-2/3})_{i=1}^{\infty} \), we have
\[
S_{2/3}(T) \geq \sum_{i=1}^{\infty} (i^{-3/2})^{2/3} = \sum_{i=1}^{\infty} i^{-1}
\]
This contradiction implies that \( T \) is not \( \ell^1 \)-factorable.

Definition 3.10 If \( T \in \mathcal{L}(E,F) \) we will say that \( T \) is \( p \)-factorable, \( p \geq 1 \), if \( T \) factors through a diagonal \( D: \ell^\infty \to \ell^1 \) having the property that \( \sum_{n=1}^{\infty} \alpha_n(T^*) \) converges. Let \( \mathcal{F}_p(E,F) \) denote the collection of \( p \)-factorable operators from \( E \) to \( F \).

It is clear from the definitions that \( \mathcal{F}_1(E,F) \) coincides with the \( \ell^1 \)-factorable operators from \( E \) to \( F \). We will next show that (3.8) can be generalized in a natural way to \( \mathcal{F}_p(E,F) \). First we
prove the following elementary lemma.

**Lemma 3.11** Let \( p \geq 1 \) then \( \sum_{i=1}^{n-1} i^{p-1} \geq K(p)n^p \) where \( K(p) = \frac{1}{p^{2-p}}. \)

**Proof:**

If \( n = 2 \) then \( \sum_{i=1}^{1} i^{p-1} = 1 \geq (p^{2-p})^2 \), so assume the lemma holds \( n \). Then \( \sum_{i=1}^{n} i^{p-1} = \sum_{i=1}^{n-1} i^{p-1} + n^{p-1} \geq p^{2-p}(n^p) + n^{p-1} \) by the induction hypothesis. We will show that \( p^{2-p}n^p + n^{p-1} \geq (n+1)p^{2-p} \). Now

\[
p^{2-p}n^p + n^{p-1} \geq (n+1)p^{2-p}
\]

if and only if

\[
n^{p-1} \geq p^{2-p}(n+1)^p - n^p)
\]

that is, if and only if

\[
(*) \quad \frac{p^2n^{p-1}}{(n+1)^p - n^p} \geq 1.
\]

To see that (*) holds let \( f(x) = x^p \). Since \( f'(x) = px^{p-1} \), by the Mean Value Theorem there is a number \( z, n \leq z \leq n+1 \), such that

\[
f(n+1) - f(n) = f'(z).
\]

But then

\[
\frac{p^2n^{p-1}}{(n+1)^p - n^p} = \frac{p^2n^{p-1}}{f(n+1) - f(n)} = \frac{p^2n^{p-1}}{f'(z)}
\]

\[
= \frac{p^2n^{p-1}}{z^{p-1}}
\]

\[
\geq 2^p \left( \frac{n}{n+1} \right)^{p-1}
\]

\[
\geq 1.
\]
Proposition 3.12 Let $p \geq 1$ and $T \in \mathcal{L}(E,F)$. Then $T$ factors through a diagonal $\mathcal{B}: \ell^\infty \to \ell^1$ having the property that $\sum_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\mathcal{B})$ converges if and only if there are sequences $(f_n)^\infty_{n=1} \subseteq U_E$, $(y_n)^\infty_{n=1} \subseteq U_F$ and scalars $(\lambda_n)^\infty_{n=1}$ with $\sum_{n=1}^{\infty} n^p |\lambda_n|$ finite such that $T = \sum_{n=1}^{\infty} \lambda_n f_n \otimes y_n$.

Proof:

If $T$ has such a representation then $T = BA$ where $B, \mathcal{B}$ and $A$ are as in (3.8). Since $\alpha_n(\mathcal{B}) = \sum_{k=n+1}^{\infty} |\lambda_k|$ for each $n$ we have $\sum_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\mathcal{B}) \leq \sum_{n=1}^{\infty} n^{p-1} \sum_{k=n}^{\infty} |\lambda_k| \leq \sum_{n=1}^{\infty} n^p |\lambda_n|$ hence $\mathcal{B}$ has the desired property.

Conversely, if $T$ factors through a diagonal $\mathcal{B}: \ell^\infty \to \ell^1$ having the property that $\sum_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\mathcal{B})$ converges then $\mathcal{B} \sim (\lambda_n)^\infty_{n=1}$ where $|\lambda_{n+1}| \geq |\lambda_n|$ for all $n$ (we can assume $(\lambda_n)^\infty_{n=1}$ is decreasing by taking a suitable permutation of the indices if necessary). Then

$$\sum_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\mathcal{B}) = \sum_{n=1}^{\infty} n^{p-1} (\sum_{k=n}^{\infty} |\lambda_k|)$$

$$= \sum_{n=1}^{\infty} \sum_{i=1}^{n} i^{p-1} |\lambda_n|$$

$$\geq K(p) \sum_{n=1}^{\infty} n^p |\lambda_n|$$

by (3.11). Let $f_i, y_i$, and $\mu_i$ be as in (3.8). Then $T = \sum_{i=1}^{\infty} \mu_i f_i \otimes y_i$ is a representation of $T$ having the desired property.
Definition 3.13 For $p \geq 1$ and $T \in \mathcal{F}_p(E,F)$ let $\mathcal{F}_p(T) = \inf \{ \sum_{n=1}^{\infty} n^{p-1} \alpha_n(B): B$ is a diagonal from $\ell^\infty$ to $\ell^1$ and $T$ factors through $B \}$. 

It follows from (3.12) that $\mathcal{F}_p(T)$ is equivalent to $\inf \{ \sum_{i=1}^{\infty} |\lambda_i|: T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i \}$. We will next show that $\mathcal{F}_p$ generates a topology on $\mathcal{F}_p(E,F)$ making $\mathcal{F}_p(E,F)$ a complete, metrizable, topological vector space.

Proposition 3.14 For all $p \geq 1$ $\mathcal{F}_p(E,F)$ is a vector space.

Proof:

Let $T_1, T_2 \in \mathcal{F}_p(E,F)$. Then $T_i = V_i \theta_i U_i$ where $\theta_i: \ell^\infty \rightarrow \ell^1$ is a diagonal with $\sum_{n=1}^{\infty} n^{p-1} \alpha_n(B)$ finite for $i=1,2$. Define $I: E \rightarrow E \oplus E$ by $Ix = (x,x)$ and $S: F \oplus \ell^\infty F \rightarrow F$ by $S(x,y) = x+y$. $I$ and $S$ are bounded linear operators and we have the commutative diagram.

\[ 
\begin{array}{c}
E \\
\downarrow I \\
E \oplus E \\
\downarrow \theta_i \oplus \theta_2 \\
\ell^\infty \oplus \ell^\infty \\
\downarrow \theta_i \oplus \theta_2 \\
\ell^\infty \oplus \ell^1 \\
\downarrow S \\
F \\
\end{array}
\]

We will first show that $\sum_{n=1}^{\infty} n^{p-1} \alpha_n(B_1 \oplus B_2)$ converges.

Indeed, if $A_i: \ell^\infty \rightarrow \ell^1$ is of rank at most $k$ with $\|\theta_i - A_i\|_{\ell^1} + \epsilon$, $i=1,2$, then the operator $A = A_1 \oplus A_2: \ell^\infty \oplus \ell^\infty \rightarrow \ell^1 \oplus \ell^1$ has rank at most $2k$. If $\xi = (\xi^1, \xi^2) \in \ell^\infty \oplus \ell^\infty$ and $\|(\xi^1, \xi^2)\|_{\ell^\infty} \leq 1$ then
\[ \| (\mathcal{B}_1 \oplus \mathcal{B}_2 - \mathcal{A}) \xi \|_\infty = \| (\varphi_{1,1} \mathcal{B}_1 \mathcal{B}_2 - \mathcal{A}) \Xi \|_\infty \]

\[ = \max \{ \| \mathcal{B}_1 \mathcal{B}_2 - \mathcal{A} \|_1, \| \mathcal{B}_2 - \mathcal{A} \|_1 \} \]

\[ \leq \max \{ \| \varphi_{1,1} \| \} \]

\[ < \max \alpha_k(\varphi_{1,1}) + \epsilon \]

Thus \( \alpha_{2k}(\mathcal{B}_1 \oplus \mathcal{B}_2) \leq \max_{i=1,2} \alpha_k(\varphi_{1,1}) \) for each \( k \). Since \( \alpha_{2k+1}(\mathcal{B}_1 \oplus \mathcal{B}_2) \)

\[ \leq \alpha_{2k}(\mathcal{B}_1 \oplus \mathcal{B}_2) \lesssim (2k+1)^{P-1} \leq C(p)(2k)^{P-1} \]

it follows that \( \sum_{k=1}^{\infty} k^{P-1}\alpha_{2k}(\mathcal{B}_1 \oplus \mathcal{B}_2) \)) converges if and only if

\[ \sum_{k=0}^{\infty} (2k)^{P-1}\alpha_{2k}(\mathcal{B}_1 \oplus \mathcal{B}_2) \]

\[ \leq 2^{P-1}\max \{ \sum_{i=1,2}^{\infty} (k+1)^{P-1}\alpha_k(\varphi_{1,1}) \} \]

We next show that \( \mathcal{B}_1 \oplus \mathcal{B}_2 \) factors through a diagonal \( \mathcal{B} : \ell^\infty \rightarrow \ell^1 \). If \( (\xi, \eta) \in \ell^\infty \oplus \ell^\infty \) define \( \mathcal{R} : \ell^\infty \oplus \ell^\infty \rightarrow \ell^\infty \) by \( \mathcal{R}(\xi, \eta) = (\gamma_{1,1}) \)

where

\[ \gamma_{1,1} = \begin{cases} 
\xi_{(i+1)/2} & \text{if } i \text{ is odd} \\
\eta_{i/2} & \text{if } i \text{ is even} 
\end{cases} \]

where \( \xi = (\xi_{1,1})_{i=1}^{\infty}, \eta = (\eta_{1,1})_{i=1}^{\infty} \in \ell^\infty \). That is,

\[ \mathcal{R}(\xi, \eta) = (\xi_1, \eta_1, \xi_2, \eta_2, \ldots, \xi_n, \eta_n, \ldots) \in \ell^\infty \]. Then \( \mathcal{R} \) is well-defined, linear, and \( \| \mathcal{R} \| = 1 \). Define \( \mathcal{B} : \ell^\infty \rightarrow \ell^1 \) by \( \mathcal{B} \sim (\delta_{1,1})_{i=1}^{\infty} \)

where

\[ \delta_{1,1} = \begin{cases} 
\lambda_{(i+1)/2} & \text{if } i \text{ is odd} \\
\mu_{i/2} & \text{if } i \text{ is even} 
\end{cases} \]
and $B_1 \sim (\lambda_i)_{i=1}^\infty$, $B_2 \sim (\mu_i)_{i=1}^\infty$. Then $B(\xi_i)_{i=1}^\infty = $ $\langle \lambda_i \xi_i, \mu_i \xi_i \rangle_{i=1}^\infty$. Then $B$ is well-defined, linear, and 

$$
||B|| \leq \sum_{i=1}^{\infty} (|\lambda_i| + |\mu_i|). \quad \text{Let } F: \ell^1 \rightarrow \ell^1 \oplus \ell^1 \text{ be defined by }
$$

$$
F(\xi_i)_{i=1}^\infty = ((\alpha_i)_{i=1}^\infty, (\beta_i)_{i=1}^\infty) \text{ where } \alpha_i = \xi_{2i-1} \text{ and } \beta_i = \xi_{2i} \text{ for each } i.
$$

Then it follows from the definitions that $B_1 \oplus B_2 = F \circ R$.

Letting $\widetilde{U} = R(U_1 \oplus U_2)I$ and $\widetilde{V} = S(V_1 \oplus V_2)F$ we have $T_1 + T_2 = \widetilde{V} \widetilde{U}$, where $\widetilde{U}: \ell^\infty \rightarrow \ell^1$ is a diagonal. We now construct operators $A: \ell^\infty \rightarrow \ell^1 \oplus \ell^\infty$ and $B: \ell^1 \oplus \ell^1 \rightarrow \ell^1$ such that $B = B_1 \oplus B_2 \circ A$. It will then follow that $\sum_{k=1}^{\infty} k^{p-1} \alpha_k-1(B) \leq ||A|| ||B|| \sum_{k=1}^{\infty} k^{p-1} \alpha_k-1(B_1 \oplus B_2)$,

hence $B$ will have the desired property. Define $A: \ell^\infty \rightarrow \ell^1 \oplus \ell^\infty$ by $A(\xi_i)_{i=1}^\infty = (x_1, x_2) \in \ell^1 \oplus \ell^\infty$ where $x_1 = (\xi_{i=1}^1)_{i=1}^\infty$, $x_2 = (\xi_{i=1}^2)_{i=1}^\infty$ and $\xi_1 = \xi_{2i-1}$, $\xi_2 = \xi_{2i}$ for each $i$. That is, $\xi^1 = (\xi_1, \xi_3, \xi_5, \ldots)$ and $\xi^2 = (\xi_2, \xi_4, \xi_6, \ldots)$. Clearly $A$ is well-defined and linear and $||A|| = 1$. Define $B: \ell^1 \oplus \ell^1 \rightarrow \ell^1$ by $B((\xi_i)_{i=1}^\infty (\eta_i)_{i=1}^\infty) = (\gamma_i)_{i=1}^\infty$

where

$$
\gamma_i = \begin{cases} 
\xi_i(i+1)/2 & \text{if } i \text{ is odd} \\
\eta_i/2 & \text{if } i \text{ is even}
\end{cases}
$$

That is $(\gamma_i)_{i=1}^\infty = (\xi_1, \eta_1, \xi_2, \eta_2, \ldots)$. $B$ is well-defined, linear and $||B|| \leq 2$. If $(\xi_i)_{i=1}^\infty \in \ell^\infty$ then

$$
B(B_1 \oplus B_2)A(\xi_i)_{i=1}^\infty = B(B_1 \oplus B_2)(\xi^1, \xi^2) = B(B_1 \xi^1, B_2 \xi^2)
$$
where $\xi^1 = (\xi_1, \xi_3, \xi_5, \ldots)$ and $\xi^2 = (\xi_2, \xi_4, \xi_6, \ldots)$. By the definition of $\mathcal{B}_1$ and $\mathcal{B}_2$ we have
\[ \mathcal{B}_1 \xi^1 = (\lambda_1 \xi_1, \lambda_2 \xi_3, \lambda_3 \xi_5, \ldots) \]
and
\[ \mathcal{B}_2 \xi^2 = (\mu_1 \xi_2, \mu_2 \xi_4, \mu_3 \xi_6, \ldots). \]
But then
\[ \mathcal{B}(\mathcal{B}_1 \xi^1, \mathcal{B}_2 \xi^2) = (\lambda_1 \xi_1, \mu_1 \xi_2, \lambda_2 \xi_3, \mu_2 \xi_4, \ldots) \]
And
\[ = \mathcal{B}(\xi_i^1) \]

**Corollary 3.15** For $p \geq 1$ and $T_i \in \mathcal{F}_p (E, F)$, $i=1,2$, there is a constant $C(p)$, depending only on $p$, such that $\ell_p (T_1 + T_2) \leq C(p) (\ell_p(T_1) + \ell_p(T_2))$. For $p = 1$ we can take $C(1) = 2$.

**Lemma 3.16** If $p \geq 1$ and $\mathcal{B}_p (\ell^\infty, \ell^1)$ denotes the space of all diagonals from $\ell^\infty$ to $\ell^1$ such that $\sigma_p (\mathcal{B}) = \sum_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\mathcal{B})$ is finite then $\mathcal{B}_p (\ell^\infty, \ell^1)$ is complete under the topology generated by $\sigma_p$.

**Proof:**
Suppose $\{\mathcal{B}_n\}_{n=1}^{\infty} \subset \mathcal{B}_p (\ell^\infty, \ell^1)$ is $\sigma_p$-cauchy. Then there is a diagonal $\mathcal{B}: \ell^\infty \to \ell^1$ such that $||\mathcal{B}_n - \mathcal{B}||_n \to 0$ (since $\sigma_p (\mathcal{B}_n - \mathcal{B}_m) \geq ||\mathcal{B}_n - \mathcal{B}_m||_n$ & the space of all diagonals from $\ell^\infty$ to $\ell^1$ is complete under the operator norm). Thus $\alpha_{k} (\mathcal{B}_n - \mathcal{B}) \to 0$ uniformly in $k$. Let $M > 0$ be such that $\sigma_p (\mathcal{B}_n) \leq M$ for all $n$. Then $\sum_{n=1}^{N} n^{p-1} \alpha_{n-1}(\mathcal{B}) = \sum_{n=1}^{N} n^{p-1}$

\[ (\alpha_{n-1}(\mathcal{B}) - \alpha_{n-1}(\mathcal{B}_m)) + \sum_{n=1}^{N} n^{p-1} \alpha_{n-1}(\mathcal{B}_m). \]
If $m$ is chosen so that

\[ k^{p-1} |\alpha_{k-1}(\mathcal{B}) - \alpha_{k-1}(\mathcal{B}_m)| < 1/N \text{ for all } k, 1 \leq k \leq N, \text{ then } \sum_{n=1}^{N} n^{p-1} \alpha_{n-1}(\mathcal{B}) < M+1. \]
That is, $\sigma_p(\beta)$ is finite. That $\sigma_p(\beta_{m,n}) \to 0$ is clear.

**Proposition 3.17** For each $p \geq 1 \mathcal{F}(E,F)$ equipped with the topology generated by $\mathcal{F}_p$ is complete.

**Proof:**

If $(T_n^i)_{n=1}^\infty \subset \mathcal{F}_p(E,F)$ is $\mathcal{F}_p$-Cauchy then there is $T \in \mathcal{L}(E,F)$ such that $\|T-T_n\| \to 0$. Choose a subsequence $(T_n^i)_{i=1}^\infty$ of $(T_n)_{n=1}^\infty$

such that $\mathcal{F}_p(T_{n_i+1}^i - T_n) < 2^{-ni}$, $i=1,2,\ldots$ and let $S_i = T_{n_i+1}^i - T_n^i$

Then $T = T_0 + \sum_{i=1}^{\infty} S_i = \sum_{i=1}^{\infty} S_i$ where $S_0 = T_n^i$. Write $S_i = V_i \beta_i U_i$

with $\|V_i\| = \|U_i\| = 1$ and $\sum_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\beta_i) = \mathcal{F}_p(S_i) + 2^{-ni}$ for each $i$.

Then we can assume $\|V_i\| = \|U_i\| = 2^{-ni}$ for each $i=1$ and $\sum_{n=1}^{\infty} \alpha_{n-1}(\beta_i) < 2^{-ni}$, $i=1,2,\ldots$ and $\|V_0\| = \|U_0\| = 1$ with $\sum_{n=1}^{\infty} n^{p-1} \alpha_{n-1}(\beta_0) < \mathcal{F}_p(S_0) + 2^{-6i}$. Then $\sum_{n=0}^{\infty} \|V_n\|$ and $\sum_{n=0}^{\infty} \|U_n\|$ converge so there are $U \in \mathcal{L}(E,\ell^\infty)$ and $V \in \mathcal{L}(\ell^1,F)$ such that $U = \sum_{n=0}^{\infty} U_n$ and $V = \sum_{n=0}^{\infty} V_n$. Also $\sum_{n=0}^{\infty} \sigma(\beta_n) \quad n=0$ converges, and so by (3.16) there is $\beta = \sum_{n=0}^{\infty} \beta_n$ so that $\beta = \sum_{n=0}^{\infty} \beta_n$.

Now

$$\|V_0 U - \sum_{i=0}^{n} V_i \beta_i U_i\| \leq \|\sum_{i=0}^{\infty} V_i P_i\| \|\sum_{i=0}^{\infty} \beta_i U_i\| \leq \|\sum_{i=0}^{\infty} V_i P_i\| \|\sum_{i=0}^{\infty} \beta_i Q_i\| \|\sum_{i=0}^{\infty} U_i\| \leq \sum_{i=0}^{\infty} \|V_i\| \sum_{i=0}^{\infty} \|\beta_i\| \sum_{i=0}^{\infty} \|U_i\| \to 0$$
where \( P_i : \bigoplus_{i=1}^{\infty} \ell_1 \to \ell_1 \) and \( Q_i : \bigoplus_{i=1}^{\infty} \ell_1 \to \ell_1 \)

are the natural projections (viewing \( \ell^1 \) and \( \ell^\infty \) as \( \bigoplus_{i=1}^{\infty} \ell_1 \) and \( \bigoplus_{i=1}^{\infty} \ell_1 \) respectively). Thus \( V \cup U = \sum_{i=1}^{\infty} s_i = T \).

In [17, § 1] Markus introduced a class of operators between Banach spaces which he called \( F_p(E,F), 0 < p \leq 1 \), and studied the relationship between the spaces \( E^p(E,F), \mathcal{A}^p(E,F), \) and \( F_p(E,F), 0 < p \leq 1 \). The space \( F_p(E,F) \) is defined to be the set of all \( T \in \mathcal{L}(E,F) \)

such that \( T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i \) where \( (f_i)^\infty_{i=1} \subseteq U_E, (y_i)^\infty_{i=1} \subseteq U_F \),

and \( \lambda_i = o(i^{-1/p}) \). \( F_p(E,F) \) is a complete, metrizable, topological vector space given the topology generated by \( F_p(T) = \inf \{ \sup_{n=1}^{\infty} |\lambda_n| : T = \sum_{n=1}^{\infty} \lambda_n f_n \otimes y_n \} \).

We next study the relationship between the spaces \( \mathcal{F}_p(E,F), F_p(E,F), L^p(E,F), \mathcal{D}^p(E,F) \) and \( \mathcal{L}_p^p(E,F) \).

It is clear from the definitions that \( \mathcal{F}_p(E,F) \subseteq F_{1/p}(E,F) \) and \( \mathcal{F}_p(T) \geq F_{1/p}(T) \) for each \( p \geq 1 \). It also follows immediately that for \( 0 < p \leq 1 \), \( F_p(E,F) \subseteq \mathcal{F}_q(E,F) \) for every \( q \), \( 0 < q < (1-p)/p \) and \( \mathcal{F}_q(T) \subseteq K_{p/q}(F_p(T)) \). In fact this is the result possible. To see this consider the diagonal \( T : \ell^1 \to \ell^1, T \sim (\lambda_n)_{n=1}^{\infty} \) where \( \lambda_n = n^{-1/p}[\lambda n(n+1)]^{-1} \) for \( 0 < p \leq 1 \). Then \( n^{1/p} \lambda_n \to 0 \) hence \( T \in F_p(\ell^1, \ell^1) \) but \( n^{(1-p)/p} \lambda_n = n^{-1}[\lambda n(n+1)]^{-1} \) hence \( T \notin \mathcal{F}_q(\ell^1, \ell^1) \), \( q = \frac{1-p}{p} \).

Proposition 3.18 Let \( 0 < q \leq 1 \) and \( T \in L^q(E,F) \). Then \( T \) factors through a diagonal from \( \ell^\infty \) to \( \ell^1 \) of type \( \mathcal{D}^p, 1/p = 1/q - 1 \). In
particular, $T$ is of type $\ell^p$, $1/p = 1/q - 1$ and this value of $p$ is the best possible. If $0 < q \leq 2/3$ then $T \in \mathcal{L}^p_{\mathcal{R}}(E,F), 1/p = 1/q - 3/2$.

Proof:

If $T \in L^q(E,F)$ then $T = \sum_{i=1}^{\infty} \lambda_i f_i \otimes y_i$ where $\|f_i\|, \|y_i\| \leq 1$, $|\lambda_i| \geq |\lambda_{i+1}|$ for all $i$, and $(\lambda_i)_{i=1}^{\infty} \in \ell^q$. Thus $T$ has the factorization

\[
\begin{array}{c}
E \\
\downarrow U \\
\downarrow \ell \\
\downarrow V \\
F
\end{array}
\xrightarrow{T}
\begin{array}{c}
E \\
\downarrow \ell \\
\downarrow \ell \\
\downarrow \ell \\
F
\end{array}
\]

where $U_x = (\langle x, f_i \rangle)^{\infty}_{i=1}$, $\ell \sim (\lambda_i)^{\infty}_{i=1}$, and $V(\xi_i)^{\infty}_{i=1} = \sum_{i=1}^{\infty} \xi_i y_i$.

Now $\mathcal{B} = \mathcal{B}_2 \mathcal{B}_1$ where

\[
\begin{array}{c}
\ell \\
\downarrow \ell \\
\downarrow \ell \\
\downarrow \ell \\
\ell
\end{array}
\xrightarrow{\mathcal{B}_1}
\begin{array}{c}
\ell \\
\downarrow \ell \\
\downarrow \ell \\
\downarrow \ell \\
\ell
\end{array}
\xrightarrow{\mathcal{B}_2}
\begin{array}{c}
\ell \\
\downarrow \ell \\
\downarrow \ell \\
\downarrow \ell \\
\ell
\end{array}
\xrightarrow{\mathcal{B}_3}
\begin{array}{c}
\ell \\
\downarrow \ell \\
\downarrow \ell \\
\downarrow \ell \\
\ell
\end{array}
\]

and $\mathcal{B}_1 \sim (\lambda_i^{\infty})^{\infty}_{i=1}$, $\mathcal{B}_2 \sim (\lambda_i^{1-q})^{\infty}_{i=1}$. By (1.27) $\alpha_k(\mathcal{B}_2) = |\lambda_i|^{1-q}_{i=1}$, hence $\mathcal{B}_2$ is of type $\ell^q/(1-q)$. Since $T = V \mathcal{B}_2 \mathcal{B}_1 U$ we have the desired result.

If $q \leq 2/3$, write $\mathcal{B} = \mathcal{B}_3 \mathcal{B}_2 \mathcal{B}_1$ where

\[
\begin{array}{c}
\ell \\
\downarrow \ell \\
\downarrow \ell \\
\downarrow \ell \\
\ell
\end{array}
\xrightarrow{\mathcal{B}_1}
\begin{array}{c}
\ell \\
\downarrow \ell \\
\downarrow \ell \\
\downarrow \ell \\
\ell
\end{array}
\xrightarrow{\mathcal{B}_2}
\begin{array}{c}
\ell \\
\downarrow \ell \\
\downarrow \ell \\
\downarrow \ell \\
\ell
\end{array}
\xrightarrow{\mathcal{B}_3}
\begin{array}{c}
\ell \\
\downarrow \ell \\
\downarrow \ell \\
\downarrow \ell \\
\ell
\end{array}
\]

\[
\begin{array}{c}
\ell \\
\downarrow \ell \\
\downarrow \ell \\
\downarrow \ell \\
\ell
\end{array}
\xrightarrow{\mathcal{B}_1}
\begin{array}{c}
\ell \\
\downarrow \ell \\
\downarrow \ell \\
\downarrow \ell \\
\ell
\end{array}
\xrightarrow{\mathcal{B}_2}
\begin{array}{c}
\ell \\
\downarrow \ell \\
\downarrow \ell \\
\downarrow \ell \\
\ell
\end{array}
\xrightarrow{\mathcal{B}_3}
\begin{array}{c}
\ell \\
\downarrow \ell \\
\downarrow \ell \\
\downarrow \ell \\
\ell
\end{array}
\]
and $\beta_3 \sim (\lambda_{1}^{q/2})_i^{\infty}$, $\beta_2 \sim (\lambda_{1}^{1-3q/2})_i^{\infty}$, and $\beta_1 \sim (\lambda_{1}^{q})_i^{\infty}$.

By (1.27) $\alpha_k(\beta_2) \leq |\lambda_{k+1}|^{1-3q/2} = |\lambda_{k+1}|^{(2-3q)/2}$, hence $\beta_2$ is of type $k^P$, $1/p = 1/q - 3/2$. Proceeding as in (2.10) gives the desired result.

We now show that the result $T \in L^q(E,F)$ implies that $T \in \mathcal{L}^P(E,F)$, $1/p = 1/q - 1$, is the best possible result. For $0 < q < 1$ we will construct a diagonal $T: E^\infty \rightarrow E^1$ which is strongly $q$-summable but not of type $k^\left[q/(1-q)\right]-\varepsilon$ for any $\varepsilon > 0$. For fixed $q$, $0 < q < 1$, choose $\beta$ such that $\beta q > 1$ (necessarily then $\beta > 1$). Let

$$\beta_n = (n+1)^{(q-1)/q} \cdot \frac{\lambda(n+1)}{[\lambda(n+1)]^q}$$

and define $T: E^\infty \rightarrow E^1$, $T \sim (\lambda_n^{n+1})_n^{\infty}$, where $\lambda_n = \beta_n \cdot n^{1-q}$.

To see that $T \in L^q(E^\infty, E^1)$ observe that

$$|\beta_n - \beta_{n+1}|^q = \left| \frac{f(n+2) - f(n+1)}{f(n+1) f(n+2)} \right|^q$$

where $f(x) = x^{(1-q)/q}$ \ [\ln x]$. By the Mean Value Theorem there exists $z, n+1 \leq z \leq n+2$, such that

$$(*) \quad \left| \frac{f'(z)}{f(n+1) f(n+2)} \right|^q \leq \left| \frac{f'(n+2)}{f(n+1)^2} \right|^q.$$ 

But $f'(x) = \frac{1-q}{q} x^{(1-2q)/q} \cdot (\ln x)^{\beta} + \frac{\beta q}{1-q} (\ln x)^{\beta-1}$

$$\leq C(q) x^{(1-2q)/q} (\ln x)^{\beta}$$

and therefore

$$\left| \frac{f'(n+2)}{f(n+1)^2} \right|^q \leq \left[ \frac{C(q) (n+2)^{(1-2q)/q} (\ln(n+2))^\beta}{(n+1)^{(2-2q)/q} \cdot [\lambda(n+1)]^{2\beta}} \right]^q.$$
\[
\begin{align*}
K(q) & = K(q) (n+1)^{-1/q} [\&n(n+1)]^{-\beta q} \\
& = K(q) (n+1)^{-1/q} [\&n(n+1)]^{-\beta q} .
\end{align*}
\]

Since \( \beta q > 1 \) and \( 1/q > 1 \), \( \sum_{n=1}^{\infty} (n+1)^{-1/q} [\&n(n+1)]^{-\beta q} \) converges, hence T is strongly \( q \)-summable.

By (1.27) \( \alpha_n(T) = \beta_{n+1} \) and so for \( \epsilon > 0 \) we have

\[
\sum_{n=0}^{\infty} \alpha_n(T) [q/(1-q)]^{-\epsilon} = \sum_{n=1}^{\infty} \beta_n [q/(1-q)]^{-\epsilon} .
\]

Now

\[
\beta_n [q/(1-q)]^{-\epsilon} = \frac{1}{(n+1)^{1-\epsilon(1-q)/q} [\&n(n+1)]^{-\beta q/(1-q)-\epsilon}} .
\]

If \( \epsilon \geq q/(1-q) \) then \( 1-\epsilon(1-q)/q < 0 \); hence, \( \sum_{n=1}^{\infty} \beta_n [q/(1-q)]^{-\epsilon} \) diverges. If \( \epsilon < q/(1-q) \) then \( \beta_n [q/(1-q)]^{-\epsilon} \) is of the form

\[
(n+1)^{\alpha(\&n(n+1))^{-\delta}} \text{ where } 0 < \alpha < 1, \text{ and } \delta > 0 , \text{ hence } \sum_{n=1}^{\infty} \beta_n [q/(1-q)]^{-\epsilon}
\]

diverges.

**Remark:** We point out that in [17] Markus proved that \( \ell^p(E,F) \subset \ell^{p/(1-p)}(E,F) \) (using completely different techniques) and remarked that \( p/(1-p) \) was the best result possible.

The following proposition is no doubt well known, but we include a proof for completeness.

**Proposition 3.19** If \( (\lambda_n) \) is any sequence of positive scalars such that \( \sum_{n=1}^{\infty} n^p \lambda_n \) converges for \( p \geq 1 \), then \( (\lambda_n) \) for every \( \epsilon > 0 \) and this is the best possible result. If \( (\lambda_n) \) for \( p \geq 1 \) then \( \sum_{n=1}^{\infty} n^p \lambda_n \) converges and this is the best possible result.
Proof:

First suppose that $\sum_{n=1}^{\infty} n^{p}\lambda_{n}$ converges for some $p \geq 1$. Write $\lambda_{n} = n^{p}\lambda_{n}^{-p}$. Then for $\epsilon > 0$ we have $\lambda_{n} = (1+\epsilon)/(p+1)$.

$[n^{p}\lambda_{n}]^{(1+\epsilon)/(p+1)-p(1+\epsilon)/(p+1)}$. Let $q = (p+1)/(1+\epsilon)$ and $q' = (p+1)/(p-\epsilon)$. Since $p(1+\epsilon)/(p-\epsilon) > 1$ the sequence $(n^{-p(1+\epsilon)/(p+1)})^{\infty}_{n=1}$ converges since $1/q + 1/q' = 1$. But then $\sum_{n=1}^{\infty} (\lambda_{n}^{(1+\epsilon)/(p+1)})$ converges since $1/q + 1/q' = 1$.

To see that this is the best possible result let $\lambda_{n} = n^{-(p+1)}(\ln(n+1))^{-(p+1)}$. Then $\sum_{n=1}^{\infty} n^{p}\lambda_{n} = \sum_{n=1}^{\infty} n^{-1}(\ln(n+1))^{-1}$ which converges; however, $\sum_{n=1}^{\infty} \lambda_{n}^{1/(p+1)} = \sum_{n=1}^{\infty} n^{-1}(\ln(n+1))^{-1}$ which is divergent.

Now suppose that $p \geq 1$ and $(\lambda_{n}^{n})^{\infty}_{n=1} \epsilon^{n}1/(p+1)$. We can assume that $\lambda_{n} \geq \lambda_{n+1}$ for all $n$ by a suitable permutation of the indices if necessary. Then $(n^{p}\lambda_{n} p/(p+1))^{\infty}_{n=1}$ is bounded. Now

$\sum_{n=1}^{\infty} n^{p}\lambda_{n} = \sum_{n=1}^{\infty} (n^{p}\lambda_{n} p/(p+1))^{1/(p+1)} \lambda_{n}^{1/(p+1)}$, hence $\sum_{n=1}^{\infty} n^{p}\lambda_{n}$ converges.

To see that this is the best result possible suppose that $\sum_{n=1}^{\infty} n^{p+\delta} \lambda_{n}$ converges for some $\delta > 0$. Then by the first part of this proposition it follows that $(\lambda_{n}^{n})^{\infty}_{n=1} \epsilon^{n}(1+\epsilon)/(p+\delta+1)$ for every $\epsilon > 0$.

Choose $\epsilon > 0$ such that $\epsilon < \delta/(p+1)$. Then $(1+\epsilon)/(p+\delta+1) < 1/(p+1)$.

We obtain a contradiction by choosing $(\lambda_{n}^{n})^{\infty}_{n=1} \epsilon^{n}1/(p+1)$ with $(\lambda_{n}^{n})^{\infty}_{n=1} \epsilon^{n}(1+\epsilon)/(p+\delta+1)$.


Proposition 3.20 If $p > 1$ and $T \in \mathcal{F}_p^1(E,F)$ then $T$ factors through a diagonal $\mathcal{D}: \ell^1 \to \ell^2$ of type $\ell^1/(p-1)$. In particular, $T \in \mathcal{F}_R^{1/(p-1)}(E,F)$. Moreover $T$ factors through a diagonal $\mathcal{D}_1: \ell^1 \to \ell^1$ of type $\ell^2/(2p-1)$. In particular, $T \in \ell^2/(2p-1)(E,F)$.

Proof:

If $T \in \mathcal{F}_p^1(E,F)$ then by (3.12) $T = \sum_{n=1}^{\infty} \lambda_n f_n \otimes y_n$ where $(f_n)_{n=1}^{\infty} \subseteq U_E$, $(y_n)_{n=1}^{\infty} \subseteq U_F$, and $\sum_{n=1}^{\infty} n^p |\lambda_n|$ converges. In particular, $(\lambda_n)_{n=1}^{\infty} \in \ell^2/(2p+1)$ by (3.19). Therefore we can factor $T$

$\begin{array}{cc}
U & \downarrow T \\
\mathcal{F} & \downarrow \cap \\
E & \downarrow F \\
\ell & \downarrow \ell
\end{array}$

where $U$, $V$, $\mathcal{F}$ are as in (3.18). Now we can write $\mathcal{F} = \mathcal{D}_2 \mathcal{D}_3$ where

$\begin{array}{ccc}
\mathcal{D}_3 & \cap & \mathcal{D}_2 \\
\ell^1 & \downarrow & \ell^2 \\
\ell & \downarrow & \ell
\end{array}$

with $\mathcal{D}_3 \sim (\lambda_n^{2/(2p+1)})_{n=1}^{\infty}$, $\mathcal{D} \sim ((\lambda_n^{2(p-1)/(2p+1)})_{n=1}^{\infty}$, and $\mathcal{D}_2 \sim (\lambda_n^{1/(2p+1)})_{n=1}^{\infty}$. Now $\alpha_k(\mathcal{D}) \leq |\lambda_{k+1}^{2(p-1)/(2p+1)}|$, hence $\mathcal{D} \in \ell^1/(p-1)(\ell^1, \ell^2)$ and it follows that $T \in \mathcal{F}_R^{1/(p-1)}(E,F)$. If we let $\mathcal{D}_1 = \mathcal{D}_2 \mathcal{D}_3$ then $\mathcal{D} \in \ell^1/(p-1)(\ell^1, \ell^2)$ and it follows that $T \in \mathcal{F}_R^{1/(p-1)}(E,F)$. If we let $\mathcal{D}_1 = \mathcal{D}_2 \mathcal{D}_3$ then $\mathcal{D}_1 \sim (\lambda_n^{(2p-1)/(2p+1)})_{n=1}^{\infty}$. Since $\alpha_k(\mathcal{D}_1) \leq |\lambda_{k+1}^{(2p-1)/(2p+1)}|$, we have the desired result.

Proposition 3.21 If $T \in \mathcal{F}_1^1(E,F)$ then $T \in \mathcal{F}_R^2(E,F)$. Moreover $T$ factors through a diagonal from $\ell^\infty$ to $\ell^2$ of type $\ell^2$. 
Proof:

For $T \in \mathcal{J}(E,F)$ choose a representation $T = \sum_{n=1}^{\infty} \lambda_n f_n \otimes y_n$

with $(f_n)_{n=1}^{\infty} \subset U_E$, $(y_n)_{n=1}^{\infty} \subset U_F$, and $\sum_{n=1}^{\infty} n|\lambda_n|$ finite. Then

$T = VU$ where

\[
\begin{array}{ccc}
E & \xrightarrow{T} & F \\
\downarrow U & & \downarrow V \\
\beta & \xleftarrow{\beta} & \gamma
\end{array}
\]

with $Ux = (\langle x, f_n \rangle)_{n=1}^{\infty}$, $\beta \sim (\lambda_n^{2/3})_{n=1}^{\infty}$, and $V(\xi_n)_{n=1}^{\infty} = \sum_{n=1}^{\infty} \lambda_n^{1/3} \xi_n y_n$.

$\beta$ and $V$ are well-defined by (3.19). Now $\beta = BA$ where

\[
\begin{array}{ccc}
& \xrightarrow{\beta} & \\
A & \xleftarrow{\beta} & B
\end{array}
\]

with $A \sim (\lambda_n^{1/3})_{n=1}^{\infty}$, $B \sim (\lambda_n^{1/3})_{n=1}^{\infty}$. $B$ is of type $\ell^2$ by (1.28) and (3.19), hence $\beta \in \ell^2(\ell^1, \ell^2)$.

Proposition 3.22 Let $p \geq 1$ and $T \in \mathcal{L}^q(E,F)$, $q = 1/(p+1)$. Then $T \in \mathcal{J}_p(E,F)$. In particular, if $T \in \mathcal{J}_q(E,F)$ then $T$ is $p$-factorable and these are the best results possible.

Proposition 3.22 is immediate from (3.19).
**Proposition 3.23** Let $p \geq 1$. If $T \in c_c(E,F)$ has a representation
\[ T = \sum_{n=1}^{\infty} \lambda_n f_n \otimes y_n \in E \otimes F, \quad (f_n)_{n=1}^{\infty} \subseteq U_E', \quad (y_n)_{n=1}^{\infty} \subseteq U_F, \quad \text{and } \sum_{n=1}^{\infty} n^p|\lambda_n| \]
converges then $\sum_{n=1}^{\infty} n^{p-1} \alpha_n^{-1}(T)$ is finite and this is the best result possible.

**Proof:**
Clearly $\alpha_n(T) \leq \sum_{i=n+1}^{\infty} |\lambda_i|$ hence $\sum_{n=1}^{\infty} n^{p-1} \alpha_n^{-1}(T) \leq \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} |\lambda_i| \leq \sum_{n=1}^{\infty} n^p|\lambda_n|$. To see that $p-1$ is the best result let $p > 1$ and let
\[ \beta_n = n^{-p} \ln^2(n+1). \]
Let $\alpha_n = \beta_n^{-1}$ and define $T: \ell^p \rightarrow \ell^1$ by
\[ T \sim (\alpha_n)_{n=1}^{\infty}. \]
If $f(x) = x^p \ln^2 x$ then
\[ |\lambda_n| = |\beta_n^{-1} \alpha_{n+1}| = \left| \frac{f(n) - f(n+1)}{f(n) f(n+1)} \right| \leq \left| \frac{f'(z)}{f(z)^2} \right| \]
where $n \leq z \leq n+1$; $f'(x) = px^{p-1}(\ln^2 x + (2/p) \ln x)$ hence
\[ |f'(x)| \leq K(p)|x^{p-1} \ln^2 x| \]
for large $x$. Then
\[ \left| \frac{f'(z)}{f(z)^2} \right| \leq \frac{K(p)(n+1)^{p-1} \ln^2(n+1)}{n^{2p} \ln n} \]
\[ \leq C(p) \frac{n^{-p} \ln^2 n}{n^{2p} \ln n} \quad \text{for large } n \]
\[ = C(p) n^{-(p+1)} \ln^{-2} n, \quad n \text{ large,} \]
and it follows that $\sum_{n=1}^{\infty} n^p|\lambda_n|$ converges. Now $\alpha_n(T) = \beta_{n+1}$ by
\[ (1.27) \text{ and so } \sum_{n=1}^{\infty} n^{(p-1)+\varepsilon} \alpha_n^{-1}(T) = \sum_{n=1}^{\infty} n^{(p-1)+\varepsilon} \beta_n = \sum_{n=1}^{\infty} \frac{n^{(p-1)+\varepsilon}}{(n+1)^{p} \ln^2(n+1)}. \]
But \(\sum_{n=1}^{\infty} n^{-(1-\varepsilon)} n^{-2(n+1)}\) diverges. Thus, for any \(\varepsilon > 0\),\n
\[\sum_{n=1}^{\infty} n^{(p-1)+\varepsilon} \alpha_{n-1}(T)\] diverges.

We now prove a partial converse to (3.23).

**Proposition 3.24** Let \(p \geq 1\) and \(T \in \mathcal{E}(E,F)\). If \(\sum_{n=1}^{\infty} n^p \alpha_{n-1}(T)\) converges then \(T\) has a representation \(T = \sum_{n=1}^{\infty} \lambda_n f_n \otimes y_n\) where \(\|f_n\|, \|y_n\| \leq 1\) and \(\sum_{n=1}^{\infty} q^n |\lambda_n|\) converges for every \(q, 0 < q < p\).

**Proof:**

Since \(\sum_{n=1}^{\infty} n^p \alpha_{n-1}(T)\) converges, \((\alpha_n(T))_{n=0}^{\infty} \in \mathcal{E}^r\) for every \(r > 1/(p+1)\) by (3.19). If \(1/(1+p) < r \leq 1\) then it follows from (0.3) that \(T = \sum_{n=1}^{\infty} \lambda_n f_n \otimes y_n\) where \(\|f_n\| = \|y_n\| = 1\), \(|\lambda_n| \geq |\lambda_{n+1}|\) for all \(n\) and \(\sum_{n=1}^{\infty} |\lambda_n|^r\) is finite. In particular, \((n^{1/r} |\lambda_n|)_{n=1}^{\infty}\) is bounded, hence \(\sum_{n=1}^{\infty} n^{1/r} |\lambda_n| n^{-(1+\varepsilon)}\) converges for every \(\varepsilon > 0\). But \(n^{1/r-(1+\varepsilon)} = n^{p-\delta}\) for \(\delta = \delta(\varepsilon, r)\) arbitrarily small. Thus \(\sum_{n=1}^{\infty} n^{p-\delta} |\lambda_n|\) converges for every \(\delta > 0\).

If \(T \in \mathcal{F}_p(E,F)\) and \(S_1 \in \mathcal{E}(G,E), S_2 \in \mathcal{E}(F,X)\) then \(TS_1 \in \mathcal{F}_p(G,F)\) and \(S_2T \in \mathcal{F}_p(F,X)\). If \(T \in \mathcal{F}_p(E,F)\) and \(S \in \mathcal{F}_q(F,G)\) then we have the composition formula given by the following proposition.

**Proposition 3.25** If \(p, q \geq 1\), \(T\) is \(p\)-factorable and \(S\) is \(q\)-factorable then \(ST\) is \(r\)-factorable for every \(r, 1 \leq r < p+q-1\).
Proof:

We can write \( T = \sum_{n=1}^{\infty} \lambda_n f_n \otimes y_n \) with \( \|f_n\| = \|y_n\| = 1 \) and 
\[ \sum_{n=1}^{\infty} n^p |\lambda_n| \text{ finite.} \]

\( S \) also has a representation \( S = \sum_{n=1}^{\infty} \mu_n g_n \otimes x_n \)

where \( \|g_n\| = \|x_n\| = 1 \) and \( \sum_{n=1}^{\infty} n^q |\mu_n| \) converges. Since
\[ \alpha_n(S) \leq \sum_{i=n+1}^{\infty} |\mu_i| \]

we have
\[ n^q \alpha_n(S) \leq n^q \sum_{i=n}^{\infty} |\mu_i| \leq n^q \sum_{i=n}^{\infty} |\mu_i|. \]

Thus \( n^q \alpha_n(S) \to 0 \). It follows from (3.23) that \( \sum_{n=1}^{\infty} n^{p-1} \alpha_n-1(T) \)
converges, but then
\[ \sum_{n=1}^{\infty} n^{p+q-1} \alpha_n-1(ST) \leq \sum_{n=1}^{\infty} (2n)^{p+q-1} \alpha_n(T) + \sum_{n=1}^{\infty} (2n+1)^{p+q-1} \alpha_{2n+1}(T) \]

\[ \leq 2^{p+q-1} \sum_{n=1}^{\infty} [n^{p-1} \alpha_n-1(T)] [n^q \alpha_n-1(S)] \]

\[ + C(p,q) \sum_{n=1}^{\infty} (2n)^{p+q-1} \alpha_{2n}(ST) \]

\[ \leq K(p,q) \sum_{n=1}^{\infty} [n^{p-1} \alpha_n-1(T)] [n^q \alpha_n-1(S)]. \]

Hence, \( \sum_{n=1}^{\infty} n^{p+q-1} \alpha_n-1(ST) \) converges. By (3.35) \( ST \) has a representation
\[ ST = \sum_{n=1}^{\infty} \beta_n h_n \otimes z_n \]

where \( \|h_n\| = \|z_n\| = 1 \) and \( \sum_{n=1}^{\infty} n^r |\beta_n| \) converges for every \( r, 1 \leq r < p+q-1 \). The proposition now follows from (3.12).

We now give a summary of this section:

Theorem 3.26

1) If \( 0 < p < 1 \) then \( L^p(E,F) \subset \ell^p/(1-p)(E,F) \) and this is
the best result possible.

ii) For $0 < p \leq 1$, $\mathcal{L}^p(E,F) \subseteq L^{(p)}(E,F)$

iii) If $p \geq 1$ then $L^{(q)}(E,F) \subseteq \mathfrak{F}_p(E,F)$, $q = 1/(p+1)$.

In particular, $\mathcal{L}^q(E,F) \subseteq \mathfrak{F}_p(E,F)$.

iv) If $p \geq 1$ then $\mathfrak{F}_p(E,F) \subseteq L^{(r)}(E,F)$ for every $r > 1/(p+1)$

and this is the best result possible.

v) If $p > 1$ then $\mathfrak{F}_p(E,F) \subseteq \mathcal{L}^{2/(2p-1)}(E,F)$ and $\mathfrak{F}_p(E,F) \subseteq \mathcal{L}^{1/(p-1)}(E,F)$. If $p = 1$, then $\mathfrak{F}_1(E,F) \subseteq \mathcal{L}^1(E,F) \cap \mathcal{L}^2(E,F)$

vi) If $p \geq 1$ then $\mathfrak{F}_p(E,F) \subseteq F_{1/p}(E,F)$

vii) If $0 < p < 1/2$ then $F_p(E,F) \subseteq \mathfrak{F}_q(E,F)$

for every $q$, $1 \leq q < (1-p)/p$ and this is the best result possible.

viii) If $0 < q \leq 1/2$ then $L^{(q)}(E,F) \subseteq F_r(E,F)$, $r = q/(1-q)$.

In particular, $\mathcal{L}^q(E,F) \subseteq F_r(E,F)$.

ix) If $0 \leq p \leq 1/2$ then $F_p(E,F) \subseteq L^{(r)}(E,F)$ for every $r$,

$1 \geq r > \frac{p}{1-p}$.

x) If $0 < q < 1/2$ then $F_q(E,F) \subseteq \mathcal{L}^r(E,F)$ for every $r$,

$r > 2p/2-3p$. 
CHAPTER IV
APPLICATION TO INTERPOLATION THEORY

In this chapter we give an application of the calculations obtained in Chapter I to interpolation between spaces of diagonal operators of type $\ell^p$ on the $\ell^q$-spaces. The results obtained in this section extend the results of V.R. Oloff [21] on interpolation between the spaces $\ell^p(\mathcal{H},\mathcal{K})$, $0<p<\infty$, $\mathcal{K}$ a separable Hilbert space, to the spaces $D_r(\ell^p,\ell^q)$, $1 \leq p,q \leq \infty$, $0 < r \leq \infty$.

Let $E$ and $F$ be a pair of Banach spaces which are continuously embedded in a topological vector space $X$. An interpolation method is a construction technique which assigns an intermediate space to the pair $E,F$ which is also continuously embedded in $X$ [1], [22]. The concept of interpolation has been extended to quasi-normed spaces by Kree [13] and Peetre [23].

By a quasi-norm on a vector space $X$ we mean a function $\rho$ from $X$ to the positive reals satisfying:

i) $\rho(x) = 0$ if and only if $x = 0$

ii) $\rho(\lambda x) = |\lambda| \rho(x)$ for all scalars $\lambda$ and $x \in X$

iii) there is a constant $K > 0$ such that $\rho(x+y) \leq K(\rho(x)+\rho(y))$

for all $x, y \in X$. In particular, the function $\rho_p$ is a quasi-norm on the space $\ell^p(E,F)$, $0 < p \leq \infty$.

If $E$ and $F$ are two quasi-normed spaces, with quasi-norms $\| \cdot \|_E$, $\| \cdot \|_F$, respectively, which are embedded in a topological vector space $X$, then for each $u \in E+F$ and each scalar $t > 0$ define a function
\[ K(t,u,E,F) = \inf \{ \|x\|_E + t\|y\|_F : u = x+y \in E+F \} \]

The interpolation space \((E,F)_{\theta,r;K}\) (K-method) [1] is defined for each \(\theta \in (0,1)\) by

\[ (E,F)_{\theta,r;K} = \{ u \in E+F : \int_0^\infty [t^{-\theta}K(t,u;E,F)]^r \frac{dt}{t} < +\infty \} \]

for \(0 < r < \infty\) and

\[ (E,F)_{\theta,\infty;K} = \{ u \in E+F : \text{Var max } K(t,u;E,F) < +\infty \} \]

with quasi-norm

\[ \|u\|_{\theta,r;K} = \begin{cases} \left(\int_0^\infty [t^{-\theta}K(t,u;E,F)]^r \frac{dt}{t}\right)^{1/r} & 0 < r < \infty \\ \text{Var max } K(t,u;E,F) & r = \infty \end{cases} \]

Using the L-method [23] replace the function \(K(t,u;E,F)\) by the function

\[ L_{p,q}(T,u;E,F) = \inf \{ \|x\|_E^p + t\|y\|_F^q : u = x+y \in E+F \} \]

where \(p, q > 0\). The space \((E,F)_{\theta,r;L_{p,q}}\) is defined as for the K-method and has quasi-norm \(\|u\|_{\theta,r;L_{p,q}}\).

By \(D_r(\ell^p,\ell^q), 1 \leq p, q \leq \infty, 0 < r \leq \infty\), we will mean the complete, metrizable, topological vector space of all diagonal operators of type \(\ell^r\) from \(\ell^p\) to \(\ell^q\) given the topology generated by \(\rho_r\). We will write \(D_r\) for \(D_r(\ell^p,\ell^q)\).

Using techniques developed by Oloff [21] and Triebel [33] we will show that \((D_p,D_{\infty})_{\theta,r;K} = D_r\) for certain \(r\) and \(\theta\) and
(D_p, D_q) = D_r for certain \( \eta \) and \( r \). First we will consider

\((D_p, D_\infty) = \theta, r; K\)

**Definition 4.1** For each \( T \in \mathcal{E}^p(E,F) \), \( 0 < p \leq \infty \), define a function \( \alpha_T \) on the positive reals by \( \alpha_T(\tau) = \alpha_T(T) \) where \( \alpha_T(T) = \alpha_{[\tau]}(T) \) [21].

**Lemma 4.2** For \( T \in \mathcal{D}_\infty(\mathcal{E}^p, \mathcal{E}^q) \), \( 1 \leq p, q \leq \infty \), and \( n > 0 \) the functions

\[ K(t, T; D_n, D_\infty) \quad \text{and} \quad \left( \int_0^t \alpha_T(\tau)^n d\tau \right)^{1/n} \]

are equivalent.

**Proof:**

If \( T \in \mathcal{D}_\infty(\mathcal{E}^p, \mathcal{E}^q) \) say \( T \sim (\lambda_i)_{i=1}^\infty \) then \( T \) can be written in the form \( T = T_n + T_\infty \), \( T_n \in D_n \) and \( T_\infty \in D_\infty \). Indeed, take \( T_n = \sum a_i e_i \otimes e_i \)

and \( T_\infty = \sum b_i e_i \otimes e_i \) where

\[
a_i = \begin{cases} 
\lambda_i - \lambda_{[t^n]+1} & \text{if } i \leq [t^n] \\
0 & \text{if } i > [t^n]
\end{cases}
\]

and

\[
b_i = \begin{cases} 
\lambda_{[t^n]+1} & \text{if } i \leq [t^n] \\
\lambda_i & \text{if } i > [t^n]
\end{cases}
\]

If \( T = T_n + T_\infty \) is an arbitrary representation of \( T \) then

\( \alpha_k(T) \leq \alpha_k(T_n) + \|T_\infty\| \) for each \( k \). Also, if \( a, b \geq 0 \) then
\[(a+b)^m \leq 2^{m-1}(a^m+b^m)\] for \(m \geq 1\) hence

\[
\int_0^{t^n} \alpha_T(\tau)^n d\tau = \sum_{i=1}^{[t^n]} \int_{i-1}^{i} \alpha_T(\tau)^{n} d\tau + \int_{i-[t^n]}^{t^n} \alpha_T(\tau)^{n} d\tau
\]

\[
= \sum_{i=1}^{[t^n]} \alpha_{i-1}(T)^n + (t^n-[t^n])\alpha_{[t^n]}(T)^n
\]

\[
\leq \max(1,2^n-1) \left\{ \sum_{i=1}^{[t^n]} \alpha_{i-1}(T)^n + [t^n]||T_\infty||^n\right\}
\]

\[
+ (t^n-[t^n])\alpha_{[t^n]}(T)^n + ||T_\infty||^n\right\}
\]

\[
\leq \max(1,2^n-1) \left\{ \sum_{i=1}^{\infty} \alpha_{i-1}(T)^n + t^n||T_\infty||^n\right\}.
\]

Thus

\[
\left( \int_0^{t^n} \alpha_T(\tau)^n d\tau \right)^{1/n} \leq 2^{n-1/n} \inf \{ \rho_T(T) + t \rho_\infty(T) : T = T_n + T_\infty \in D_n + D_\infty \}
\]

Let \(T \in D_\infty(\ell^p, \ell^q)\) with \(T \sim (\lambda_i)_i\). By a permutation of the indices and, if necessary, by replacing \(\xi_i\) by \(\xi_i\), \(\xi = (\xi_i)_i\) \(\in \ell^p\), we can assume that \(\lambda_i \geq \lambda_{i+1} \geq 0\) for all \(i\). Define diagonals \(T_n, T_\infty : \ell^p \rightarrow \ell^q\) by \(T_n \sim (a_i)_i\) and \(T_\infty \sim (b_i)_i\) where

\[
a_i = \begin{cases} 
\lambda_i - \lambda_{[t^n]} & \text{if } i \leq [t^n] \\
0 & \text{if } i > [t^n]
\end{cases}
\]
and

\[ b_i = \begin{cases} 
\lambda_i^{2^{-i/r}} & \text{if } i \leq [t^n] \\
\lambda_i & \text{if } i > [t^n] 
\end{cases} \]

where \( r = 1 \) if \( p \leq q \) and \( 1/r = 1/q - 1/p \) if \( p > q \). Then \( T = T_n + T_\infty \).

To compute \( \rho_n(T_n) = \left( \sum_{k=0}^{\infty} \alpha_k(T_n)^n \right)^{1/n} \) we must consider three cases:

**case (i):** \( p = q \)

Then \( \rho_n(T_n)^n = \sum_{k=0}^{[t^n]-1} |\lambda_{k+1} - \lambda|^{2^{-n}(k+1)} \)

\[ \leq \sum_{k=0}^{[t^n]-1} \lambda_{k+1} = \sum_{k=0}^{[t^n]-1} \alpha_k(T)^n \]

\[ \leq \int_0^{t^n} \alpha_T(\tau)^n d\tau. \]

**case (ii):** \( p > q \)

Then \( \rho_n(T_n)^n = \sum_{k=0}^{[t^n]-2} \left( \sum_{i=k+1}^{[t^n]-1} |\lambda_{i+1} - \lambda_i|^{2^{-i/r}} \right)^{n/r} \)

\[ \leq \sum_{k=0}^{[t^n]-2} \sum_{i=k+1}^{[t^n]-1} \lambda_i^{n/r} \]

\[ \leq \sum_{k=0}^{[t^n]-2} \left( \sum_{i=k+1}^{[t^n]-1} \lambda_i^r \right)^{n/r} = \sum_{k=0}^{[t^n]-2} \alpha_k(T)^n \]

\[ \leq \int_0^{t^n} \alpha_T(\tau)^n d\tau. \]
case (iii): $p < q$

Then $\rho_n(T_n)^n \leq \sum_{k=0}^{[t^n]} |\lambda_{k+1} - \lambda| \cdot 2^{-(k+1)|n}$

$$\leq 2^n \sum_{k=0}^{[t^n]-1} \alpha_k(T_n)^n \leq 2^n \int_0^{t^n} \alpha_T(\tau)^n d\tau.$$ 

Thus in all cases, $\rho_n(T) \leq 2 \left( \int_0^{t^n} \alpha_T(\tau)^n d\tau \right)^{1/n}$.

Now $t^n \rho_\omega(T_\omega)^n = t^n ||T_\omega||^n$ and if

case (i): $p = q$

$$i^n ||T_\omega||^n \leq t^n \lambda^n \leq \int_0^{t^n} \alpha_T(\tau)^n d\tau.$$ 

case (ii): $p > q$

$$t^n ||T_\omega||^n \leq t^n \left( \sum_{i=1}^{[t^n]+1} \lambda_i^n \right) + \int_0^{t^n} \alpha_T(\tau)^n d\tau.$$ 

$$\leq \left( t \lambda^n \right) \left[ \lambda^{n+1} \right] \leq \left( \int_0^{t^n} \alpha_T(\tau)^n d\tau \right)^{1/n} + \left( \int_0^{t^n} \alpha_T(\tau)^n d\tau \right)^{1/n}.$$ 

$$\leq \max\{1,2^n\} \int_0^{t^n} \alpha_T(\tau)^n d\tau.$$ 

case (iii) $p < q$

$$t^n ||T_\omega|| \leq t^n \lambda^n \leq \int_0^{t^n} \alpha_T(\tau)^n d\tau.$$ 

Thus in any case, $t\rho_\omega(T_\omega) \leq 2 \left( \int_0^{t^n} \alpha_T(\tau)^n d\tau \right)^{1/n}$.
hence
\[ \inf\{p_n(T_n) + t\rho_\infty(T_\infty): T = T_n + T_\infty \in D_n + D_\infty\} \leq 4\left(\int_0^{t^n} T_\infty(\tau)d\tau\right)^{1/n}.\]

The next lemma is due to Hardy, Littlewood, and Polya [7].

**Lemma 4.3** If \( f \) is a positive, measurable function on \([0, \infty)\) and if \( \lambda > 1 \) then
\[ \int_{0}^{\infty} \left( \frac{1}{x} \int_{0}^{x} f(t)dt \right)^{\lambda} dx \leq \frac{\lambda}{(\lambda - 1)} \int_{0}^{\infty} f(x)^{\lambda} dx. \]

**Theorem 4.4** For \( p > 0 \) and \( \theta \in (0,1) \), \( 1/\lambda = (1-\theta)/p \) \((D_p, D_\infty) = \theta, r; K\) and their quasi-norms are equivalent.

**Proof:**

Let \( T \in D_r \). By applying (4.3) one obtains from (4.2)
\[
\left( \int_{0}^{\infty} \left[ t^{-\theta} K(t, T; D_p, D_\infty) \right]^{\lambda} \frac{dt}{t} \right)^{1/\lambda} \leq c^{-1} \left( \int_{0}^{\infty} t^{p-\lambda} \left[ \int_{0}^{t} \alpha_T(\tau) d\tau \right]^{p} \frac{dt}{t} \right)^{1/\lambda} \\
= c^{-1} \left( \int_{0}^{\infty} \frac{1-p}{p} \left[ \int_{0}^{x} \alpha_T(\tau) d\tau \right]^{p \lambda} \frac{dx}{p} \right)^{1/\lambda} \\
= c^{-1} \left( \frac{1}{p} \left[ \int_{0}^{\infty} \alpha_T(\tau) d\tau \right]^{p} \right)^{1/\lambda} \\
\leq c^{-1} \left( \frac{1}{p} \left[ \int_{0}^{\infty} \alpha_T(x) dx \right] \right)^{1/\lambda} \\
= c^{-1} \left( \frac{1}{p} \left[ \sum_{i=0}^{\infty} \alpha_i(T)^{1/p} \right] \right)^{1/\lambda} < + \infty
\]

Thus \( T \in (D_p, D_\infty) = \theta, r; K \). Now let \( T \in (D_p, D_\infty) = \theta, r; K \).
Then
\[ + \infty > \left( \int_{0}^{\infty} \left[ t^{-\theta} K(t,T;D_{p},D_{\infty}) \right]^{r} \frac{dt}{t} \right)^{1/r} \]
\[ \geq A_{p}^{-1} \left( \int_{0}^{\infty} t^{p-r} \left[ \int_{0}^{t} \alpha_{T}(\tau)^{p} d\tau \right]^{r/p} \frac{dt}{t} \right)^{1/r} \]
\[ \geq A_{p}^{-1} \left( \int_{0}^{\infty} t^{p-r} \alpha_{T}(t^{p})^{r/p} \frac{dt}{t} \right)^{1/r} \]
\[ = A_{p}^{-1} \left( \int_{0}^{\infty} t^{p-1} \alpha_{T}(t^{p})^{r} dt \right)^{1/r} \]
\[ = A_{p}^{-1} \left( \int_{0}^{\infty} \alpha_{T}(x) \frac{dx}{px} \right)^{1/r} \]
\[ = A_{p}^{-1} \left( \int_{0}^{\infty} \alpha_{T}(x) dx \right)^{1/r} \]
\[ = A_{p}^{-1} \left( \sum_{i=0}^{\infty} \alpha_{i}(T)^{r} \right)^{1/r} \]

hence \( T \in D_{r} \).

We next prove some results concerning interpolation between \( D_{p} \) and \( D_{r} \), but first we state the following lemma due to Oloff [21].

**Lemma 4.5** For positive numbers \( a, p, q, t \) the functions \( f(t) = \inf\{b^{p} + tc^{q}: a=b+c; b,c \geq 0\} \) and \( g(t) = \min\{a^{p}, t^{a^{q}}\} \) are equivalent for \( 0 < p, q \leq 1 \).

**Lemma 4.6** For \( T \in D_{\max(p,q)}(L^{r}, L^{s}) \) with \( 0 < p, q < \infty \) and \( 0 < r, s \leq \infty \) the functions \( \min\{\alpha_{i}(T)^{p}, t\alpha_{i}(T)^{q}\} \) are equivalent.

**Proof:**

If \( T \in D_{\max(p,q)}(L^{r}, L^{s}) \) then \( T \sim (\lambda_{i})_{i=1}^{\infty} \). Without loss of generality we can assume that \( \lambda_{i} \geq \lambda_{i+1} \geq 0 \) for all \( i \). Let \( T_{p} + T_{q} \) be a decomposition of \( T \) where \( T_{j} \in D_{j}(L^{r}, L^{s}) \) for \( j = p, q \) with
\[ T_p \sim (a_i)_{i=1}^{\infty}, T_q \sim (b_i)_{i=1}^{\infty} \text{ with } a_i, b_i \geq 0 \text{ and } a_i + b_i = \lambda_i. \]

Then \[ L_{p,q}(t,T;D_p,D_q) \leq \inf_{i=1}^{\infty} \{ a_i^p + b_i^q : \lambda_i = a_i + b_i \} \]

\[ = \sum_{i=1}^{\infty} \inf \{ a_i^p + t b_i^q : \lambda_i = a_i + b_i \} \]

\[ \leq \sum_{i=1}^{\infty} \min \{ \lambda_i^p, t \lambda_i^q \} \text{ by } (4.5). \]

Consider the three cases:

\textbf{case (i): } r = s

\[ \sum_{i=1}^{\infty} \min \{ \lambda_i^p, t \lambda_i^q \} \leq \sum_{i=1}^{\infty} \min \{ \alpha_i(T)^p, t \alpha_i(T)^q \} \]

since \[ \alpha_i(T) = \lambda_{i+1} \]

\textbf{case (ii): } r > s

\[ \sum_{i=1}^{\infty} \min \{ \lambda_i^p, t \lambda_i^q \} \leq \sum_{i=1}^{\infty} \min \{ \alpha_i(T)^p, t \alpha_i(T)^q \} \]

since \[ \lambda_{i+1} \leq \left( \sum_{k=i+1}^{\infty} \lambda_k \right)^{1/j} = \alpha_i(T), \frac{1}{r+1/j} = \frac{1}{s}. \]

\textbf{case (iii): } r < s

\[ \sum_{i=1}^{\infty} \min \{ \lambda_i^p, t \lambda_i^q \} \leq \max (2^p, 2^q) \sum_{i=1}^{\infty} \min \{ \alpha_i(T)^p, t \alpha_i(T)^q \} \]

since \[ \lambda_{i+1} \leq 2 \alpha_i(T) \]

In all cases \[ L_{p,q}(t,T;D_p,D_q) \leq \max (2^p, 2^q) \sum_{i=0}^{\infty} \min \{ \alpha_i(T)^p, t \alpha_i(T)^q \}. \]

On the other hand, if \[ T = T_p + T_q \in D_p + D_q \] then \[ \alpha_{2i}(T) \leq \alpha_i(T_p) + \alpha_i(T_q) \text{ for all } i. \] Therefore
\[ \sum_{i=0}^{\infty} \min\{\alpha_i(T)^P, t\alpha_i(T)^q\} \leq 2 \sum_{i=0}^{\infty} \min\{\alpha_{2i}(T)^P, t\alpha_{2i}(T)^q\} \]
\[ \leq 2 \sum_{i=0}^{\infty} (\alpha_i(T)_p + \alpha_i(T)_q)^p, \]
\[ t(\alpha_i(T)_p + \alpha_i(T)_q)^q) \]
\[ \leq 2 \max(2^p, 2^q) \sum_{i=0}^{\infty} (\alpha_i(T)_p + t\alpha_i(T)_q)^q]. \]

Thus
\[ \sum_{i=0}^{\infty} \min\{\alpha_i(T)^P, t\alpha_i(T)^q\} \leq 2 \max(2^p, 2^q) L_p, q(t, T; D_p, D_q). \]

**Theorem 4.7** For positive numbers \( p, q \) and \( \eta \in (0, 1) \) we have \((D_p, D_q) = D, r = (1-\eta)p + \eta q \) and the quasi-norms are equivalent.

**Proof:**

By using the function equivalent to \( L_p, q \) (see 4.6) for \( T \in (D_p, D_q) \) we have
\[ ||T||_{\eta, 1; L_p, q} = \int_0^{\infty} t^{-\eta} L_p, q(t, T; D_p, D_q) \frac{dt}{t} \]
\[ \leq C_{p, q} \int_0^{\infty} t^{-\eta-1} \sum_{i=0}^{\infty} \min(\alpha_i(T)^P, t\alpha_i(T)^q) dt \]

and
\[ ||T||_{\eta, 1; L_p, q} \geq B_{p, q} \int_0^{\infty} t^{-\eta-1} \sum_{i=0}^{\infty} \min(\alpha_i(T)^P, t\alpha_i(T)^q) dt \]

where \( C_{p, q} \) and \( B_{p, q} \) are constants depending only on \( p \) and \( q \).

Now
\[ \int_0^{\infty} t^{-\eta-1} \min(\alpha_i(T)^P, t\alpha_i(T)^q) dt = \int_0^{\alpha_i(T)^P} t^{-\eta-1} t^{-1} \alpha_i(T)^q dt \]
\[ + \int_0^{\alpha_i(T)^P} t^{-\eta-1} \alpha_i(T)^p dt \]

Thus
\[ \sum_{i=0}^{\infty} \min\{\alpha_i(T)^P, t\alpha_i(T)^q\} \leq 2 \max(2^p, 2^q) L_p, q(t, T; D_p, D_q). \]
Thus

$$\frac{1}{B_{P,q}(1-\eta)^n} \sum \alpha_i(T)^r \leq \|T\|_{1;L_{p,q}} \leq \frac{C_{p,q}}{(1-\eta)^n} \sum \alpha_i(T)^r.$$ 

In [23] Peetre proved the following Equivalence Theorem.

**Theorem 4.8** For positive numbers $p,q$, and $r$ and for $\eta \in (0,1)$ and $s = (1-\eta)p + \eta q$ and $1/s = (1-\eta)/p + \eta/q$ we have $(E,F)_{r,s} = (E,F)_{\eta,r} \leq L_{p,q}$ and the quasi-norms are equivalent.

Theorem 4.7 together with the Equivalence Theorem of Peetre gives the following corollary.

**Corollary 4.9** For positive numbers $p,q$ and for $\theta \in (0,1)$

$$(D_p(\ell^r,\ell^s), D_q(\ell^r,\ell^s))_{\theta,n} = D_n(\ell^r,\ell^s)$$

where $1 \leq r,s \leq \infty$ and $1/n = (1-\theta)/p + \theta/q$ and the quasi-norms are equivalent.


11. M. Kadec (to appear)


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