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Nonlinear Equations and wavelets. Multi-Scale Analysis

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Abstract

We use a multi-scale similarity analysis which gives specific relations between the velocity, amplitude and width of localized solutions of nonlinear differential equations, whose exact solutions are generally difficult to obtain.

In this paper, wavelet-inspired approaches for localized solutions of NPDE are explored [1,2]. A first method provides relations between the characteristics of such solutions (amplitude, width and velocity) without the need of solving the corresponding NPDE. The method uses the multi-resolution analysis [2] instead of the traditional tools like the Fourier integrals or linear harmonic analysis which are inadequate for describing such systems. This scale approach has the advantage that it does not need the explicit form of the exact solutions. Hence, it is useful especially in situations when such solutions are unknown. The self-similar character of the fission process of fluid drops is an example where the same type of singularity occurs in any scale [3,4]. In the following we introduce a one-scale analysis (OSA) for the NPDE, in terms of their localized traveling solutions. The so called OSA analysis, described and applied here for localized traveling solutions belonging to any type of NPDE, provides algebraic connections between the width L, amplitude A, and the velocity V of the solution, without actually solving the equation. The procedure consists in the substitution of all the terms in the NPDE, according to the rules:

i (substitution valid for traveling waves) $u_t \to -V u_x$

ii $u \to +A, \qquad u_x \to \pm A/L^2,$

and so forth for higher order of derivatives. This substitution in eq.(ii) is possible only for localized (finite extended support) solutions having at least one local maximum (like solitons or Gauss functions), if they exist. Since we are interested in traveling solutions, the first substitution reduces the number of variables from 2 to 1 so that we are now dealing with an ordinary differential equation instead of a PDE. Then, the second substitution transforms the ordinary differential equation into an equation in the parameters describing the amplitude, width and velocity. Consequently, the NPDE is mapped into an algebraic equation in A, L and V. The proof of the method follows from the expansion of the soliton-like solution $u(x,t) = u(s)$ with$s = x - Vt$, in a Gaussian family of wavelets $\Psi(s) = Ne^{Q(s)}$, where $Q(s)$ is a polynomial and N the normalization constant [2]. If we choose $Q = -is - \{\{s\hat{}2\}\backslash over 2\}$ we obtain a very particular wavelet with the support mainly confined in the $(-1, 1)$ interval, namely $\Psi(s) = \exp\left[-is - \frac{s^2}{2}\right]/\pi^{1/4}$. We have the discrete wavelet expansion of u

$$u(s) = \sum_k C_{j,k} \Psi\left(2^j s - k\right) = \sum_k C_{j,k} \Psi_{j,k}(s) \tag{1}$$

in terms of integer translations k of Ψ, which provide the analysis of localization, and in terms of dyadic dilations 2^j of Ψ which provide the description of different scales. The idea of the proof is to choose a point where the expansion in eq.(1) can be well approximated by one single scale such that $u(s)$ can be approximated with a sum of phases in $s$the OSA approach is given by the convective-dispersive equations, for which the most celebrated example is provided by the KdV equation

$$u_t + uu_x + u_{xxx} = 0, \tag{2}$$

In Table 1 we present examples of pure dispersive NPDE, identified in the first column by the form of the equation, and in the second column by a corresponding traveling localized solution, if the analytical form is available. Such exact solutions provide special relations between L, A and V, which are given in the third column of Table 1. In the last column we introduce the results of OSA, namely the relations between these three parameters, provided by eqs.(ii). The usefulness of the approach may be checked, by a quick comparison between the second and the third columns. The case of the KdV and MKdV equations, eq(2,3), are described in the first and second rows of the Table 1, and also are analyzed in previous papers of the same authors. Moreover, the same relations like in the KdV case remain valid even for the solutions of the "compacton" type [6]

$$u(x,t) = \frac{\sqrt{32}k\cos\left[k\left(x - 4k^2\right)t\right]}{3\left(1 - \frac{2}{3}\cos\left[k\left(x - 4k^2 t\right)^2\right]\right)}, \tag{3}$$

where $L = \pi/6k$, that is $L\tilde{}1/A$, like in the Table 1. Next example (third row) is provided by a generalised KdV equation, in which the dispersion term is quadratic

$$u^t + (u^2)_x + (u^2)_{xxx} = 0. \tag{4}$$

Eq.(3), known as K(2,2) equation because of the two quadratic terms, admits compact supported traveling solutions, named compactons [1,5,7-9]. In general, the compactons are obtained in the form of a power of some trigonometric function defined only on its half-period, and zero otherwise, in such a way that the solution is enough smooth for the NPDE in discussion. In the above example the square of the solution has to be continuous up to its third derivative with respect to x. Different from solitons, the compacton width is independent of the amplitude and this fact provides the special connection with the wavelet bases. The compactons are characterize by a unique scale, and it is this feature that makes it possible to introduce a nonlinear basis starting from a unique generic function. For eq.(3) the compacton solution is given by

$$\eta_c(x - Vt) = \frac{4V}{3}\cos^2\left[\frac{x - Vt}{4}\right], \qquad (5)$$

if $|x - Vt| < 2\pi$ and 0 otherwise. Here we notice that the velocity is proportional to the amplitude and the width of the wave is independent of the amplitude, $L = 4$. As a field of application we mention that the quadratic dispersion term is characteristic for the dynamics of a chain with nonlinear coupling. The general compacton solution for eq.(3) is actually a "dilated" version of eq.(4). Actually, this combination is just a kink compacton joined smoothly with an antikink one

$$\begin{aligned} \eta_{KAK}(x - VT; \lambda) &= \qquad (6) \\ & 0..., \\ \frac{4V}{3}\cos^2\left[\frac{x - Vt}{4}\right] \text{ for } -2\pi &< x - Vt < 0, \\ \frac{4V}{3} \text{ for } 0 &< x - Vt < \lambda, \\ \frac{4V}{3}\cos^2\left[\frac{x - Vt - \lambda}{4}\right] \text{ for } \lambda &< x - Vt < \lambda + 2\pi, \\ & 0... \end{aligned}$$

Finally, we can construct solutions by placing a compacton on the top of a KAK. Such a solution exists only for a short interval of time (λ/V), since the two structures have different velocities. The analytic expression of the solution is given by

$$\eta(x,t) = \eta_{KAK}(x - Vt; \lambda) + \left(\eta_c\left(x - V't - 2\pi\right) + \frac{4V}{3}\right)\frac{\chi(x - V't - 2\pi)}{2\pi}, \qquad (7)$$

for $0 < t < (\lambda - 4\pi)/(V' - V)$ and zero in the rest. Here $\chi(x)$ is the support function, equal with 1 for $|x < 1$ and 0 in the rest, and $V' = 3\max\{\eta_c/4\} + 2V$. For the K(2,2) compacton eq.(4) fulfills some relations between the parameters: $A = 4V/3$ and $L = 4$ [7]. The relations provided by OSA in the last column of the third row, predict such relations, and hence also proove the existence of the compacton. Another good example of the predictive power of the method is exemplified in the case of a general convection-nonlinear dispersion equations, denoted by K(n,m)

$$\eta_t + \eta_x^n + \eta_{xxx}^m = 0, \tag{8}$$

Compacton solution for any $n \backslash neqm$ are not known in general, except for some particular cases. In this case we find a general relation among the parameters, for any n,m , shown in the fourth and fifth rows. These general relations L(A,V) approach the known relations for the exact solutions, in the particular cases like n=m (fourth row), n=m=2 (third row), n=m=3 (first reference in [1]) and n=3, m=2 ; n=2, m=3 (fifth row). These results can be used to predict the behavior of solutions for all values of n,m .

The situation is different in the case of compactons, which allow also stationary solutions. When linear and nonlinear disspersion occur simultaneously, like in the so called K(2,1,2) equation

$$u_t + u_x^2 + u_{xxx} + \epsilon u_{xxx}^2 = 0, \tag{9}$$

where ϵ is a control parameter, the OSA yields a dependence of the form $L = \sqrt{\frac{\pm A+\epsilon}{V \pm A}}$, which still provides a constant width if $V = \pm A + 2\epsilon$. In this case, the speed is proportional to the amplitude, but can change its sign even at non-zero amplitude. Solutions with larger amplitude than a critical one ($A_{crit} = \mp 2\epsilon$)move to the right, solutions having the critical amplitude are at rest, and solutions smaller than the critical amplitude move to the left. This behavior was explored in [7], too. A compacton of amplitude A on the top of a infinite-length KAK solution of amplitude δ

$$u(x,t) = Acos^2 \left(\frac{x - Vt}{4} \right) + \delta, \tag{10}$$

is still a solution of the K(2,2) equation, with the velocity given by $V = \frac{3}{4}(2\delta + A)$. For $A = -2\delta$ the solution becomes an anti-compacton moving together with the KAK. In the case of a slow-scale time-dependent amplitude the oscillations in amplitude can transform into oscillations in the velocity. The key to such a conversion of oscillations is the coupling between the traditional nonlinear picture (convection-dispersion-diffusion) and the typical Schrodinger terms. In Table 2 we present another class of NPDE, namely the dissipative ones. These equations generalize the linear wave equation (first row) where there is no typical length of the traveling solutions. The wavelet analysis provides the correct expression for the dispersion relation ($V = c \rightarrow k^2 = \omega^2/c^2$) with no constraint on either the amplitude A or on the width L . In the second row we introduce the Burgers equation which represents the simplest model for the convective-dissipative interaction. Dissipative systems are to a large extent indifferent to how they were initialized, and follow their own intrinsic dynamics. We provide in the second column an analytic solution of the Burgers equation.

For some special of values of the integration constants ($2C < V^2, D = 0$)the solution becomes a traveling kink

$$u(x,t) = V + \sqrt{V^2 - 2C}\tanh(\sqrt{V^2 - 2C}(x - Vt)). \tag{11}$$

By applying the OSA approach to the Burger equation (third column) we obtain the same relation between amplitude and half-width, like in the case of the exact solution eq.(9), providing the velocity is proportional with the amplitude. In the following we apply the OSA approach to investigate a nonlinear Burgers equation

$$u_t + au_x^{\beta} - \mu u_{xx}^{k} + cu^{\gamma} = 0, \tag{12}$$

called quasi-linear parabolic equation [9], and used to describe the flow of fluids in porous media or the transport of thermal energy in plasma. The existence and stability of waves or patterns is strongly dependent on the coefficients $a, \mu, c, \beta,$ and γ , and at this point the OSA can be useful again since there is no general analytic solution for eq.(10). The result of the OSA approach is presented in the third row of Table 2. The typical scale of patterns depends on the parameters in the equations and the amplitude of the excitations, in a complicated way. However, in order to test OSA again, we found a simple class of exact solutions when $c = 0$, presented in the fourt row in Table 2, and expressed as the inverse of a degenerated hypergeometric function. In this expression we have $\mathcal{A}= \mu k V^{\alpha-1} F[((k-1)a^{\alpha})^{-1}, z] = (a/V)u^{m-1}$ and $\alpha = (k-1)/(m-1)$. The asymptotic behavior of the left hand side of the solution given in fourth row, second column, is described by

$$\Gamma(\alpha - 1)\left[(-1)^{\alpha} + \frac{1}{\Gamma(\alpha)} z^{(\alpha-1)} e^{\alpha}\right] + \mathcal{O}(1/z). \tag{13}$$

If z approaches + ∞ the solution increases indefinitely like an exponential. For α >1 (strong difussion effects), for even k and for even β , the traveling wave $u(x - Vt)$ has a negative singularity towards -∞ at $x + x_0 = \Gamma(\alpha+1)(-1)^{\alpha} < 0$. For k odd there is also a singularity at $x + x_0 > 0$. If k is even and β is odd (the singularity is pushed towards imaginary x), or if $0 < \alpha < 1$, the singularity is eliminated and the solution becomes semi-bounded, like in the particular situations investigated in the article [9]. In this case, OSA provides again the correct relations, since we obtaine the special behavior of the solution if the velocity is proportional to the power $\beta - 1$ of the amplitude A . Also, we predict the space scale of these semi-compact pulses, namely the length $L = \frac{\mu k^2 A^{k-1}}{V \pm a\beta A^{m-1}}$. The OSA analysis can be applied in the case of sine-Gordon equation, fifth row of Table 2. The solutions with the velocity proportional with L^2 are characterized through the OSA approach by a transcendental equation in A, identical with the equation fulfilled by the amplitude A of the exact sine-Gordon soliton. In the sixth row, we present the cubic nonlinear Schrodinger equation (NLS3) which has a soliton solution. In the sixth row of Table 2 we present the NLS3 equation together with its one soliton solution of amplitude η_0

, obtained by the inverse scattering method. In the last column we also show the relation between the parameters of a localized solution, obtained by OSA. The equation for $L(A,V)$ is more general than that one fulfilled by the soliton, and hence is related to more general localized solutions. By choosing the velocity proportional to the amplitude, we reobtain the $L \sim 1/A = 1/\eta_0 \sim 1/V$ typical relations for the soliton given in the second column.

$$-\frac{\hbar^2}{2m}\Psi_{xx} + (E - V)\Psi + a\Psi^3 = 0, \tag{14}$$

then the L parameter gives an estimation for the wavelength of the wavefunctions, or for the correlation length in a Bose model

$$L = \frac{\hbar}{\sqrt{2m(E-V) + aA^2}} \simeq \frac{\hbar}{\sqrt{2m(E-V)}} \text{for small } A. \tag{15}$$

For the general case of a NLS equation of order n (seventh row), where a general analytical solution is unknown, the method predicts a special $L = L(A,V)$ dependence, shown in the third column. Contrary to third order NLS, where the dependence of L with A is monotonous for $V =\sim \pm A$ $(n = 3)$, at higher orders than 3, the $L(A)$ function has discontinuities in the first derivative. This wiggle of the function $n = 4$ hold at a critical width, possibly producing bifurcations in the solutions and scales. As a consequence, initial data close to this width can split into doublet (or even triplet, for higher order NLS) solutions, with different amplitudes. Such phenomena have been put into evidence in several numerical experiments for quintic nonlinear equations [8-10]. The final example of Table 2 is provided by the Gross-Pitaevski (GP) mean field equation, which is used to describe the dilute Bose condensate [11]. The scalar field (or order parameter) governed by this equation was shown to behave in a particle manner, too, since it can contain topological deffects, namely dark solitons. The space scale L of such solutions is important, for both the theory and experiment, since is related to the trap dimensions and to the scattering length. In the last row of the Table 2 we give one particular solution of a simplified one-dimensional version of the GP equation [12]

$$i\hbar\frac{\partial\Psi(\overrightarrow{x},t)}{\partial t} = \left(\frac{-\hbar^2}{2m}\triangle + V_{ext}(\overrightarrow{x}) + \frac{4\pi\hbar^2 a}{m}\Psi(\overrightarrow{x},t)^2\right)\Psi(\overrightarrow{x},t), \tag{16}$$

where a is the s-wave scattering length and V_{ext} is the confining potential. In the solution provided in the table, the half-width of the exact nonstationary solution is $L = 1/\sqrt{\nu_c^2 - p^2}$, where $v_c = \sqrt{1 - aV}$ is the Landau critical velocity, and $p = dq(t)/dt$ is the momentum associated with the motion of this disturbance. It is easy to check that the OSA provides a good match with this exact solution, and also L fits the correlation length $l_0 = \sqrt{\frac{m}{4\pi\hbar^2 a}}$. We stress that such estimation of the length is also important in nuclear physics where one can explain the fragmentation process as a bosonization in α-particles, inside the nucleus. Such systems are coherent if the wavelength associated with the cluster (the resulting

L in the GP equation) is comparable with the distance between the α -clusters.

$$u_t + au_x^m + \mu u_{xx}^k + cu_{xxx}^n = 0. \tag{17}$$

Here m, k and n are integers and the corresponding terms are responsible of the nonlinear interaction (convective term), dissipation and dispersion [9]. The above equation is related to weakly nonlinear phenomena, and it occurs in modeling porous medium, magma, interfacial phenomena in fluids (and hence applications to drop physics), etc. Thi OSA approach maps this equation into

$$amA^{m-1} - VL^2 - \mu k^2 A^{k-1} L + n^3 A^{n-1} = 0, \tag{18}$$

Table 3, first row. The most symmetric case is obtained when either $V = 0$ (stationary patterns) or $V \sim A^{m-1}$. In this situation the condition to have a monotonous dependence of L as a function of A is $2k = m + n$ which yields a scale structure

$$L = A^{k-m} \left(\frac{\mu k^2 \pm \sqrt{\mu^2 k^4 - 4mn^3(a - V_0)}}{2m(a - V_0)} \right) \sim A^{k-m}, \tag{19}$$

where we put $V = mV_0 A^{m-1}$. The condition $2k = m + n$ is just the condition obtained in [9] from a scaling approach. This condition assures the universality of the corresponding patterns, and it is the unique case in which L depends on a power of A. In the above cited paper, the author finds out the condition for mass invariance at scaling transformations as $m = n + 2 = k + 1$. In our case we just have to request the product AL (which gives a measure of the mass, or volume of the pattern, like in the case of one-dimensional solitons) to be a constant. This gives the condition $k - m + 1 = 0$ which, together with the general invariance condition $2k = m + n$, reproduces $m = n + 2 = k + 1$. In this case we have patterns characterized by a width

$$L = \frac{\mu k^2 \pm \sqrt{\mu^2 k^4 - 4mn^3(a - V_0)}}{2mA(a - V_0)} \rightarrow \frac{n^3}{A\mu k^2}. \tag{20}$$

If $a \sim V_0$ the width approaches $n^3/A\mu k^2$. In order to make L independen of A , like in the compacton case, we need $m = k$, which together with the first invariance condition $2k = m+n$, yields $m = n = k$. This is the exceptional case when the dissipative and dispersive processes have the same scaling, resulting form the invariance of the eq.(12) under the group of scales. Finaly, if we choose $L \sim V$ we obtain the condition $k + 1 = 2m$ which (together with $2k = m + n$) is the condition for spiral symmetry and occurence of similarity structures [9]. The next example is provided by one of the most generalized KdV equation, which is generated from the Lagrangian [5]

$$\mathcal{L}(n, l, m, p) = \int \left[\frac{\phi_x \phi_t}{2} + \alpha \frac{\phi_x^{p+2}}{(p+1)(p+2)} - \beta \phi_x^m \phi_{xx}^2 + \frac{\gamma}{2} \phi_x^n \phi_{xx}^l \phi_{xxx}^2 \right] dx, \tag{21}$$

where α, β and γ are parameters adjusting the relative strentgh of the interactions, and n, l, m, p are integers. For example, for $\gamma = 0$ one re-obtains the K(2,2) equation, and for $\gamma = m = 0, p = 1$ one obtains the KdV equation. The associated Euler-Lagrange equation in the function $\phi_x = u(x,t) \rightarrow u(x - Vt) = u(y)$, reads after one integration

$$\begin{aligned} Vu &= \frac{\alpha}{p+1}u^{p+1} - \beta m u^{m-1}u_y^2 + 2\beta(u^m u_y)_y + \frac{\gamma n}{2}u^{n-1}(u_y)^l(u_{yy})^2 \\ & -\frac{\gamma l}{2}(u^n(u_y)^{l-1}(u_{yy})^2)_y + \gamma(u^n(u_y)^l u_{yy})_{yy} + C, \end{aligned} \quad (22)$$

where C is the integration constant. By using the OSA we obtain the following important result, expressed in the second row of Table 3: The unique case when such an equation allows compact supported traveling solutions is when $m = p = n + r$, $C = 0$ and $V = V_0 A^m$. This result is in full agrement with the variational calculation in [5]. Both eqs.(12) and (16) are rather more qualitative than capable of modeling measurable phenomena. That is why we introduce now a more general model equation, in the form

$$u_t + f(u)_x + g(u)_{xx} + h(u)_{xxx} = 0, \quad (23)$$

where f, g and h are differentiable functions of the the function $u(x,t)$ itself. The OSA approach gives the equation

$$-V + f'(A) + \frac{Ag''(A) + g'(A)}{L} + \frac{A^2h'''(A) + 3Ah''(A) + h'(A)}{L^2}. \quad (24)$$

A general analysis of eq.(17) is difficult, and the best ways are numerical investigations obtained for particular choices of the three functions. We confine ourselves here only to show that the class of solutions which have similarity properties are those for which $V = V_0 f'(A)$. In this case eq.(18) can be reduced to

$$L^2 f'(1 - V_0) + L(Ag'' + g') + A^2h''' + 3Ah'' + h' = 0, \quad (25)$$

case which is presented in the third row of Table 3. This last relation can be used for different purposes. For example, given a certain type of dispersion and difusion (g, h fixed), we can estimate for what types of nonlinearity (f) the width L will have a given dependence with A . Or, if we know for instance $f(u) = f_0 u^{q_1}$ and $h(u) = h_0 u^{q_2}$, we can ask what type of diffusion g we need, to have constant scale (width) of the patterns (waves), no matter of the magnitude of the amplitude A . In other words, which is the compatible diffusion term, for given nonlinearity-dispersion terms, which provides fixed scale solutions. The result is obtained by integration eq.(19) with respect to $g(u)$

$$g(u) = -\frac{h_0}{L}\left(1 + q^2 + \frac{1}{q^2 - 1}\right)u^{q_2} - \frac{Lf_0(1 - V_0)}{q_1 - 1}u^{q_1} + C_3 \text{Log } u + C_{4,} \quad (26)$$

where $C_{3,4}$ are constants of integration. In a similar way one can check the existence of different other configurations by solving eq.(19), or more general, eq.(18). A last application of this method, occurs if the KdV equation has an additional term depending on the square of the curvature

$$u_t + u_x + u_{xxx} + \epsilon \left(u_{xx}^2\right)_x = 0. \tag{27}$$

This is the case for extremely sharp surfaces (surface waves in solids or granular materials) when the hydrodynamic surface pressure cannot be linearized in curvature. Such a new term yields a new type of localized solution fulfilling the relations

$$L = \sqrt{\frac{4\epsilon A}{-1 \pm \sqrt{1 - 8\epsilon A(A \pm V)}}}. \tag{28}$$

If we look for a constant half-width solution (compacton of $1/L = \alpha$) we need a dependence of velocity of the form $V = (1 + \alpha^2 \epsilon/8)A + 1/8\epsilon A + \alpha/4$. There are many new effects in this situation. The non-monoton dependence of the speed on A introduces again bifurcations of a unique pulse in dublets and triplets. Also, there is an upper bound for the amplitude at some critical values of the width. Pulses narrower than this critical width drop to zero. Such bumps can exist in pairs of identical amplitude at different widths. They may be related with the recent observed "oscillations" in granular materials [7]. The examples presented in Tables 1-3 prove that the above method provides a reliable criterion for finding compact suported solutions.

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Table 1: Traveling localized solutions for nonlinear dispersive equations.

NPDE	Analytic solution and the relations among parameters	OSA approach
$u_t + 6uu_x + u_{xxx} = 0$	$A \text{ sech}^2 \frac{x-Vt}{L}$; $L = \sqrt{2/A}$, $V = 2A$	$L = \lvert V \pm 6A \rvert^{-1/2}$ If $V \sim A$, $L \sim A^{-1/2}$
$u_t + u^2 u_x + u_{xxx} = 0$	$A \text{ sech } \frac{x-Vt}{L}$; $L = 1/A$, $A = \sqrt{V}$	$L = \lvert V \pm 6A^2 \rvert^{-1/2}$ If $V \sim A^2, L \sim A^{-1}$
$u_t + (u^2)_x + (u^2)_{xxx} = 0$	$A\cos^2 \frac{x-Vt}{L}$, if $\lvert (x - Vt)/4 \rvert \leq \pi/2$; L=4	$L = \left(\frac{8A}{\lvert V \pm 2A \rvert} \right)^{1/2}$
$u_t + (u^n)_x + (u^n)_{xxx} = 0$	$\left[Acos^2 \left(\frac{x-Vt}{L} \right) \right]^{\frac{1}{n-1}}$, if $\lvert x - Vt \rvert \leq \frac{2n\pi}{n-1}$ and 0 else; $L = \frac{4n}{(n-1)}$, $A = \frac{2Vn}{n+1}$	$L = \left(\frac{n(n^2+1)}{\alpha \pm n} \right)^{1/2}$ if $V = \alpha A^{n-1}$
$u_t + (u^n)_x + (u^m)_{xxx} = 0$ $n \neq m$	unknown in general	$L = \left(\frac{n(n^2+1)A^{n-1}}{V \pm mA^{m-1}} \right)^{1/2}$

Table 2: Traveling localized solutions for nonlinear diffusive equations.

NPDE	Analytic solution and the relations among parameters	OSA approach
$u_{xx} - \frac{1}{c^2}u_{tt} = 0$	$\sum C_k e^{i(kx \pm \omega t)}$; $k^2 = \omega^2/c^2$	$V = c$ $A,\ L$ arbitrary
$u_t + uu_x - u_{xx} = 0$	$\sqrt{C - V^2}\tan(\sqrt{C - V^2}\frac{x - Vt}{2} + D)$ $+V$	$L = (A \pm V)^{-1}$ If $V \sim A$, $L \sim 1/A$
$u_t + a(u^m)_x - \mu(u^k)_{xx} + cu^\gamma$ $=0$	only particular cases known	$cA^\gamma L^2 + (V \pm amA^{m-1})L$ $\pm \mu k^2 A^{k-1} = 0$
$u_t + a(u^m)_x - \mu(u^k)_{xx} = 0$	$-\mathcal{A}z^\alpha\ {}_1F_1(\alpha, \alpha+1, z) = x + x_0$	$L = \frac{\mu k^2}{am - \alpha}A^{k-m}$, if $V = \alpha A^{m-1}$
$u_{xt} - \sin u = 0$	$A\ \tan^{-1}\gamma\ e^{\frac{x - Vt}{L}}$	$\pm\frac{VA}{L^2} = sinA$ If $V = L^2, A = sinA$
$i\Psi_t + \Psi_{xx} + 2\|\Psi\|^2\Psi = 0$	$\eta_0 e^{i(\omega t + kx)} sech[\eta_0(x - Vt)]$; $L = 1/\eta_0$	$L = \frac{\pm V \pm \sqrt{\|V^2 - 4A^2\|}}{2A^2}$ If $A \sim V$, $L = 1/A$
$i\Psi_t + \Psi_{xx} + \|\Psi\|^{n-1}\Psi = 0$	unknown in general	$L = \frac{\pm V \pm \sqrt{\|V^2 - 4A^n\|}}{2A^n}$
$i\Psi_t = -\frac{1}{2}\triangle\Psi$ $+[a\|\Psi\|^2 + V(x) - 1]\Psi$	$ip + \sqrt{v_c^2 - p^2}\times$ $\tanh[a\sqrt{v_c^2 - p^2}(x - q(t))]$	$L = (aA^2 \pm V - 1)^{-1/2}$ If $V \sim \pm 1, L \sim 1/(A\sqrt{a})$

Table 3: Traveling localized solutions for dissipative-dispersive equations.

The NPDE equation	OSA approach
$u_t + a(u^m)_x + b(u^k)_{xx} + c(u^n)_{xxx} = 0;$	$L = A^{m-k} \cdot \frac{\mu k^2 \pm \sqrt{\mu^2 k^4 - 4mn^3(a-V_0)}}{2m(a-V_0)}$ if $V = mV_0 A^{k-1}$
$Vu = \frac{\alpha}{p+1} u^{p+1} - \beta m u^{m-1} (u_y)^2 + 2\beta(u^m u_y)_y$ $+ \frac{\gamma n}{2} u^{n-1} (u_y)^l (u_{yy})^2 - \frac{\gamma l}{2} (u^n (u_y)^{l-1} (u_{yy})^2)_y$ $+ \gamma (u^n (u_y)^l u_{yy})_{yy} + C$	$2L^{l+4}((n+l+1)V_0 - \alpha)$ $-2L^{l+2}(l+n+1)(l+n+2)\beta$ $-(l+n+1)(2+2n^2+3l+l^2+n(5+3l))\gamma$ if $C = 0$, $V = V_0 A^m$ and $m = p = n + l$
$u_t + f(u)_x + g(u)_{xx} + h(u)_{xxx} = 0$	$L = -\Bigg[g' + Ag'' \mp \Big((Ag'' + g')^2$ $-4f'(1-V_0)(A^2 h''' + 3Ah'' + h') \Big)^{1/2} \Bigg]$ $\times (2A^2 h''' + 6Ah'' + 2h')^{-1}$ if $V = V_0 f'(A)$