12-7-1997

A new Young diagrammatic method for Kronecker products of O(n) and Sp(2m)

Pan Feng
Louisiana State University

Shihai Dong
Liaoning Normal University

J. P. Draayer
Louisiana State University

Recommended Citation

Feng, P., Dong, S., & Draayer, J. (1997). A new Young diagrammatic method for Kronecker products of O(n) and Sp(2m). *Journal of Physics A: Mathematical and General, 30* (23), 8279-8287. https://doi.org/10.1088/0305-4470/30/23/024

A New Young Diagrammatic Method For Kronecker Products of O(n) and Sp(2m)

Feng Pan[1†], Shihai Dong[2] and J. P. Draayer[1]

[1]*Department of Physics & Astronomy, Louisiana State University,*
Baton Rouge, LA 70803-4001

[2]*Department of Physics, Liaoning Normal University, Dalian 116029, P. R. China*

Abstract

A new simple Young diagrammatic method for Kronecker products of O(n) and Sp(2m) is proposed based on representation theory of Brauer algebras. A general procedure for the decomposition of tensor products of representations for O(n) and Sp(2m) is outlined, which is similar to that for U(n) known as the Littlewood rules together with trace contractions from a Brauer algebra and some modification rules given by King.

PACS numbers: 02.20.Qs, 03.65.Fd

[†] On leave from Department of Physics, Liaoning Normal Univ., Dalian 116029, P. R. China

1. Introduction

Representation theory of orthogonal and symplectic groups plays an important role in many areas of physics and chemistry. It arises, for example, in the description of symmetrized orbitals in quantum chemistry[1], fermion many-body theory[2], grand unification theories for elementary particles[3], supergravity[4], interacting boson and fermion dynamical symmetry models for nuclei[5−8], nuclear symplectic models[9−10], and so on.

Reductions of Kronecker products of representations of O(n) and Sp(2m) groups were outlined in a series of works of King and his collaborators[11−15] based on the pioneering work of Murnaghan[16], Littlewood[17−18], and Newell[19] on character theory and Schur functions. A similar approach was then revisited by Koike and Terada[20], in which some main points were rigorously proved. On the other hand, a Young diagrammatic method for Kronecker products for Lie groups of B, C, and D types were proposed by Fischer[21]. However, as pointed out by Girardi et al[22−23], rules for the decomposition of tensor products for SO(n) and Sp(2m) given in [21] are numerous; and some of them are even ambiguous. After introducing generalized Young tableaux, with negative rows for describing SO(2m), Girardi et al gave a formula to compute the Kronecker products for SO(n) and Sp(2m) in [22-23]. The formula can be used to compute both tensor and spinor representations of SO(n) and Sp(2m). However, no proof was given for their formula. Littelmann in [24] proposed another Young tableau method to compute Kronecker product of some simply connected algebraic groups based on the character theory. The feature of the method is that it does not use the representation theory of symmetric group. later, Nakashima proposed a Crystal graph base [25] together with the generalized Young diagrams for the same problem. This method applies equally well for the q-analogue of the universal enveloping algebras of A, B, C, and D types[26].

In addition to the usefulness of these groups in many applications, the decomposition of the Kronecker products of orthogonal and symplectic groups have long been an interesting problem in mathematics, which was first considered by Weyl[27] and Brauer[28]. Besides the works of mentioned above, there are many other similar works. For example, Berele discussed a similar problem for the symplectic case in [29], and Sundaram for the orthogonal case in [30].

In this paper, we will outline a new simple Young diagrammatic method for the Kronecker products of O(n) and Sp(2m). Our procedure is mainly based on the induced representation of the Brauer algebra $D_f(n)$, which applies to O(n) and Sp(2m) because of the well-known Brauer-Schur-Weyl duality relation between $D_f(n)$ and O(n) or Sp(2m). This relation has already enabled us to derive Clebsch-Gordan coefficients and Racah coefficients of the quantum group $U_q(n)$ from induction and subduction coefficients of Hecke algebras[31−32], and Racah coefficients of O(n) and Sp(2m) from subduction coefficients of Brauer algebra[33].

In Section 2, we will give a brief introduction to Brauer algebras. Induced representations of the Brauer algebra $S_{f_1} \times S_{f_2} \uparrow D_f(n)$ will be discussed in Section 3, which are important for our purpose. In Section 4, we will outline a new simple Young diagrammatic method for the decomposition of the Kronecker products for O(n) and Sp(2m). A concluding remark will be given in Section 5.

2. Brauer algebra $D_f(n)$

The Brauer algebra $D_f(n)$ is defined algebraically by $2f-2$ generators $\{g_1,\ g_2,\ \cdots,\ g_{f-1},\ e_1,\ e_2,\ \cdots,\ e_{f-1}\}$ with the following relations

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},$$

$$g_i g_j = g_j g_i, \quad |i-j| \geq 2, \tag{1a}$$

$$e_i g_i = e_i,$$

$$e_i g_{i-1} e_i = e_i. \tag{1b}$$

Using these defining relations and via drawing pictures of link diagrams [34-35], one can also derive other useful ones. For example,

$$e_i e_j = e_j e_i, \quad |i-j| \geq 2,$$

$$e_i^2 = n e_i,$$

$$(g_i - 1)^2 (g_i + 1) = 0. \tag{1c}$$

It is easy to see that $\{g_1,\ g_2,\ \cdots,\ g_{f-1}\}$ generate a subalgebra $\mathbf{C}S_f$, which is isomorphic to the group algebra of the symmetric group; that is, $D_f(n) \supset \mathbf{C}S_f$. The properties of $D_f(n)$ have been discussed by many authors[34−35]. Based on these results, it is known that $D_f(n)$ is semisimple, i. e. it is a direct sum of full matrix algebra over $\mathbf{C}$, when n is not an integer or is an integer with $n \geq f-1$, otherwise $D_f(n)$ is no longer semisimple. In the following, we assume that the base field is $\mathbf{C}$ and n is an integer with $n \geq f-1$. In this case, $D_f(n)$ is semisimple, and irreducible representations of $D_f(n)$ can be denoted by a Young diagram with $f,\ f-2,\ f-4,\ \cdots,\ 1$ or 0 boxes. An irrep of $D_f(n)$ with $f-2k$ boxes is denoted as $[\lambda]_{f-2k}$. The branching rule of $D_f(n) \downarrow D_{f-1}(n)$ is

$$[\lambda]_{f-2k} = \oplus_{[\mu]\leftrightarrow[\lambda]} [\mu], \tag{2}$$

where $[\mu]$ runs through all the diagrams obtained by removing or (if $[\lambda]$ contains less than f boxes) adding a box to $[\lambda]$. Hence, the basis vectors of $D_f(n)$ in the standard basis can be denoted by

$$\left| \begin{array}{cc} [\lambda]_{f-2k} & D_f(n) \\ [\mu] & D_{f-1}(n) \\ \vdots & \vdots \\ [p] & D_{f-p+1}(n) \\ [\nu] & D_{f-p}(n) \end{array} \right) = \left| \begin{array}{c} [\lambda]_{f-2k} \\ [\mu] \\ \vdots \\ [p] \\ Y_M^{[\nu]} \end{array} \right), \tag{3}$$

where $[\nu]$ is identical to the same irrep of S_{f-p}, $Y_M^{[\nu]}$ is a standard Young tableau, and M can be understood either as the Yamanouchi symbols or indices of the basis vectors in the so-called decreasing page order of the Yamanouchi symbols. Procedures for evaluating matrix elements of g_i, and e_i with $i = 1,\ 2,\ \cdots,\ f-1$ in the standard basis (3) have been given in [36], and [37]. It is obvious that (3) is identical to the standard basis vectors of S_f when $k = 0$. In this case, all matrix elements of e_i are zero, while matrix elements of g_i can be obtained by the well-known formula for S_f.

3. Induced representations of $D_f(n)$

From the early work of Brauer [28] and recent studies [34-35] one knows that there is an important relation, the so-called Brauer-Schur-Weyl duality relation between the Brauer algebra $D_f(n)$ and O(n) or Sp(2m). If G is the orthogonal group O(n) or symplectic group Sp(2m), the corresponding centralizer algebra $B_f(G)$ are quotients of Brauer's $D_f(n)$ or $D_f(-2m)$, respectively. We also need a special class of Young diagrams, the so-called n-permissible Young diagrams defined in [31]. A Young diagram $[\lambda]$ is said to be n-permissible if $P_\mu(n) \neq 0$ for all subdiagrams $[\mu] \leq [\lambda]$, where the subdiagrams $[\mu]$ can be obtained from $[\lambda]$ by taking away appropriate boxes, and $P_{[\mu]}(n)$ is the dimension of O(n) or $Sp(2m)$ for the irrep $[\mu]$. A Young diagram $[\lambda]$ is n-permissible if and only if

(a) Its first 2 columns contain at most n boxes for n positive,
(b) It contains at most m columns for $n = -2m$ a negative even integer,
(c) Its first 2 rows contain at most $2 - n$ boxes for n odd and negative.

If these conditions are satisfied, $D_f(n)$ is isomorphic to $B_f(O(n))$ for n positive, to $B_f(O(2-n))$ for n negative and odd, and to $B(Sp(2m))$ for $n = -2m < 0$. In the following, we assume that all irreps to be discussed are n-permissible with $n \leq f-1$ for $n > 0$ or $-n \leq f-1$ for negative n. These condition imply that the $D_f(n)$ being considered is semisimple.

Therefore, an irrep of $B_f(O(n))$ or $B_f(Sp(2m))$ is simultaneously the same irrep of O(n) or Sp(2m). But the space of $B(G)$ and G are different. The former is labelled by its Brauer algebra indices which is operating among $B_f(G)$ space, while the later is labelled by its tensor components of group G. This is the so-called Brauer-Schur-Weyl duality relation between $B_f(G)$ and G, where $G = O(n)$ or $Sp(2m)$.

Hence, in order to discuss the Kronecker products of $O(n)$ and $Sp(2m)$ for general cases

$$[\lambda_1]\times[\lambda_2]\downarrow\sum_{\lambda}\{\lambda_1\lambda_2\lambda\}[\lambda], \tag{4}$$

where $\{\lambda_1\lambda_2\lambda\}$ is the number of occurrence of irrep $[\lambda]$ in the decomposition $[\lambda_1]\times[\lambda_2]$, we can switch to consider induced representations of the Brauer algebra, $S_{f_1}\times S_{f_2}\uparrow D_f(n)$ for the same decomposition given by (4). In this case, we only need to study irreps of $D_f(n)$ induced by irreps of $S_{f_1}\times S_{f_2}$. The standard basis vectors of $[\lambda_1]_{f_1}$, and $[\lambda_2]_{f_2}$ for S_{f_1} and S_{f_2} can be denoted by $|Y^{[\lambda_1]}_{m_1}(\omega^0_1)>$, and $|Y^{[\lambda_2]}_{m_2}(\omega^0_2)>$, respectively, where

$$(\omega^0_1)=(1,\ 2,\ \cdots,\ f_1),\quad (\omega^0_2)=(f_1+1,\ f_1+2,\ \cdots,\ f_1+f_2) \tag{5}$$

are indices in the standard tableaux $Y^{[\lambda_1]}_{m_1}$ and $Y^{[\lambda_2]}_{m_2}$, respectively. The product of the two basis vectors are denoted by

$$|Y^{[\lambda_1]}_{m_1},\ Y^{[\lambda_2]}_{m_2},\ (\omega^0_1),\ (\omega^0_2)>\equiv|Y^{[\lambda_1]}_{m_1}(\omega^0_1)>|Y^{[\lambda_2]}_{m_2}(\omega^0_2)>, \tag{6}$$

which is called primitive uncoupled basis vector.[31−32,34]

The left coset decomposition of $D_f(n)$ with respect to the subalgebra $S_{f_1}\times S_{f_2}$ is denoted by

$$D_f(n)=\sum_{\omega k}\oplus Q^k_\omega(S_{f_1}\times S_{f_2}), \tag{7}$$

where the left coset representatives $\{Q^k_\omega\}$ have two types of operations. One is the order-preserving permutations, which is the same as that for symmetric group,[31−32]

$$Q^{k=0}_\omega(\omega^0_1,\ \omega^0_2)=(\omega_1,\ \omega_2), \tag{8}$$

where

$$(\omega_1)=(a_1,\ a_2,\ \cdots,\ a_{f_1}),\ \ (\omega_2)=(a_{f_1+1},\ a_{f_1+2},\ \cdots,\ a_f) \tag{9}$$

with $a_1<a_2<\cdots<a_{f_1}$, $a_{f_1+1}<a_{f_1+2}<\cdots<a_f$, and a_i represents any one of the numbers $1,\ 2,\ \cdots,\ f$. The other, $\{Q^{k\geq1}_\omega\}$ contains k-time trace contractions between two sets of indices (ω_1) and (ω_2). For example, in $S_2\times S_1\uparrow D_3(n)$ for the outer product $[2]\times[1]$, there are six elements in $\{Q^k_\omega\}$ with

$$\{Q^0_\omega\}=\{1,\ g_2,\ g_1g_2\},\ \ \{Q^1_\omega\}=\{e_2,\ g_1e_2,\ e_1g_2\}. \tag{10}$$

Let the number of operators in $\{Q^k_\omega\}$ be h, and the dimensions of the irreps $[\lambda_1]_{f_1}\times[\lambda_2]_{f_2}$ be $h_{[\lambda_1]}h_{[\lambda_2]}$, where $h_{[\lambda_i]}$ with $i=1,\ 2$, can be computed, for example, by using Robinson's formula for symmetric group S_f. It is obvious that the total dimension including multi-occurrence of the same irrep in the decomposition (4) is given by $hh_{[\lambda_1]}h_{[\lambda_2]}$; that is,

$$hh_{[\lambda_1]}h_{[\lambda_2]}=\sum_{\lambda}\{\lambda_1\lambda_2\lambda\}\dim([\lambda];\ D_f(n)), \tag{11}$$

where $\dim([\lambda]; D_f(n))$ is the dimension of $[\lambda]$ for $D_f(n)$, which was given in [29]. Hence, applying the h Q^k_ω's to the primitive uncoupled basis vector (6), we obtain all the uncoupled basis vectors needed in construction of the coupled basis vectors of $[\lambda]$ for $D_f(n)$, which can be denoted as

$$Q^k_\omega|Y^{[\lambda_1]}_{m_1},\ Y^{[\lambda_2]}_{m_2},\ (\omega^0_1),\ (\omega^0_2)>=|Y^{[\lambda_1]}_{m_1},\ Y^{[\lambda_2]}_{m_2},\ \overbrace{(\omega_1),\ (\omega_2)}^{k}>, \tag{12}$$

where $\overbrace{(\omega_1),\ (\omega_2)}^{k}$ stands for k-time contractions between indices in (ω_1) and (ω_2). However, all contractions among (ω_1) or (ω_2) will be zero because $[\lambda_i]$ with $i = 1,\ 2$, has exact f_i boxes, i.e., in this case, the irrep $[\lambda_i]$ of S_{f_i} is the same irrep of $D_{f_i}(n)$. Therefore, $S_{f_1}\times S_{f_2}$ can also be denoted as $D_{f_1}(n) \times D_{f_2}(n)$ when the irreps $[\lambda_i]$ for $i = 1,\ 2$, has exactly f_i boxes only. In the following, we will always discuss this situation, and denote $S_{f_1} \times S_{f_2}$ as $D_{f_1}(n) \times D_{f_2}(n)$ without further explanation.

Finally, basis vectors of $[\lambda]_{f-2k}$ can be expressed in terms of the uncoupled basis vectors given by (12).

$$|[\lambda]_{f-2k},\ \tau;\ \rho>=\sum_{m_1\ m_2\ \omega} C^{[\lambda]\rho;\tau}_{m_1m_2;\omega}Q^k_\omega|Y^{[\lambda_1]}_{m_1}(\omega^0_1),\ Y^{[\lambda_2]}_{m_2}(\omega^0_2)>, \tag{13}$$

where ρ is the multiplicity label needed in the outer-product $[\lambda_1]_{f_1} \times [\lambda_2]_{f_2} \uparrow [\lambda]_{f-2k}$, τ stands for other labels needed for the irrep $[\lambda]_{f-2k}$, and the coefficient $C^{[\lambda]\rho;\tau}_{m_1m_2;\omega}$ is $[\lambda_1]_{f_1} \times [\lambda_2]_{f_2} \uparrow [\lambda]_{f-2k}$ induction coefficient (IDC) or the outer-product reduction coefficient (ORC).

4. A Young diagrammatic method for Kronecker products of O(n) and Sp(2m)

Analytical derivation or algorithm for the IDCs discussed in Section 3 is not necessary if only outer-products of $D_{f_1}(n)\times D_{f_2}(n)$ for irreps $[\lambda_1]_{f_1}\times[\lambda_2]_{f_2}$ are considered. It is obvious in (12) that irreps with $f-2k$ boxes of $D_f(n)$ can be induced from irreps of $D_{f_1}(n)\times D_{f_2}(n)$. When $k = 0$, (12) is identical to that for symmetric groups. An important operation in (12) is performed by $\{Q^k_\omega\}$ with $k \neq 0$. After k-time contraction the uncoupled primitive basis vector of $[\lambda_1]_{f_1}\times[\lambda_2]_{f_2}$ will be equivalent to basis vectors of $[\lambda'_1]_{f_1-k}\times[\lambda'_2]_{f_2-k}$, where $[\lambda'_i]_{f_i-k}$ with $i = 1,\ 2$ is any possible standard Young diagrams with $f_i - k$ boxes, which can be obtained from $[\lambda_i]_{f_i}$ by deleting k boxes from $[\lambda_i]$ in all possible ways. Therefore, as far as representations are concerned, the irrep $\{[\lambda]_{f-2k}\}$ of $D_f(n)$ can be obtained from the outer-product $\{[\lambda'_1]_{f_1-k} \times [\lambda'_2]_{f_2-k}\}$ of the symmetric group $S_{f_1-k} \times S_{f_2-k}$. Thus, we obtain the following rules for the outer-products of $D_{f_1}(n) \times D_{f_2}(n)$.

Lemma 1. The outer-product rule for $D_{f_1}(n) \times D_{f_2}(n) \uparrow D_f(n)$ for the decomposition

$$[\lambda_1]_{f_1} \times [\lambda_2]_{f_2} \uparrow \sum_\lambda \{\lambda_1\lambda_2\lambda\}[\lambda]$$

can be obtained diagrammatically by

(1) Removing k boxes, where $k = 0,\ 1,\ 2,\ \cdots, \min(f_1, f_2)$, from $[\lambda_1]_{f_1}$ and $[\lambda_2]_{f_2}$ simultaneously in all possible ways under the following restrictions:

(a) Always keep the resultant diagrams $[\lambda'_i]_{f_i-k}$ with $i = 1,\ 2$ standard Young diagrams;

(b) No more than two boxes in the same column (row) in $[\lambda_1]$ with those in the same row (column) in $[\lambda_2]$ can be removed simultaneously.

(2) Applying the Littlewood rule of the outer-product reduction for symmetric group to the outer-product $[\lambda'_1]_{f_1-k} \times [\lambda'_2]_{f_2-k}$, and repeatedly doing so for each k.

What we need to explain is restriction (b). Consider a simple example which is representative of the general case. Let $[\lambda_1] = [2]$, $[\lambda_2] = [1^2]$, and a k-time trace contraction operator be Q^k. According to our procedure, we have

$$Q^1(\ \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \times \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}\ ) = (\ \begin{array}{|c|c|}\hline \ & \alpha \\ \hline\end{array} \times \begin{array}{|c|}\hline \ \\ \hline \alpha \\ \hline\end{array}\ ) = (\ \boxed{\ } \times \boxed{\ }\ ), \tag{14a}$$

while

$$Q^2(\ \begin{array}{|c|c|}\hline \ & \ \\ \hline\end{array} \times \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}\ ) = (\ \begin{array}{|c|c|}\hline \beta & \alpha \\ \hline\end{array} \times \begin{array}{|c|}\hline \beta \\ \hline \alpha \\ \hline\end{array}\ ). \tag{14b}$$

Because trace contraction occurs in pairs, the indices α, and β labelled in the boxes indicate that those with the same indices are contracted with each other. It is known that trace contraction of two vectors results in the symmetrization of the tensor components. Therefore, trace contraction of anti-symmetric tensors is zero. However, the indices of α part is not only symmetric but also anti-symmetric with those of β part in (14b). Hence, restriction (b) holds.

Finally, using the Brauer-Schur-Weyl duality relation between $D_f(n)$ and O(n) or Sp(2m), one knows that Lemma 1 applies to the decompositions of the Kronecker products of O(n) or Sp(2m) as well. Thus, we have the following lemma.

Lemma 2. The Kronecker product of O(n) or Sp(2m) for the decomposition given by (4) can be obtained by using procedures (1) and (2) given by Lemma 1 together with the following modification rules:

For the group O(n), where $n = 2l$ or $2l + 1$, (Sp(n), where $n = 2l$), the resulting irrep $[\lambda] = [\lambda_1,\ \lambda_2,\ \cdots, \lambda_p,\ \dot{0}]$ is nonstandard if $p > l$. In this case, we need to remove boxes from $[\lambda]$ along a continuous boundary with hook of length $2p - n$ $(2p - n - 2)$ and depth x, where x is counted by starting from the first column of $[\lambda]$ to the right-most column that the boundary hook reaches.[12] The resultant Young diagram will be admissible or set to zero if, at any stage, the removal of the required hook leaves an irregular Young diagram. Then, the resultant irrep $[\lambda]_{\text{allowed}}$ can be denoted symbolically as

$$[\lambda]_{\text{allowed}} \begin{cases} = (-)^{x}[\sigma], & \text{for O(n)}, \\ \\ = (-)^{x+1}[\sigma], & \text{for Sp(2m)}, \end{cases}$$

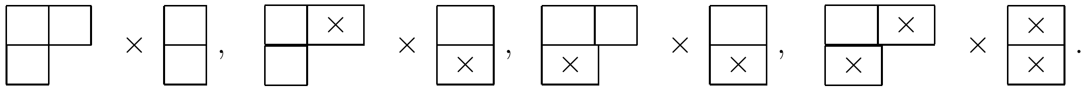
where $[\sigma]$ is obtained from $[\lambda]$ by using the above modification rules. For example,

$$[3^3,\ 1] = \begin{cases} = [3^3] & \text{for O(7)}, \\ \\ = [3^2] & \text{for O(4)}, \\ \\ = -[20] & \text{for O(2)}, \\ \\ = 0 & \text{for O(6), O(5), and O(3)}, \end{cases} \tag{15}$$

which was illustrated in [12] by King. In the following, we give an example to show how this method works.

Example. Find the Kronecker product $[21] \times [11]$ for O(n) or Sp(2m).

First, we consider all possible diagrams with 0, 1, and $\min(f_1,\ f_2) = 2$ -time trace contractions, which are



(16)

Then, we need to compute the Kronecker products $[21] \times [11]$, $[11] \times [1]$, $[2] \times [1]$, and $[1] \times [0]$, which can be obtained by using the Littlewood rule for U(n). We get

$$[21] \times [11] = [32] + [221] + [2111] + [311], \tag{17a}$$

$$[20] \times [1] = [30] + [21], \tag{17b}$$

$$[11] \times [1] = [21] + [111], \tag{17c}$$

$$[1] \times [0] = [1]. \tag{17d}$$

Finally, summing up all the irreps appearing on the rhs. of (17), we obtain

$$[21] \times [11] = [32] + [221] + [2111] + [311] + [30] + 2[21] + [111] + [10], \tag{18}$$

which is valid for O(n) when $n \geq 8$ and Sp(2m) when $m \geq 4$. Using the modification rules given in Lemma 2, we can easily obtain the following results

$$[210] \times [110] = [320] + [221] + [211] + [311] + [300] + 2[210] + [111] + [100] \ \text{ for O(7)}, \tag{19a}$$

$$[210]\times[110]=[320]+[221]+3[210]+[311]+[300]+[111]+[100]\ \ \text{for O(6)},\tag{19b}$$

$$[21]\times[11]=[32]+[22]+[20]+[31]+[30]+2[21]+[11]+[10]\ \ \text{for O(5)},\tag{19c}$$

$$[21]\times[11]=[32]+2[30]+2[21]+2[10]\ \ \text{for O(4)}.\tag{19d}$$

In the above computation, the following results have been used

$$[2111]=\begin{cases}[211]\ \ \text{for}\ \ \text{O(7)},\\ [21]\ \ \text{for}\ \ \text{O(6)},\\ [20]\ \ \text{for}\ \ \text{O(5)},\\ [10]\ \ \text{for}\ \ \text{O(4)},\end{cases}\tag{20a}$$

$$[221]=\begin{cases}[22]\ \ \text{for}\ \ \text{O(5)},\\ 0\ \ \ \ \ \text{for}\ \ \text{O(4)},\end{cases}\tag{20b}$$

$$[311]=\begin{cases}[31]\ \ \text{for}\ \ \text{O(5)},\\ [30]\ \ \text{for}\ \ \text{O(4)},\end{cases}\tag{20c}$$

which are obtained from modification rules given in Lemma 2. While

$$[210]\times[110]=[320]+[221]+[311]+[300]+2[210]+[111]+[100]\ \ \text{for}\ \ \text{Sp(6)},\tag{21a}$$

$$[21]\times[11]=[32]+[30]+[21]+[10]\ \ \text{for}\ \ \text{Sp(4)},\tag{20b}$$

where the following modification rule have been used:

$$[2111]=\begin{cases}0\ \ \text{for}\ \ \ \ \text{Sp(6)},\\ -[21]\ \ \text{for}\ \ \text{Sp(4)}\end{cases},\tag{22a}$$

$$[221]=[311]=[111]=0\ \ \text{for Sp(4)}.\tag{22b}$$

5. Concluding Remarks

In this paper, a new simple Young diagrammatic method for the decomposition of the Kronecker products of O(n) and Sp(2m) is outlined based on the induced representation theory of $D_f(n)$. This algebra was proposed by Brauer at the end of thirties. His aim was indeed to solve the decomposition problem of the Kronecker products of O(n) and Sp(2m). On the other hand, because the representations of $D_f(n)$ are the same as those of Birman-Wenzl algebras $C_f(r,q)$ when r, and q are not a root of unity, the method applies to quantum groups $O_q(n)$ and $Sp_q(2m)$ as well for q being not a root of unity. The induced representations of $D_f(n)$ presented in Section 3 can also be used to derive Clebsch-Gordan coefficients of SO(n) when IDCs of $D_{f_1}(n) \times D_{f_2}(n)$ are evaluated, which will be discussed in our next paper.

It should be stated that though our Young diagrammatic method for decomposition of $O(n)$ and $Sp(2m)$ Kronecker products is derived from induced representation theory of Brauer algebra with the help of Brauer-Schur-Weyl duality relation, the final results are the same as those derived by Littlewood and Newell based on character theory and Schur functions[18−19]. In[18], the main results on how to obtain the Kronecker product of $O(n)$ and $Sp(2m)$ were achieved through the combinatorials of certain type of S-functions. However, in [18], only $p \geq r$ cases were considered, where $n = 2p$ or $2p + 1$ for $O(n)$, and $p = m$ for $Sp(2m)$, and r is the number of rows for the corresponding irrep. In this case, no modification rule is needed, which is the same as ours. When $p \leq r$ in a Young diagram, the final diagram with number of rows greater than p will become non-standard irrep, the correspondence between these non-standard diagrams and the corresponding standard ones with signs in the front of the diagrams was first studied by Newell in [19], which gives just the so-called modification rules proposed by King in a much simper manner.[12] This fact is now summarized by Lemma 2 in this paper.

On the other hand, the Young tableau method proposed by Littelmann [24] and crystal graph base given in [25] are related to the weight space of the corresponding Lie groups (algebras). Therefore, these methods do not use the representation theory of symmetric groups at all. But the final results on the decomposition of the Kronecker product of $O(n)$ and $Sp(2m)$ are the same as those obtained by our Young diagrammatic method derived from Brauer algebras.

Furthermore, this method can also be applied to the Kronecker products of $\mathrm{SO}(2l+1)$ for any irreps and $\mathrm{SO}(2l)$ for their irreps $[\lambda_1,\ \lambda_2,\ \cdots, \lambda_k, \dot{0}]$ for $k < l$. If $k = l$, the irrep of O(2l) $[\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_k]$ with $\lambda_k \neq 0$ reduces into irreps of SO(2l) denoted by $[\lambda_1,\ \lambda_2,\ \cdots,\ \lambda_k]$ and $[\lambda_1,\ \lambda_2,\ \cdots,\ -\lambda_k]$, of which the dimensions are the same. In this case, one should be cautious and use this method. The dimension formula for SO(n) is always helpful in checking final results.

Finally, it should be noted that the method applies to tensor or “true” representations of O(n) only. The spinor representations of O(n) are related to spinor representations of Brauer algebras according to the Brauer-Schur-Weyl duality relation, which still need to be further studied.

Acknowledgment

The authors are very much grateful to our referees for their helpful suggestions and comments, especially for providing us with references [18-19], and [24-26] which we formerly let unnoticed. The project was supported by National Natural Science Foundation of China, and a grant from US National Science Foundation through LSU.

[1] B. G. Wybourne, Int. J. Quant. Chem., **7**(1973) 117
[2] H. Fukutome, M. Yamamura, and A. Nishiyama, Prog. Theor. Phys., **57** (1977) 1554
[3] M. Gell-Mann, Rev. Mod. Phys., **50**(1978) 721
[4] M. Gell-Mann, P. Ramond, and R. Slansky, Nucl. Phys., **B159**(1979) 141
[5] A. Arima, and F. Iachello, Ann. Phys. (N. Y.), **99**(1976) 253
[6] C. L. Wu, Da Hsuan Feng, M. Guidry, Adv. Nucl. Phys. **21** (1994) 227
[7] Feng Pan, Zeng-Yong Pan, and Yu-Fang Cao, Chin. Phys. Lett., **8** (1991) 56
[8] P. Navratil, H. B. Geyer, and J. Dobaczewski, Ann. Phys. (N. Y.), **243** (1995) 218
[9] G. Rosensteel, and D. J. Rowe, Phys. Rev. Lett., **38** (1977) 10
[10] O. Castaños, P. Hess, J. P. Draayer, and P. Pochford, Nucl. Phys. **A524** (1991) 469
[11] R. C. King and N. G. I. El-Sharkaway, J. Phys. **A16** (1983) 3157
[12] R. C. King, J. Math. Phys. **12**(1971) 1588
[13] R. C. King, Lect. Notes in Phys. **50**(1975) 481
[14] R. C. King, Luan Dehuai, and B. G. Wybourne, J. Phys. **A14** (1981) 2509
[15] R. C. King, J. Phys. **A8**(1975) 429
[16] F. D. Murnaghan, The Theory of Gourp Representations, (Johns Hopkings, Baltimore,1938)
[17] D. E. Littlewood, The Theory of Gourp Characters, 2nd edn. (Oxford, Claredom, 1950)
[18] D. E. Littlewood, Can. J. Math. **10**(1958) 17
[19] M. J. Newell, Proc. Royal Irish Acad. **54A**(1951) 153
[20] K. Koike, and I. Terada, J. Alg., **107**(1987) 466
[21] M. Fischer, J. Math. Phys. **22** (1981) 637
[22] G. Girardi, A. Sciarrino, and P. Sorba, J. Phys. **A15** (1982) 1119
[23] G. Girardi, A. Sciarrino, and P. Sorba, J. Phys. **A16** (1983) 2069
[24] P. Littelmann, J. Alg. **130**(1990) 328
[25] T. Nakashima, Commun. Math. Phys. **154**(1993) 215
[26] M. Kashiwara, J. Alg. **165**(1994)295
[27] H. Weyl, The Classical Groups, (Princeton U. P., Princeton, N. J., 1939)
[28] R. Brauer, Ann. Math., **63** (1937) 854
[29] A. Berele, J. Comb. Th. Series A, **43** (1986) 320
[30] S. Sundaram, J. Comb. Th. Series A, **53** (1990) 239
[31] Feng Pan, and J. Q. Chen, J. Math. Phys. **34** (1993) 4305; 4316
[32] Feng Pan, J. Phys. **A26**(1993) 4621
[33] Feng Pan, and Lianrong Dai, J. Phys., **A29** (1996) 5079; 5093
[34] H. Wenzl, Ann. Math., **128**(1988) 173
[35] H. Wenzl, Commun. Math. Phys., **313**(1990) 383
[36] Feng Pan, J. Phys., **A28** (1995) 3139
[37] R. Leduc, A. Ram, 1996, to appear in Adv. Math.