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H. G. Ganev

Institute for Nuclear Research and Nuclear Energy Bulgarian Academy of Sciences

A. I. Georgieva

Institute for Nuclear Research and Nuclear Energy Bulgarian Academy of Sciences

J. P. Draayer

Louisiana State University

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Six-dimensional Davidson potential as a dynamical symmetry of the symplectic Interacting Vector Boson Model

H. G. Ganev¹, A. I. Georgieva^{1,2}, and J. P. Draayer²

¹*Institute for Nuclear Research and Nuclear Energy,*

Bulgarian Academy of Science, Sofia 1784, Bulgaria and

²*Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA 70803*

A six-dimensional Davidson potential, introduced within the framework of the Interacting Vector Boson Model (IVBM), is used to describe nuclei that exhibit transitional spectra between the purely rotational and vibrational limits of the theory. The results are shown to relate to a new dynamical symmetry that starts with the $Sp(12, R) \supset SU(1, 1) \times SO(6)$ reduction. Exact solutions for the eigenstates of the model Hamiltonian in the basis defined by a convenient subgroup chain of $SO(6)$ are obtained. A comparison of the theoretical results with experimental data for heavy nuclei with transitional spectra illustrates the applicability of the theory.

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I. INTRODUCTION

The interaction between competing collective modes in atomic nuclei is very important in determining their structure. The collective modes that manifest themselves most strongly [1] are rotations and vibrations. These modes are characterized by very specific energy level spacings and electromagnetic transition strengths. There are various models that give exact algebraic solutions in these limits, one being the interacting boson model (IBM) [2] that contains both as special symmetry limits of the overarching theory. Nonetheless, systems that exhibit a strongly mixed rotational-vibrational character are neither easy to model nor understand, even within an IBM-type algebraic picture that claims exact analytic results in each of these symmetry limits.

The desire to have an algebraically solvable theory that can describe systems with rotational-vibrational interactions has led nuclear physicist to consider the Davidson potential [3], which has known algebraic solutions when applied to diatomic molecules. In an algebraic approach for either the nuclear many-body problem or the Bohr-Mottelson collective model, the addition of the Davidson potential to the Hamiltonian requires the consideration of a dynamical subgroup chain that contains the direct product $Sp(2, R) \otimes SO(n) \subset Sp(2n, R)$, with $n = 3$ and 5 , respectively. A short overview of these cases and their application in nuclear physics is given in Section 2. In Section 3 the more general case of a 6-dimensional Davidson potential is considered within the framework of the phenomenological interacting vector boson model (IVBM) [4]. The latter has $Sp(12, R)$ as its dynamical symmetry group. This model has been applied successfully to a description of various collective phenomena [5, 6, 7]. In the present paper, a new reduction chain of the dynamical group $Sp(12, R)$ through the direct product subgroup $Sp(2, R) \otimes SO(6)$ is reported. As shown in Section 4, this innovation extends the applicability of the theory to include rotational-vibrational interactions.

In short, the present study leads to a better understanding of and provides motivation for an algebraic

IVBM for two interacting many-particle systems. The two systems of interest for nuclei are comprised of protons and neutrons. The application of this dynamical symmetry to nuclei confirms the ability of the Davidson potential to reproduce nuclear rotational-vibrational behavior found in nature.

II. ALGEBRAIC APPROACHES IN APPLICATIONS OF THE DAVIDSON POTENTIAL TO NUCLEAR STRUCTURE

A. The Davidson potential

The need for a description of nuclei in which rotational-vibrational interactions dominate has led to a search for algebraically solvable potentials and a meaningful set of basis states that make the transitional nature of these systems more transparent.

Davidson proposed such a potential,

$$V(r) = \chi(r^2 + \frac{\varepsilon}{r^2}), \quad (1)$$

for diatomic molecules [3]. The Hamiltonian (including its kinetic part) for a system with a strong rotational-vibrational interaction in harmonic oscillator units takes the following form

$$H_\varepsilon = \frac{1}{2} \hbar \omega^2 (-\nabla^2 + r^2 + \frac{\varepsilon}{r^2}). \quad (2)$$

Both ∇^2 and r^2 are $SO(3)$ scalars, and hence H itself is a $SO(3)$ invariant. On the other hand, H can be expressed in terms of the $SU(1, 1)$ generators defined in the following way [8]

$$Z_1 = -\nabla^2 + \frac{\varepsilon}{r^2}, \quad Z_2 = r^2, \quad Z_3 = \frac{1}{2}(r \cdot \nabla + \nabla \cdot r).$$

This means the eigenstates of the system can be classified according to the direct product $SU(1, 1) \times SO(3)$. Using the latter, algebraic solutions (eigenvalues and wavefunctions) can be obtained. Indeed, as shown in [8],

the many-body system with the Davidson potential has $SU(1,1) \otimes SO(3)$ as its spectrum generating algebra.

It is well known in nuclear physics that the most successful description of rotations and vibrations are obtained within the framework of the Bohr-Mottelson collective model (CM) [9], which in many respect is the geometrical equivalent of the IBM [2]. It is a liquid-drop model with 5 collective coordinates ($\nu = 0, \pm 1, \pm 2$)

$$q_\nu = \beta \cos \gamma \mathcal{D}_{0\nu}^2(\Omega) + \frac{1}{\sqrt{2}} \beta \sin \gamma (\mathcal{D}_{2\nu}^2(\Omega) + \mathcal{D}_{-2\nu}^2(\Omega)), \quad (3)$$

and canonical momenta $\{p_\nu = -i\hbar\partial/\partial q_\nu\}$, expressed in terms of intrinsic (β, γ) coordinates and rotational angles Ω of $SO(3)$. The model Hamiltonian

$$H_0 = \frac{1}{2B} p \cdot p + \frac{1}{2} B \omega^2 q \cdot q,$$

gives a harmonic vibrational spectrum and with the addition of the quadrupole Davidson potential

$$H_\varepsilon = \frac{1}{2B} p \cdot p + \frac{1}{2} B \omega^2 \left(q \cdot q + \frac{\varepsilon}{q \cdot q} \right), \quad (4)$$

yields a rotational-vibrational spectrum, characteristic of the so-called γ -soft Wilets-Jean rotor [10]. This limit of the model corresponds to the $O(6)$ -symmetry limit of the IBM [2]. The potentials used in these limits are independent of γ (and Ω) and so are functions only of the variable $\beta^2 = q \cdot q$, where both p^2 and β^2 are scalar products of 5-dimensional vectors, so they are $SO(5)$ invariants. Another nice feature of H_ε (4), is that it has $SU(1,1)$ dynamical symmetry. The energy spectrum for a 5-dimensional collective Hamiltonian with a Davidson potential was given by Elliott *et al* [11].

An orthonormal basis for the CM with the Davidson interaction is given [12] by a set of states $\{|n\nu\alpha LM\rangle\}$ that is labelled by the quantum numbers of the subgroup chain

$$\begin{array}{ccccccc} SU(1,1) & \otimes & SO(5) & \supset & U(1) & \otimes & SO(3) & \supset & SO(2) \\ n & & v & & \alpha & & n & & L & & M \end{array}, \quad (5)$$

where n labels the lowest weight $SU(1,1)$ state, v is the highest weight $SO(5)$ irreducible representation (irrep) label, and α is a running index that is used to distinguish multiple occurrences of an $SO(3)$ irrep that is labelled by the angular momentum L and its projection M .

Just as there is a correspondence between the description of the geometrical collective model and the IBM for the γ -unstable transitional region, an exact solution has also been obtained for the $O(6) \leftrightarrow U(5)$ transitional region using an infinite-dimensional algebraic technique [13]. In [13], the complementarity of the $U(5) \supset SO(5)$ and $SO(6) \supset SO(5)$ bases to the $SU^d(1,1) \supset U(1)$ and $SU^{sd}(1,1) \supset U(1)$, respectively, with corresponding relations between the Casimir invariants and quantum numbers labeling representations in the corresponding chains,

was used to obtain analytical results for the energy eigenvalues and eigenstates of isotopic chains for transitional nuclei lying between the γ -soft and vibrational limits of this theory.

B. Collective behavior of many-body systems and symplectic geometry

Notwithstanding these considerations, a microscopic description of the collective behavior of a many-particle system in 3-dimensional space requires higher symmetries. Specifically, such a complicated system is usually characterized by an irrep of $Sp(6n, R)$ and its $Sp(6, R) \otimes O(n)$ subgroup, where $n = A - 1$ and A is equal to the total number of nucleons in the system. Collective effects emerge within this structure when the system is constrained to a specific irrep of $O(n)$ which in turn determines the $Sp(6, R)$ irrep [14]. When this is done, the Hamiltonian falls within the enveloping algebra of $Sp(6, R)$ rather than $Sp(6n, R)$.

The coordinates x_{is} and momenta p_{is} of the n particle system $s = 1, 2, \dots, n$ in a 3-dimensional space $i = 1, 2, 3$ defined by the only nonzero commutator

$$[x_{is}, p_{jt}] = i\delta_{ij}\delta_{st}, \quad i, j = 1, 2, 3; \quad s, t = 1, 2, \dots, n \quad (6)$$

are the elements of a $3n$ -dimensional Weyl Lie algebra $W(3n)$. The Hermitian quadratic expressions in the coordinates and momenta

$$x_{is}x_{jt}, \quad x_{is}p_{jt} + p_{jt}x_{is}, \quad p_{is}p_{jt}, \quad (7)$$

that close under commutation [15] yield the $3n(2 \times 3n + 1)$ generators of the $Sp(6n, R)$ symplectic group.

The generators of the subgroups in the reduction $Sp(6, R) \times O(n)$ of $Sp(6n, R)$ can be obtained from (7) by means of contractions with respect to the indices s and i , respectively. For the $O(n)$ group the infinitesimal operators have the [16] well-known antisymmetrized form

$$L_{st} = \sum_{i=1}^3 (x_{is}p_{it} - x_{it}p_{is}). \quad (8)$$

For $Sp(6, R)$ there are $3(2 \times 3 + 1) = 21$ Hermitian generators [15],[17], [18] of the form

$$q_{ij} = \sum_{s=1}^n (x_{is}x_{js}), \quad (9)$$

$$S_{ij} = \frac{1}{2} \sum_{s=1}^n (x_{is}p_{js} + x_{js}p_{is} + p_{js}x_{is} + p_{is}x_{js}), \quad (10)$$

$$T_{ij} = \sum_{s=1}^n (p_{is}p_{js}), \quad (11)$$

$$L_{ij} = \sum_{s=1}^n (x_{is}p_{js} - x_{js}p_{is}). \quad (12)$$

The L_{ij} generate the $SO(3)$ subgroup of $Sp(6, R)$ (12). As we are dealing only with the irreps $[\frac{1}{2}^6]$ or $[\frac{1}{2}^5 \frac{3}{2}]$ of $Sp(6, R)$, the irreps of $Sp(6, R)$ and $O(n)$ are complementary [19], i.e., if we fix the irrep of $O(n)$ that of $Sp(6, R)$ is specified or vice versa. From shell-model considerations it has been shown [20] that fixing the $O(n)$ irrep isolates the collective effects [21]. Thus $Sp(6, R)$ can be used to solve the collective part of the many-body problem.

A complete set of states is provided by the eigenstates of the Hamiltonian

$$H_0 = \sum_{s=1}^n \sum_{i=1}^3 H_{is}, \quad H_{is} = \frac{1}{2}(x_{is}^2 + p_{is}^2), \quad (13)$$

where the operators H_{is} are special combinations of the generators of $Sp(6, R)$ that commute among themselves, i.e. $[H_{is}, H_{jt}] = 0$ and therefore are the weight generators [16] of this group. Furthermore, in this basis we can calculate the matrix elements of all the generators of the $Sp(6, R)$ group. This means that an arbitrary Hamiltonian H involving central forces is in the enveloping algebra of $Sp(6, R)$ and can be written as a function of the quadratic expressions in (7) which are invariant under space reflections. This also applies to other integrals of motion such as the square of components of the total angular momentum (12) and functions thereof. Subgroups of the dynamical group $Sp(6, R)$ can be used to further classify the eigenstates of H_0 .

The challenge is to define a basis in terms of irrep labels of groups in a physically meaningful chain of subgroups of $Sp(6, R)$. The basis characterized by the chain $Sp(6, R) \supset U(3) \supset SO(3)$, where $U(3)$ is the group of the quadrupole momentum introduced by Elliott, is well known and obtained in both an abstract way [18, 22] as well as in terms of shell-model states [23]. Many successful applications of this theory have been made for deformed nuclear systems.

Another relevant chain for developing collective basis states is $Sp(6, R) \supset Sp(2, R) \otimes SO(3)$ that was considered by Moshinsky and his collaborators in an effort to obtain a simple description of vibrational collective nuclear motion [14]. Indeed if the local isomorphism of the $sp(2, R) \approx su(1, 1)$ algebras is taken into account its relation to the spectrum generating algebra of the many body nuclear system with the Davidson interaction becomes explicit. This provides motivation for considering this reduction in seeking a description of a more complex modes that includes rotational-vibrational interactions.

III. NEW DYNAMICAL SYMMETRY IN THE IVBM

A. Group theoretical background of the model

On the basis of the above considerations, namely the use of a symplectic geometry in investigating nuclear collective motion, a further elaboration of the problem can be achieved if we consider the nuclear many-body system as consisting of interacting proton and neutron subsystems. This leads to the phenomenological IVBM [4], where $Sp(12, R)$ – the group of linear canonical transformation in a 12-dimensional phase space [19] – appears as the dynamical symmetry of the model. The $sp(12, R)$ algebra is realized in terms of creation (annihilation) operators $u_m^\pm(\alpha)(u_m(\alpha))$, in a 3-dimensional oscillator potential $m = 0, \pm 1$ of two types of bosons differing by the value of the “pseudo-spin” projection $\alpha = p = 1/2$ and $\alpha = n = -1/2$. These are related with the cyclic coordinates $x_{\pm 1}(\alpha) = \mp \frac{1}{\sqrt{2}}(x_1(\alpha) \pm ix_2(\alpha))$, $x_0(\alpha) = x_3(\alpha)$ and their associated momenta $q_m(\alpha) = -i\partial/\partial x^m(\alpha)$ in the standard way

$$u_m^\pm(\alpha) = \frac{1}{\sqrt{2}}(x_m(\alpha) - iq_m(\alpha)), \quad (14)$$

$$u_m(\alpha) = (u_m^\pm(\alpha))^\dagger,$$

where $x_i(\alpha)$ $i = 1, 2, 3$ like in (6) are the Cartesian coordinates of a quasi-particle vectors with an additional index, namely the projection of the “pseudo-spin” $\alpha = \pm \frac{1}{2}$. The bilinear products of the creation and annihilation operators of the two vector bosons (14) generate the boson representations of the non-compact symplectic group $Sp(12, R)$ [4]:

$$F_M^L(\alpha, \beta) = \sum_{k,m} C_{1k1m}^{LM} u_k^+(\alpha) u_m^+(\beta),$$

$$G_M^L(\alpha, \beta) = \sum_{k,m} C_{1k1m}^{LM} u_k(\alpha) u_m(\beta), \quad (15)$$

$$A_M^L(\alpha, \beta) = \sum_{k,m} C_{1k1m}^{LM} u_k^+(\alpha) u_m(\beta), \quad (16)$$

where C_{1k1m}^{LM} are the usual Clebsch-Gordon coefficients and $L = 0, 1, 2$ and $M = -L, -L+1, \dots, L$ define the transformation properties of (15) and (16) under rotations. The commutation relations between the pair creation and annihilation operators (15) and the number preserving operators (16) are calculated in [4]. The set of operators (16) close under commutation to form the algebra of the maximal compact subgroup of $Sp(12, R) \supset U(6)$. The linear invariant of $U(6)$ is the number operator,

$$N = \sqrt{3}(A^0(p, p) + A^0(n, n)) = N_+ + N_-, \quad (17)$$

that counts the total number of bosons. The action space of the operators (15) and (16) is in general reducible and

the invariant operator $(-1)^N$ decomposes it into even H_+ with $N = 0, 2, 4, \dots$, and odd H_- with $N = 1, 3, 5, \dots$, subspaces, so in the reduction $Sp(12, R) \supset U(6)$ both the even and odd irreps of the $Sp(12, R)$ decompose into an infinite sum of finite fully symmetric irreps of $U(6)$, $[N]_6 = [N, 0^5]_6$, where N is the eigenvalue of the operator (17).

B. Reduction through the non-compact $Sp(2, R)$

In order to relate the IVBM to the 6-dimensional Davidson potential, we introduce another reduction of the $Sp(12, R)$ group through its non-compact subgroup [14], [19], [20]:

$$Sp(12, R) \supset Sp(2, R) \otimes SO(6). \quad (18)$$

As can be deduced from the considerations given above, this construction obviously survives the addition of Davidson potential. The infinitesimal generators of the $Sp(2, R)$ algebra

$$F = \sum_{k,m,\alpha} C_{1k1m}^{00} u_k^+(\alpha) u_m^+(\alpha) = 2S^+,$$

$$G = \sum_{k,m,\alpha} C_{1k1m}^{00} u_k(\alpha) u_m(\alpha) = 2S^-, \quad (19)$$

$$A = \sum_{k,m,\alpha} C_{1k1m}^{00} u_k^+(\alpha) u_m(\alpha) = \frac{1}{\sqrt{3}} N = 2S^0 - 1,$$

are obtained from the $Sp(12, R)$ generators (15) and (16) by means of contraction with respect to both the spatial $m = 0, \pm 1$ and the "pseudospin" $\alpha = p = 1/2, \alpha = n = -1/2$ indices. It is straightforward to show that the operators $S^\tau, \tau = 0, \pm$ commute in a standard way for the $SU(1, 1)$ algebra generators [13]

$$[S^0, S^\pm] = S^\pm, [S^+, S^-] = -2S^0,$$

so the $sp(2, R)$ and the $su(1, 1)$ algebras are locally isomorphic with a Casimir operator written as $C_2(SU(1, 1)) = S^0(S^0 - 1) - S^+S^-$.

By construction, the operators (19) are scalars with respect to 6-dimensional rotations and they commute with the components of the 6-dimensional momentum operators [4],

$$\Lambda_M^L(\alpha, \beta) = A_M^L(\alpha, \beta) - (-1)^L A_M^L(\beta, \alpha), \quad (20)$$

which obey the property

$$\Lambda_M^L(\alpha, \beta) = (-1)^L \Lambda_M^L(\beta, \alpha)$$

and generate the $SO(6)$ algebra. When $\alpha = \beta$, from (20) one obtains

$$\Lambda_M^L(\alpha, \alpha) = A_M^L(\alpha, \alpha) - (-1)^L A_M^L(\alpha, \alpha),$$

which are different from zero only for $L = 1$. Indeed, the 6 operators $A_M^1(p, p)$ and $A_M^1(n, n)$ are rank one tensors with respect to $O(3)$ and represent, respectively, the angular momenta of the p and n boson systems. In the case when $\alpha \neq \beta$ the operators (20) are the $5 + 3 + 1$ components of tensors of rank 2, 1, and 0, respectively. In this way, the direct product of the two groups (18) is realized. The second order invariant for the $SO(6)$ group is

$$\Lambda^2 = \sum_{L,\alpha,\beta} (-1)^M \Lambda_M^L(\alpha, \beta) \Lambda_{-M}^L(\beta, \alpha), \quad (21)$$

and it is related to the second order invariant of the $Sp(2, R)$, as in the direct product (18) the two groups are complementary [19], which means that the irreps of the group $SO(6)$ determine those of $Sp(2, R) \approx SU(1, 1)$ and vice versa.

C. Labeling of the basis

In order to define the basis of the system with (18) as a dynamical symmetry that allows one to include the 6-dimensional Davidson potential, we consider the reduction of the $SO(6)$ algebra to the $SO(3)$ algebra of the angular momentum through the following chain [4], [24]

$$SO(6) \supset SU(3) \otimes O(2) \supset SO(3) \\ \omega \quad (\lambda, \mu) \quad \nu \quad L, \quad (22)$$

which defines the γ -unstable limit of the IVBM. The single infinitesimal operator of $O(2)$ is proportional to the scalar operator $\Lambda^0(\alpha, \beta)$ from the $SO(6)$ generators (20),

$$M_{\alpha\beta} = -\sqrt{3} \Lambda^0(\alpha, \beta) = -\sqrt{3} [A^0(\alpha, \beta) - A^0(\beta, \alpha)], \quad (23)$$

and the generators of $SU(3)$ [4] are

$$X_M^2 = i(A_M^2(p, n) - A_M^2(n, p)), M = 0, \pm 1, \pm 2,$$

$$Y_M^1 = A_M^1(p, p) + A_M^1(n, n) = -\frac{1}{\sqrt{2}} L_M, M = 0, \pm 1. \quad (24)$$

Note, that in this case the quadrupole moment X (24) is the proton-neutron interaction. The second-order Casimir invariants of the two groups in the direct product in (22) can be written as

$$2C_2(O_2) = M^2 = \sum_{\alpha,\beta} M_{\alpha\beta} M_{\beta\alpha},$$

$$C_2(SU(3)) = \sum_M (-1)^M (X_M X_{-M} + Y_M Y_{-M}).$$

For $SO(6) \subset U(6)$, the symmetric representation $[N]_6$ of $U(6)$ decomposes into fully symmetric $(\omega, 0, 0)_6 \equiv (\omega)_6$ irreps of $SO(6)$ [25] according to the rule

$$[N]_6 = \bigoplus_{\omega=N, N-2, \dots, 0(1)} (\omega, 0, 0)_6 = \bigoplus_{i=0}^{\langle \frac{N}{2} \rangle} (N-2i)_6, \quad (25)$$

where $\langle \frac{N}{2} \rangle = \frac{N}{2}$ if N is even and $\frac{N-1}{2}$ if N is odd. Furthermore, the following relation between the quadratic Casimir operators $C_2(SU(3))$, M^2 of $O(2)$ and Λ^2 (21) of $SO(6)$ holds [26]:

$$\Lambda^2 = 2C_2(SU(3)) - \frac{1}{3}M^2, \quad (26)$$

which means that the reduction from $SO(6)$ to the rotational group $SO(3)$ is carried out through the complementary groups $O(2)$ and $SU(3)$ [19]. As a consequence, the irrep labels $[f_1, f_2, 0]_3$ of $SU(3)$ are determined by $(\omega)_6$ of $SO(6)$ and by the integer label $(\nu)_2$ of the associated irrep of $O(2)$ i.e.

$$(\omega)_6 = \bigoplus [f_1, f_2, 0]_3 \otimes (\nu)_2. \quad (27)$$

Using the relation (26) of the Casimir operators, for their respective eigenvalues one obtains:

$$\omega(\omega+4) = \frac{4}{3}(f_1^2 + f_2^2 - f_1 f_2 + 3f_1) - \frac{\nu^2}{3}. \quad (28)$$

Then if $f_2 = 0$ and $\nu = f_1$, from (28) written as $(f_1 - \omega)(f_1 + \omega + 4) = 0$, it follows that $f_1 = \omega$. If $f_2 = i$ in (28) we obtain the relation

$$\nu = \pm(\omega - 2i), \quad i = 0, 1, \dots, \omega. \quad (29)$$

Hence (27) can be rewritten as

$$(\omega)_6 = \bigoplus_{i=0}^{\omega} [\omega, i, 0]_3 \otimes (\omega - 2i)_2 = \bigoplus_{\nu=\omega, \omega-2, \dots, 0(1)} [\omega, \frac{\omega-\nu}{2}, 0]_3 \otimes (\nu)_2,$$

or in terms of the Elliott's notation (λ, μ)

$$(\omega)_6 = \bigoplus_{\nu=\omega, \omega-2, \dots, 0(1)} \left(\frac{\omega+\nu}{2}, \frac{\omega-\nu}{2} \right) \otimes (\nu)_2. \quad (30)$$

Finally, the convenience of this reduction can be further enhanced through the use of the standard rules for the reduction of the $SU(3) \supset SO(3)$ chain:

$$\begin{aligned} K &= \min(\lambda, \mu), \min(\lambda, \mu) - 2, \dots, 0 \quad (1) \\ L &= \max(\lambda, \mu), \max(\lambda, \mu) - 2, \dots, 0 \quad (1); K = 0 \quad (31) \\ L &= K, K + 1, \dots, K + \max(\lambda, \mu); K \neq 0. \end{aligned}$$

The latter is the usual reduction of an irrep (λ, μ) of $SU(3)$ into irreps L of $SO(3)$ where the multiplicity number K is used to label the rotational bands in the energy spectra of the system.

D. A 6-dimensional Davidson potential and its algebraic structure

As discussed in Sec.2, basis states of the collective model are classified according to irrep labels (quantum numbers) of the reductions [8], [27]

$$SU(1,1) \otimes SO(5) \supset U(1) \otimes SO(5) \supset SO(3).$$

For spherical nuclei, $SU(1,1)$ is realized when $U(1) \otimes SO(5)$ is a subgroup of $U(5)$; on the other hand, for deformed nuclei a transformation of $SU(1,1)$ gives eigenstates of the Davidson potential. With the extension of the collective model to six dimensions, the states can be classified by the quantum numbers [27] provided by the chain

$$U(3) \supset U(1) \otimes SU(3) \supset SO(3)$$

which are eigenstates of the spherical vibrational Hamiltonian

$$H_0 = h\omega N.$$

The $U(1) \otimes SU(3)$ subalgebra, generating a rotational type of spectrum, appears also in the chain

$$SU(1,1) \otimes SU(3) \supset U(1) \otimes SU(3) \supset SO(3).$$

It should be clear that one can make the same transformation of $SU(1,1)$ as for the 5-dimensional collective model, and in so doing obtain basis states which are eigenstates of a 6-dimensional analog of the Davidson potential. The former reduction chain is contained naturally in the group-theoretical structure of the Interacting Vector Boson Model [14], [19], [20]:

$$\begin{array}{l} Sp(12, R) \supset Sp(2, R) \otimes SO(6) \\ \quad \quad \quad N \quad \quad \quad \omega \\ \quad \quad \quad \Downarrow \quad \quad \quad \cup \\ \quad \quad \quad SU(1,1) \otimes SU(3) \otimes O(2) \supset SO(3), \\ \quad \quad \quad N \quad \quad \quad (\lambda, \mu) \quad \quad \nu \quad K \quad L \end{array} \quad (32)$$

where below the different subgroups the quantum numbers which characterize their irreducible representations are shown. In the last line of (32) we take into account the fact that the group $Sp(2, R)$ is locally isomorphic to the group $SU(1,1)$. The basis, labeled by the quantum numbers classified by the group-subgroup chain (32), can be written as

$$|N\omega; (\lambda, \mu)\nu; K, L\rangle \quad (33)$$

where the reduction rules for obtaining specific values for each state are the same as given earlier. By means of these labels, the basis states can be classified in each of the two irreducible even H_+ with $N = 0, 2, 4, \dots$, and odd H_- with $N = 1, 3, 5, \dots$ representations of $Sp(12, R)$. We illustrate this in Table 1 for the even H_+ irreducible representation, where N with the set of ω contained in it

(25) label the rows and the values of the quantum number ν (29) label the columns. The $SU(3)$ quantum numbers (λ, μ) define the cells of the Table 1 as they are obtained with the help of ω and ν (30).

The Hamiltonian with the dynamical symmetry corresponding to the chain (32) is expressed in terms of the first and second order Casimir operators of the different subgroups in it:

$$H = aN + bN^2 + \alpha_6\Lambda^2 + \alpha_2M^2 + \beta_3L^2. \quad (34)$$

and it is obviously diagonal in the basis (33) labeled by

the quantum numbers of their representations. The second order invariant of $SU(3)$ is dropped in (34), because of its linear dependence on the Casimir operators of the $SO(6)$ and $O(2)$ (26). Then the eigenvalues of the Hamiltonian (34) that yield the spectrum of a system interacting with 6-dimensional Davidson potential are:

$$E(N, \omega, \nu, L) = aN + bN^2 + \alpha_6\omega(\omega + 4) + \alpha_2\nu^2 + \beta_3L(L + 1). \quad (35)$$

Table 1.

N	ω	$\nu \dots$	6	4	2	0	-2	-4	-6	\dots
0	0					(0, 0)				
2	2					(1, 1)				
	0				(2, 0)	(0, 0)	(0, 2)			
4	4					(2, 2)				
	2			(4, 0)	(3, 1)	(1, 1)	(1, 3)	(0, 4)		
	0				(2, 0)	(0, 0)	(0, 2)			
6	6					(3, 3)				
	4		(6, 0)	(5, 1)	(4, 2)	(2, 2)	(2, 4)	(1, 5)	(0, 6)	
	2			(4, 0)	(3, 1)	(1, 1)	(1, 3)	(0, 4)		
	0				(2, 0)	(0, 0)	(0, 2)			
\vdots		\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\ddots	

IV. APPLICATION TO REAL NUCLEI

In applications of this new dynamical symmetry (32) of the IVBM to real nuclear systems that follows, we exploit the “algebraic” definition of yrast states as introduced in [6]. This means that we consider those states with maximal value of the angular momentum L for a given number of bosons N to be yrast. With this definition, the states of the ground band, which are the yrast states of the nucleus, are basis states with $\omega = N$, $\lambda = \mu = \frac{N}{2}$ where $N = 0, 4, 8, \dots$ ($\Delta N = 4$) from the $\nu = 0$ column of Table 1.

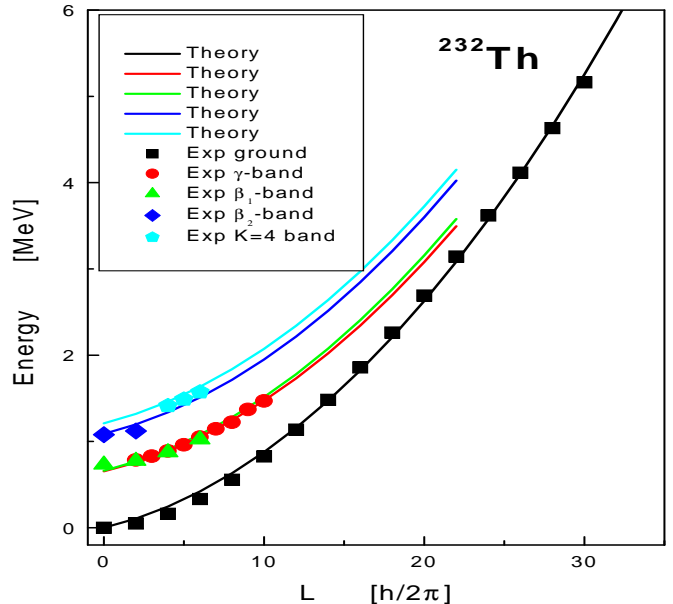
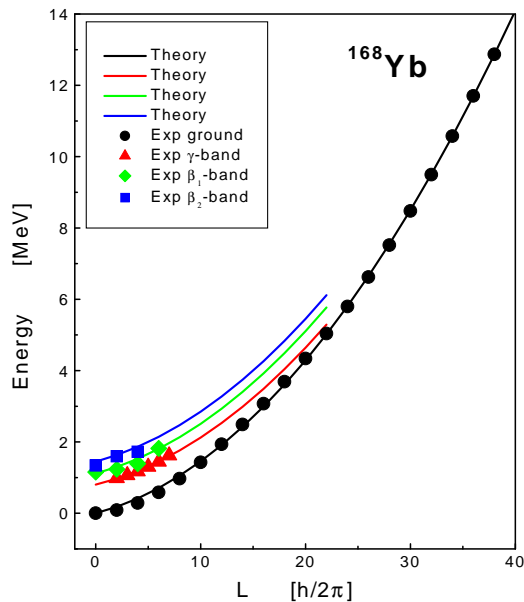
The states of excited bands are not yrast states. The correct placement of the bands in the spectrum strongly depends on their band-heads’ configuration, and in particular, on the number of bosons, $N = N_{\min}$, from which they are built [7]. For the excited bands considered, we choose first a corresponding N_{\min} for the band-head state and then the bands are developed by changing N_{\min} with $\Delta N = 4$ and $\Delta\omega = 2$, so that the lowest L of the band-head is taken from the N_{\min} multiplet and $\Delta L = 2$ for the β_i -bands with $K_i^\pi = 0_i^+$, where i enumerates the 0^+ excited states in the order of increasing energy and $\Delta L = 1$ for $K^\pi = L^+$ ($L \neq 0$, $L = 2, 4, \dots$), as prescribed by the reduction rules (31). The values of the (λ, μ) of the $su(3)$ multiplets to which the excited bands corre-

spond are obtained by fixing the value of the $o(2)$ label ν . For example, the states of first excited β -band and/or γ -band may belong to two different diagonals $(\lambda, \mu = 0)$, $K = 0$ and/or to $(\lambda, \mu = 2)$, $K = 0$ and/or 2, so that $\nu = L$ or/and $\nu = L - 2$ for L -even and $\nu = L - 1$ for L -odd respectively and $\Delta\nu = 2$ for each neighboring $su(3)$ multiplets in a band under consideration. This variety of possible choices for the excited bands allows us to reproduce correctly the behavior of these bands with respect to one another, which can change a lot even in neighboring nuclei because of the mixing of the vibrational and rotational collective modes [28].

From (35) it is obvious that there are 5 free parameters, which we determine by fitting the theoretical predictions for the energies of the ground and few excited bands to the experimental data [1], using a χ^2 -procedure. The parameters that were obtained, the number s of experimental states, χ^2 , and N_{\min} are all given in Table 2 for four different nuclei.

Table 2.

Nucleus	s	N_{\min}	bands	χ^2	parameters
^{168}Yb $R_4 = 3.26$	20	0	ground	0.0047	$a = 0.03493$ $b = 0.00059$
	6	20	γ	0.0006	$\alpha_6 = 0.00055$
	4	20	β_1	0.0073	$\alpha_2 = -0.00245$
	3	24	β_2	0.0125	$\beta_3 = 0.00229$
^{232}Th $R_4 = 3.28$	15	0	ground	0.0050	$a = 0.01986$ $b = 0.00037$
	9	24	γ	0.0002	$\alpha_6 = 0.00035$
	4	20	β_1	0.0022	$\alpha_2 = -0.00096$
	2	28	β_2	0.0054	$\beta_3 = 0.00145$
	3	38	$K = 4$	0.0020	
^{150}Sm $R_4 = 2.32$	11	0	ground	0.0057	$a = 0.09705$ $b = 0.00016$
	5	12	γ	0.0089	$\alpha_6 = 0.00042$
	3	6	β_1	0.0083	$\alpha_2 = 0.00064$
	3	12	β_2	0.0198	$\beta_3 = 0.00123$
^{152}Sm $R_4 = 3.01$	9	0	ground	0.0022	$a = 0.02971$ $b = 0.00057$
	7	24	γ	0.0023	$\alpha_6 = 0.00056$
	7	14	β_1	0.0105	$\alpha_2 = 0.00229$
	3	28	β_2	0.0088	$\beta_3 = 0.00229$

FIG. 2: (Color online) The same as in Fig. 1, but for the nucleus ^{232}Th .FIG. 1: (Color online) Comparison of the theoretical and experimental energies for ground and excited bands of ^{168}Yb .

The first two, ^{168}Yb and ^{232}Th , are rather well deformed nuclei from the rare earth and light actinide region. We have chosen these particular nuclei as they have

rather long spin sequences going up to very high values of the angular momentum in their ground bands. From the experimental ratios $R_4 = \frac{E_4}{E_2}$ of the ground band energies, which is $R_4 = 3.26$ for the ^{168}Yb and $R_4 = 3.28$ for the ^{232}Th , we see that these two nuclei are very close to rotators, especially in their ground state bands. The values of N_{\min} that determine the start of the band-heads for the excited bands considered are rather high. One can see the good agreement between theory and experiment for the ground, first two β and γ bands for ^{168}Yb , and the ground, first two β , γ and $K = 4$ bands for ^{232}Th , respectively, in Figure 1 and Figure 2. The parameters (see Table 2) in both cases are of the same order of magnitude, but for ^{168}Yb they are somewhat larger. In both cases the 0^+ band-head of the β -band is placed below the 2^+ band-head of the γ -band. Additionally, one can clearly see the different degree of degeneracy of the L -even states of the γ and first β bands, which is a typical property for well-deformed nuclei.

Additionally, in Table 2 we give the results for the ^{150}Sm and ^{152}Sm , which are considered to be transitional nuclei. The first one with $R_4 = 2.32$ is transitional between the γ -soft ($R_4 = 2.5$) and vibrational nuclei with ($R_4 = 2.0$) and the second one is a nuclei at the critical point of phase/shape transition [29] with so-called $X(5)$ symmetry. As shown in Figure 3 and 4, the experimental data is reproduced remarkably well, especially for ^{152}Sm where the β and γ bands are well distinguished. The values of N_{\min} varies more for these two nuclei and the

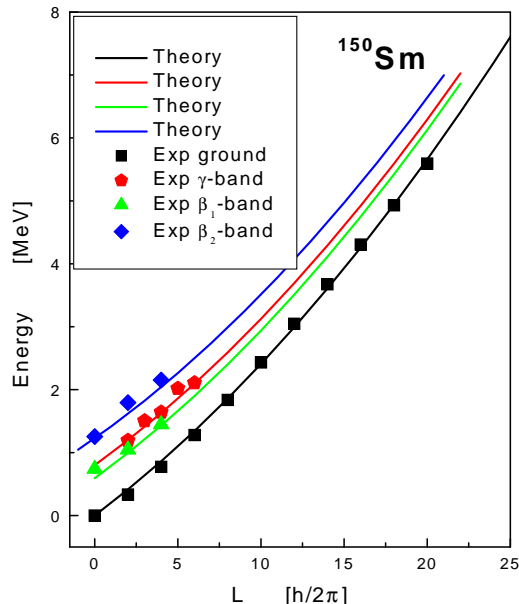


FIG. 3: (Color online) Comparison of the theoretical and experimental energies for ^{150}Sm .

parameter α_2 changes its sign.

The values of χ^2 are rather good (small) for all of the examples considered. This suggests that the model is appropriate for the description of a rather broad range of nuclei, and most importantly nuclei that display mixed rotational and vibrational degrees of freedom.

V. SUMMARY AND CONCLUSIONS

In this work we introduce a new reduction of the dynamical group $Sp(12, R)$ of the algebraic Interacting Vector Boson Model. It is based on the fact, that the rotational-vibrational spectrum of the nuclear system can be generated from a Hamiltonian with a Davidson interaction which has as a spectrum generating algebra the direct product $SU(1, 1) \otimes SO(n)$. The three- and five-dimensional cases ($n = 3$ and 5 , respectively) have already been explored in the nuclear structure physics and are related to the many-body problem in 3-dimensional space and the γ -soft limit of the geometrical Collective Model, respectively. Based on the algebraic approaches to these problems, we introduce an extension of the spectrum generating algebra to $SU(1, 1) \otimes SO(6)$ which includes the 6-dimensional Davidson potential. It is naturally contained in the group of dynamical symmetry $Sp(12, R)$ of the IVBM (32). Further, the reduction of the boson representations of $SU(1, 1) \otimes SO(6)$ to the angular momentum group $SO(3)$ is obtained in order to

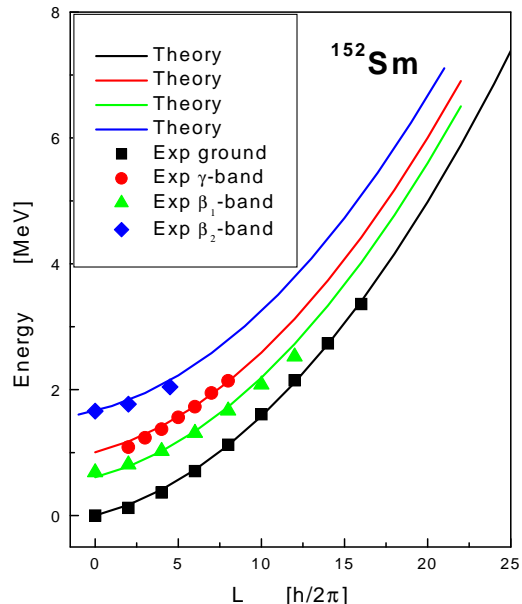


FIG. 4: (Color online) The same as on Fig. 1 for ^{152}Sm .

provide for a complete labeling of the basis states of the system and the model Hamiltonian is written in terms of the first and second order invariants of the groups from the corresponding reduction chain. Hence the problem is exactly solvable within the framework of the IVBM which, in turn, yields a simple and straightforward application to real nuclear systems.

We present results that were obtained through a phenomenological fit of the models' predictions for the spectra of collective states to the experimental data for two nuclei from the rare-earth and actinide major shells exhibiting rotational spectra, as well as two with transitional character between γ -soft and vibrational spectra. The good agreement between the theoretical predictions and the experiment results confirms the applicability of the IVBM to a broad range of nuclei with quite different collective properties. These features could be further developed to study the phase/shape transitions in nuclear systems [30], which of late has been a subject of high interest from a theoretical [31] as well as from experimental [29] point of view.

The most important feature of the model, from a physical point of view, is that it leads to a successful description of different types of nuclear collective spectra as well as mixed-mode results with the proton and neutron substructures and interactions between them taken into account explicitly. This is accomplished within the framework of a symplectic symmetry that allows for a change in the number of bosons of each type.

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- [1] Mitsuo Sakai, Atomic Data and Nuclear Data Tables 31 (1984), 399; Level Retrieval Parameters, <http://iaeand.iaea.or.at/nudat/levform.html>
- [2] F. Iachello, and A. Arima, *The Interacting Boson Model*, Cambridge University Press, Cambridge, (1987).
- [3] P. M. Davidson, Proc. Roy. Soc. **A135**, 459 (1932).
- [4] A. Georgieva, P. Raychev and R. Roussev, J. Phys. G: Nucl. Phys., **8**, 1377 (1982).
- [5] A. Georgieva, P. Raychev and R. Roussev, J. Phys. G: Nucl. Phys. **9**, 521 (1983).
- [6] H. Ganev, V. Garistov and A. Georgieva, Phys. Rev. **C 69**, 014305 (2004).
- [7] H. Ganev, V. P. Garistov, A. I. Georgieva, and J. P. Draayer, Phys. Rev. **C 70**, 010411 (2004).
- [8] D. J. Rowe, J. Math. Phys. **35**, 3178 (1994); D. J. Rowe and C. Bahri, J. Phys. A: Math. Gen **31**, 4947 (1998).
- [9] A. Bohr and B. R. Mottelson, *Nuclear Structure*, Benjamin, New York, (1975).
- [10] L. Wilets and M. Jean, Phys. Rev. **102**, 788 (1956).
- [11] J. P. Elliott, J. A. Evans and P. Park, Phys. Lett. **B169**, 309 (1986); J. P. Elliott, P. Park and J. A. Evans, Phys. Lett. **B171**, 145 (1986).
- [12] D. J. Rowe and K. T. Hecht, J. Math. Phys. **36**, 223 (1981).
- [13] Feng Pan and J. P. Draayer, Nucl. Phys. **A636**, 156 (1998).
- [14] M. Moshinsky, J. Math. Phys. **25**, 1555 (1984).
- [15] M. Moshinsky, SIAM, J. Appl. Math. **25**, 193 (1973).
- [16] M. Moshinsky, *Group Theory and the Many Body Problem*, Gordon and Breach, New York, (1968).
- [17] G. N. Afanas'ev, I. N. Mikhailov and P. Raychev, Yad. Fiz. **14**, 734 (1971). [Sov. J. Nucl. Phys.]; P. P. Raychev, Yad. Fiz. **16**, 1171 (1972). [Sov. J. Nucl. Phys.]
- [18] G. Rosensteel and D. J. Rowe, Phys. Rev. Lett. **38**, 10 (1977); Ann. Phys. (N.Y.) **126**, 343 (1980); G. Rosensteel, J. Math. Phys. **21**, 924 (1980); G. Rosensteel and D. J. Rowe, Phys. Rev. Lett. **47**, 223 (1981); D. J. Rowe, Rep. Prog. Phys. **48**, 1419 (1985).
- [19] M. Moshinsky and C. Quesne, J. Math. Phys. **11**, 1631 (1970); J. Math. Phys. **12**, 1772 (1971).
- [20] V. V. Vanagas, *Algebraic foundations of microscopic nuclear theory*, Nauka, Moscow (in russian) (1988).
- [21] A. Ya. Dzublik, V. I. Ovcharenko, A. I. Steshenko and G. F. Filippov, Sov. J. Nucl. Phys. **15**, 487 (1972)
- [22] O. Castanos, E. Chacon and M. Moshinsky, J. Math. Phys. **25**, 1211 (1984).
- [23] J. P. Draayer, in Algebraic approaches to nuclear structure: Interacting boson and fermion models, Contemporary Concepts in Physics VI, edited by R. F. Casten, Harwood academic, Chur, Switzerland, (1993) p. 423
- [24] E. Chacon, O. Castanos and A. Frank, J. Math. Phys. **25**, 1442 (1984).
- [25] V. V. Vanagas, *Algebraic methods of microscopic nuclear theory*, Vilnius (in russian) (1971).
- [26] A. J. Dragt, J. Math. Phys. **6**, 533 (1965).
- [27] D. J. Rowe, Prog. Part. Nucl. Phys. **37**, 4711 (1995).
- [28] G. Popa, A. Georgieva and J. P. Draayer, Phys. Rev. **C 69**, 064307 (2004).
- [29] R. F. Casten and N. V. Zamfir, Phys.Rev. Lett, **85**, 3584 (2000); M. A. Caprio, Experiments on critical point nuclei - Mapping the Triangle, AIP Conference proceedings, **638**, eds; A. Aprahamian, J. A. Cizewski, S. Pittel, and N. V. Zamfir, 17 (2002)
- [30] D. J. Rowe, Phys. Rev. Lett. **93**, 122502 (2004)
- [31] F. Iachello, Phys.Rev. Lett. **85**, 3580 (2000); F. Iachello, Phys.Rev. Lett. **87**, 052502 (2001); J. Jolie et. al., Phys. Rev. Lett. **89**, 182502 (2002); F. Iachello, Phys. Rev. Lett. **91**, 132502 (2003)