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Paley-Wiener Theorem for Line Bundles over Compact Symmetric Spaces

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PALEY-WIENER THEOREM FOR LINE BUNDLES OVER COMPACT SYMMETRIC SPACES

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Abstract

We generalize a Paley-Wiener theorem to homogeneous line bundles L_χ on a compact symmetric space U/K with χ a nontrivial character of K. The Fourier coefficients of a χ-bi-coinvariant function f on U are defined by integration of f against the elementary spherical functions of type χ on U, depending on a spectral parameter μ, which in turn parametrizes the χ-spherical representations π of U. The Paley-Wiener theorem characterizes f with sufficiently small support in terms of holomorphic extendability and exponential growth of their χ-spherical Fourier transforms. We generalize Opdam's estimate for the hypergeometric functions in a bigger domain with the multiplicity parameters being not necessarily positive, which is crucial to the proof of Paley-Wiener theorem in our case.

Chapter 1
Introduction

In the first chapter I intend to give some motivation and background information.

One of the fundamental questions in harmonic analysis is to determine the image of different function spaces under the Fourier transform. Paley-Wiener theorem characterizes the image of the space of compactly supported smooth functions or distributions under the Fourier transform in terms of holomorphic extendibility and growth condition.

The classical Paley-Wiener theorem identifies the space $C_c^\infty(\mathbb{R}^n)$ of smooth compactly supported functions on $\mathbb{R}^n$ with certain classes of holomorphic functions on $\mathbb{C}^n$ of exponential growth via the usual Fourier transform on $\mathbb{R}^n$. The exponent is determined by the size of the support. Precisely, recall that the space $C_c^\infty(\mathbb{R}^n)$ equipped with the Schwartz topology is denoted by $\mathcal{D}(\mathbb{R}^n)$. For a compact set $K \subset \mathbb{R}^n$, $\mathcal{D}_K(\mathbb{R}^n)$ denotes the space of smooth functions on $\mathbb{R}^n$ with compact support in K. For a positive number $r > 0$ we let $\mathcal{D}_r(\mathbb{R}^n) = \mathcal{D}_{B_r(0)}(\mathbb{R}^n)$. Then $\mathcal{D}_r(\mathbb{R}^n)$ is the space of smooth functions on $\mathbb{R}^n$ with support contained in a closed ball of radius r centered at zero. To each $f \in \mathcal{D}(\mathbb{R}^n)$ we associate its Fourier transform

$$\mathcal{F}(f)(\lambda) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)\, e^{-i\, x\cdot\lambda}\, dx.$$

Let $r > 0$ and $f \in \mathcal{D}_r(\mathbb{R}^n)$. Then $\lambda \mapsto \mathcal{F}(f)(\lambda)$ has a holomorphic extension to $\mathbb{C}^n$. The image of $\mathcal{D}_r(\mathbb{R}^n)$ under the (extended) Fourier transform is $\mathrm{PW}_r(\mathbb{C}^n)$ which is the space of entire functions F on $\mathbb{C}^n$ of exponential type r, that is, for every $k \in \mathbb{Z}^+$ there exists a constant C_k such that

$$|F(\lambda)| \leq C_k\, (1 + \|\lambda\|)^{-k}\, e^{r\,\|\mathrm{Im}\,(\lambda)\|}, \qquad \forall\, \lambda \in \mathbb{C}^n.$$

The vector space $\mathrm{PW}_r(\mathbb{C}^n)$ is topologized by the family of seminorms

$$\rho_k(F) := \sup_{\lambda\in\mathbb{C}^n} (1 + \|\lambda\|)^k\, e^{-r\,\|\mathrm{Im}\,(\lambda)\|}\, |F(\lambda)|, \qquad k \in \mathbb{Z}^+.$$

The classical Paley-Wiener theorem states that the Fourier transform $\mathcal{F}$ is a linear topological isomorphism of $\mathcal{D}_r(\mathbb{R}^n)$ onto $\mathrm{PW}_r(\mathbb{C}^n)$ for any $r > 0$. The Fourier inversion is

$$f(x) = \mathcal{F}^{-1}(F)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} F(\lambda)\, e^{i\,\lambda\cdot x}\, d\lambda. \tag{1.1}$$

An important aspect of this theorem is that the smallest exponent r in the estimates coincides with the radius of the smallest closed ball $\overline{B}_r(0) = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$. The polynomial factor $(1 + \|\lambda\|)^{-k}$ is related to the smoothness of f.

There are several generalizations of this theorem to settings where $\mathbb{R}^n$ is replaced by a locally compact Hausdorff topological group, by a Lie group, or by a homogeneous space. Among the homogeneous spaces, the symmetric spaces play an important role for their many applications to other branches of mathematics and to physics.

In fact, the above is a basic example of noncompact Lie group if we consider $\mathbb{R}^n$ as an additive group. Also, $\mathbb{R}^n$ is viewed as a homogeneous space G/H with $G = \mathbb{R}^n$ and $H = \{0\}$, the trivial subgroup. Here, G acts on $\mathbb{R}^n$ by translations. Recall that all irreducible unitary representations of $\mathbb{R}^n$ are one dimensional and given by the exponential map $x \mapsto e^{i\,\lambda\cdot x}$ with $x \in \mathbb{R}^n$. Let ℓ be the regular representation of G on $L^2(\mathbb{R}^n)$. We see that the inversion formula (1.1) provides a decomposition of ℓ as the direct integral over $\lambda \in \mathbb{R}^n$ of one dimensional irreducible representations of $\mathbb{R}^n$:

$$(\ell,\ L^2(\mathbb{R}^n)) \cong \int_{\mathbb{R}^n}^{\oplus} (\chi_\lambda,\ \mathbb{C})\, d\,\lambda$$

where $\chi_\lambda(x) = e^{i\,\lambda\cdot x}$ and the Plancherel measure on the unitary dual $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$ is the Lebesgue measure. This is the Plancherel decomposition for $\mathbb{R}^n$ with respect to the group action of G. Moreover, $\mathbb{R}^n$ can be interpreted as a noncompact symmetric space by $(\mathbb{R}^n \rtimes \mathrm{SO}(n))/\mathrm{SO}(n)$[1].

The most general results have been obtained for Riemannian symmetric spaces G/K of the non-compact type by Gangolli [8] and Helgason [14, 16] for smooth functions, and by Eguchi [5] (also see Dadok [4]) for distributions, to semisimple Lie groups by Arthur [1], and to pseudo-Riemannian reductive symmetric spaces by van den Ban and Sclichtkrull [33]. More recently, Ólafsson and Sclichtkrull extended the result to Riemannian symmetric spaces U/K of the compact type, [22] for K-invariant functions, [23] for K-finite functions, and [24] for distributions.

In contrast to the noncompact case G/K, the results obtained for compact case U/K are local in the sense that they are only valid for functions supported in sufficiently small balls with an explicit upper bound for the radius (the upper bound need not to be optimal). In the compact case U/K, it is clear that the support is compact because a closed subset of a compact set is compact. The essential issue is then to determine the size of the support of a smooth function from the growth property of its Fourier transform.

As an example, consider the torus $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ which is a compact Lie group and also a compact symmetric space. Suppose $f \in \mathbb{C}_r^\infty(\mathbb{T})$, i.e. f has support in $\exp(i\,[-r,\,r])$ with $0 < r < \pi$ (one can also write $\mathbb{T} = \mathbb{R}/2\pi\,\mathbb{Z}$ and view f as a periodic function on $\mathbb{R}$ by $t \mapsto f(e^{i\,t})$ with $\mathrm{supp}(f) \subseteq [-r,\,r] + 2\pi\,\mathbb{Z}$). The

[1] $\mathbb{R}^n \rtimes \mathrm{SO}(n)$ is the orientation preserving Euclidean motion group which is the semidirect product of the group of rotations and the group of translations. Here a topological group G is the semidirect product of two closed subgroups M and N if M is normal in G and the map $M \times N \to N$, $(m,\,n) \mapsto m\,n$ is a homomorphism. In this case $G = M \rtimes N$.

Fourier transform of f is $n \mapsto \widehat{f}\,(n)$ on $\mathbb{Z}$ where

$$\widehat{f}\,(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\,(e^{i\,t})\, e^{-i\,n\,t}\, dt.$$

It has a holomorphic extension to $\mathbb{C}$, defined by the same formula, say $\lambda \mapsto \widehat{f}\,(\lambda)$, $\lambda \in \mathbb{C}$. By the Paley-Wiener theorem on $\mathbb{R}^n$ we see that this extension has at most exponential growth of type r, and each holomorphic function on $\mathbb{C}$ of this type arises in this way from a unique $f \in C_r^{\infty}\,(\mathbb{T})$. The inversion

$$f\,(t) = \sum_{n=-\infty}^{\infty} \widehat{f}\,(n)\, e^{i\,n\,t}$$

gives a decomposition of $L^2\,([-\pi,\, \pi])$ as the direct sum (over $\mathbb{Z}$) of all one dimensional irreducible representations of $\mathbb{T}$ defined by $t \mapsto \chi_n\,(t) = e^{i\,n\,t}$ with $n \in \mathbb{Z}$,

$$(\ell,\, L^2\,([-\pi,\, \pi])) \cong \bigoplus_{n\in\mathbb{Z}} (\chi_n,\, \mathbb{C}).$$

The result of [22] generalized this local Paley-Wiener theorem on $\mathbb{T}$ to any arbitrary Riemannian symmetric space U/K of the compact type. In [22] they proved a Paley-Wiener theorem which characterized the Fourier image of the space $C_r^{\infty}\,(U/K)^K$ of smooth compactly supported K-biinvariant functions f on U. The compactness of U/K is reflected by the discreteness of its dual space, which is the set Λ_0^+ of irreducible spherical unitary representations of U. This set parametrizes the set of elementary spherical functions on U. In this case, the spherical representations of U (and thus the spherical functions on U) correspond to the trivial one dimensional representation of K (the trivial K-type). The spherical Fourier transform of f are functions on the discrete set Λ_0^+. The Fourier coefficients of f are defined by integration of f against the spherical functions on U. Likewise, the Fourier inversion formula, which recovers f in terms of spherical functions, is thus given by a series. Under the (extended) spherical Fourier transform there is a bijection from $C_r^{\infty}\,(U/K)^K$ onto the relevant Paley-Wiener space, which is the space of holomorphic functions of exponential type r plus satisfying certain Weyl group translation law.

Our work is motivated by [22]. We generalizes their result to the settings where the trivial K-type is replaced by the nontrivial K-types. Instead of $C_r^{\infty}\,(U/K)^K$, we consider $C_r^{\infty}\,(U//K,\, \chi)$ of χ-bi-coinvariant functions on U where χ is a nontrivial character of K. These functions can be viewed geometrically as smooth sections in a homogeneous line bundle over U/K. In our case, the elementary spherical functions on U correspond to the nontrivial K-types χ (thus called spherical functions of type χ). The χ-spherical Fourier transform and Fourier inversion are defined in a similar way. The χ-bi-coinvariant functions f with small support will be characterized in terms of holomorphic extendibility and exponential growth of their χ-spherical Fourier transform with the exponent linked to the size of the support of f. This is the content of Paley-Wiener theorem for line bundles over U/K. Let G/K be the noncompact dual symmetric space of U/K. Our proof relies on the fact that the spherical functions of type χ on U are connected to the spherical functions of type χ on G by holomorphic continuation, while there are many known results on the latter which we can use. Notice that the spherical functions of type χ on G are linked to the hypergeometric functions, but whose multiplicity parameters are not necessarily positive. We thus need to generalize Opdam's estimate (see [27]) for the hypergeometric functions to meet our situation.

The Paley-Wiener theorem is also known for many particular cases (refer to [22, §1] for more information on further developments of this subject). A Paley-Wiener type theorem for central functions on a compact Lie group U was proved by Gonzalez [9] where U is viewed as a symmetric space $(U \times U)/\mathrm{diag}\,(U)$. He basically reduced the proof to the Euclidean case by using the Weyl character formula. The proof of one aspect of our case relies on this result. An analogue of Paley-Wiener theorem for line bundles over noncompact symmetric spaces was obtained by [31].

This manuscript is organized as follows. In Chapter 2 we introduce basic notations and structure theory on Riemannian symmetric spaces. In Chapter 3 we discuss harmonic analysis related to line bundles over compact symmetric spaces, including the theory of highest weights for χ-spherical representations, elementary spherical functions of type χ, and χ-spherical Fourier transforms. In Chapter 6 we define the relevant Paley-Wiener space and state the Paley-Wiener theorem for line bundles over compact symmetric spaces (Theorem 6.6), to prove which we need some tools of differential operators (Chapter 4) and hypergeometric functions (Chapter 5). Sections 6.2 and 6.4.2 contains the main body of the proof. Section 6.2 shows the χ-spherical Fourier transform maps into the Paley-Wiener space, relied on a uniform estimate of the hypergeometric function (not requiring all multiplicity parameters to be positive) in a suitably big tubular domain (see Proposition 5.10). Section 6.4.2 proves the bijectivity of the χ-spherical Fourier transform. Finally, in Chapter 7, we treat rank one case and give an alternative method to prove the χ-spherical Fourier transform maps into the Paley-Wiener space.

Chapter 2

Riemannian Symmetric Spaces and Related Structure Theory

This introductory chapter is divided into four parts. In section 2.1 we recall some standard notations and facts related to differential geometry, Lie groups, and symmetric spaces with some emphasis on topics needed for the later treatment. In particular, a short review on compact symmetric spaces and their noncompact duals is given in Section 2.2. Then we discuss homogeneous line bundles over compact symmetric spaces in Section 2.3. The study of symmetric spaces leads naturally to semisimple Lie algebras. The last section is devoted to study root structures of these Lie algebras. The main references of this chapter are [11], [30], and [29].

2.1 Differential Geometry, Lie Groups, and Symmetric Spaces

Lie Groups: A Lie group G is a group and a manifold so that the map

$$G \times G \longrightarrow G, \qquad (x,\, y) \longmapsto x\, y^{-1}$$

is smooth. Roughly speaking, a Lie group is an analytic manifold with a group structure such that the group operations are analytic. A homogeneous space is a manifold M with a transitive action[1] of a Lie group G. Equivalently, it is a manifold of the form G/H where G is a Lie group and H a closed subgroup of G.

Theorem 2.1. *Assume that G acts transitively on M. Let $p \in M$ and $\mathrm{stab}_p\,(G) = \{g \in G \mid g \cdot p = p\}$ the stabilizer of p. Then the map*

$$G/\mathrm{stab}_p\,(G) \longrightarrow M$$

is a G-isomorphism (i.e. a diffeomorphic G-map).

A class of homogeneous spaces, for which the program of harmonic analysis via spectral decomposition of invariant differential operators is particularly compelling, is the class of symmetric spaces. We will work on Riemannian geometry and begin a study of Riemannian symmetric spaces.

Riemannian Symmetric Spaces: For a Riemannian manifold M denote by $I\,(M)$ the isometry group of M, that is, the set of all isometries[2] of M. A Riemannian homogeneous space is a Riemannian manifold M on which $I\,(M)$ acts transitively. It is shown that a Riemannian homogeneous space is diffeomorphic to a homogeneous space G/K where $G = I\,(M)$ and K is the isotropy subgroup of a point in M[3].

[1] The action is transitive if any point of M can be transformed into another point by an element of G.
[2] An isometry of M is a diffeomorphism that preserves the metric on M.
[3] Let a group G act on M. The isotropy subgroup of G at a point $p \in M$ is $\{g \in G \mid g \cdot p = p\}$.

A Riemannian manifold M is called a Riemannian symmetric space, if for any $p \in M$, there exists an involutive[4] isometry s_p of M such that p is an isolated fixed point of s_p. In this case, s_p is the symmetry of M at p.

Let G be a Lie group and K a closed subgroup. A symmetric pair may be defined as a pair (G, K) for which there is an involutive automorphism θ of G such that $G_0^\theta \subset K \subset G^\theta$, where

$$G^\theta = \{g \in G \mid \theta(g) = g\}$$

is the subgroup of fixed points for θ and G_0^θ denotes its identity component containing the identity element e of G. If, in addition, the image $\mathrm{Ad}_G(K)$ under the map $\mathrm{Ad}_G : G \to \mathrm{GL}(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G, is compact, then (G, K) is said to be a Riemannian symmetric pair.

Theorem 2.2. *There is a correspondence between Riemannian symmetric spaces and Riemannian symmetric pairs. Precisely,*

1. *If (G, K) is a Riemannian symmetric pair, then there is a G-invariant metric g on $M = G/K$ which makes (M, g) a Riemannian symmetric space*[5]*;*
2. *Let M be a Riemannian symmetric space and $p \in M$. Let $G = I_0(M)$ (the identity component of $I(M)$), K the isotropy subgroup of G at p, and s_p the symmetry of M at p. Then $(I_0(M), K)$ is a Riemannian symmetric pair with the involution θ of G given by $\theta(x) = s_p\, x\, s_p$.*

A symmetric pair (G, K) is called a semisimple symmetric pair if G is semisimple. A symmetric space X is semisimple if and only if there is a semisimple symmetric pair (G, K) with G acting on X by affine transformations, such that X is the symmetric space G/K. Note that the same space X with the same symmetries may correspond to several symmetric pairs (G, K), among which only some are semisimple. When we speak of a semisimple symmetric space G/K, it is to be understood that (G, K) is a semisimple symmetric pair.

Let G be a locally compact group and K a compact subgroup. Then the pair (G, K) is a Gelfand pair if the convolution algebra

$$L^1(K\backslash G/K) \cong \{f \in L^1(G) \mid f(k_1\, g\, k_2) = f(g),\ \forall\, k_1, k_2 \in K,\ g \in G\ \text{a.e.}\}$$

is commutative. Let M be a compact Riemannian symmetric space. Let $U = I_0(M)$ and K the isotropy subgroup of U at a point in M. Then (U, K) is a Riemannian symmetric pair and $M = U/K$. Since M is compact, then U is compact. We call (U, K) a compact Riemannian symmetric pair. In fact, the pair (U, K) is a Gelfand pair.

Vector Bundles: A complex vector bundle of rank k over a manifold M is a manifold $\mathcal{V}$ together with a smooth map $\pi : \mathcal{V} \to M$ (called the projection) so that

[4]A map $\theta : M \to M$ is an involutive map (or an involution) if $\theta(a\,b) = \theta(a)\,\theta(b)$, $\theta \neq \mathrm{id}$, and $\theta^2 = \mathrm{id}$.

[5]In this case, if θ is the involution for the pair (G, K) then the map $gK \mapsto \theta(g)\,K$ from X into itself is then the symmetry around the origin $o = e\,K$. By parallel transport there are symmetries around all other points of G/K as well.

1. For each $x \in M$ the fiber over x, $\pi^{-1}(x) =: \mathcal{V}_x$, is a vector space of dimension k;

2. For each $x \in M$ there is a neighborhood U of x and a diffeomorphism

$$\Phi : \pi^{-1}(U) \longrightarrow U \times \mathbb{C}^k$$

so that $\Phi(\mathcal{V}_y) = (y, \mathbb{C}^k)$ for $y \in U$.

A vector bundle $\mathcal{V}$ over M is a homogeneous vector bundle if G acts on $\mathcal{V}$ in such a way that

1. If $g \in G$ and $x \in M$ then $a \cdot \mathcal{V}_x = \mathcal{V}_{g \cdot x}$;

2. The map $\mathcal{V}_x \to \mathcal{V}_{g \cdot x}$, $v \mapsto g \cdot v$, is linear.

A section $s : M \to \mathcal{V}$ is a smooth map such that $\pi \circ s = \mathrm{id}_M$ or equivalently, $s(x) \in \mathcal{V}_x$ for all $x \in M$.

Representation Theory: Let G be a locally compact Hausdorff topological group. Let V, V' be topological vector spaces and $\mathrm{Hom}(V, V')$ be the set of continuous linear transformations from V to V'. Let $\mathrm{GL}(V)$ be the set of invertible elements in $\mathrm{Hom}(V, V)$. A representation of G on V is a pair (π, V) where $\pi : G \to \mathrm{GL}(V)$ is a homomorphism and the map

$$G \times V \longrightarrow V, \qquad (g, v) \longmapsto \pi(g)\, v$$

is continuous. If $n = \dim V < \infty$ then $\pi : G \to \mathrm{GL}(n, \mathbb{C})$ is a continuous homomorphism and hence analytic. A subspace $W \subset V$ is G-invariant if $\pi(g)\, W \subseteq W$ for all $g \in G$. A nonzero representation (π, V) is irreducible if the only closed G-invariant subspaces are $\{0\}$ and V. Let (π, V) and (π', V') be representations of G. Then $T \in \mathrm{Hom}(V, V')$ is an intertwining operator (or G-map) if $T \circ \pi = \pi' \circ T$. Denote by $\mathrm{Hom}_G(V, V')$ (or $\mathrm{Hom}_G(\pi, \pi')$) the set of all G-maps. We say V and V' are equivalent, $V \cong V'$ (or $\pi \cong \pi'$) if there is a bijective G-map from V to V'. If V is a Hilbert space we say the representation (π, V) is unitary when $\pi(g)$ is unitary for all $g \in G$. Two unitary representations π and π' are unitarily equivalent if there is a unitary isomorphism $T \in \mathrm{Hom}_G(V, V')$. Unitary equivalence is an equivalence relation on the set of all unitary representations of G. Let $\widehat{G}$ denote the set of all equivalence classes of irreducible unitary representations of G. We call $\widehat{G}$ the unitary dual of G.

Measures: For any locally compact group G there is a left (or right) Haar measure on G, which is a nonzero Radon measure[6] μ on G satisfying $\mu(g\,E) = \mu(E)$

[6] If X is any topological space, the σ-algebra of Borel sets in X is the smallest σ-algebra containing all open subsets of X. A measure defined on the Borel sets is called a Borel measure if it is finite for each compact set. The σ-algebra of Baire sets in X is the smallest σ-algebra of subsets of X such that each $f \in C_c(X)$ is measurable. It is the smallest σ-algebra containing every compact G_δ set. A Radon measure on X is a measure μ on the Baire sets such that $\mu(Y) < \infty$ for each compact G_δ set Y, and for each Baire set E, $\mu(E)$ is the supremum of all $\mu(Y)$ where Y is a compact G_δ subset of E.

(or $\mu(E\,g) = \mu(E)$) for any Borel set $E \subset G$ and any $g \in G$. If G is compact, then there is a left Haar measure which is also a right Haar measure, called a Haar measure.

A type I group is a second countable locally compact Hausdorff group whose unitary representations are all type I. We do not go into this here, but remark that abelian, connected nilpotent and semisimple Lie groups, and compact groups are type I. Let G be a unimodular locally compact type I group. Then if μ is a Haar measure on G, there is a unique measure σ, called the Plancherel measure on $\widehat{G}$, such that

$$\int_G |f(g)|^2 \, d\mu(g) = \int_{\widehat{G}} \mathrm{Tr}\,(\pi(f)\,\pi(f)^*)\, d\sigma(\pi).$$

The left regular representation of G with a Haar measure is the representation ℓ on $L^2(G)$ given by $\ell\,(g)\,f(x) = f\,(g^{-1}\,x)$. The right regular representation of G on $L^2(G)$ is defined by $\rho(g)\,f(x) = f(x\,g)$.

2.2 Compact Symmetric Spaces U/K and Their Noncompact Duals G/K

Consider a Riemannian symmetric space of the compact type which can be realized by U/K where U is a connected semisimple compact Lie group and $K \subseteq U$ a closed symmetric subgroup. Thus, there exists a nontrivial involution $\theta : U \to U$ such that $U_0^\theta \subseteq K \subseteq U^\theta$ where U^θ is the subgroup of θ-fixed points, and U_0^θ is its connected component containing the identity element e of U. Then $(U,\,K)$ is the compact Riemannian symmetric pair associated with θ. As K is closed in U and U is compact, it follows that K is compact. For simplicity, assume U/K is irreducible. We can further assume U is simply connected because the spherical harmonic analysis on a general compact symmetric space U/K can be reduced to the simply connected case (see [21, p.7]). Since U is simply connected, then K is connected and U/K is simply connected.

Let $\mathfrak{u}$ be the Lie algebra of U. Then θ induces an involution (automorphism) of $\mathfrak{u}$ which is the differential of θ, also denoted by θ. We have the Cartan decomposition

$$\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q}$$

where

$$\mathfrak{k} = \{X \in \mathfrak{u} \mid \theta\,(X) = X\}, \qquad \mathfrak{q} = \{X \in \mathfrak{u} \mid \theta\,(X) = -X\}$$

are the eigenspaces of θ with the eigenvalues 1 and -1, respectively. Then $\mathfrak{k}$ is the Lie algebra of K, and $\mathfrak{q} \cong T_o\,(U/K)$ where $o = e\,K$ is the identity coset of U/K. Every element $u \in U$ can be written as $u = k\,\exp\,X$ for some $k \in K$ and $X \in \mathfrak{q}$. However, in general, this decomposition is not unique. The exponential map $\exp : \mathfrak{u} \to U$ is surjective. Define Exp: $\mathfrak{q} \to U/K$ by $\mathrm{Exp}(X) = \exp\,(X) \cdot o$.

Let $\langle\ ,\ \rangle$ be the inner product on $\mathfrak{u}$ given by

$$\langle X,\ Y\rangle\ = -\mathcal{K}\,(X,\,Y) = -\mathrm{Tr}\,(\mathrm{ad}\,X \circ \mathrm{ad}\,Y),\ \forall X,\,Y \in \mathfrak{u}$$

where $\mathcal{K}$ is the Cartan-Killing form on $\mathfrak{u}$. Since $\mathfrak{u}$ is compact and semisimple, $\mathcal{K}$ is negative definite on $\mathfrak{u}$. We assume that the Riemannian metric of U/K is normalized such that it agrees with $\langle\ ,\ \rangle$ on the tangent space $\mathfrak{q}$. The inner product on $\mathfrak{u}$ gives an inner product on the dual space $\mathfrak{u}^*$ in a canonical way, and by sesquilinear extensions they induce inner products on $\mathfrak{u}_{\mathbb{C}} = \mathfrak{u} \oplus i\,\mathfrak{u}$ and the complex dual space $\mathfrak{u}^*_{\mathbb{C}}$. Here and in the following the subscript $\mathbb{C}$ denotes complexification. All of these involved inner products are denoted by the same symbol.

Let $\mathfrak{b}$ be a maximal abelian subspace of $\mathfrak{q}$, called a Cartan subspace of the compact symmetric pair $(U,\, K)$, which is unique up to conjugacy by K. In particular, all Cartan subspaces have the same dimension. The number $n = \dim\, \mathfrak{b}$ is called the real rank of $\mathfrak{u}$ and of U. This is, by definition, also the rank of the compact symmetric space U/K. Let $\mathfrak{b}^*$ be the (real) dual space of $\mathfrak{b}$ and $\mathfrak{b}^*_{\mathbb{C}}$ its complexified dual space. Let $\mathfrak{h}$ be a Cartan subalgebra[7] of $\mathfrak{u}$ containing $\mathfrak{b}$. Then $\mathfrak{h}$ is θ-stable and

$$\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus \mathfrak{b}.$$

Let B be the analytic subgroups of U with Lie algebras $\mathfrak{b}$. Then $B = \exp\, \mathfrak{b}$ is a connected abelian closed subgroup of U. We call B the corresponding Cartan subgroup of $(U,\, K)$.

Since U is compact, it admits a finite dimensional faithful unitary representation. Thus if p is the dimension of this representation, we can assume that $U \subset U\,(p) \subset \mathrm{GL}\,(p,\, \mathbb{C})$. So there is a unique connected complex Lie group $U_{\mathbb{C}}$ with the Lie algebra $\mathfrak{u}_{\mathbb{C}}$ such that $U \subseteq U_{\mathbb{C}}$ is a real subgroup. For any $g \in U_{\mathbb{C}}$ there are a unique $u \in U$ and a unique $X \in \mathfrak{u}$ such that $g = u \exp\,(i\,X)$. Note that $U_{\mathbb{C}}$ is a closed analytic subgroup of $\mathrm{GL}\,(p,\, \mathbb{C})$ and is a complex submanifold of $\mathrm{GL}\,(p,\, \mathbb{C})$.

Let $G = K\, \exp\,(i\,\mathfrak{q})$. Then G is the analytic subgroup of $U_{\mathbb{C}}$ with the Lie algebra

$$\mathfrak{g} := \mathfrak{k} \oplus i\,\mathfrak{q}.$$

The decomposition $G = K\, \exp\,(i\,\mathfrak{q}) \cong K \times (i\,\mathfrak{q})$ implies that G/K as a manifold is diffeomorphic via the exponential map to the Euclidean space $i\,\mathfrak{q}$. Note that G is a noncompact semisimple Lie group and is connected, closed in $U_{\mathbb{C}}$, and $K \subset G$ maximal compact. Also, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{u}_{\mathbb{C}}$ as complex vector spaces, and $U_{\mathbb{C}}$ complexifies both U and G. So $U_{\mathbb{C}} = G_{\mathbb{C}}$. The involution θ can be extended to a holomorphic involution on $U_{\mathbb{C}}$, also denoted by θ. Then $\theta\,|_G$ is a Cartan involution on G. The pair $(G,\, K)$ is a noncompact Riemannian symmetric pair associated with $\theta\,|_G$. We still denote $\theta\,|_G$ by θ. The symmetric space G/K is called the noncompact dual of U/K. We have $K = G^\theta = U \cap G$ is connected and G/K is simply connected. Let $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus i\,\mathfrak{k}$ and $K_{\mathbb{C}}$ be the connected subgroup of $G_{\mathbb{C}}$ with the Lie algebra $\mathfrak{k}_{\mathbb{C}}$. Then $(U_{\mathbb{C}},\, K_{\mathbb{C}})$ is non-Riemannian symmetric pair with respect to the involution θ. The symmetric spaces U/K and G/K embed in the complex homogeneous space $U_{\mathbb{C}}/K_{\mathbb{C}}$ as totally real submanifolds.

[7] A subalgebra $\mathfrak{h}$ of $\mathfrak{u}$ is called a Cartan subalgebra if $\mathfrak{h}$ is nilpotent and $\mathfrak{h} = N_{\mathfrak{u}}\,(\mathfrak{h}) = \{X \in \mathfrak{u} \mid [\mathfrak{h},\, X] \subset \mathfrak{h}\}$.

Let $\mathfrak{a} = i\,\mathfrak{b}$ and $A = \exp \mathfrak{a}$. Then $\mathfrak{a} \subset i\,\mathfrak{q}$ is maximal abelian, called a Cartan subspace for the symmetric pair $(G,\, K)$, and $A \subset G$ is the corresponding Cartan subgroup. The elements of $\operatorname{ad}\mathfrak{a}$ can be simultaneously diagonalized with real eigenvalues. Let $\mathfrak{a}_{\mathbb{C}} := \mathfrak{a} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{a} \oplus i\,\mathfrak{a}$ and $A_{\mathbb{C}}$ the connected subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{a}_{\mathbb{C}}$. We have $\mathfrak{b}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}}$ and $B_{\mathbb{C}} = A_{\mathbb{C}} = A\,B$ is the polar decomposition of $B_{\mathbb{C}}$. The exponential map $\exp : \mathfrak{b}_{\mathbb{C}} \to B_{\mathbb{C}}$ is the canonical projection of $\mathfrak{b}_{\mathbb{C}}$ onto $B_{\mathbb{C}}$ whose multi-valued inverse is log.

By restriction, $\langle\ ,\ \rangle$ defines an inner product on $\mathfrak{b}$, which in turn induces an inner product on $\mathfrak{b}^*$ by duality. By sesquilinear extension we obtain inner products on $i\,\mathfrak{b}^*$ and $\mathfrak{b}^*_{\mathbb{C}}$.

It is convenient to establish some conventions about the normalization of invariant measures on the groups and symmetric spaces considered in this manuscript. We normalize the Haar measures du, dk, and db on compact groups U, K, and B, respectively, such that the total measure is one. In general, we normalize any compact group in this way. Moreover, if L is a Lie group and Q is a closed subgroup of L, with left Haar measures dl and dq, respectively, then the homogeneous space L/Q (when it exists) possesses a unique left L-invariant Borel measure $d\,(l\,Q)$[8]. We normalize it so that

$$\int_L f\,(l)\,dl = \int_{L/Q} \left(\int_Q f\,(l\,q)\,dq \right) d\,(l\,Q) \tag{2.1}$$

where $f \in L^1(L)$. In this case,

$$\int_{L/Q} F\,(l\,Q)\,d\,(l\,Q) = \int_L F \circ \iota(l)\,dl$$

where $\iota : L \to L/Q$ is the canonical projection and $F \in L^1(L/Q)$. This condition (2.1) fixes the U-invariant measure $d\,(uK)$ on U/K to have total mass one. We shall use the notation $dx = d\,(uK)$ for the invariant measure on U/K.

2.3 Line Bundles over Compact Symmetric Spaces U/K

Given a representation $(\chi,\, V_\chi)$ of K, let K act on $U \times V_\chi$ from the right by

$$(g,\, v) \cdot k = (g\,k,\ \chi\,(k)^{-1}\,(v)).$$

It is clear that the left U-action on $U \times V_\chi$, $u \cdot (g,\, v) = (u\,g,\ v)$ and the right K action commutes. Define an equivalence relation $\sim$ on $U \times V_\chi$ by $(u,\, v) \sim (g,\, w)$ if and only if they are in the same K-orbit, i.e. there is a $k \in K$ such that

$$u\,k = g \qquad \text{and} \qquad \chi(k)^{-1}\,(v) = w.$$

[8]The measure $d(l\,Q)$ is obtained by taking the push forward measure on L/Q of the Haar measure dl on L via the canonical projection $\iota : L \to L/Q$ given by $l \mapsto l\,Q$.

Denote by $[u, v]$ the equivalence class of (u, v). Then the homogeneous vector bundle over U/K is defined by

$$U \times_K V_\chi = (U \times V_\chi)/\sim = \{[u, v] \mid u \in U,\ v \in V_\chi\}.$$

The projection map $\pi : U \times_K V_\chi \to U/K$ is given by $\pi(u, v) = uK$.

If dim $V_\chi = 1$ then $V_\chi \cong \mathbb{C}$ and $\chi : K \to \mathbb{T}$ is a character of K. In this case,

$$L_\chi := U \times_K V_\chi \cong U \times_K \mathbb{C}$$

is then a homogeneous line bundle over U/K.

Denote by $Z(K)$ and $\mathfrak{z}(\mathfrak{k})$ the centers of K and $\mathfrak{k}$. Let $K_1 = [K, K]$ be the commutator subgroup generated by $\{aba^{-1}b^{-1} \mid a, b \in K\}$. As K is a compact connected Lie group, K_1 is a connected closed normal Lie subgroup of K with the Lie algebra $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$. We have

$$\chi(a\,b\,a^{-1}\,b^{-1}) = \chi(a)\,\chi(b)\,\chi(a^{-1})\,\chi(b^{-1}) = 1.$$

This implies that $\chi|_{K_1} = 1$. Since $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{z}(\mathfrak{k})$, if K is semisimple, then $\mathfrak{z}(\mathfrak{k}) = \{0\}$ and $\mathfrak{k}_1 = \mathfrak{k}$. Thus, $K = K_1$ and $\chi \equiv 1$, i.e. the trivial representation is the only one dimensional representation of K. But if K is not semisimple, or equivalently if U/K is of Hermitian type, there are nontrivial one dimensional K-types to occur. It is for these groups U we take into consideration.

In the following table we give the classification of possible spaces U/K where K is not semisimple so that a nontrivial one dimensional K-types exists and so that over such a space a nontrivial homogeneous line bundle exists (cf. [11, p.516, 518])

Table 1: Classification of irreducible Riemannian symmetric spaces:
U compact, G noncompact, K connected

	class	G	U	K	n	d	
1	$AIII$	$\mathrm{SU}(p, q)$	$\mathrm{SU}(p+q)$	$\mathrm{S}(\mathrm{U}_p \times \mathrm{U}_q)$	q	$2pq$	$p \geq q \geq 1$
2	BDI	$\mathrm{SO}_o(p, 2)$	$\mathrm{SO}(p+2)$	$\mathrm{SO}(p) \times \mathrm{SO}(2)$	2	$2p$	$p \geq 3$
3	$DIII$	$\mathrm{SO}^*(2j)$	$\mathrm{SO}(2j)$	$\mathrm{U}(j)$	$[\frac{j}{2}]$	$j(j-1)$	$j \geq 4$
4	CI	$\mathrm{Sp}(j, \mathbb{R})$	$\mathrm{Sp}(j)$	$\mathrm{U}(j)$	j	$j(j+1)$	$j \geq 2$
5	$EIII$	$\mathfrak{e}_{6(-14)}$	$\mathfrak{e}_{6(-78)}$	$\mathfrak{so}(10) + \mathbb{R}$	2		
6	$EVII$	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{7(-133)}$	$\mathfrak{e}_6 + \mathbb{R}$	3		

Remark 2.3. In Table 1, $n = \dim \mathfrak{b}$ is the rank of U/K and $d = \dim U/K$. The conditions listed in the last column are given to prevent coincidence due to lower dimensional isomorphisms, which might give redundant candidates, so we should take them out. Precisely,

1. Case 2 with $p = 1$ is a redundant candidate since

$$\mathrm{SO}(3)/\mathrm{SO}(2) \cong S^2 \cong \mathrm{SU}(2)/\mathrm{S}(\mathrm{U}_1 \times \mathrm{U}_1)$$

(cf. [11, p.519]) which coincides with Case 1 with $p = q = 1$.

2. Because $\mathrm{U}(1) \cong \mathrm{SO}(2)$ and $\mathrm{U}(2) \cong \mathrm{U}(1) \times \mathrm{U}(1)$, we see that Case 2 with $p = 2$ coincides with Case 3 with $j = 2$, they are

$$\mathrm{SO}(4)/\mathrm{SO}(2) \times \mathrm{SO}(2) \cong \mathrm{SO}(4)/\mathrm{U}(2).$$

   Since $\mathrm{SO}(4) = \mathrm{SO}(3) \times \mathrm{SO}(3)$, this space is isomorphic to $S^2 \times S^2$. In view of [19, p.424], the root system is of type $D_2 = A_1 \times A_1$, a split case. It is not in our consideration since we only consider irreducible root systems. Therefore, this case is not listed in Table 1.

3. In case 3, $j \neq 1$ because $[\frac{1}{2}] = 0$.

4. Since $\mathrm{SO}(6) \cong \mathrm{SU}(4)$, then

$$\mathrm{SO}(6)/\mathrm{U}(3) \cong \mathrm{SU}(4)/\mathrm{U}_3$$

   which implies that Case 3 with $j = 3$ coincides with Case 1 with $p = 3$ and $q = 1$.

5. Since $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ and $\mathrm{U}(1) \cong \mathrm{S}(\mathrm{U}_1 \times \mathrm{U}_1)$, we see that

$$\mathrm{Sp}(1)/\mathrm{U}(1) \cong \mathrm{SU}(2)/\mathrm{S}(\mathrm{U}_1 \times \mathrm{U}_1).$$

   So Case 4 with $j = 1$ coincides with Case 1 with $p = q = 1$.

It is helpful to discuss the geometric meanings of such U/K. In Case 1,

$$U/K = \mathrm{SU}(p+q)/\mathrm{S}(\mathrm{U}_p \times \mathrm{U}_q)$$

is the space of p-planes in $\mathbb{C}^{p+q}$, known as the complex Grassmann manifolds. In Case 2,

$$U/K = \mathrm{SO}(p+2)/(\mathrm{SO}(p) \times \mathrm{SO}(2))$$

is a covering of $\mathrm{SO}(p+2)/\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(2))$. Note that $\mathrm{SO}(p+2)/\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(2))$ is the space of p-planes in $\mathbb{R}^{p+2}$, known as the real Grassmann manifolds.

In the case U/K of Hermitian type, the set of such χ's are parametrized by $\mathbb{Z}$, precisely,

Proposition 2.4. *Let $l \in \mathbb{Z}$. Define $\chi_l : K \to \mathbb{T}$ by*

$$\begin{cases} \chi_l(k) = 1, & \forall\, k \in K_1 \\ \chi_l(e^{tZ}) = e^{ilt}, & \forall\, t \in \mathbb{R}. \end{cases}$$

Here, $Z \in \mathfrak{z}(\mathfrak{k}) \setminus \{0\}$ is defined as [29, (3.1)] so that $e^{tZ} \in Z(K)$, and $e^{tZ} \in K_1$ if and only if $t \in 2\pi\mathbb{Z}$. This is a well defined one dimensional representation of K. Moreover, if χ is an one dimensional representation of K, then there is a unique $l \in \mathbb{Z}$ such that $\chi = \chi_l$.

Proof. Refer to Proposition 3.4 in [29] and its following comment. ☐

Since all one dimensional representations χ of K have this form, hereafter, we parametrize $\chi = \chi_l$ for $l \in \mathbb{Z}$. If $l = 0$, then χ_0 is trivial. The character χ_l satisfies

$$\chi_l\,(k)^{-1} = \chi_l\,(k^{-1}) = \chi_{-l}\,(k).$$

Denote by $L^2\,(U/K,\,\chi_l)$ the space of functions f on U satisfying

$$f(u\,k) = \chi_l(k)^{-1}\,f(u) \qquad \text{and} \qquad \int_U |f\,(u)|^2\,du < \infty,$$

where the latter condition is equivalent to $\int_{U/K} |f\,(x)|^2\,dx < \infty$ because $|\chi_l\,(k)| = 1$, for all $k \in K$. Consider

$$C^\infty\,(U/K,\ \chi_l) = \{f \in C^\infty\,(U)\ \mid\ f\,(u\,k) = \chi_l\,(k)^{-1}\,f\,(u),\,\forall\,k \in K\}$$

the space of smooth χ_l-right-coinvariant functions on U. When $l = 0$, this space becomes $C^\infty\,(U/K)$. Let $\Gamma^\infty\,(L_{\chi_l})$ be the space of smooth sections of L_{χ_l}. The group U acts on $\Gamma^\infty\,(L_{\chi_l})$ by $(u \cdot s)(x) = u \cdot s(u^{-1}\,x)$. It is well known that

Proposition 2.5. *There is a U-isomorphism (linear U-intertwining bijection) Φ : $C^\infty\,(U/K,\ \chi_l) \to \Gamma^\infty\,(L_{\chi_l})$, given by*

$$\Phi\,(f)\,(u\,K) = [u,\ f\,(u)] = s_f\,(u\,K),$$

whose inverse is $\Phi^{-1}\,(s)\,(u) = u^{-1} \cdot (s(u\,K))$.

Hence we may think of functions in $C^\infty\,(U/K,\ \chi_l)$ geometrically as smooth sections in a homogeneous line bundle $L_{\chi_l} \to U/K$.

2.4 Root Structures of Semisimple Lie Algebras

Let $\Delta = \Delta\,(\mathfrak{u},\,\mathfrak{h})$ be the set of roots of $\mathfrak{u}$ with respect to $\mathfrak{h}$, that is, it contains all nonzero $\beta \in \mathfrak{h}_{\mathbb{C}}^*$ for which the vector space

$$\mathfrak{u}_{\mathbb{C}}^{\beta} = \{X \in \mathfrak{u}_{\mathbb{C}} \mid \forall\,H \in \mathfrak{h}_{\mathbb{C}},\,[H,\,X] = \beta\,(H)\,X\}$$

is nonzero. Since $\mathfrak{u}$ is compact, all elements of Δ take purely imaginary values on $\mathfrak{h}$. So $\Delta \subset i\,\mathfrak{h}^*$. For $\alpha \in \mathfrak{b}_{\mathbb{C}}^*$, let

$$\mathfrak{u}_{\mathbb{C},\,\alpha} = \{X \in \mathfrak{u}_{\mathbb{C}}\,|\,\forall\,H \in \mathfrak{b}_{\mathbb{C}},\,[H,\,X] = \alpha\,(H)\,X\}.$$

If $\mathfrak{u}_{\mathbb{C},\,\alpha} \neq \{0\}$, then α is called a (restricted) root. The spaces $\mathfrak{u}_{\mathbb{C},\,\alpha}$ are called root spaces. Denote by $\Sigma = \Sigma\,(\mathfrak{u},\,\mathfrak{b})$ the set of nonzero restricted roots of the pair $(\mathfrak{u},\,\mathfrak{b})$. Then Σ is a root system[9]. We do not assume Σ is a reduced root system[10]. If $\beta \in \Delta$, then $\beta\,|_{\mathfrak{b}}$ either equals zero or is in Σ. Since all the elements of Σ are

[9]That means Σ satisfies the axioms of an abstract root system.
[10]A root system Σ is said to be reduced if $\alpha, \beta \in \Sigma$ and $\beta = r\,\alpha$ implies that $r = \pm\,1$.

purely imaginary on $\mathfrak{b}$, we see that $\Sigma \subset i\,\mathfrak{b}^* = \mathfrak{a}^*$, and $\mathfrak{u}_{\mathbb{C},\,\alpha} \cap \mathfrak{u} = \{0\}$. Note that Cartan duality is a bijection between the classes of simply connected symmetric spaces of compact type and noncompact type: $U/K \longleftrightarrow G/K$. On the Lie algebra level this bijection is given by

$$\mathfrak{u} = \mathfrak{k} \oplus \mathfrak{q} \;\longleftrightarrow\; \mathfrak{g} = \mathfrak{k} \oplus i\,\mathfrak{q}. \tag{2.2}$$

Let $\Sigma\,(\mathfrak{g},\,\mathfrak{a}) \subset \mathfrak{a}^*$ be the set of nonzero restricted roots of the pair $(\mathfrak{g},\,\mathfrak{a})$, i.e. the set of nonzero α such that

$$\mathfrak{g}_{\mathbb{C},\,\alpha} = \{X \in \mathfrak{g}_{\mathbb{C}} \mid \forall\, H \in \mathfrak{a}_{\mathbb{C}},\, [H,\,X] = \alpha\,(H)\,X\} \neq \{0\}.$$

It is clear that $\mathfrak{g}_{\mathbb{C},\,\alpha} = \mathfrak{u}_{\mathbb{C},\,\alpha}$, $\mathfrak{g}_{\mathbb{C},\,\alpha} = \mathfrak{g}_\alpha + i\,\mathfrak{g}_\alpha$ where

$$\mathfrak{g}_\alpha = \mathfrak{g}_{\mathbb{C},\,\alpha} \cap \mathfrak{g} = \{X \in \mathfrak{g} \mid \forall\, H \in \mathfrak{a},\, [H,\,X] = \alpha\,(H)\,X\}.$$

Since $\mathfrak{b}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}}$, the set of roots is preserved under duality, i.e. $\Sigma\,(\mathfrak{u},\,\mathfrak{b}) = \Sigma\,(\mathfrak{g},\,\mathfrak{a})$, where we view these roots as $\mathbb{C}$-linear functionals on $\mathfrak{b}_{\mathbb{C}}$. The eigenspace $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$. Let $\mathfrak{m} = \{X \in \mathfrak{k} \mid [\mathfrak{a},\,X] = 0\}$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Then $\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m}$ and hence

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha.$$

A point $X \in \mathfrak{a}$ is called regular if $\alpha(X) \neq 0$ for all $\alpha \in \Sigma$, otherwise singular. The subset $\mathfrak{a}^{\mathrm{reg}} \subset \mathfrak{a}$ of regular elements in $\mathfrak{a}$ consists of the complement of finitely many hyperplanes, and its connected components are called Weyl chambers. Fix a Weyl chamber $\mathfrak{a}^+$ and call a root α positive if α has positive values on $\mathfrak{a}^+$. Fix a set $\Sigma^+ \subset \Sigma$ of positive restricted roots and fix a compatible set $\Delta^+ \subset \Delta$ of positive roots. Then $\Sigma = \Sigma^+ \cup (-\Sigma^+)$. The positive Weyl chamber is given by

$$\mathfrak{a}^+ := \{H \in \mathfrak{a} :\; \alpha\,(H) > 0,\, \forall\, \alpha \in \Sigma^+\}.$$

This is an open polyhedral cone. Let $\mathfrak{b}^+ = i\,\mathfrak{a}^+$. Let $A^+ = \exp\,\mathfrak{a}^+$ and $B^+ = \exp\,\mathfrak{b}^+$.

If $\alpha \in \Sigma$, it can happen that either $\alpha/2 \in \Sigma$ or $2\,\alpha \in \Sigma$, but not both. A root $\alpha \in \Sigma$ is said to be unmultiplicable if $2\,\alpha \notin \Sigma$ and indivisible if $\alpha/2 \notin \Sigma$. Denote by

$$\Sigma_* = \{\alpha \in \Sigma \;\mid\; 2\,\alpha \notin \Sigma\}, \qquad \Sigma_{\mathrm{i}} = \{\alpha \in \Sigma \;\mid\; \frac{1}{2}\,\alpha \notin \Sigma\}.$$

Both Σ_* and Σ_i are reduced root systems. Set $\Sigma_*^+ = \Sigma_* \cap \Sigma^+$. Note that U/K is irreducible if and only if Σ_i is irreducible[11].

A root $\alpha \in \Sigma^+$ is called simple if it can not be written as a sum $\alpha = \beta + \gamma$ where β and γ are positive roots. Recall that

$$n = \mathrm{rank}\; U/K = \dim\,\mathfrak{b} = \text{the number of simple roots}.$$

[11] A root system is called irreducible if it can not be decomposed into two nonempty disjoint orthogonal subsets.

Let $\Pi = \{\alpha_j\}_{j=1}^n$ be the fundamental system of simple roots associated with Σ^+ which is a basis of $i\,\mathfrak{b}^*$. For $j = 1, \ldots, n$ choose

$$\beta_j = \begin{cases} \alpha_j, & \text{if } 2\,\alpha_j \notin \Sigma \\ 2\,\alpha_j, & \text{if } 2\,\alpha_j \in \Sigma. \end{cases}$$

Then $\Pi_* = \{\beta_j\}_{j=1}^n$ consists of simple roots in Σ_*^+ which is also a basis of $i\,\mathfrak{b}^*$. Define the dual basis $\{\omega_j\}_{j=1}^n$ to be the linear functionals $\omega_j \in i\,\mathfrak{b}^*$ satisfying

$$\frac{\langle \omega_i,\, \beta_j \rangle}{\langle \beta_j,\, \beta_j \rangle} = \delta_{i,\,j}, \quad 1 \le i,\, j \le n. \tag{2.3}$$

The weights ω_j are the class 1 fundamental weights for $(\mathfrak{u},\, \mathfrak{k})$.

For any $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ and $\alpha \in i\,\mathfrak{b}^*$ with $\alpha \neq 0$, set

$$\lambda_\alpha := \frac{\langle \lambda,\, \alpha \rangle}{\langle \alpha,\, \alpha \rangle}.$$

We have $2\,\lambda_\alpha = \lambda_{\alpha/2}$. The restricted integral weight lattice of Σ is the set

$$P = \{\lambda \in i\,\mathfrak{b}^* \;\mid\; \lambda_\alpha \in \mathbb{Z},\ \forall\, \alpha \in \Sigma\}.$$

An element in $\mathfrak{b}_{\mathbb{C}}^* \setminus P$ is said to be generic. The set of dominant restricted integral weights is

$$P^+ = \{\lambda \in i\,\mathfrak{b}^* \;\mid\; \lambda_\alpha \in \mathbb{Z}^+,\ \forall\, \alpha \in \Sigma^+\}.$$

In fact, it is enough to use Σ_* instead of Σ in the constraint of P and P^+.

For $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$, we define the function e^λ on $B_{\mathbb{C}}$ by

$$e^\lambda\,(a) = a^\lambda := e^{\lambda\,(\log\, a)} \in \mathbb{C}^*, \qquad a \in B_{\mathbb{C}}.$$

If $\lambda \in P$, e^λ is single-valued on $B_{\mathbb{C}}$. So $a \mapsto a^\lambda$ is a character on $B_{\mathbb{C}}$ if $\lambda \in P$. Denote by $\mathbb{C}\,[P]$ the group algebra over $\mathbb{C}$ generated by e^λ with $\lambda \in P$. An element of $\mathbb{C}\,[P]$ is an exponential polynomial on $B_{\mathbb{C}}$ of the form $\sum_{\lambda \in P} c_\lambda\, e^\lambda$ where $c_\lambda \in \mathbb{C}$ and $c_\lambda \neq 0$ for only finitely many $\lambda \in P$. It satisfies

$$e^\lambda \cdot e^\mu = e^{\lambda+\mu}, \qquad (e^\lambda)^{-1} = e^{-\lambda}, \qquad e^0 = 1.$$

For any $0 \neq \alpha \in \Sigma$, define a linear transformation $r_\alpha : i\,\mathfrak{b}^* \to i\,\mathfrak{b}^*$ by

$$r_\alpha\,(\lambda) := \lambda - 2\,\frac{\langle \lambda,\, \alpha \rangle}{\langle \alpha,\, \alpha \rangle}\,\alpha = \lambda - 2\,\lambda_\alpha\,\alpha,\ \forall\, \lambda \in i\,\mathfrak{b}^*.$$

Denote the hyperplane with normal α by

$$\alpha^\perp = \{\beta \in i\,\mathfrak{b}^* \;\mid\; \langle \beta,\, \alpha \rangle = 0\}.$$

If $\beta \in \alpha^\perp$ then $r_\alpha\,(\beta) = \beta$ (this says that the fixed points of r_α constitute a hyperplane in $i\,\mathfrak{b}^*$) and if $\beta = k\,\alpha$ with $k \in \mathbb{R}$ then $r_\alpha\,(\beta) = -\beta$. Thus r_α is the reflection in the hyperplane $\alpha^\perp$. In particular, r_α is an orthogonal transformation[12], $\det\, r_\alpha = -1$, and $r_\alpha^2 = \mathrm{id}$.

Let $W = W(\Sigma)$ be he finite group of orthogonal transformations of $i\,\mathfrak{b}^*$ generated by r_α with $\alpha \in \Sigma$. Then W is called the Weyl group associated to Σ. Note that $W(\Sigma) = W(\Sigma_i)$. Let $W(\mathfrak{h}) = W(\mathfrak{h}, \Delta)$ be the Weyl group of $\mathfrak{u}$ relative to $\mathfrak{h}$, i.e. it is generated by the reflections along the roots in Δ. Let

$$\begin{aligned} Z_K(\mathfrak{b}) &= \{k \in K \mid \mathrm{Ad}(k)\, X = X,\ \forall X \in \mathfrak{b}\}, \\ N_K(\mathfrak{b}) &= \{k \in K \mid \mathrm{Ad}(k)\, \mathfrak{b} = \mathfrak{b}\} \end{aligned}$$

be the centralizer and normalizer of $\mathfrak{b}$ in K, respectively. Similar for $N_U(\mathfrak{b})$ and $Z_U(\mathfrak{b})$. It is known that

$$W \cong N_K(\mathfrak{b})/Z_K(\mathfrak{b}) \cong N_U(\mathfrak{b})/Z_U(\mathfrak{b}).$$

The Weyl group action extends to $i\,\mathfrak{b}$ by duality[13], and then to $\mathfrak{b}_{\mathbb{C}}$ and $\mathfrak{b}_{\mathbb{C}}^*$ by $\mathbb{C}$-linearity, and to B and $B_{\mathbb{C}}$ by the exponential map. Moreover, W acts on functions f on any of these spaces by $(w\,f)(x) := f(w^{-1}x)$ for $w \in W$. The lattice P is W-invariant, and W acts on $\mathbb{C}[P]$ by $w(e^\lambda) := e^{w\,\lambda}$ for all $w \in W$.

Note that

$$W\,\mathfrak{a}^+ = \{H \in \mathfrak{a} :\ \alpha(H) \neq 0,\ \forall\, \alpha \in \Sigma^+\} = \mathfrak{a}^{\mathrm{reg}}$$

is open and dense in $\mathfrak{a}$. If $w \neq e$ then $w\,\mathfrak{a}^+ \cap \mathfrak{a}^+ = \emptyset$. The set

$$\begin{aligned} \mathfrak{b}_{\mathbb{C}}^{\mathrm{reg}} &:= \{X \in \mathfrak{b}_{\mathbb{C}} \mid e^{2\,\alpha(X)} \neq 1,\ \forall\, \alpha \in \Sigma^+\} \\ &= \{X \in \mathfrak{b}_{\mathbb{C}} \mid \alpha(X) \notin \pi\, i\,\mathbb{Z},\ \forall\, \alpha \in \Sigma^+\} \end{aligned}$$

consists of the regular points of $\mathfrak{b}_{\mathbb{C}}$ for the action of W. Note that $\mathfrak{a}^+$ and $\mathfrak{b}^+$ are subsets of $\mathfrak{b}_{\mathbb{C}}^{\mathrm{reg}}$. We can also define the set of regular points in Lie group level by

$$B_{\mathbb{C}}^{\mathrm{reg}} := \exp\, \mathfrak{b}_{\mathbb{C}}^{\mathrm{reg}}, \qquad A^{\mathrm{reg}} = B_{\mathbb{C}}^{\mathrm{reg}} \cap A = \exp\, \mathfrak{a}^{\mathrm{reg}}.$$

Then $A^+ \subset A^{\mathrm{reg}}$, $W\,A^+ = A^{\mathrm{reg}}$ is open and dense in A, and $A \setminus A^{\mathrm{reg}}$ has measure zero.

For the set $\{\omega_1, \ldots, \omega_n\} \subset P^+$, we put

$$z_j = \sum_{w \in W/W^{\omega_j}} e^{w\,\omega_j}, \quad j = 1, \ldots, n$$

where W^{ω_j} is the subgroup of W that stabilizes ω_j. The z_j are called the fundamental W-invariant exponential polynomials, and it is well known from [2, p.188] (or by Chevalley's theorem) that

$$\mathbb{C}[P]^W = \mathbb{C}[z_1, \ldots, z_n].$$

[12] A linear transformation from a vector space V into itself is orthogonal if it preserves the inner product on V.

[13] This means we define the Weyl group action on $i\,\mathfrak{b}$ by $\langle w \cdot X, \lambda\rangle = \langle X, w^{-1} \cdot \lambda\rangle$ for $w \in W$ and $X \in i\,\mathfrak{b}$. It is shown that this action is same as $w(X) = \mathrm{Ad}(w)\, X$.

A multiplicity function on Σ is a W-invariant function $m : \Sigma \to \mathbb{C}$. Set $m_\alpha := m(\alpha)$. Thus, $m_{w\alpha} = m_\alpha$ for all $\alpha \in \Sigma$ and $w \in W$. Denote by

$$\mathcal{M} = \{m = (m_\alpha) \mid m_\alpha \in \mathbb{C},\, m_{w\alpha} = m_\alpha,\, \forall\, \alpha \in \Sigma,\, w \in W\}$$

the $\mathbb{C}$-vector space of $\mathbb{C}$-valued multiplicity functions on Σ. It is a finite-dimensional subspace of $\mathbb{C}^\Sigma$, and $\mathcal{M} \cong \mathbb{C}^k$, k equal to the number of conjugacy classes of roots in Σ. A multiplicity function m is said to be positive if $m_\alpha \geq 0$ for all $\alpha \in \Sigma$. Given $m \in \mathcal{M}$ we define the following functions on $B_\mathbb{C}$:

$$\begin{aligned} \rho(m) &:= \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha\, \alpha \in i\, \mathfrak{b}^* \\ \upsilon(m) &:= \prod_{\alpha \in \Sigma^+} (e^\alpha - e^{-\alpha})^{m_\alpha} = e^{2\rho(m)} \prod_{\alpha \in \Sigma^+} (1 - e^{-2\alpha})^{m_\alpha} \\ \delta(m) &:= \prod_{\alpha \in \Sigma^+} |e^\alpha - e^{-\alpha}|^{m_\alpha}. \end{aligned}$$

Since we are working on compact symmetric spaces, the multiplicity function m in our case is geometric[14], that is, $m : \Sigma \to \mathbb{R}^+$ is defined by

$$m_\alpha := \dim_\mathbb{R} \mathfrak{g}_\alpha = \dim_\mathbb{C} \mathfrak{u}_{\mathbb{C},\alpha} \geq 0, \qquad \forall\, \alpha \in \Sigma.$$

In contrast to the diagonalization of a Cartan subalgebra of a complex Lie algebra where the root spaces are always one dimensional, the root $\alpha \in \Sigma$ has a multiplicity m_α which may exceed 1.

In chapters 4 and 5 we will discuss the various spectral problems associated with commutative algebra $\mathbb{D}_l$. For that we impose the restriction on m (always satisfied for group values[15]):

$$m_\alpha + m_{\alpha/2} \geq 0 \quad \text{and} \quad m_\alpha \geq 0, \qquad \forall\, \alpha \in \Sigma_*. \tag{2.4}$$

Because of this constraint the function $\delta(m)$ is a nonnegative continuous function on all of $B_\mathbb{C}$ and thus $\delta(m, b)\, db$ is a positive measure on B, whereas the function $\upsilon(m)$ is viewed as a multivalued holomorphic function on $B_\mathbb{C}^{\text{reg}}$ obtained by analytic continuation of $\delta(m)$ on $A^+ \subset B_\mathbb{C}^{\text{reg}}$.

Similarly for a multiplicity function m on Δ we define

$$\rho(m, \mathfrak{h}) := \frac{1}{2} \sum_{\beta \in \Delta^+} m_\beta\, \beta \in i\, \mathfrak{h}^*.$$

Note that $\dim \mathfrak{u}_\mathbb{C}^\beta = 1$ for any $\beta \in \Delta$ (as $\mathfrak{h}$ is a Cartan subalgebra), so in the geometric case $m_\beta = 1$ and

$$\rho(m, \mathfrak{h}) = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta.$$

---

[14] A multiplicity function m is geometric if there is a Riemannian symmetric space with restricted root system Σ such that m_α is the multiplicity of the root α for all $\alpha \in \Sigma$.

[15] It means m_α is an integer for all $\alpha \in \Sigma$.

When U/K is fixed, m is fixed and no need to underline their dependence on m, we then write $\delta(m) = \delta$, $\rho(m) = \rho$, and $\rho(m, \mathfrak{h}) = \rho(\mathfrak{h})$.

Proposition 2.6. *We have the relation*

$$\rho(\mathfrak{h})|_{\mathfrak{b}} = \rho. \tag{2.5}$$

Proof. Note that

$$\Sigma = \{\beta|_{\mathfrak{b}} \mid \beta \in \Delta,\, \beta|_{\mathfrak{b}} \neq 0\}.$$

For $\alpha \in \Sigma$, let

$$\Delta(\alpha) := \{\beta \in \Delta \mid \beta|_{\mathfrak{b}} = \alpha\}.$$

Then $\dim_{\mathbb{R}} \mathfrak{g}_\alpha$ equals the number of elements in the set $\Delta(\alpha)$, i.e. $m_\alpha = \#(\Delta(\alpha))$. Since Δ^+ is compatible with Σ^+, it follows that if $\beta \in \Delta^+$ then $\beta|_{\mathfrak{b}}$ is either zero or is in Σ^+. So

$$2\rho(\mathfrak{h}) = \sum_{\beta\in\Delta^+} \beta = \sum_{\alpha\in\Sigma^+} \sum_{\beta\in\Delta(\alpha)} \beta + \sum_{\substack{\beta\in\Delta^+ \\ \beta|_{\mathfrak{b}}=0}} \beta.$$

It follows that

$$2\rho(\mathfrak{h})|_{\mathfrak{b}} = \sum_{\alpha\in\Sigma^+} \sum_{\beta\in\Delta(\alpha)} \beta|_{\mathfrak{b}} = \sum_{\alpha\in\Sigma^+} (\#\Delta(\alpha))\,\alpha = \sum_{\alpha\in\Sigma^+} m_\alpha\,\alpha = 2\rho.$$

This gives the desired result. ∎

That U/K is irreducible and K is not semisimple is equivalent to the fact that Σ is of type BC_n (or C_n) and the multiplicity of long roots is 1. This is the content of the following result due to either the theory of strongly orthogonal roots (cf. Moore [20, Theorem 5.2] or [11, p. 528]) or the classification of root systems (cf. [11, p.532]):

Theorem 2.7. *Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be an orthogonal basis of $i\,\mathfrak{b}^*$. There are two possibilities for the root system Σ^+:*

Case I: $\Sigma^+ = \{\varepsilon_j \pm \varepsilon_i\,(1 \le i < j \le n),\ 2\,\varepsilon_j\,(1 \le j \le n)\}$

Case II: $\Sigma^+ = \{\varepsilon_j\,(1 \le j \le n),\, \varepsilon_j \pm \varepsilon_i\,(1 \le i < j \le n),\, 2\,\varepsilon_j\,(1 \le j \le n)\}.$

Remark 2.8. To coincide with the setting of [29], our choice of Σ^+ is consistent with the positive roots of $(\mathfrak{g}, \mathfrak{t})$ in [29] through the Cayley transform where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. The choice of strongly orthogonal roots[16] is also the same as that in [29], that is, $\{2\varepsilon_j\}_{j=1}^n$ is a maximal strongly orthogonal subset in Σ^+.

Since U/K is irreducible, Σ is a disjoint union of three W-orbits in Σ corresponding to short, medium, and long roots[17], respectively, that is,

$$\Sigma = \mathcal{O}_s \cup \mathcal{O}_m \cup \mathcal{O}_l.$$

[16] Two roots $\alpha, \beta \in \Sigma$ are called strongly orthogonal if $\alpha \neq \pm\beta$ and $\alpha \pm \beta \notin \Sigma$

[17] They are determined by the lengths of roots. If α is a root, the length of α is given by $\|\alpha\| = \langle \alpha, \alpha \rangle^{1/2}$.

Let $\mathcal{O}_s^+ = \mathcal{O}_s \cap \Sigma^+$, $\mathcal{O}_m^+ = \mathcal{O}_m \cap \Sigma^+$, and $\mathcal{O}_l^+ = \mathcal{O}_l \cap \Sigma^+$. Thus

$$\mathcal{O}_s^+ = \{\varepsilon_i\}, \qquad \mathcal{O}_m^+ = \{\varepsilon_j \pm \varepsilon_i\}, \qquad \mathcal{O}_l^+ = \{2\varepsilon_i\}.$$

Adopt the notation

$$m = (m_\alpha) = (m_s,\ m_m,\ m_l)$$

for root multiplicities of short, medium, and long roots, respectively, where

$$\begin{array}{llll} \text{case I:} & m_s = 0 & m_m = m_{\varepsilon_j \pm \varepsilon_i}\ (i \neq j) & m_l = m_{2\,\varepsilon_i} \equiv 1 \\ \text{case II:} & m_s = m_{\varepsilon_i} & m_m = m_{\varepsilon_j \pm \varepsilon_i}\ (i \neq j) & m_l = m_{2\,\varepsilon_i} \equiv 1, \end{array}$$

given by dimensions of root spaces. The case I is actually of type C_n. We consider it as being of type BC_n with $m_s = 0$ in BC_n. In this way, the root system Σ is of type BC_n in both cases. In view of [11, p. 518, table V, and p. 532, table VI], we give the root structures and root multiplicities of each space listed in the Table 1:

Table 2: Root structures and multiplicities

compact symmetric spaces U/K				
	U	K	Σ	$(m_s,\, m_m,\, m_l)$
1	$\mathrm{SU}\,(p+q)$	$\mathrm{S}\,(\mathrm{U}_p \times \mathrm{U}_q)$	case I $\ p = q$ case II $\ p > q$	$(0,\, 2,\, 1)$ $(2\,(p-q),\, 2,\, 1)$
2	$\mathrm{SO}\,(p+2)$	$\mathrm{SO}\,(p) \times \mathrm{SO}\,(2)$	case I	$(0,\, p-2,\, 1)$
3	$\mathrm{SO}\,(2\,j)$	$\mathrm{U}\,(j)$	case I $\ j$ is even case II $\ j$ is odd	$(0,\, 4,\, 1)$ $(4,\, 4,\, 1)$
4	$\mathrm{Sp}\,(j)$	$\mathrm{U}\,(j)$	case I	$(0,\, 1,\, 1)$
5	$\mathfrak{e}_{6\,(-78)}$	$\mathfrak{so}\,(10) + \mathbb{R}$	case II	$(8,\, 6,\, 1)$
6	$\mathfrak{e}_{7\,(-133)}$	$\mathfrak{e}_6 + \mathbb{R}$	case I	$(0,\, 8,\, 1)$

Remark 2.9. Our multiplicity notation is different from the one used by Heckman and Opdam. The root system R they use is related to our Σ by $R = \{2\,\alpha \mid \alpha \in \Sigma\}$. The multiplicity function k they deal with is related to our m by $k_{2\,\alpha} = m_\alpha/2$. Let R_+ be a set of positive roots in R. We can easily see that the definition of our $\rho(m)$ coincides with their $\rho(k)$. The reason is

$$\begin{aligned} \rho(m) = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \cdot \alpha \ &= \ \frac{1}{2} \sum_{\alpha \in \Sigma^+} 2(k_{2\alpha}) \cdot \alpha \\ &= \ \frac{1}{2} \sum_{\alpha \in \Sigma^+} (k_{2\alpha}) \cdot (2\alpha) \\ &= \ \frac{1}{2} \sum_{\beta \in R_+} k_\beta \cdot \beta \\ &= \ \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \cdot \alpha = \rho(k). \end{aligned}$$

For any $l \in \mathbb{Z}$ define $m_\pm(l) \in \mathcal{M} \cong \mathbb{C}^3$ by

$$\begin{aligned} m_+(l) = m(l) &= (m_s - 2\,|l|,\ m_m,\ m_l + 2\,|l|), \\ m_-(l) &= (m_s + 2\,|l|,\ m_m,\ m_l - 2\,|l|), \end{aligned}$$

with $m_l = 1$. It is clear that $m_\pm(l)$ satisfy (2.4). Let $\rho(l) := \rho(m(l))$. Define

$$\rho_s = \frac{1}{2} \sum_{\alpha \in \mathcal{O}_s^+} \alpha.$$

We then come up with a nice formula in the following proposition which tells the relation between ρ and $\rho(l)$. It plays a role in the description of the restricted highest weights of χ_l-spherical representations of U (see (3.4)).

Proposition 2.10. *We have*

$$\rho(l) = \rho + 2|l|\,\rho_s. \tag{2.6}$$

Proof. Note that $\Sigma^+ = \mathcal{O}_s^+ \cup \mathcal{O}_m^+ \cup \mathcal{O}_l^+$, so

$$\begin{aligned} \rho(l) &= \frac{1}{2} \sum_{\alpha \in \Sigma^+} (m(l))_\alpha \cdot \alpha \\ &= \frac{1}{2} \left[\sum_{\alpha \in \mathcal{O}_s^+} (m_s - 2|l|)\,\alpha + \sum_{\alpha \in \mathcal{O}_m^+} m_m\,\alpha + \sum_{\alpha \in \mathcal{O}_l^+} (m_l + 2|l|)\,(2\alpha) \right] \\ &= \frac{1}{2} \left[\sum_{\alpha \in \mathcal{O}_s^+} m_s\,\alpha + \sum_{\alpha \in \mathcal{O}_m^+} m_m\,\alpha + \sum_{\alpha \in \mathcal{O}_l^+} m_l\,(2\alpha) \right] + \frac{1}{2} \sum_{\alpha \in \mathcal{O}_s^+} 2|l|\,(\alpha) \\ &= \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha\,\alpha + 2|l| \cdot \frac{1}{2} \sum_{\alpha \in \mathcal{O}_s^+} \alpha \\ &= \rho + 2|l|\,\rho_s. \end{aligned}$$

□

Since $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is an orthogonal basis of $i\,\mathfrak{b}^*$, any $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ can be written as

$$\lambda = \sum_{j=1}^{n} \lambda_j\,\varepsilon_j$$

where $\lambda_j = \lambda_{\varepsilon_j} \in \mathbb{C}$. It follows obviously that

$$\lambda_\alpha \geq 0,\ \forall\,\alpha \in \Sigma^+ \quad \Longleftrightarrow \quad \lambda_n \geq \cdots \geq \lambda_1 \geq 0.$$

Choose $H_{\varepsilon_j} \in [\mathfrak{g}_{\varepsilon_j},\, \mathfrak{g}_{-\varepsilon_j}]$ such that $\{H_{\varepsilon_j}\}$ is a basis of $\mathfrak{a}$ which is dual to $\{\varepsilon_j\}$, i.e. $\varepsilon_j\,(H_{\varepsilon_i}) = \delta_{i\,j}$. So any $Z \in \mathfrak{b}_{\mathbb{C}}$ can be written as $Z = \sum z_j\,H_{\varepsilon_j}$ for $z_j \in \mathbb{C}$. We then identify

$$\begin{aligned} \mathfrak{b}_{\mathbb{C}}^* \cong \mathbb{C}^n, &\qquad \lambda \longmapsto (\lambda_1, \ldots, \lambda_n) \\ \mathfrak{b}_{\mathbb{C}} \cong \mathbb{C}^n, &\qquad Z \longmapsto (z_1, \ldots, z_n). \end{aligned}$$

Last, we introduce the Cartan decomposition $U = K\,B\,K$. It says that every $u \in U$ can be written as $u = k\,b\,h$ with k, $h \in K$ and $b \in B$. We have

$$B = \uplus_{w \in W}\ w\,\overline{B^+}$$

and $K\,B^+\,K \subset U$ is open dense. The element b is uniquely determined up to W-invariance, and thus can be chosen in $\overline{B^+}$. So the Cartan decomposition of U is

$$U = K\,\overline{B^+}\,K.$$

As a consequence we get a kind of a polar coordinate decomposition of U/K:

Theorem 2.11. *We have $U/K = K\,\overline{B^+} \cdot o$ and the map $(K/M) \times B^+ \to U/K$ given by*

$$(k\,M,\ b) \longmapsto k\,b\,K$$

is an analytic diffeomorphism from $(K/M) \times B^+$ onto an open dense subset of U/K.

Proof. See, for instance, Corollary 1.2 in [11, Chapter IX]. ☐

This theorem is true for any connected semisimple Lie group U. It guarantees we can write the radial part of the Laplace-Beltrami operator on U/K (or G/K) in terms of coordinates in B^+ (or A^+) as it guarantees the manifold B^+ (or A^+) satisfies the transversality condition (see chapters 4 and 5 for more discussion).

Chapter 3

Fourier Analysis Related to Line Bundles over Compact Symmetric Spaces U/K

This chapter is devoted to a preliminary study of harmonic analysis related to line bundles over compact symmetric spaces, including the theory of highest weights for χ_l-spherical representations, elementary spherical functions of type χ_l, χ_l-spherical Fourier transforms, and the Plancherel formula. The central result is the properties (3.4) and (3.12). We refer to [29] and [18] as the main source for this chapter.

3.1 Harmonic Analysis on Compact Groups

In this section we review some well-known facts for harmonic analysis on compact groups. This material will be presented mostly without proofs, which can be found in [30] and [7].

Proposition 3.1. *Let π be a representation of a compact group U on a finite dimensional vector space V. There exists a Euclidean inner product on V for which π is unitary.*

Theorem 3.2 (Schur's Lemma)**.** *Let V and W be unitary representations of a Lie group G on Hilbert spaces. If V and W are irreducible, then*

$$\dim \operatorname{Hom}_G(V, W) = \begin{cases} 1 & \text{if} \quad V \cong W \\ 0 & \text{if} \quad V \ncong W. \end{cases}$$

In general, the representation V is irreducible if and only if $\operatorname{Hom}_G(V, V) = \mathbb{C}\, I$.

Theorem 3.3 (Schur's Orthogonal Relations)**.** *Let V and W be irreducible finite dimensional unitary representations of a compact Lie group U with U-invariant inner products $(\,\cdot\,,\,\cdot\,)_V$ and $(\,\cdot\,,\,\cdot\,)_W$. If $v_1, v_2 \in V$ and $w_1, w_2 \in W$,*

$$\int_U (gv_1,\, v_2)_V\, \overline{(gw_1,\, w_2)_W}\, dg = \begin{cases} 0 & \text{if} \quad V \ncong W \\ \frac{1}{\dim V}\,(v_1,\, w_1)_V\, \overline{(v_2,\, w_2)_W} & \text{if} \quad V = W. \end{cases}$$

Definition 3.4. Let G be a locally compact Hausdorff group and $\varphi : G \to \mathbb{C}$ a continuous function. φ is said to be *positive definite* if for all finite sets $\{g_1,\, \ldots,\, g_n\}$ of elements in G,

$$\sum_{i,\,j=1}^{n} c_i\, \overline{c_j}\, \varphi\,(g_i^{-1}\, g_j) \geq 0,$$

for all $n \in \mathbb{N}$ and all $c_j \in \mathbb{C}$, $j = 1, \ldots, n$. Such a function φ satisfies

$$\varphi\,(e) \geq 0, \qquad \varphi\,(g^{-1}) = \overline{\varphi\,(g)}, \qquad |\varphi\,(g)| \leq \varphi\,(e).$$

Theorem 12.1 in [13, Chapter III] proved

Proposition 3.5. *We have*

1. *Let π be a unitary representation of locally compact group G on a Hilbert space V. For each vector $v \in V$ the function $g \mapsto (v, \pi(g)\, v)$ on G is positive definite.*

2. *If $\varphi \not\equiv 0$ is a positive definite function on G, then there is a unitary representation π of G on a Hilbert space V such that $\varphi(g) = (v, \pi(g)\, v)$ for a suitable vector $v \in V$. These can be chosen so that v is a cyclic vector[1].*

Let U be a compact group. For each $\pi \in \widehat{U}$ we choose a representative (π, V_π). Let $d(\pi) = \dim V_\pi$. Write $\mathrm{End}\,(V_\pi) = \mathrm{Hom}\,(V_\pi, V_\pi)$ for the set of endomorphisms on V_π. The space $\mathrm{End}\,(V_\pi)$ is a Hilbert space with respect to the Hilbert-Schmidt inner product

$$(T, S)_{\mathrm{HS}} = \mathrm{Tr}\,(S^* \circ T),$$

and $\dim \mathrm{End}\,(V_\pi) = d(\pi)^2$. If $f \in L^1(U)$, we define the operator-valued Fourier transform of f to be $f \mapsto \widehat{f}(\pi) = \pi(f)$ where

$$\pi(f) = \int_U f(u)\, \pi(u)\, du \in \mathrm{End}\,(V_\pi).$$

Since U is compact, it has finite volume. By Hölder inequality $L^2(U) \subseteq L^1(U)$. Thus $\pi(f)$ is well defined when $f \in L^2(U)$. It follows from the Peter-Weyl Theorem and from Schur's orthogonal relations that

Theorem 3.6 (Plancherel's Theorem)**.** *Let U be a compact group and $f \in L^2(U)$. Then f equals the sum of its Fourier series (in L^2-sense):*

$$f(u) = \sum_{\pi \in \widehat{U}} d(\pi)\, \mathrm{Tr}\,(\pi(u)^{-1} \pi(f)) = \sum_{\pi \in \widehat{U}} d(\pi) \sum_{j=1}^{d(\pi)} (\pi(u)^{-1} \pi(f)\, e_j,\ e_j)$$

where for each $\pi \in \widehat{U}$, $\{e_j\}_{j=1}^{d(\pi)}$ is an orthonormal basis for V_π. Moreover, we have the following Plancherel formula

$$\begin{aligned} \|f\|_2^2 &= \sum_{\pi \in \widehat{U}} d(\pi)\, \|\pi(f)\|_{\mathrm{HS}} \\ &= \sum_{\pi \in \widehat{U}} d(\pi)\, \mathrm{Tr}\,(\pi(f)^* \pi(f)) \\ &= \sum_{\pi \in \widehat{U}} d(\pi) \sum_{j=1}^{d(\pi)} \|\pi(f)\, e_j\|^2. \end{aligned}$$

[1] A vector v is a cyclic vector for π if the vector space spanned by $\pi(G)\, v$ is dense in V.

3.2 Representations of Compact Semisimple Lie Groups

3.2.1 Theory of Highest Weights

For each $\pi \in \widehat{U}$ we choose in the equivalence class a concrete representation π on a complex vector space V_π and $d(\pi) = \dim V_\pi$. Since U is compact, π automatically extends to a holomorphic representation $\pi_{\mathbb{C}}$ of $U_{\mathbb{C}}$. If we assume U is a closed subgroup of some unitary group as in Section 2.2 then

$$\pi_{\mathbb{C}}\,(g)^* = \pi_{\mathbb{C}}\,(g^*)$$

where g^* is the complex conjugate transpose of g. The derived representation $(d\,\pi_{\mathbb{C}},\, V_\pi)$ is a Lie algebra representation of $\mathfrak{u}_{\mathbb{C}}$ satisfying $\exp\,(d\,\pi_{\mathbb{C}}\,Z) = \pi_{\mathbb{C}}\,(\exp\,Z)$ where the differential of $\pi_{\mathbb{C}}$ is defined by

$$d\,\pi_{\mathbb{C}}\,(Z) = \frac{d}{d\,t}\,\pi_{\mathbb{C}}\,(\exp\,(t\,Z))\Big|_{t=0}, \quad \forall\, Z \in \mathfrak{u}_{\mathbb{C}}.$$

Recall that a triangular decomposition of $\mathfrak{u}_{\mathbb{C}}$ is given by

$$\mathfrak{u}_{\mathbb{C}} = \widetilde{\mathfrak{n}}_{\mathbb{C}}^{-} \;\oplus\; \mathfrak{h}_{\mathbb{C}} \;\oplus\; \widetilde{\mathfrak{n}}_{\mathbb{C}}^{+}$$

where

$$\widetilde{\mathfrak{n}}_{\mathbb{C}}^{\pm} \;=\; \bigoplus_{\beta \in \Delta^{\pm}} \mathfrak{u}_{\mathbb{C}}^{\beta}.$$

Also recall that the weight space decomposition of V_π is given by

$$V_\pi = \bigoplus_{\lambda \in \Delta\,(V_\pi)} V_{\pi,\,\lambda}$$

where the weight space

$$V_{\pi,\,\lambda} = \{v \in V_\pi \;\mid\; d\pi_{\mathbb{C}}\,(H) = \lambda\,(H)\,v,\; \forall\, H \in \mathfrak{h}_{\mathbb{C}}\}$$

is of dimension 1, and $\Delta\,(V_\pi) \subset \mathfrak{h}_{\mathbb{C}}^*$ is the set of the weights of V_π, that is, it consists of those λ such that $V_{\pi,\,\lambda}$ is nonzero. A $\mu \in \mathfrak{b}_{\mathbb{C}}^*$ is called a restricted weight if there exists a $0 \neq w \in V_\pi$ such that

$$d\,\pi(H)\,w = \mu(H)\,w, \quad \forall\, H \in \mathfrak{b}.$$

The restricted weights of V_π coincide with the restrictions to $\mathfrak{b}$ of the weights of V_π (see proposition 4.21 in [13]).

By $\mathbb{C}$-linearity, $\lambda \in \Delta\,(V_\pi)$ is completely determined by its restriction to either $\mathfrak{h}$ or $i\,\mathfrak{h}$. Thus we can interchangeably view λ as an element of any of the dual spaces $\mathfrak{h}_{\mathbb{C}}^*$, $(i\,\mathfrak{h})^*$ (real valued), or $i\,\mathfrak{h}^*$ (purely imaginary valued).

Let $\lambda \in \Delta(V_\pi)$. A nonzero vector $v \in V_{\pi,\lambda}$ is called a highest weight vector of the representation (π, V_π) with the weight λ if $\widetilde{\mathfrak{n}}_{\mathbb{C}}^{+} v = 0$, i.e.

$$d\pi_{\mathbb{C}}(Z)\, v = 0, \quad \forall\, Z \in \widetilde{\mathfrak{n}}_{\mathbb{C}}^{+}.$$

The corresponding linear functional λ is called a highest weight of the representation (π, V_π).

3.2.2 Spherical Representations

Let $\Lambda^{+}(U) \subset i\,\mathfrak{h}^*$ be the set of highest weights of irreducible representations of U. For any general compact group (not necessarily semisimple nor simply connected) there is a one-to-one correspondence between $\widehat{U}$ and $\Lambda^{+}(U)$, given by each $\pi \in \widehat{U}$ being sent to its highest weight λ_π. Let $\Lambda^{+}(\mathfrak{h})$ be the set of dominant integral weights on $\mathfrak{h}$. Then

$$\Lambda^{+}(\mathfrak{h}) = \left\{ \lambda \in \mathfrak{h}_{\mathbb{C}}^{*} \;\middle|\; \frac{2\langle \lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle} \in \mathbb{Z}^{+},\ \forall\, \alpha \in \Delta^{+} \right\}$$

Note that all elements of $\Lambda^{+}(\mathfrak{h})$ take purely imaginary values on $\mathfrak{h}$. So $\Lambda^{+}(\mathfrak{h}) \subset i\,\mathfrak{h}^*$. For any compact group, $\Lambda^{+}(U) \subseteq \Lambda^{+}(\mathfrak{h})$ with equality holds if and only if U is simply connected and semisimple. Since we assume U is simply connected and semisimple, $\Lambda^{+}(U) = \Lambda^{+}(\mathfrak{h})$ and so $\Lambda^{+}(\mathfrak{h})$ is a parametrization of $\widehat{U}$ in our case.

For $\lambda \in \Lambda^{+}(U)$ choose an irreducible unitary representation (π_λ, V_λ) of U and define the space of K-fixed vectors in V_λ by

$$V_\lambda^K = \{ v \in V_\lambda \mid \pi_\lambda(k)\, v = v,\ \forall\, k \in K \}.$$

If $V_\lambda^K \neq \{0\}$, then π_λ is said to be a spherical representation of U. In this case, $\dim V_\lambda^K = 1$. Otherwise, $\dim V_\lambda^K = 0$. Denote by $\widehat{U}_0$ the set of all irreducible spherical representations of U. Thus

$$\widehat{U}_0 = \{ \pi_\lambda \in \widehat{U} \mid V_\lambda^K \neq 0 \}$$

Let $\Lambda_0^{+}(\mathfrak{h})$ denote the subset of $\Lambda^{+}(U)$ which consists of highest weights of irreducible spherical representations of U, i.e.

$$\Lambda_0^{+}(\mathfrak{h}) = \{ \lambda \in \Lambda^{+}(U) \mid V_\lambda^K \neq 0 \} \subset i\,\mathfrak{h}^*.$$

So there is a bijection $\Lambda_0^{+}(\mathfrak{h}) \cong \widehat{U}_0$ given by $\lambda \mapsto \pi_\lambda$.

Theorem 3.7. *Let (π_λ, V_λ) be an irreducible unitary representations of U with the highest weight λ and v a nonzero highest weight vector with weight λ. Then the followings are equivalent:*

1. *π_λ is a spherical representation.*

2. *v is invariant under $M = Z_K(\mathfrak{b})$, that is $\pi_\lambda(m)\, v = v$ for $m \in M$.*

3. *For $H \in \mathfrak{h} \cap \mathfrak{k}$, $\lambda(H) = 0$ so that $\lambda \in i\,\mathfrak{b}^*$ and for*

$$H \in \mathfrak{b}_K := \{H \in \mathfrak{b} \ \mid \ \exp H \in K\},$$

   we have $\lambda(H) \in 2\pi\, i\, \mathbb{Z}$.

Proof. See Theorem IV.4.2 in [6]. □

In the following we recall an identification of $\Lambda_0^+(\mathfrak{h})$ which was proved by Helgason (see [12, p.535 and p.538]). Also see [22, p.205].

Theorem 3.8 (Helgason)**.** *Let U be a compact simply connected semisimple Lie group and K the fixed point group of an involutive automorphism of U. Then*

$$\Lambda_0^+(\mathfrak{h}) = \left\{ \lambda \in i\,\mathfrak{h}^* \ \middle|\ \lambda|_{\mathfrak{h}\cap\mathfrak{k}} = 0 \quad \textit{and} \quad \frac{\langle \lambda,\, \alpha\rangle}{\langle \alpha,\, \alpha\rangle} \in \mathbb{Z}^+,\ \forall \alpha \in \Sigma^+ \right\}.$$

Recall that $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) \oplus \mathfrak{b}$. For $\lambda \in \Lambda_0^+(\mathfrak{h})$, write $\mu = \lambda|_{\mathfrak{b}} \in i\,\mathfrak{b}^*$. For any $\alpha \in \Sigma^+$,

$$\frac{\langle \lambda,\, \alpha\rangle}{\langle \alpha,\, \alpha\rangle} \in \mathbb{Z}^+ \iff \frac{\langle \mu,\, \alpha\rangle}{\langle \alpha,\, \alpha\rangle} \in \mathbb{Z}^+$$

because $\lambda|_{\mathfrak{h}\cap\mathfrak{k}} = 0$. Denote by

$$\Lambda_0^+ = \{\mu \in i\,\mathfrak{b}^* \ \mid \ \mu = \lambda|_{\mathfrak{b}},\, \lambda \in \Lambda_0^+(\mathfrak{h})\} \subset i\,\mathfrak{b}^*$$

the set of the restrictions on $\mathfrak{b}$ of highest weights of irreducible spherical representations of U. Thus this set is in bijective correspondence with $\Lambda_0^+(\mathfrak{h})$ via $\mu \mapsto \lambda = (\mu,\, 0)$. This allows us to view μ as a linear form on $\mathfrak{h}_{\mathbb{C}}^*$ with 0 on $\mathfrak{h} \cap \mathfrak{k}$. Therefore,

$$\Lambda_0^+ \cong \widehat{U}_0, \qquad \mu \longmapsto \pi_\mu.$$

We then come to the following fact which gives a parametrization of Λ_0^+.

Corollary 3.9. *Let U and K be the same as in the previous theorem. Then*

$$\Lambda_0^+ = \left\{ \mu \in i\,\mathfrak{b}^* \ \middle|\ \frac{\langle \mu,\, \alpha\rangle}{\langle \alpha,\, \alpha\rangle} \in \mathbb{Z}^+,\ \forall\, \alpha \in \Sigma^+ \right\}. \tag{3.1}$$

This corollary tells us that the highest restricted weights of irreducible spherical representations of U are the dominant restricted integral weights, that is,

$$\Lambda_0^+ = P^+.$$

In general, if we do not assume that U is simply connected, more work should be taken (see [23, p.614] or [25, Remark 5.3]).

Recall (2.3) for the definition of the fundamental weights ω_j. It follows from [13, Chapter II, Proposition 4.23], that we can also parametrize Λ_0^+ in terms of $\mathbb{Z}^+$-combinations of ω_j:

Proposition 3.10. *Let $\mu \in i\,\mathfrak{b}^*$. Then $\mu \in \Lambda_0^+$ if and only if*

$$\mu = \sum_{j=1}^{n} k_j\,\omega_j, \qquad k_j \in \mathbb{Z}^+.$$

Furthermore, according to Theorem 3.7, we obtain the following parametrization of Λ_0^+:

Proposition 3.11. *Let $\mu \in i\,\mathfrak{b}^*$. Then $\mu \in \Lambda_0^+$ if and only if*

$$\mu\,(\mathfrak{b}_K) \subset 2\pi\,i\,\mathbb{Z}, \qquad \textit{and} \qquad \langle \mu,\, \alpha\rangle \geq 0,\, \forall\,\alpha \in \Sigma^+.$$

In summary, we set up a bijection from the set Λ_0^+ of highest restricted weights of irreducible spherical representations onto the set $\widehat{U}_0$ of equivalence classes of irreducible spherical representations via $\mu \mapsto \pi_\mu$.

3.2.3 χ_l-spherical Representations

We study spherical representations in the previous section. They are irreducible unitary representations of U which have nonzero K-fixed vectors. In this section we will discuss χ_l-spherical representations where $l \in \mathbb{Z}$. They are those which have nonzero K_1-fixed vectors. When $l = 0$, χ_0-spherical representations are exactly spherical representations. The necessary and sufficient condition on the highest weight λ of a finite dimensional irreducible representation π of U in order for π has 1-dimensional K-types is similar to that of Helgason's theorem. We will discuss it in the following.

Definition 3.12. Let $(\pi,\, V)$ be an irreducible unitary representation of U and χ_l a nontrivial character of K. Let

$$V^l := \{v \in V \ \mid\ \pi\,(k)\,v = \chi_l\,(k)\,v,\ \forall\,k \in K\}.$$

be the subspace of V consisting of χ_l-coinvariant vectors. The representation $(\pi,\, V)$ is said to be *χ_l-spherical* if $V^l \neq \{0\}$. Fix $l \in \mathbb{Z}$ denote by $\widehat{U}_l$ the set of all irreducible χ_l-spherical representations of U.

The spherical representations are those where χ_l is a trivial character, i.e. $l = 0$. In this case, $V^0 = V^K$ consists of K-fixed vectors.

Proposition 3.13. *There is a $l \in \mathbb{Z}$ such that π is a χ_l-spherical representation of U if and only if π has a K_1-fixed vector.*

Proof. $\Rightarrow$: Recall that $\chi_l\,|_{K_1} = 1$. If $\pi \in \widehat{U}_l$, then there is $0 \neq v \in V^l$ such that

$$\pi\,(k)\,v = \chi_l\,(k)\,v = 1 \cdot v = v, \qquad \forall\,k \in K_1.$$

So π has v as a K_1-fixed vector. This implies that $V^K \subseteq V^l \subseteq V^{K_1}$, where $V^{K_1} = \{v \in V \ \mid\ \pi(k)\,v = v,\, \forall\,k \in K_1\}$.

$\Leftarrow$: If π has a nonzero K_1-fixed vector then $V^{K_1} \neq \{0\}$. Since $V^{K_1} \subset V$, it is finite dimensional. Recall that $K = T\,K_1$ where $T = \exp(\mathbb{R}\,Z)$ is a one dimensional abelian compact subgroup of K. Since T commutes with K_1, we see that T leaves V^{K_1} invariant. So V^{K_1} is a representation of T (or even a representation of $T/(T \cap K_1)$ since V^{K_1} is stable under the action of K_1). But $T/(T \cap K_1) \cong K/K_1$ which are compact. Thus V^{K_1}, as a finite dimensional representation of a compact group, can be written as a finite sum of irreducibles:

$$(\pi\,|_K,\ V^{K_1}) \cong \bigoplus_{j=1}^{d} (\chi_{l_j},\ \mathbb{C})$$

where χ_{l_j} are some characters of K and $\{1, \ldots, d\}$ is some finite index set. Hence a $v \in V^{K_1}$ can be written as $v = v_1 + \cdots + v_n$ such that $\pi(k)\,v_j = \chi_{l_j}(k)\,v_j$ for $k \in K$ and for all j. So π is a χ_{l_j}-spherical representation for some $l_j \in \mathbb{Z}$. For this direction also see Theorem 7.2 in [29] for a different proof. $\square$

Proposition 3.14. *For the compact symmetric pair $(U,\,K)$, if $(\pi,\,V)$ is a χ_l-spherical representation of U, then* $\dim V^l = 1$.

Proof. Consider π as a representation of the commutative algebra

$$L^1(U//K,\,\chi_l) \cong \left\{ f \in L^1(U) \;\middle|\; \begin{array}{c} f(k_1\,u\,k_2) = \chi_l(k_1k_2)^{-1}\,f(u), \\ \forall k_1, k_2 \in K, \text{ and a.e. } u \in U \end{array} \right\}.$$

Since V^l is irreducible under the action π of $L^1(U//K,\,\chi_l)$, it is one-dimensional by Schur's lemma. Also see Theorem 7.2 in [29] for a different proof. $\square$

We have the canonical decomposition (see Proposition 3.20)

$$(\ell,\ L^2(U/K,\,\chi_l)) \cong_U \bigoplus_{\pi \in \widehat{U}_l} (\pi,\ V_\pi) \tag{3.2}$$

where ℓ stands for the left regular representation of U on $L^2(U/K,\,\chi_l)$ given by $\ell(u)\,f(x) = f(u^{-1}\,x)$ for all $u \in U$, and π occurs with multiplicity one.

For $\lambda \in \Lambda^+(U)$ choose an irreducible unitary representation $(\pi_\lambda,\,V_\lambda)$ of U. Then

$$\widehat{U}_l = \{\pi_\lambda \in \widehat{U} \ \mid\ V_\lambda^l \neq 0\}.$$

Let $\Lambda_l^+(\mathfrak{h})$ be the set of highest weights of irreducible χ_l-spherical representations of U. Then

$$\Lambda_l^+(\mathfrak{h}) = \{\lambda \in \Lambda^+(U) \ \mid\ V_\lambda^l \neq 0\} \subset \mathfrak{h}_{\mathbb{C}}^*.$$

So there is a bijection $\Lambda_l^+(\mathfrak{h}) \cong \widehat{U}_l$ given by $\lambda \mapsto \pi_\lambda$.

Recall that $\mathfrak{b} \subseteq \mathfrak{q}$ is a Cartan subspace and we extend it to a Cartan subalgebra $\mathfrak{h}$ with $\mathfrak{h} = \mathfrak{b} \oplus (\mathfrak{h} \cap \mathfrak{k})$. Following from [29], we decompose $\mathfrak{h} \cap \mathfrak{k}$ as

$$\mathfrak{h} \cap \mathfrak{k} = (\mathfrak{h} \cap \mathfrak{k}) \cap \mathfrak{k}_1 \,\oplus\, \mathbb{R}\,X$$

where X is defined as in [29, p.285, (4.4)] so that

1. $e^{tX} \in K_1$ if and only if $t \in 2\pi\, i\, \mathbb{Z}$,

2. $Z - X \in \mathfrak{k}_1$ where Z is the same as in Proposition 2.4 (see Lemma 4.3 in [29]).

Let $\lambda\,(i\,X) = \mu_0$, a fixed integer (cf. [29, p.290]). When χ_l is fixed for some $l \in \mathbb{Z}$, μ_0 is then fixed. According to the possible two types of Σ (cf. Table 2), there are two cases:

$$\begin{aligned} \text{Case I:} \quad & \mathfrak{h} = \mathfrak{b} \,\oplus\, (\mathfrak{h} \cap \mathfrak{k}) \cap \mathfrak{k}_1 \\ \text{Case II:} \quad & \mathfrak{h} = \mathfrak{b} \,\oplus\, (\mathfrak{h} \cap \mathfrak{k}) \cap \mathfrak{k}_1 \,\oplus\, \mathbb{R}\, X, \end{aligned}$$

where in Case I, $X = 0$ (whence $\mu_0 = 0$) and $\mathfrak{h} \cap \mathfrak{k} \subseteq \mathfrak{k}_1$. The condition on λ for one-dimensional K-types to occur was given by [29]:

1. $\lambda\,\big|_{(\mathfrak{h}\cap\mathfrak{k})\cap\mathfrak{k}_1} = 0$,

2. λ has to satisfy a certain integrality condition (see below).

Let $\mu := \lambda|_{\mathfrak{b}}$. If $\lambda \in \Lambda_l^+\,(\mathfrak{h})$, we have

$$\begin{aligned} \text{Case I:} \quad & \lambda = (\mu,\ 0,\ 0) \\ \text{Case II:} \quad & \lambda = (\mu,\ 0,\ \mu_0). \end{aligned}$$

Hence, λ is uniquely determined by its restriction $\mu \in i\,\mathfrak{b}^*$. Let Λ_l^+ denote the set of the restrictions on $\mathfrak{b}$ of highest weights of irreducible χ_l-spherical representations of U. Then

$$\Lambda_l^+ = \{\mu \in i\,\mathfrak{b}^* \ \mid\ \mu = \lambda|_{\mathfrak{b}},\ \lambda \in \Lambda_l^+\,(\mathfrak{h})\}.$$

Thus there is a bijective correspondence $\Lambda_l^+\,(\mathfrak{h}) \cong \Lambda_l^+$ via $\lambda = (\mu, 0, \mu_0) \mapsto \mu$.

Recall that $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is an orthogonal basis of $i\,\mathfrak{b}^*$. For $\lambda \in \Lambda_l^+\,(\mathfrak{h})$ we have $\lambda = (\mu,\ 0,\ \mu_0)$ (these three components are mutually orthogonal), so

$$\frac{\langle\lambda,\ \varepsilon_j\rangle}{\langle\varepsilon_j,\ \varepsilon_j\rangle} = \frac{\langle\mu,\ \varepsilon_j\rangle}{\langle\varepsilon_j,\ \varepsilon_j\rangle} =: \mu_j, \quad j = 1, \ldots, n.$$

Proposition 7.1 in [29] gives the following parametrization of $\Lambda_l^+(\mathfrak{h})$:

Proposition 3.15. *Let $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ satisfy $\lambda|_{(\mathfrak{h}\cap\mathfrak{k})\cap\mathfrak{k}_1} = 0$. Then $\lambda \in \Lambda_l^+(\mathfrak{h})$ if and only if*

$$\begin{cases} \mu_0 \in \mathbb{Z},\ \mu_j \in \mathbb{Z}^+\,(1 \le j \le n),\ |\mu_0| \le \mu_1 \\ \mu_j - \mu_i \in 2\,\mathbb{Z}^+\,(\textit{Case I: } 1 \le i < j \le n,\ \textit{Case II: } 0 \le i < j \le n). \end{cases}$$

Let $M = Z_K\,(\mathfrak{b})$. The following fact was proved in [29, Theorem 7.2]:

Theorem 3.16. *For $\lambda \in \Lambda^+(U)$ the followings are equivalent:*

1. *$\lambda|_{(\mathfrak{h}\cap\mathfrak{k})\cap\mathfrak{k}_1} = 0$ and $\mu_j - \mu_i \in 2\,\mathbb{Z}^+$, $1 \le i < j \le n$.*

2. π_λ *has a nonzero* K_1*-fixed vector.*

3. $\pi_\lambda|_M$ *is trivial on* $M \cap K_1$.

If any of these three conditions holds, then $\lambda \in \Lambda_l^+(\mathfrak{h})$*. Precisely,* π_λ *contains the following one-dimensional* K*-types* χ_l*, each contained once,*

$$\begin{aligned} \textit{In Case I}: &\quad l = -\mu_1, -\mu_1 + 2, \ldots, \mu_1 - 2, \mu_1; \\ \textit{In Case II}: &\quad l = \mu_0. \end{aligned}$$

Note that $\Lambda^+(\mathfrak{h}) = \Lambda^+(U)$ in our case, so these above two facts determine the condition for λ to be the highest weight of a χ_l spherical representation, that is,

$$\Lambda_l^+(\mathfrak{h}) = \left\{ \lambda \in \Lambda^+(U) \;\middle|\; \begin{array}{l} \lambda|_{(\mathfrak{h}\cap\mathfrak{k})\cap\mathfrak{k}_1} = 0,\ \mu_j - \mu_i \in 2\,\mathbb{Z}^+ \ (1 \le i < j \le n) \\ \mu_1 \in |l| + 2\,\mathbb{Z}^+,\ \mu_0 = 0 \text{ (Case I)};\ \mu_0 = l \text{ (Case II)} \end{array} \right\}.$$

It thus follows that

Theorem 3.17. *Let* U *be a compact simply connected semisimple Lie group and* K *the fixed point group of an involution of* U*. Let* χ_l *be a nontrivial character of* K *for some* $l \in \mathbb{Z}$*. The set of highest restricted weights of irreducible* χ_l*-spherical representations of* U *is*

$$\Lambda_l^+ = \left\{ \mu \in i\,\mathfrak{b}^* \;\middle|\; \begin{array}{l} \mu_j - \mu_i \in 2\,\mathbb{Z}^+ \ (1 \le i < j \le n) \\ \mu_0 = 0 \textit{ (Case I)};\ \mu_0 = l \textit{ (Case II)} \\ \mu_1 \in |l| + 2\,\mathbb{Z}^+ \end{array} \right\}. \tag{3.3}$$

Lemma 3.18. *When* $l = 0$*, (3.3) agrees with (3.1).*

Proof. Substituting $l = 0$ in (3.3) gives $\mu_j \in 2\,\mathbb{Z}^+$ for $j = 1, \ldots, n$. For any μ satisfies (3.3), $\mu_1 \in 2\,\mathbb{Z}^+$ and so $\mu_j = \mu_{\varepsilon_j} \in 2\,\mathbb{Z}^+$ for all j. Then

$$\frac{\langle \mu,\, 2\varepsilon_j\rangle}{\langle 2\varepsilon_j,\, 2\varepsilon_j\rangle} = \frac{1}{2}\,\frac{\langle \mu,\, \varepsilon_j\rangle}{\langle \varepsilon_j,\, \varepsilon_j\rangle} = \frac{\mu_j}{2} \in \mathbb{Z}^+,$$

and for $1 \le i < j \le n$,

$$\frac{\langle \mu,\, \varepsilon_j \pm \varepsilon_i\rangle}{\langle \varepsilon_j \pm \varepsilon_i,\, \varepsilon_j \pm \varepsilon_i\rangle} = \frac{\langle \mu,\, \varepsilon_j \pm \varepsilon_i\rangle}{2\langle \varepsilon_j,\, \varepsilon_j\rangle} = \frac{\langle \mu,\, \varepsilon_j\rangle}{2\langle \varepsilon_j,\, \varepsilon_j\rangle} \pm \frac{\langle \mu,\, \varepsilon_i\rangle}{2\langle \varepsilon_i,\, \varepsilon_i\rangle} = \frac{\mu_j}{2} \pm \frac{\mu_i}{2} \in \mathbb{Z}^+.$$

This is enough to prove μ satisfies (3.1). So (3.3) agrees with (3.1). ☐

The next proposition gives an even simpler parametrization of Λ_l^+ and sets up a connection between Λ_0^+ and Λ_l^+ for $l \in \mathbb{Z}$. They differ by a factor depending on l.

Proposition 3.19. *For* $l \in \mathbb{Z}$ *we have*

$$\Lambda_l^+ = \Lambda_0^+ + 2|l|\,\rho_s = P^+ + 2|l|\,\rho_s. \tag{3.4}$$

Proof. Let $\mu \in \Lambda_l^+$. To obtain (3.4) we want to show $\mu - 2|l|\,\rho_s \in P^+$. Recall that

$$\rho_s = \frac{1}{2} \sum_{\alpha \in \mathcal{O}_s^+} \alpha = \frac{1}{2}\,(\varepsilon_1 + \cdots + \varepsilon_n),$$

so $2|l|\,\rho_s = |l|\,(\varepsilon_1 + \cdots + \varepsilon_n)$. We then need to show

$$\frac{\langle \mu - |l|\,(\varepsilon_1 + \cdots + \varepsilon_n),\, \alpha \rangle}{\langle \alpha,\, \alpha \rangle} \in \mathbb{Z}^+, \ \forall\, \alpha \in \Sigma^+.$$

In view of Theorem 2.7, $\{2\varepsilon_j\}$ are the long roots in Σ^+, so it is enough to check

$$\frac{\langle \mu - |l|\,(\varepsilon_1 + \cdots + \varepsilon_n),\, 2\varepsilon_j \rangle}{\langle 2\varepsilon_j,\, 2\varepsilon_j \rangle} \in \mathbb{Z}^+, \quad j = 1, \ldots, n.$$

Note that $\varepsilon_i \perp \varepsilon_j$ for $i \neq j$. Thus

$$\frac{\langle \mu - |l|\,(\varepsilon_1 + \cdots + \varepsilon_n),\, 2\varepsilon_j \rangle}{\langle 2\varepsilon_j,\, 2\varepsilon_j \rangle} = \frac{\langle \mu,\, 2\varepsilon_j \rangle}{\langle 2\varepsilon_j,\, 2\varepsilon_j \rangle} - \frac{\langle |l|\,\varepsilon_j,\, 2\varepsilon_j \rangle}{\langle 2\varepsilon_j,\, 2\varepsilon_j \rangle} = \frac{1}{2}\,(\mu_j - |l|).$$

Since $\mu \in \Lambda_l^+$, μ satisfies (3.3). For case I we have $\mu_1 - |l| \in 2\mathbb{Z}^+$. So $(\mu_1 - |l|)/2 \in \mathbb{Z}^+$. For $j = 2, \ldots, n$, $\mu_j = \mu_1 + 2\,k_j$ with some $k_j \in \mathbb{Z}^+$. Then

$$\frac{1}{2}\,(\mu_j - |l|) = \frac{1}{2}\,(\mu_1 - |l| + 2\,k_j) = \frac{1}{2}\,(\mu_1 - |l|) + k_j \in \mathbb{Z}^+.$$

For case II, $l = \mu_0$ and $\mu_j - |\mu_0| \in 2\mathbb{Z}^+$ for $j = 1, \ldots, n$. So

$$\frac{1}{2}\,(\mu_j - |l|) = \frac{1}{2}\,(\mu_j - |\mu_0|) \in \mathbb{Z}^+$$

as desired. □

So far we see that there is a bijection from the set Λ_l^+ of highest restricted weights of irreducible χ_l-spherical representations of U onto the set $\widehat{U}_l$ of equivalence classes of irreducible χ_l-spherical representations:

$$\Lambda_l^+ \xrightarrow{\cong} \widehat{U}_l, \qquad \mu \longmapsto \pi_\mu.$$

For $\pi \in \widehat{U}$ we choose a representative irreducible unitary representation (π_μ, V_μ) with the highest weight μ for some $\mu \in \Lambda^+(U)$. For brevity we write V_μ instead of V_{π_μ} as the representation space on which π_μ acts. Write V_μ^l as the subspace of V_μ consisting of χ_l-coinvariant vectors. Put $d(\mu) = d(\pi_\mu) = \dim V_\mu$. Let $(\,\cdot\,,\,\cdot\,)$ be the inner product in the space V_μ of π_μ for which π_μ is unitary, i.e. $(\,\cdot\,,\,\cdot\,)$ is $\pi_\mu(U)$-invariant. Fix $e_\mu \in V_\mu^l$ with $\|e_\mu\| = \sqrt{(e_\mu,\, e_\mu)} = 1$.

Proposition 3.20. *Let U be compact and simply connected. The followings are equivalent:*

1. *π_μ is a χ_l-spherical representation of U,*

2. *$V_\mu^l \neq \{0\}$,*

3. *$\mu \in \Lambda_l^+$,*

4. *π_μ is a subrepresentation of the representation ℓ of U on $L^2(U/K, \chi_l)$.*

If these hold, $\dim V_\mu^l = 1$ *and π_μ occurs with multiplicity 1 in the representation of U on $L^2(U/K, \chi_l)$.*

Proof. That $1 \Leftrightarrow 2$ follows from the definition of a χ_l-spherical representation. The above discussion has proved $2 \Leftrightarrow 3$. It remains to prove $2 \Leftrightarrow 4$.

For $2 \Rightarrow 4$: Assume $V_\mu^l \neq \{0\}$ and pick $e_\mu \in V_\mu^l$ with $\|e_\mu\| = 1$. To prove 4, we simply need to prove there is a nontrivial intertwining operator $T : V_\mu \to L^2(U/K, \chi_l)$. Define

$$\varphi_v := (v, \pi_\mu(\,\cdot\,)\, e_\mu), \qquad v \in V_\mu,$$

For $k \in K$ we have

$$\varphi_v\,(u\,k) = (v, \pi_\mu(u\,k)\, e_\mu) = (v, \chi_l(k)\, \pi_\mu(u)\, e_\mu) = \chi_l(k)^{-1}\, \varphi_v(u).$$

This implies $\varphi_v \in L^2(U/K, \chi_l)$. Define $T \in \mathrm{Hom}_U(\pi, \ell)$ by $T(v) = \varphi_v$. We want to show $T \neq 0$ and

$$T \circ \pi_\mu\,(h)\, v = \ell\,(h) \circ T\,(v), \qquad h \in U,\, v \in V_\mu.$$

Since $T(e_\mu)\,(e) = (e_\mu, \pi_\mu(e) e_\mu) = \|e_\mu\|^2 = 1$, it follows that $T \neq 0$. Next, T intertwins π_μ and ℓ because for $u \in U$,

$$\begin{aligned} T(\pi_\mu(h)v)\,(u) &= \varphi_{h\cdot v}\,(u) \\ &= (\pi_\mu(h)v, \pi_\mu(u)e_\mu) \\ &= (v, \pi_\mu(h^{-1}\,u)e_\mu) \\ &= \varphi_v\,(h^{-1}\,u) \\ &= \ell\,(h)\,\varphi_v\,(u) \\ &= \ell\,(h)\,T(v)\,(u). \end{aligned}$$

Let $W_\mu := T(V_\mu) = \{\varphi_v \mid v \in V_\mu\}$. Then $W_\mu \subset L^2(U/K, \chi_l)$ and T is a bijective U-map from V_μ onto W_μ. So $V_\mu \cong_U W_\mu$. Since $\pi_\mu(h)v \in V_\mu$, then $\ell\,(h)\,\varphi_v(u) = \varphi_{\pi_\mu\,(h)v}\,(u) \in W_\mu$. This implies that W_μ is a subrepresentation of $L^2(U/K, \chi_l)$ and thus so is V_μ.

For $4 \Rightarrow 2$: Assume $W_\mu \subset L^2(U/K, \chi_l)$ is an irreducible subrepresentation. To prove 2, It is enough to prove

$$W_\mu^l := \{f \in W_\mu \mid \ell\,(k)\, f = \chi_l(k)\, f, \forall\, k \in K\} \neq \{0\}.$$

Let $0 \neq f \in W_\mu$. Then there is a $x \in U$ such that $f(x) \neq 0$. Let $g = \ell(x^{-1}) f$. Since W_μ is U-invariant, then $g \in W_\mu$. Also, $g(e) = f(x) \neq 0$. Let

$$F(u) := \int_K \chi_l(k)^{-1} g(k^{-1} u) \, dk = \int_K \chi_l(k)^{-1} \ell(k) \, g(u) \, dk.$$

Since $g \in W_\mu \subset L^2(U/K, \chi_l)$, we have $g(k^{-1}) = g(e\, k^{-1}) = \chi_l(k) \, g(e)$ and so

$$F(e) = \int_K \chi_l(k)^{-1} \chi_l(k) \, g(e) \, dk = \int_K g(e) \, dk = g(e) \neq 0.$$

Thus, $F \neq 0$. As $g \in W_\mu$, the left translation of g is still in W_μ. Integrating it over a compact set the outcome is also in W_μ. So from the construction of F we see that $F \in W_\mu$. Last, we show $F \in W_\mu^l$:

$$\begin{aligned} \ell(h) \, F(u) &= F(h^{-1} u) \\ &= \int_K \chi_l(k)^{-1} g(k^{-1} h^{-1} u) \, dk \\ (hk \mapsto k) &= \int_K \chi_l(h^{-1} k)^{-1} g(k^{-1} u) \, dk \\ &= \chi_l(h) \int_K \chi_l(k)^{-1} g(k^{-1} u) \, dk \\ &= \chi_l(h) \, F(u). \end{aligned}$$

Therefore, $F \in W_\mu^l$, as desired. □

Remark 3.21. (1) The space W_μ contains a unique χ_l-coinvariant function $\psi_{\mu, l}$ satisfying $\psi_{\mu, l}(e) = 1$, namely the χ_l-spherical function

$$\psi_{\mu, l}(u) = (e_\mu, \ \pi_\mu(u) \, e_\mu)$$

which we will discuss in more details in the next section.

(2) We have the decomposition

$$L^2(U/K, \chi_l) = \bigoplus_{\mu \in \Lambda_l^+} W_\mu.$$

By the equivalence 2 ⇔ 3, the canonical decomposition (3.2) thus takes the form

$$(\ell, \ L^2(U/K, \ \chi_l)) \cong_U \bigoplus_{\mu \in \Lambda_l^+} (\pi_\mu, \ V_\mu). \tag{3.5}$$

The equivalence 3 ⇔ 4 implies that Λ_l^+ runs over the highest weights of the representations showing up in the representation of U on $L^2(U/K, \chi_l)$. Therefore, there exists a measure, the Plancherel measure, say $d\,\sigma(\mu)$ on $\Lambda_l^+ \subseteq \widehat{U}$ given by $d\,\sigma(\mu) = d\,(\mu)\,d\,\mu$ where $d\,\mu$ is the counting measure[2] on Λ_l^+, such that

$$(\ell,\, L^2(U/K,\, \chi_l)) \cong \int_{\Lambda_l^+}^{\oplus} (\pi_\mu,\, V_\mu)\, d\sigma(\mu). \tag{3.6}$$

Since Λ_l^+ is a discrete set, the decomposition (3.6) is nothing but the direct sum (3.5).

3.3 Harmonic Analysis on Line Bundles over U/K

In this section we will define the spherical functions of type χ_l on the compact group U and the relevant spherical functions of type χ_l on the noncompact group G, and the χ_l-spherical Fourier transform of $f \in C^\infty\,(U//K,\, \chi_l)$, where

$$C^\infty\,(U//K,\, \chi_l) = \{f \in C^\infty\,(U) \ \mid \ f\,(k_1\, u\, k_2) = \chi_l\,(k_1\, k_2)^{-1}\, f\,(u),\, \forall\, k_1, k_2 \in K\}$$

is the subspace of $C^\infty\,(U/K,\, \chi_l)$ of χ_l-bi-coinvariant functions on U.

Definition 3.22. Let G be a locally compact Hausdorff group and $K \subset U$ a compact subgroup. A continuous function $\varphi : G \to \mathbb{C}$ is an *elementary spherical function of type* χ_l if it is not identically 0,

$$\varphi\,(k_1\, g\, k_2) = \chi_l\,(k_1\, k_2)^{-1}\, \varphi\,(g), \quad \forall\, g \in G,\, \forall\, k_1, k_2 \in K,$$

and

$$\int_K \varphi\,(g\, k\, h)\, \chi_l\,(k)\, dk = \varphi\,(g)\, \varphi\,(h), \quad \forall\, g,\, h \in U. \tag{3.7}$$

Note that the equation (3.7) ensures that φ is χ_l-bi-invariant and $\varphi(e) = 1$.

3.3.1 Spherical Functions of type χ_l on U

Let $f \in L^2\,(U/K,\, \chi_l)$. For $\pi \in \widehat{U}$ we choose an irreducible unitary representation $(\pi_\mu,\, V_\mu)$ with the highest weight $\mu \in \Lambda^+(U)$. For all $v \in V_\mu$ and all $k \in K$,

$$\begin{aligned} \pi\,(f)\, v &= \int_U f\,(u)\, \pi\,(u)\, v\, du \\ &= \int_U \chi_l\,(k)\, f\,(u\, k)\, \pi\,(u)\, v\, du \\ &= \int_U \chi_l\,(k)\, f\,(u)\, \pi\,(u\, k^{-1})\, v\, du. \end{aligned}$$

[2] A counting measure is a measure which takes the value one at each point.

As this holds for all $k \in K$, integrating each side over K gives

$$\begin{aligned} \pi(f)\,v &= \int_K \int_U \chi_l(k)\, f(u)\, \pi(u\,k^{-1})\, v\, du\, dk \\ &= \int_U f(u)\, \pi(u) \int_K \chi_l(k^{-1})\, \pi(k)\, v\, d\,k\, d\,u. \end{aligned}$$

Define an operator $P_l : V_\mu \to V_\mu^l$ by

$$P_l(v) = \int_K \chi_l(k^{-1})\, \pi(k)\, v\, d\,k.$$

Since π is finite dimensional, P_l is well-defined. It has the property that

$$\pi(f)\, v = \pi(f)\, P_l(v), \qquad \forall\, v \in V_\mu. \tag{3.8}$$

Proposition 3.23. *P_l is an orthogonal projection of V_μ onto V_μ^l, and*

$$\pi(f) = \pi(f)\, P_l$$

for $f \in L^2(U/K, \chi_l)$.

Proof. For any $h \in K$ and $v \in V_\mu$ we have

$$\begin{aligned} \pi(h)\, P_l(v) &= \int_K \chi_l(k^{-1})\, \pi(h)\, \pi(k)\, v\, d\,k \\ &= \int_K \chi_l(k^{-1})\, \pi(h\,k)\, v\, d\,k \\ &= \int_K \chi_l(k^{-1}\,h)\, \pi(k)\, v\, d\,k \\ &= \chi_l(h) \int_K \chi_l(k^{-1})\, \pi(k)\, v\, d\,k \\ &= \chi_l(h)\, P_l(v). \end{aligned}$$

This implies that $P_l(v) \in V_\mu^l$. Also,

$$\begin{aligned} P_l^2(v) &= \int_K \chi_l(h^{-1})\, \pi(h) \int_K \chi_l(k^{-1})\, \pi(k)\, v\, d\,k\, dh \\ &= \int_K \int_K \chi_l(\underbrace{h^{-1}\, h\, k^{-1}}_{=k^{-1}})\, \pi(k)\, v\, d\,k\, dh \\ &= P_l(v), \end{aligned}$$

where $\int_K dh = 1$. So P_l is idempotent. For $v,\, w \in V_\mu$,

$$\begin{aligned} (P_l(v),\ w) &= \int_K (\chi_l(k^{-1})\, \pi(k)\, v,\ w)\, dk \\ &= \int_K (v,\ \chi_l(k)\, \pi(k^{-1})\, w)\, dk \\ &= (v, \int_K \chi_l(k^{-1})\, \pi(k)\, w\, dk) \\ &= (v,\ P_l(w)), \end{aligned}$$

So P_l is self-adjoint. Thus P_l is an orthogonal projection. $\square$

If $\mu \notin \Lambda_l^+$ then $V_\mu^l = \{0\}$ and so $\pi(f)\, v = 0$ for all $v \in V_\mu$. If $\mu \in \Lambda_l^+$ then $\dim V_\mu^l = 1$. Fix $e_\mu \in V_\mu^l$ with $\|e_\mu\| = 1$. Then $V_\mu^l = \mathbb{C}\, e_\mu$. The equation (3.8) implies that $\pi(f)\big|_{(V_\mu^l)^\perp} = 0$ and thus $\pi(f)$ is determined by $\pi(f)\big|_{V_\mu^l}$. It follows that $\pi(f)$ is a rank one operator[3]. Hence, for χ_l-spherical representations π_μ (i.e. $\mu \in \Lambda_l^+$) it is natural to define the vector valued Fourier transform of f to be

$$\widetilde{f}_l(\mu) := \pi_\mu(f)\, e_\mu \in V_\mu,$$

where $\widetilde{f}_l : \Lambda_l^+ \to \oplus_{\mu\in\Lambda_l^+} V_\mu$. This vector-valued Fourier transform extends to the unitary isomorphism (3.6).

Note that we have

$$\begin{aligned} P_l(v) &= (v,\, e_\mu)\, e_\mu \\ \pi(f)\, v &= (v,\, e_\mu)\, \pi(f)\, e_\mu \\ \mathrm{Tr}\,(\pi(f)) &= (\pi(f)\, e_\mu,\, e_\mu). \end{aligned}$$

We extend $\{e_\mu\}$ to an orthonormal basis for V_μ, say $\{e_1, \ldots, e_{d(\mu)}\}$ with $e_1 = e_\mu$. Recall Theorem 3.6 for the Plancherel formulas for $f \in L^2(U)$. They are reduced to the Plancherel formulas for $f \in L^2(U/K,\, \chi_l)$:

$$\begin{aligned} f(u) &= \sum_{\mu\in\Lambda_l^+} d(\mu)\, (\pi(u^{-1})\, \pi(f)\, e_\mu,\, e_\mu); \\ \|f\|_2^2 &= \sum_{\mu\in\Lambda_l^+} d(\mu)\, \|\pi(f)\, e_\mu\|^2. \end{aligned}$$

Assume $f \in L^2(U//K,\, \chi_l)$ where

$$L^2(U//K,\, \chi_l) = \{f \in L^2(U) \;\mid\; f(k_1\, u\, k_2) = \chi_l(k_1\, k_2)^{-1} f(u),\, \forall\, k_1, k_2 \in K\}.$$

For $k \in K$ we have

$$\begin{aligned} \pi(k)\, \pi(f)\, e_\mu &= \int f(u)\, \pi(k\, u)\, e_\mu\, du \\ &= \int f(k^{-1} u)\, \pi(u)\, e_\mu\, du \\ &= \chi_l(k) \int f(u)\, \pi(u)\, e_\mu\, du \\ &= \chi_l(k)\, \pi(f)\, e_\mu. \end{aligned}$$

[3] A finite rank operator is a bounded linear transformation from a Hilbert space into another Hilbert space which has a finite dimensional range. A rank one operator is a finite rank operator whose range is one dimensional.

So $\pi(f)\, e_\mu \in V_\mu^l$. Since $V_\mu^l = \mathbb{C}\, e_\mu$, there is a scalar $\widetilde{f}_l(\mu) \in \mathbb{C}$ such that $\pi(f)\, e_\mu = \widetilde{f}_l(\mu)\, e_\mu$. Then the Plancherel formulas for $f \in L^2(U//K,\, \chi_l)$ become

$$\begin{aligned} f(u) &= \sum_{\mu \in \Lambda_l^+} d(\mu)\, \widetilde{f}_l(\mu)\, (e_\mu,\ \pi(u)\, e_\mu); \\ \|f\|_2^2 &= \sum_{\mu \in \Lambda_l^+} d(\mu)\, \|\widetilde{f}_l(\mu)\|^2. \end{aligned}$$

We compute $\widetilde{f}_l(\mu)$ by

$$\widetilde{f}_l(\mu) = (\widetilde{f}_l(\mu)\, e_\mu,\ e_\mu) = (\pi(f)\, e_\mu,\ e_\mu) = \int_U f(u)\, (\pi(u)\, e_\mu,\ e_\mu)\, du.$$

This is named the scalar valued Fourier transform of f.

We therefore define $\psi_{\mu,l} = \psi_{\mu,\chi_l}$ on U to be the matrix coefficient of the χ_l-spherical representation π_μ, i.e.

$$\psi_{\mu,l}(u) := (e_\mu,\ \pi_\mu(u)\, e_\mu),\ \forall\, u \in U,\ \mu \in \Lambda_l^+.$$

Lemma 3.24. *The function $\psi_{\mu,l}$ is an (elementary) spherical function of type χ_l on U associated with π_μ.*

Proof. We have $\psi_{\mu,l}(e) = 1$, and

$$\begin{aligned} \psi_{\mu,l}(k_1\, u\, k_2) &= (e_\mu,\ \pi_\mu(k_1\, u\, k_2)\, e_\mu) \\ &= (\pi_\mu(k_1^{-1})\, e_\mu,\ \pi_\mu(u)\, \pi_\mu(k_2)\, e_\mu) \\ &= \chi_l(k_1\, k_2)^{-1}\, \psi_{\mu,l}(u) \end{aligned}$$

for all $k_1, k_2 \in K$. Moreover, for $k \in K$ and $g, h \in U$,

$$\begin{aligned} \int_K \psi_{\mu,l}(g\, k\, h)\, \chi_l(k)\, dk &= \int_K (e_\mu,\ \pi(g\, k\, h)\, e_\mu)\, \chi_l(k)\, dk \\ &= \int_K (\pi(g^{-1})\, e_\mu,\ \pi(k)\, \pi(h)\, e_\mu)\, \chi_l(k)\, dk \\ &= (\pi(g^{-1})\, e_\mu,\ \int_K \chi_l(k^{-1})\, \pi(k)\, \pi(h)\, e_\mu\, dk) \\ &= (\pi(g^{-1})\, e_\mu,\ (\pi(h)\, e_\mu,\ e_\mu)\, e_\mu) \\ &= (\pi(g^{-1})\, e_\mu,\ e_\mu)\, \overline{(\pi(h)\, e_\mu,\ e_\mu)} \\ &= (e_\mu,\ \pi(g)\, e_\mu)\, (e_\mu,\ \pi(h)\, e_\mu) \\ &= \psi_{\mu,l}(g)\, \psi_{\mu,l}(h). \end{aligned}$$

$\square$

Note that $\psi_{\mu,l}$ is smooth (by its definition), positively definite (see Proposition 3.5), and independent of the choice of e_μ. Each spherical function of type χ_l on U is of the form $\psi_{\mu,l}$ for some $\mu \in \Lambda_l^+$. When $l = 0$, $\psi_{\mu,0} = \psi_\mu$ is exactly the spherical function on U (see [22]).

Let $f \in C^\infty(U//K,\, \chi_l)$. The χ_l-spherical Fourier transform $\mathcal{S}_l$ of f is a function $\mathcal{S}_l(f) = \widetilde{f}_l : \Lambda_l^+ \to \mathbb{C}$ defined by

$$\widetilde{f}_l(\mu) = (f,\ \psi_{\mu,l}) = \int_U f(u)\,\overline{\psi_{\mu,l}(u)}\,du. \tag{3.9}$$

Note that $\widetilde{f}_l$ is uniquely determined by f. From Schur's orthogonality relations,

$$\|\psi_{\mu,l}\|^2 = \frac{1}{d(\mu)}, \quad \mathcal{S}_l(\psi_{\nu,l})(\mu) = \frac{1}{d(\mu)}\,\delta_{\nu,\mu},$$

for all $\nu, \mu \in \Lambda_l^+$. The χ_l-spherical Fourier series of f is given by

$$f(u) = \sum_{\mu \in \Lambda_l^+} d(\mu)\,\widetilde{f}_l(\mu)\,\psi_{\mu,l}(u). \tag{3.10}$$

Proposition 3.25. *If $f \in C^\infty(U//K,\, \chi_l)$ and $\widetilde{f}_l = 0$ then $f = 0$.*

The sum (3.10) is convergent absolutely and uniformly due to the smoothness of f. Note that f is smooth if and only if $\widetilde{f}_l$ is rapidly decreasing, i.e. for each $k \in \mathbb{Z}^+$, there is a constant C_k such that

$$|\widetilde{f}_l(\mu)| \le C_k\,(1 + |\mu|)^{-k},\ \forall\, \mu \in \Lambda_l^+.$$

By using the classification of symmetric spaces, we can derive an explicit formula for $d(\mu)$ from Weyl dimension formula (also see [18, Proposition 5.2.10] for details). For $\mu \in \Lambda_l^+$, $\mu \mapsto d(\mu)$ extends to a polynomial function on $\mathfrak{b}_{\mathbb{C}}^*$. Define

$$\ell^2(\Lambda_l^+,\, d(\mu)\,d\mu) = \{(a_\mu)_{\mu \in \Lambda_l^+} \ \mid\ a_\mu \in \mathbb{C},\ \sum_{\mu \in \Lambda_l^+} d(\mu)\,|a_\mu|^2 < \infty\},$$

where $d\mu$ is the counting measure. This is a Hilbert space with the inner product

$$((a_\mu)_\mu,\ (b_\mu)_\mu) = \sum_{\mu \in \Lambda_l^+} d(\mu)\,a_\mu\,\overline{b_\mu}.$$

We see that the sequence $\mathcal{S}_l(f) = (\widetilde{f}_l(\mu))_\mu \in \ell^2(\Lambda_l^+)$, and

$$\mathcal{S}_l : C^\infty(U//K,\, \chi_l) \longrightarrow \ell^2(\Lambda_l^+)$$

is an isometry, i.e. $\|f\|^2 = \|\mathcal{S}_l(f)\|^2$. Since

$$|\psi_{\mu,l}(u)| = |(e_\mu,\ \pi_\mu(u)\,e_\mu)| \le \|e_\mu\|\,\|\pi_\mu(u)\|\,\|e_\mu\| \le 1,$$

then if $f \in L^p(U//K,\, \chi_l)$ we have $|\widetilde{f}_l\,(\mu)| \leq \|f\|_p$. In particular,

$$|\widetilde{f}_l\,(\mu)| \leq \|f\|_\infty$$

if f is continuous (and hence bounded).

Notice that the set

$$\{\sqrt{d\,(\mu)}\,\psi_{\mu,\,l} \;\mid\; \mu \in \Lambda_l^+\}$$

is an orthonormal basis for the Hilbert space $L^2\,(U//K,\, \chi_l)$. The above discussion gives

Theorem 3.26 (Plancherel Theorem)**.** *The χ_l-spherical Fourier transform $\mathcal{S}_l$ extends by continuity to an unitary isomorphism*

$$\mathcal{S}_l : L^2\,(U//K,\, \chi_l) \xrightarrow{\cong} \ell^2\,(\Lambda_l^+).$$

If $f \in L^2\,(U//K,\, \chi_l)$, then the sum in (3.10) is understood in L^2 sense, $|\widetilde{f}_l\,(\mu)| \leq \|f\|_2$, and

$$\|f\|_2^2 = \sum_{\mu \in \Lambda_l^+} d\,(\mu)\, |\widetilde{f}_l\,(\mu)|^2.$$

Lemma 3.27. *Let $\mu \in \Lambda_l^+$. Then $\overline{\psi_{\mu,\,l}\,(u)} = \psi_{\mu,\,l}\,(u^{-1})$ for $u \in U$.*

Proof. This comes from the fact that $\psi_{\mu,\,l}$ is positive definite. ☐

Recall that π_μ extends to a holomorphic representation $(\pi_\mu)_\mathbb{C}$ of $U_\mathbb{C}$. This gives

Lemma 3.28. *Let $\mu \in \Lambda_l^+$. Then the χ_l-type spherical function $\psi_{\mu,\,l} : U \to \mathbb{C}$ extends to a holomorphic function on $U_\mathbb{C}$. The extension is given by*

$$\psi_{\mu,\,l}\,(g) = ((\pi_\mu)_\mathbb{C}\,(g^{-1})\,e_\mu,\; e_\mu), \qquad g \in U_\mathbb{C}.$$

3.3.2 Spherical Functions of type χ_l on G

Recall that the pair $(G,\, K)$ is noncompact dual of $(U,\, K)$. We shall discuss the spherical functions of type χ_l on G which are the matrix coefficients of the (nonunitary) spherical principal series representations of G.

Before we get into this topic we first review a property: the group G and its Lie algebra $\mathfrak{g}$ admit the Iwasawa decomposition. Let $\mathfrak{n} = \oplus_{\alpha\in\Sigma^+}\, \mathfrak{g}_\alpha$ and $N = \exp\, \mathfrak{n}$ be the analytic subgroup of G with Lie algebra $\mathfrak{n}$. Then N is closed in G and nilpotent. The Iwasawa map

$$K \times A \times N \ni (k,\, a,\, n) \longmapsto k\,a\,n \in G$$

is a holomorphic diffeomorphism. Let $H : G \to \mathfrak{a}$ be the Iwasawa projection given by $k\,(\exp\, X)\,n \mapsto X$. Thus for each $g \in G$ we see that

$$g = \kappa\,(g)\, \exp\,(H\,(g))\,n\,(g)$$

is uniquely determined for $\kappa(g) \in K$, $H(g) \in \mathfrak{a}$, and $n(g) \in N$. Write $a(g) = \exp(H(g)) \in A$. The Iwasawa decomposition of Lie algebras $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{C}}$ are

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}, \qquad \mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} + \mathfrak{a}_{\mathbb{C}} + \mathfrak{n}_{\mathbb{C}}.$$

Adopt the notation

$$a^{\lambda} := e^{\lambda(\log(a))} = e^{\lambda(X)}, \qquad \lambda \in \mathfrak{b}_{\mathbb{C}}^{*}$$

where $a = \exp(X) \in A$. Notice that the homomorphisms $A \to \mathbb{C}^{*}$ are exactly $a \mapsto a^{\lambda}$. So this is a character of A. Define $\varphi_{\lambda,l} : G \to \mathbb{C}$ by

$$\varphi_{\lambda,l}(g) = \int_K a(g^{-1}k)^{\lambda-\rho}\,\chi_l(\kappa(g^{-1}k)\,k^{-1})\,dk. \tag{3.11}$$

Proposition 3.29. *The corresponding (elementary) spherical function of type χ_l on G with spectral parameter $\lambda \in \mathfrak{b}_{\mathbb{C}}^{*}$ has the integral form (3.11). Any spherical function of type χ_l on G has this form for some $\lambda \in \mathfrak{b}_{\mathbb{C}}^{*}$. Moreover, $\varphi_{\lambda,l}$ is real analytic in $g \in G$, and holomorphic in $(\lambda, l) \in \mathfrak{b}_{\mathbb{C}}^{*} \times \mathbb{Z}$.*

Proof. See Proposition 5.4.1 in [18] and Proposition 3.3 in [31]. □

Notice that the formula (3.11) differs from the one defined in [18, P.82, (5.4.1)] by an inverse sign (due to a technical reason).

Proposition 3.30. *The definition (3.11) for $\varphi_{\lambda,l}(g)$ is equivalent to*

$$\varphi_{\lambda,l}(g) = \int_K a(g\,k)^{-\lambda-\rho}\,\chi_l(\kappa(g\,k)^{-1}\,k)\,dk.$$

Proof. Using the integral formula

$$\int_K f(\kappa(g^{-1}k))\,a(g^{-1}k)^{-2\rho}\,dk = \int_K f(k)\,dk,$$

for an integrable function f on K and the fact that $a(g^{-1}k)^{-1} = a(g\,\kappa(g^{-1}k))$, we obtain

$$\begin{aligned}
\varphi_{\lambda,l}(g) &= \int_K a(g^{-1}k)^{\lambda-\rho}\,\chi_l(\kappa(g^{-1}k)\,k^{-1})\,dk \\
&= \int_K a(g^{-1}k)^{\lambda+\rho}\,\chi_l(\kappa(g^{-1}k)\,k^{-1})\,a(g^{-1}k)^{-2\rho}\,dk \\
&= \int_K a(g\,\kappa(g^{-1}k))^{-\lambda-\rho}\,\chi_l(\kappa(g^{-1}k)\,\kappa(g\,\kappa(g^{-1}k))^{-1})\,a(g^{-1}k)^{-2\rho}\,dk \\
&= \int_K a(g\,k)^{-\lambda-\rho}\,\chi_l(k\,\kappa(g\,k)^{-1})\,dk.
\end{aligned}$$

□

When $l = 0$ we see that

$$\varphi_\lambda(g) = \varphi_{\lambda,0}(g) = \int_K a(g\,k)^{-\lambda-\rho}\,dk = \int_K a(g^{-1}\,k)^{\lambda-\rho}\,dk$$

is exactly the well-known Harish-Chandra spherical function on G.

Proposition 3.31. *The function $\varphi_{\lambda,l}$ satisfies*

1. $\varphi_{\lambda,l} = \varphi_{\mu,l}$ *if and only if there is a $w \in W$ such that $\lambda = w\,\mu$.*
2. $\varphi_{\lambda,l}(a) = \varphi_{\lambda,-l}(a) = \varphi_{-\lambda,-l}(a) = \varphi_{-\lambda,l}(a)$, $\forall\, a \in A$.
3. $\varphi_{\lambda,l}(g) = \varphi_{-\lambda,-l}(g^{-1}) = \varphi_{\lambda,-l}(g^{-1}) = \varphi_{-\lambda,l}(g)$, $\forall\, g \in G$.

Proof. Part 1 follows from Proposition 3.3 in [31]. Observe that $\varepsilon_1, \ldots, \varepsilon_n$ are roots so $r_{\varepsilon_1} \cdots r_{\varepsilon_n} = -1 \in W$. Then part 2 comes from this fact, part 1, and Corollary 3.7 in [31]. Compare the above two equivalent definitions of $\varphi_{\lambda,l}$ and we get

$$\varphi_{\lambda,l}(g) = \int_K a(g\,k)^{-\lambda-\rho}\,\chi_l(\kappa(g\,k)^{-1}\,k)\,dk = \varphi_{-\lambda,-l}(g^{-1}).$$

Apply the fact $-1 \in W$ and part 1 again. It completes the proof of part 3. □

Note: We will extend $\varphi_{\lambda,l}$ holomorphically to a complex domain $\mathcal{V}$ in $G_{\mathbb{C}}$ (see Remark 3.34 and Chapter 5). These properties of $\varphi_{\lambda,l}$ are thus also valid on the extended domain.

The χ_l-type spherical functions $\psi_{\mu,l}$ on U are defined on a discrete set Λ_l^+. They can be extended continuously to $\mathfrak{b}_{\mathbb{C}}^*$. How we do it relies on the fact that the functions $\psi_{\mu,l}$ are related to the corresponding χ_l-type spherical functions $\varphi_{\lambda,l}$ on G by holomorphic continuation.

Lemma 3.32. *Let $\mu \in \Lambda_l^+$ and $\psi_{\mu,l}$ the holomorphic extension to $U_{\mathbb{C}}$ of the χ_l-type spherical function $\psi_{\mu,l}$ on U. Then $\psi_{\mu,l}|_G = \varphi_{\mu+\rho,l}$.*

Proof. Let $\mu \in \Lambda_l^+$. For any $v \in V_\mu$, $P_l(v) = (v,\, e_\mu)\,e_\mu$. So we can fix a highest weight vector v for π_μ such that $(v,\, e_\mu) = 1$. Then

$$e_\mu = \int_K \chi_l(k^{-1})\,\pi_\mu(k)\,v\,dk.$$

For $h \in B$,

$$\psi_{\mu,l}(h) = (\pi_\mu(h^{-1})\,e_\mu,\; e_\mu) = \int_K (\chi_l(k^{-1})\,\pi_\mu(h^{-1}\,k)\,v,\; e_\mu)\,dk.$$

Since K is compact, the above equality is still true for the holomorphic extension of $\psi_{\mu,l}$ to $U_{\mathbb{C}}$. In particular, it is true for $h \in A$. Since $(v,\, e_\mu) = 1$ and since v is a highest weight vector of weight μ, we have

$$\begin{aligned}
(\chi_l(k^{-1})\,\pi_\mu(h^{-1}k)\,v,\ e_\mu) &= (\chi_l(k^{-1})\,\pi_\mu(\kappa(h^{-1}k)\,a(h^{-1}k)\,n(h^{-1}k))\,v,\ e_\mu) \\
&= \chi_l(k^{-1})\,(\pi_\mu(a(h^{-1}k))\,v,\ \pi_\mu(\kappa(h^{-1}k)^{-1})\,e_\mu) \\
&= \chi_l(k^{-1})\,(a(h^{-1}k)^\mu\,v,\ \chi_l(\kappa(h^{-1}k)^{-1})\,e_\mu) \\
&= \chi_l(\kappa(h^{-1}k)\,k^{-1})\,a(h^{-1}k)^\mu\,(v,\ e_\mu) \\
&= \chi_l(\kappa(h^{-1}k)\,k^{-1})\,a(h^{-1}k)^{(\mu+\rho)-\rho}.
\end{aligned}$$

where we use the fact that $\pi_\mu(n(h^{-1}k))\,v = v$ and

$$\pi_\mu(a(h^{-1}k))\,v = \pi_\mu(e^{H(h^{-1}k)})\,v = e^{\pi_\mu(H(h^{-1}k))}\,v = e^{\mu(H(h^{-1}k))}\,v = a(h^{-1}k)^\mu\,v.$$

Therefore,

$$\psi_{\mu,l}(h) = \int_K a(h^{-1}k)^{(\mu+\rho)-\rho}\,\chi_l(\kappa(h^{-1}k)\,k^{-1})\,dk = \varphi_{\mu+\rho,l}(h).$$

This is enough to show $\psi_{\mu,l}(g) = \varphi_{\mu+\rho,l}(g)$ for $g \in G$ because of the Cartan decomposition $G = K\,A\,K$. □

Lemma 3.33. *The elementary χ_l-type spherical function $\varphi_{\lambda,l}$ which is analytic on G extends to a holomorphic function on $G_{\mathbb{C}}$, and by restriction gives an elementary χ_l-type spherical function on U if and only if λ belongs to the W-orbit of $\Lambda_l^+ + \rho$.*

Proof. Recall (2.6) and (3.4). We have

$$\Lambda_l^+ + \rho = P^+ + \rho + 2|l|\,\rho_s = P^+ + \rho\,(l).$$

Thus the proof follows from [18, corollary 5.2.3]. □

Remark 3.34. Lemma 3.32 says for $\mu \in \Lambda_l^+$ the restriction on G of the holomorphic extension of $\psi_{\mu,l}$ equals $\varphi_{\mu+\rho,l}$, i.e. $\psi_{\mu,l}|_G = \varphi_{\mu+\rho,l}$. Lemma 3.33 states that we may think of $\psi_{\mu,l}$ as the restriction on U of the holomorphic extension of the corresponding $\varphi_{\mu+\rho,l}$, i.e. $\psi_{\mu,l} = \varphi_{\mu+\rho,l}\big|_U$ for $\mu \in \Lambda_l^+$. Since both $\psi_{\mu,l}$ and $\varphi_{\mu+\rho,l}$ are holomorphic on $G_{\mathbb{C}}$, they agree everywhere on $G_{\mathbb{C}}$ by holomorphic continuation, that is,

$$\psi_{\mu,l}(g) = \varphi_{\mu+\rho,l}(g), \qquad \forall\, g \in G_{\mathbb{C}},\ \mu \in \Lambda_l^+. \tag{3.12}$$

By (3.9) and (3.12), for $\mu \in \Lambda_l^+$ we have

$$\widetilde{f}_l(\mu) = \int_U f(u)\,\psi_{\mu,l}(u^{-1})\,du = \int_U f(u)\,\varphi_{\mu+\rho,l}(u^{-1})\,du.$$

Therefore $\mu \mapsto \widetilde{f}_l(\mu)$ can (formally) have an extension to $\mathfrak{b}_{\mathbb{C}}^*$, which has the same form, $\lambda \mapsto \widetilde{f}_l(\lambda)$ given by

$$\widetilde{f}_l(\lambda) = \int_U f(u)\,\varphi_{\lambda+\rho,l}(u^{-1})\,du, \qquad \lambda \in \mathfrak{b}_{\mathbb{C}}^*. \tag{3.13}$$

The extension of the χ_l-spherical Fourier transform of f is still denoted by $\widetilde{f_l}$ or $\mathcal{S}_l\,(f)$. Since $\varphi_{\lambda,\,l}$ is W-invariant in λ, then the function $\widetilde{f_l}\,(\lambda)$ is W-invariant, too.

Recall that $\varphi_{\lambda,\,l}\,(g)$ is holomorphic for $(\lambda,\,l,\,g) \in \mathfrak{b}_{\mathbb{C}}^* \times \mathbb{Z} \times G$. Thus $\widetilde{f_l}\,(\lambda)$ is holomorphic in this set. However, for every $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ the (extended) $\varphi_{\lambda,\,l}$ might not be defined on all of $G_{\mathbb{C}}$. To prove a Paley-Wiener type theorem, we need to verify (3.13) is well-defined and holomorphic for all $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$, that is, to determine for which spaces of functions f this extension is meaningful. So we have to find a suitable subset in $G_{\mathbb{C}}$, say $G \subseteq \mathcal{V} \subseteq G_{\mathbb{C}}$, such that $\varphi_{\lambda,\,l}\,(g)$ is well-defined for every $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ and $g \in \mathcal{V}$.

Since K is compact, χ_l extends holomorphically to a representation $\chi_{l,\,\mathbb{C}}$ of $K_{\mathbb{C}}$. In view of (3.11), it suffices to see for which space $\mathcal{V}$, $a\,(g^{-1}\,k)^{\lambda-\rho}$ and $\kappa\,(g^{-1}\,k)$ are well-defined for $g \in \mathcal{V}$. From the Iwasawa map $G \cong K\,A\,N$, we see that the A-part $a\,(g)$ and K-part $\kappa\,(g)$ of an element $g \in G$ are uniquely determined. However,

$$\{e\} \subsetneqq K_{\mathbb{C}} \cap A_{\mathbb{C}}, \quad K_{\mathbb{C}}\,A_{\mathbb{C}}\,N_{\mathbb{C}} \cap G \neq G.$$

Thus, the $A_{\mathbb{C}}$-part and $K_{\mathbb{C}}$-part of $g \in G_{\mathbb{C}}$ are not uniquely determined in the complexified Iwasawa decomposition of $G_{\mathbb{C}}$. In order for $a\,(g^{-1}\,k)^{\lambda-\rho}$ and $\kappa\,(g^{-1}\,k)$ being well-defined, we need $g^{-1}\,k \in K_{\mathbb{C}}\,A_{\mathbb{C}}\,N_{\mathbb{C}}$, i.e. $a\,(g^{-1}\,k) \in A_{\mathbb{C}}$ and $\kappa\,(g^{-1}\,k) \in K_{\mathbb{C}}$ are uniquely determined. There is, by [23, lemma 2.1], a $K_{\mathbb{C}} \times K$ invariant domain $G \subseteq \mathcal{W} \subseteq G_{\mathbb{C}}$ such that the map

$$\mathcal{W} \longrightarrow K_{\mathbb{C}} \times A_{\mathbb{C}} \times N_{\mathbb{C}}, \qquad g \mapsto (\kappa\,(g),\,a\,(g),\,n\,(g))$$

is well-defined, holomorphic in g, and agrees with the Iwasawa decomposition on $\mathcal{W} \cap G$. Let $\mathcal{V} = \mathcal{W}^{-1}$. Then the map $(\lambda,\,g) \mapsto \varphi_{\lambda,\,l}\,(g)$ extends holomorphically to $\mathfrak{b}_{\mathbb{C}}^* \times \mathcal{V}$. So the holomorphic extension $\lambda \mapsto \widetilde{f_l}\,(\lambda)$ is well-defined for all $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ if $f \in C_c^{\infty}\,(\mathcal{V} \cap U)$. A particular $\mathcal{V}$ can be constructed by using Lemma 5.8.

Chapter 4
Invariant Differential Operators

The aim of this chapter is to present some fundamental facts about invariant differential operators on $\Gamma^{\infty}(L_{\chi_l})$ and their radial components. We will give a generalization of the well-known Harish-Chandra isomorphism. Then the χ_l-type spherical functions can also be defined via eigenequations whose eigenvalues are derived from the generalized Harish-Chandra map. In particular, we shall consider Laplace-Beltrami operator and need an explicit form for its eigenvalues later. A good discussion in this aspect can be found in [18].

4.1 The Harish-Chandra Isomorphism

Let $\mathbb{D}_l = \mathbb{D}_l(U/K)$ be the (commutative) algebra of U-invariant differential operators on U which maps $C^{\infty}(U/K, \chi_l)$ into itself (these operators commute with the action of U, hence called invariant differential operators). We can identify $\mathbb{D}_l$ as the algebra of all U-invariant differential operators on smooth sections in the line bundle L_{χ_l}. If $l = 0$, then $\mathbb{D}_0 = \mathbb{D}(U/K)$ is the algebra of U-invariant differential operators on U/K. Let $\mathbb{D}_l^d = \mathbb{D}_l(G/K)$ for the noncompact symmetric space G/K. We will see that $\mathbb{D}_l \cong \mathbb{D}_l^d$ by means of Cartan duality which is a useful and important technique throughout this manuscript. Recall that duality gives bijections $U/K \cong G/K$ and $\mathfrak{u} \cong \mathfrak{g}$ (see (2.2)), and ensures that $\Sigma(\mathfrak{u}, \mathfrak{b})$ is essentially the same as $\Sigma(\mathfrak{g}, \mathfrak{a})$.

Let $\mathcal{U}(\mathfrak{u})$ be the universal enveloping algebra[1] of $\mathfrak{u}_{\mathbb{C}}$ and

$$\mathcal{U}(\mathfrak{u})^K = \{z \in \mathcal{U}(\mathfrak{u}) \mid \mathrm{Ad}(K)\, z = z\}.$$

If $\{X_1, \ldots, X_k\}$ is a basis of $\mathfrak{u}$, then

$$\{X_1^{n_1} \cdots X_k^{n_k} \mid n_j \in \mathbb{Z}^+\}$$

is a basis of $\mathcal{U}(\mathfrak{u})$ over $\mathbb{C}$. For any $f \in C^{\infty}(U)$ we let

$$(X_1^{n_1} \cdots X_k^{n_k} f)(u) = \left(\frac{\partial}{\partial t_1}\right)^{n_1} \cdots \left(\frac{\partial}{\partial t_k}\right)^{n_k} f(u\, e^{t_1 X_1} \cdots e^{t_k X_k})\Big|_{t_1 = \cdots = t_n = 0}$$

(this action is extended to $\mathfrak{u}_{\mathbb{C}}$ by $\mathbb{C}$-linearity). This means the elements of $\mathfrak{u}_{\mathbb{C}}$ and $\mathcal{U}(\mathfrak{u})$ are considered as left-invariant differential operators on U.

Lemma 4.1. *For $f \in C^{\infty}(U/K, \chi_l)$ and $z \in \mathcal{U}(\mathfrak{u})^K$ we have*

$$z\, f \in C^{\infty}(U/K, \chi_l).$$

[1] If $\mathfrak{u}$ is any Lie algebra, the symbol $\mathcal{U}(\mathfrak{u})$ denotes the universal enveloping algebra of $\mathfrak{u}_{\mathbb{C}}$. The meanings of the notations such as $\mathcal{U}(\mathfrak{g})$, $\mathcal{U}(\mathfrak{a})$, $\mathcal{U}(\mathfrak{b})$, and so on should be clear without further explanation.

Proof. If $X \in \mathfrak{u}$ then

$$\begin{aligned}(X f)(u k) &= \frac{d}{dt}\Big|_{t=0} f(u k \exp(t X)) \\ &= \frac{d}{dt}\Big|_{t=0} f(u \exp(t \operatorname{Ad}(k) X) k) \\ &= \chi_l(k)^{-1} \frac{d}{dt}\Big|_{t=0} f(u \exp(t \operatorname{Ad}(k) X)) \\ &= \chi_l(k)^{-1} [(\operatorname{Ad}(k) X) f](u).\end{aligned}$$

So if $z \in \mathcal{U}(\mathfrak{u})^K$ then

$$(z f)(u k) = \chi_l(k)^{-1} [(\operatorname{Ad}(k) z) f](u) = \chi_l(k)^{-1} (z f)(u).$$

This implies that $z f \in C^\infty(U/K, \chi_l)$. □

It follows that there is a natural action of the elements of $\mathcal{U}(\mathfrak{u})^K$ on $C^\infty(U/K, \chi_l)$. This action is exactly the action of differential operators in $\mathbb{D}_l$. Therefore, we obtain a homomorphism of algebras $\tau : \mathcal{U}(\mathfrak{u})^K \to \mathbb{D}_l$. Let

$$\mathfrak{t}_l := \{X + \chi_l(X) \mid X \in \mathfrak{k}_\mathbb{C}\}.$$

It is clear that $\mathcal{U}(\mathfrak{u})^K \cap \mathcal{U}(\mathfrak{u})\,\mathfrak{t}_l$ is a two-sided ideal in $\mathcal{U}(\mathfrak{u})^K$, and that it is annihilated by τ. We thus come to the fact below:

Proposition 4.2. *The kernel of the natural homomorphism τ from $\mathcal{U}(\mathfrak{u})^K$ onto $\mathbb{D}_l$ is $\mathcal{U}(\mathfrak{u})^K \cap \mathcal{U}(\mathfrak{u})\,\mathfrak{t}_l$. Hence, there is a natural isomorphism*

$$\mathbb{D}_l \cong \mathcal{U}(\mathfrak{u})^K / (\mathcal{U}(\mathfrak{u})^K \cap \mathcal{U}(\mathfrak{u})\,\mathfrak{t}_l).$$

Proof. Proposition 5.1.1 in [18, P.71] proves $\mathbb{D}_l^d \cong \mathcal{U}(\mathfrak{g})^K / (\mathcal{U}(\mathfrak{g})^K \cap \mathcal{U}(\mathfrak{g})\,\mathfrak{t}_l)$. Simply apply it on the compact dual U/K of the space G/K as the above discussion and we get the desired isomorphism. □

Since $\mathcal{U}(\mathfrak{u}) = \mathcal{U}(\mathfrak{g})$ and $\mathcal{U}(\mathfrak{u})^K = \mathcal{U}(\mathfrak{u})^\mathfrak{k} = \mathcal{U}(\mathfrak{g})^K$, in the terminology of the proof above we have actually that

$$\mathbb{D}_l \cong \mathbb{D}_l^d.$$

For any $z \in \mathcal{U}(\mathfrak{u})^K$, $D_z \in \mathbb{D}_l$ is simply the class of z modulo $\mathcal{U}(\mathfrak{u})^K \cap \mathcal{U}(\mathfrak{u})\,\mathfrak{t}_l$. But for any $z \in \mathcal{U}(\mathfrak{u})^K \cap \mathcal{U}(\mathfrak{u})\,\mathfrak{t}_l$, we have $D_z = 0$. Therefore, $D_z f = z f$ for $f \in C^\infty(U/K, \chi_l)$. It is obvious that $D_z f \in C^\infty(U//K, \chi_l)$ for $f \in C^\infty(U//K, \chi_l)$ and $D_z \in \mathbb{D}_l$.

Let $S(\mathfrak{b})$ be the symmetric algebra over the vector space $\mathfrak{b}_\mathbb{C}$, and

$$S(\mathfrak{b})^W = \{p \in S(\mathfrak{b}) \mid w\,p = p,\ \forall\, w \in W\}.$$

For $p \in S(\mathfrak{b})$, it is considered to be a $\mathbb{C}$-valued polynomial function on $\mathfrak{b}_\mathbb{C}^* = \operatorname{Hom}_\mathbb{C}(\mathfrak{b}_\mathbb{C}, \mathbb{C})$. Also, write $\partial_p \in S(\mathfrak{b})$ if we consider ∂_p as the corresponding constant coefficient differential operator on $\mathfrak{b}_{\mathbb{C}}$ (or on $B_{\mathbb{C}}$). Write $\mathcal{U}(\mathfrak{b})$ for the translation invariant differential operators on $B_{\mathbb{C}}$. Since $\mathfrak{b}$ is abelian,

$$S(\mathfrak{b}) \cong \mathcal{U}(\mathfrak{b})$$

are canonically isomorphic.

Recall the Cartan decomposition $U = K\,B\,K$. It follows that any smooth χ_l-bi-coinvariant function f on U is essentially determined by $f|_B$. Let $B^{\mathrm{reg}} = B_{\mathbb{C}}^{\mathrm{reg}} \cap B$. Theorem 5.1.4 in [18] proves (applied on the dual space U/K)

Proposition 4.3. *For each $D \in \mathbb{D}_l$ there exists a unique differential operator $\triangle_l\,(D) : C^\infty(B^{\mathrm{reg}}) \to C^\infty(B^{\mathrm{reg}})$ such that for $f \in C^\infty(U//K, \chi_l)$,*

$$\triangle_l\,(D)\,(f\,\big|_B) = (D\,f)\,\big|_B.$$

Definition 4.4. The operator $\triangle_l\,(D) \in (\mathcal{R} \otimes S(\mathfrak{b}))^W$ is called the *χ_l-radial part* of D along B.[2] We simply write it as $\triangle\,(D)$ and call it the radial part of D if no confusion arises. Let $\mathrm{rad}\,(\mathbb{D}_l)$ be the commutative algebra of radial components along B of the differential operators in $\mathbb{D}_l$. Similarly, $\mathrm{rad}\,(\mathbb{D}_l^d)$ denotes the algebra of radial components along A of the differential operators in $\mathbb{D}_l^d$.

We will give the description of $\mathbb{D}_l^d$ which is based on the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and the Poincaré-Birkhoff-Witt theorem. From these we have the direct sum decomposition

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{a}) \oplus (\mathfrak{n}_{\mathbb{C}}\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\,\mathfrak{t}_l).$$

Identify $\mathcal{U}(\mathfrak{a}) \cong S(\mathfrak{a})$. Define a map $\varrho_l : \mathcal{U}(\mathfrak{g}) \to S(\mathfrak{a})$ as the projection with respect to this decomposition. It follows from Proposition 4.2 that this map give rise to a homomorphism from $\mathbb{D}_l^d$ into $S(\mathfrak{a})$. It depends on the choice of Σ^+ because $\mathfrak{n}$ depends on it. Let ς be the automorphism of $S(\mathfrak{a})$ generated by

$$\varsigma\,(X) = X + \rho(X), \qquad \forall\, X \in \mathfrak{a}.$$

Define $\gamma_l^d : \mathcal{U}(\mathfrak{g})^K \to S(\mathfrak{a})$ by $\gamma_l^d = \varsigma \circ \varrho_l$. This map is called the (generalized) Harish-Chandra homomorphism. The construction of the Harish-Chandra homomorphism can be generalized to the setting of the dual space U/K, say $\gamma_l : \mathcal{U}(\mathfrak{u})^K \to S(\mathfrak{b})$.

Also see Definition 5.1.9 in [18] where they use radial parts of differential operators in $\mathbb{D}_l^d$ to construct the map γ_l^d.[3] But these two ideas match.

The following theorem in the case $l = 0$ is due to Harish-Chandra [17], for the proof see [12, Chapter II, Theorem 5.17].

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[2] Let $\mathcal{R}$ be the algebra generated by the constant function 1 (the unit) and $1/(1-e^{-2\alpha})$ with $\alpha \in \Sigma$. Since $(1-e^{2\alpha})^{-1} = 1-(1-e^{-2\alpha})^{-1}$, $\mathcal{R}$ is generated by the functions $(1-e^{-2\alpha})^{-1}$ with $\alpha \in \Sigma^+$. The Weyl group W acts on $\mathcal{R}$ and $\mathcal{R}$ is invariant under $S(\mathfrak{b})$. Hence $\mathcal{R} \otimes S(\mathfrak{b})$ represents the algebra of differential operators on $B_{\mathbb{C}}$ with coefficients in $\mathcal{R}$ and $(\mathcal{R} \otimes S(\mathfrak{b}))^W$ consists of those which are invariant under the action of W.

[3] Here is a review of Heckman's construction of γ_l^d. Write $(1-e^{-2\alpha})^{-1} = 1 + e^{-2\alpha} + e^{-4\alpha} + \cdots$ for $\alpha \in \Sigma^+$. We can expand a $P \in \mathcal{R} \otimes S(\mathfrak{a})$ in the form $P = \gamma'(P) + \cdots$ with $\gamma'(P) \in S(\mathfrak{a})$. View $\mathcal{R} \otimes S(\mathfrak{a})$ as a subalgebra of $\mathbb{C}[e^{-2\alpha_1}, \ldots, e^{-2\alpha_n}] \otimes S(\mathfrak{a})$ with $\{\alpha_j\} \in \Pi$. Then these formal expansion in $\mathcal{R} \otimes S(\mathfrak{a})$ are convergent on A^+.

Theorem 4.5 (Harish-Chandra Isomorphism)**.** *The Harish-Chandra map* $\gamma_l : \mathcal{U}(\mathfrak{u})^K \to S(\mathfrak{b})$ *is a homomorphism onto* $S(\mathfrak{b})^W$ *with kernel* $\mathcal{U}(\mathfrak{u})^K \cap \mathcal{U}(\mathfrak{u})\,\mathfrak{t}_l$*. Hence it induces an isomorphism*

$$\gamma_l : \mathbb{D}_l \longrightarrow S(\mathfrak{b})^W$$

of commutative algebras. As a consequence of W*-invariance, it is independent of the choice of* Σ^+*.*

Proof. Refer to Theorem 5.1.10 in [18] or Proposition 2.2 in [31] where $\gamma_l^d : \mathbb{D}_l^d \to S(\mathfrak{a})^W$ was proved to be an algebra isomorphism. Since $S(\mathfrak{b})^W \cong S(\mathfrak{a})^W$, from the construction we see that the isomorphism γ_l is actually identical with γ_l^d. Thus the proof of this theorem reduces to that case, again by duality. □

It follows immediately from the above theorem that $\mathbb{D}_l$ is commutative. Moreover, it is a polynomial algebra of n independent generators (recall $n = \dim \mathfrak{b}$). From Proposition 4.2 and Theorem 4.5 we have the commutative diagram:

$$\begin{array}{ccc}
\mathcal{U}(\mathfrak{u})^K & \xrightarrow{\quad\tau\quad} & \mathbb{D}_l \\
\downarrow & \nearrow\cong & \downarrow \gamma_l\ \cong \\
\mathcal{U}(\mathfrak{u})^K/(\mathcal{U}(\mathfrak{u})^K \cap \mathcal{U}(\mathfrak{u})\,\mathfrak{t}_l) & \xrightarrow{\quad\cong\quad} & S(\mathfrak{b})^W
\end{array}$$

We define functions

$$\eta_l = \eta_{+l} := \prod_{\alpha \in \mathcal{O}_s^+} \left(\frac{e^{\alpha} + e^{-\alpha}}{2} \right)^{|l|}, \qquad \eta_{-l} := \prod_{\alpha \in \mathcal{O}_s^+} \left(\frac{e^{\alpha} + e^{-\alpha}}{2} \right)^{-|l|}.$$

They are holomorphic on $B_{\mathbb{C}}$ and is W-invariant (thus it is an even function).

Corollary 4.6. *For* χ_l *nontrivial we have a bijection*

$$\triangle_l\colon\ \mathbb{D}_l \xrightarrow{\cong} \eta_{\pm l} \circ \mathrm{rad}(\mathbb{D}_l) \circ \eta_{\mp l}.$$

Proof. This follows from Corollary 5.1.11 in [18, p.75]. □

The element $\gamma'(P) \in S(\mathfrak{a})$ is called the constant term of P along A^+. For $m \in \mathcal{M}$ define $\gamma(m) : \mathcal{R} \otimes S(\mathfrak{a}) \to S(\mathfrak{a})$ by $\gamma(m)(P) = e^{\rho(m)} \circ \gamma'(P) \circ e^{-\rho(m)}$ and call it the m-constant term along A^+. Both γ' and $\gamma(m)$ are algebra homomorphisms. Let

$$\mathbb{D}(m) = \{P \in \mathcal{R} \otimes S(\mathfrak{a}) \ \mid\ [P,\, L(m)] = 0,\ w\,P\,w^{-1} = P, \forall\, w \in W\}$$

where $L(m)$ is the radial part of the Laplacian on A^+. Then the m-constant term $\gamma(m) : \mathbb{D}(m) \to S(\mathfrak{a})^W$ is an isomorphism of commutative algebras. The Harish-Chandra map $\gamma_l^d : \mathbb{D}_l^d \to S(\mathfrak{a})^W$ is defined by $\gamma_l^d(D) = \gamma(m_{\pm})(\triangle\ (D))$. It is a homomorphism since both m-constant term $\gamma(m)$ and the radial part are algebra homomorphisms. Note that the algebra $\mathbb{D}(m_{\pm})$ agrees with $\mathrm{rad}(\mathbb{D}_l)$ in the geometric case.

4.2 The Hypergeometric Differential Equations

Elementary spherical functions can be defined in various ways: by integral form or differential equations, or by representation theory. We have seen the integral form of a χ_l-type spherical function in Chapter 3 (recall (3.11)). In the following we will introduce the approach with differential equations.

For $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ and $l \in \mathbb{Z}$ write

$$\gamma_l (D) (\lambda) = \gamma_l (D, \lambda) \in \mathbb{C}, \qquad \forall\, D \in \mathbb{D}_l.$$

This is the value at $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ of the polynomial $\gamma_l (D)$, which is the image under the Harish-Chandra isomorphism γ_l of the differential operator $D \in \mathbb{D}_l$. We can view $D \mapsto \gamma_l (D, \lambda)$ as a character (an algebra homomorphism) from $\mathbb{D}_l$ to $\mathbb{C}$. Any character from $\mathbb{D}_l$ to $\mathbb{C}$ has this form for some $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$. We have $\gamma_l (D, \lambda) = \gamma_l (D, \mu)$ if and only if there is a $w \in W$ such that $\lambda = w\, \mu$.

Definition 4.7. For $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ and $l \in \mathbb{Z}$ the system of differential equations

$$D\, \varphi = \gamma_l (D, \lambda)\, \varphi, \qquad \forall\, D \in \mathbb{D}_l, \tag{4.1}$$

is called the system of hypergeometric differential equations with spectral parameter $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$, where $\varphi \in C^\infty (U/K, \chi_l)$.

Since $S (\mathfrak{b})^W$ is itself a polynomial algebra, the system of hypergeometric differential equations (4.1) is just the simultaneous eigenvalue problem for the commuting algebra $\mathbb{D}_l$.

Proposition 4.8. *A χ_l-bi-coinvariant function φ is an elementary χ_l-type spherical function on G if and only if φ satisfies (4.1) for some $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ and $\varphi (e) = 1$.*

Proof. See [18, Definition 5.2.1] or [31, Theorem 3.2]. □

Corollary 4.9. *The function $\varphi_{\lambda, l}$ defined in (3.11) is the unique χ_l-bi-coinvariant function satisfying (4.1) and $\varphi (e) = 1$. Any spherical function of type χ_l on G has this form for some $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$.*

Proof. See [31, Proposition 3.3]. □

The above proposition states that the χ_l-type spherical functions are the normalized joint eigenfunctions of $\mathbb{D}_l$. In fact, the integrand in (3.11) is already an eigenfunction of $\mathbb{D}_l$ with this eigenvalue $\gamma_l(D, \lambda)$ where $D \in \mathbb{D}_l$. Since $\mathbb{D}_l$ contains the Laplace-Beltrami operator[4] (which is an elliptic operator), all χ_l-type spherical functions on G are indeed real analytic[5].

[4] The Laplace-Beltrami operator exists on any semisimple symmetric space due to the pseudo-Riemannian structure. It is an invariant differential operator since the pseudo-Riemannian structure is invariant.

[5] An eigenfunction of an elliptic operator on an analytic Riemannian manifold is analytic.

The formal transpose (or adjoint) D^* of $D \in \mathbb{D}_l$ is the differential operator defined by

$$\int_U (D\, f)(u)\, \overline{g\,(u)}\, du = \int_U f\,(u)\, \overline{(D^*\, g)\,(u)}\, du$$

with respect to Haar measure du on U for $f,\, g \in C^\infty\,(U/K,\, \chi_l)$ with at least of them having compact support. Then $D^* \in \mathbb{D}_l$ and it has the properties that

$$(D_1\, D_2)^* = D_2^*\, D_1^*, \qquad (\partial_p)^* = \partial_{p^*}$$

where for $p \in S\,(\mathfrak{b}_{\mathbb{C}})$, $p^* \in S\,(\mathfrak{b}_{\mathbb{C}})$ is defined by $p^*(\lambda) = p\,(-\lambda)$. The latter formula is same as $\gamma_l\,(D^*) = \gamma_l\,(D)^*$ for $D \in \mathbb{D}_l$. If $D^* = D$ then D is called formally self-adjoint, which means D is a symmetric operator.

If D is a differential operator on $B_{\mathbb{C}}^{\mathrm{reg}}$ we denote by D^* the formal transpose of D as a differential operator on B with respect the Haar measure $d\,b$ on B:

$$\int_B (D\, f\,(b))\, \overline{g\,(b)}\, db = \int_B f\,(b) \overline{(D^*\, g)\,(b)}\, db$$

for all $f,\, g \in C^\infty\,(B^{\mathrm{reg}})$ with at least one of them having compact support.

Proposition 4.10. *The χ_l-type spherical function $\psi_{\mu,\,l}$ on U satisfies the joint eigenequation*

$$D\, \psi_{\mu,\,l} = \gamma_l\,(D,\, \mu + \rho)\, \psi_{\mu,\,l}, \quad \forall\, D \in \mathbb{D}_l.$$

Hence,

$$\mathcal{S}_l\,(D\, f) = \overline{\gamma_l\,(D^*,\, \cdot + \rho)}\, \mathcal{S}_l\, f, \qquad \forall f \in C^\infty\,(U/K,\, \chi_l).$$

Proof. By (3.12) and (4.1) we have

$$D\, \psi_{\mu,\,l} = D\, \varphi_{\mu+\rho,\,l} = \gamma_l\,(D,\, \mu + \rho)\, \varphi_{\mu+\rho,\,l} = \gamma_l\,(D,\, \mu + \rho)\, \psi_{\mu,\,l}.$$

It follows that for $\mu \in \Lambda_l^+$

$$\begin{aligned}
\mathcal{S}_l\,(D\, f)\,(\mu) &= (D\, f,\ \psi_{\mu,\,l}) \\
&= (f,\ D^*\, \psi_{\mu,\,l}) \\
&= (f,\ \gamma_l\,(D^*,\, \mu + \rho)\, \psi_{\mu,\,l}) \\
&= \int_U f(u)\, \overline{\gamma_l\,(D^*,\, \mu + \rho)\, \psi_{\mu,\,l}\,(u)}\, du \\
&= \overline{\gamma_l\,(D^*,\, \mu + \rho)}\, \mathcal{S}_l\,(f)\,(\mu).
\end{aligned}$$

□

Let $\{\xi_j\}_{j=1}^n$ be an orthonormal basis of $\mathfrak{a}$. Then $\sum_{j=1}^n \partial_{\xi_j}^2 \in S(\mathfrak{a})^W$[6] is the Laplace operator on A. For $m \in \mathcal{M}$ define a differential operator

$$L(m) := \sum_{j=1}^{n} \partial_{\xi_j}^2 + \sum_{\alpha\in\Sigma^+} m_\alpha \frac{1+e^{-2\alpha}}{1-e^{-2\alpha}} \partial_\alpha$$

which generalizes to arbitrary multiplicity function m the radial part on A^+ of the Laplace-Beltrami operator on G/K, independent of the choice of A^+[7]. This is the standard second order hypergeometric operator. It is well defined on $\mathfrak{b}_{\mathbb{C}}^{\mathrm{reg}}$ (or $B_{\mathbb{C}}^{\mathrm{reg}}$) and is invariant under the Weyl group W.

Recall that we have the integral formula for Cartan decomposition $U = K\,B\,K$ (see [12, theorem 5.10, p.190]):

$$\int_U f(u)\,du = \frac{1}{|W|} \int_K \int_B \int_K f(k_1\,b\,k_2)\,\delta(m,\,b)\,d\,k_2\,d\,b\,d\,k_1$$

where $f \in C^\infty(U/K,\,\chi_l)$, m is given associated with the symmetric space U/K, and $|W|$ denotes the cardinality of W. Let f_1, f_2 be in $C^\infty(U//K,\,\chi_l)$ with at least one of them having compact support. Then

$$\begin{aligned}
& (f_1,\,f_2) \\
:= & \int_U f_1(u)\,\overline{f_2(u)}\,du \\
= & \frac{1}{|W|} \int_K \int_B \int_K f_1(k_1\,b\,k_2)\,\overline{f_2(k_1\,b\,k_2)}\,\delta(m,\,b)\,d\,k_2\,d\,b\,d\,k_1 \\
= & \frac{1}{|W|} \int_K \int_K \underbrace{\chi_l(k_1\,k_2)^{-1}\,\overline{\chi_l(k_1\,k_2)^{-1}}}_{=1}\,dk_1\,dk_2 \int_B f_1(b)\,\overline{f_2(b)}\,\delta(m,\,b)\,db \\
= & \frac{1}{|W|} \int_B f_1(b)\,\overline{f_2(b)}\,\delta(m,\,b)\,db. \qquad (4.2)
\end{aligned}$$

Let $m \in \mathcal{M}$ satisfy (2.4). Then $\delta(m,\,b)\,db$ is a positive measure on B. Define an inner product $(\,\cdot\,,\,\cdot\,)_m$ on $C^\infty(B)^W$ by (4.2).

Proposition 4.11. *The operator $L(m)$ is symmetric with respect to $\delta(m,\,b)\,db$ for any $m \in \mathcal{M}$ subject to (2.4).*

---

[6]For any $X \in \mathfrak{a}$ let ∂_X denote the directional derivative in $\mathfrak{a}$ with respect to X, that is,

$$(\partial_X\,\phi)(Y) = \frac{d}{d\,t}\,\phi(Y + t\,X)\Big|_{t=0}, \qquad Y \in \mathfrak{a}.$$

We identify $\mathfrak{a}^*$ with $\mathfrak{a}$ via $\alpha \mapsto A_\alpha$ and then view $\partial_\alpha = \partial_{A_\alpha}$ for any $\alpha \in \Sigma^+$.

[7]Here K acts on G/K with $A^+ \cdot o = \operatorname{Exp}\mathfrak{a}^+$ as a transversal manifold, that is, it is a submanifold of G/K which meets each K-orbit $K \cdot p$ at only one point $p \in A^+ \cdot o$, and

$$T_p(G/K) = T_p(A^+ \cdot o) \oplus T_p(K \cdot p).$$

See Propositions 3.9 and 3.11 of Chapter II in [12].

Proof. The idea comes from the proof of Proposition 5.1.5 in [18, p.72]. Write $L = L(m)$ and we want to show L is invariant under taking adjoint with respect to $(\,\cdot\,,\,\cdot\,)_m$. Let $ML := L + \langle \rho, \rho \rangle$ with $\rho = \rho(m)$. Since $\langle \rho, \rho \rangle$ is just a number, it is automatically symmetric with respect to any measure. If ML is symmetric with respect to $\delta(m, b)\, db$, then so is L. So it is enough to show

$$\int_B [ML\, f_1(b)]\, \overline{f_2(b)}\, \delta(m, b)\, db = \int_B f_1(b)\, \overline{[ML\, f_2(b)]}\, \delta(m, b)\, db.$$

Note that

$$\delta(m) = \upsilon(m)^{1/2}\, \overline{\upsilon(m)}^{1/2} \in \mathbb{C}[P]^W,$$

and from Theorem 2.1.1 in [18] we have

$$\upsilon(m)^{1/2} \circ ML \circ \upsilon(m)^{-1/2} = \sum_{j=1}^{n} \partial_{\xi_j}^2 + \sum_{\alpha \in \Sigma^+} \frac{m_\alpha (2 - m_\alpha - 2m_{2\alpha}) \langle \alpha, \alpha \rangle}{(e^\alpha - e^{-\alpha})^2}$$

where the first term of the right-hand side is the second-order directional derivatives, it is symmetric with respect to both Haar measures $d\,b$ on B and $d\,a$ on A (a natural choice could be the continuation of $d\,b$); and the second term is the first-order term, which is a potential function and no differentiation to deal with there, and hence symmetric with $d\,b$ and $d\,a$. This implies that the right-hand side (and hence the left-side) is symmetric with respect to the measure $d\,b$ and $d\,a$. It follows that

$$\begin{aligned}
&\int_B [ML\, f_1(b)]\, \overline{f_2(b)}\, \delta(m, b)\, db \\
&= \int_B [ML\, f_1(b)]\, \overline{f_2(b)}\, \upsilon(m, b)^{1/2}\, \overline{\upsilon(m, b)}^{1/2}\, db \\
&= \int_B [(\upsilon(m)^{\frac{1}{2}} \circ ML \circ \upsilon(m)^{-\frac{1}{2}})\, \upsilon(m)^{\frac{1}{2}}(b)\, f_1(b)]\, \overline{\upsilon(m)^{\frac{1}{2}}(b)\, f_2(b)}\, db \\
&= \int_B \upsilon(m)^{\frac{1}{2}}(b)\, f_1(b)\, \overline{(\upsilon(m)^{\frac{1}{2}} \circ ML \circ \upsilon(m)^{-\frac{1}{2}})\, \upsilon(m)^{\frac{1}{2}}(b)\, f_2(b)}\, db \\
&= \int_B f_1(b)\, \overline{\upsilon(m, b)}^{-\frac{1}{2}}\, \overline{[(\upsilon(m)^{\frac{1}{2}} \circ ML \circ \upsilon(m)^{-\frac{1}{2}})\, \upsilon(m)^{\frac{1}{2}}(b)\, f_2(b)]}\, \upsilon(m, b)^{\frac{1}{2}}\, \overline{\upsilon(m, b)}^{\frac{1}{2}}\, db \\
&= \int_B f_1(b)\, \overline{[ML\, f_2(b)]}\, \delta(m, b)\, db
\end{aligned}$$

as desired. □

Chapter 5
The Hypergeometric Functions

The geometry constrains the root multiplicities m_α to assume certain specific values. The spherical functions are determined by the geometry as well because the system (5.1) of differential equations originates from the algebra of G-invariant differential operators on G/K. Heckman and Opdam aimed to construct, for arbitrary complex values of multiplicities, the systems (5.1) of differential equations. As analytic continuations in the multiplicity parameters of Harish-Chandra spherical functions, the unique (up to normalization) simultaneous eigenfunctions of these new systems would have provided a class of multivariable generalized spherical functions which are nowadays known as hypergeometric functions associated with root systems. The hypergeometric function we will study is a generalization of the χ_l-type spherical function for a real semisimple Lie group in a sense which will be explained in this chapter. The main work we have done is to obtain a nice exponential growth estimate for such a hypergeometric function (see Proposition 5.10). This is of crucial importance for the proof of the Paley-Wiener Theorem for line bundles over symmetric spaces. Our study below is based mainly on [18], [27], [12, Chapter IV], and [31], to which we refer for details.

5.1 The Harish-Chandra Expansion

In this section we will give a short introduction of the Harish-Chandra asymptotic expansion for χ_l-type spherical functions $\varphi_{\lambda,\,l}$. Refer to [18, Part I, Chapter 4] and [31, Theorem 3.6] for details.

If a χ_l-bi-coinvariant function φ is an eigenfunction for $\mathbb{D}_l$ (i.e. it satisfies (4.1)) then

$$\triangle\,(D)\,(\varphi|_{A^+}) = (D\,\varphi)|_{A^+} = (\gamma_l\,(D,\,\lambda)\,\varphi)|_{A^+} = \gamma_l\,(\triangle\,(D),\,\lambda)\,\varphi|_{A^+}.$$

Abusing notation by writing $\varphi = \varphi|_{A^+}$ the above equation is

$$\triangle\,(D)\,\varphi = \gamma_l\,(\triangle\,(D),\,\lambda)\,\varphi, \qquad \forall\; D \in \mathbb{D}_l^d.$$

This tells us that φ is an eigenfunction for $\mathbb{D}_l$ if and only if $\varphi|_{A^+}$ is an eigenfunction for $\mathrm{rad}\,(\mathbb{D}_l^d)$, i.e. it satisfies the system of differential equations

$$D\,\varphi = \gamma_l\,(D,\,\lambda)\,\varphi, \qquad \forall\; D \in \mathrm{rad}\,(\mathbb{D}_l^d). \tag{5.1}$$

Therefore the restriction of χ_l-type spherical function $\varphi_{\lambda,\,l}$ on A^+ satisfies (5.1). Moreover, the function $\eta_{-l}\,\varphi_{\lambda,\,l}$ is a joint eigenfunction of a commutative algebra of differential operators $\eta_{-l} \circ D \circ \eta_l$ (this can also be seen from Corollary 4.6), that is, it satisfies the system of equations

$$(\eta_{-l} \circ D \circ \eta_l)\, \varphi = \gamma_l\,(D,\, \lambda)\, \varphi, \qquad \forall\, D \in \mathrm{rad}\,(\mathbb{D}_l^d). \tag{5.2}$$

Note that the radial part of the Laplace-Beltrami operator on G/K (acting on χ_l-bi-coinvariant functions) is exactly the operator

$$L\,(l) = L\,(m(l)) := \sum_{j=1}^{n} \partial_{\xi_j}^2 + \sum_{\alpha\in\Sigma^+} (m(l))_\alpha \, \frac{1+e^{-2\alpha}}{1-e^{-2\alpha}}\, \partial_\alpha \in \mathrm{rad}\,(\mathbb{D}_l^d)$$

associated with the root system Σ and the multiplicity $m(l)$. This operator is actually defined on $B_{\mathbb{C}}^{\mathrm{reg}}$. It is shown that

$$\gamma_l\,(L\,(l),\, \lambda) = \langle \lambda,\, \lambda\rangle - \langle \rho(l),\, \rho(l)\rangle$$

and thus the differential equation (5.2) corresponding to the operator $L(l)$ can be written as

$$L\,(l)\,(\varphi) = (\langle \lambda,\, \lambda\rangle - \langle \rho(l),\, \rho(l)\rangle)\, \varphi, \qquad \lambda \in \mathfrak{b}_{\mathbb{C}}^*, \tag{5.3}$$

(see [31, (3.8)]). The function $\eta_{-l}\, \varphi_{\lambda,\, l}$ on A^+ satisfies (5.3).

We briefly review the Harish-Chandra series for $\varphi_{\lambda,\, l}$ in the below. We look for solutions of (5.3) with spectral parameter λ which are of the form

$$\Phi\,(\lambda,\, m(l);\, a) = \Phi_{\lambda,\, l}\,(a) = a^{\lambda-\rho(l)} \sum_{\mu\in\Xi} \Gamma_{\mu,\, l}\,(\lambda)\, a^{-\mu}, \qquad a \in A^+ \tag{5.4}$$

where $\Xi := \{\sum_{j=1}^n n_j\, \alpha_j \mid n_j \in \mathbb{Z}^+,\, \alpha_j \in \Pi\}$ (recall Π is the set of simple roots in Σ^+). The coefficients $\Gamma_{\mu,\, l}\,(\lambda) \in \mathbb{C}$ are uniquely determined by $\Gamma_{0,\, l}\,(\lambda) \equiv 1$ and the recurrence relation

$$\langle \mu,\, \mu - 2\lambda\rangle \Gamma_{\mu,\, l}\,(\lambda) = 2 \sum_{\alpha\in\Sigma^+} (m(l))_\alpha \sum_{\substack{k\in\mathbb{N}\\ \mu-2k\,\alpha\in\Xi}} \Gamma_{\mu-2k\,\alpha,\, l}\,(\lambda)\, \langle \mu + \rho(l) - 2\,k\,\alpha - \lambda,\, \alpha\rangle$$

provided $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ satisfies $\langle \mu,\, \mu - 2\lambda\rangle \neq 0$ for all $\mu \in \Xi \setminus \{0\}$. For such λ the Harish-Chandra series (5.4) converges to a meromorphic function of λ and $\Phi_{\lambda,\, l}$ is real analytic on A^+. In fact, there is a tubular neighborhood of A^+ in $A_{\mathbb{C}}$ on which this series converges to a holomorphic function. If $D \in \mathrm{rad}\,(\mathbb{D}_l^d)$ then $D\, \Phi_{\lambda,\, l} = \gamma_l\,(D,\, \lambda)\, \Phi_{\lambda,\, l}$. It is remarkable that $\Phi_{\lambda,\, l}$ turns out to solve the entire system (5.2) and allows to construct a basis for the smooth solutions of (5.2) on A^+.

Theorem 5.1 (The Harish-Chandra Expansion). *Let the notation be as above. There is a meromorphic function $c\,(\lambda,\, l) = c\,(\lambda,\, m(l))$ on $\mathfrak{b}_{\mathbb{C}}^* \times \mathcal{M}$ so that $\varphi_{\lambda,\, l}$ admits on A^+ the expansion*

$$\varphi_{\lambda,\, l} = \eta_l \sum_{w\in W} c\,(w\lambda,\, l)\, \Phi_{w\lambda,\, l}. \tag{5.5}$$

The function $c(\lambda, l)$ is the known Harish-Chandra c-function which governs the asymptotic behavior of $\varphi_{\lambda, l}$ on A^+ and is given by

$$c(\lambda, l) = \int_{\overline{N}} e^{-(\lambda+\rho)(H(\overline{n}))} \chi_l(\kappa(\overline{n}))^{-1} d\overline{n}$$

where $\overline{N} := \theta(N)$ and the Haar measure $d\overline{n}$ on $\overline{N}$ is normalized by the condition $c(\rho, 0) = 1$ (see [29, Remark 7.3]). It admits an explicit form [31, (3.15)] by using Gindikin-Karpelevič product formula.

5.2 The Hypergeometric Functions

Heckman and Opdam used the right-hand side of (5.5) with arbitrary multiplicity function $m \in \mathcal{M}$ to define their generalized spherical functions known as the hypergeometric functions:

Definition 5.2. Define meromorphic functions $\widetilde{c}, c : \mathfrak{b}^*_{\mathbb{C}} \times \mathcal{M} \to \mathbb{C}$ by

$$c(\lambda, m) = \frac{\widetilde{c}(\lambda, m)}{\widetilde{c}(\rho, m)}, \quad \widetilde{c}(\lambda, m) = \prod_{\alpha\in\Sigma^+} \frac{\Gamma(\lambda_\alpha + \frac{m_{\alpha/2}}{4})}{\Gamma(\lambda_\alpha + \frac{m_{\alpha/2}}{4} + \frac{m_\alpha}{2})}.$$

The function

$$F(\lambda, m; a) = \sum_{w\in W} c(w\lambda, m)\, \Phi(w\lambda, m; a)$$

is called the *hypergeometric function* associated with the triple $(\mathfrak{a}, \Sigma, \mathcal{M})$ or simply Σ. Here $\Phi(\lambda, m; a)$ is the Harish-Chandra series.

Heckman and Opdam proved the fundamental result of the hypergeometric functions (see Theorem 4.4.2 in [18]):

Theorem 5.3. *Let $\mathcal{P} \subset \mathcal{M}$ be defined by*

$$\mathcal{P} = \{m \in \mathcal{M} \mid \widetilde{c}(\rho, m) = 0\}.$$

Then $F(\lambda, m; a)$ is W-invariant and holomorphic in $\mathfrak{b}^*_{\mathbb{C}} \times (\mathcal{M} \setminus \mathcal{P}) \times T$, where T is a W-invariant tubular neighborhood of A in $B_{\mathbb{C}}$.

Remark 5.4. The open set $\mathcal{M} \setminus \mathcal{P}$ contains the closed subset

$$\{m \in \mathcal{M} \mid \operatorname{Re}(m_{\alpha/2} + m_\alpha) \geq 0, \ \forall\, \alpha \in \Sigma_*\}.$$

It is easy to see that $m_\pm(l)$ is contained in this subset and thus in $\mathcal{M}\setminus\mathcal{P}$. It follows that $F(\lambda, m_\pm(l); \cdot)$ is holomorphic in $\mathfrak{b}^*_{\mathbb{C}} \times (\mathcal{M} \setminus \mathcal{P}) \times T$.

By construction the hypergeometric functions reduce to Harish-Chandra's spherical functions of type χ_l if m is a geometric multiplicity. This is the content of the following fact:

Proposition 5.5. *Let F be the hypergeometric functions associated with Σ. Then*

$$\varphi_{\lambda,\,l}\,|_A = \eta_{\pm\,l}\,F\,(\lambda,\ m_\pm(l);\ \cdot)$$

where $\lambda \in \mathfrak{b}^*_{\mathbb{C}}$, and the $\pm$ sign indicates that both possibilities are valid.

Proof. See [18, p.76, theorem 5.2.2]. □

Proposition 5.6. *The hypergeometric function $F\,(\lambda,\ m;\ \cdot)$ has a holomorphic extension to $\mathfrak{b}^*_{\mathbb{C}} \times \mathcal{M} \times (\mathfrak{a} + V)$ where V is a neighborhood of 0 in $\mathfrak{b}$.*

Proof. This is well known, see theorem 3.15 in [27]. □

Write $\varphi_{\lambda,\,l} = \varphi_{\lambda,\,l}|_A$. We therefore have $\varphi_{\lambda,\,l} = \eta_l\,F\,(\lambda,\ m(l);\ \cdot)$ on $\mathfrak{a} + V$ by holomorphic continuation for some neighborhood V of 0 in $\mathfrak{b}$. Since F are the solutions of (5.3), we have

$$L\,(l)\;F\,(\lambda,\ m(l);\ Z) = (\langle\lambda,\ \lambda\rangle - \langle\rho(l),\ \rho(l)\rangle)\,F\,(\lambda,\ m(l);\ Z),\quad \forall\, Z \in \mathfrak{a} + V. \tag{5.6}$$

Remark 5.7. Consider the domains

$$\Omega := \{X \in \mathfrak{b}\ \mid\ |\alpha\,(X)| < \pi,\ \forall\,\alpha \in \Sigma\}$$

and for $\varepsilon > 0$,

$$\Omega_\varepsilon := \{X \in \mathfrak{b}\ \mid\ |\alpha\,(X)| < \pi - \varepsilon,\ \forall\,\alpha \in \Sigma\}.$$

Recall from Remark 3.34 that the χ_l-type spherical function $\varphi_{\lambda,\,l}\,(g)$ is holomorphic in $(\lambda,\ g) \in \mathfrak{b}^*_{\mathbb{C}} \times \mathcal{V}$. Let F be the hypergeometric function associated with Σ. It follows from [3, Remark 3.17] that F has a holomorphic extension to $\mathfrak{b}^*_{\mathbb{C}} \times \mathcal{M} \times (\mathfrak{a} + \Omega)$, so

$$\varphi_{\lambda,\,l}\,(a) = \eta_l\,F\,(\lambda,\ m(l);\ a),\qquad a \in \exp\,(\mathfrak{a} + \Omega).$$

Then a natural choice of the complex domain $\mathcal{V}$ could be

$$\mathcal{V} = K_{\mathbb{C}}\,\exp\,(\mathfrak{a} + \Omega)\,K_{\mathbb{C}}. \tag{5.7}$$

Lemma 5.8. *For $X \in \Omega$ and $\mu \in \Lambda^+_l$, the map $\mu \mapsto \psi_{\mu,\,l}\,(e^X)$ has an analytic continuation to $\mathfrak{b}^*_{\mathbb{C}}$, say $\lambda \mapsto \psi_{\lambda,\,l}\,(e^X)$. Moreover, the map $X \mapsto \psi_{\lambda,\,l}\,(e^X)$ is analytic, and*

$$\psi_{w\,(\lambda+\rho)-\rho,\,l}\,(e^X) = \psi_{\lambda,\,l}\,(e^X),\qquad \forall\, w \in W.$$

Proof. Let $\mathcal{V}$ be as in (5.7). For each $g \in \mathcal{V}$ and $\lambda \in \mathfrak{b}^*_{\mathbb{C}}$ we define

$$\psi_{\lambda,\,l}\,(g) = \varphi_{\lambda+\rho,\,l}\,(g)$$

and therefore attain an analytic continuation to $\mathfrak{b}^*_{\mathbb{C}}$ of the map $\mu \mapsto \psi_{\mu,\,l}\,(g)$ where $\mu \in \Lambda^+_l$, which has the same form, $\lambda \mapsto \psi_{\lambda,\,l}\,(g)$. Let $f \in C^\infty_r\,(U//K,\ \chi_l)$. This gives a well-defined holomorphic extension of the spherical Fourier transform $\lambda \mapsto \widetilde{f}_l\,(\lambda)$ with $\widetilde{f}_l\,(\lambda)$ given by (3.13) if f has compact support in $U \cap \mathcal{V}$. In particular, for $X \in \Omega$, the map $(\lambda,\ X) \mapsto \psi_{\lambda,\,l}\,(e^X)$ is holomorphic (or analytic) in $\mathfrak{b}^*_{\mathbb{C}} \times \Omega$.

By the fact that $\varphi_{\lambda,\,l}$ is W-invariant in $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$, we have

$$\psi_{w\,(\lambda+\rho)-\rho,\,l}\,(e^X) = \varphi_{w\,(\lambda+\rho),\,l}\,(e^X) = \varphi_{(\lambda+\rho),\,l}\,(e^X) = \psi_{\lambda,\,l}\,(e^X).$$

□

Remark 5.9. For $X \in \Omega$ and $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$, we have $\varphi_{\lambda,\,l}\,(e^X) = \eta_l\,(e^X)\,F\,(m(l),\,\lambda,\,X)$ where

$$\eta_l\,(e^X) = \prod_{\alpha\in\mathcal{O}_s^+} \left(\frac{e^{\alpha\,(X)} + e^{-\alpha\,(X)}}{2}\right)^{|l|}.$$

Since $\alpha\,(\mathfrak{b}) \in i\,\mathbb{R}$, then

$$0 < |\eta_l\,(e^X)| = \prod_{\alpha\in\mathcal{O}_s^+} |\cos\,\mathrm{Im}\,\alpha\,(X)|^{|l|} \leq 1.$$

Therefore, η_l is holomorphic on $A\,(\exp\,\Omega)$ and bounded on $\exp\,\Omega$.

In Chapter 6 we will show the spherical Fourier transform of a χ_l-bi-coinvariant function actually ends up into the Paley-Wiener space, for which we need to have a nice control over the growth behaviour in $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ of $F\,(\lambda,\,m(l);\;X)$ where X is in some neighborhood of 0 in $\mathfrak{b}$. Proposition 6.1 in [27] gives a uniform estimate both in $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$ and in $X \in \mathfrak{a}+\overline{\Omega/2}$, but requires all multiplicities involved in $F\,(\lambda,\,k;\;X)$ to be positive, i.e. $k_\alpha \geq 0$ for all $\alpha \in R$ (recall Remark 2.9 for the meaning of these notations used in [27]). Clearly, the multiplicity $m(l)$ might be negative. So we can not use this result. However, we can generalize it to a similar but still nice growth estimate for $F\,(\lambda,\,m;\;X)$ where X is in a bigger domain $\mathfrak{a} + \overline{\Omega_\varepsilon}$, even though we have a weaker assumption (2.4) on the multiplicity parameters.

Proposition 5.10. *Let F be the hypergeometric function associated with Σ. Let $m \in \mathcal{M}$ satisfy $m_\alpha + m_{\alpha/2} \geq 0$ and $m_\alpha \geq 0$ for all $\alpha \in \Sigma_*$. Let $\varepsilon > 0$. Then there is a constant $C = C_\varepsilon > 0$ depending on ε such that*

$$|F\,(\lambda,\,m;\;Z)| \leq |W|^{\frac{1}{2}}\,\exp\,(-\min_{w\in W}\,\mathrm{Im}\,(w\,\lambda\,(Y)) + \sqrt{C}\,\max_{w\in W}\,w\,\rho\,(Y) + \max_{w\in W}\,\mathrm{Re}\,(w\,\lambda\,(X)))$$

where $Z = X + i\,Y$ with $X,\,Y \in \mathfrak{a}$ and $|\alpha\,(Y)| \leq \pi - \varepsilon$ for all $\alpha \in \Sigma$, and $\rho = \rho(m)$.

Proof. Let $\phi_w(Z) = G\,(\lambda, m,\,w^{-1}Z)$ where G is the nonsymmetric hypergeometric function defined as in [27, Theorem 3.15], so that $F = |W|^{-1}\sum_w G^w$ where $G^w\,(\lambda, m, Z) = G\,(\lambda, m, w^{-1}Z)$. By Definition 3.1 and Lemma 3.2 in [27] this implies that

$$\partial_\xi\phi_w = -\frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha\,\alpha(\xi)\,\left[\frac{1+e^{-2\alpha\,(Z)}}{1-e^{-2\alpha\,(Z)}}\,(\phi_w - \phi_{r_\alpha\,w}) - \mathrm{sgn}\,(w^{-1}\,\alpha)\,\phi_{r_\alpha\,w}\right] + (w\lambda,\,\xi)\,\phi_w.$$

Assume $m_\alpha + m_{\alpha/2} \geq 0$ and $m_\alpha \geq 0$ for all $\alpha \in \Sigma_*$. Taking complex conjugates,

$$\partial_\xi\overline{\phi}_w = -\frac{1}{2}\sum_{\alpha\in\Sigma^+} m_\alpha\,\alpha(\overline{\xi})\,\left[\frac{1+e^{-2\alpha\,(\overline{Z})}}{1-e^{-2\alpha\,(\overline{Z})}}\,(\overline{\phi}_w - \overline{\phi}_{r_\alpha\,w}) - \mathrm{sgn}\,(w^{-1}\,\alpha)\,\overline{\phi}_{r_\alpha\,w}\right] + (w\,\overline{\lambda},\,\overline{\xi})\,\overline{\phi}_w.$$

It follows that

$$
\begin{aligned}
&\partial_\xi \sum_w |\phi_w|^2 \\
= \;& \sum_w [(\partial_\xi \phi_w)\,\overline{\phi}_w + \phi_w\,(\partial_\xi \overline{\phi}_w)] \\
= \;& -\frac{1}{2} \sum_{\alpha\in\Sigma^+,\,w} [m_\alpha\,\alpha(\xi) \left(\frac{1+e^{-2\alpha\,(Z)}}{1-e^{-2\alpha\,(Z)}}\,(\phi_w - \phi_{r_\alpha\,w})\,\overline{\phi}_w - \operatorname{sgn}(w^{-1}\,\alpha)\,\phi_{r_\alpha\,w}\,\overline{\phi}_w\right) \\
& + m_\alpha\,\alpha(\overline{\xi}) \left(\frac{1+e^{-2\alpha\,(\overline{Z})}}{1-e^{-2\alpha\,(\overline{Z})}}\,(\overline{\phi}_w - \overline{\phi}_{r_\alpha\,w})\,\phi_w - \operatorname{sgn}(w^{-1}\,\alpha)\,\overline{\phi}_{r_\alpha\,w}\,\phi_w\right)] \\
& + 2\sum_w \operatorname{Re}(w\,\lambda\,(\xi))\,|\phi_w|^2.
\end{aligned}
$$

For fixed α, we add the terms with index w and $r_\alpha\,w$. Then

$$
\begin{aligned}
&\partial_\xi \sum_w |\phi_w|^2 \\
= \;& -\frac{1}{4} \sum_{\alpha\in\Sigma^+,\,w} m_\alpha \left[\alpha\,(\xi)\,\frac{1+e^{-2\alpha\,(Z)}}{1-e^{-2\alpha\,(Z)}} + \alpha\,(\overline{\xi})\,\frac{1+e^{-2\alpha\,(\overline{Z})}}{1-e^{-2\alpha\,(\overline{Z})}}\right] |\phi_w - \phi_{r_\alpha\,w}|^2 \\
& + \sum_{\alpha\in\Sigma^+,\,w} m_\alpha \operatorname{sgn}(w^{-1}\alpha)\operatorname{Im}(\alpha(\xi))\operatorname{Im}(\overline{\phi}_w\,\phi_{r_\alpha\,w}) + 2\sum_w \operatorname{Re}(w\,\lambda\,(\xi))\,|\phi_w|^2.
\end{aligned}
$$

Observe that

$$
|1-e^{-2\alpha\,(Z)}|^2 = (1-e^{-2\alpha\,(Z)})\,\overline{1-e^{-2\alpha\,(Z)}} = (1-e^{-2\alpha\,(Z)})\,(1-e^{-2\alpha\,(\overline{Z})}),
$$

which gives

$$
\begin{aligned}
&\alpha\,(\xi)\,\frac{1+e^{-2\alpha\,(Z)}}{1-e^{-2\alpha\,(Z)}} + \alpha\,(\overline{\xi})\,\frac{1+e^{-2\alpha\,(\overline{Z})}}{1-e^{-2\alpha\,(\overline{Z})}} \\
= \;& \frac{\alpha(\xi)\,(1+e^{-2\alpha\,(Z)})\,(1-e^{-2\alpha\,(\overline{Z})}) + \alpha\,(\overline{\xi})\,(1+e^{-2\alpha\,(\overline{Z})})\,(1-e^{-2\alpha\,(Z)})}{|1-e^{-2\alpha\,(Z)}|^2}.
\end{aligned}
$$

Write $Z = X + i\,Y$ with $X,\, Y \in \mathfrak{a}$ and $|\alpha\,(Y)| \le \pi - \varepsilon,\ \forall\,\alpha \in \Sigma$. Let $\alpha\,(X) = t \in \mathbb{R}$ and $\alpha\,(Y) = s \in \mathbb{R}$. Then $\alpha\,(Z) = t + i\,s$. Write $\alpha\,(\xi) = a + i\,b$ with $a = \operatorname{Re}\alpha\,(\xi)$ and $b = \operatorname{Im}\alpha\,(\xi)$. Note that

$$
\begin{aligned}
(1+e^{-2\alpha\,(Z)})\,(1-e^{-2\alpha\,(\overline{Z})}) &= 1 + e^{-2\alpha\,(Z)} - e^{-2\alpha\,(\overline{Z})} - e^{-2\alpha\,(Z) - 2\alpha\,(\overline{Z})} \\
&= 1 + e^{-2t}\,\cos(2s) - i\,e^{-2t}\,\sin(2s) - e^{-2t}\,\cos(2s) \\
&\quad - i\,e^{-2t}\,\sin(2s) - e^{-2t - i\,2s - 2t + i\,2s} \\
&= 1 - e^{-4t} - 2\,i\,e^{-2t}\,\sin(2s).
\end{aligned}
$$

Similarly,

$$
(1+e^{-2\alpha\,(\overline{Z})})\,(1-e^{-2\alpha\,(Z)}) = 1 - e^{-4t} + 2\,i\,e^{-2t}\,\sin(2s).
$$

Therefore,

$$\begin{aligned}
& \alpha(\xi)\,(1+e^{-2\alpha\,(Z)})\,(1-e^{-2\alpha\,(\overline{Z})}) + \alpha\,(\overline{\xi})\,(1+e^{-2\alpha\,(\overline{Z})})\,(1-e^{-2\alpha\,(Z)}) \\
= \; & (a+i\,b)\,(1-e^{-4t}-2\,i\,e^{-2t}\,\sin{(2s)}) + (a-i\,b)\,(1-e^{-4t}+2\,i\,e^{-2t}\,\sin{(2s)}) \\
= \; & 2\,a\,(1-e^{-4t}) + 4\,b\,e^{-2t}\,\sin{(2s)} \\
= \; & 2\operatorname{Re}(\alpha\,(\xi))\,(1-e^{-4\alpha\,(X)}) + 4\operatorname{Im}(\alpha\,(\xi))\,e^{-2\alpha(X)}\,\sin{(2\alpha\,(Y))}.
\end{aligned}$$

Hence,

$$\begin{aligned}
& \partial_\xi \sum_w |\phi_w|^2 \qquad (5.8) \\
&= -\frac{1}{2} \sum_{\alpha\in\Sigma^+,\,w} m_\alpha \left[\frac{\operatorname{Re}(\alpha(\xi))\,(1-e^{-4\alpha\,(X)}) + 2\operatorname{Im}(\alpha\,(\xi))\,e^{-2\alpha(X)}\,\sin{(2\alpha\,(Y))}}{|1-e^{-2\alpha\,(Z)}|^2}\right] |\phi_w - \phi_{r_\alpha w}|^2 \\
&\quad + \sum_{\alpha\in\Sigma^+,\,w} m_\alpha \operatorname{sgn}(w^{-1}\alpha) \operatorname{Im}(\alpha(\xi)) \operatorname{Im}(\overline{\phi}_w\,\phi_{r_\alpha w}) + 2\sum_w \operatorname{Re}(w\,\lambda\,(\xi))\,|\phi_w|^2.
\end{aligned}$$

We first take $X, \xi \in \mathfrak{a}^{\mathrm{reg}}$ such that they are in the same Weyl chamber. Let $\mu \in \{w \operatorname{Re}\lambda\}_{w\in W}$ be such that $\mu\,(\xi) = \max_w \operatorname{Re}(w\,\lambda)\,(\xi)$. Then $(w\operatorname{Re}\lambda - \mu)\,(\xi) \leq 0$. The formula (5.8) gives

$$\begin{aligned}
& \partial_\xi \,(e^{-2\mu\,(X)} \sum_{w\in W} |\phi_w\,(Z)|^2) \\
= \; & -\frac{1}{2} \sum_{\alpha\in\Sigma^+,\,w} m_\alpha \frac{\alpha\,(\xi)\,(1-e^{-4\alpha\,(X)})}{|1-e^{-2\alpha\,(Z)}|^2}\,|\phi_w - \phi_{r_\alpha w}|^2 e^{-2\mu\,(X)} \qquad (5.9) \\
& +2 \sum_{w\in W} (w\operatorname{Re}\lambda - \mu)\,(\xi)\,|\phi_w|^2\,e^{-2\mu\,(X)}, \qquad (5.10)
\end{aligned}$$

Observe that the term (5.10) is clearly less than or equal to zero. In the term (5.9), the factor $|\phi_w - \phi_{r_\alpha w}|^2\,e^{-2\mu\,(X)} \geq 0$. Consider

$$\begin{aligned}
& \sum_{\alpha\in\Sigma^+} m_\alpha \frac{\alpha\,(\xi)\,(1-e^{-4\alpha\,(X)})}{|1-e^{-2\alpha\,(Z)}|^2} \\
= \; & \sum_{\alpha\in\Sigma^+_*} \left[m_\alpha \frac{\alpha(\xi)\,(1-e^{-4\alpha(X)})}{|1-e^{-2\alpha(Z)}|^2} + m_{\alpha/2}\,\frac{(1/2)\,\alpha(\xi)\,(1-e^{-2\alpha(X)})}{|1-e^{-\alpha(Z)}|^2}\right] \\
= \; & \sum_{\alpha\in\Sigma^+_*} \left[m_\alpha \frac{\alpha(\xi)\,(1-e^{-2\alpha(X)})\,(1+e^{-2\alpha(X)})}{|1-e^{-\alpha(Z)}|^2\,|1+e^{-\alpha(Z)}|^2} + m_{\alpha/2}\,\frac{(1/2)\,\alpha(\xi)\,(1-e^{-2\alpha(X)})}{|1-e^{-\alpha(Z)}|^2}\right] \\
= \; & \sum_{\alpha\in\Sigma^+_*} \alpha\,(\xi)\,\frac{1-e^{-2\alpha\,(X)}}{|1-e^{-\alpha(Z)}|^2} \left[m_\alpha \frac{1+e^{-2\alpha\,(X)}}{|1+e^{-\alpha(Z)}|^2} + \frac{1}{2}\,m_{\alpha/2}\right]. \qquad (5.11)
\end{aligned}$$

Since X, ξ are in the same Weyl chamber, $\alpha(\xi)\,(1-e^{-2\alpha(X)}) \geq 0$ for all $\alpha \in \Sigma^+$. Since $m_{\alpha/2} \geq -m_\alpha$ and $m_\alpha \geq 0$, then

$$m_\alpha \frac{1+e^{-2\alpha\,(X)}}{|1+e^{-\alpha(Z)}|^2} + \frac{1}{2}\,m_{\alpha/2} \geq m_\alpha \frac{1+e^{-2\alpha\,(X)}}{|1+e^{-\alpha(Z)}|^2} - \frac{1}{2}\,m_\alpha = m_\alpha \left[\frac{1+e^{-2\alpha\,(X)}}{|1+e^{-\alpha(Z)}|^2} - \frac{1}{2}\right] \geq 0$$

The reason is as follows:

$$\frac{1+e^{-2\alpha(X)}}{|1+e^{-\alpha(Z)}|^2}-\frac{1}{2}=\frac{2\,(1+e^{-2t})-|1+e^{-t}\,e^{-i\,s}|^2}{2\,|1+e^{-t}\,e^{-i\,s}|^2}\geq 0$$

if and only if the numerator is great than or equal to zero, which is clearly

$$\begin{aligned}
& 2\,(1+e^{-2t})-|1+e^{-t}\,e^{-i\,s}|^2 \\
= \;& 2\,(1+e^{-2t})-|1+e^{-t}\,\cos(s)-i\,e^{-t}\,\sin(s)|^2 \\
= \;& 2\,(1+e^{-2t})-[(1+e^{-t}\,\cos(s))^2+(e^{-t}\,\sin(s))^2] \\
= \;& 2\,(1+e^{-2t})-[1+e^{-2t}\,\cos^2(s)+2\,e^{-t}\,\cos(s)+e^{-2t}\,\sin^2(s)] \\
= \;& 1+e^{-2t}-2\,e^{-t}\,\cos(s/2) \\
\geq \;& 1+e^{-2t}-2\,e^{-t}\;=\;(1-e^{-t})^2\geq 0.
\end{aligned}$$

It follows that (5.11) is great than or equal to zero. Thus the term (5.9) is non-positive and hence

$$\partial_\xi\,(e^{-2\mu\,(X)}\sum_{w\in W}|\phi_w\,(Z)|^2)\leq 0.$$

This implies that the function is decreasing. So

$$\begin{aligned}
e^{-2\max_w\;\mathrm{Re}\,(w\,\lambda(X))}\sum_w|\phi_w\,(Z)|^2 \;&\leq\; e^{-2\max_w\;\mathrm{Re}\,(w\,\lambda(0))}\sum_w|\phi_w\,(0+i\,Y)|^2 \\
&=\;\sum_w|\phi_w\,(i\,Y)|^2
\end{aligned}$$

if $X\in\mathfrak{a}^{\mathrm{reg}}$, and by continuity it holds for all $X\in\mathfrak{a}$. Note that

$$|\phi_e\,(Z)|=|G\,(\lambda,\,m,\,e^{-1}Z)|=|G\,(\lambda,\,m,\,Z)|$$

and $|\phi_e\,(Z)|^2\leq\sum_w|\phi_w\,(Z)|^2$ which implies $|\phi_e\,(Z)|\leq(\sum_w|\phi_w\,(Z)|^2)^{1/2}$, we have

$$|G\,(\lambda,\,m,\,X+i\,Y)|\leq e^{\max_w\;\mathrm{Re}\,(w\,\lambda(X))}\left(\sum_w|\phi_w\,(i\,Y)|^2\right)^{1/2}. \tag{5.12}$$

Substituting $Y=0$ yields

$$|G\,(\lambda,\,m,\,X)|\leq|W|^{1/2}\,e^{\max_w\;\mathrm{Re}\,(w\,\lambda(X))},$$

where we use the fact that $G\,(\lambda,\,m,\,0)=1$ (cf. [27, Theorem 3.15]).

Next, we take $Y\in\mathfrak{a}^{\mathrm{reg}}$ such that $|\alpha\,(Y)|\leq\pi-\varepsilon$ for all $\alpha\in\Sigma$, and $\eta\in\mathfrak{a}^{\mathrm{reg}}$ belonging to the same Weyl chamber, and let $\xi=i\,\eta$. Then

$$\mathrm{Re}\,(w\,\lambda(\xi))=-\mathrm{Im}\,(w\,\lambda(\eta))\qquad\text{and}\qquad\mathrm{Im}\,(\alpha\,(\xi))=\mathrm{Re}\,(\alpha\,(\eta)).$$

Take $\mu \in \{w \operatorname{Im}\lambda\}_{w\in W}$ such that $-\operatorname{Im}(w\,\lambda(\eta)) \leq -\mu(\eta)$ for all $w \in W$. This is to say, $\mu = \min_w \operatorname{Im}(w\,\lambda)$. Observe that

$$2\,\rho = \sum_{\alpha\in\Sigma^+} m_\alpha\,\alpha = \sum_{\alpha\in\Sigma^+_*} (m_\alpha + \frac{1}{2}\,m_{\alpha/2})\,\alpha,$$

and

$$\sum_{\alpha\in\Sigma^+_*} (m_\alpha + \frac{1}{2}\,m_{\alpha/2})\,|\alpha(\eta)| \leq \max_w \sum_{\alpha\in\Sigma^+_*} (m_\alpha + \frac{1}{2}\,m_{\alpha/2})\,\alpha(w\,\eta).$$

We have

$$\begin{aligned}
&\left|\sum_{\alpha\in\Sigma^+,\,w} m_\alpha \operatorname{sgn}(w^{-1}\,\alpha) \operatorname{Im}(\alpha(\xi)) \operatorname{Im}(\overline{\phi_w}\,\phi_{r_\alpha w})\right| \\
=&\left|\sum_{\alpha\in\Sigma^+_*,\,w} (m_\alpha + \frac{1}{2}\,m_{\alpha/2}) \operatorname{sgn}(w^{-1}\,\alpha) \operatorname{Im}(\alpha(\xi)) \operatorname{Im}(\overline{\phi_w}\,\phi_{r_\alpha w})\right|
\end{aligned}$$

Since $m_\alpha + \frac{1}{2}\,m_{\alpha/2} \geq 0$, then

$$\begin{aligned}
&\leq \sum_{\alpha\in\Sigma^+_*,\,w} (m_\alpha + \frac{1}{2}\,m_{\alpha/2})\,|\alpha(\eta)|\,|\phi_w|\,|\phi_{r_\alpha w}| \\
&\leq 2\,\max_w (w\,\rho,\,\eta) \sum_w |\phi_w|^2,
\end{aligned}$$

Choose $\nu \in \{w\,\rho\}_{w\in W}$ such that $(\nu,\,\eta) = \max_w (w\rho,\,\eta)$. Let $C > 2$ be a constant to be determined and let

$$H(i\,Y) = e^{2\mu(Y)}\,e^{-C\nu(Y)} \sum_w |\phi_w(i\,Y)|^2.$$

Using the formula (5.8) we obtain

$$\begin{aligned}
&(\partial_\xi\,H)(i\,Y) \qquad (5.13)\\
=&-\sum_{\alpha\in\Sigma^+,\,w} m_\alpha \frac{\alpha(\eta)\,\sin 2\alpha(Y)}{|1-e^{-2\alpha(i\,Y)}|^2}\,|\phi_w - \phi_{r_\alpha w}|^2 \cdot e^{(2\mu - C\,\nu)(Y)} \\
&-(C-2)\,(\nu,\,\eta) \sum_w |\phi_w|^2\,e^{(2\mu-C\,\nu)(Y)} \\
&+\left[\sum_{\alpha\in\Sigma^+,\,w} m_\alpha \operatorname{sgn}(w^{-1}\,\alpha) \operatorname{Im}(\alpha(\xi)) \operatorname{Im}(\overline{\phi_w}\,\phi_{r_\alpha w}) - 2\,(\nu,\,\eta) \sum_w |\phi_w|^2\right] e^{(2\mu-C\,\nu)(Y)} \\
&+\left[2 \sum_w (\mu - w \operatorname{Im}\lambda)(\eta)\,|\phi_w|^2\right] e^{(2\mu-C\,\nu)(Y)},
\end{aligned}$$

where the last three terms on the right-hand side are clearly non-positive. For the first term, observe that, recalling $\alpha(Y) = s$,

$$|1 - e^{-2\alpha(iY)}|^2 = |1 - e^{-2is}|^2 = \left(|e^{-is}|\,|e^{is} - e^{-is}|\right)^2 = 4\sin^2(s).$$

It follows that

$$\begin{aligned}
&\sum_{\alpha\in\Sigma^+} m_\alpha \frac{\alpha(\eta)\,\sin 2\alpha(Y)}{|1 - e^{-2\alpha(iY)}|^2} \\
&= \sum_{\alpha\in\Sigma_*^+} \left[m_\alpha \frac{\alpha(\eta)\,\sin 2\alpha(Y)}{|1 - e^{-2\alpha(iY)}|^2} + \frac{1}{2} m_{\alpha/2} \frac{\alpha(\eta)\,\sin(\alpha(Y))}{|1 - e^{-\alpha(iY)}|^2} \right]
\end{aligned}$$

Since $m_{\alpha/2} \geq -m_\alpha$ and $m_\alpha \geq 0$, then

$$\begin{aligned}
&\geq \sum_{\alpha\in\Sigma_*^+} \left[m_\alpha \frac{\alpha(\eta)\,\sin 2\alpha(Y)}{|1 - e^{-2\alpha(iY)}|^2} - \frac{1}{2} m_\alpha \frac{\alpha(\eta)\,\sin(\alpha(Y))}{|1 - e^{-\alpha(iY)}|^2} \right] \\
&= \sum_{\alpha\in\Sigma_*^+} \left[m_\alpha \frac{\alpha(\eta)\,\sin(2s)}{4\sin^2(s)} - \frac{1}{2} m_\alpha \frac{\alpha(\eta)\,\sin(s)}{4\sin^2(s/2)} \right] \\
&= \sum_{\alpha\in\Sigma_*^+} \frac{m_\alpha\,\alpha(\eta)}{4} \left[\frac{\sin(2s)}{\sin^2(s)} - \frac{1}{2}\frac{\sin(s)}{\sin^2(s/2)} \right]
\end{aligned}$$

Using the formulas from trigonometry: $\sin(2s) = 2\sin(s)\cos(s)$ and $\cos(2s) = \cos^2(s) - \sin^2(s)$, we have

$$\begin{aligned}
&= \sum_{\alpha\in\Sigma_*^+} \frac{m_\alpha\,\alpha(\eta)}{4} \left[\frac{2\sin(s)\cos(s)}{\sin^2(s)} - \frac{1}{2}\frac{2\sin(s/2)\cos(s/2)}{\sin^2(s/2)} \right] \\
&= \sum_{\alpha\in\Sigma_*^+} \frac{m_\alpha\,\alpha(\eta)}{4} \left[\frac{2\cos(s)}{\sin(s)} - \frac{\cos(s/2)}{\sin(s/2)} \right] \\
&= \sum_{\alpha\in\Sigma_*^+} \frac{m_\alpha\,\alpha(\eta)}{4} \left[\frac{2(\cos^2(s/2) - \sin^2(s/2))}{2\sin(s/2)\cos(s/2)} - \frac{\cos(s/2)}{\sin(s/2)} \right] \\
&= \sum_{\alpha\in\Sigma_*^+} \frac{m_\alpha\,\alpha(\eta)}{4} \left[\frac{\cos(s/2)}{\sin(s/2)} - \frac{\sin(s/2)}{\cos(s/2)} - \frac{\cos(s/2)}{\sin(s/2)} \right] \\
&= \sum_{\alpha\in\Sigma_*^+} -\frac{m_\alpha\,\alpha(\eta)}{4} \tan(s/2).
\end{aligned}$$

Since η, Y belong to the same chamber, we see that $\alpha(\eta)\tan(s/2) \geq 0$. Since $|s| = |\alpha(Y)| \leq \pi - \varepsilon$, then there is a constant $C_1 = C_1(\varepsilon) > 0$ depending on ε such that $|\tan(s/2)| \leq C_1$. Also,

$$\sum_w |\phi_w - \phi_{r_\alpha w}|^2 \leq \sum_w (|\phi_w| + |\phi_{r_\alpha w}|)^2 = \sum_w 4\,|\phi_w|^2.$$

It follows that the first term of the right-hand side of (5.13) is

$$\begin{aligned} &\leq \sum_{\alpha \in \Sigma_*^+,\, w} m_\alpha\, |\alpha\,(\eta)|\, C_1\, |\phi_w|^2\, e^{(2\mu - C\,\nu)\,(Y)} \\ &\leq 2\, C_1 \max_w\,(w\,\rho,\, \eta) \sum_w |\phi_w|^2\, e^{(2\mu - C\,\nu)\,(Y)}. \end{aligned}$$

Hence the sum of the first two terms of the right-hand side of (5.13) is less than or equal to zero, by taking $C = 2 + 2\,C_1$ (C thus depends on ε).

Thus, $(\partial_\xi\, H)\,(i\,Y) \leq 0$. We see that $H\,(i\,Y) \leq H\,(0) = |W|$. So

$$\sum_w |\phi_w\,(i\,Y)|^2 \leq |W|\, e^{(C\,\nu - 2\mu)\,(Y)}.$$

Together with (5.12), we get

$$|G\,(\lambda,\, m,\, Z)| \leq |W|^{\frac{1}{2}}\, \exp\,(-\min_{w \in W}\, \mathrm{Im}\,(w\,\lambda\,(Y)) + \sqrt{C} \max_{w \in W}\, w\,\rho\,(Y) + \max_{w \in W}\, \mathrm{Re}\,(w\,\lambda\,(X))).$$

But $|G| = |F|$, we therefore prove the desired estimate for F. $\square$

Since $\rho(\,\cdot\,)$ and $|W|$ are constants, we restate Proposition 5.10 as

Proposition 5.11. *Let $m \in \mathcal{M}$ satisfy (2.4). Let $\varepsilon > 0$. Then there exists a constant $C = C_\varepsilon$ such that*

$$|F\,(\lambda,\, m;\, X + i\,Y)| \leq C\, \exp\,(\max_{w \in W}\, \mathrm{Re}\, w\,\lambda\,(X) - \min_{w \in W}\, \mathrm{Im}\, w\,\lambda\,(Y)),$$

for all $X \in \overline{\Omega_\varepsilon}$, $Y \in \mathfrak{b}$, and $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$.

Chapter 6

Paley-Wiener Theorem for Line Bundles over Compact Symmetric Spaces

With the preparations of the previous chapters we will present our main result in this chapter. In the first section we will introduce the Paley-Wiener spaces and state Paley-Wiener Theorem for line bundles over compact symmetric spaces. The detailed proof of this theorem, which is motivated by [22], will be given in the consecutive three sections.

6.1 Paley-Wiener Space and Paley-Wiener Theorem

Let $\| \ \|$ be the norm on $\mathfrak{u}$ with respect to the U-invariant inner product $\langle \, , \rangle$, i.e. $\| \ \| = \langle \, , \rangle^{1/2}$. Recall that we give U/K the Riemannian structure induced by the restriction of $\langle \, , \rangle$ to $\mathfrak{q} \times \mathfrak{q}$. For $r > 0$ let

$$B_r(0) = \{X \in \mathfrak{q} \ | \ \|X\| < r\}$$

be the geodesic open ball in $\mathfrak{q}$ centered at 0 with radius r. Define a function d on U by $d(u, e) := d_1(u \cdot o, o)$ where d_1 is the associated Riemannian distance function on U/K. Then $d(k, e) = 0$, for all $k \in K$. The injectivity radius R of U/K is the supremum of the values r for which the restriction of Exp to $B_r(0)$ is a diffeomorphism of onto its image.

All functions in $C^\infty(U//K, \chi_l)$ have compact supports since U is compact. We consider those with small supports. Assume that $r \leq R$. Let

$$\begin{aligned} C_r^\infty(U//K, \chi_l) &= \{f \in C^\infty(U//K, \chi_l) \ | \ \operatorname{supp} f \subseteq K \exp \overline{B}_r(0) K\} \\ &= \{f \in C^\infty(U//K, \chi_l) \ | \ \operatorname{supp} s_f \subseteq \operatorname{Exp} \overline{B}_r(0)\} \end{aligned}$$

where we identify f with the smooth section s_f (recall Proposition 2.5) and then $C_r^\infty(U//K, \chi_l)$ is also well defined using the support of s_f. The subscript r indicates that the support is contained in a metric ball of radius r. If $l = 0$ this space is $C_r^\infty(U/K)^K$. A Paley-Wiener theorem for this case was proved in [22].

The topology on $C^\infty(U)$ is defined by the seminorms

$$\nu_z(f) := \|\ell(z) f\|_\infty, \qquad z \in \mathcal{U}(\mathfrak{u}) \tag{6.1}$$

where ℓ is the left regular representation. This is the topology of uniform convergence of functions and their derivatives over compact sets. The same family of seminorms (6.1) also defines the topology on a closed subspace of $C^\infty(U)$. This applies to the following closed subspaces: $C^\infty(U/K, \chi_l)$, $C^\infty(U//K, \chi_l)$, and $C_r^\infty(U//K, \chi_l)$. The same topology can be defined by the seminorms

$$\sigma_k(f) := \|L_{U/K}^k f\|_\infty, \qquad k \in \mathbb{Z}^+$$

where $L_{U/K}$ is the negative definite Laplacian on U/K (see Theorem 4 in [32]). The topology on $C^{\infty}(B \cdot o)$ is given by the seminorms ν_z with $z \in \mathcal{U}(\mathfrak{b})$. Also, $C^{\infty}(B \cdot o)^W$ is a closed subspace whose topology is given by the same family of seminorms.

Let $\mathbf{S}(\Lambda_l^+)$ be the space of sequences $a = (a_\mu)_{\mu \in \Lambda_l^+}$ of complex numbers such that for each $k \in \mathbb{N}$,

$$\tau_k(a) := \sup_{\mu \in \Lambda_l^+} (1 + \|\mu\|)^k |a_\mu| < \infty.$$

The τ_k are seminorms. Sequences $(a_\mu)_{\mu \in \Lambda_l^+}$ satisfying $\tau_k(a) < \infty$ for all $k \in \mathbb{N}$ are said to be rapidly decreasing. The topology on $\mathbf{S}(\Lambda_l^+)$ defined by the seminorms τ_k makes it into a locally convex complete topological vector space. It follows that the χ_l-spherical Fourier transform

$$\mathcal{S}_l : C^{\infty}(U//K, \chi_l) \xrightarrow{\cong} \mathbf{S}(\Lambda_l^+)$$

is a topological isomorphism.

Lemma 6.1. *The restriction map $C^{\infty}(U//K, \chi_l) \to C^{\infty}(B)^W$ defines a bijection*

$$\mathrm{res} : C^{\infty}(U//K, \chi_l) \xrightarrow{\cong} \eta_l \cdot C^{\infty}(B \cdot o)^W.$$

Moreover, it is a topological isomorphism.

Proof. Since $B \cdot o \cong B/B \cap K$, it follows from [18, p.77] that the restriction map is bijective. Thus, $\mathcal{S}_l$ also gives a topological isomorphism from $\eta_l \cdot C^{\infty}(B \cdot o)^W$ onto $\mathbf{S}(\Lambda_l^+)$. Hence, we have the following commutative diagram

$$\begin{array}{ccc}
C^{\infty}(U//K, \chi_l) & \xrightarrow{\mathrm{res}} & \eta_l \cdot C^{\infty}(B \cdot o)^W \\
\mathcal{S}_l \downarrow \cong & & \mathcal{S}_l \downarrow \cong \\
\mathbf{S}(\Lambda_l^+) & \xrightarrow[\mathrm{id}]{\cong} & \mathbf{S}(\Lambda_l^+)
\end{array}$$

It follows that the restriction map is a topological isomorphism. $\square$

Lemma 6.2. *Let $0 < r < R$. Then*

$$C_r^{\infty}(B)^W \cong C_r^{\infty}(B \cdot o)^W$$

is a linear isomorphism of vector spaces (also of algebras).

Proof. We have $B \cdot o = B/(B \cap K) =: T$ where $B \cap K$ is a discrete finite subgroup and it is not necessarily $\{e\}$. Let $\iota : B \to B \cdot o$ which is a finite covering map[1].

[1] It means ι is continuous and surjective, and every point $x \in B \cdot o$ has a neighborhood V such that the inverse image $\iota^{-1}(V)$ can be written as the union of disjoint open sets V_j in B, and for each j the restriction of ι to V_j is a homeomorphism of V_j onto V. Note that the map ι is not injective because $\ker \iota = B \cap K \neq \{e\}$.

Since $0 < r < R$, we can find a $\epsilon > 0$ so that $r + \epsilon < R$. Denote by

$$D_r^B(e) = \exp\, B_r^{\mathfrak{b}}(0) \subseteq B$$

the metric ball in B of radius r centered at e, and by

$$D_r^T(o) = \operatorname{Exp} B_r^{\mathfrak{b}}(0) \subseteq B \cdot o$$

the metric ball in $B \cdot o$ of radius r centered at o. Similarly for $D_{r+\epsilon}^B(e)$ and $D_{r+\epsilon}^T(o)$. Then $\iota : D_{r+\epsilon}^B(e) \to D_{r+\epsilon}^T(o)$ is a diffeomorphism. Thus

$$C^\infty(D_{r+\epsilon}^B(e)) \xrightarrow{\cong} C^\infty(D_{r+\epsilon}^T(o))$$

is a linear isomorphism given by $f \mapsto f \circ \iota$ (this map is linear and bijective). We extend this map to the desired isomorphism as follows. If $f \in C_r^\infty(B \cdot o)^W$, we define $F \in C^\infty(D_{r+\epsilon}^B(e))$ by $F = f \circ \iota$. Then F is smooth on $D_{r+\epsilon}^B(e)$ since both f and ι are smooth, and F is W-invariant since f is so. Also, F has compact support in $\overline{D}_r^B(e)$ since f has compact support in $\overline{D}_r^T(o)$ and ι is a diffeomorphism from $D_{r+\epsilon}^B(e)$ onto $D_{r+\epsilon}^T(o)$. We see that

$$F = 0 \qquad \text{on} \qquad D_{r+\epsilon}^B(e) \setminus D_r^B(e).$$

Extend F to all of B by taking zero outside $D_{r+\epsilon}^B(e)$. Therefore $F \in C_r^\infty(B)^W$. On the other hand, if $F \in C_r^\infty(B)^W$, then $\operatorname{supp} F \subseteq \overline{D}_r^B(e)$. We construct a function f on $B \cdot o$ by

$$f|_{D_{r+\epsilon}^T(o)} := F \circ (\iota^{-1}|_{D_{r+\epsilon}^T(o)}),$$

and zero elsewhere. In a similar way we see that $f \in C_r^\infty(B \cdot o)^W$. It follows that $C_r^\infty(B)^W \cong C_r^\infty(B \cdot o)^W$. ☐

Proposition 6.3. *Let* $0 < r \le R$. *Let* $f \in C^\infty(U//K,\, \chi_l)$. *It follows that* $f \in C_r^\infty(U//K,\, \chi_l)$ *if and only if* $f|_B \in C_r^\infty(B)^W$. *That is,* $\operatorname{supp} f \subseteq K \exp \overline{B}_r(0)\, K$ *if and only if* $\operatorname{supp} f|_B \subseteq \exp \overline{B}_r^{\mathfrak{b}}(0)$ *where*

$$B_r^{\mathfrak{b}}(0) = B_r(0) \cap \mathfrak{b} = \{X \in \mathfrak{b} \mid \|X\| < r\}.$$

Proof. $\Rightarrow$: Assume $\operatorname{supp} f \subseteq K \exp \overline{B}_r(0)\, K$. Let $X \in \mathfrak{b}$ with $\|X\| < R$ be so that

$$f(\exp X) = f|_B(\exp X) \neq 0.$$

So $\exp X \in \operatorname{supp} f$. Then there are $k_1, k_2 \in K$ and $Y \in \overline{B}_r(0)$ (i.e. $Y \in \mathfrak{q}$ and $\|Y\| \le r < R$) such that

$$\exp(X) = k_1 \exp(Y)\, k_2.$$

Since $B_r(0) = \operatorname{Ad}(K)\, B_r^{\mathfrak{b}}(0)$, then there is a $h \in K$ so that $\operatorname{Ad}(h)\, Y \in \mathfrak{b}$ (this is actually valid for any $Y \in \mathfrak{q}$). Because the Killing form $\mathcal{K}$ is $\operatorname{Ad}(U)$-invariant, $\|\operatorname{Ad}(h)\, Y\| = \|Y\| \le r$. We then get

$$\exp(X) = (k_1\, h^{-1}) \exp(\operatorname{Ad}(h)\, Y)\, (h\, k_2).$$

It follows from the Cartan decomposition of U that there is a $w \in W$ such that $\mathrm{Ad}\,(h)\,Y = w \cdot X$. Hence,

$$\|X\| = \|\mathrm{Ad}\,(h)\,Y\| \le r.$$

This implies that $\mathrm{supp}\, f|_B \subseteq \exp\, \overline{B}^{\mathfrak{b}}_r\,(0)$.

$\Leftarrow$: Let $f \in C^\infty\,(U//K,\, \chi_l)$ be such that $\mathrm{supp}\, f|_B \subseteq \exp\, \overline{B}^{\mathfrak{b}}_r\,(0)$. Let $Y \in \mathfrak{q}$ with $\|Y\| < R$ and assume that there are $k_1,\, k_2 \in K$ such that $f\,(k_1\, \exp\,(Y)\, k_2) \neq 0$. But we have

$$f\,(k_1\, \exp\,(Y)\, k_2) = \chi_l\,(k_1\, k_2)^{-1}\, f\,(\exp\, Y),$$

and since $\chi_l\,(k_1\, k_2)^{-1}$ is never zero, then $f\,(k_1\, \exp\,(Y)\, k_2) \neq 0$ if and only if $f\,(\exp\, Y) \neq 0$. Let $h \in K$ be so that $X := \mathrm{Ad}\,(h)\,Y \in \mathfrak{b}$. Then

$$0 \neq f\,(\exp\, Y) = f\,(\exp\,(\mathrm{Ad}\,(h^{-1})\,X)) = f\,(h\, \exp\,(X)\, h^{-1}) = f\,(\exp\, X).$$

Since $\mathrm{supp}\, f|_B \subseteq \exp\, \overline{B}^{\mathfrak{b}}_r\,(0)$, then $\|X\| \le r$ and therefore

$$\|Y\| = \|\mathrm{Ad}\,(h^{-1})\,X\| = \|X\| \le r.$$

This implies that $\mathrm{supp}\, f \subseteq K\, \exp\, \overline{B}_r\,(0)\, K$. □

Corollary 6.4. *Let $0 < r < R$. Then the restriction map*

$$\mathrm{res} : C^\infty_r\,(U//K,\, \chi_l) \xrightarrow{\cong} \eta_l \cdot C^\infty_r\,(B)^W$$

is a topological isomorphism.

Proof. This follows immediately from the above two lemmas and the previous proposition. □

Definition 6.5. Let $r > 0$. Denote by $\mathrm{PW}_r\,(\mathfrak{b}^*_{\mathbb{C}})$ the space of holomorphic functions φ on $\mathfrak{b}^*_{\mathbb{C}}$ satisfying

1. φ is of exponential type r, i.e. for every $k \in \mathbb{N}$, there is a constant C_k such that
$$|\varphi\,(\lambda)| \le C_k\,(1 + \|\lambda\|)^{-k}\, e^{r\,\|\mathrm{Re}\,\lambda\|}, \qquad \forall\, \lambda \in \mathfrak{b}^*_{\mathbb{C}}.$$
2. φ satisfies the W-transformation law: for all $w \in W$ and $\lambda \in \mathfrak{b}^*_{\mathbb{C}}$,
$$\varphi\,(w\,(\lambda + \rho) - \rho) = \varphi\,(\lambda).$$

We now represent our main result, Paley-Wiener Theorem for homogeneous line bundles over compact symmetric space, in the following:

Theorem 6.6 (Paley-Wiener Theorem)**.** *There is a $S > 0$ such that the (extended) χ_l-spherical Fourier transform $\mathcal{S}_l$ gives a linear bijection*

$$\mathcal{S}_l : C^\infty_r\,(U//K,\ \chi_l) \ \xrightarrow{\cong}\ \mathrm{PW}_r\,(\mathfrak{b}^*_{\mathbb{C}})$$

for each $0 < r < S$. Precisely,

1. *If* $f \in C_r^\infty(U//K,\ \chi_l)$*, then* $\mathcal{S}_l(f) : \Lambda_l^+ \to \mathbb{C}$ *extends to a function in* $\mathrm{PW}_r(\mathfrak{b}_\mathbb{C}^*)$*;*

2. *Let* $\varphi \in \mathrm{PW}_r(\mathfrak{b}_\mathbb{C}^*)$*. Then there exists a unique* $f \in C_r^\infty(U//K,\ \chi_l)$ *such that* $\mathcal{S}_l(f)(\mu) = \varphi(\mu)$*, for all* $\mu \in \Lambda_l^+$*;*

3. *The functions in* $\mathrm{PW}_r(\mathfrak{b}_\mathbb{C}^*)$ *are uniquely determined by their values on* Λ_l^+*.*

Remark 6.7. By Theorem 6.6, the extension of the χ_l-spherical Fourier transform of a function $f \in C_r^\infty(U//K,\ \chi_l)$ satisfies

$$|\mathcal{S}_l(f)(\lambda)| \leq C_k (1 + \|\lambda\|)^{-k} e^{r\,\|\mathrm{Re}\,\lambda\|}, \qquad \forall\, \lambda \in \mathfrak{b}_\mathbb{C}^*,\ k \in \mathbb{N}.$$

We see that essentially the polynomial factor $(1+\|\lambda\|)^{-k}$ in the estimate is related to the property of f, i.e. f is smooth. The exponential factor $e^{r\,\|\mathrm{Re}\,\lambda\|}$ results in the compact support of f, i.e. the exponent r is characterized by the size of the support. The value of r has to be less than an upper bound S. As discussed in [22, Remark 4.3], the explicit values of S could be distinct in each part of Theorem 6.6.

1. In part 1, S has to be so that $S \leq R$ and $K \exp \overline{B}_S(0)\, K \subset U \cap \mathcal{V}$ for a suitable domain $\mathcal{V}$ (see Remark 3.34). When we make a particular choice for $\mathcal{V}$ (see (5.7)), the latter condition becomes $S \leq \pi/(\|\alpha\|)$ for all $\alpha \in \Sigma$ (see Section 6.2).

2. For part 2, we need $S \leq R$ (see Section 6.4.2).

3. For part 3, it can be proved that $S \leq \pi/\max \|\omega_j\|$, $j = 1, \ldots, n$.

Finally, S is taken to be the minimum of these three values.

Corollary 6.8. *There is a bijection* $C_r^\infty(U/K)^K \cong C_r^\infty(U//K, \chi_l)$*. If* U *is a classical Lie group, then there is a surjective map* $C_r^\infty(U)^U \twoheadrightarrow C_r^\infty(U//K, \chi_l)$*, for* $l \in \mathbb{Z}$*, which is given by*

$$F \longmapsto f, \quad f(u) = \int_K \chi_l(k)\, F(u\,k)\, dk.$$

Proof. The first statement follows from Theorem 6.6 and [22, Theorem 4.2]:

$$\begin{array}{ccc} C_r^\infty(U//K, \chi_l) & \xrightarrow{\ \cong\ } & C_r^\infty(U/K)^K \\ {\scriptstyle \mathcal{S}_l,\ \cong} \searrow & & \nearrow {\scriptstyle \mathcal{S}_0^{-1},\ \cong} \\ & \mathrm{PW}_r(\mathfrak{b}_\mathbb{C}^*) & \end{array}$$

The second statement follows from Theorem 6.6, Theorem 6.10, and Remark 6.14:

$$\begin{aligned} C_r^\infty(U)^U \ &\cong\ \mathrm{PW}_r^{\rho(\mathfrak{h})}(\mathfrak{h}_\mathbb{C}^*)^{W(\mathfrak{h})} \ \cong\ \mathrm{PW}_r(\mathfrak{h}_\mathbb{C}^*)^{W(\mathfrak{h})} \\ &\twoheadrightarrow\ \mathrm{PW}_r(\mathfrak{b}_\mathbb{C}^*)^W \ \cong\ \mathrm{PW}_r(\mathfrak{b}_\mathbb{C}^*) \ \cong\ C_r^\infty(U//K, \chi_l). \end{aligned}$$

□

6.2 χ_l-Spherical Fourier Transform Maps Into Paley-Wiener Space

In this section we shall prove part 1 of Theorem 6.6, that is, for $f \in C_r^\infty(U//K,\ \chi_l)$ we want to show the extension to $\mathfrak{b}_\mathbb{C}^*$ of $\widetilde{f}_l$ is in the Paley-Wiener space $\mathrm{PW}_r(\mathfrak{b}_\mathbb{C}^*)$. Precisely, we want to show the extended map $\widetilde{f}_l$ has the right exponential growth and polynomial decay, and satisfies Weyl group transformation law.

Let $m = (m_s,\ m_m,\ 1) \in \mathcal{M}^+$ is given associated with the symmetric space U/K and $\delta = \delta(m)$. Using the same pattern as (4.2) we obtain the following integral formula for $f \in C^\infty(U//K,\ \chi_l)$, up to a constant,

$$\widetilde{f}_l(\mu) = (f,\ \psi_{\mu,l}) = \int_B f(b)\,\psi_{\mu,l}(b^{-1})\,\delta(b)\,db.$$

Let $S \leq R$ be small enough. If $0 < r < S$ and $f \in C_r^\infty(U//K,\ \chi_l)$, in view of Corollary 6.4 (it ensures that the restriction of f on B still has the right size of the support) , we have

$$\begin{aligned}\widetilde{f}_l(\mu) &= \int_{\overline{D}_r^B(e)} f(b)\,\psi_{\mu,l}(b^{-1})\,\delta(b)\,d\,b \\ &= \int_{B_r^{\mathfrak{b}}(0)} f(e^X)\,\psi_{\mu,l}(e^{-X})\,\delta(e^X)\,dX \end{aligned} \tag{6.2}$$

where we lift up the invariant measure $d\,b$ on B to the measure $d\,X$ on $\mathfrak{b}$ in the sense that (6.2) holds. If $X \in B_r^{\mathfrak{b}}(0)$, then $|\alpha(X)| \leq \|\alpha\| \cdot \|X\| < \|\alpha\| \cdot r$ for all $\alpha \in \Sigma$. So

$$\|\alpha\| \cdot r < \pi \iff r < \frac{\pi}{\|\alpha\|}.$$

So if $S \leq \pi/\|\alpha\|$ for all $\alpha \in \Sigma$, then $|\alpha(X)| < \pi$. In order for $B_r^{\mathfrak{b}}(0) \subset \Omega$ we need

$$r < S \leq \frac{\pi}{\max_{\alpha\in\Sigma}\ \|\alpha\|}.$$

By Lemma 5.8, $\mu \mapsto \widetilde{f}_l(\mu)$ has an analytic continuation to $\mathfrak{b}_\mathbb{C}^*$, which is given by the same formula,

$$\widetilde{f}_l(\lambda) = \int_{B_r^{\mathfrak{b}}(0)} f(e^X)\,\psi_{\lambda,l}(e^{-X})\,\delta(e^X)\,dX.$$

It follows that

$$\begin{aligned}\widetilde{f}_l(\lambda) &= \int_B f(b)\,\psi_{\lambda,l}(b^{-1})\,\delta(b)\,d\,b \\ &= \int_B f(b)\,\varphi_{\lambda+\rho,l}(b^{-1})\,\delta(b)\,d\,b \\ &= \int_B f(b)\,\varphi_{\lambda+\rho,l}(b)\,\delta(b)\,d\,b \\ &= \int_B f(b)\,\eta_l(b)\,F(\lambda+\rho,\ m(l);\ b)\,\delta(b)\,d\,b\end{aligned}$$

for all $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$. Since $m(l) \in \mathcal{M}$ satisfies (2.4), it follows from Proposition 5.11 that

$$\begin{aligned} |\widetilde{f}_l(\lambda)| &\leq \int_B \big|f(b)\,\eta_l(b)\,\delta(b)\big|\,|F(\lambda+\rho,\, m(l);\; b)|\,d\,b \\ &\leq C \max_{b\in B}\{|f(b)\,\eta_l(b)\,\delta(b)|\}\,\exp\big(\max_{w\in W} \operatorname{Re} w\,(\lambda+\rho)\,(X)\big), \end{aligned}$$

where $X \in B_r^{\mathfrak{b}}(0)$ satisfies

$$\begin{aligned} |\exp\big(\max_{w\in W} \operatorname{Re} w\,(\lambda+\rho)\,(X)\big)| &= |\exp\big(\max_{w\in W} \operatorname{Re} w\,\lambda\,(X)\big)| \\ &\leq \exp\big(\|\max_{w\in W} \operatorname{Re} w\,\lambda\| \cdot \|X\|\big) \\ &= \exp\big(\|\operatorname{Re}\lambda\| \cdot \|X\|\big) \\ &< \exp\big(r\,\|\operatorname{Re}\lambda\|\big) \end{aligned}$$

due to the fact that $\rho \in i\,\mathfrak{b}^*$ and so $\operatorname{Re}\rho\,(X) = 0$, and the fact that a Weyl group element preserves the norm. It follows that

$$|\widetilde{f}_l(\lambda)| \leq C \max_{b\in B}\{|f(b)\,\eta_l(b)\,\delta(b)|\}\,e^{r\,\|\operatorname{Re}\lambda\|}.$$

Here and in the following C is a constant subject to change. But f has compact support, i.e. supp $(f\,|_B) \subset \exp \overline{B}_r^{\mathfrak{b}}(0)$. Also, η_l is bounded by 1, and δ is bounded by some constant. Thus $\max_{b\in B}\{|f(b)\,\eta_l(b)\,\delta(b)|\}$ is actually a constant depending on r.

We next show the polynomial decay of $\widetilde{f}_l$. Write $p_l(\lambda) = \langle\lambda,\,\lambda\rangle - \langle\rho(l),\,\rho(l)\rangle$. Using Proposition 4.11 and the formula (5.6) we obtain

$$\begin{aligned} &p_l(\lambda+\rho)\,\widetilde{f}_l(\lambda) \\ = &\int_B f(b)\,\eta_l(b)\;p_l(\lambda+\rho)\,F(\lambda+\rho,\,m(l);\,b)\,\delta(m,\,b)\,d\,b \\ = &\int_B f(b)\,\eta_l(b)\,\delta(2|l|,0,-2|l|)\,(b)\,L(l)\big[F(\lambda+\rho,\,m(l);\,b)\big]\,\delta(m(l),\,b)\,d\,b \\ = &\int_B L(l)\,\big[f(b)\,\eta_l(b)\,\delta(2|l|,0,-2|l|)\,(b)\big]\,F(\lambda+\rho,\,m(l);\,b)\,\delta(m(l),\,b)\,d\,b, \end{aligned}$$

where

$$\delta(2|l|,0,-2|l|) = \prod_{\alpha\in\mathcal{O}_s^+}\left|\frac{e^{\alpha}-e^{-\alpha}}{e^{2\alpha}-e^{-2\alpha}}\right|^{2|l|} = \prod_{\alpha\in\mathcal{O}_s^+}\left|\frac{1}{e^{\alpha}+e^{-\alpha}}\right|^{2|l|}$$

so $\delta(2|l|,0,-2|l|)\,(\,\cdot\,) \leq C$ which will not blow up since the denominator is never zero (note: for any $\alpha \in \mathcal{O}_s^+$, $|\alpha(\,\cdot\,)| < \pi/2$). We also use the fact

$$\delta(m(l)) = \delta(m)\,\delta(-2|l|,0,2|l|)$$

where

$$\delta(-2|l|,0,2|l|)\,(\,\cdot\,) = \prod_{\alpha\in\mathcal{O}_s^+}|e^{\alpha}+e^{-\alpha}|^{2|l|} = \prod_{\alpha\in\mathcal{O}_s^+}|2\cos(\operatorname{Im}\alpha(\,\cdot\,))|^{2|l|} \leq C,$$

and thus the term $\delta\,(m(l),\,\cdot\,)$ will not blow up too. It follows from Proposition 5.11 that

$$|p_l(\lambda+\rho)|\,|\widetilde{f}_l\,(\lambda)| \le C\,\max_{b\in B}\left\{\left|L(l)\left[f\,(b)\,\eta_l\,(b)\,\delta(2|l|,0,-2|l|)\,(b)\right]\right|\right\}\,e^{r\,\|\mathrm{Re}\,\lambda\|}.$$

Applying suitably high powers of $L(l)$ will yield the right polynomial decay. Let $k\in\mathbb{N}$. Then

$$|p_l(\lambda+\rho)|^k\,|\widetilde{f}_l\,(\lambda)| \le \underbrace{C\,\max_{b\in B}\left\{\left|L(l)^k\left[f\,(b)\,\eta_l\,(b)\,\delta(2|l|,0,-2|l|)\,(b)\right]\right|\right\}}_{=:\,C_k}\,e^{r\,\|\mathrm{Re}\,\lambda\|}.$$

Note that both ρ and $\rho(l)$ are constants (independent of the variable λ), so

$$p_l\,(\lambda+\rho) = \langle\lambda+\rho,\,\lambda+\rho\rangle - \langle\rho(l),\,\rho(l)\rangle \sim \|\lambda\|^2 \sim 1+\|\lambda\|^2.$$

Therefore,

$$|\widetilde{f}_l\,(\lambda)| \le C_k\,(1+\|\lambda\|^2)^{-k}\,e^{r\,\|\mathrm{Re}\,\lambda\|},\qquad \forall\,\lambda\in\mathfrak{b}^*_{\mathbb{C}}.$$

This is equivalent to

$$|\widetilde{f}_l\,(\lambda)| \le C_k\,(1+\|\lambda\|)^{-k}\,e^{r\,\|\mathrm{Re}\,\lambda\|},\qquad \forall\,\lambda\in\mathfrak{b}^*_{\mathbb{C}}.$$

In the end, the Weyl group translation law follows easily from Lemma 5.8:

$$\begin{aligned}
\widetilde{f}_l\,(w\,(\lambda+\rho)-\rho) &= \int_B f\,(b)\,\psi_{w\,(\lambda+\rho)-\rho,\,l}\,(b^{-1})\,\delta\,(b)\,db\\
&= \int_B f\,(b)\,\psi_{\lambda,\,l}\,(b^{-1})\,\delta\,(b)\,db\\
&= \widetilde{f}_l\,(\lambda).
\end{aligned}$$

Hence, $\widetilde{f}_l\in\mathrm{PW}_r\,(\mathfrak{b}^*_{\mathbb{C}})$.

6.3 Central Functions on Compact Lie Groups

The proof of surjectivity of χ_l-spherical Fourier Transform $\mathcal{S}_l$ is reduced to the special group case where the symmetric space is the compact semisimple Lie group U itself. In this case the local Paley-Wiener theorem was proved by Gonzalez [9]. We briefly review his result as follows.

Consider the transitive action of $U\times U$ on U given by $(g,\,h)\cdot u = g\,u\,h^{-1}$ and the involution $\tau: U\times U\to U\times U$, $(g,\,h)\mapsto(h,\,g)$. Then

$$\mathrm{stab}_e\,(U\times U) = (U\times U)^\tau = \mathrm{diag}\,U = \{(u,\,u)\ \mid\ u\in U\}\ \cong\ U.$$

So U is viewed as a symmetric space by $(U\times U)/\mathrm{diag}\,U$ or $(U\times U)/U$ via the map

$$(U\times U)/\mathrm{diag}\,U \longrightarrow U,\qquad (g,\,h)\,\mathrm{diag}\,U\longmapsto g\,h^{-1}.$$

The diag U-invariant functions on U are the central functions, i.e. $F \circ \mathrm{Ad}\,(g) = F$ for all $g \in U$, where $\mathrm{Ad}\,(g)\,(u) = g\,u\,g^{-1}$. Denote

$$\begin{aligned} C^{\infty}(U)^{U} &= C^{\infty}(U \times U/\mathrm{diag}\,U)^{\mathrm{diag}\,U} \\ &= \{F \in C^{\infty}(U) \mid F(g\,u\,g^{-1}) = F(u), \forall\, g \in U\}. \end{aligned}$$

Recall from Section 3.2.2 that we have $\Lambda^{+}(U) \cong \widehat{U}$. For $\mu \in \Lambda^{+}(U)$, (π_{μ}, V_{μ}) denotes a unitary irreducible representation with the highest weight μ. Let $d(\mu) = \dim V_{\mu}$. Then $\mu \mapsto d(\mu)$ is a polynomial function on $\mathfrak{h}_{\mathbb{C}}^{*}$. Let $\chi_{\mu^{\vee}}$ denote the character of the dual representation $\pi_{\mu}^{\vee}$. Then

$$\chi_{\mu^{\vee}}(u) = \mathrm{Tr}\,(\pi_{\mu}(u^{-1})), \qquad \forall\, u \in U.$$

The function $d(\mu)^{-1}\,\chi_{\mu^{\vee}}$ takes value 1 at e, and it is exactly the spherical function on the symmetric space U associated with $\pi_{\mu} \otimes \pi_{\mu^{\vee}}$. The set of $d(\mu)^{-1}\,\chi_{\mu^{\vee}}$ exhausts the class of (elementary) spherical function on U. However, $\chi_{\mu^{\vee}}$ is a unit vector in $L^{2}(U)$ and the set $\{\chi_{\mu^{\vee}}\}_{\mu \in \Lambda^{+}(U)}$ forms a complete orthonormal basis for

$$L^{2}(U)^{U} = \{F \in L^{2}(U) \mid F \circ \mathrm{Ad}\,(g) = F, \forall\, g \in U\}.$$

For $F \in C^{\infty}(U)^{U}$, define the Fourier transform of F as $\widehat{F} : \Lambda^{+}(U) \to \mathbb{C}$,

$$\widehat{F}(\mu) = (F,\ \chi_{\mu^{\vee}}) = \int_{U} F(u)\,\chi_{\mu}(u)\,du, \quad \mu \in \Lambda^{+}(U)$$

where $(\ ,\ )$ is the L^{2} inner product. The Fourier transform extends to a unitary isomorphism $\widehat{\ }: L^{2}(U)^{U} \to \ell^{2}(\Lambda^{+}(U))$. The corresponding Fourier series is

$$\sum_{\mu \in \Lambda^{+}(U)} \widehat{F}(\mu)\,\chi_{\mu^{\vee}}. \tag{6.3}$$

It converges to F in L^{2} sense. If $F \in C^{\infty}(U)^{U}$, then the convergence is absolute and uniform (cf. [12, p. 534]). Let $R > 0$ be the injective radius of U.

Definition 6.9. Let $\mathrm{PW}_{r}^{\rho(\mathfrak{h})}(\mathfrak{h}_{\mathbb{C}}^{*})^{W(\mathfrak{h})}$ be the space of holomorphic functions Φ on $\mathfrak{h}_{\mathbb{C}}^{*}$ such that

1. for all $k \in \mathbb{N}$, there is a $C_{k} > 0$ such that

$$|\Phi(\lambda)| \le C_{k}(1 + \|\lambda\|)^{-k}\,e^{r\,\|\mathrm{Re}\,\lambda\|}, \quad \forall\, \lambda \in \mathfrak{h}_{\mathbb{C}}^{*}.$$

2. for all $w \in W(\mathfrak{h})$ and $\lambda \in \mathfrak{h}_{\mathbb{C}}^{*}$,

$$\Phi(w(\lambda + \rho(\mathfrak{h})) - \rho(\mathfrak{h})) = \det(w)\,\Phi(\lambda).$$

Theorem 6.10 (Gonzalez Theorem)**.** *Let U be a compact semisimple Lie group. Let $F \in C^{\infty}(U)^{U}$ and $0 < r < R$. Then $F \in C_{r}^{\infty}(U)^{U}$ if and only if $\mu \mapsto \widehat{F}(\mu)$ extends to a holomorphic function $\Phi = \Phi_{F}$ on $\mathfrak{h}_{\mathbb{C}}^{*}$ such that $\Phi \in \mathrm{PW}_{r}^{\rho(\mathfrak{h})}(\mathfrak{h}_{\mathbb{C}}^{*})^{W(\mathfrak{h})}$.*

Note that the extension Φ is unique when r is small enough. In that case, the holomorphic extension of Fourier transform gives an isomorphism

$$C_{r}^{\infty}(U)^{U} \cong \mathrm{PW}_{r}^{\rho(\mathfrak{h})}(\mathfrak{h}_{\mathbb{C}}^{*})^{W(\mathfrak{h})}.$$

6.4 Bijectivity of χ_l-Spherical Fourier Transform

6.4.1 Preliminaries

In this section we shall prove part 2 and part 3 of Theorem 6.6. We start with a few simple lemmas which enable us to construct $f \in C_r^\infty(U//K, \chi_l)$ by averaging $F \in C_r^\infty(U)^U$.

Lemma 6.11. *Let $\lambda \in \Lambda^+(U)$ and $\mu = \lambda|_{\mathfrak{b}}$. If $\lambda \in \Lambda_l^+(\mathfrak{h})$, then*

$$\int_K \chi_l(k)\,\chi_\lambda((u\,k)^{-1})\,dk = \psi_{\mu,l}(u), \qquad \forall\, u \in U,$$

and otherwise equals zero.

Proof. This lemma says that we can express the χ_l-spherical functions on U by means of characters of irreducible χ_l-spherical representation of U. The proof goes as follows. Recall that the operator $P_l = \int_K \chi_l(k^{-1})\,\pi(k)\,dk$ is an orthogonal projection from V_λ onto V_λ^l. Since $\chi_\lambda(k^{-1}\,u^{-1}) = \chi_\lambda(u^{-1}\,k^{-1})$, we have

$$\begin{aligned}\int_K \chi_l(k)\,\chi_\lambda((u\,k)^{-1})\,dk &= \int_K \chi_l(k)\,\mathrm{Tr}\,(\pi_\lambda(u^{-1}\,k^{-1}))\,dk \\ &= \mathrm{Tr}\,(\pi_\lambda(u^{-1})\int_K \chi_l(k)\,\pi_\lambda(k^{-1})\,dk) \\ &= \mathrm{Tr}\,(\pi_\lambda(u^{-1})\,P_l).\end{aligned}$$

There are two possibilities:

1. If $\lambda \notin \Lambda_l^+(\mathfrak{h})$, then $V_\lambda^l = \{0\}$ and so $P_l = 0$. Thus,

$$\int_K \chi_l(k)\,\chi_\lambda((u\,k)^{-1})\,dk = 0.$$

2. If $\lambda \in \Lambda_l^+(\mathfrak{h})$, let e_λ be a unit χ_l-coinvariant vector. Then $V_\lambda^l = \mathbb{C}\,e_\lambda$, and we can extend $v_1 = e_\lambda$ to an o.n.b. $v_1, \ldots, v_d$ for V_λ where $d = d(\lambda)$. So

$$P_l\,v_1 = v_1, \quad P_l\,v_j = 0,\ \forall\, j > 1.$$

It follows that

$$\int_K \chi_l(k)\,\chi_\lambda((u\,k)^{-1})\,dk = (\pi_\lambda(u^{-1})\,v_1,\ v_1) = \psi_{\mu,l}(u).$$

$\square$

Lemma 6.12. *Let $F \in C^{\infty}(U)^{U}$ and define $f : U \to \mathbb{C}$ by*

$$f(u) = \int_K \chi_l(k)\, F(u\,k)\, dk.$$

Then $f \in C^{\infty}(U//K,\, \chi_l)$ and $d(\mu)\, \widetilde{f_l}(\mu) = \widehat{F}(\mu)$ for all $\mu \in \Lambda_l^+$.

Proof. Since $F \in C^{\infty}(U)^{U}$, then for $h, g \in K$,

$$\begin{aligned} f(h\,u\,g) &= \int_K \chi_l(k)\, F(h\,u\,g\,k)\, dk \\ &= \int_K \chi_l(g^{-1}\,k\,h^{-1})\, F(u\,k)\, dk \\ &= \chi_l(h\,g)^{-1}\, f(u), \end{aligned}$$

which implies that $f \in C^{\infty}(U//K,\, \chi_l)$. Since F is smooth, the series (6.3) converges uniformly, and in view of Lemma 6.11, we have

$$\begin{aligned} f(u) = \int_K \chi_l(k) \sum_{\lambda \in \Lambda^+(U)} \widehat{F}(\lambda)\, \chi_\lambda(u\,k)^{-1}\, dk \\ = \sum_{\lambda \in \Lambda^+(U)} \widehat{F}(\lambda) \int_K \chi_l(k)\, \chi_\lambda(u\,k)^{-1}\, dk \\ = \sum_{\lambda \in \Lambda_l^+(\mathfrak{h})} \widehat{F}(\lambda) \int_K \chi_l(k)\, \chi_\lambda(u\,k)^{-1}\, dk \\ = \sum_{\mu \in \Lambda_l^+} \widehat{F}(\mu)\, \psi_{\mu,\, l}(u), \end{aligned}$$

where $\mu = \lambda\,|_{\mathfrak{b}}$. Compare this to the series (3.10), we have $d(\mu)\, \widetilde{f_l}(\mu) = \widehat{F}(\mu)$. ☐

Let $0 < r < R$. Let $B_r^{\mathfrak{u}}(0) = \{X \in \mathfrak{u} : \ \|X\| < r\}$. Then

$$C_r^{\infty}(U)^{U} = \{F \in C^{\infty}(U)^{U} \ \mid \ \operatorname{supp} F \subseteq \exp\, B_r^{\mathfrak{u}}(0)\}.$$

Lemma 6.13. *Let F and f be as in Lemma 6.12. If $F \in C_r^{\infty}(U)^{U}$ for some $r > 0$, then $f \in C_r^{\infty}(U//K,\, \chi_l)$.*

Proof. Recall that via the identification of f with a smooth section S_f of L_{χ_l} we see that $f \in C_r^{\infty}(U//K,\, \chi_l)$ means $\operatorname{supp} S_f \subseteq \operatorname{Exp} \overline{B}_r(0)$. Then the proof goes the same as [22, Lemma 9.3], except for $f(u) = \int_K \chi_l(k)\, F(u\,k)\, dk$, there is a factor $\chi_l(k) \in \mathbb{T}$ when averaging F, but which does not affect anything. ☐

6.4.2 Proof of Bijectivity of $\mathcal{S}_l$

We first introduce the following notations. Let $\mathrm{PW}_r\,(\mathfrak{b}_{\mathbb{C}}^*)^W$ be the space of W-invariant holomorphic functions on $\mathfrak{b}_{\mathbb{C}}^*$ of exponential type r, and $\mathrm{PW}_r\,(\mathfrak{h}_{\mathbb{C}}^*)^{W\,(\mathfrak{h})}$ the space of $W\,(\mathfrak{h})$-invariant holomorphic functions on $\mathfrak{h}_{\mathbb{C}}^*$ of exponential type r.

To see the χ_l-spherical Fourier transform

$$\mathcal{S}_l : C_r^\infty\,(U//K,\,\chi_l) \longrightarrow \mathrm{PW}_r\,(\mathfrak{b}_{\mathbb{C}}^*),$$

is a bijection, we shall first show $f \mapsto \mathcal{S}_l\,(f)|_{\Lambda_l^+}$ is a bijection. This is the content of part 2 of Theorem 6.6. Precisely, let $\varphi \in \mathrm{PW}_r\,(\mathfrak{b}_{\mathbb{C}}^*)$, we need to find a $f \in C_r^\infty\,(U//K,\,\chi_l)$ such that $\widetilde{f}_l\,(\mu) = \varphi\,(\mu)$, $\forall\,\mu \in \Lambda_l^+$. Moreover, such a f is unique.

Proof I: If U is any classical group in Table 1, the proof of surjectivity of $\mathcal{S}_l$, follows easily from [25, Lemma 4.4] and Theorem 2.2 in [26].

Define $\varphi_1\,(\lambda) := \varphi\,(\lambda - \rho)$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$. Then for $w \in W$,

$$\varphi_1\,(w\,\lambda) = \varphi\,(w\,\lambda - \rho) = \varphi\,(w\,[(\lambda - \rho) + \rho] - \rho) = \varphi\,(\lambda - \rho) = \varphi_1\,(\lambda).$$

So $\varphi_1 \in \mathrm{PW}_r\,(\mathfrak{b}_{\mathbb{C}}^*)^W$. From Remark 6.14 we see that the restricted map

$$\mathrm{PW}_r\,(\mathfrak{h}_{\mathbb{C}}^*)^{W\,(\mathfrak{h})} \longrightarrow \mathrm{PW}_r\,(\mathfrak{b}_{\mathbb{C}}^*)^W$$

is surjective (in general this map is not injective), so there is a $\psi_1 \in \mathrm{PW}_r\,(\mathfrak{h}_{\mathbb{C}}^*)^{W\,(\mathfrak{h})}$ such that

$$\psi_1\,|_{\mathfrak{b}_{\mathbb{C}}^*} = \varphi_1.$$

By Remark 6.14 there is a linear isomorphism

$$T : \mathrm{PW}_r^{\rho\,(\mathfrak{h})}\,(\mathfrak{h}_{\mathbb{C}}^*)^{W\,(\mathfrak{h})} \stackrel{\cong}{\longrightarrow} \mathrm{PW}_r\,(\mathfrak{h}_{\mathbb{C}}^*)^{W\,(\mathfrak{h})}.$$

Thus, there exists a $\psi \in \mathrm{PW}_r^{\rho\,(\mathfrak{h})}\,(\mathfrak{h}_{\mathbb{C}}^*)^{W\,(\mathfrak{h})}$ such that $T\,\psi = \psi_1$, i.e. $\psi = T^{-1}\,\psi_1$ where

$$(T^{-1}\,\psi_1)\,(\lambda) = \frac{\varpi\,(\lambda + \rho\,(\mathfrak{h}))}{\varpi\,(\rho\,(\mathfrak{h}))}\,\psi_1\,(\lambda + \rho\,(\mathfrak{h})) = d\,(\lambda)\,\psi_1\,(\lambda + \rho\,(\mathfrak{h})).$$

where $\varpi\,(\lambda) = \prod_{\beta \in \Delta^+} \langle \lambda,\,\beta \rangle$. Moreover, by Theorem 6.10 there exists a $F \in C_r^\infty\,(U)^U$ such that $\widehat{F}\,(\lambda) = \psi\,(\lambda)$ for all $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$. We then construct a $f : U \to \mathbb{C}$ by

$$f\,(u) = \int_K \chi_l\,(k)\,F\,(u\,k)\,dk.$$

By Lemma 6.12 and Lemma 6.13, $f \in C_r^\infty(U//K,\, \chi_l)$. For $\mu \in \Lambda_l^+$,

$$\begin{aligned}
\widetilde{f}_l(\mu) &= \frac{1}{d(\mu)}\, \widehat{F}\Big|_{\mathfrak{b}^*_{\mathbb{C}}}(\mu) \\
&= \frac{1}{d(\mu)}\, \psi\Big|_{\mathfrak{b}^*_{\mathbb{C}}}(\mu) \\
&= \frac{1}{d(\mu)}\, (T^{-1}\psi_1)\Big|_{\mathfrak{b}^*_{\mathbb{C}}}(\mu) \\
&= \frac{1}{d(\mu)}\, d(\mu)\, \psi_1\Big|_{\mathfrak{b}^*_{\mathbb{C}}}(\mu + \rho(\mathfrak{h})\,|_{\mathfrak{b}}) \\
&= \psi_1\Big|_{\mathfrak{b}^*_{\mathbb{C}}}(\mu + \rho) \\
&= \varphi_1(\mu + \rho) \\
&= \varphi(\mu + \rho - \rho) \\
&= \varphi(\mu),
\end{aligned}$$

where we use the fact that $\rho(\mathfrak{h})\,|_{\mathfrak{b}} = \rho$. This shows the surjectivity of the map $f \mapsto \mathcal{S}_l(f)|_{\Lambda_l^+}$.

Remark 6.14. (1) Let Π_i be the set of simple roots in Σ_i^+. Let $\widetilde{W}$ be the extension of W defined as in [25, §2]:

$$\widetilde{W} = \begin{cases} W & \text{if } \Pi_i \text{ is not of type } D \\ W \rtimes \{\pm 1\} & \text{otherwise,} \end{cases}$$

and similarly for $\widetilde{W}(\mathfrak{h})$. By [25, §2], if U are the classical Lie groups, the only two cases where Π_i is of type D are

- $\mathrm{SO}(2n,\mathbb{C})/\mathrm{SO}(2n)$;
- $\mathrm{SO}_o(p,\,p)/\mathrm{SO}(p)\times\mathrm{SO}(p)$ (a split case);

however, which are not in our consideration (see the Table 1), since we assume K is not semisimple. Therefore, we have $\widetilde{W} = W$ (also $\widetilde{W}(\mathfrak{h}) = W(\mathfrak{h})$) for the classical case. Hence, by [25, Lemma 4.4], there is a linear isomorphism

$$T : \mathrm{PW}_r^{\rho(\mathfrak{h})}(\mathfrak{h}^*_{\mathbb{C}})^{W(\mathfrak{h})} \xrightarrow{\cong} \mathrm{PW}_r(\mathfrak{h}^*_{\mathbb{C}})^{W(\mathfrak{h})}.$$

(2) Write

$$W_{\mathfrak{b}}(\mathfrak{h}) = \{w \in W(\mathfrak{h}) \mid w(\mathfrak{b}) = \mathfrak{b}\}.$$

It is well known that for all semisimple Lie algebras $\mathfrak{u}$ we have

$$W_{\mathfrak{b}}(\mathfrak{h})\,|_{\mathfrak{b}} = W. \tag{6.4}$$

Helgason proved, in [15], that for all classical semisimple Lie algebras,

$$S(\mathfrak{h})^{W(\mathfrak{h})}\Big|_{\mathfrak{b}} = S(\mathfrak{b})^W. \tag{6.5}$$

If both (6.4) and (6.5) are satisfied then the restriction map

$$\mathrm{PW}_r\left(\mathfrak{h}_{\mathbb{C}}^*\right)^{W(\mathfrak{h})} \longrightarrow \mathrm{PW}_r\left(\mathfrak{b}_{\mathbb{C}}^*\right)^{W} \tag{6.6}$$

is surjective (see [26, Theorem 2.2] or [25, Theorem 6.9]). However, if U is of exceptional type, for example, the last two listed in Table 1,

$$S(\mathfrak{h})^{W(\mathfrak{h})}\Big|_{\mathfrak{b}} \neq S(\mathfrak{b})^{W}$$

according to [13, Theorem 10.3, p.327] and [14, Proposition 2.1]. Therefore, the surjectivity of (6.6) is unknown for the exceptional case. Hence, the above proof is not applicable to the exceptional case. Refer to Theorem 1.4, Theorem 1.5, Theorem 1.6, and Theorem 6.9 in [25], and Theorem 2.2 in [26] for more details.

Proposition 6.15. *Let $\varphi \in \mathrm{PW}_r\left(\mathfrak{b}_{\mathbb{C}}^*\right)^{W}$. Let $k = |W(\mathfrak{h})|$. Then there exist a collection of polynomials $p_j \in S(\mathfrak{b})^W$ and $\phi_j \in \mathrm{PW}_r\left(\mathfrak{h}_{\mathbb{C}}^*\right)^{W(\mathfrak{h})}$, $j = 1, \ldots, k$, such that*

$$\varphi_1 = p_1\left(\phi_1\big|_{\mathfrak{b}_{\mathbb{C}}^*}\right) + \cdots + p_k\left(\phi_k\big|_{\mathfrak{b}_{\mathbb{C}}^*}\right).$$

Proof. This is an application of the result of Rais [28]. For proof see [22, Corollary 10.2]. ☐

Proof II: The method we use in the following comes from [22, Section 10, 11]. Notice that it is applicable to both the classical case and exceptional case.

Define $\varphi_1(\lambda) := \varphi(\lambda - \rho)$, for all $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$. Then for $w \in W$,

$$\varphi_1(w\,\lambda) = \varphi(w\,\lambda - \rho) \;=\; \varphi(w\,[(\lambda-\rho)+\rho]-\rho) = \varphi(\lambda-\rho) \;=\; \varphi_1(\lambda).$$

So $\varphi_1 \in \mathrm{PW}_r\left(\mathfrak{b}_{\mathbb{C}}^*\right)^{W}$. By Proposition 6.15 there are collection of $p_j \in S(\mathfrak{b})^W$ and $\phi_j \in \mathrm{PW}_r\left(\mathfrak{h}_{\mathbb{C}}^*\right)^{W(\mathfrak{h})}$, such that

$$\varphi_1 = p_1\left(\phi_1\big|_{\mathfrak{b}_{\mathbb{C}}^*}\right) + \cdots + p_k\left(\phi_k\big|_{\mathfrak{b}_{\mathbb{C}}^*}\right).$$

By Theorem 4.5 there are $D_j \in \mathbb{D}_l$ such that

$$p_j(\lambda) = \overline{\gamma_l\left(D_j^*,\, \lambda\right)}, \qquad \forall\, \lambda \in \mathfrak{b}_{\mathbb{C}}^*.$$

By Weyl dimension formula, $\mu \mapsto d(\mu)$ extends to a polynomials on $\mathfrak{h}_{\mathbb{C}}^*$ and satisfies for $w \in W(\mathfrak{h})$,

$$\begin{aligned} d\left(w\left(\lambda+\rho(\mathfrak{h})\right)-\rho(\mathfrak{h})\right) &= \frac{\varpi\left(w\left(\lambda+\rho(\mathfrak{h})\right)\right)}{\varpi\left(\rho(\mathfrak{h})\right)} \\ &= \frac{\det(w)\,\varpi\left(\lambda+\rho(\mathfrak{h})\right)}{\varpi\left(\rho(\mathfrak{h})\right)} \\ &= \det(w)\, d(\lambda). \end{aligned}$$

Define Φ_j on $\mathfrak{h}_{\mathbb{C}}^*$ by

$$\Phi_j(\lambda) = d(\lambda)\,\phi_j(\lambda + \rho(\mathfrak{h})).$$

Then Φ_j is of exponential type r because ϕ_j is so, and satisfies

$$\begin{aligned}\Phi_j(w(\lambda + \rho(\mathfrak{h})) - \rho(\mathfrak{h})) &= d(w(\lambda + \rho(\mathfrak{h})) - \rho(\mathfrak{h}))\,\phi_j(w(\lambda + \rho(\mathfrak{h}))) \\ &= \det(w)\,d(\lambda)\,\phi_j(\lambda + \rho(\mathfrak{h})) \\ &= \det(w)\,\Phi_j(\lambda),\end{aligned}$$

where we use the fact that ϕ_j is $W(\mathfrak{h})$-invariant. Thus

$$\Phi_j \in \mathrm{PW}_r^{\rho(\mathfrak{h})}(\mathfrak{h}_{\mathbb{C}}^*)^{W(\mathfrak{h})}.$$

By Theorem 6.10, there are $F_j \in C_r^\infty(U)^U$ such that $\widehat{F}_j(\mu) = \Phi_j(\mu)$ for all $\mu \in \Lambda_l^+$. Let

$$f_j(u) = \int_K \chi_l(k)\,F_j(u\,k)\,dk.$$

By Lemma 6.12 and Lemma 6.13, $f_j \in C_r^\infty(U//K,\,\chi_l)$, so that

$$\mathrm{supp}(f_j) \subset K\ \exp \overline{B}_r(0)\ K.$$

Define $f = \sum_j D_j\,f_j$. The construction of f makes sense, and $f \in C_r^\infty(U//K,\,\chi_l)$. This is because differentiation does not increase the support, whence

$$\mathrm{supp}(f) \subset \mathrm{supp}(f_j) \subset K\ \exp \overline{B}_r(0)\ K.$$

It follows that

$$\begin{aligned}\widetilde{f}_l(\mu) &= (f,\,\psi_{\mu,l}) = \sum (D_j\,f_j,\,\psi_{\mu,l}) = \sum (f_j,\,D_j^*\,\psi_{\mu,l}) \\ \text{(Proposition 4.10)} &= \sum (f_j,\,\gamma_l(D_j^*,\,\mu+\rho)\,\psi_{\mu,l}) \\ &= \sum \overline{\gamma_l(D_j^*,\,\mu+\rho)}\,\widetilde{f}_j(\mu) \\ \text{(Lemma 6.12)} &= \sum \overline{\gamma_l(D_j^*,\,\mu+\rho)}\,d(\mu)^{-1}\,\widehat{F}_j\Big|_{\mathfrak{b}_{\mathbb{C}}^*}(\mu) \\ &= \sum \overline{\gamma_l(D_j^*,\,\mu+\rho)}\,d(\mu)^{-1}\,\Phi_j\Big|_{\mathfrak{b}_{\mathbb{C}}^*}(\mu) \\ &= \sum p_j(\mu+\rho)\,\phi_j\big|_{\mathfrak{b}_{\mathbb{C}}^*}(\mu+\rho) \\ &= \varphi_1(\mu+\rho) \;=\; \varphi(\mu).\end{aligned}$$

where we use the relation $\rho(\mathfrak{h})|_{\mathfrak{b}} = \rho$ (recall (2.5)) and whence

$$\Phi_j\big|_{\mathfrak{b}_{\mathbb{C}}^*}(\mu) = d(\mu)\,\phi_j\big|_{\mathfrak{b}_{\mathbb{C}}^*}(\mu + \rho(\mathfrak{h})|_{\mathfrak{b}}) = d(\mu)\,\phi_j\big|_{\mathfrak{b}_{\mathbb{C}}^*}(\mu+\rho).$$

This proves that $f \mapsto \mathcal{S}_l(f)|_{\Lambda_l^+}$ is surjective.

The uniqueness of such a f is clear. If there exist $f, g \in C_r^\infty(U//K,\,\chi_l)$ such that $\widetilde{f}_l(\mu) = \widetilde{g}_l(\mu) = \varphi(\mu)$, then $\mathcal{S}_l(f-g)(\mu) = 0$. By the property of the Fourier transform, we must have $f - g \equiv 0$. Therefore, $f \mapsto \mathcal{S}_l(f)|_{\Lambda_l^+}$ is injective and hence bijective.

6.4.3 Unique Extension of $\mathcal{S}_l$

The (extended) χ_l-spherical Fourier transform

$$\mathcal{S}_l : C_r^\infty(U//K,\, \chi_l) \longrightarrow \mathrm{PW}_r(\mathfrak{b}_{\mathbb{C}}^*),$$

is a bijection means for any $\varphi \in \mathrm{PW}_r(\mathfrak{b}_{\mathbb{C}}^*)$ there is a unique $f \in C_r^\infty(U//K,\, \chi_l)$ such that $\mathcal{S}_l(f)(\lambda) = \varphi(\lambda)$ for all $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$. The injectivity of $\mathcal{S}_l$ is obvious. If $\mathcal{S}_l(f)(\lambda) = 0$ for all $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$, then $\mathcal{S}_l(f)(\mu) = 0$ for all $\mu \in \Lambda_l^+$ which implies $f = 0$ immediately from the property of Fourier transform (see Proposition 3.25).

We have shown in the previous section that there is a unique $f \in C_r^\infty(U//K,\, \chi_l)$ such that $\mathcal{S}_l(f)(\mu) = \varphi(\mu)$ for all $\mu \in \Lambda_l^+$. By part 1 of Theorem 6.6, the function $\mu \mapsto \mathcal{S}_l(f)(\mu)$ extends holomorphically to a function in $\mathrm{PW}_r(\mathfrak{b}_{\mathbb{C}}^*)$, given by the same formula $\lambda \mapsto \mathcal{S}_l(f)(\lambda)$. So far we know both $\mathcal{S}_l(f) \in \mathrm{PW}_r(\mathfrak{b}_{\mathbb{C}}^*)$ and $\varphi \in \mathrm{PW}_r(\mathfrak{b}_{\mathbb{C}}^*)$ are holomorphic in $\mathfrak{b}_{\mathbb{C}}^*$, and they agree with each other on the subset $\Lambda_l^+ \subset \mathfrak{b}_{\mathbb{C}}^*$. It remains to show they match on all of $\mathfrak{b}_{\mathbb{C}}^*$. In other words, we need make sure the holomorphic extension to $\mathfrak{b}_{\mathbb{C}}^*$ of $\mu \mapsto \mathcal{S}_l(f)(\mu)$ is unique. This is what we present in part 3 of Theorem 6.6. This part is intended to complete the proof of surjectivity of $f \mapsto \mathcal{S}_l(f)$.

Therefore we want to show for any $\varphi \in \mathrm{PW}_r(\mathfrak{b}_{\mathbb{C}}^*)$ with $\varphi(\mu) = 0$ for all $\mu \in \Lambda_l^+$, then $\varphi = 0$.

Let $\varphi_0(\lambda) := \varphi(\lambda + 2|l|\,\rho_s)$ for all $\lambda \in \mathfrak{b}_{\mathbb{C}}^*$. Then $\varphi_0 \in \mathrm{PW}_r(\mathfrak{b}_{\mathbb{C}}^*)$ and

$$\varphi_0(\mu) = \varphi(\mu + 2|l|\,\rho_s) = 0,\ \forall\, \mu \in \Lambda_0^+.$$

because $\mu + 2|l|\,\rho_s \in \Lambda_l^+$ for $\mu \in \Lambda_0^+$. Based on a generalization of Carlson's theorem[2], [22, §7] proved $\varphi_0 = 0$ if $r < \pi/(\max \|\omega_j\|)$ where $\{\omega_j\}$ are the fundamental weights given by (2.3). So $\varphi = 0$ if r satisfies the same condition. Hence, if $S \leq \pi/(\max \|\omega_j\|)$ and $0 < r < S$, then $\mathcal{S}_l$ is surjective and hence bijective.

[2]The generalization of Carlson's theorem in [22] states: Let $f : \mathbb{C}^n \to \mathbb{C}$ be holomorphic. Assume (1) there are a constant $c < \pi$ and for each $z \in \mathbb{C}^n$ a constant C such that $|f(z + \zeta\, e_j)| \leq C\, e^{c\,|\zeta|}$ for all $\zeta \in \mathbb{C}$, $j = 1, \ldots, n$; (2) $f(k) = 0$ for all $k \in (\mathbb{Z})^n$. Then $f = 0$.

Chapter 7
Rank One Compact Symmetric Spaces

The rank one case corresponds to $n = 1$, that is, $\mathfrak{b}$ is one-dimensional. From Table 1, we see that the only rank one symmetric spaces U/K for which K is not semisimple are

$$\mathrm{SU}\,(p+1)/\mathrm{U}\,(p),\ p \geq 1.$$

These are the spaces of complex lines in $\mathbb{C}^{p+1}$, known as the Grassmann manifolds of one-dimensional subspaces of $\mathbb{C}^{p+1}$. The root system $\Sigma = \{\pm\,\alpha,\, \pm\, 2\,\alpha\}$ of type BC_1, and we fix $\Sigma^+ = \{\alpha,\, 2\alpha\}$. Then $P^+ = 2\,\mathbb{Z}_+\,\alpha$. Set $k_1 = m_\alpha$ and $k_2 = m_{2\,\alpha}$. From Table 2, we have $k_1 = 2\,(p-1)$, $p \geq 1$, and $k_2 = 1$. We identify $\mathfrak{b}$ and $i\,\mathfrak{b}^*$ with $i\,\mathbb{R}$, and $\mathfrak{b}_{\mathbb{C}}$ and $\mathfrak{b}^*_{\mathbb{C}}$ with $\mathbb{C}$. So $B = \exp\,\mathfrak{b} \cong \mathbb{T}$. The Weyl group $W = \{\pm\,1\}$ acting on $i\,\mathbb{R}$ and $\mathbb{C}$ by multiplication. The χ_l-spherical Fourier transform becomes, up to a constant,

$$\widetilde{f_l}\,(\lambda) = \int_{\mathbb{T}^+} f\,(x)\,\eta_l\,(x)\,F\,(\lambda+\rho,\, m(l);\, x)\,\delta\,(x)\,d\,x$$

where $d\,x$ is the invariant measures on the torus $\mathbb{T}$.

It is well-known that $\mathbb{C}\,[P] = \mathbb{C}\,[x,\, x^{-1}]$ with $x = e^{2\alpha}$, and $\mathbb{C}\,[x,\, x^{-1}]^W = \mathbb{C}\,[s]$ with $s = \frac{1}{2}(x + x^{-1})$. The weight measure $\delta\,(x)\,d\,x$ becomes

$$2^p\,(1-s)^{p-1}\,ds,$$

where ds is the invariant measure on $\mathbb{R}$. Since $f|_B$ is W-invariant and f has compact support in $B_r\,(0) \cap \mathfrak{b}^+ \subset (\Omega/2) \cap \mathfrak{b}^+$, it reduces $\widetilde{f_l}\,(\lambda)$ to an integral with respect to s over $0 \leq s \leq 1$.

We give an alternative method to get the exponential growth of $\widetilde{f_l}$. Recall that Proposition 6.1 in [27] gives an estimate of the hypergeometric function F, but requires all multiplicity parameters in F are positive. In our case, the multiplicity parameter $m(l)$ in $F\,(\lambda+\rho,\, m(l);\ \cdot\ )$ might be negative. With the help of shift operators (cf. Chapter 3 in [18]), we can shift up or down certain multiplicities as needed. Applying a suitable shift operator to F we can move multiplicities from negative to positive so that we are free to use that estimate. For rank one case, the shift operator we use has a simple form, derived from [18, (3.3.5)],

$$E_- = (s-1)\,\frac{d}{d\,s} + C, \quad C = k_1 + k_2 - 1.$$

This is a first order differential operator. Using integration by parts, followed by Proposition 6.1 in [27], gives the desired exponential growth of $\widetilde{f_l}$.

Remark 7.1. Let $\mathfrak{u}$ be a rank n ($n > 1$) Lie algebra associates with the multiplicity $(m_s, m_m, 1)$ with m_m even. Let $\mathfrak{u}_j$ be rank one Lie algebras with $(m_s,\, 0,\, 1)$, $j = 1, \ldots, n$. Then $\mathfrak{b} = \mathfrak{b}_1 \oplus \cdots \oplus \mathfrak{b}_n$ where $\mathfrak{b}_j \subset \mathfrak{q}_j$ is maximal abelian, and $\mathfrak{q}_j$ is the -1-eigenspace of $\mathfrak{u}_j$. So

$$B = B_1 \times B_2 \times \cdots \times B_n, \quad B_j = \exp\, \mathfrak{b}_j.$$

Let $f \in C_r^\infty(U//K,\, \chi_l)$. Using a shift operator to move m_m down to 0, we can then write $\widetilde{f}_l(\lambda)$ as a n-fold iterated integral of rank one cases with which we have done. This is a different proof of an exponential growth of $\widetilde{f}_l$, but only for the case m_m is even.

References

[1] J. Arthur, *A Paley-Wiener theorem for real reductive groups*, Acta. Math. **150** (1983), 1-89.

[2] Bourbaki, *Groupes et algèbres de Lie*, Chapter 4, 5, et 6, Masson, Paris, 1981.

[3] T. Branson, G. Ólafsson, and A. Pasquale, *The Paley-Wiener theorem and the local Huygens' principle for compact symmetric spaces: the even multiplicity case*, Indag. Mathem., N.S., **16 (3-4)** (2005), 393 - 428.

[4] J. Dadok, *Paley-Wiener theorem for singular support of K-finite distributions on symmetric spaces*, J. Funct. Anal. **31** (1979), 341-354.

[5] M. Eguchi, M. Hashizume, and K. Okamoto, *The Paley-Wiener theorem for distributions on symmetric spaces*, Hiroshima Math. J. **3** (1973), 109-120.

[6] J. Faraut, *Espaces Hilbertiens invariant de fonctions holomorphes*, Semin. Congr., Volume **7**, Soc. Math. de France, Paris, 2003, 101-167.

[7] R. Fabec and G. Ólafsson, *Noncommutative Harmonic Analysis*, To appear.

[8] R. Gangolli, *On the Plancherel formula and the Paley-Wiener theorem for spherical functions on semisimple Lie groups*, Ann. of Math. **93** (1971), 150-165.

[9] F. B. Gonzalez, *A Paley-Wiener theorem for central functions on compact Lie groups*, Contemporary Math. **278** (2001), 131-136.

[10] R. Gangolli and V. S. Varadarajan, *Harmonic Analysis of Spherical Functions on Real Reductive Groups*, Ergebnisse der Mathematik 101, Springer-Verlag, 1988.

[11] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic press, New York, 1978.

[12] S. Helgason, *Groups and Geometric Analysis*, Mathematical Surveys and Monographs, Volume 83, AMS, 2000.

[13] S. Helgason, *Geometric Analysis on Symmetric Spaces*, Mathematical Surveys and Monographs, Volume 39, AMS, 2008.

[14] S. Helgason, *An analogue of the Paley-Wiener theorem for the Fourier transform on certain symmetric spaces*, Math. Ann. **165** (1966), 297-308.

[15] S. Helgason, *Fundamental solutions of invariant differential operators on symmetric spaces*, Amer. J. Math. **86** (1964), 565-601.

[16] S. Helgason, *Paley-Wiener theorems and surjectivity of invariant differential operators on symmetric spaces and Lie groups*, Bull. Amer. Math. Soc. **79** (1973), 129-132.

[17] Harish-Chandra, *Spherical Functions on A Semisimple Lie Group*, I-II, Amer. J. Math. **80** (1958), 241-310, 553-613.

[18] G. Heckman, H. Schlichtkrull, *Harmonic Analysis and Special Functions on Symmetric Spaces*, Perspectives in Mathematics, Volume 16, Academic Press, 1994.

[19] A. W. Knapp, *Lie groups: Beyond an introduction*, Progress in Mathematics, Volume 140, Birkhäuser, 2002.

[20] C. C. Moore, *Compactifications of symmetric spaces II: the Cartan domains*, Amer. J. Math. **86** (1964), 358-378.

[21] G. Ólafsson and A. Pasquale, *Ramanujan's master theorem for Riemannian symmetric spaces*, 2011.

[22] G. Ólafsson and H. Schlichtkrull, *A local Paley-Wiener theorem for compact symmetric spaces*, Advance in Mathematics **218** (2008), 202-215.

[23] G. Ólafsson and H. Schlichtkrull, *Fourier series on compact symmetric spaces: K-finite functions of small support*, J. Fourier Anal. Appl. **16** (2010), 609-628.

[24] G. Ólafsson and H. Schlichtkrull, *Fourier transforms of spherical distributions on compact symmetric spaces*, Math. Scand. **109** (2011), 93-113.

[25] G. Ólafsson and J. A. Wolf, *The Paley-Wiener theorem and limits of symmetric spaces*, 2011.

[26] G. Ólafsson and J. A. Wolf, *Extension of symmetric spaces and restriction of Weyl groups and invariant polynomials*, Contemporary Mathematics, 2011.

[27] E. M. Opdam, *Harmonic analysis for certain representations of graded Hecke algebras*, Acta. Math. **175** (1995), 75-121.

[28] M. Raid, *Groupes linéaires compacts et fonctions C^∞ covariants*, Bull. Sci. Math., **107** (1983), 93-111.

[29] H. Schlichtkrull, *One-dimensional K-types in finite dimensional representations of semisimple Lie groups: a generalization of Helgason's theorem*, Math. Scand. **54** (1984), 279-294.

[30] M. R. Sepanski, *Compact Lie Groups*, Graduate Texts in Mathematics **235**, Springer, 2007.

[31] N. Shimeno, *The Plancherel formula for spherical functions with one dimensional K-type on a simply connected simple Lie group of hermitian type*, J. Funct. Anal. **121** (1994), 330-388.

[32] M. Sugiura, *Fourier series of smooth functions on compact Lie groups*, Osaka Math. J. **8** (1971), 33-47.

[33] E. van den Ban and H. Sclichtkrull, *A Paley-Wiener theorem for reductive symmetric spaces*, Ann. of Math. **164** (2006), 879-909.

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