2015

Well-Quasi-Ordering by the Induced-Minor Relation

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WELL-QUASI-ORDERING BY THE INDUCED-MINOR RELATION

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by
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December 2015

Acknowledgments

First and foremost I would like to thank my advisor, Professor Guoli Ding, for his incredible amount of help, support, and guidance. This work would not be possible without him. He always points me to the right direction and encourages me to move forward. I really appreciate his kindness and patience to me throughout these years. Thank you sir.

I would like to thank my committee members, professors: Richard Litherland, Bogdan Oporowski, Milen Yakimov, James Oxley, and Thomas Corbitt, for their thoughtful comments and suggestions regarding this dissertation. I owe my deepest gratitude to Professor Oxley. He always provide feedback which helped me to grow as a teacher and a researcher. I can not thank Professor Oporowski enough for his guidance on my path as an international graduate student.

I would like to thank all of my mathematics teachers who have inspired me over the years and introduced me to some wonderful mathematics, among them: Professors Kunlaya Sriligo, Somsak Orankitjaroen, Chaiwat Maneesawarng, Bogdan Oporowski, James Oxley, and Gouli Ding.

Many thanks to the Department of Mathematics of Louisiana State University for all the supports provided. I would like to thank the faculty and staff members here for their help and advice. I also want to thank Professors Leonard Richardson and William Adkins, and Mrs. Phoebe Rouse for giving me a chance to work for the department. Thank you all my students for being patient with my teaching.

I would also like to thank my wonderful friends that I have made here at LSU, especially, Ryan Bowman and his family, Anuwat Sae-Tang, Jun Peng, Kwang Ju Choi, and Bir Kafle for their help and support. I would also like to thank the Thai community in Baton Rouge for their warm welcome and support. I cannot thank everyone personally, but overall, one of my wonderful memories are made here.

I would like to thank the Development and Promotion of Science and Technology Talents Project (DPST), Royal Thai Government, for providing me the financial support. I also want to thank the Office of Educational Affairs (OEA) staff at the Royal Thai Embassy in Washington DC for their help and care.

Last, buy not least, I want to thank my family for their constant love and support. Most importantly, my wife, Phakaporn Lewchalermvongs, thank you for always being there for me through everything, being on my side, and believing in me. Thank you for her love and understanding.

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Abstract

Robertson and Seymour proved Wagner's Conjecture, which says that finite graphs are well-quasi-ordered by the minor relation. Their work motivates the question as to whether any class of graphs is well-quasi-ordered by other containment relations. This dissertation is concerned with a special graph containment relation, the induced-minor relation.

This dissertation begins with a brief introduction to various graph containment relations and their connections with well-quasi-ordering. In the first chapter, we discuss the results about well-quasi-ordering by graph containment relations and the main problems of this dissertation. The graph theory terminology and preliminary results that will be used are presented in the next chapter. The class of graphs that is considered in this research is the class $\mathscr{W}$ of graphs that contain neither W_4 (a wheel graph with five vertices) and $K_5\backslash e$ (a complete graph on five vertices minus an edge) as an induced minor. Chapter 3 is devoted to studying the structure of this class of graphs. A class of graphs is well-quasi-ordered by a containment relation if it contains no infinite antichain, so infinite antichains are important. We construct in Chapter 4 an infinite antichain of $\mathscr{W}$ with respect to the induced minor relation and study its important properties in Chapter 5. These properties are used in determining all well-quasi-ordered subclasses of $\mathscr{W}$ to reach the main result of Chapter 6.

Chapter 1
Introduction

The graph theory terminology used here generally follows Diestel [5] except where otherwise noted. Most terminology is formally define in Section 2.2.

An *antichain* in a partially ordered set $(Q, \leqslant)$ is a subset of Q for which no two distinct elements are comparable. One of the most important results in graph theory is Robertson and Seymour's proof of Wagner's Conjecture [16], which says that the class of all finite graphs has no infinite antichain under the minor relation. In other words, a graph is a minor of another if the first is obtained from the second by a (possibly empty) sequence of vertex deletions, edge deletions, and edge contractions (where the order of the graph operations is irrelevant). Their work leads to the question as to whether the class of all finite graphs has no infinite antichain for other containment relations.

An induced minor relation is a special minor relation that allows only vertex deletion and edge contraction. The class of all finite graphs has an infinite antichain for the induced minor relation, which is the set of the complement of cycles on at least three vertices. However, if we restrict to some smaller classes of graphs, there could be no infinite antichain for this relation. The following result of Thomas [18] is the first result on the induced minor with this property and it is also one of the motivation behind our research. A *series-parallel graph* is a graph that does not contain a subdivision of K_4.

Theorem 1.1. *[18] The class of series-parallel graphs has no infinite antichain for the induced minor relation.*

Thomas proved this result by studying the labeled rooted version of this class of graphs. He also gave an example of infinite antichain in the class of planar graphs for the induced minor relation. He proposed the problem whether there is an infinite antichain in the class of graphs that cannot be contracted onto $K_5\backslash e$ (a complete graph on five vertices minus an edge) for the induced minor relation or not. Our research is concerned with the induced minor relation. We can also use our result to answer this question and generalize Thomas' result.

Remark that a formal definition of *well-quasi-ordering* (or *wqo*) is given in Chapter 2, and for a natural graph containment relation, it is wqo on a class of graphs if and only if there is no infinite antichain. The remainder of this chapter is devoted to briefly discussing results on wqo and graph containment relations, properties of induced minor, and the main results of this dissertation.

1.1 Well-Quasi-Ordering and Graph Containment Relations

Natural operations in graphs include vertex deletion, edge deletion , and edge contraction. Table 1.1 [1] shows graph containment relations obtained by combining these graph operations. For example, a graph H is an induced minor of a graph G if H is obtained from G by a (possibly empty) sequence of vertex deletions and edge contractions. All relations in Table 1.1, except the minor relation, are not wqo. For instance, the set of the complement of complete graphs on at least 1 vertex is an infinite antichain for the spanning subgraph and isomorphism relations.

The topological minor is a graph relation that was the background of the earliest results in the area of wqo. A *subdivision* of H is a graph obtained from H by replacing edges by paths. A graph H is a *topological minor* of a graph G if G has a subgraph that is a subdivision of H. The topological minor is not well-quasi-ordering on the class of finite graphs, see examples of infinite antichains in [7] and [9]. However, this relation is wqo on some smaller classes of graphs. Kruskal proved the positive results of this relation.

Table 1.1: Containment relations obtained by vertex deletion (VD), edge deletion (ED), or edge contraction (EC) [1].

Containment Relation	VD	ED	EC
Minor	Yes	Yes	Yes
Induced Minor	Yes	No	Yes
Contraction	No	No	Yes
Subgraph	Yes	Yes	No
Induced Subgraph	Yes	No	No
Spanning Subgraph	No	Yes	No
isomorphism	No	No	No

Theorem 1.2. *[5] The finite trees are well-quasi-ordered by the topological minor relation.*

For a positive integer n, a *double path* B_n of length n is the graph obtained from a n-edge path by doubling each edge in parallel. Robertson conjectured the following statement, which was proved later by Liu [14].

Theorem 1.3. *[14] (Robertson's Conjecture) For every positive integer k, graphs that do not have a topological minor B_n are well-quasi-ordered by the topological minor relation.*

Ding [7] proved Robertson's Conjecture in the special case of a minor-closed class of graphs. A class $\mathscr{G}$ of graphs is minor-closed if every minor of a graph in $\mathscr{G}$ is in $\mathscr{G}$.

Theorem 1.4. *[7] A minor-closed class $\mathscr{G}$ of graphs is well-quasi-ordered by the topological minor relation if and only if some B_n is not in $\mathscr{G}$.*

The class of finite graphs has an infinite antichain for the edge contraction relation, which is the set of graphs with two vertices and n edges, for $n = 1, 2, \ldots$. Kamiński, Raymond, and Trunk [11] proved the following result on the wqo by this relation for the class of multigraphs (parallel edges are allowed) with some restrictions. A *bond* in a graph is a minimal set of edges whose removal increases the number of connected components in the graph.

Theorem 1.5. *[11] The class of multigraphs with at most p connected components and bonds of size at most k is well-quasi-ordered by the edge contraction relation for all positive integers p, k.*

We next discuss some results of wqo by subgraph relation,induced subgraph relation, and induced minor relation.

1.1.1 Well-Quasi-Ordering by subgraph relation and induced subgraph relation

A graph H is a *subgraph* of a graph G if H is obtained from G by a (possibly empty) sequence of vertex deletions and edge deletions. An *induced subgraph* can be constructed by only a (possibly empty) sequence of vertex deletions. In general, these two relations are not wqo. For example, cycles with different length do not contain another as a subgraph or an induced subgraph so we can form an infinite antichain by cycles with different length. We will discuss the positive results of these two relations by excluding some graphs as subgraphs or induced subgraphs. Damaschke [4] proved the following results.

Theorem 1.6. *[4] Then the following classes of graphs are well-quasi-ordered by the induced subgraph relation.*

(i) The class of cographs, graphs that do not have an induced subgraph a path on four vertices.

(ii) The class of graphs that do not have an induced subgraph P_5 or a complete graph on three vertices K_3.

(iii) The class of graphs that do not have an induced subgraph K_3 or K_2+2K_1 (the disjoint union of K_2 and two copies of K_1).

We call a class of graphs $\mathscr{F}$ an *ideal* with respect to the subgraph relation, $\subseteq$, if $G \subseteq G' \in \mathscr{F}$ implies that $G \in \mathscr{F}$. Ding [6] characterized these graph ideals in terms of excluding subgraphs. Let C_n be a cycle on n vertices and F_n be a graph obtained from a path on n vertices by attaching two leaves to each end of the path.

Theorem 1.7. *[6] Let $\mathscr{F}$ be an ideal of graphs with respect to the subgraph relation. Then the following are equivalent.*

(i) $\mathscr{F}$ is well-quasi-ordered by the subgraph relation.

(ii) $\mathscr{F}$ is well-quasi-ordered by the induced subgraph relation.

iii $\mathscr{F}$ contains only finitely many graphs C_n and F_n.

The following theorem of Ding [6] shows the positive results of the induced subgraph relation by excluding some graphs as subgraphs or induced subgraphs.

Theorem 1.8. *[6] The class of graphs that do not have a subgraph P_n is well-quasi-ordered by the induced subgraph relation.*

Ding also studied the class of bipartite graphs. A graph G is called *bipartite* if its vertex set $V(G)$ can be partitioned into two sets X, Y such that every edge in its edge set $E(G)$ connects a vertex in X to a vertex in Y. Let $\bar{G}$ denote the *bipartite complement* of G which is a bipartite graph with the partition sets X, Y and the edge set $X \times Y \setminus E(G)$. He proved the following results.

Theorem 1.9. *[6] The following classes of graphs are well-quasi-ordered by the induced subgraph relation.*

(i) The class of bipartite graphs that do not have an induced subgraph P_7, J_1, or J_2 illustrated in Figure 1.1.

(ii) The class of bipartite graphs that do not have an induced subgraph P_6 or $\bar{P}_6$.

Ding [6] also proved the same result as 1.9(i) for digraphs (or directed graphs). On the other hand, he gave an example of a class of graphs obtained by excluding some graphs as

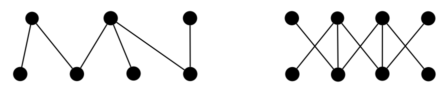


Figure 1.1: Graphs J_1 and J_2 in [6].

induced sugraphs that is not well-quasi-ordered by the induced minor relation. He found an infinite antichain of the class of bipartite graphs that do not have an induced subgraph P_8 or $\bar{P}_8$.

From [6], the induced subgraph on the class of bipartite graphs that do not have an induced subgraph P_6 or $\bar{P}_6$ is wqo, but it is not wqo on the class of bipartite graphs that do not have an induced subgraph P_8 or $\bar{P}_8$. The question is whether it is wqo on the class of bipartite graphs that do not have an induced subgraph P_7 or not. Notice that P_7 and $\bar{P}_7$ are isomorphic. Korpelainen and Lozin [12] proved that this class is not wqo by showing an infinite antichain. They found that this antichain does not have an induced subgraph Sun_4 illustrated in Figure 1.2.

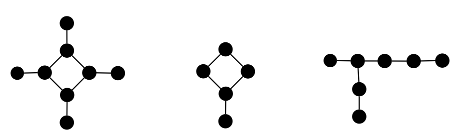


Figure 1.2: Graphs Sun_4, Sun_1, and $Sun_{1,2,3}$ in [12].

Notice that $Sun_{1,2,3}$ and Sun_4 in Figure 1.2 are isomorphic to J_1 and J_2 in Figure 1.1, respectively. Korpelainen and Lozin [12] extended the results in Theorem 1.9 to the larger class of bipartite graphs that do not have an induced subgraph P_7 or $Sun_{1,2,3}$. They proved the following positive results.

Theorem 1.10. *[12] The following classes of graphs are well-quasi-ordered by the induced subgraph relation.*

(i) The class of bipartite graphs that do not have an induced subgraph P_7 or $Sun_{1,2,3}$.

(ii) The class of bipartite graphs that do not have an induced subgraph P_7 or Sun_1.

They also gave a negative example on the induced subgraph relation that they provided is the class of biconvex graphs that do not have an induced subgraph P_8 or $\bar{P}_8$, where they defined a biconvex graph as a bipartite graph such that its vertices can be linearly ordered so that the neighborhood of each vertex u (the set of vertices adjacent to u) consists of consecutive vertices in the order.

In the same research, they proved the following positive result on the classes of bipartite permutation graphs, which are the intersection of bipartite graphs and permutation graphs.

Theorem 1.11. *[12] The class of bipartite permutation graphs that do not have an induced subgraph P_n is well-quasi-ordered by the induced subgraph relation.*

Korpelainen and Lozin [13] studied a class of graphs obtained by excluding two graphs as induced subgraphs. They called this class of graphs *bigenic*. They characterized many bigenic classes of graphs that are well-quasi-ordered or are not well-quasi-ordered by the induced subgraph relation. Let $G + H$ denote the disjoint union of graphs G and H, and let nG denote union of n copies of G. For example, they proved that the following bigenic classes of graphs are well-quasi-ordered by the induced subgraph relation; $\{K_3, P_3 + 2K_1\}$, $\{K_3, P_4 + K_1\}$, $\{K_3, P_3 + P_2\}$, and $\{K_n, mK_1\}$. On the other hand the following bigenic classes of graphs are not well-quasi-ordered by the induced subgraph relation by revealing their infinite antichains; $\{C_4, 2K_2\}$, $\{K_3, 2P_3\}$, $\{K_3, K_2 + 3K_1\}$, and $\{K_4, 2K_2\}$. More details on bigenic classes of graphs can be found in [13].

1.1.2 Well-Quasi-Ordering by the induced minor relation

A graph H is an *induced minor* of a graph G if H is obtained from G by a (possibly empty) sequence of vertex deletions and edge contractions. Ding [8] studied wqo by the induced minor relation on the class of chordal graphs and the class of interval graphs. An *intersection graph*, which is a graph whose vertex set consists of nonempty sets and there is an edge connecting two vertices if and only if the intersection of the corresponding sets of those two vertices is not empty. A *chordal graph* is an intersection graph with vertex set consisting of vertex sets of finite subtrees of an infinite tree. In other words, it is a graph for which every cycle on at least four vertices has a chord, which is an edge that is not in the cycle but connects two vertices in the cycle. Two vertices of this graph are adjacent if the intersection of corresponding vertex sets of finite subtrees is not empty. An *interval graph* is a special chordal graph, which is the intersection graph with vertex set consisting of vertex sets of subpaths of an infinite path. We can think of the interval graph as the intersection graph of intervals of the real line. These classes are closed under induced minor. Ding proved that the class of interval graphs is not well-quasi-ordered by the induced minor relation. He constructed an antichain in this class with respect to the relation. The antichain consists of graphs G_n, which is an intersection graph of intervals in $S_n \cup T_n$ where S_n is the set of closed intervals $[i, j]$ for $i = \pm 1, \pm 2, \ldots, \pm 2n$, and T_n is the set of the following closed intervals:

- $[-2, 2], [-4, 1], [-2n + 3, 2n], [-2n + 1, 2n - 1]$;
- $[-2i + 1, 2i + 1]$ for $i = 1, 2, \ldots, n - 2$; and
- $[-2i, 2i - 2]$ for $i = 3, 4, \ldots, n$.

Ding proved that the class of chordal graphs of bounded clique size is well-quasi-ordered by the induced minor relation by proving the stronger result on the Q-labeled fully oriented version. Note that a fully oriented graph is a graph that its vertex set of every clique is linearly ordered, and the label is put on the set of all cliques of a graph.

Theorem 1.12. *[8] Let $(Q, \leq)$ be a wqo and let t be a positive integer. Then the class of all Q-labeled fully oriented chordal graphs without cliques of size $t+1$ is well-quasi-ordered by the induced minor relation.*

The dichotomy result on induced minors and well-quasi-ordering was proved by Blasiok, Kamiński, Raymond, and Trunk [2].

Theorem 1.13. *[2] Let H be a graph. The class of graphs that do not have an induced minor H is well-quasi-ordered by the induced minor relation if and only if H is an induced minor of the gem or $\hat{K}_4$ illustrated in Figure 1.3.*

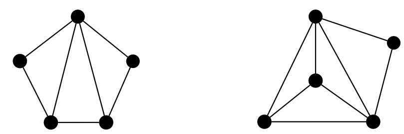


Figure 1.3: The gem and the graph $\hat{K}_4$ in [2].

Notice that K_4 is an induced minor of $\hat{K}_4$ and $K_5\backslash e$ is not an induced minor of both the gem and $\hat{K}_4$. By this theorem, the class of graphs with no K_4-induced minor is well-quasi-ordered by the induced minor relation, but the class of graphs with no $K_5\backslash e$-induced minor is not. This also generalizes Thomas' result and answers the question that he proposed in [18].

1.2 Related Results

Cicalese and Milanic [3] studied graphs of *separability* at most k, which are graphs such that every two nonadjacent vertices are separated by a set of at most k other vertices. Note that complete graphs have separability at most 0. The main results are on graphs of separability at most 2. They proved the result on the structure of these graphs as follow.

Theorem 1.14. *[3] A graph G has separability at most 2 if and only if G can be built from complete graphs and cycles by an iterative application of the disjoint union operation and of pasting two disjoint graphs along a vertex or along an edge.*

They characterized graphs of separability at most 2 in terms of graphs that have common properties involving induced subgraph and induced minor relations.

Theorem 1.15. *[3] A graph G has separability at most 2 if and only if G does not have an induced subgraph $K_5\backslash e$, H_0, H_1, H_2, or H_3, which are graphs illustrated in Figure 1.4.*

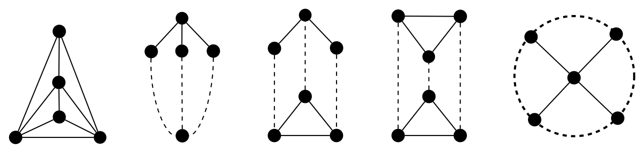


Figure 1.4: Graphs $K_5\backslash e$, H_0, H_1, H_2, and H_3 (wheel), where a dotted indicates a chordless path containing one or more edges [3].

Theorem 1.16. *[3]*

(i) Graphs of separability 0 are precisely graphs that do not have an induced minor P_3.

(ii) Graphs of separability at most 1 are precisely graphs that do not have an induced minor C_4 or diamond (K_4 minus an edge).

(iii) Graphs of separability at most 2 are precisely graphs that do not have an induced minor $K_{2,3}$, F_5, W_4, or $K_5\backslash e$, illustrated in Figure 1.5.

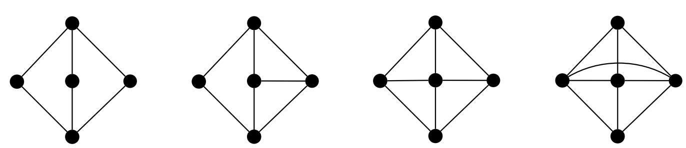


Figure 1.5: Graphs $K_{2,3}$, F_5, W_4, and $K_5\backslash e$ in [3].

The problem of testing that a graph H is an induced minor of a graph G or not is called a decision problem. All relations in Table 1.1 have their corresponding decision problems, and all these problem, except the isomorphism problem, are NP-complete when both G and H are input [1]. By fixing H and inputting only G, we state these problems by adding $H-$ in front of the relations. For any graph H, the H-minor problem can be solved in cubic time [15], and the H-subgraph, H-spanning subgraph, H-induced subgraph, and H-graph isomorphism can be solved in polynomial time [1]. However, for the H-induced minor problem and H-contractibility problem, there exist graphs H such that these two problems are NP-complete [1]. Fellows, Kratochvil, Middendorf, and Pfeiffer [10] proved the following theorem by showing that the induced minor testing of the graph in Figure 1.6 is NP-complete.

Theorem 1.17. *[10] There is a graph H such that the H-induced minor problem is NP-complete.*

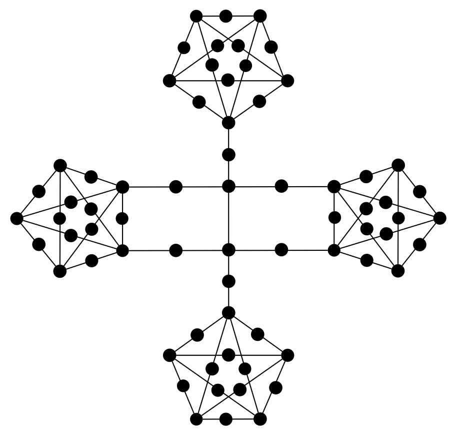


Figure 1.6: A graph H such that the H-induced minor testing is NP-complete [10].

1.3 Main Result

We are now discuss the main result of this dissertation. By a graph we mean a finite, undirected, simple graph. This research concerns with $\{W_4, K_5\backslash e\}$-free graphs, which are graphs that contain neither W_4 (a wheel graph with five vertices) nor $K_5\backslash e$ as an induced minor. This class of graphs will be denoted by $\mathscr{W}$. In order to study $\mathscr{W}$, we introduce a composite graph, which is obtained from a graph in $\mathscr{W}$ by assigning directions to some edges and declaring some vertices as special. Then we create a labelled rooted directed graph from a composite graph by fixing two vertices as roots and assigning labels on directed edges and special vertices. We defined the terminology in Chapter 2. Using these new notations will produce a stronger result, but the main reason for using them is to make the process in our proof work. In Chapter 3, we study the structure theorem for graphs in $\mathscr{W}$ and prove that the graphs can be constructed from cliques (complete graphs) by repeatedly applying the disjoint union operation and combining two graphs by identifying a vertex or an edge. We call these operations sums of graphs. Note that there is a specific condition for identifying an edge, that will be defined in Chapter 3.

As part of our goal to characterize subclasses of $\mathscr{W}$ that are well-quasi-ordered by the induced minor relation, we study the structure of an antichain in $\mathscr{W}$. Let $\mathscr{D}^\Gamma$ be the class of graphs illustrated in Figure 1.7. In Chapter 4, we prove the following result.

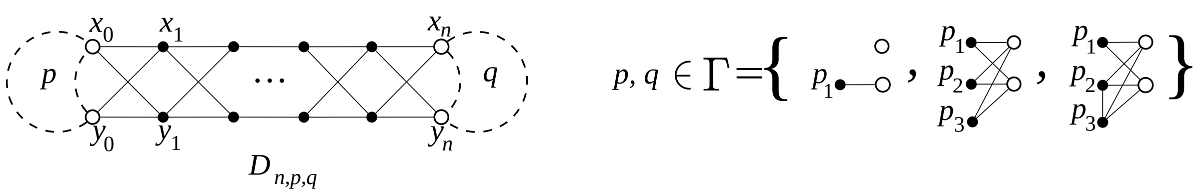


Figure 1.7: A graph $D_{n,p,q}$ in $\mathscr{D}^\Gamma$.

Theorem 1.18. *$\mathscr{D}^\Gamma \subseteq \mathscr{W}$ and $\mathscr{D}^\Gamma$ is an antichain.*

In order to prove this statement, we have to consider $\bar{\mathscr{D}}^{\Gamma+}$, which consists of labeled rooted directed version of $\mathscr{D}^\Gamma$. This theorem implies that $\mathscr{W}$ is not well-quasi-ordered by the induced minor relation. Notice that the main part of a graph in $\mathscr{D}^\Gamma$ can be obtained from K_3's and K_4's by identifying edges. In Chapter 5, we focus on the subclass of $\mathscr{W}$ whose members can be constructed using such method. We prove important properties of an infinite antichain in this subclass. Then we study the structure of a graph in this class in term of tree structure and prove the results on labeled rooted directed version. Finally, in Chapter 6, we prove the main result of the research. We define a *closed* subclass $\mathscr{X}$ of $\mathscr{W}$ as a subclass of $\mathscr{W}$ such that every induced minor of any $G \in \mathscr{X}$ is in $\mathscr{X}$.

Theorem 1.19. *For any closed subclass $\mathscr{X}$ of $\mathscr{W}$, $\mathscr{X}$ contains an infinite antichain if and only if $\mathscr{X} \cap \mathscr{D}^\Gamma$ is infinite.*

In terms of wqo, this says a closed subclass $\mathscr{X}$ of $\mathscr{W}$ is well-quasi-ordered by the induced minor relation if and only if $\mathscr{X} \cap \mathscr{D}^{\Gamma}$ is finite. This implies that $\mathscr{X}$ is well-quasi-ordered by the induced minor relation if $\mathscr{X}$ contains only finitely many graphs $D_{n,p,q}$ in $\mathscr{D}^{\Gamma}$.

Let $\mathscr{K}_4$ be the class of series-parallel graphs. Then every graph in $\mathscr{K}_4$ does not have a minor K_4 and an induced minor K_4. Since K_4 is an induced minor of W_4 and $K_5\backslash e$, $\mathscr{K}_4 \subseteq \mathscr{W}$. Since K_4 is an induced minor of $D_{n,p,q}$ for all $n \geq 3$, $\mathscr{K}_4 \cap \mathscr{D}^{\Gamma}$ is finite. From Theorem 1.19, $\mathscr{K}_4$ is well-quasi-ordered by the induced minor relation. This implies the result of Thomas in [18]. From the implication of Theorem 1.18 in terms of wqo, the class of graphs with no $K_5\backslash e$-induced minor is not because it contains $\mathscr{W}$ as a subclass. We answer the question that Thomas proposed in [18].

From Theorem 1.16(iii), every graph of separability at most 2 is in $\mathscr{W}$. Since $K_{2,3}$ is an induced minor of $D_{n,p,q}$ for all $n \geq 2$, the class of graphs separability at most 2 contains only finitely many graphs $D_{n,p,q}$. So this class of graphs is well-quasi-ordered by the induced minor relation.

Chapter 2
Preliminaries

In this chapter we introduce some standard terminology that will be use throughout the dissertation, and important previous results in graph theory.

2.1 Well-Quasi-Ordering

Let X be a set and $\leqslant$ be a binary relation on X. The relation $\leqslant$ on X is called a *quasi-ordering* if it is reflexive and transitive. A sequence $x_1, x_2, \ldots$ of members of X is called a *good sequence* if there are indices $i < j$ such that $x_i \leqslant x_j$. The ordered pair (x_i, x_j) is called a *good pair*. It is a *bad sequence* if otherwise. We call $(X, \leqslant)$ a *well-quasi-ordering* (or a *wqo*) if every infinite sequence $x_1, x_2, \ldots$ in X is a good sequence, in other words, there is no infinite bad sequence.

Recall that Two elements x and y of X are *comparable* if $x \leqslant y$ or $y \leqslant x$. A subset A of X is called an *antichain* of X if no two distinct elements are comparable. The following is one of the key lemmas that is used in this research.

Lemma 2.1. *[5] $(X, \leqslant)$ is not a well-quasi-ordering if and only if there is either an infinite antichain or an infinite strictly decreasing sequence.*

Note that since a class of finite graphs ordered by the induced minor relation has no infinite strictly decreasing sequence, it is well-quasi-ordered by the induced minor relation if and only if it contains no infinite antichain.

An element a of a subset A of X is a *minimal* element of A if $x \leqslant a$ implies $a \leqslant x$ for all x in A. The relation $\leqslant$ on X is called *well-founded* if every nonempty subset of X has a minimal element, which means there is no an infinite strictly decreasing sequence.

For any antichain A of X, let $A^< = \{x \in X : x < a \text{ for some } a \in A\}$. We say that antichain A is *fundamental* if $A^<$ has no infinite antichains. This definition implies the following lemma.

Lemma 2.2. *If B is a subset of a fundamental antichain A, then B is also fundamental.*

We call A a *maximal antichain* if no proper superset of A is an antichain.

Lemma 2.3. *[9] If a well-founded partial order $(X, \leqslant)$ has an infinite antichain, then $(X, \leqslant)$ has an infinite maximal antichain A such that every infinite antichain of $A \cup A^<$ is a subset of A.*

Observe that the infinite maximal antichain A determined by this lemma is fundamental. This lemma implies the following lemma.

Lemma 2.4. *If a well-founded quasi-order $(X, \leqslant)$ has an infinite antichain, then there is a fundamental infinite antichain A of X.*

Let A and B be two subsets of $(X, \leqslant)$. We define $A \leqslant_* B$ if there is a one to one mapping ϕ from A to B such that $x \leqslant \phi(x)$ for all $x \in A$. We define $A <_* B$ if there is a one to one mapping ϕ from A to B such that $x < \phi(x)$ for all $x \in A$. If $(X, \leqslant)$ is a quasi-ordering, we can extend $\leqslant$ to a quasi-ordering $\leqslant_*$ on $[X]^{<\omega}$, the set of all finite subsets of X.

Lemma 2.5. *[5] (Higman's theorem) If $(X, \leqslant)$ is a wqo, then $([X]^{<\omega}, \leqslant_*)$ is a wqo.*

Let $(X_1, \leqslant_1), (X_2, \leqslant_2), \ldots, (X_n, \leqslant_n)$ be wqo. The Cartesian product of these n sets can be represented by an array of n dimensions, where each element is an n-tuple, $X_1 \times X_2 \times \ldots \times X_n = \{(x_1, x_2, \ldots, x_n) | x_i \in X_i \text{ for all } i = 1, \ldots, n\}$. Let $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ be two elements of this set. We define $(x_1, x_2, \ldots, x_n) \leqslant_\times (y_1, y_2, \ldots, y_n)$ if $x_i \leqslant_i y_i$ for all $i = 1, \ldots, n$. A sequence $x_1, x_2, \ldots$ of members of X is *increasing* if $x_1 \leqslant x_2 \leqslant \ldots$. Corollary 12.1.2 in [5] says that every infinite sequence of a wqo set contains an infinite increasing subsequence. This corollary imply the following lemma.

Lemma 2.6. *The Cartesian product of finite number of wqo sets is wqo.*

2.2 Graphs

A *graph* G is an ordered pair (V, E), where V is a finite set and E is a finite multiset whose elements are unordered pairs of elements of V. We call the elements of V the *vertices* of G, and the elements of E the *edges* of G. The *order* of G, $|G|$, is the number of vertices. If $u, v \in V$ and $e = (u, v) \in E$, then u and v are called the *endvertices* or *ends* of e and we write $e = uv$. An edge is *incident* with each of its ends and vice versa. If $uv \in E$ then vertices $u, v \in V$ are *adjacent* or *neighbors*. The *neighborhood* of a vertex v in V is the set of neighbors of v, written as $N_G(v)$. Two edges with a common end are *adjacent*, and two edges with the same ends are *parallel*. An edge with identical ends is called a *loop*.

A graph with no loops or parallel edges is *simple*. A *multigraph* is a graph that can have loops and parallel edges. A *simplification* of a graph G is a simple graph obtained from G by deleting all loops an parallel edges. In this research, by a graph we mean a simple graph. The *complement* $\bar{G}$ of a graph G is the graph with the vertex set V such that two vertices in $\bar{G}$ are adjacent if they are not adjacent in G. The *degree* of a vertex v, $deg_G(v)$, in a graph G is the number of edges incident with v, which is equal to $|N_G(v)|$.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. Then G is *isomorphic* to H, $G \simeq H$, if there is a bijection $\phi : V(G) \to V(H)$ so that $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. Let $G \cup H$ be the *union of graphs* G and H, $(V(G) \cup V(H), E(G) \cup E(H))$, and let $G \cap H$ be the *intersection of graphs* G and H, $(V(G) \cap V(H), E(G) \cap E(H))$. If $V(G) \cap V(H) = \emptyset$, then G and H are *disjoint*. We say that H is a *subgraph* of G, denoted as $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $H \subseteq G$ and $H \neq G$, then H is a *proper subgraph* of G. If $H \subseteq G$ and $V(H) = V(G)$, then H is a *spanning subgraph* of G. We say H is an *induced subgraph* of G if $H \subseteq G$ and H contains every edge in G whose ends belong to $V(H)$. If H is an induced subgraph of G with vertex set $X \subseteq V(G)$, then H is the subgraph of G *induced by* X, written as $H = G[X]$. If X is any set of vertices (usually of G), we denote $G[V(G) \setminus X]$ by $G \setminus X$. If G' is a graph, we simply write $G \setminus G'$ instead of $G \setminus V(G')$.

A *complete graph* or *clique* K_n is a simple graph on n vertices such that each distinct pair of vertices are adjacent. A *triangle* is the complete graph on 3 vertices. A *path* $P = (V(P), E(P))$ is a graph with $V(P) = \{v_0, v_1, \ldots, v_n\}$ and $E(P) = \{v_0v_1, v_1v_2, \ldots, v_{n-1}v_n\}$, where v_i's are all distinct. The vertices v_0 and v_n are called the *endvertices* of P, and the vertices $v_1, \ldots, v_{n-1}$ are called the *inner* vertices. We call P a $v_0 - v_n$ *path*. The *length of path* P is n, and P can also be denoted by P_n. If A and B are sets of vertices such that $V(P) \cap A = \{v_0\}$ and $V(P) \cap B = \{v_n\}$, then we say P is an $A - B$ *path*. Two or more paths are *independent* if their inner vertices are disjoint.

A *cycle* $C = (V(C), E(C))$ is a graph with $V(C) = \{v_1, \ldots, v_n\}$ and $E(C) = \{v_1v_2, \ldots, v_nv_1\}$, where v_i's are all distinct. The *length of cycle* C is n, and we call C an *n-cycle*, denoted by C_n. A *wheel* W_n is a graph on $n + 1$ vertices obtained from a cycle C_n by adding a vertex connecting to all vertices on the cycle.

A graph G is *connected* if for any two distinct vertice $u, v \in V(G)$, there is a $u - v$ path in G. We say G is *disconnected* if G is not connected. A maximal connected subgraph of G is a *component* of G. A *forest* is a graph with no cycles. A *tree* is a connected forest. A vertex of degree 1 in a tree is called a *leaf*.

We say that a subset X of $V(G) \cup E(G)$ *separates* sets $A, B \subseteq V(G)$ if every $A - B$ path in G contains a vertex or an edge from X. We say X *separates* G if $G \setminus X$ is disconnected, in other words, X separates some two vertices in G that are not in X. A *cutvertex* in G is a vertex that separates two other vertices of the same component in G. A *bridge* is an edge that is not contained in any cycle. A *separation* of a graph G is an unordered pair $\{A, B\}$ such that $A \cup B = V(G)$ and there is no edge between $A \setminus B$ and $B \setminus A$. If

$A \setminus B \neq \emptyset$ and $B \setminus A \neq \emptyset$, then $\{A, B\}$ is a *proper separation*. The order of the separation $\{A, B\}$ is $|A \cap B|$, and $\{A, B\}$ is called a *k-separation* if $|A \cap B| = k$. The set $A \cap B$ *separates* A from B. A graph G is *k-connected* for some $k \in \mathbb{N}$ if $|G| > k$ and every proper separation of G has order at least k. Notice that every non-empty graph is 0-connected, and the non-trivial connected graphs are 1-connected. The following theorem is a classical result of Menger [5].

Theorem 2.7. *[5] (Menger's theorem) Let G be a graph and $A, B \subseteq V(G)$. Then the minimum number of vertices separating A from B in G is equal to the maximum number of independent $A - B$ paths in G.*

If $e = uv$ is an edge of G, let G/e be the graph obtained from $G \setminus u \setminus v$ by adding a new vertex and connecting it to all vertices that were adjacent to u or v in G without creating parallel edges. This operation will be referred to as *edge contraction*. Notice that this definition is slightly different from the ordinary definition of edge contraction, under which parallel edges could be created. As a matter of fact, our edge contraction is exactly the simplification of the corresponding ordinary edge contraction. The following lemma is a result of the structure of 3-connected graphs, which can be found in [5].

Lemma 2.8. *[5] If G is 3-connected and $G \neq K_4$ then there is an edge e in G such that G/e is 3-connected.*

2.3 Directed Graphs, Mixed Graphs, and Composite Graphs

This section contains some new graph terminology that will be used throughout the dissertation. A *directed graph* is a pair $D = (V, A)$, where V is a finite set and $A \subseteq V \times V$ such that $(v, v) \notin A$ for all $v \in V$, and at most one of $(v_1, v_2), (v_2, v_1)$ is in A for all distinct $v_1, v_2 \in V$. Members of V are *vertices* and members of A are *directed edges*, which are also called *arcs*. An arc (u, v) can be written as uv. If $a = uv$ is an arc, we will say that

a is *directed from* u to v. If the direction is irrelevant in the context, we use the similar terminology as graphs: *between, incident, and end*. The *underlying graph* of $D = (V, A)$ is the graph $G = (V, E)$ such that G has an edge between $u, v \in V$ if and only if D has an arc between u and v. We will call D an *orientation* of G. That is, we can think of D as a result of orienting edges of G. A directed graph D is *connected* if its underlying graph is connected.

A *mixed graph* is a triple $M = (V, E, A)$ such that (V, E) is a graph, (V, A) is a directed graph, and there is at most one (directed or undirected) edge between any two vertices. Equivalently, a mixed graph M is obtained from a simple graph G by orienting some of its edge. We will call G the *underlying graph* of M. An edge of M with ends u, v will be denoted by uv, where we assume implicitly that if the edge is directed then it is directed from u to v. We call a mixed graph $M' = (V', E', A')$ a *subgraph* of M, denoted $M' \subseteq M$, if $V' \subseteq V$, $E' \subseteq E$, and $A' \subseteq A$. Notice that M' is obtained from the subgraph $G' = (V', E')$ of G by inheriting the orientation from A. If $M' \subseteq M$ and $M' \neq M$, then M' is a *proper subgraph*. A mixed graph M is *connected* if its underlying graph is connected.

A *composite graph* is a pair $C = (G, D)$, where G is a graph and D is an orientation of a subgraph of G. The way to think of a composite graph is to consider it as a graph G together with an extra structure $D = (U, A)$, where A declares a direction on some edges of G and U declares some vertices of G as *special*. Therefore, we can equivalently define C as a pair (M, U), where M is a mixed graph and U is a set of vertices that contains all ends of all directed edges of M. For convenience, we will use both (G, D) and (M, U) to represent a composite graph $C = (G, D)$. We call a composite graph $C' = (M', U')$ a subgraph of C if $M' \subseteq M$ and $U' \subseteq U$. Let $\bar{\mathscr{G}}$ denote the class of all composite graphs for which the underlying graph is in $\mathscr{G}$.

Let $C = (M, U)$ be a composite graph with $M = (V, E, A)$. For any distinct $u, v \in V$, let $C + uv$ be the composite graph obtained as follows. If C has a directed edge from u

to v, then $C + uv := C$; else $C + uv := ((V, E \cup \{uv\}, A), U \cup \{u, v\})$. We call the triple (C, u, v) a *rooted composite graph*.

Let Q be a set and let $C = (G, D)$ be a composite graph with $D = (U, A)$. A *Q-labeling* of C is a mapping $g : U \cup A \to Q$. For any class $\mathscr{C}$ of composite graphs, let $\mathscr{C}(Q) = \{(C, g) : C \in \mathscr{C}$ and g is a Q-labeling of $C\}$. We call (C, g) a *Q-labeled composite graph* and (C, u, v, g) a *Q-labeled rooted composite graph*.

2.4 Induced Minor Relation

In this section, we introduce the definition of the induced minor relation. If u is a vertex of G, as usual, let $G \setminus u$ be the graph obtained from G by deleting u and all edges incident with u, and this operation will be referred to as *vertex deletion*. Recall from Section 2.2 the edge contraction is the result of an ordinary contraction plus a simplification.

A graph H is an *induced minor* of a graph G if H is obtained from G by repeatedly applying a vertex deletion or an edge contraction. Because the order in which the sequence of vertex deletions and edge contractions does not affect the resulting graph, we can think of H can be obtained from an induced subgraph G' of G by contracting edges. For convenience, we say *G' is contracted to H*. Assume that a vertex x of H is the result of contracting some edges of G. Then these edges form a connected induced subgraph X of G, and for distinct vertices these induced subgraphs are disjoint. We say *X is contracted to x*. We say that H is a *proper induced minor* of G if H is an induced minor of G but $H \neq G$.

We call G *H-$free$* if H is not an induced minor of G. For a set $\mathscr{H}$ of graphs, G is called *$\mathscr{H}$-$free$* if G is *H-$free$* for all $H \in \mathscr{H}$. We say that a subclass $\mathscr{X}$ of a class of graphs $\mathscr{G}$ is a *closed* subclass of $\mathscr{G}$ if for every induced minor of any $G \in \mathscr{X}$, if it is in $\mathscr{G}$, then it is in $\mathscr{X}$.

The concept of induced minor can be naturally extended to mixed graphs and composite graphs. Let $M = (V, E, A)$ be a mixed graph. For any vertex v, let $M \setminus v$ be the mixed graph obtained from M by deleting v from its vertex set and also deleting edges incident with v from $E \cup A$. For any edge uv of M, let M/uv be a mixed graph obtained from M as follows:

(1) delete uv from $E \cup A$;

(2) identify u with v, and let w be the new vertex;

(3) for each $z \in V \setminus \{u, v\}$, if there are two edges between z and w, delete exactly one of them.

Notice that step (3) could produce different mixed graphs since different edges could be deleted. Therefore, notation M/uv presents any one of these mixed graphs. Finally, an *induced minor* of M is a mixed graph obtained from M by repeatedly applying:

(1) a vertex deletion;

(2) an edge contraction;

(3) an arc unmarked operation, which removes an arc from A (turning a directed edge into an undirected edge).

Let $C = (M, U)$ be a composite graph. For any vertex v of C, let $C \setminus v = (M \setminus v, U \setminus v)$. We call a composite graph $C' = (M', U')$ an *induced subgraph* of C if $M' = M \setminus X$ and $U' = U \setminus X$ for some subset X of $V(G)$, where G is the underlying graph of M. For any edge uv of C, let $C/uv = (M/uv, U/uv)$, where $U/uv = (U \setminus \{u, v\}) \cup \{w\}$ if $U \cap \{u, v\} \neq \emptyset$, or $U/uv = U$ if otherwise. We remark that w is the new vertex of M/uv. Finally, an *induced minor* of C is a composite graph obtained from C by repeatedly deleting vertices, contracting edges, and unmarking arcs. A *closed subclass* $\mathscr{X}$ of a class $\mathscr{G}$ of composite graphs is a subclass of $\mathscr{G}$ such that if for every $C = (G, D) \in \mathscr{X}$, all induced minors of C are in $\mathscr{X}$, and every composite graph $C' = (G, D')$ is also in $\mathscr{X}$ as long as D' is a subgraph of D.

From the definitions of a vertex deletion and an edge contraction, the order in which a sequence of vertex deletions and edge contractions does not affect the resulting graph. Equivalently, a composite graph $C_1 = (M_1, U_1)$ is an induced minor of a composite graph $C_2 = (M_2, U_2)$ if there is a map f with domain $V_1 \cup E_1 \cup A_1$ satisfying:

(i) for every $v \in V_1$, $f(v)$ is a connected induced subgraph of G_2 (the underlying graph of M_2); and if $v \in U_1$, $f(v) \cap U_2 \neq \emptyset$;

(ii) for any distinct $u, v \in V_1$, $f(u) \cap f(v) = \emptyset$;

(iii) for any distinct $u, v \in V_1$, there is an edge in M_1 between u and v if and only if there is an edge in M_2 between a vertex in $V(f(u))$ and a vertex in $V(f(v))$;

(iv) for each $e \in E_1$, $f(e)$ is an edge in E_2; and if $e \in A_1$ directed from u to v, $f(e)$ is an arc in A_2 directed from a vertex in $V(f(u))$ to a vertex in $V(f(v))$.

We say C_1 is a *proper induced minor* of C_2 if C_1 is an induced minor of C_2 but $C_1 \neq C_2$. Then (C_1, u_1, v_1) is an induced minor of (C_2, u_2, v_2) if C_1 is an induced minor of C_2 such that $f(u_2) = u_1$ and $f(v_2) = v_1$.

Next, we define the notations for the induced minor relation on a class of Q-labeled rooted composite graphs. Suppose that $(Q, \leqslant)$ is quasi-ordering. For any two labeled composite graphs $(C_1, g_1), (C_2, g_2)$, we define $(C_1, g_1) \preceq (C_2, g_2)$ if C_1 is an induced minor of C_2 and we require the induced minor relation to respect the labels. When an edge is contracted, the label of the new vertex is the one of label of the two old vertices. For the label of the arcs incident to the new vertex is the one of label of the arcs in parallel (before simplification). Equivalently, $(C_1, g_1) \preceq (C_2, g_2)$ if there is a map f with domain

$V_1 \cup E_1 \cup A_1$ satisfying (i)-(iv) in the definition of the induced minor of composite graphs and the following conditions:

(i) for each $x \in U_1$, there is $y \in f^{-1}(x)$ such that $g_1(x) \leq g_2(y)$,

(ii) for each $e \in A_1$, $g_1(e) \leq g_2(f^{-1}(e))$.

We say (C_1, g_1) is a proper induced minor of (C_2, g_2), written $(C_1, g_1) \prec (C_2, g_2)$, if $(C_1, g_1) \preceq (C_2, g_2)$ and $(C_1, g_1) \neq (C_2, g_2)$. We define ϵ as a special element such that for any quasi-ordering $(Q, \leqslant)$, $\epsilon \leq q$ for all $q \in Q$.

When Robertson and Seymour proved Wagner's Conjecture, they proved something stronger. In fact they proved the minor relation on a class of directed graphs with the label on the vertices or edges. We first introduce the definition of the minor relation. If u is an edge of a graph G, as usual, let $G \setminus e$ be the graph obtained from G by deleting e, and this operation will be called *edge deletion*. A graph H is a *minor* of a graph G if H is obtained from G by repeatedly applying a vertex deletion, an edge deletion, and an edge contraction. Equivalently, H can be obtained from a subgraph of G by contracting edges. From Section 2.3, we can think of a directed graph $D = (V, A)$ as a composite graph such that every edge in the underlying graph is declared a direction by A and every vertex in V is special. Let Q be a set. Then a Q-labeling of D is a mapping $g : V(D) \cup A \to Q$. We call (D, g) a *Q-labeled directed graph*, and we can also denote the class of such graphs as $\mathscr{C}(Q)$. Suppose that $(Q, \leqslant)$ be a quasi-ordering. Then a Q-labeled directed graph D_1 is a minor of a Q-labeled directed graph D_2 if there is a map η with domain $V(D_1) \cup A_1$, satisfying:

- for each $v \in V(D_1)$, $\eta(v)$ is a connected subgraph of D_2, and there exists $w \in V(\eta(v))$ with $g_1(v) \leqslant g_2(w)$; and $\eta(v) \cap \eta(v') = \emptyset$ for all distinct $v, v' \in V(D_1)$;

- for each $e \in A_1$ directed from u to v, $\eta(e)$ is an arc of A_2 with $g_1(e) \leqslant g_2(\eta(e))$ directed from a vertex in $V(\eta(u))$ to a vertex in $V(\eta(v))$.

From the existence of the map η, it follows that there is a subgraph D_2' of D_2 corresponding to D_1 such that each vertex v in D_1 corresponds to a connected subgraph $\eta(v)$ of D_2' and each edge in D_1 corresponds to an edge in D_2'. By contracting every edge in each $\eta(v)$, the resulted graph is isomorphic to D_1. So D_1 is isomorphic to a minor of D_2 respecting the directions and labels. The following is a simplified version of the Robertson and Seymour result since they allow multiple edges and loops which we do not allow.

Theorem 2.9. *The class $\mathscr{C}(Q)$ of labeled directed graphs is well-quasi-ordered by the minor relation if $(Q, \leqslant)$ is a wqo.*

A *labeled composite clique* is a labeled composite graph such that its underlying graph is a clique. Let $\bar{\mathscr{K}}(Q)$ denote the class of labeled composite cliques.

Corollary 2.10. *$(\bar{\mathscr{K}}(Q), \preceq)$ is a wqo if $(Q, \leqslant)$ is.*

Proof. Let $(C_1, g_1), (C_2, g_2), \ldots$ be an infinite sequence in $\bar{\mathscr{K}}(Q)$, where $C_i = (G_i, D_i)$ with the orientation $D_i = (U_i, A_i)$. For all $i = 1, 2, \ldots$, let $D_i' = (U_i', A_i')$ be a directed graph obtained from C_i by declaring a direction on every undirected edge in $E(G_i$ and turning every vertex in $V(G_i) \setminus U_i$ be special. Then $U_i' = V(G_i)$ and A_i' consists of all new arcs and all arcs in A_i. Let $Q' = Q \cup \{\epsilon\}$. Then $(Q', \leqslant)$ is still wqo. Let (D_i', g_i') be the Q-labeled directed graph obtained by defining $g_i'(x) = g_i(x)$ if $x \in U_i \cup A_i$ or $g_i'(x) = \epsilon$ if otherwise. By Theorem 2.9, there exist $1 \leq i \leq j$ such that (D_i', g_i') is a minor of (D_j', g_j'). Since the underlying graph of these two composite graphs are cliques, for any distinct $u, v \in U_i'$, there is an edge in A_i' between u and v if and only if there is an edge in A_j' between a vertex in $V(\eta(u))$ and a vertex in $V(\eta(v))$. So (D_i', g_i') is an induced minor of (D_j', g_j'). By using the arc unmarked operation, we have that (C_i, g_i) is an induced minor of (C_j, g_j). Hence, $(\bar{\mathscr{K}}(Q), \preceq)$ is a wqo. $\square$

Chapter 3
$\{W_4, K_5\backslash e\}$-Free Graphs

In this chapter, we study the structure of a $\{W_4, K_5\backslash e\}$-free graph. Let $\mathscr{W}$ be the class of such graphs. We first introduce the sum operation of graphs, 0-, 1-, 2-sums. Then we show that a graph in $\mathscr{W}$ can be constructed from cliques by repeatedly applying 0-, 1-, and 2-sums with specific conditions on 2-sum.

3.1 Sums of Graphs

The *0-sum* is an operation to combining two graphs by disjoint union them to produce a new graph, which is called a *0-sum*. Note that 0-sum is an operation and a result of this operation. Every graph G can be constructed via 0-sums starting from connected graphs. These connected graphs are precisely connected components of G.

A clarification. Suppose O is an operation that produces a graph for any pair of input graphs (for example, O could be 0-sum). Let $\mathcal{G}_0$ be a class of graphs. When we say "a graph G can be constructed via operation O starting from graphs in $\mathcal{G}_0$", we means that G can be constructed from graphs in $\mathcal{G}_0$ by repeatedly applying operation O. To be more precise, for each positive integer i, let $\mathcal{G}_i$ be the union of $\mathcal{G}_{i-1}$ and the class of graphs obtained by applying O to all possible pairs of graphs from $\mathcal{G}_{i-1}$. Let $\mathcal{G}_\infty$ be the union of $\mathcal{G}_i$ over all integers $i \geq 0$. So when we say "G can be constructed from graphs in $\mathcal{G}_0$ by repeatedly applying operation O" we mean G belongs to $\mathcal{G}_\infty$.

The *1-sum* is an operation to combining two graphs by identifying a vertex of one graph with a vertex of the other graph to produce a new graph, which is called a 1-sum. Every connected graph G of order ≥ 2 is the 1-sum of its *blocks* (maximal 2-connected subgraphs or bridges). Let $B(G)$ be the *block graph* of G which is a bipartite graph on $A \cup \mathcal{B}$ where A

is the set of *cutvertices* of G, $\mathcal{B}$ is the set of its blocks, and $a \in A$ and $B \in \mathcal{B}$ are adjacent if $a \in V(B)$. By the maximality of blocks, we obtain the following result, which is known as *block-tree theorem* [5].

Lemma 3.1. *[5] The block graph of a connected graph is a tree.*

The *2-sum* is an operation to combining two graphs by identifying an edge, where the common edge may or may not be deleted. If the common edge is not deleted, then the result is called a 2^{I}*-sum*; otherwise, if the common edge is deleted, then the result is called a 2^{II}*-sum*. The 2-sum depends on if the common edge is deleted and how the ends of the identified edges are paired. So there are four possible different results when 2-sum two graphs.

Notice that the resulting graph of each sum is not unique; 1-sum depends on how the two vertices are chosen; 2-sum depends on how the two edges are chosen, as well as how the two edges are identified, and if the identified edge is deleted.

Lemma 3.2. *Every 2-connected G can be constructed via 2-sums starting from K_3 and 3-connected graphs.*

Proof. To prove this, we first show that if a 2-connected graph H has a proper 2-separation then H is a 2-sum of two smaller 2-connected graphs. Let $\{A, B\}$ be a 2-proper separation of H. Then $A \cup B = V(H)$, $A \setminus B \neq \emptyset$, $B \setminus A \neq \emptyset$, and $A \cap B = \{u, v\}$ for some $u, v \in V(H)$. Let e be an edge joining u and v, which may or may not be an edge of H. Let H_A and H_B be induced subgraphs of $H^+ = (V(H), E(H) \cup \{e\})$ on A and B, respectively, which both are smaller than H. So H is either a 2^{I}-sum or a 2^{II}-sum of H_A and H_B performing over e. If H_A has a proper 0- or 1-separation then we may replace e with H_B to obtain a proper 0- or 1-separation of H, which contradicts with the 2-connectivity of H. By the same argument, H_A and H_B are 2-connected. Suppose that G is 2-connected but neither 3-connected nor K_3. Then G has a 2-separation $\{A, B\}$. By the previous statement, G is a 2-sum of 2-connected graphs G_A and G_B. By induction, G can be constructed a 2-sum of K_3 or 3-connected graphs. □

Let k be a nonnegative integer, and let G_1 and G_2 be vertex disjoint graphs. A graph G is a k-sum of G_1 and G_2, $G = G_1 \oplus_k G_2$, means G is obtained from G_1 and G_2 by identifying a complete subgraph of G_1 on k vertices with a complete subgraph of G_2 on k vertices and deleting a (possibly empty) set of identified edges. Then 0-, 1-, and 2- sums are the cases when $k = 0, 1$, and 2, respectively. Next, we show that when we talk about constructing graphs by repeatedly k-summing, where k is a nonnegative integer, the order of performing the operations do not affect the result.

Proposition 3.3. *Suppose $G = G_1 \oplus_{k_1} (H_1 \oplus_{k_2} H_2)$. Then $G = (G_1 \oplus_{k_1} H_i) \oplus_{k_2} H_{3-i}$, for some $i \in \{1, 2\}$.*

Proof. For $i = 1, 2$, let C_i be the complete subgraph of H_i over which the k_2-sum took place. Let C be the complete subgraph of $H_1 \oplus_{k_2} H_2$ over which the k_1-sum took place. Then edges of C consist of some identified edges and some edges from only one of H_1 and H_2. Then the result follows. □

3.2 0-, 1-, 2-Sums of Cliques

Let $\mathscr{S}$ be the class of graphs constructed from cliques by repeatedly applying 0-, 1-, and 2-sums with the condition that a 2^{II}-sum is performed over an edge e only when every clique containing e has size 3 or 4. We will prove the following statement.

Theorem 3.4. $\mathscr{W} = \mathscr{S}$

To do so, we need the following results.

Lemma 3.5. *Let H be a connected graph, and let G be a disjoint union of graphs A and B. Then H is an induced minor of G if and only if H is an induced minor of A or B.*

Proof. ($\Leftarrow$) Suppose that H is an induced minor of A. Since deleting $V(B)$ makes A as induced minor of G, H is an induced minor of G. ($\Rightarrow$) Suppose not, there are subgraphs A' of A and B' of B that are contracted to two different vertices of H, a and b, respectively. Since H is connected, H has a path P_{ab} from a to b. So there is a path P in G from A' to B' that is contracted to P_{ab}, contradicting to the fact that there is no path from A to B. □

Lemma 3.6. *Let H be a 2-connected graph, and let G be a 1-sum of graphs A and B. Then H is an induced minor of G if and only if H is an induced minor of A or B.*

Proof. Let x be the common vertex of A and B over which the 1-sum is performed. ($\Leftarrow$) Suppose that H is an induced minor of A. Since deleting $V(B) - \{x\}$ makes A an induced minor of G, H is an induced minor of G. ($\Rightarrow$) Suppose on the contrary that H is not an induced minor of either A or B. Since H is an induced minor of G, there are induced connected subgraphs A' of $A \setminus x$ and B' of $B \setminus x$ that are contracted to two different vertices of H, a and b, respectively. Since $V(A) \cap V(B) = \{x\}$, every path from A' to B' contains x. So G does not have two independent paths between A' and B', and thus H does not have two independent paths between a and b since H is obtained from an induced subgraph of G after contraction. By Menger's theorem, this contradicts with the fact that H is 2-connected. □

Lemma 3.7. *If G is a 2-sum of 2-connected graphs A and B, then A and B are induced minors of G.*

Proof. Let $e = uv$ be the common edge of A and B over which the 2-sum is performed. So e may or may not be in G. Since B is 2-connected, B has a $u - v$ path P not containing e. Then P is also a path in G. By deleting vertices of G in $B \setminus P$ and contracting all but one edges of P, we obtain A an induced minor of G. □

Lemma 3.8. *Let H be a 3-connected graph. The following statements are true.*

(i) If G is a 2^I-sum of 2-connected graphs A and B over the common edge $e = uv$, and H is an induced minor of G, then H is an induced minor of A or B.

(ii) If G is a 2^{II}-sum of 2-connected graphs A and B over the common edge $e = uv$, and H is an induced minor of G, then H is an induced minor of A or B or $A \setminus e$ or $B \setminus e$.

Proof. Part (i). Suppose on the contrary that H is not an induced minor of either A or B. We consider the following cases.

Case 1. There are subgraphs A' of $A \setminus \{u, v\}$ and B' of $B \setminus \{u, v\}$ that are contracted to two different vertices of H, a and b, respectively. Since $V(A) \cap V(B) = \{u, v\}$, every path between A' and B' contains u or v. So G has at most two independent paths between A' and B', and H also has at most two independent paths between a and b since H is obtained from an induced subgraph of G after contraction. By Menger's theorem, this contradicts with the fact that H is 3-connected.

Case 2. If there are no such A' and B' as described in Case 1, so by symmetry we may assume that for each connected subgraph C of G contracted to a vertex of H, $C \cap A \neq \emptyset$. There are five cases: neither u nor v is contained in any C; u is contained in some C but v is not contained in any C; u is not contained in any C but v is contained in some C; u and v are contained in the same C; u and v are contained in different subgraphs, C_1 and C_2, respectively. In the first case, since H is an induced minor of G but not an induced minor of either A or B, this leads to Case 1. In the second case, since $C \setminus V(B \setminus \{u, v\})$ is a connected subgraph of A, there is an induced subgraph of A contracted to H. So H is an induced minor of A, contradiction. In the third case, we can use the same argument as the second case to obtain a contradiction. In the fourth case, since we can replace a uv-path in G by e, $C \setminus V(B \setminus \{u, v\})$ is a connected subgraph of A; so H is an induced minor of A, contradiction. In the last case, since $e \in E(G)$, C_1 and C_2 are adjacent, and so do $C_1 \setminus V(B \setminus \{u, v\})$ and $C_2 \setminus V(B \setminus \{u, v\})$, which are connected subgraphs of A. Thus H is an induced minor of A, contradiction.

Conclusion (ii) can be proved by the same argument as (i), except the last part in case 2 because $e \notin E(G)$. If C_1 and C_2 are adjacent in G, then $C_1 \setminus V(B \setminus \{u, v\})$ and $C_2 \setminus V(B \setminus \{u, v\})$ are adjacent in A by e; so H is an induced minor of A, contradiction. Otherwise, $C_1 \setminus V(B \setminus \{u, v\})$ and $C_2 \setminus V(B \setminus \{u, v\})$ are not adjacent in $A \setminus e$; so H is an induced minor of $A \setminus e$, contradiction. □

Remark that Lemma 3.8(i) is not true if G is a 2^{II}-sum of 2-connected graphs. For example, G is a 2^{II}-sum of two K_5's, then $K_5 \backslash e$ is an induced minor of G but not an induced minor of K_5.

We now prove the main result of this chapter.

Prove of Theorem 3.4. First, we show that $\mathscr{S} \subseteq \mathscr{W}$. Let $G \in \mathscr{S}$. Then G is constructed by 0, 1, or 2-sums of cliques. Suppose on the contrary that W_4 or $K_5 \backslash e$ is an induced minor of G. Note that W_4 and $K_5 \backslash e$ are 3-connected. From Proposition 3.3, we know that the order of the 0- and 1-sums to construct G is irrelevant. So G is the 0-sum of graphs, that are cliques or 2-sum of cliques. By Lemmas 3.5, 3.6, and 3.8(i), W_4 or $K_5 \backslash e$ is an induced minor of some K_n or a graph that is a 2^{II}-sum of copies of K_3 and K_4. Since deleting a vertex or contracting an edge of K_n gives a clique K_{n-1}, all induced minors of K_n are cliques. By Lemma 3.8(ii), K_4 is the only 3-connected induced minor of a 2^{II}-sum of copies of K_3 and K_4. So W_4 and $K_5 \backslash e$ are not induced minors in both cases. Hence, G is $\{W_4, K_5 \backslash e\}$-free graph, and $\mathscr{S} \subseteq \mathscr{W}$.

Next, we show that $\mathscr{W} \subseteq \mathscr{S}$. We first show that if G is a 3-connected graph in $\mathscr{W}$, then $G = K_n$ for some $n \in \mathbb{N}$. Equivalently, if $G \neq K_n$ for any $n \in \mathbb{N}$, then G has W_4 or $K_5 \backslash e$ as an induced minor. We will prove this statement by induction on $|G| = n$. At $|G| = 5$, $G \neq K_5$. Since G is 3-connected, each vertex in G has degree at least 3 and $|E(G)| \geq 8$.

So G is either $K_5\backslash e$ or W_4. Suppose that this statement is true for $|G| = n-1$. We will show that it is true for $|G| = n$. Suppose that $G \neq K_n$. From Lemma 2.8, we have that there is an edge e in G such that G/e is 3-connected. If $G/e \neq K_{n-1}$, then we are done by the induction hypothesis. Suppose that $G/e = K_{n-1}$. Let u and v be incident vertices of e. Let $N(u)$ and $N(v)$ denote the set of vertices that are adjacent to u and v, respectively. Then $G \setminus \{u, v\} = K_{n-2}$.

Case 1. $(N(u) - \{v\}) \cap (N(v) - \{u\}) = \emptyset$. Since G is 3-connected, there are $w, x \in N(u) - \{v\}$ and $y, z \in N(v) - \{u\}$. By deleting all vertices in $V(G) - \{u, v, w, x, y, z\}$ and contracting xy, we obtain W_4, see Figure 3.1.

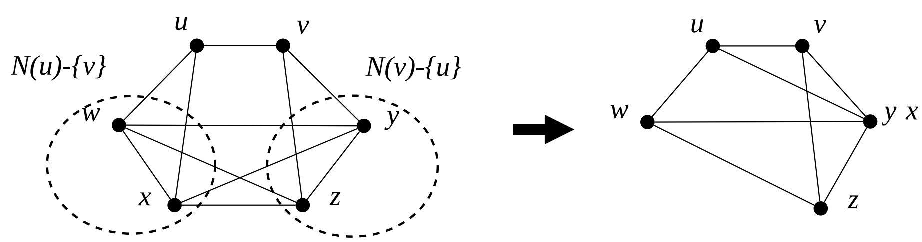


Figure 3.1: $(N(u) - \{v\}) \cap (N(v) - \{u\}) = \emptyset$.

Case 2. $(N(u) - \{v\}) \cap (N(v) - \{u\}) \neq \emptyset$. Let $x \in (N(u) - \{v\}) \cap (N(v) - \{u\})$. If there are $w \in N(u) - N(v) - \{v\}$ and $y \in N(v) - N(u) - \{u\}$. By deleting all vertices in $V(G) - \{u, v, w, x, y\}$, we obtain W_4, see Figure 3.2. If either $N(u) - N(v) - \{v\}$ or $N(v) - N(u) - \{u\}$ is not empty, we suppose that there is $z \in N(v) - N(u) - \{u\}$. Then v is adjacent to all vertices in $V(G)$. Since G is 3-connected, there are $w, x \in N(u) - \{v\}$. By deleting all vertices in $V(G) - \{u, v, w, x, z\}$, we obtain $K_5\backslash e$, see Figure 3.3.

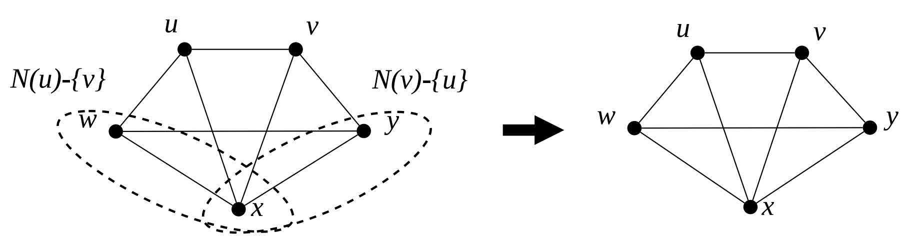


Figure 3.2: $N(u) - N(v) - \{v\}$ and $N(v) - N(u) - \{u\}$ are not empty.

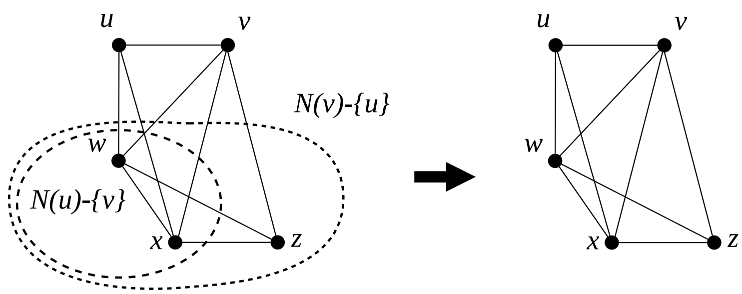


Figure 3.3: Either $N(u) - N(v) - \{v\}$ or $N(v) - N(u) - \{u\}$ is empty.

Hence, all 3-connected graphs in $\mathscr{W}$ are in $\mathscr{S}$. Now, we consider a graph $G \in \mathscr{W}$ which is 2-connected but not 3-connected. We will prove that $G \in \mathscr{S}$, by doing the induction on $|G|$. Since the smallest 2-connected but not 3-connected graph is K_3, which is in $\mathscr{W}$, the statement is true when $|G| = 3$. Suppose that the statement is true when $|G| = n-1$. For $|G| = n$, since G is 2-connected but not 3-connected, there are 2-connected graphs A and B such that G is a 2-sum of these two graphs on the common edge e. By Lemma 3.7, A and B are in $\mathscr{W}$. By the induction hypothesis, A and B are in $\mathscr{S}$. If the 2-sum between A and B to construct G is a 2^{II}-sum, and e is contained in a clique K with order greater than 4 in the constructions of A or B, then $K \backslash e$ contains $K_5 \backslash e$ as an induced minor. Since G contains $K \backslash e$ as an induced minor, G contains $K_5 \backslash e$ as an induced minor, contradiction. Thus either the 2-sum is a 2^I-sum or all cliques that contain e have order least than 5. So $G \in \mathscr{S}$. For a graph G which is 1-connected but not 2-connected, we use the same argument together with Lemma 3.6, except we start the induction at $n = 2$, and obtain the result. For a graph G which is 0-connected (every graph), it consists of components which are 1-connected and in $\mathscr{S}$. So $G \in \mathscr{S}$. Hence, $\mathscr{W} \subseteq \mathscr{S}$. $\square$

Chapter 4
Infinite Antichain

In this chapter, we prove that $\mathscr{D}^{\Gamma}$ is an antichain in $\mathscr{W}$, which implies that $\mathscr{W}$ is not well-quasi-ordered by the induced minor relation. To prove this result, we consider the class of composite graphs with underlying graphs in $\mathscr{D}^{\Gamma}$ and some specific graphs, which will be explained later. We show that this class of composite graphs is an antichain in $\bar{\mathscr{W}}$.

We define D_n for each $n \in \mathbb{N}$ to be a graph such that $V(D_n) = \{x_0, \ldots, x_n, y_0, \ldots, y_n\}$ and $E(D_n) = \{x_0x_1, x_0y_1, y_0x_1, y_0y_1, \ldots, x_{n-1}x_n, x_{n-1}y_n, y_{n-1}x_n, y_{n-1}y_n\}$, see Figure 4.1. Notice that D_n can be constructed from K_3's and K_4's by repeatedly applying 2^{II}-sum. For $m < n$, D_m is an induced minor of D_n by deleting vertices $x_{m+1}, \ldots, x_n, y_{m+1}, \ldots, y_n$.

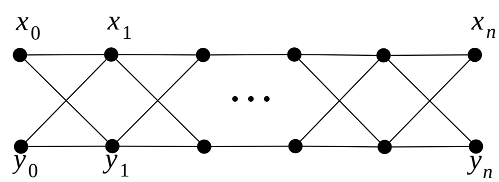


Figure 4.1: Graph D_n

Let Γ_i, $i = 1, \ldots, 4$, be a graph illustrated in Figure 4.2, and let Γ^+ be the class of such graphs.

Figure 4.2: Graphs Γ_1, Γ_2, Γ_3, Γ_4

We define $D_{n,p,q}$, where $n \in \mathbb{N}$ and $p, q \in \Gamma^+$, to be a graph obtained from D_n, p, and q by identifying a pair $\{a, b\}$ of p to a pair $\{x_0, y_0\}$ of D_n and identifying a pair $\{a, b\}$ of q to a pair $\{x_n, y_n\}$ of D_n. A pair of degree-4 or degree-5 vertex in $D_{n,p,q}$ are called *twins* if they have the same set of neighbors. Notice that x_0 and y_0 are not twins if p is Γ_1 or Γ_4, which is the same as x_n and y_n. Let $\mathscr{D}^{\Gamma+} = \{D_{n,p,q} | n \in \mathbb{N} \text{ and } p, q \in \Gamma^+\}$. Notice that $D_{n,p,q}$ can be obtained from K_2's, K_3's, and K_4's by repeatedly applying 1- and 2^{II}-sums; D_n is a member in $\mathscr{D}^{\Gamma+}$ where p and q are Γ_4. By Theorem 3.4, $\mathscr{D}^{\Gamma+} \subseteq \mathscr{W}$.

Let $\bar{\mathscr{D}}^{\Gamma+}$ be the class of composite graphs with underlying graphs in $\mathscr{D}^{\Gamma+}$ such that for each composite graph $C = (D_{n,p,q}, S)$, its orientation $S = (U, A)$ consists of $A = \emptyset$ and $U \in \{\{u\}, \{v\}, \{u, v\}\}$, where $u \in \{x_0, y_0\}$ and $v \in \{x_n, y_n\}$, if p or q is Γ_4; otherwise $U = \emptyset$, see Figure 4.3. Then $\mathscr{D}^{\Gamma} \subseteq \bar{\mathscr{D}}^{\Gamma+}$ and $\bar{\mathscr{D}}^{\Gamma+} \subseteq \bar{\mathscr{W}}$.

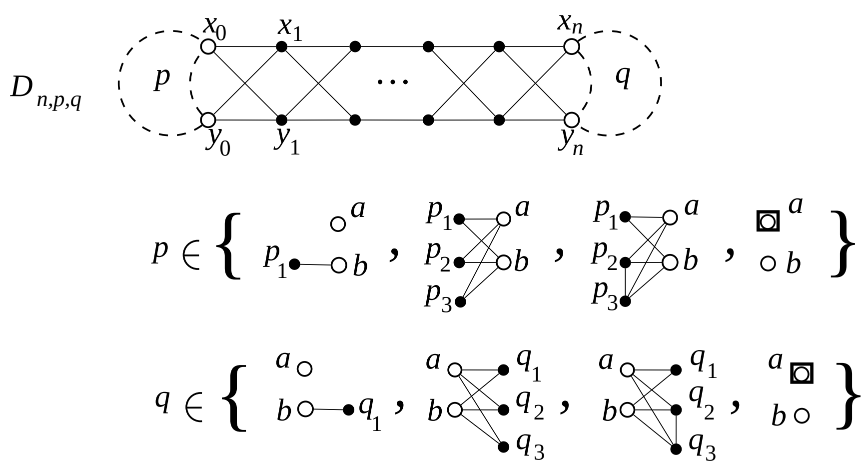


Figure 4.3: A composite graph $C = (D_{n,p,q}, S)$, where a vertex in a box is special.

We show that $\bar{\mathscr{D}}^{\Gamma+}$ is an antichain in $\bar{\mathscr{W}}$. To prove this result, we need the following lemma.

Lemma 4.1. *For all $m, n \geq 3$, if $(D_{m,p,q}, S^1)$ is an induced minor of $(D_{n,p,q}, S^2)$, then $(D_{m,p,q}, S^1) = (D_{n,p,q}, S^2)$.*

Proof. Suppose that $(D_{m,p,q}, S^1)$ is an induced minor of $(D_{n,p,q}, S^2)$. Then there is a map f from $D_{m,p,q}$ to $D_{n,p,q}$. We begin with the following claims.

Claim 1: If u is a vertex in $D_{m,p,q}$ with degree greater than 3, then $I_u = \{i | x_i$ or $y_i \in f(u)$ for some $0 \leq i \leq n\}$ is not empty. Suppose on the contrary that $I_u = \emptyset$. Then

$f(u)$ is contained in either p or q, say $f(u) \subseteq p$. Since $f(u)$ is connnected, p is Γ_1, Γ_2, or Γ_3, contradicting the fact that the degree of u is greater than 3.

Claim 2: If u and v are twins in $D_{m,p,q}$, then the following are true.

(2.1) $I_u \cap I_v \neq \emptyset$. Suppose otherwise that $I_u \cap I_v = \emptyset$. Notice that I_u and I_v are consecutive sets because $f(u)$ and $f(v)$ are connected. Then we assume that $\max I_u < \min I_v$. Since u and v are not adjacent, there is an index i_0 such that $\max I_u < i_0 < \min I_v$. So there are at most two independent $f(u) - f(v)$ paths in $D_{n,p,q}$, contradicting with the fact that $D_{m,p,q}$ has four independent $u - v$ paths.

(2.2) $I_u = I_v = \{i\}$ for some $0 \leq i \leq n$ because u and v are not adjacent.

(2.3) If u has degree four in $D_{m,p,q}$, then $I_u = \{i\}$ for some $0 < i < n$. If $f(u)$ contains x_0, then $f(v)$ contains y_0 because of (2.2). Since u has degree four, p in $D_{n,p,q}$ is Γ_2 or Γ_3. Then $f(u) = \{x_0\}$ and $f(u) = \{y_0\}$ because u and v are not adjacent. If p in $D_{n,p,q}$ is Γ_2, since u has degree four, $f(u)$ is adjacent to at least two degree-2 vertices in p, say p_1 and p_2, such that these two vertices are in two different connected subgraphs $f(u')$ and $f(v')$ of $D_{n,p,q}$ for some u' and v' in $D_{m,p,q}$. Then $f(u') = \{p_1\}$ and $f(u') = \{p_2\}$. So u' and v' are degree-2 vertices in $D_{m,p,q}$ that are not special. Since u has degree four, p and q in $D_{m,p,q}$ are Γ_1 or Γ_4. We may assume that $u' \in p$ and $v' \in q$. This implies that $m < 3$, contradiction. If p in $D_{n,p,q}$ is Γ_3, since u has degree four, $f(u)$ is adjacent to at least two vertices in p, either $\{p_1, p_2\}$ or $\{p_2, p_3\}$ such that these two vertices are in two different connected subgraphs $f(u')$ and $f(v')$ of $D_{n,p,q}$ for some u' and v' in $D_{m,p,q}$. In the first case, we can obtain a contradiction by using the same result as p in $D_{n,p,q}$ is Γ_2. In the second case, there is an edge between $f(u')$ and $f(v')$, contradicting with the fact that all neighbors of a degree-4 vertex in $D_{m,p,q}$ are pairwise nonadjacent.

(2.4) If u has degree four, then $f(u) = \{x_i\}$ or $\{y_i\}$ for some $0 < i < n$.

Claim 3: Let $u_1, \ldots, u_{m-1}, v_1, \ldots, v_{m-1}$ be degree-4 vertices in $D_{m,p,q}$ for all $1 \leq j \leq m-1$. We will show that $f(u_1) = \{x_1\}$ and $f(u_{m-1}) = \{x_{n-1}\}$. Suppose that $f(u_1) = \{x_i\}$

for some $i > 1$. By Claim 2, $f(v_1) = \{y_i\}$. Then $x_{i-1} \in f(u_0)$ and $y_{i-1} \in f(v_0)$. Notice that if $x_{i-2} \in f(u_0)$ or $y_{i-2} \in f(v_0)$, then $f(u_0)$ and $f(v_0)$ are adjacent, so are u_0 and v_0, contradiction. So we may assume $f(u_0) = \{x_{i-1}\}$ and $f(v_0) = \{y_{i-1}\}$, where $i - 1 \geq 1$. Since x_{i-1} and y_{i-1} have degree four in $D_{n,p,q}$, u_0 and v_0 cannot be degree-5 vertices in $D_{m,p,q}$. If u_0 or v_0 has degree three in $D_{m,p,q}$, see Γ_1, then $f(p_1)$ contains x_{i-2}. Thus, $f(p_1)$ is adjacent to $f(v_0)$. So p_1 is adjacent to v_0 and u_0, contradiction. If u_0 and v_0 have degree two in $D_{m,p,q}$, then one of these vertices is a special vertex in $D_{m,p,q}$, which contradicts with the fact that both x_{i-1} and y_{i-1} are not special vertices in $D_{n,p,q}$. So $f(u_1) = \{x_1\}$, $f(u_{m-1}) = \{x_{n-1}\}$, $f(v_1) = \{y_1\}$, and $f(v_{m-1}) = \{y_{n-1}\}$.

Hence, $(D_{m,p,q}, S^1) = (D_{n,p,q}, S^2)$. □

From Lemma 4.1, we obtain the following.

Lemma 4.2. *$\bar{\mathscr{D}}^{\Gamma+}$ is an antichain in $\bar{\mathscr{W}}$ with respect to the induced minor relation.*

For all $m, n \geq 3$, if $D_{m,p,q}$ is an induced minor of $D_{n,p,q}$ in $\mathscr{D}^{\Gamma}$, then $(D_{m,p,q}, S^1)$ is an induced minor of $(D_{n,p,q}, S^2)$ in $\bar{\mathscr{D}}^{\Gamma+}$. By Lemma 4.2, we obtain the following lemma.

Lemma 4.3. *$\mathscr{D}^{\Gamma}$ is an antichain in $\mathscr{W}$ with respect to the induced minor relation.*

This lemma implies the following.

Corollary 4.4. *$\mathscr{W}$ is not well-quasi-ordered by the induced minor relation.*

Chapter 5

2^{II}-sum of K_3 and K_4

We notice from the previous Chapter that the main part of a graph in $\mathscr{D}^{\Gamma}$ is constructed from K_3's and K_4's by repeatedly applying 2^{II}-sums. This chapter concerns with the class of graphs constructed by such method. Let $\mathscr{L}$ be the class of graphs that are 2^{II}-sums of copies of K_3 and K_4. Then $\bar{\mathscr{L}}$ consists of composite graphs whose underlying graphs are in $\mathscr{L}$.

5.1 Tails

In this section, we investigate a part of a graph in $\bar{\mathscr{L}}$, which is called *tail*. We can think of the tail as a part that is attached to the main body of the graph. The main part of the tail is constructed from K_4's by repeatedly applying 2^{II}-sums. However, the tail is not formed a graph in $\mathscr{D}^{\Gamma+}$.

Let $\mathscr{B}^r$ be the class of rooted composite graphs (B_n, x_0, y_0), where $B_n = (G_{B_n}, D_{B_n})$, illustrated in Figure 5.1, $\mathscr{B}^r = \{(B_n, x_0, y_0) | n \in \mathbb{N}\}$. Notice that in some B_n, x_n or y_n is a special vertex. We call a graph in $\mathscr{B}^r$ a *tail*, where n represents the *length of the tail.* We call a vertex in $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ an *inner vertex of the tail* and an edge in $\{x_1x_2, \ldots x_{n-1}x_n, y_1y_2, \ldots y_{n-1}y_n\}$ an *inner edge of the tail.* Notice that for any tail (B_n, x_0, y_0) in $\mathscr{B}^r$, where $B_n = (G_{B_n}, D_{B_n})$, the orientation D_{B_n} consists of $U_{B_n} \subseteq \{x_0, y_0, x_n, y_n\}$ and A_{B_n} is either an empty set or a set $\{x_ny_n\}$. Two tails in $\mathscr{B}^r$ have the *same type* if they have the same orientation. There are seven types as illustrated in Figure 5.1. By fixing roots x_0, y_0, for any tails (B_m, x_0, y_0) and (B_n, x_0, y_0) in $\mathscr{B}^r$ such that $m < n$ and both have the same type, (B_m, x_0, y_0) is an induced minor of (B_n, x_0, y_0). For type (a), we obtain the result by deleting all vertices in $\{x_{m+1}, \ldots, x_n, y_{m+1}, \ldots, y_n\}$. For types (b)

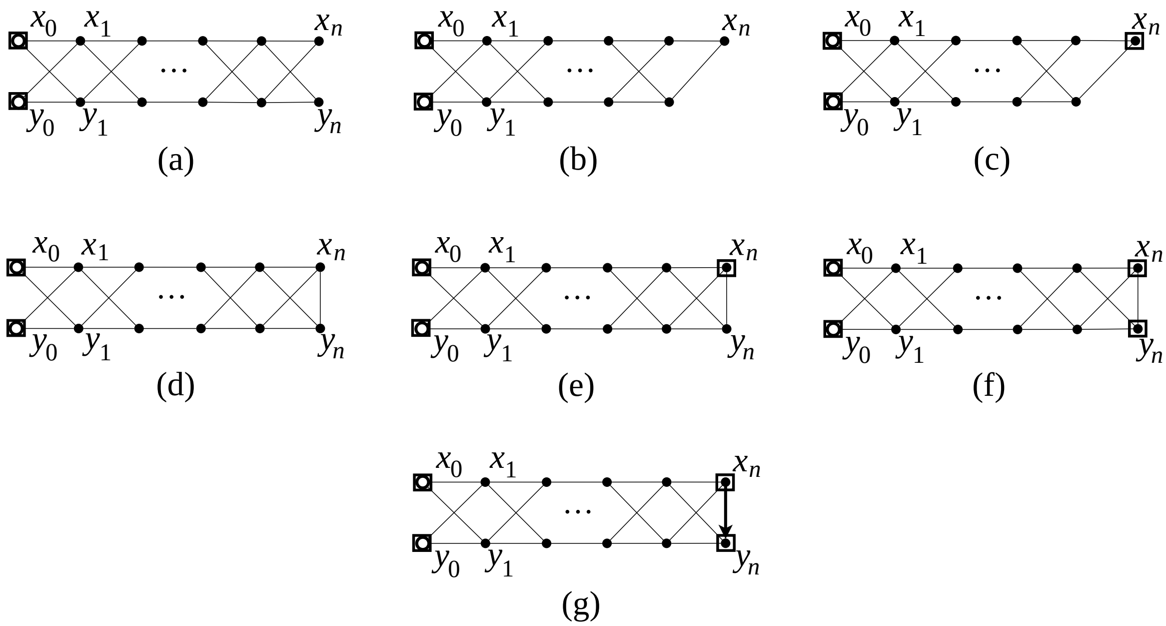


Figure 5.1: Graphs in $\mathscr{B}^r$ (tails)

and (c), we obtain the result by deleting all vertices in $\{y_{m+1}, \ldots, y_{n-1}\}$ and contracting all edges in $\{x_i x_{i+1} | m \leq i < n\}$. For types (d), (e), (f), and (g), we obtain the result by contracting all edges in $\{x_i x_{i+1} | m \leq i < n\} \cup \{y_i y_{i+1} | m \leq i < n\}$. This implies the following.

Lemma 5.1. *$\mathscr{B}^r$ is well-quasi-ordered by the induced minor relation.*

We also obtain a similar result for the labeling version.

Lemma 5.2. *If $(Q, \leq)$ is a wqo, then $\mathscr{B}^r(Q)$ is well-quasi-ordered by the induced minor relation.*

We call a connected composite induced subgraph $B = (G_B, D_B)$ of a composite graph $C = (G, D)$ a *tail of C* if

(i) there is a connected subgraph H of G such that $G = H \cup G_B$, $V(H) \cap V(G_B) = \{x_0, y_0\}$, and $E(H) \cap E(G_B) = \emptyset$, and

(ii) after making x_0 and y_0 to be roots of B, $(B, x_0, y_0) \in \mathscr{B}^r$.

5.2 Well-Quasi-Ordering of A Subclass of $\bar{\mathscr{L}}$

In this section, we study a subclass of $\bar{\mathscr{L}}$ that its members are 2-connected graphs with no long tail. We prove that this subclass is well-quasi-ordered by the induced minor relation.

Let $C = (G, D)$ be a composite graph in $\bar{\mathscr{L}}$. We define C^{cut} as an induced minor of C obtained by contracting all inner edges of every tail of C. Let $\bar{\mathscr{L}}^r = \{(C, u, v) : C + uv$ is 2-connected and belongs to $\bar{\mathscr{L}}\}$. Let K_C be a maximal subgraph of (C, u, v) such that

(i) $u, v \in K_C$, and

(ii) K_C was a clique K_i for some i which is in a construction of the underlying graph of C.

We call K_C a K_i*-structure*. For each $n \in \mathbb{N}$, we define D'_n in the same way as D_n, except $x_0y_0 \in E(D'_n)$, see Figure 5.2. For $k = 0, 1, 2, \ldots$, let $\bar{\mathscr{L}}^r_k$ be the class of graphs in $\bar{\mathscr{L}}^r$ such that (C^{cut}, u, v) of $C + uv$ does not contain (D_k, x_0, y_0) or (D'_k, x_0, y_0) as an induced minor. Our main result in this section is to prove that the labeled version of $\bar{\mathscr{L}}^r_k$ is well-quasi-ordered by the induced minor relation.

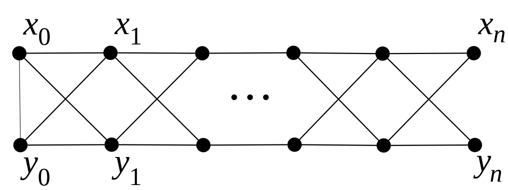


Figure 5.2: Graph D'_n

Theorem 5.3. *Let $(Q, \leq)$ be a well-quasi-ordering. For $k = 0, 1, 2, \ldots$, $\bar{\mathscr{L}}^r_k(Q)$ is well-quasi-ordered by the induced minor relation.*

To prove this theorem, we first find the important properties of a fundamental infinite antichain in $\bar{\mathscr{L}}^r_k$ by considering the adjacency of the roots.

5.2.1 Properties of a Fundamental Infinite Antichain in $\bar{\mathscr{L}}_k^r$ Part I

We consider a graph $(C, u, v) \in \bar{\mathscr{L}}_k^r$ such that $uv \notin E(G)$, where G is the underlying graph of C.

Lemma 5.4. *Let $(Q, \leq)$ be a well-quasi-ordering. If $\mathcal{A}$ is an infinite subset of $\bar{\mathscr{L}}_k^r(Q)$ such that for each $(C, u, v, g) \in \mathcal{A}$, there is a 2^{II}-sum performing over uv (u and v contain in at least two K_C's which have K_3 or K_4-structure), then $\mathcal{A}$ is not a fundamental infinite antichain.*

Proof. Let $\mathcal{A}$ be an infinite subset of $\bar{\mathscr{L}}_k^r(Q)$ satisfying the condition in the lemma. Suppose on the contrary that $\mathcal{A}$ is a fundamental infinite antichain. Let M_C denote the set of maximal connected subgraphs of (C, u, v, g) performing 2^{II}-sum over uv, that are made into labeled rooted composite graphs in $\bar{\mathscr{L}}_k^r(Q)$ by choosing u and v as their roots and inheriting label and orientation from (C, u, v, g). Let $M = \cup_{(C,u,v,g)\in\mathcal{A}} M_C$. Since for each C every graph in M_C is a proper induced minor of (C, u, v, g) by deleting all vertices which are not in the vertex set of that graph, we have that $M \subseteq \mathcal{A}^<$. Since A is fundamental, M is wqo. By Lemma 2.5, $[M]^{<\omega}$ is wqo. Then there is a good pair $(M_C, M_{C'})$. Let $m : M_C \to M_{C'}$ be an injection map such that $X \preceq m(X)$ for all $X \in M_C$. Then there is a map f_X from X to $m(X)$. We extend the union of these maps to a map f from $V \cup E \cup A$ to $V' \cup E' \cup A'$ by letting $f(u) = \cup_{X\in M_C} f_X(u)$ and $f(v) = \cup_{X\in M_C} f_X(v)$. This map f shows that $(C, u, v, g) \preceq (C', u', v', g')$. Hence $((C, u, v, g), (C', u', v', g'))$ is a good pair in the antichain $\mathcal{A}$, a contradiction. □

Lemma 5.5. *Let $(Q, \leq)$ be a well-quasi-ordering. If $\mathcal{A}$ is an infinite subset of $\bar{\mathscr{L}}_k^r(Q)$ such that for each $(C, u, v, g) \in \mathcal{A}$,*

(i) $uv \notin E(G)$, where G is the underlying graph of C,

(ii) there is only one K_C containing u and v, and K_C is a K_3-structure,

(iii) there is a clique performing 2-sum with K_C,

then $\mathcal{A}$ is not a fundamental infinite antichain.

Proof. Let $\mathcal{A}$ be an infinite subset of $\bar{\mathscr{L}}_k^r(Q)$ satisfying the condition in the lemma. Suppose on the contrary that $\mathcal{A}$ is a fundamental infinite antichain. For each $(C, u, v, g) \in \mathcal{A}$, let $V(K_C)$ consist of u, v, and w. By Lemma 2.2, we can consider the case that uw and wv are not in $E(G)$.

Let M_1 denote the set of maximal connected subgraphs $H_{1,C}$ of C performing 2^{II}-sum with K_C over edge uw, that are made into labeled rooted composite graphs $(H_{1,C}, u, w, g|_{H_{1,C}})$ in $\bar{\mathscr{L}}_k^r(Q)$ by choosing u and w as their roots and inheriting label and orientation from (C, u, v, g). By the condition of a graph in $\bar{\mathscr{L}}_k^r(Q)$, there is a maximal connected subgraphs $H_{2,C}$ of C performing 2^{II}-sum over wv. We define M_2 consisting of $(H_{2,C}, w, v, g|_{H_{2,C}})$'s made into labeled rooted composite graphs $(H_{2,C}, w, v, g|_{H_{2,C}})$ in $\bar{\mathscr{L}}_k^r(Q)$ by choosing w and v as their roots and inheriting label and orientation from (C, u, v, g) So for each $(C, u, v, g) \in \mathcal{A}$, we have that $(H_{1,C}, u, w, g|_{H_{1,C}}) \preceq (C, u, v, g)$ by contracting every edge in $H_{2,C}$; similarly, we have that $(H_{2,C}, w, v, g|_{H_{2,C}}) \preceq (C, u, v, g)$ by contracting every edge in $H_{1,C}$. By Lemma 2.4, M_1 and M_2 are wqo. Hence, $M_1 \times M_2$ are wqo (by Lemma 2.6). There is a good pair in a chain $B = \{((H_{1,C}, u, w, g|_{H_{1,C}}), (H_{2,C}, w, v, g|_{H_{2,C}})) \in M_1 \times M_2 : (C, u, v, g) \in \mathcal{A}\}$. Let $((H_{1,C}, u, w, g|_{H_{1,C}}), (H_{2,C}, w, v, g|_{H_{2,C}}))$ and $((H_{1,C'}, u', w', g'|_{H_{1,C'}}), (H_{2,C'}, w', v', g'|_{H_{2,C'}}))$ form a good pair in B from (C, u, v, g) and (C', u', v', g'), respectively. Then there are a map f_1 from $(H_{1,C}, u, w, g|_{H_{1,C}})$ to $(H_{1,C'}, u', w', g'|_{H_{1,C'}})$ and a map f_2 from $(H_{2,C}, w, v, g|_{H_{2,C}})$ to $(H_{2,C'}, w', v', g'|_{H_{2,C'}})$. We extend the union of these maps to a map f from $V \cup E \cup A$ to $V' \cup E' \cup A'$ by letting $f(w) = f_1(w) \cup f_2(w)$. This map f shows that $(C, u, v, g) \preceq (C', u', v', g')$. Hence $((C, u, v, g), (C', u', v', g'))$ is a good pair in the antichain $\mathcal{A}$, a contradiction. $\square$

Lemma 5.6. *Let $(Q, \leq)$ be a well-quasi-ordering. If $\bar{\mathscr{L}}^r_{k-1}(Q)$ is well-quasi-ordered by the induced minor relation and $\mathcal{A}$ is an infinite subset of $\bar{\mathscr{L}}^r_k(Q)$ such that for each $(C, u, v, g) \in \mathcal{A}$,*

(i) $uv \notin E(G)$, where G is the underlying graph of C,

(ii) there is only one K_C containing the roots u and v, and K_C is a K_4-structure,

(iii) there is a clique performing 2-sum with K_C,

then $\mathcal{A}$ is not a fundamental infinite antichain.

Proof. Suppose that $\bar{\mathscr{L}}^r_{k-1}(Q)$ is well-quasi-ordered the induced minor relation and $\mathcal{A}$ is an infinite subset of $\bar{\mathscr{L}}^r_k(Q)$ satisfying the condition in the lemma. Suppose on the contrary that $\mathcal{A}$ is a fundamental infinite antichain. For each $(C, u, v, g) \in \mathcal{A}$, let $V(K_C)$ consist of u, v, w, and z. If there is a pair of vertices in K_C, which is not $\{u, v\}$, such that there is no edge in E connecting them and there is no 2^{II}-sum on it, then we can consider the clique containing the roots u and v as in Lemmas 5.5 or 5.4.

By Lemma 2.2, we can find a fundamental infinite antichain $\mathcal{B}$, which is a subset of $\mathcal{A}$, such that for each $(C, u, v, g) \in \mathcal{B}$, every pair of vertices in K_C, which is not $\{u, v\}$, there is a 2^{II}-sum performing over them. We can use the same argument as Lemma 5.5 to obtain a contradiction. Let M_1, M_2, M_3, M_4, and M_5 denote the same kind of set as M_i in Lemma 5.5 on edges uw, wv, uz, zv, and wz, respectively. First, we consider $(H_{1,C}, u, w, g|_{H_{1,C}}) \in M_1$. By deleting all vertices which are not in $V(H_{1,C}) \cup V(H_{2,C})$, and contracting all edges in $H_{2,C}$, we have that $(H_{1,C}, u, w, g|_{H_{1,C}}) \prec (C, u, v, g)$. By Lemma 2.4, M_1 is wqo. Similarly, we have that M_2, M_3, and M_4 are wqo. Since $M_5 \subseteq \bar{\mathscr{L}}^r_{k-1}(Q)$ and $\bar{\mathscr{L}}^r_{k-1}(Q)$ is wqo, M_5 is wqo. Hence, $M_1 \times M_2 \times M_3 \times M_4 \times M_5$ are wqo by Lemma 2.6. By the same argument as Lemma 5.5, we can form a good pair in a chain from (C, u, v, g) and (C', u', v', g'). Then for each $j = 1, \ldots, 5$ there is a map f_j from $(H_{j,C}, x, y, g|_{H_{j,C}})$ to $(H_{j,C'}, x', y', C'|_{H_{j,C'}})$.

We extend the union of these maps to a map f from $V \cup E \cup A$ to $V' \cup E' \cup A'$ by letting $f(u) = f_1(u) \cup f_3(u)$, $f(v) = f_3(v) \cup f_4(v)$, $f(w) = f_1(w) \cup f_2(w) \cup f_5(w)$, and $f(z) = f_3(z) \cup f_4(z) \cup f_5(z)$. This map f shows that $(C, u, v, g) \preceq (C', u', v', g')$. Hence $((C, u, v, g), (C', u', v', g'))$ is a good pair in the antichain $\mathcal{A}$, a contradiction. $\square$

Lemma 5.7. *Let $(Q, \leq)$ be a well-quasi-ordering. If $\bar{\mathscr{L}}^r_{k-1}(Q)$ is well-quasi-ordered by the induced minor relation and a closed subclass $\mathscr{C}$ of $\bar{\mathscr{L}}^r_k(Q)$ is not well-quasi-ordered by the induced minor relation, then there is a fundamental infinite antichain $\mathcal{A}$ of $\mathscr{C}$ such that for all $(C, u, v, g) \in \mathcal{A}$, an edge $uv \in E(G)$, where G is the underlying graph of C.*

Proof. Suppose that $\bar{\mathscr{L}}^r_{k-1}(Q)$ is well-quasi-ordered by the induced minor relation and a subclass $\mathscr{C}$ of $\bar{\mathscr{L}}^r_k(Q)$, which is closed under taking induced minor, is not well-quasi-ordered by the induced minor relation . By Lemma 2.4 there is a fundamental infinite antichain $\mathcal{A}$. By Lemma 2.2, we may assume either $uv \in E$ for all $(C, u, v, g) \in \mathcal{A}$ or $uv \notin E$ for all $(C, u, v, g) \in \mathcal{A}$. In the first case we are done, and in the second case we will find a contradiction. By Lemmas 2.2 and 5.4, we only need to consider an infinite subset $\mathcal{B}$ of $\mathcal{A}$ with the condition that for each $(C, u, v, g) \in \mathcal{B}$, there is only one K_C containing the roots u and v. Then K_C is either K_3 or K_4-structure. If for each $(C, u, v, g) \in \mathcal{B}$, $G = K_C$, then it is wqo because $\mathcal{B}$ is a finite subset, a contradiction. So for each $(C, u, v, g) \in \mathcal{B}$, there is a clique performing 2-sum with K_C. By Lemmas 5.5 and 5.6, we are done. $\square$

5.2.2 Properties of a Fundamental Infinite Antichain in $\bar{\mathscr{L}}^r_k$ Part II

We now consider a graph $(C, u, v) \in \bar{\mathscr{L}}^r_k$ such that $uv \in E(G)$, where G is the underlying graph of C. We follow the same process as in 5.2.1 to prove the following lemmas.

Lemma 5.8. *Let $(Q, \leq)$ be a well-quasi-ordering. If $\mathcal{A}$ is an infinite subset of $\bar{\mathscr{L}}^r_k(Q)$ such that for each $(C, u, v, g) \in \mathcal{A}$,*

(i) $uv \in E(G)$, where G is the underlying graph of C,

(ii) there is only one K_C containing the roots u and v, and K_C is a K_3-structure,

(iii) there is a clique performing 2-sum with K_C,

then $\mathcal{A}$ is not a fundamental infinite antichain.

Proof. Suppose that $\mathcal{A}$ is an infinite subset of $\bar{\mathscr{L}}_k^r(Q)$ satisfying the condition in the lemma. Suppose on the contrary that $\mathcal{A}$ is a fundamental infinite antichain. For each $(C, u, v, g) \in \mathcal{A}$, let $V(K_C)$ consist of u, v, and w. Let M_1 denote the set of maximal connected subgraphs $H_{1,C}$ of C performing 2^{II}-sum with K_C on edge uw that are made into labeled rooted composite graphs $(H_{1,C}, u, w, g|_{H_{1,C}})$ in $\bar{\mathscr{L}}_k^r(Q)$ by choosing u and w as their roots and inheriting label and orientation from (C, u, v, g). By the condition of a graph in $\bar{\mathscr{L}}_k^r(Q)$, there is a maximal connected subgraphs $H_{2,C}$ of C performing 2^{II}-sum over wv. We define M_2 consisting of $H_{2,C}$'s made into labeled rooted composite graphs $(H_{2,C}, w, v, g|_{H_{2,C}})$ in $\bar{\mathscr{L}}_k^r(Q)$ by choosing w and v as their roots and inheriting label and orientation from (C, u, v, g).

By Lemma 2.2, we can find a fundamental infinite antichain $\mathcal{B}$, which is a subset of $\mathcal{A}$, for each $(C, u, v, g) \in \mathcal{B}$, $H_{1,C}$ and $H_{2,C}$ perform 2^{II}-sum with K_C. Then $(H_{1,C}, u, w, g|_{H_{1,C}})$ is not an induced minor of its original graphs (C, u, v, g). However, if M_1 is not wqo, we can use Lemmas 2.4 and 5.7 to find a fundamental infinite antichain $\mathcal{B}'$ which is $<_* \mathcal{B}$. So $\mathcal{B}' \subseteq \mathcal{B}^<$, contradicting with the definition of $\mathcal{B}^<$ that has no infinite antichain.

Hence M_1 and M_2 are wqo, and $M_1 \times M_2$ is wqo by Lemma 2.6. By the same argument as Lemma 5.5, we can form a good pair in a chain from (C, u, v, g) and (C', u', v', g'). Then there are a map f_1 from $(H_{1,C}, u, w, g|_{H_{1,C}})$ to $(H_{1,C'}, u', w', g'|_{H_{1,C'}})$ and a map f_2 from $(H_{2,C}, w, v, g|_{H_{2,C}})$ to $(H_{2,C'}, R'|_{H_{2,C'}}, C'|_{H_{2,C'}})$. We extend the union of these maps to a map f from $V \cup E \cup A$ to $V' \cup E' \cup A'$ by letting $f(w) = f_1(w) \cup f_2(w)$. This map f shows that $(C, u, v, g) \preceq (C', u', v', g')$. Hence $((C, u, v, g), (C', u', v', g'))$ is a good pair in the antichain $\mathcal{A}$, a contradiction. $\square$

Lemma 5.9. *Let $(Q, \leq)$ be a well-quasi-ordering. If $\bar{\mathscr{L}}^r_{k-1}(Q)$ is well-quasi-ordered by the induced minor relation and $\mathcal{A}$ is an infinite subset of $\bar{\mathscr{L}}^r_k(Q)$ such that for each $(C, u, v, g) \in \mathcal{A}$,*

(i) $uv \in E(G)$, where G is the underlying graph of C,

(ii) there is only one K_C containing the roots u and v, and K_C is a K_4-structure,

(iii) there is a clique doing 2-sum with K_C,

then $\mathcal{A}$ is not a fundamental infinite antichain.

Proof. Suppose that $\bar{\mathscr{L}}^r_{k-1}(Q)$ is well-quasi-ordered by the induced minor relation and $\mathcal{A}$ is an infinite subset of $\bar{\mathscr{L}}^r_k(Q)$ satisfying the condition in the lemma. Suppose on the contrary that $\mathcal{A}$ is a fundamental infinite antichain. For each $(C, u, v, g) \in \mathcal{A}$, let $V(K_C)$ consist of u, v, w, and z. If for each $(C, u, v, g) \in \mathcal{A}$, there is a pair of vertices in K_C, which is not $\{u, v\}$, such that there is no edge in E connecting them and there is no 2^{II}-sum on them, then we can consider this case as a K_3-structure, see Lemma 5.8, and obtain a contradiction. Suppose this case cannot happen. By Lemma 2.2, we can find a fundamental infinite antichain $\mathcal{C} \subseteq \mathcal{A}$ such that for each $(C, u, v, g) \in \mathcal{C}$ every edge in K_C, that is not uv, is performed 2^{II}-sum over. We will use the same argument as Lemma 5.8 to obtain a contradiction.

Let M_1, M_2, M_3, M_4, and M_5 denote the same kind of set as M_1 in Lemma 5.8 on edges uw, wv, uz, zv, and wz, respectively. By Lemma 2.2, we can find a fundamental infinite antichain $\mathcal{B} \subseteq \mathcal{C}$ such that for each $(C, u, v, g) \in \mathcal{B}$ and for some $j = 1, \ldots, 5$, $H_{j,C}$ performs 2^{II}-sum with K_C. Then each $H_{j,C}$ with root and label is not an induced minor of its original graphs (C, u, v, g). However, if M_j is not wqo, we can use Lemmas 2.4 and 5.7 to find a fundamental infinite antichain $\mathcal{B}'$ which is $<_* \mathcal{B}$. So $\mathcal{B}' \subseteq \mathcal{B}^<$, contradicting with the definition of $\mathcal{B}^<$ that has no infinite antichain.

Hence, $M_1 \times M_2 \times M_3 \times M_4 \times M_5$ is wqo by Lemma 2.6. By the same argument as Lemma 5.5, we can form a good pair in a chain from (C, u, v, g) and (C', u', v', g'). Then for each $j = 1, \ldots, 5$ there is a map f_j from $(H_{j,C}, x, y, g|_{H_{j,C}})$ to $(H_{j,C'}, x', y', C'|_{H_{j,C'}})$. We extend the union of these maps to a map f from $V \cup E \cup A$ to $V' \cup E' \cup A'$ by letting $f(u) = f_1(u) \cup f_3(u)$, $f(v) = f_3(v) \cup f_4(v)$, $f(w) = f_1(w) \cup f_2(w) \cup f_5(w)$, and $f(z) = f_3(z) \cup f_4(z) \cup f_5(z)$. This map f shows that $(C, u, v, g) \preceq (C', u', v', g')$. Hence $((C, u, v, g), (C', u', v', g'))$ is a good pair in the antichain $\mathcal{A}$, a contradiction. $\square$

Proof of Theorem 5.3. We will prove by induction on k. When $k = 0$, $\bar{\mathscr{L}}_0^r(Q)$ is empty set, so it is well-quasi-ordered by the induced minor relation. Suppose that $\bar{\mathscr{L}}_{k-1}^r(Q)$ is well-quasi-ordered by the induced minor relation, we will prove that $\bar{\mathscr{L}}_k^r(Q)$ is well-quasi-ordered by the induced minor relation. Suppose not, by subsection 5.2.1 , there is a fundamental infinite antichain $\mathcal{A}$ such that for each $(C, u, v, g) \in \mathcal{A}$, $uv \in E$. By Lemmas 2.2, we only have to consider a fundamental infinite antichain $\mathcal{B} \subseteq \mathcal{A}$ such that for each $(C, u, v, g) \in \mathcal{B}$ there is only one K_C containing the roots u and v. Suppose that for each $(C, u, v, g) \in \mathcal{B}$, K_C is a K_3- or K_4-structure. If for each $(C, u, v, g) \in \mathcal{B}$, $G = K_C$, then $\mathcal{B}$ is wqo since it is a finite subset, a contradiction. So for some $(C, u, v, g) \in \mathcal{B}$, there is a clique performing 2-sum with K_C. By Lemmas 5.8 and 5.9, we are done. $\square$

5.3 Tree-Representation

In this section we prove an analog of the block-tree theorem for the 2-sum operation and state properties of graphs in $\bar{\mathscr{L}}$. The main advantage of such a theorem is to show that the order of the 2-sum operations to construct a graph is irrelevant by looking at the structure of the graph. We formally define a tree-representation as follow. Let G be a graph. *A tree-representation* of G is a triple $(T, \{e_x : x \in X\}, \{G_y : y \in Y\})$ that satisfies the following properties.

(i) T is a tree with $|V(T)| \neq 2$ and X, Y are the two color classes of T such that all leaves of T are in Y;

(ii) For each $x \in X$, $e_x = uv$ is an edge, where $u, v \in V(G)$. If u, v are adjacent in G then e_x is the edge of G joining u and v; if u, v are not adjacent in G then e_x is a new edge;

(iii) For each $y \in Y$, G_y is an induced subgraph of $G^+ = (V(G), E(G) \cup \{e_x : x \in X\})$ with $|V(G_y)| > 2$ and moreover, the union of G_y over all $y \in Y$ is G^+;

(iv) For any $x \in X$ and any two distinct components T_1, T_2 of $T \setminus x$, $V_1 \cap V_2 = \{u, v\}$, where $uv = e_x$ and V_i $(i = 1, 2)$ is the union of $V(G_y)$ over all $y \in Y \cap V(T_i)$.

Lemma 5.10. *A 2-connected graph G can be constructed via 2-sums from a class $\mathcal{G}$ of graphs if and only if G admits a tree-representation such that every G_y belongs to $\mathcal{G}$.*

Proof. ($\Rightarrow$) Suppose that G is a 2-connected graph which can be constructed via 2-sums from a class $\mathcal{G}$ of graphs. If G is a 3-connected graph, then the triple $(T, \emptyset, \{G\})$ satisfies all properties, and it is a tree-representation of G. Suppose that G is a 2-sum of G_1 and G_2 over the common edge $e = uv$, which both are smaller than G. By induction, both have tree-representations. Let $(T_1, \{e_x : x \in X_1\}, \{G_y : y \in Y_1\})$ and $(T_2, \{e_x : x \in X_2\}, \{G_y : y \in Y_2\})$ be tree-representations of G_1 and G_2, respectively. We construct a tree T from T_1 and T_2 by considering the following cases.

Case 1. If for all $x \in X_1 \cup X_2$, $e \neq e_x$, then we connect T_1 to T_2 by adding edges y_1x_0 and x_0y_2, where x_0 is a new vertex such that $e = e_{x_0}$, and $y_1 \in Y_1$, $y_2 \in Y_2$ such that $e \in G_{y_1} \cup G_{y_2}$. We let $X = X_1 \cup X_2 \cup \{x_0\}$.

Case 2. If there is $x_1 \in X_1$ such that $e = e_{x_1}$ but for all $x_2 \in X_2$, $e \neq e_{x_2}$, then we connect T_1 to T_2 by adding an edge x_1y_2 where $y_2 \in Y_2$ such that $e \in G_{y_2}$. We let $X = X_1 \cup X_2$.

Case 3. If there are $x_1 \in X_1$ and $x_2 \in X_2$ such that $e_{x_1} = e_{x_2} = e$, then we perform 1-sum between T_1 and T_2 by identifying x_1 with x_2. We let $X = X_1 \cup (X_2 - x_2)$.

Let $Y = Y_1 \cup Y_2$, and let $x_0 \in X$ such that $e_{x_0} = e$. Since T_1 and T_2 are tree-representations, the triple $(T, \{e_x : x \in X\}, \{G_y : y \in Y\})$ satisfies properties (i), (ii), and (iii) of a tree-representation, and it also satisfies property (iv) for any $x \in X - x_0$. Let T_1 and T_2 be distinct components of $T \setminus x_0$. From the construction of T in those three cases, $V_1 \cap V_2 = \{u, v\}$, and V_i $(i = 1, 2)$ is the union of $V(G_y)$ over all $y \in Y \cap V(T_i)$. So $(T, \{e_x : x \in X\}, \{G_y : y \in Y\})$ is a tree-representation of G.

($\Leftarrow$) Suppose that G admits a tree-representation $(T, \{e_x : x \in X\}, \{G_y : y \in Y\})$ such that every G_y belongs to $\mathcal{G}$. If T has only one vertex then it is y in Y because all leaves of T are in Y (property (i)), and $G = G^+ = G_y \in \mathcal{G}$ by property (iii). Notice that if $X \neq \emptyset$, then from property (i) $|Y| \geq 2$ and $|V(T)| > 2$. We consider a leaf $y \in Y$ of T. Then $G_y \in \mathcal{G}$. By the definition of T, y is adjacent to a vertex $x \in X$, and $e_x \in E(G_y)$. By property (iv), $T \setminus x$ has two components T_1 and T_2 where $T_1 = \{y\}$. Let $G' = (V_2, E(G[V_2]) \cup \{e_x\})$, where $G[V_2]$ is an induced subgraph of G on V_2. If x has degree two in T, then the tree-representation of G' inherits from $T \setminus \{x, y\}$ with additional condition $e_x \in E(G')$; if x has degree greater than two in T, then the tree-representation of G' inherits from $T \setminus \{y\}$ with additional condition $e_x \in E(G')$. Then G is a 2^{I}-sum over e_x of G_y and G' if $e_x \in E(G)$; G is a 2^{II}-sum over e_x of G_y and G' if $e_x \notin E(G)$. By induction, G can be constructed via 2-sums from $\mathcal{G}$. $\square$

Lemmas 5.10 and 3.2 imply the following.

Lemma 5.11. *Every 2-connected graph admits a tree-representation such that each G_y is either 3-connected or isomorphic to K_3.*

The following lemma is a property of a graph in $\bar{\mathscr{L}}$.

Lemma 5.12. *Let $C = (G, D)$ be a composite graph such that $|V| > 3$, C is a 2-sum of copies of K_3 and K_4, and C contains K_4 as an induced minor. By the definition of induced minor relation, there are connected subgraphs X_1, X_2, X_3, and X_4 of C mapped to those 4 vertices in K_4 and adjacent to each other. Then there is a K_4 graph denoted by K_4^t and represented by a vertex in the tree-representation $(T, \{e_x : x \in X\}, \{G_y : y \in Y\})$ of C, such that $K_4^t \cap X_i \neq \emptyset$ for all $i = 1, 2, 3, 4$.*

Proof. Let H be such K_4. We consider the underlying graph G of C. If G is K_4, we are done. Suppose that G is not K_4, then G has a proper 2-separation $\{A, B\}$. Since G is 2-connected, by Lemma 3.2, G is a 2-sum of smaller 2-connected graphs G_A and G_B performing on edge $e = xy$, and both are 2-sums of copies of K_3 and K_4. Since for all $X_i \neq X_j$ there is an edge in G connecting them, there is no X_i and X_j such that $X_i \subseteq G_A \setminus \{x, y\}$ and $X_j \subseteq G_B \setminus \{x, y\}$. Thus, we can suppose that $G_A \cap X_i \neq \emptyset$ for all $i = 1, 2, 3, 4$. If there are i and j such that X_i contains x, X_j contains y, then they are connected by e in G_A. So G_A contains H as an induced minor. By induction, we continue this process until we have a 3-connected graph G_m containing H as an induced minor such that $G_m \cap X_i \neq \emptyset$ for all $i = 1, 2, 3, 4$. Since G_m is 3-connected and it is a 2-sum of copies of K_3 and K_4, G_m is K_4^t. □

Let $(T, \{e_x : x \in X\}, \{G_y : y \in Y\})$ be the tree-representation of a composite graph C in $\bar{\mathscr{L}}$, and let P be a path in T. Let $\{y_1, y_2, \ldots, y_n\}$ be all vertices in P representing a K_4. We call the K_4 represented by y_i a *good* K_4 *in* P if $i = 1$ or n; otherwise, every edge in this K_4 represented by a vertex in P does not have a common vertex. It is a *bad* K_4 in P if otherwise. The vertex y_i in P represented a good K_4 is called a *good vertex in* P. We define the *length of* P, $\|P\|$, as the number of good vertices (good K_4's) in P. The *longest path in* T is a path P_C such that $\|P_C\| \geq \|P\|$ for every path P in T. The *distance between two vertices* u and v in C is the number of good K_4's between u and v, which is the length of the shortest st-path in T where s and t represent cliques containing u and v, respectively.

In the following lemma, we use the tree-representation of a composite graph to study its structure involving $D_{n,p,q}$.

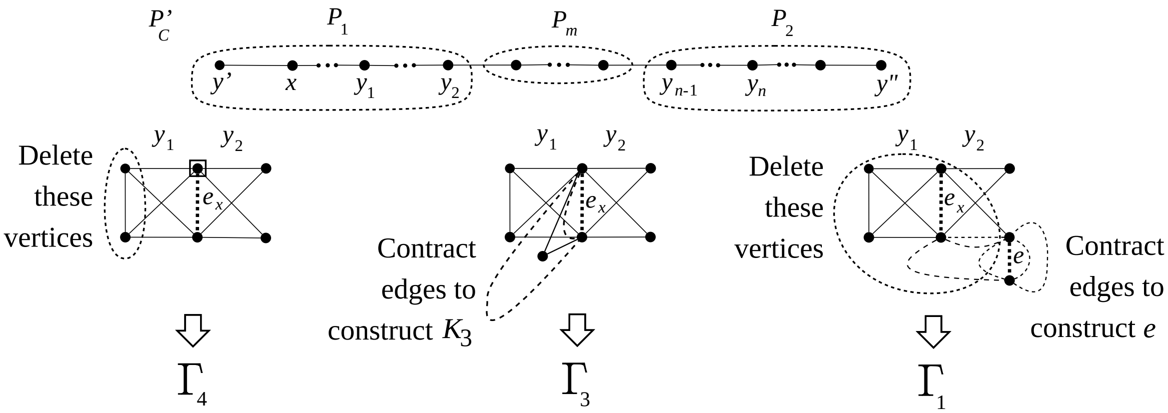
Lemma 5.13. *Let $C = (G, D)$ be a composite graph in $\bar{\mathscr{L}}$ such that every tail in C has length less than 2 and $\|P_C\| = n$ where $n \geq 5$. Then C contains $D_{i,p,q}$ as an induced minor for some $i \geq n-4$.*

Proof. Let $\{y_1, \ldots, y_n\}$ be the set of good vertices (represented good K_4's) in P_C. We extend P_C from y_1 and y_n along T until both reach leaves y' and y'' of T, respectively. Then y' and y'' represent K_3 or K_4. Let P'_C be this extended path. We divide P'_C into three paths as follows. Let P_1 be the path in P'_C connecting y' to y_2, let P_2 be the path in P'_C connecting y_{n-1} to y'', and let $P_m = P'_C \setminus P_1 \setminus P_2$. We first construct a graph in Γ^+ by considering P_1. Let x be a vertex in P'_C adjacent to y', and let e_x be an edge represented by x on which is performed a 2^{II}-sum in C. We consider two cases.

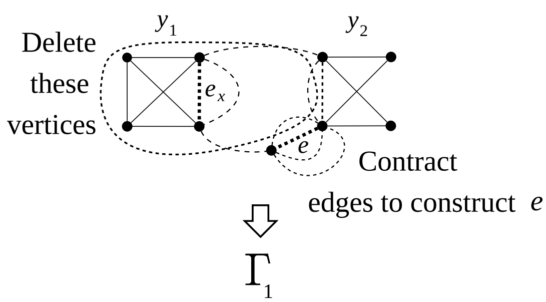
For the first case we suppose that $y' = y_1$. If x is adjacent to y_2, since the length of tail in C is less than 2, we can construct Γ^3 or Γ_4 by performing vertex deletion and edge contraction operations as shown in Figure 5.3(a). If x is not adjacent to y_2 then we perform the operations as shown in Figure 5.3(b) to construct Γ_1.

Next, we consider the case when $y' \neq y_1$. Then y' represents a K_3. If x is adjacent to y_1, since the length of tail in C is less than 2, we construct a graph in Γ^+ by using the same method as in the previous case. There are two other cases which are shown in Figure 5.4(a). If x is not adjacent to y_1 then we perform the operations as shown in Figure 5.4(b).

We obtain a graph in Γ^+ from P_2 by using the same consideration. To construct D_{n-4} we will consider P_m. Let C_H denote the subgraph of C corresponding to vertices of a subgraph



(a)



(b)

Figure 5.3: Case $y' = y_1$

H of T. For any component H, which is adjacent to a vertex in P_m representing an edge in C, we delete all vertices in $V(C_H)\backslash V(C_{P_m})$. For any component H, which is adjacent to a vertex in P_m not representing an edge in $\bar{G}$, we contract all edges in C_H to obtain the edge that is represented by a vertex in T connecting H to P_m. For every K_3 represented by a vertex y in P_m, we contract an edge in K_3 which are not in any 2^{II}-sum in C_{P_G}. For every bad K_4 represented by a vertex y in P_m, we delete a vertex in K_4 which is not in any 2^{II}-sum performing in C_{P_G}. Then there is only one edge not in 2^{II}-sum performing in C_{P_G} left, and we contract this edge. The resulting graph is a D_{n-4}. We apply this consideration to the parts in C_{P_1} and C_{P_2} that are not the graphs in Γ^+ constructed above. The resulting graph is $D_{i,p,q}$ for some $i \geqslant n-4$. Therefore, $D_{i,p,q}$ is an induced minor of C. $\square$

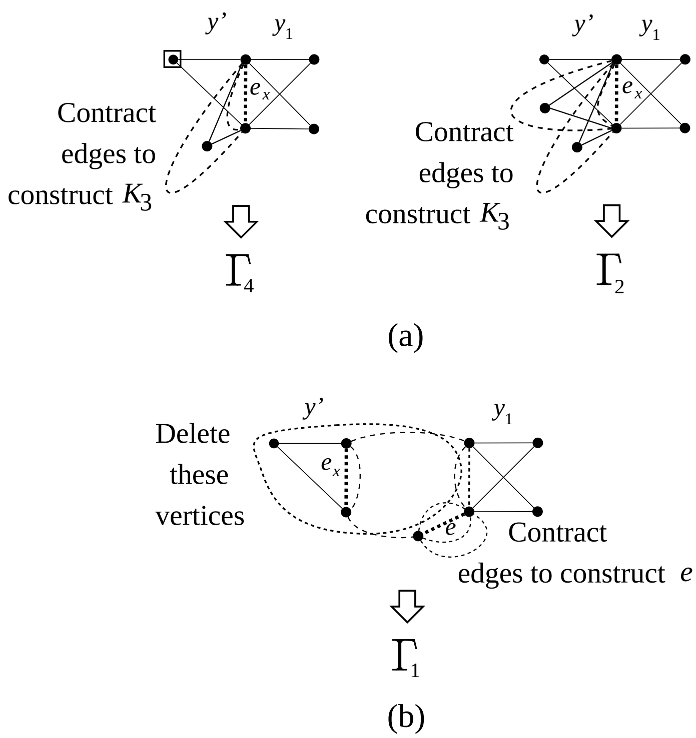


Figure 5.4: Case $y' \neq y_1$

From Chapter 4, we know that $\bar{\mathscr{D}}^{\Gamma+}$ is an infinite antichain of $\bar{\mathscr{L}}$. We now characterize all closed subclasses of $\bar{\mathscr{L}}$ which are well-quasi-ordered by the induced minor relation.

Lemma 5.14. *The following are equivalent for any closed subclass $\bar{\mathscr{Z}}$ of $\bar{\mathscr{L}}$.*

(i) $(\bar{\mathscr{Z}}(Q), \preceq)$ *is a wqo;*

(ii) $\bar{\mathscr{Z}}$ *is well-quasi-ordered by the induced minor relation;*

(iii) $\bar{\mathscr{Z}} \cap \mathscr{D}^{\Gamma+}$ *is finite.*

Proof. The implication (i)⇒(ii) is clear. To prove (ii)⇒(iii), if $\bar{\mathscr{Z}} \cap \mathscr{D}^{\Gamma+}$ is infinite, then by Lemma 4.3, $\bar{\mathscr{Z}}$ contains an infinite antichain. So $\bar{\mathscr{Z}}$ is not well-quasi-ordered by the induced minor relation.

To prove (iii)⇒(i), we assume that $\bar{\mathscr{Z}} \cap \mathscr{D}^{\Gamma+}$ is finite. Suppose on the contrary that $\mathcal{A}$ is a fundamental infinite antichain. By Lemma 2.2, we can consider the following cases.

Case 1. There is a fundamental infinite antichain $\mathcal{B}$, which is a subset of $\mathcal{A}$, such that for each (C, g) in $\mathcal{B}$, (C, g) has a tail (B, x_0, y_0) with length greater than 1. We define (C, x_1, y_1, g) be a labeled rooted composite graph. Let M_C denote the set of maximal connected subgraphs of (C, x_1, y_1, g) performing 2^{II}-sum over x_1y_1, that are made into labeled rooted composite graphs by choosing x_1 and y_1 as their roots and inheriting label and orientation from (C, x_1, y_1, g). Let $M = \cup_{(C,x_1,y_1,g)\in\mathcal{B}} M_C$. Since for each C every graph in M_C is a proper induced minor of (C, x_1, y_1, g) by deleting all vertices which are not in the vertex set of that graph, we have that $M \subseteq \mathcal{B}^{<}$. Since B is fundamental, M is wqo. By Lemma 2.5, $[M]^{<\omega}$ is wqo. Then there is a good pair $(M_C, M_{C'})$. Let $m : M_C \to M_{C'}$ be an injection map such that $H \preceq m(H)$ for all $H \in M_C$. Then there is a map f_H from H to $m(H)$ for all $H \in M_C$. We extend the union of these maps to a map f from $V \cup E \cup A$ to $V' \cup E' \cup A'$ by letting $f(x_1) = \cup_{H\in M_C} f_H(x_1)$ and $f(y_1) = \cup_{H\in M_C} f_H(y_1)$. This map f shows that $(C, x_1, y_1, g) \preceq (C', x'_1, y'_1, g')$. So (C, g) and $(C', g'))$ form a good pair in the antichain $\mathcal{A}$, a contradiction.

Case 2. There is no such fundamental infinite antichain. Then there is a fundamental infinite antichain $\mathcal{B}$, which is a subset of $\mathcal{A}$, such that for each (C, g) in $\mathcal{B}$, every tail in (C, g) has length less than 2. If we can show that $\bar{\mathscr{Z}}(Q) \subseteq \bar{\mathscr{L}}_n(Q)$ for some n, we are done. Let $\|\bar{\mathscr{Z}}(Q)\| = \{\|P_C\| : (C, g) \in \bar{\mathscr{Z}}(Q)\}$. Then $\|\bar{\mathscr{Z}}(Q)\|$ is bounded below by 0, and $\|\bar{\mathscr{Z}}(Q)\|$ is either bounded above or unbounded above. We consider the following cases.

Case 2.1. $\|\bar{\mathscr{Z}}(Q)\|$ is bounded above. Then there is k such that $\|P_C\| < k$ for all $(C, g) \in \bar{\mathscr{Z}}(Q)$. We will show that $\bar{\mathscr{Z}}(Q) \subseteq \bar{\mathscr{L}}_{3k}(Q)$. Suppose on the contrary that (D_{3k}, d_{3k}) is an induced minor of (C, g). Then for each $i = 1, \ldots, k$, the induced subgraph of D_{3k} with vertex set $\{x_{3i-1}, x_{3i-2}, x_{3i-3}, y_{3i}, y_{3i-1}, y_{3i-2}\}$ contains K_4 as an induced minor by contracting $x_{3i-2}x_{3i-3}$ and $y_{3i}y_{3i-1}$. For each $i = 1, \ldots, k$, we denote this K_4 by K_4^i. Let $(T, \{e_x : x \in X\}, \{G_y : y \in Y\})$ be the tree-representation of C. By Lemma 5.12, there are k K_4^t's represented by vertices in T mapped to these k K_4^i's in D_{3k}. Next, we will show that all of these vertices are in the same path of T. Suppose not, there are three vertices, t_1, t_2, t_3, lying in three different branches. Let s be a vertex in T linking these three branches together. Let K_4^1, K_4^2, and K_4^3 be K_4's which are induced minor of induced subgraphs of D_{3k}, and they are corresponding to $K_4^{t_1}$, $K_4^{t_2}$, and $K_4^{t_3}$ represented by t_1, t_2, and t_3, in T, respectively. In D_{3k}, which is 2-connected, there are two disjoint paths connecting K_4^1 and K_4^2, and others two disjoint paths connecting K_4^2 and K_4^3. Moreover, these four paths are adjacent to K_4^2 in four different vertices.

According to the induced minor relation, connected subgraphs of C which are corresponding to different vertices in D_{3k} are all disjoint, and two vertices in D_{3k} are adjacent if and only it there is an edge in C incident with their corresponding connected subgraphs. So a path in D_{3k}, which consists of a set of vertices and a set of edges in D_{3k}, is mapped to a set of disjoint connected subgraphs of C corresponding to those vertices and a set of edges in C corresponding to those edges. Since for every two edges which are incident with a connected subgraph in C there is a path in this subgraph connecting these two edges together. So we can find a path in C corresponding to a path in D_{3k}, and for every two paths in D_{3k}, which are disjoint, their corresponding paths in C are also disjoint. So we can find four disjoint corresponding paths in C such that two of them connect $K_4^{t_1}$ and $K_4^{t_2}$, and the other two of them connect $K_4^{t_2}$ and $K_4^{t_3}$. These four paths are adjacent to $K_4^{t_2}$ in four different vertices. Since C is constructed by 2-sum of K_3's or K_4's, there is a vertex t in T representing an edge e_t and lying between t_2 and s, and this t can be s too. Then C is constructed by 2-sum of two 2-connected subgraphs A and B on e_t, where $K_4^{t_2}$ is an induced minor of A, and $K_4^{t_1}$ and $K_4^{t_3}$ are an induced minor of B. Therefore, every path from either $K_4^{t_1}$ or $K_4^{t_3}$ to $K_4^{t_2}$ passes through endpoints of e_t, which means we can find only two disjoint corresponding paths in C, contradiction.

Thus, all vertices in T representing k K_4^t's are in the same path in T. Since each K_4^t lying between other two K_4^t's has two pairs of disjoint paths adjacent to its four different vertices linking it to the other K_4^T's, we have that all these k K_4^T are good. So the length of D_{3k}, which is the shortest path from x_0 to x_{3x-2}, is $3k-2 \leqslant \|P_G\| < k$, contradiction. Therefore, $\bar{\mathscr{Z}}(Q) \subseteq \bar{\mathscr{L}}_{3k}(Q)$, and $(\bar{\mathscr{Z}}(Q), \preceq)$ is a wqo.

Case 2.2. $\|\bar{\mathscr{Z}}(Q)\|$ is unbounded above. Then for all k, there is C such that $\|P_C\| \geq k$. By Lemma 5.13, we can conclude that $\bar{\mathscr{Z}} \cap \bar{\mathscr{D}}^{\Gamma^+}$ is infinite, contradiction.

Hence, $(\bar{\mathscr{Z}}(Q), \preceq)$ is a wqo. $\square$

Chapter 6
Main Result

The goal of this chapter is to prove the main result that for any closed subclass $\mathscr{Z}$ of $\mathscr{W}$, $\mathscr{Z}$ is well-quasi-ordered by the induced minor relation if and only if $\mathscr{Z} \cap \mathscr{D}^{\Gamma}$ is finite. From Chapter 3, we study the structure of a graph G in $\mathscr{W}$, which is $\{W_4, K_5\backslash e\}$-free, and we know that G can be constructed from cliques in $\mathscr{K}$ by repeatedly applying 0-, 1-, 2^{I}-, and 2^{II}-sums. In Chapters 4 and 5, we study the structure of an antichain and the subclass $\mathscr{L}$ of $\mathscr{W}$, which contains the antichain D^{Γ}. Notice that we can decompose a graph in $\mathscr{W}$ where 0-, 1-, and 2^{I}-sums are performed in the graph. Since a graph in $\mathscr{L}$ can be constructed from K_3's and K_4's be repeatedly applying 2^{II}-sums, a graph in $\mathscr{W}$ can be constructed from graphs in $\mathscr{L} \cup \mathscr{K}$, by repeatedly applying 0-, 1-, and 2^{I}-sums. We approach the result by considering the wqo of the class of such graphs.

6.1 0-, 1-, 2^{I}-sum of graphs in a wqo class of graphs

First we prove the following lemmas that are tools to preserve the wqo of graphs. Let X be a class of composite graphs, and let X_2^* be the class of graphs constructed from graphs in X by repeatedly applying 2^{I}-sum on arcs.

Lemma 6.1. *If $(X(Q), \preceq)$ is a wqo for all wqo $(Q, \leq)$, then $(X_2^*(Q), \preceq)$ is a wqo.*

Proof. Suppose on contrary that there is a fundamental infinite antichain $\mathcal{A}$ of $X_2^*(Q)$. For each (C, g) in $\mathcal{A}$, let $C_1 = (G_1, D(C_1)), C_2 = (G_2, D(C_2)), \ldots, C_k = (G_k, D(C_k))$ be the maximal connected subgraphs of C performing 2^{I}-sum on an arc uv of C, where $D(C_i) = (U_i, A_i)$ for all $i = 1, \ldots, k$. Let $C_0 = C$, where $U_0 = U$ and $A_0 = A$. We define g_i for all $i = 0, \ldots, k$ to be the Q'-labeling of C_i, where $Q' = Q \times \{0, 1, 2\}$ and

$g_i(x) = (g(x), h_i(x))$ (for all $x \in U_i \cup A_i$) such that $h_i(u) = 1$ and $h_i(v) = 2$. Then all the labeled composite graphs (C_i, g_i) and (C, g_0) are members of $X_2^*(Q')$. For any $(q,h),(q',h') \in Q'$, we define $(q,h) \leq' (q',h')$ if $q \leq q'$ and $h = h'$. Then $(Q', \leq')$ is a wqo provided $(Q, \leq)$ is. Let M_C be the class of all (C_i, g_i), $i = 1, \ldots, k$, and let $M = \cup_{(C,g)\in\mathcal{A}} M_C$. Clearly, for all $i = 1, \ldots, k$, (C_i, g_i) is a proper induced minor of (C, g_0) by deleting all vertices which are not in V_i. So $M \subseteq \mathcal{A}^{<}$. Since $\mathcal{A}$ is fundamental, M is wqo. By Lemma 2.5, $[M]^{<\omega}$ is wqo. Then there is a good pair $(M_C, M_{C'})$ in $[M]^{<\omega}$. Let $m : M_C \to M_{C'}$ be an injection map such that $H \preceq m(H)$ for all $H \in M_C$. Then there is a map f_H from H to $m(H)$ for all $H \in M_C$. We extend the union of these maps to a map f from $V \cup E \cup A$ to $V' \cup E' \cup A'$ by letting $f(u) = \cup_{H\in M_C} f_H(u)$ and $f(v) = \cup_{H\in M_C} f_H(v)$. This map f shows that $(C, g_0) \preceq (C', g_0')$. So $((C,g),(C',g'))$ is a good pair in the antichain $\mathcal{A}$, a contradiction. Hence, $(X_2^*(Q'), \preceq)$ is a wqo, and so is $(X_2^*(Q), \preceq)$. □

Let X_1^* be the class of graphs constructed from graphs in X by repeatedly applying 1-sum on special vertices and 2^I-sum on arcs.

Lemma 6.2. *If $(X(Q), \preceq)$ is a wqo for all wqo $(Q, \leq)$, then $(X_1^*(Q), \preceq)$ is a wqo.*

Proof. To prove this result, we use the same argument as Lemma 6.1 by decomposing graphs, where 1-sum appears, and putting an extra label on a vertex on which 1-sum is performed. Suppose on contrary that there is a fundamental infinite antichain $\mathcal{A}$ of $X_1^*(Q)$. For each (C, g) in $\mathcal{A}$, let $C_1 = (G_1, D(C_1)), C_2 = (G_2, D(C_2)), \ldots, C_k = (G_k, D(C_k))$ be the maximal connected subgraphs of C performing 1-sum on a special vertex u of C, where $D(C_i) = (U_i, A_i)$ for all $i = 1, \ldots, k$. Let $C_0 = C$, where $U_0 = U$ and $A_0 = A$. We define g_i for all $i = 0, \ldots, k$ to be the Q'-labeling of C_i, where $Q' = Q \times \{0, 1\}$ and $g_i(x) = (g(x), h_i(x))$ (for all $x \in U_i \cup A_i$) such that $h_i(u) = 1$. Then all the labeled composite graphs (C_i, g_i) and (C, g_0) are members of $X_1^*(Q')$. For any $(q,h),(q',h') \in Q'$, we define $(q,h) \leq' (q',h')$ if $q \leq q'$ and $h = h'$. Then $(Q', \leq')$ is a wqo provided $(Q, \leq)$ is.

Let M_C be the class of all (C_i, g_i), $i = 1, \ldots, k$, and let $M = \cup_{(C,g)\in\mathcal{A}} M_C$. Clearly, for all $i = 1, \ldots, k$, (C_i, g_i) is a proper induced minor of (C, g_0) by deleting all vertices which are not in V_i. So $M \subseteq \mathcal{A}^<$. Since $\mathcal{A}$ is fundamental, M is wqo. By Lemma 2.5, $[M]^{<\omega}$ is wqo. Then there is a good pair $(M_C, M_{C'})$ in $[M]^{<\omega}$. Let $m : M_C \to M_{C'}$ be an injection map such that $H \preceq m(H)$ for all $H \in M_C$. Then there is a map f_H from H to $m(H)$ for all $H \in M_C$. We extend the union of these maps to a map f from $V \cup E \cup A$ to $V' \cup E' \cup A'$ by letting $f(u) = \cup_{H\in M_C} f_H(u)$. This map f shows that $(C, g_0) \preceq (C', g_0')$. So $((C, g), (C', g'))$ is a good pair in the antichain $\mathcal{A}$, a contradiction. Hence, $(X_1^*(Q'), \preceq)$ is a wqo, and so is $(X_1^*(Q), \preceq)$. □

Let X^* be the class of graphs constructed from graphs in X by repeatedly applying 0-sum, 1-sum on special vertices, and 2^I-sum on arcs.

Lemma 6.3. *If $(X(Q), \preceq)$ is a wqo for all wqo $(Q, \leq)$, then $(X^*(Q), \preceq)$ is a wqo.*

Proof. Suppose on contrary that there is a fundamental infinite antichain $\mathcal{A}$ of $X^*(Q)$. For each (C, g) in $\mathcal{A}$, let M_C be the class of all connected components of C. Then $M_C \subseteq X_1^*(Q)$. From Lemma 6.2, $(X_1^*(Q), \preceq)$ is a wqo. So $[X_1^*(Q)]^{<\omega}$ is wqo by Higman's Theorem. Then there is a good pair $(M_C, M_{C'})$ in $\{M_C|(C, g) \in \mathcal{A}\}$. Let $\alpha : M_C \to M_{C'}$ be an injection map with $H \preceq \alpha(H)$ for all $H' \in M_C$. We define a map f from $V \cup E \cup A$ to $V' \cup E' \cup A'$ by extending the union of the maps $H \to \alpha(H)$. This map shows that $(C, g) \preceq (C', g')$. So $((C, g), (C', g'))$ is a good pair in the antichain $\mathcal{A}$, a contradiction. Hence, $(X^*(Q), \preceq)$ is a wqo. □

6.2 Proof of the Main Result

In order to prove Theorem 1.19, we consider a graph constructed from graphs in $\bar{\mathscr{L}}(Q) \cup \bar{\mathscr{K}}(Q)$, by repeatedly applying 0-, 1-, and 2^I-sums on special vertices and arcs. Let $\bar{\mathscr{W}}'(Q)$

be the class of such graphs. We will prove that for any closed subclass $\bar{\mathscr{Z}}$ of $\bar{\mathscr{W}}'$, $(\bar{\mathscr{Z}}(Q), \preceq)$ is a wqo if and only if $\bar{\mathscr{Z}} \cap \mathscr{D}^{\Gamma+}$ is finite. This result is stronger than Theorem 1.19. From 3.4, each graph G in $\mathscr{W}$ is constructed from cliques by repeatedly applying 0-, 1-, 2^I-, and 2^{II}-sum. We can make a composite graph $C = (G, D)$, where $D = (U, A)$, in $\bar{\mathscr{W}}'$ by letting A declare a direction on all edges of G over which 2^I-sums are performed, and U consist of all vertices of G which are ends of an arc or vertices performed 1-sum.

Theorem 6.4. *The followings are equivalent for any closed subclass $\bar{\mathscr{Z}}$ of $\bar{\mathscr{W}}'$.*

(i) $(\bar{\mathscr{Z}}(Q), \preceq)$ *is a wqo;*

(ii) $\bar{\mathscr{Z}}$ *is well-quasi-ordered by the induced minor relation;*

(iii) $\bar{\mathscr{Z}} \cap \mathscr{D}^{\Gamma+}$ *is finite.*

Proof. The implication (i)⇒(ii) is clear. To prove (ii)⇒(iii), if $\bar{\mathscr{Z}} \cap \mathscr{D}^{\Gamma+}$ is infinite, then by Lemma 4.3 $\bar{\mathscr{Z}}$ contains an infinite antichain. So $\bar{\mathscr{Z}}$ is not well-quasi-ordered by the induced minor relation.

To prove (iii)⇒(i), we assume that $\bar{\mathscr{Z}} \cap \mathscr{D}^{\Gamma+}$ is finite. Since $\bar{\mathscr{Z}} \subseteq \bar{\mathscr{W}}'$, every graph in $\bar{\mathscr{Z}}$ is constructed from graphs in $\mathscr{X}_1 \cup \mathscr{X}_2$ by repeatedly applying 0-sum, 1-sum on special vertices, and 2^I-sum on arcs, where $\mathscr{X}_1$ and $\mathscr{X}_2$ are subclasses of $\bar{\mathscr{L}}$ and $\bar{\mathscr{K}}$, respectively. Then $\mathscr{X}_1 \cap \mathscr{D}^{\Gamma+}$ is finite. By Lemmas 5.14 and 2.10, $(\mathscr{X}_1(Q), \preceq)$ and $(\mathscr{X}_2(Q), \preceq)$ are wqo. Hence, $(\bar{\mathscr{Z}}(Q), \preceq)$ is a wqo by Lemma 6.3. □

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