H('infinity) Rings of Operators.

Willis J. Bourque Jr
Louisiana State University and Agricultural & Mechanical College

Follow this and additional works at: https://repository.lsu.edu/gradschool_disstheses

Recommended Citation
https://repository.lsu.edu/gradschool_disstheses/1905

This Dissertation is brought to you for free and open access by the Graduate School at LSU Scholarly Repository. It has been accepted for inclusion in LSU Historical Dissertations and Theses by an authorized administrator of LSU Scholarly Repository. For more information, please contact gradetd@lsu.edu.
BOURQUE, Jr., Willis J., 1942-
H∞ RINGS OF OPERATORS.
The Louisiana State University and Agricultural
and Mechanical College, Ph.D., 1971
Mathematics

University Microfilms, A XEROX Company, Ann Arbor, Michigan

THIS DISSERTATION HAS BEEN MICROFILMED EXACTLY AS RECEIVED
H∞ RINGS OF OPERATORS

A Dissertation

Submitted to the Graduate Faculty of the Louisiana State University and Agricultural and Mechanical College in partial fulfillment of the requirements for the degree of Doctor of Philosophy

in

The Department of Mathematics

by

Willis J. Bourque, Jr.
B.S., University of Southwestern Louisiana, 1964

January, 1971
ACKNOWLEDGEMENT

The author wishes to express his gratitude to Professor Pasquale Porcelli for his patience and for the inspiration and guidance that he furnished during the writing of this dissertation.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>CHAPTER</th>
<th>PAGE</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACKNOWLEDGEMENT</td>
<td>ii</td>
</tr>
<tr>
<td>ABSTRACT</td>
<td>iv</td>
</tr>
<tr>
<td>I INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>II $\mathbb{H}$ RINGS</td>
<td>7</td>
</tr>
<tr>
<td>III ISOMETRIES IN $\mathbb{H}^\infty(U)$</td>
<td>22</td>
</tr>
<tr>
<td>IV ANALYTIC DIRECT INTEGRAL THEORY</td>
<td>44</td>
</tr>
<tr>
<td>BIBLIOGRAPHY</td>
<td>55</td>
</tr>
<tr>
<td>BIOGRAPHY</td>
<td>56</td>
</tr>
</tbody>
</table>
ABSTRACT

The ring of all bounded analytic functions in the unit polydisc in complex n-dimensional space is examined for all \( n \geq 1 \). More precisely, the fact that these rings are algebraically isomorphic to certain rings of operators is exploited to obtain information concerning the structure of these rings.

In the first chapter, notation is established, and the more specialized theorems from Hilbert space theory which are used in subsequent sections are briefly reviewed.

In Chapter II, special properties of these \( H^\infty \) rings of operators are catalogued. These properties are then used to show that the \( H^\infty \) rings corresponding to the various dimensions are distinct. In particular, the one dimensional case is classified by use of some of these properties. The Theorems are also used to obtain specific information concerning the power series expansions of bounded analytic functions in the n-dimensional unit polydisc.

In Chapter III, information concerning specific operators in the one dimensional case is obtained. Specifically, all of the isometries are classified in terms of properties of the functions they correspond to. Also, the matrices of certain operators are studied and this information is used to describe the commutant of
various subrings of the one dimensional $H^0$ ring of operators. Also, the fact that only trivial compact operators are contained in this ring is demonstrated.

In Chapter IV, a theory is developed for the two dimensional case. While this theory is related to the classical direct integral theory, it differs in that certain analytic conditions must be imposed to achieve the desired results. In the process of developing this theory, information concerning certain subseries of given power series is obtained.
CHAPTER I
INTRODUCTION

We assume in this paper that the reader is familiar with the elementary aspects of the theory of rings of operators on a Hilbert space. All Hilbert spaces to be considered will be complex Hilbert spaces. If $H$ is a Hilbert space and $x, y \in H$, then $(x, y)$ will denote the inner product of $x$ and $y$, and $\|x\|$ will denote the norm of $x$. $\mathbb{C}$ will denote the complex numbers.

We shall consider only bounded linear operators in this paper and therefore we shall use the term "operator" solely to mean a bounded linear operator. The set of all operators on a Hilbert space $H$ will be denoted by $B(H)$. By a ring of operators we shall mean a subset of $B(H)$, which forms a ring in the usual algebraic sense and in addition is a vector space over the complex numbers. In the literature, this structure is also referred to as an algebra of operators. While all rings of operators discussed in this paper are non-symmetric, we will occasionally need to discuss the adjoint, $A^*$, of an operator $A$. Also we employ the concepts of normal, isometric, partial isometric, hermitian and compact operator.

Two rings of operators $R_1$ and $R_2$ on Hilbert spaces $H_1$ and $H_2$, respectively, are said to be spatially isomorphic if and only if there exists a unitary operator
U from $H_1$ onto $H_2$ such that the induced mapping of 
$B(H_1)$ onto $B(H_2)$ defined by
$$T \mapsto UTU^{-1}$$
maps $R_1$ onto $R_2$.

From the theory of Banach algebras, we know that a semisimple commutative Banach algebra is isometric isomorphic to a ring of continuous functions on a compact Hausdorff space, namely the algebra's maximal ideal space.

This isometric isomorphism is called the Gelfand transform and is denoted by $a \mapsto \hat{a}$. We use the following special theorem from [2], page 71.

**Theorem 1.1** Let $R$ be a commutative Banach algebra with identity. Suppose $h$ is a continuous function on $M$, the maximal ideal space of $R$, such that
$$\sum_{k=0}^{n} \hat{a}_k h^k = 0, \quad \sum_{k=1}^{n} \hat{a}_k h^{k-1} \neq 0$$
at every point of $M$, where $a_1, \ldots, a_k$ are elements of $R$. Then there exists $w \in R$ such that $\hat{w} = h$ and
$$\sum_{k=0}^{n} a_k w^k = 0.$$

We let $\ell^2_+ = \{(a_0, a_1, \ldots) | \sum_{i=0}^{\infty} |a_i|^2 < \infty; a_i \neq 0, 1, 2, \ldots\}$, and the shift operator $S$ be defined by the equation:
$$s(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots).$$

Suppose $H$ is a direct sum, $H = \bigoplus_{i=1}^{\infty} H_i$, of Hilbert
spaces $H_i$, where $H_i$ is isomorphic to $H_j$, for all $i$ and $j$, and $U_i$ is an isometry from $H_i$ onto $H_{i+1}$. A generalized shift, $S$, on $H$ is defined as follows:

$$Sx = \sum_{i=1}^{\infty} U_i x_i.$$ 

We assume knowledge of the weak, strong and norm topologies for any subring of $B(H)$, and of their interrelationships. We shall say that a ring $R_1$ of operators is generated in the weak operator topology by an operator $U$ if and only if the set of polynomials in $U$ are dense in the weak operator topology in $R$. Note that such a ring of operators need not be symmetric.

If $R \subseteq B(H)$, $R$ is a ring and $\xi \in H$, then $R\xi = \{A\xi \mid A \in R\}$ is a linear subspace of $R$. When $R\xi = H$, then we say that $\xi$ is a cyclic vector for $R$. If $K$ is a subspace of $H$ and $RK \subseteq K$ we say that $K$ is an invariant subspace of $R$ or simply that $K$ is invariant with respect to $R$. The commutant of a ring $R$ is the set of operators which commute with all of $R$.

Most of the rings studied in this paper are rings of analytic functions; more precisely, they are rings of operators which are algebraically isomorphic to certain rings of analytic functions. We assume the reader is familiar with the basic aspects of the theory of several complex variables. Of fundamental concern
in this approach are certain basic aspects of $L^p$ theory. In particular, all such spaces in this paper are $p$-spaces over the product of circles and hence are $L^p$ spaces of locally compact commutative groups with respect to their Haar measures. Specifically, we assume the reader is familiar with Hölder's inequality and the following theorem from Hoffman [1], page 19.

**Theorem 1.2** If $f \in L^\infty(T)$, $T = \{z | z \in \Phi, |z| = 1\}$ and $\Sigma_n$ is the $n$th Cesaro mean of $f$, then for $g$ in $L^1(T)$,

$$\int_T |f(w) - \Sigma_n(w)|g(w)dm(w) \to 0$$

where $dm$ denotes normalized Haar measure.

For the most part, whenever we use the theory of several complex variables we use the notation used by Rudin in his monograph [6]. For convenience we collect pertinent notation in this chapter and list, without proof, some of the results used in this paper.

We let $\Phi^n$ be the $n$-Cartesian product of the complex numbers and $U^n = \{(z_1, \ldots, z_n) \in \Phi^n \mid |z_i| < 1, i = 1, 2, 3, \ldots\}$. We let $T^n$, the distinguished boundary of $U^n$, be the points $(z_1, \ldots, z_n)$ such that $|z_i| = 1$ for $i = 1, \ldots, n$. Under pointwise multiplication and the inherited topology from $\Phi^n$ (which is of course homeomorphic to $2n$ Euclidean space), $T^n$ is a locally compact Hausdorff commutative group. We let $dm_n$ represent normalized Haar measure on $T_n$. $H^\infty(U^n)$ shall denote the set of bounded analytic functions on
Let $\text{dm}_n$ denote the normalized Haar measure on $T_n$. We let $H^2(U^n)$ be the set of all analytic functions $f$ on $U^n$ such that $\int_{T^n} |f(rw)|^2 \text{dm}_n(w)$ is uniformly bounded for $0 \leq r < 1$.

We use the following rather famous theorem concerning the boundary values for a function in $H^2(U^n)$

**Theorem 1.3** If $f$ is in $H^2(U^n)$ for $n = 1, 2, 3, \ldots$ then $F(w) = \lim_{\rho \to 1} f(rw)$ exists for almost all $w$ in $T^n$ with respect to $\text{dm}_n$, and is, in fact, in $L^2(T^n)$. When $f$ is in $H^\infty(U^n)$, then $F \in L^\infty(T^n)$.

$F$ is called, for obvious reasons, the boundary value function for $f$. In the case $n = 1$, see Porcelli [5] page 63 for a proof of this theorem. For $n > 1$, the proof in [5] does not generalize. For a development of these ideas see Rudin [6], chapters two and three.

While it is true that $L^2(T^n)$ and $L^2(T^m)$ are isometric isomorphic for all $n$ and $m \geq 1$, we still distinguish amongst them due to Theorem 1.2. For each $n$, the mapping $f \mapsto F$ yields a rather natural imbedding of $H^2(U^n)$ in $L^2(T^n)$ and of $H^\infty(U^n)$ in $L^\infty(T^n)$.

For $z = (z_1, \ldots, z_n)$ in $U^n$ and $w = (w_1, \ldots, w_n)$ in $T^n$, let $c(z, w) = c(z_1, w_1) \ldots c(z_n, w_n)$ where $c(z_i, w_i) = (1 - z_i\overline{w_i})^{-1}$ for $i = 1, \ldots, n$. $c(z, w)$ is, of course, the Cauchy kernel in complex $n$-space. We utilize the following form of the Cauchy Integral theorem.
Theorem 1.3  For $f$ in $H^2(U^\mathbb{N})$ and $z$ in $U^\mathbb{N}$, we have

$$f(z) = \int_{T^n} F(w)c(z, w)\,dm_n(w).$$

We note here that for $z$ fixed, $c(z, w)$ is in $L_p(T^n)$ when considered as a function of $w$.

We assume the reader is familiar with the factorization theory for $H^p(U)$. For a development of these ideas, including definitions of inner, outer and singular functions, see chapter 5 of Hoffman [1].

We also use one very well known theorem concerning separable metric dimension of Hurewicz and Wallman [3]. This is the fact that dimension is a topological property.
CHAPTER II

In this chapter, we study $H^\infty(U^n)$ as a ring of operators. More precisely, for each $f$ in $H^\infty(U^n)$, we let $A_f$ be the operator on $H^2(U^n)$ defined by $A_f g = fg$ for all $g$ in $H^2(U^n)$; note that the pairing between $f$ and $A_f$ is unique. We let $R^n$ denote the ring of all operators $A_f$ with $f$ in $H^\infty(U^n)$. We begin by cataloging seven properties shared by the rings $R^n$ for $n = 1, 2, 3, \ldots$. Then we present a property unique to $R^1$ and finally we show that in some sense of the word, the various rings $R^n$, $n = 1, 2, 3, \ldots$ are distinct.

For convenience, we present the first basic properties in one theorem.

Theorem 2.1  For $n = 1, 2, 3, \ldots$, $R^n$ satisfies the following properties:

P_1)  For $A_f$ in $R^n$, $\|f\|_\infty = \|A_f\|$

P_2)  $R^n$ is maximal commutative; i.e. if $B$ is an operator on $H^2(U^n)$ such that $A_f B = B A_f$ for all $A_f$ in $R^n$, then $B = A_g$ for some $g$ in $H^\infty(U^n)$.

P_3)  $R^n$ is closed in the norm, weak operator and strong operator topologies.

P_4)  $R^n$ contains no normal operators, other than multiples of the identity.

P_5)  $R^n$ is semisimple.

P_6)  $R^n$ has a cyclic vector.
P.7) The maximal ideal space of $\mathbb{R}^n$ contains no isolated points.

**Proof of P.7:** (For $A_f$ in $\mathbb{R}^n$, $||f||_\infty = ||A_f||_\infty$)

We begin by showing that if $F$ represents the boundary value function of $f$ on $T^n$, then $||f||_\infty = ||F||_\infty$. Since $c(z, w) \in L^p(T^n)$ for $p \geq 1$ and $F \in L^p(T^n)$ for $p \geq 1$, we have by Hölder's inequality that for $z$ in $U^n$,

$$|f(z)| = \left| \int_{T^n} F(w) c(z, w) dm_n(w) \right| \leq \left\{ \int_{T^n} |F(w)|^p dm_n(w) \right\}^{1/p} \left\{ \int_{T^n} |c(z, w)|^q dm_n(w) \right\}^{1/q}$$

for $p \geq 1$ and $1/p + 1/q = 1$. But since

$$\lim_{p \to \infty} ||F||_p \leq ||F||_\infty$$

and $q$ and $1/q$ converge to 1 as $p$ converges to $\infty$, we have that $|f(z)| \leq ||F||_\infty$ for all $z$ in $U^n$. Hence $||f||_\infty \leq ||F||_\infty$. The reverse inequality is true since the values of $F$ are contained in the closure of $f(U^n)$ in $\mathcal{C}$.

It is therefore sufficient to show that $||F||_\infty = ||A_f||$. The fact that $||A_f|| \leq ||f||_\infty$ follows from the following short computation:

$$||A_f|| = \sup \{ ||A_f g||_2 : g \in H^2(U^n) ; ||g||_2 \leq 1 \}
= \sup \{ \int_{T^n} |F(w)|^2 |G(w)|^2 dm_n(w) \}^{1/2} \leq ||F||_\infty \sup \{ ||g||_2 \leq 1 \} ||g||_2 \leq ||F||_\infty$$

We would now like to see that $||F||_\infty \leq ||A_f||$. For each positive integer $k$, we let

$$E_k = \{ w \in T^n | |F(w)| \geq (||F||_\infty - 1/k) \}.$$
Set
\[ g_0(w) = \begin{cases} \left[ m_n(E_k) \right]^{-1/2} & w \in E_k, \\ 0 & w \notin E_k. \end{cases} \]

We notice that
\[ \{ \int_{T^n} |g_0(w)|^2 d\mu_n(w) \}^{1/2} = \| g_0 \|_2 = 1. \]
Since \( g_0 \in L^2(T^n) \), for every \( \varepsilon > 0 \), there exists a trigonometric polynomial \( g \) such that \( \| g_0 - g \|_2 < \varepsilon \).

Now, if \( \| g \|_2 - \| g_0 \|_2 \geq 0 \), we have that
\[ 0 \leq \| g \|_2 - \| g_0 \|_2 = \| g + g_0 \| - \| g_0 \|_2 \leq \| g - g_0 \|_2 + \| g_0 \|_2 - \| g_0 \|_2 < \varepsilon. \]
Similarly if \( \| g \|_2 - \| g_0 \|_2 \geq 0 \) we have
\[ 0 \leq \| g \|_2 - \| g_0 \|_2 \leq \| g_0 - g \|_2 < \varepsilon. \]
Therefore \( |g_0|_2^2 - |g|_2^2 < \varepsilon. \) Since \( g \) is continuous, we have by Theorem 1, there exists \( h \in H^\infty(U^n) \subset H^2(U^n) \) such that \( |h| = |g| \) a.e. on \( T^n \).

Then \( \| A_f \| = \text{Sup} \{ |A_f \psi|_2 \psi \in H^2(U^n), \| \psi \|_2 \leq 1 \} \geq \| A_f h \|_2 = \| F \cdot h \|_2 = \| F \cdot g \|_2 = \| F \cdot g_0 \|_2 - \| F \|_\infty \varepsilon = \left( \int_{T^n} |F(w)|^2 g_0(w) |2 d\mu_n(w) \right)^{1/2} - \| F \|_\infty \varepsilon \geq \left( \int_{E_k} |F(w)|^2 g_0(w) |2 d\mu_n(w) \right)^{1/2} - \| F \|_\infty \varepsilon > \left( \left( |F|_\infty - 1/k \right) \left[ m_n(E_k) \right]^{-1} \left[ m_n(E_k) \right] \right)^{1/2} - \| F \|_\infty \varepsilon = |F|_\infty - 1/k - \| F \|_\infty \varepsilon \)

Letting \( \varepsilon \rightarrow 0 \), we have \( |A_f| \geq |F|_\infty - 1/k \) for all \( k \). Hence \( |A_f| \geq |F|_\infty \).

Proof of \( P_2 \). (\( R^n \) is maximal commutative.)

Suppose \( T \) is an operator on \( H^2(U^n) \) and \( T \) commutes
with \( \mathbb{R}^n \). Let \( b(z) = T_e(z) \) for all \( z \) in \( U^n \), where \( e \) is the function which is identically 1 on \( U^n \). We first note that \( b \) is analytic since, in fact, \( b \) is in \( H^2(U^n) \). Also for \( f \) in \( H^\infty(U^n) \), we have \( T(f) = T \cdot A_f(e) = A_f T(e) = A_f b = f \cdot b = b \cdot f \); i.e. if we restrict \( T \) to \( H^\infty(U^n) \) (considered as a vector subspace of \( H^2(U^n) \)), then \( T \) is exactly multiplication by the function \( b \). Hence if we show that \( b \in H^\infty(U^n) \), it will follow, since \( H^\infty(U^n) \) is dense in \( H^2(U^n) \) in the 2-norm, that \( T \) is actually multiplication by \( b \) on all of \( H^2(U^n) \). We will actually show that \( \|b\|_\infty \leq \|T\| \). It is then clear that we may assume \( \|T\| = 1 \).

We proceed by contradiction. Suppose there exists \( \eta > 0 \) such that if
\[
E = \{ w \in T^n | \, |B(w)| > (1 + \eta) \}, \text{ then } m_n(E) = \delta > 0.
\]
Let \( \varepsilon > 0 \) and set \( h(w) = 1 \) for \( w \in E \) and \( \varepsilon \) for \( w \notin E \). Then \( h \in L^\infty(T^n) \) and for \( \gamma > 0 \), there exists a trigonometric polynomial \( g \) such that \( \|h - g\|_2 < \gamma \). Since \( g \) is a trigonometric polynomial, \( |g| \) is a positive lower-semi-continuous function on \( T^n \). Hence we have by Theorem 1., that there exists \( f \in H^\infty(U^n) \) such that \( |f| = |g| \) almost everywhere on \( T^n \) with respect to \( m_n \). Then we have
\[
(1 + \eta) \delta < \int_E |B(w)| \, |h(w)| \, dm_n(w)
\]
\[
\leq \int_E |B(w)| \, |(h - g)(w)| \, dm_n(w) + \int_E |B(w)| \, |g(w)| \, dm_n(w)
\]
\[ \leq \int_E |B(w)| |(h - g)(w)| \, dm_n(w) + \int_E |B(w)| |g(w)| \, dm_n(w) \]
\[ \leq \left\{ \int_E |B(w)|^2 \, dm_n(w) \right\}^{1/2} \left\{ \int_E |(h - g)(w)|^2 \, dm_n(w) \right\}^{1/2} \]
\[ + \left\{ \int_E 1 \, dm_n(w) \right\}^{1/2} \left\{ \int_E |(h - g)(w)|^2 \, dm_n(w) \right\}^{1/2} \]
\[ \leq \gamma + \delta^{1/2} \left\| b \cdot g \right\|_2 = \gamma + \delta^{1/2} \left\| b \cdot f \right\|_2 \leq \gamma + \delta^{1/2} \left( \delta + (1 - \delta)\varepsilon \right)^{1/2} \]

Letting \( \gamma + 0 \) we obtain
\[ (1 + \eta)\delta = \delta + \eta\delta \leq \delta^{1/2} (\delta + (1 - \delta)\varepsilon)^{1/2} \text{ for every } \varepsilon > 0. \]
Letting \( \varepsilon + 0 \) we obtain \( \delta + \eta\delta \leq \delta \). But this implies that \( \eta = 0 \) and we obtain the desired result.

Proof of P_3. (\( R^n \) is closed in the norm, weak operator and strong operator topologies.)

It is well known that \( H^\infty(U^n) \) is closed in the usual \( \infty \)-norm topology. Because of P_1, it follows immediately that \( R^n \) is closed in the operator norm topology. We now let \( B \) be an operator on \( H^2(U^n) \) such that \( B \) is in the weak closure of \( R^n \). Then there exists a net \( \{ B_\alpha \} \) contained in \( R^n \) such that \( B_\alpha \) converges to \( B \) in the weak operator topology. For each \( f \) in \( H^\infty(U^n) \) and for each \( \alpha, A_f B_\alpha = B_\alpha A_f \). But \( A_f B \) converges to \( A_f B \) and \( B_\alpha A_f \) converges to \( B A_f \) in the weak operator topology, since multiplication is continuous in each variable separately for this topology. The weak operator topology being Hausdorff, we obtain that \( B \) commutes with \( R^n \) and hence
by $P_2$ must actually be in $R^n$. The proof for the strong operator topology is similar since multiplication is continuous in each variable separately in the strong operator topology.

**Proof of $P_4$.** ($R^n$ contains no normal operators, other than multiples of the identity.)

Let $P$ be the natural projection of $L^2(T^n)$ onto $H^2(U^n)$. We first show that $A_f^* = PA_f$, where $A_f$ is multiplication by $f$ on $L^2(T^n)$. Let $f \in H^\infty(U^n)$ and $g$ and $h$ be in $H^2(U^n)$. Then since $1 - P$ is the projection on the orthogonal complement of $H^2(U^n)$ in $L^2(T^n)$ we have that:

\[
\langle A_f g, h \rangle = \int_{T^n} F(w)G(w)\overline{H(w)}\,dm_n(w) = \int_{T^n} G(w)\overline{F(w)h(w)}\,dm_n(w) = \int_{T^n} G(w)[P\overline{f}h](w)\,dm_n(w) + \int_{T^n} G(w)[(1 - P)\overline{f}h](w)\,dm_n(w)
\]

\[
= \int_{T^n} G(w)[P\overline{f}h](w)\,dm_n(w) = \langle g, P\overline{f}h \rangle = \langle g, PA_f h \rangle.
\]

Hence $A_f^* = PA_f$.

We now suppose that $f \in H(U^n)$ and that $A_f$ is a normal operator. Then $A_f PA_f = PA_f A_f$. Let $f(z) = \sum_\alpha a_\alpha z^\alpha$ be the series expansion for $f$. Then evaluating $A_f PA_f$ and $PA_f A_f$ at the function $e$ which is identically 1 on $U^n$, we obtain

\[
(A_f PA_f)e = A_f P\overline{f} = f(P\overline{f})
\]

and
\[(PA^e_\bar{f}A^e_\bar{f})e = P(\bar{f}\bar{f}).\]

Therefore \((A^e_\bar{f}PA^e_\bar{f})(e)(0) = |a_0, 0, \ldots, 0|^2\) and
\[(PA^e_\bar{f}A^e_\bar{f})(e)(0) = \sum |a_\alpha|^2.\] Note that \(\sum |a_\alpha|^2\) is simply the constant term in the series expansion of \(P\bar{f}\bar{f}\).

But these last two expressions being equal, we must have that \(a_\alpha = 0\) for \(\alpha \neq (0, 0, \ldots, 0)\) and hence \(f\) is a constant. Therefore \(A^e_\bar{f}\) is a multiple of the identity.

**Proof of P5.** (\(\mathbb{R}^n\) is semisimple.)

For each point \(z\) in \(\mathbb{U}^n\), \(M_z = \{A^e_\bar{f} \in \mathbb{R}^n | f(z) = 0\}\) is a maximal ideal in \(\mathbb{R}^n\). But, rather clearly
\[\bigcap_{z \in \mathbb{U}^n} M_z = (0)\] and hence the radical of \(\mathbb{R}^n\) is trivial.

**Proof of P6.** (\(\mathbb{R}^n\) has a cyclic vector.)

In particular the function \(e(z) = 1\) behaves as a cyclic vector for \(\mathbb{R}^n\) since the polynomials in \(z\) and hence \(H^\infty(\mathbb{U}^n)\) are dense in \(H^2(\mathbb{U}^n)\) in the 2-norm.

**Proof of P7.** (The maximal ideal space of \(\mathbb{R}^n\) has no isolated points.)

Let \(M_0 \in M^n\), the maximal ideal space of \(H^\infty(\mathbb{U}^n)\) and suppose that \(M_0\) is isolated. Then if we set \(h\) equal to the characteristic function of \(\{M_0\}\), \(h\) is a continuous function. Since \(h^2 = h\), it follows that \(h - h^2 \equiv 0\) on \(M^n\). But \(1 - 2h\) takes on only the values 1 and -1 and is hence never zero on \(M^n\). Hence by Theorem 1. \(\hat{\omega} = h\) has a solution with \(w \in H^\infty(\mathbb{U}^n)\).

Since \(w^2 = w\), \(w\) can take on only values of 0 or 1 on \(\mathbb{U}^n\).
But \( w \) being analytic on \( U^n \), it is either identically 0 or identically 1. In either case \( \hat{w} \neq \chi_{\{m_0\}} \), since \( \hat{w} \equiv 0 \) or \( \hat{w} \equiv 1 \) on \( M^n \).

We now focus our attention on \( H^\infty(U^1) \) and the corresponding ring of operators \( R^1 \) and demonstrate a property for it which we eventually show is not shared by the other \( R^n \) for \( n > 1 \).

**Theorem 2.2** \( R^1 \) is generated in the weak operator topology by an isometry which is spatially isomorphic to the unilateral shift on \( \lambda^2_+ \).

**Proof:** Since multiplication by \( z \) is spatially isomorphic to the unilateral shift on \( \lambda^2_+ \), it is sufficient to show that \( R^1 \) is generated in the weak operator topology by \( A_z \). Since \( R^1 \) is closed in this topology it is sufficient to see that polynomials in \( A_z \) are dense in \( R_1 \). Let \( f \in H^\infty(U^1) \). Then if we let \( \sigma_n = n^{th} \) Caesaro mean of the function \( f \), and \( \Sigma_n \) and \( F \) their corresponding boundary value functions, we have by Theorem 1.2 that \( \Sigma_n \to F \) in the weak-star topology on \( L^\infty \). But then for \( g, h \in H^2(U) \), \( G\bar{H} \in L^1(T) \) and hence:

\[
| \langle (A_f - A_{\sigma_n})g, h \rangle | = \\
| \int_T (F - \Sigma_n)(w)G(w)\overline{H(w)} \, dm_n(w) | \to 0.
\]

Hence, \( A_{\sigma_n} \) converges to \( A_f \) in the weak operator topology.
We now show that this property together with a variant of property $P_4$ characterizes $H^\infty(U^1)$.

**Theorem 2.3** Let $H$ be a Hilbert space of dimension larger than 1 and $R$ be a subring of $B(H)$, the set of all bounded linear operators on $H$. Suppose that no normal operators, other than multiples of the identity, commute with $R$ and that $R$ is generated in the weak operator topology by an isometry, $U$. Then $R$ is spatially isometric to $H^\infty(U^1)$.

**Proof:** Let $M = (R_U)$ and $S = \sum_{n=0}^{\infty} U^n M$, where $R_U$ denotes the range of $U$. We note that $S$ is actually a direct sum since for $x, y \in M$ and $n > m$,

$$(U^m x, U^n y) = (x, U^{n-m} y) = 0.$$ 

Also, $S$ is obviously invariant with respect to $U$. Suppose now that $x \perp S$; i.e. $(x, U^n y) = 0$ for every $n \geq 0$, $y \perp R_U$. Then for $y \perp R_U$, $(Ux, y) = 0$; also for $n \geq 1$ $(Ux, U^n y) = (x, U^{n-1} y) = 0$ and hence $Ux \perp S$. Letting $P$ be the projection on $S$, $P$ commutes with $U$ since $S$ is invariant with respect to $U$ and $U^*$. But $P$ is then trivial by hypothesis for $P$ must commute all of $R$ since $R$ is generated by $U$. Hence $S = (0)$ or $S = H$. If $S = (0)$ then $M = (0)$ and $U$ must be a unitary operator since its range must then be all of $H$. But then $U$ is normal and hence must be equal to the identity, for it commutes with $R$ and is an isometry.
In this case R is simply all multiples of the identity, and the commutant is actually B(H) which contains non-trivial normal operators since \( \text{dim} \ (H) > 1 \). Hence, the conclusion \( S = \{0\} \) leads to an absurdity and we conclude that \( S = H \).

At this stage we notice that U is spatially isomorphic to a generalized shift on \( \bigoplus_{i=0}^{\infty} H_i \), where each \( H_i \) is a carbon copy of M. For since U is an isometry \( U^*M = U^mM \) for all \( n, m \geq 0 \) and the spatial isometry is obvious.

We now claim that \( \text{dim} \ M = 1 \), and hence that U is spatially isomorphic to the unilateral shift on \( l^2_+ \). Suppose, however, that \( \text{dim} \ (M) > 1 \). Then we choose \( \xi_1, \xi_2 \in M \) such that \( \xi_1, \perp \xi_2 \). Setting 

\[
S = \bigoplus_{i=0}^{\infty} U^i \{ \lambda \xi_1 | \lambda \text{complex} \},
\]

we have that S is invariant with respect to U and \( U^* \). Also S is non-trivial since obviously \( \xi_2 \) is not in S. Recall \( H = \bigoplus_{n=0}^{\infty} U^nM \). But as before, this leads to the conclusion that a non-trivial normal operator commutes with R.

We note that we may replace the condition that R is generated in the weak operator topology by an isometry with either of the following two properties:

a) There exists an isometry U in R such that if A is in B(H) then A is in R if and only if A commutes with U.

b) There exists an isometry, U, in R such that
dim $(R_q)^{-1} = 1$ and $U$ commutes with $R$.

The proof of the theorem using condition (a) is obvious. Using condition (b) we simply let $R' = \text{weakly closed ring generated by } U$. Then $R'$ is spatially isomorphic to $H^\infty(U^1)$ and is hence maximal commutative. But then $R' = R$ and the conclusion follows.

At first glance, this appears to be an extremely weak characterization of $H^\infty(U^1)$ mainly since the property of being generated by an isometry is a tremendously strong criterion. However, we note that $H^\infty(U^n)$ satisfies, for any $n \geq 1$, all of the properties $P_1$ through $P_7$ in Theorem 2.1. Provided we could show that Theorem 2.2 applies only to $H^\infty(U^1)$ that the characterization given is the strongest possible, at least with respect to the properties mentioned above.

We now show that $H^\infty(U^n)$ is not spatially isomorphic to $U^\infty(U^1)$. First we prove a technical lemma needed for this result.

**Lemma 2.4** If $E = \{\phi_1, \ldots, \phi_n\}$ is a generating system for $H^\infty(U^m)$ (i.e. $H^\infty(U^m)$ is the closure in the weak operator topology of the set of all polynomials in $\phi_1, \ldots, \phi_n$) then $E$ separates points in $U^m$ (i.e. given $z, w \in U^m$, $z \neq w$ then there exists $i, 1 \leq i \leq n$ such that $\phi_i(z) \neq \phi_i(w)$).

**Proof:** We suppose that $E$ does not separate points in $U^m$. Then there exists $z, w \in U^m$, $z \neq w$ such that
\( \phi_i(z) = \phi_i(w) \) for \( i = 1, \ldots, n \). But then if
\[ P = P(\phi_1, \ldots, \phi_n) \]
is a polynomial in \( \phi_1, \ldots, \phi_n \), then \( P(z) = P(w) \). But each function \( f \) in \( H^\infty(U^n) \) is a weak limit point of a sequence of polynomials \( \{P_i\}_{i=1}^\infty \) in \( \phi_1, \ldots, \phi_n \). Since convergence in the weak operator topology implies convergence in the topology of pointwise convergence, we have that \( f(z) = f(w) \) for all \( f \in H^\infty(U^n) \). But this is absurd.

**Theorem 2.5** A minimal generating system for \( H^\infty(U^n) \) has exactly \( n \) elements.

**Proof:** Let \( m = \) number of elements in a minimal generating system for \( H^\infty(U^n) \). With \( e_i(z) = z_i^{i} \) for \( i > 1, \ldots, n \), we will now show that \( \{e_i\}_{i=1}^n \) generates \( H^\infty(U^n) \) in the weak operator topology. To this end let \( f \) be in \( H^\infty(U^n) \) and let \( f_r(w) = f(rw) \) for all \( w \) in \( T^n \) and for all \( r \) such that \( 0 < r < 1 \). Since \( \{f_r\}_{0 < r < 1} \) are uniformly bounded in \( \infty \)-norm (as functions in \( L^\infty(T^n) \)), it follows that there exists a sequence \( \{r_i\}_{i=1}^\infty \) such that \( r_i \to 1 \) and \( f_{r_i} \) converges to \( f \) in the weak-star topology on \( L^\infty(T^n) \). Since each of the \( f_{r_i} \) can be approximated in the weak-star topology by trigonometric polynomials, it follows that \( F \) is the limit, in this same topology, of trigonometric polynomials. Now, as in the proof of Theorem 2.2, it follows that \( f \) is the limit of the extensions of these
polynomials to $U^n$ in the weak operator topology. Hence \( \{e_i\}_{i=1}^n \) is a generating system for $H^\infty(U^n)$.

Suppose now that $H^\infty(U^n)$ can be generated by a system $E = \{\phi_1, \ldots, \phi_m\}$ where $m < n$. By Lemma 2.4, we have that the $\phi_i$'s separate points in $U^n$. We define a mapping from $U^n$ into $\phi^m$ by: for $z = (z_1, \ldots, z_n) \in U^n$, let $\phi(z) = (\phi_1(z_1, \ldots, z_n), \ldots, \phi_m(z_1, \ldots, z_n))$.

$\phi$ is obviously well defined since each $\phi_i$ is well defined. $\phi$ is one to one, since, by the way $\phi$ is defined, this exactly means that $\phi$ separates points in $U^n$. If we restrict the mapping $\phi$ to any compact subset of $U^n$ which has dimension $n$ (say the closure of the polydisc of radius $1/2$ centered at the origin), then $\phi$ is still one to one and continuous. But then since the domain space is compact and the image space is Hausdorff then the restriction of $\phi$ is also an open mapping. Since $m < n$, this yields a homeomorphism which lowers dimension. Consequently our assumption that $H^\infty(U^n)$ can have $m$ generators where $m < n$ is erroneous.

Under spatial isometries, the minimal number of elements in a generating system is left invariant. Consequently we also have the following theorem.

**Theorem 2.6** For $n \neq m$, $H^\infty(U^n)$ is not spatially isomorphic to $H^\infty(U^m)$.

We conclude this chapter with a description of the
finitely generated ideals in $M^1$, the maximal ideal space of $H^\infty(U)$. We adopt the following conventions.

For each $M$ in $M^1$, we let $\Pi(M) = e(M)$ where $e(z) = z$ for all $z$ in $U$. It is well known, see Hoffman [1] for example, that $\Pi$ maps $M^1$ onto $\overline{U}$ and that $\Pi^{-1}|U$ is a topological homeomorphism of $U$ into $M^1$. In view of this, we shall refer to $\Pi^{-1}(U)$ as the unit ball in $M^1$. Note that for $z$ in $U$, $\Pi^{-1}(z) = M_z = \{f|f \in H^\infty(U), f(z) = 0\}$.

**Theorem 2.7** If the unit ball in $M^1$ is dense in $M^1$, then all finitely generated ideals in $M^1$ are contained in the unit ball.

**Proof:** Suppose $M$ is finitely generated in $H^\infty(U)$ by the functions $f_1, \ldots, f_n$ and $M$ is not in $\Pi^{-1}(U)$. For each integer $k > 1$, let $U_k = U(M; \{f_i\}_{i=1}^k, 1/k) = \{N|N \in M^1, \hat{f_i}(M) < 1/k \text{ for } i = 1, \ldots, k\}$. Then for each $k$ there exists $M_{z_k}$ in $U$ since the unit ball is dense, and it follows that

$$|\hat{f_i}(M_{z_k})| = |f_i(z_k)| < 1/k \text{ for } i = 1, \ldots, n.$$  

We have immediately then that $\lim_{k \to \infty} f_i(z_k) = 0$ for $i = 1, \ldots, n$. Since $\{f_i\}_{i=1}^n$ generates $M$, each $f$ in $M$ has the form $f = \sum_{i=1}^n f_i g_i$ where $g_i$ is in $H^\infty(U)$ $i = 1, \ldots, n$. We then have that $\lim_{k \to \infty} f(z_k) = 0$ for every $f$ in $M$. By resorting to choosing a subsequence if necessary, we may assume that $\sum_{i=1}^\infty (1 - |z_i|) < \infty$. We let
I = \{ f \mid f \in H^\infty(U) \text{ and } \lim_{k \to \infty} f(z_{2k}) = 0 \}. Then I is an ideal in \( H^\infty(U) \) which contains M. We show that in fact I properly contains M. Since \( \sum_{i=1}^{\infty} (1 - |z_i|) < \infty \), the sequence \( \{z_i\}_{i=1}^{\infty} \) is an interpolation sequence; i.e. if \( \{w_i\}_{i=1}^{\infty} \) is any bounded sequence, then there exists \( f \) in \( H^\infty(U) \) such that \( f(z_i) = w_i \) \( i = 1, 2, 3, \ldots \). For a proof of this result see Hoffman [1], page 197. In particular there exists \( f \) in \( H^\infty(U) \) such that \( f(z_i) = 0 \) whenever \( i \) is even and \( f(z_i) = 1 \) whenever \( i \) is odd. But then \( f \) is in I but is not in M. Hence M is not a maximal ideal and our initial assumption that M was not in \( H^{-1}(U) \) is necessarily invalid and the proof is concluded.

While it is known due to the work of Lennart Carleson that the unit ball of \( M^1 \) is indeed dense, we avoid using this fact and instead use a conditional formulation for Theorem 2.7 in order to emphasize the close relationship between the denseness of the unit ball and the position of the finitely generated ideals in \( M^1 \).
In this chapter, we concentrate almost exclusively on $H^\infty(U^1)$ and $R^1$. In particular we obtain a characterization of the matrices of the operators in $R^1$, with respect to a fixed orthonormal basis, of course. Using this result, we obtain a numerical characterization of those sequences $\{a_n\}$ such that $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^\infty(U^1)$. While this does have a generalization to $H^\infty(U^n)$, the remainder of the chapter is devoted to the study of certain operators in $R^1$.

We begin by observing that if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a function in $H^\infty(U^1)$, then the matrix of $A_f$ with respect to the orthonormal basis $\{1, z, z^2, \ldots\}$ for $H^2(U^1)$ is:

$$
[A_f] = \begin{bmatrix}
a_0 & 0 & 0 & \cdots \\
a_1 & a_0 & 0 & \cdots \\
a_2 & a_1 & a_0 & \cdots \\
a_3 & a_2 & a_1 & \cdots \\
a_4 & a_3 & a_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
$$

This is easily seen by observing that

$$
A_f z^n = z^n \sum_{n=0}^{\infty} a_n z^n = \sum_{p=0}^{\infty} a_p z^{n+p} = \sum_{p=n}^{\infty} a_{p-n} z^p.
$$

Hence, if $[A_f] = [a_{ij}]$, we have that $a_{i,0} = a_i$ and
\[ a_{i,j} = a_{i-j}, \text{ where for } n \text{ positive } a_{-n} = 0. \]

In the development which follows, all matrices are obtained with respect to the same basis used previously; i.e. \{1, z, z^2, \ldots\}.

**Theorem 3.1** Let \([a_{ij}]\) be the matrix of a bounded linear operator on \(H^2(U)\). Then \([a_{ij}] = [A_f]\) where \(f \in H^\infty(U^1)\) if and only if for \(j > 0\), \(a_{i,j} = a_{i-j}, 0\).

(Here again, we use the convention that \(a_n, 0 = 0\) when \(n\) is negative.)

**Proof:** If \(f\) is in \(H^\infty(U^1)\), then the result follows by the observation made above.

On the other hand, suppose \([A_{ij}]\) has the form \(a_{i,j} = a_{i-j}, 0\) for \(j > 0\). Letting \(a_i = a_i, 0\) we have

\[
[A_{ij}] = \begin{bmatrix}
a_0 & 0 & 0 & \cdots \\
a_1 & a_0 & 0 & \cdots \\
a_2 & a_1 & a_0 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

Let \(g(z) = \sum_{p=0}^{\infty} b_p z^p\) be an arbitrary function in \(H^\infty(U^1)\). Then if \([b_{ij}] = [A_g]\) we have that \([A_{ij}] [b_{ij}] = \]

\([b_{ij}] [a_{ij}] = [c_{ij}]\) where \(c_i, 0 = \sum_{k=0}^{\infty} a_k b_{k-i} = \sum_{k=0}^{\infty} a_k b_{-i} b_i.\)
\[
\begin{bmatrix}
a_0 & 0 & 0 & \ldots \\ a_1 & a_0 & 0 & \ldots \\ \vdots & a_1 & a_0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\quad =
\begin{bmatrix}
a_0 & 0 & 0 & \ldots \\ b_1 & b_0 & 0 & \ldots \\ \vdots & b_1 & b_0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\quad =
\begin{bmatrix}
a_0b_0 & 0 & 0 & \ldots \\ a_0b_1 + a_1b_0 & a_0b_0 & 0 & \ldots \\ \vdots & a_0b_1 + a_1b_0 & a_0b_0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \\
\end{bmatrix}
\]

Hence for each \( g \in H^\infty(U) \), \([a_{ij}]\) commutes with \([A_g]\). But \([A_f]|f \in H^\infty(U^\perp)\) is algebraically isomorphic to \(R^1\) and hence it is also maximal commutative (property \(P_2\) of
Theorem 2.1).  

Theorem 3.2 Let \( \{a_n\} \) be a sequence of complex numbers. Then \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is the power series expansion for a function in \( H^\infty(U) \) if and only if there exists \( C > 0 \) such that
\[
(1) \quad \sum_{n=0}^{\infty} \left| \sum_{\ell=0}^{n} a_\ell b_{n-\ell} \right|^2 \leq C \sum_{n=0}^{\infty} \left| b_n \right|^2
\]
for all sequences \( \{b_n\}_{n=0}^{\infty} \) such that \( \sum_{n=0}^{\infty} \left| b_n \right|^2 < \infty \). In this case, the greatest lower bound of the set of all \( C \) satisfying condition (1) is the square of the sup-norm of \( f \).

Proof: If \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is a function in \( H^\infty(U) \), then setting \( C = \{\|f\|_\infty\}^2 \), the conclusion is simply the statement that \( A_f \) is a bounded linear operator on \( H^2(U) \). For letting \( \{b_n\}_{n=0}^{\infty} \) be a sequence such that \( \sum_{n=0}^{\infty} \left| b_n \right|^2 < \infty \), then \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) is a function in \( H^2(U) \) and \( f(z)g(z) = \sum_{n=0}^{\infty} (\sum_{\ell=0}^{n} a_\ell b_{n-\ell}) z^n \). Since \( A_f \) is a bounded linear operator on \( H^2(U) \), we have by P1 of Theorem 2.1 that
\[
\|A_f g\|_2 \leq \|f\|_\infty \|g\|_2 = C \|g\|_2
\]
But \( \|A_f g\|_2 = \|f \cdot g\|_2 = \sum_{n=0}^{\infty} \left| \sum_{\ell=0}^{n} a_\ell b_{n-\ell} \right|^2 \).

Therefore \( \sum_{n=0}^{\infty} \left| \sum_{\ell=0}^{n} a_\ell b_{n-\ell} \right| \leq C \sum_{n=0}^{\infty} \left| b_n \right|^2 \).

Conversely, if \( \{a_n\} \) is a sequence satisfying (1), then \( [A] = [a_{ij}] \) where \( a_{ij}, 0 = a_i \) and \( a_{ij}, j = a_{i-j}, 0 \).
for $j > 0$, represents a bounded linear operator on $H^2(U)$. But since it is of the correct form, Theorem 3.1 implies that it must be the matrix of a function in $H^\infty$. Hence $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a function in $H^\infty(U^n)$.

We now present a generalization of this theorem to $H^\infty(U^n)$. To this end, we consider the following computations. Let $f(z) = \sum_{\alpha} a_\alpha z^\alpha$ be the series expansion for $f$ in $H^\infty(U^n)$ and $g(z) = \sum_{\alpha} b_\alpha z^\alpha$ be the series expansion for $g$ in $H^2(U^n)$. Then

$$f \cdot g(z) = \sum_{\alpha} (\sum_{\beta \leq \alpha} a_\beta b_{\alpha - \beta}) z^\alpha,$$

where if $\alpha = (\alpha_1, ..., \alpha_k)$, $\beta = (\beta_1, ..., \beta_k)$ then $\alpha - \beta = (\alpha_1 - \beta_1, ..., \alpha_k - \beta_k)$ and $\beta \leq \alpha$ means $\beta_1 \leq \alpha_1$, ..., and $\beta_n \leq \alpha_n$. Then since $H^\infty(U^n)$ is maximal commutative we obtain by methods similar to those in Theorem 3.2, the following theorem.

**Theorem 3.3** If $\{a_\alpha\}_\alpha \in \mathbb{N}_+^n$ is a function from $\mathbb{N}_+^n$ into the complex numbers, then $f(z) = \sum_{\alpha} a_\alpha z^\alpha$ is the power series expansion of a function in $H^\infty(U^n)$ if and only if there exists $C > 0$ such that

$$\sum_{\alpha} |\sum_{\beta \leq \alpha} a_\beta b_{\alpha - \beta}|^2 \leq C \sum_{\alpha} |b_\alpha|^2$$

for all functions $\{b_\alpha\}_\alpha \in \mathbb{N}_+^n$ such that $\sum_{\alpha} |b_\alpha|^2 < \infty$.

By using matrix techniques we now obtain a factorization theorem for the commutant of certain subrings of $H^\infty(U^n)$. We first introduce some necessary notation.

In particular, let $S$ denote multiplication by $z$ on $H^2(U)$, or equivalently, the unilateral shift on $l^2_+$. 
Let $F_n$ be the ring of operators generated by $S^n$ in the weak operator topology. Let $\{e_i\}_{i=0}^\infty$ be the usual basis for $H^2(U)$; i.e. $e_i(z) = z^i$. Finally, let $M^2_j$ be the space generated by $\{e_i \mid i \equiv j(n)\}$ and $T_j^2$ be the projection on $M^2_j$.

**Theorem 3.4** $F'_n = H^\infty + H^\infty T_1^n + \ldots + H^\infty T_{n-1}^n$, where $F'_n$ is the commutant of $F_n$, i.e. the set of all operators which commute with $F^n$.

**Proof:** We first do some preliminary computations.

$$[S^n] = \begin{bmatrix} 0 & 0 & 0 & \ldots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots \\ 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ \vdots & 0 & 1 & \ldots \\ \vdots & \vdots & \ddots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

Here the first nonzero entries appear in the $n^{th}$ row.

Now let $A = [a_{ij}]$ be the matrix of an operator in $F'_n$. Then $A[S^n] = [S^n]A$, and
Here again, first nonzero entries occur in the \( n \)th row.
\[
[S^n][A] = \begin{bmatrix}
a_{00} & a_{01} & \cdots \\
a_{10} & a_{11} & \cdots \\
\vdots & \vdots & \ddots \\
\end{bmatrix} = \\
\begin{bmatrix}
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots \\
1 & 0 & 0 & \cdots \\
0 & 1 & 0 & \cdots \\
\vdots & \vdots & \ddots & \ddots \\
\end{bmatrix}
\]

Hence \(A[S^n]\) is simply the matrix \(A\) moved \(n\)-rows down and \([S^n]A\) is the submatrix of \(A\) obtained by eliminating the first \(n\)-rows of \(A\). Since \(A[S^n] = [S^n]A\), this simply means that \(a_{i,j} = a_{i-n, j-n}\) for all \(i, j \geq n\). Hence \(A\) is determined by its first \(n\)-columns and it repeats these columns lowered \(n\)-rows for each repetition. Therefore \(A\) has the following form:
Since $A$ is the matrix of an operator in $F_n$ if and only if $A$ commutes with the matrix of $S^n$, we have that $A$ is the matrix of an operator in $F_n$ if and only if $A$ has the above form.

Let $A$ be the matrix of an operator in $F_n$. Then with $I =$ identity operator, $I = \sum_{i=0}^{n-1} P_i^n$, and hence

$$A = A[I] = A \sum_{i=0}^{n-1} [P_i^n] = \sum_{i=0}^{n-1} A[P_i^n].$$

We note that $A[P_j^n]$ is the matrix which agrees with $A$ on columns $C_j$ such that $j = i(n)$ and is 0 elsewhere. Let $R_i$ be the $i$th column of $A$ so that

$$A = [R_0, R_1, \ldots, R_{n-1}, R_0, R_1, \ldots, R_{n-1}, \ldots].$$

For $j = 0, \ldots, n-1$, let
\[ A_j = \begin{bmatrix} a_{0,j} & 0 & 0 & \cdots \\ a_{1,j} & a_{0,j} & 0 & \cdots \\ a_{2,j} & a_{1,j} & a_{0,j} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ a_{2,j} & \cdots & \cdots & \cdots \end{bmatrix} \]

\( A_j \), then, is the matrix of a bounded linear operator since it is the sum of \( n \) simple transformations of \([A^p_j]\). Therefore, by Theorem 3.1, \( A_j \) is the matrix of an \( H^\infty \) function \( f_j \). Further since \( A_j \) agrees with \( A[P^n_j] \) on columns \( R^\ell \) such that \( \ell \equiv j(n) \), then \( A_j P^n_j = A[P^n_j] \).

Therefore

\[ A = \sum_{i=0}^{n-1} A[P^n_i] = \sum_{i=0}^{n-1} A_i[P^n_i] = \sum_{i=0}^{n-1} [A_{f_i}][P^n_i] \]

and hence if \([T] = A\) we have

\[ T = \sum_{i=0}^{n-1} A_{f_i}P^n_i \]

Now \( P^n_0 = I - \sum_{i=1}^{n-1} P^n_i \) and hence

\[ T = A_{f_0} + A_{f_1}P^n_1 + \ldots + A_{f_{n-1}}P^n_{n-1} \]

and letting \( g_i = f_i - f_0 \) for \( n - 1 \geq i \geq 1 \) and \( g_0 = f_0 \) we obtain

\[ A = A_{g_0} + A_{g_1}P^n_1 + \ldots + A_{g_{n-1}}P^n_{n-1}. \]

It follows that

\[ F_n \subseteq H^\infty + H^\infty P^n_1 + \ldots + H^\infty P^n_{n-1}. \]
The reverse containment follows since $H^\infty$ commutes
with $S^n$ and $P^n_i$, $i = 0, 1, \ldots, n - 1$ are projections
commuting with $S^n$.

Again using matrix techniques, we now give a
numerical characterization of inner functions in
$H^\infty(U^1)$.

**Theorem 3.5** Let $f(z) = \sum_{i=0}^{\infty} \lambda_i z^i$ be in $H^\infty(U^1)$. Then
$f$ is inner if and only if $A_f$ is an isometry on $H^2(U^1)$ if
and only if $\sum_{i=0}^{\infty} |\lambda_i|^2 = 1$ and $\sum_{i=0}^{\infty} \lambda_i \lambda_{i+k} = 0$ for
for $k = 1, 2, 3, \ldots$.

**Proof:** If $f$ is an inner function, then $|F(w)| = 1$
for almost all $w \in \mathbb{T}$ and hence for $g$ in $H^2(U^1)$ we have
that

$$||A_f g||_2 = ||f \cdot g||_2 = \left\{ \int_{\mathbb{T}} |F(w)|^2 |G(w)|^2 dm(w) \right\}^{1/2} =$$

$$\left\{ \int_{\mathbb{T}} |G(w)|^2 dm(w) \right\}^{1/2} = ||g||_2.$$

Therefore $A_f$ is an isometry.

Conversely, if $A_f$ is an isometry, then for any
$g$ and $h$ in $H^2(U^1)$, we have that $(g, h) = (A_f g, A_f h)$ and
hence that

$$\int_{\mathbb{T}} g(w) \overline{h(w)} dm(w) = \int_{\mathbb{T}} |F(w)|^2 G(w) H(w) dm(w)$$

Therefore

$$\int_{\mathbb{T}} (1 - |F(w)|^2) G(w) \overline{H(w)} dm(w) = 0.$$

Letting $g$ be identically 1 and $h(z) = z^n$ for
$n = 1, 2, \ldots$ and then alternately letting $h$ be
identically 1 and $g(w) = z^n$ for $n = 0, 1, 2, \ldots$ we
obtain $\int_T (1 - |F(w)|^2) w^n = 0$ for every integer $n$. Hence $(1 - |F|^2) = 0$ a.e. on $T$ and $f$ is an inner function.

Now, suppose as before that $f(z) = \sum_{i=0}^{\infty} \lambda_i z^i$ is in $H^\infty(U)$. Then

$$[A_f] = \begin{bmatrix}
\lambda_0 & 0 & 0 & \ldots \\
\lambda_1 & \lambda_0 & 0 & \ldots \\
\lambda_2 & \lambda_1 & \lambda_0 & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}$$

and

$$[A_f^*] = \begin{bmatrix}
\lambda_0 & \lambda_1 & \lambda_2 & \ldots \\
0 & \lambda_0 & \lambda_1 & \lambda_2 & \ldots \\
0 & 0 & \lambda_0 & \lambda_1 & \lambda_2 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \ddots
\end{bmatrix}$$

But $A_f$ is an isometry if and only if $A_f^* A_f = I$ and hence if and only if $[A_f]^* [A_f] = [\delta_{ij}]$. But
And hence $A_f$ is an isometry if and only if $\sum_{i=0}^{\infty} |\lambda_i|^2 = 1$

and $\sum_{i=0}^{\infty} \lambda_i \bar{\lambda}_{i+j} = 0$ for $j = 1, 2, \ldots$ or equivalently

$\sum_{i=0}^{\infty} \lambda_i \bar{\lambda}_{i+j} = \delta_{0,j}$.

If we let $u$ be an inner function in $H^\infty(U^1)$ then $A_u$ is an isometry $H^2(U^1)$. Let $N$ be the orthogonal complement of the span of the spaces $A_u^i(N)$ for $i = 0, 1, \ldots$. Then it is easy to see that $H^2(U) = M \oplus \sum_{i=0}^{\infty} A_u^i(N)$. We propose to show that for all inner functions, $M$ is trivial and hence it will follow that $A_u$ is a generalized shift. Further we will show that the dimension of $N$ depends on the factorization of the inner function into its Blaschke product and its singular part. We first consider the case where $u$ is
Lemma 3.6  Let \( u(z) = z - a/1 - \overline{a}z \), where \( a \) is in \( U \)
and let \( f \in H^2(U) \). Then \( f \) is orthogonal to \( 1 + \overline{a}u \) if
and only if \( f(a) = 0 \). Hence, \( 1 + \overline{a}u \) is orthogonal to
the range of \( A_u \).

Proof: Suppose \( f \) is orthogonal to \( 1 + \overline{a}u \). Then
\[
0 = (f, 1 + \overline{a}u) = \int_T F(w)(1 + \overline{a}u)(w)dm(w). \quad \text{But}
\]

\[
(1 + \overline{a}u)(w) = (1 + a\overline{u})(w) = 1 + a(\overline{w} - \overline{a}/1 - a\overline{w})
\]

\[
= (1 - \overline{aw} + \overline{aw} + |a|^2)/(1 - a\overline{w}) =
\]

\[
(1 - |a|^2)(1 - a\overline{w})^{-1}. 
\]

Therefore
\[
0 = \int_T F(w)(1 - |a|^2)(1 - a\overline{w})^{-1}dm(w)
\]
or equivalently
\[
\int_T F(w)(1 - a\overline{w})^{-1}dm(w) = -|a|^2 \int_T F(w)(1 - a\overline{w})^{-1}dm(w).
\]

But
\[
f(a) = \int_T F(w)c(a, w)dm(w) =
\]

\[
\int_T F(w)(1 - a\overline{w})^{-1}dm(w) = -|a|^2 \int_T F(w)(1 - a\overline{w})^{-1}dm(w) =
\]

\[
-a^2 f(a). 
\]

Hence \( f(a) = 0 \). (Note: if \( a = 0 \), the conclusion is
clear without this argument.) Conversely, it is clear
that these steps are reversible.

For the remaining part, we need only note that if
\( g \) is in the range of \( A_u \), then \( g = uf \) for some \( f \) in \( H^2 \)
and hence \( g(a) = u(a)f(a) = 0 \) and \( g \) is orthogonal to
1 + au by the previous argument.

We note now that this Lemma says exactly that the dimension of the orthogonal complement of the range of $A_u$ is 1. This follows since the range of $A_u$ is $uH^2(U) = \{uf | f \in H^2(U)\} = \{g | g \in H^2(U), g(a) = 0\}$.

**Lemma 3.7** $H^2(U) = \sum_{i=0}^{\infty} A_u^i(N)$ where $N =$ space generated by $1 + au$ and $u$ is as in previous Lemma.

**Proof:** By Lemma 3.6, $N$ is the orthogonal complement of the range of $A_u$. By Theorem 1 we know that $f$ is orthogonal to $A_u^i(N)$ for $i = 0, 1, \ldots$ if and only if $f$ is infinitely divisible by $A_u$; i.e. $f$ is in the range of $A_u^i$ for $i = 0, 1, \ldots$. Hence, if $f$ is orthogonal to $A_u^i(N)$ for $i = 0, 1, 2, \ldots$, and $f$ is not identically zero, then for each nonnegative integer $n$, there exists $g_n$ in $H^2(U)$ such that $f = u^n g_n$. Hence the order of the zero of $f$ at $a$ is larger than $n$ for all $n$, but this is a contradiction. Therefore $H^2(U) = \sum_{i=0}^{\infty} A_u^i(N) = \{0\}$ and hence $H^2(U) = \sum_{i=0}^{\infty} A_u^i(N)$.

**Theorem 3.8** Let $\{a_i\}_{i=1}^{n} \subset U$ such that $a_i \neq a_j$ for $i \neq j$ and set $u_i(z) = z - a_i/1 - \bar{a}_i z$ for $i = 1, \ldots, n$. If $f = u_1 \ldots u_n$ and $N = (\text{Range of } A_f)^{\perp}$, then
dimension $N = \sum_{j=1}^{n} i_j$. 
To simplify our argument, we first prove the following lemma.

**Lemma 3.9** Let \( u(z) = z - a/1 - \bar{az} \), and \( e = 1 + \bar{au} \). Then if \( N = \text{Range of } A_k \), \( N \) is generated by \( E = \{ u^i e \}_{i=0}^{k-1} \) and the elements of \( E \) are linearly independent.

**Proof:** If \( g \in H^2(U) \), then since \( A_k \) is an isometry we have using Lemma 3.6 that
\[
0 = (1 + \bar{au}, ug) = (e, ug) = (u^j e, u^{j+1} g).
\]
Hence \( u^i e \perp \text{range of } A_k \) for \( i = 1, \ldots, k-1 \).

Conversely, let \( f \perp \text{range of } A_k \). Then we may write \( f = u^{k-1} h_1 + h_2 \), where \( h_2 \perp \text{range of } A_{k-1} \). But then
\[
0 = (h_2, A_{k-1} \ell) = (u^{k-1} \ast h_2, \ell) \text{ for all } \ell \in H^2(U)
\]
and hence \( (u^{k-1}) \ast h_2 = 0 \). Now for \( g \in H^2(U) \),
\[
0 = (f, u^k g) = (u^{k-1} h_1 + h_2, u^k g) =
((u^{k-1}) \ast (u^{k-1}) h_1 + (u^{k-1}) \ast h_2, (u^{k-1}) \ast u^k g) = (h_1, u^k g).
\]
Hence \( h_1 = \lambda_{k-1} (1 + \bar{au}) = \lambda_{k-1} e \) for some complex number \( \lambda_{k-1} \) and \( f = \lambda_{k-1} u^{k-1} e + h_2 \) where \( h_2 \perp \text{range of } A_{k-1} \).

By continuing this process we obtain \( f = \sum_{i=0}^{k-1} \lambda_i u^i e \) and hence \( N \) is the space generated by \( E \).

We would now like to see that the elements of \( E \) are linearly independent. Suppose, now, that
Let \( k \), \( m \) be the largest integers such that \( \lambda_i \) and \( \lambda_m \) are non-zero. At least two of the \( \lambda_i \)'s are non-zero since otherwise, either \( u = 0 \) or \( e = 0 \), which is a contradiction. Also we assume \( k < m \). Then we have

\[
0 = \sum_{i=0}^{k-1} \lambda_i u^i e
\]

and therefore \( u^m e \perp \text{range } A_u \); i.e. \( 0 = (u^m e, u^j g) \) for all \( g \in H^2(u) \) or equivalently \( 0 = (u^{m-j} e, g) \) for all \( g \in H^2(U) \). But this again leads to the contradiction that either \( e = 0 \) or \( u = 0 \).

**Proof:** Let \( e_j = l + \overline{a}_j u_j \) for \( j = 1, \ldots, n \),

\[
E_j = \{e_j, u_j e_j, \ldots, u_j^{i-1} e_j\} \quad \text{and } E = \bigcup_{i=1}^{n} E_i.
\]

It follows readily now by Lemma 3.7, that if \( f \) is in the subspace generated by \( E \) then \( f \perp \text{range } A_u \). By an argument similar to that used in the proof of Lemma 3.9, we may conclude that \( f \) is in the subspace generated by \( E \).

We now claim that \( E \) is linearly independent. We suppose there exists scalars \( \{\lambda_i, j\} \) not all zero such that:

\[
0 = \sum_{i=0}^{n} \lambda_i u^i e_1 + \sum_{j=0}^{n} \lambda_j u^j e_2 + \ldots + \sum_{j=0}^{n} \lambda_n u^n e_n.
\]

Reindexing the functions \( u_1, \ldots, u_n \) if necessary we may assume that there exists an integer \( \ell \geq 2 \) such that

i) if \( k < \ell \), there exists \( \lambda_k, j \) such that
ii) if $k > \lambda$, $\lambda_k, j=0$ for $j = 1, \ldots, i_k$.

If not, then $\lambda_k, j = 0$ except for some fixed value of $k$ and in this case we have the collection $E_i$ being a linearly dependent set of vectors for the same value of $i$. For $k \leq \lambda$, let $j_k$ be the largest integer $p < i_k - 1$ such that $\lambda_k, p \neq 0$. Then

$$0 = \sum_{i=1}^{j_i} \lambda_i, i^1 + \ldots + \sum_{i=1}^{j_\lambda} \lambda_i, i^\lambda$$

or equivalently

$$j_i \leq \lambda_i, j_i^1 + \ldots + \sum_{i=1}^{j_\lambda} \lambda_i, j_i^\lambda$$

where $\lambda_i, j_i = \lambda_i, j_i^1, j_i^\lambda$.

But then $u^1_{j_1} \in \text{range } \Lambda_{j_1} \ldots u^1_{j_\lambda}$, or equivalently

the range of $\Lambda_{j_1} \ldots u^1_{j_\lambda}$ is contained in the range of $u^1_{j_1} \ldots u^1_{j_\lambda}$.

$A_{u_{j_1}}$. In particular this says that $u_{j_1}, \ldots, u_{j_\lambda}$ appear in the factorization of $u_1$, but this is impossible since $u_{j_\lambda}$ has a zero at $a_{j_\lambda}$ but $u_1$ does not. We now have the desired conclusion, namely: the dimension of the orthogonal complement of the range of $A_u$ is $\sum_{j=1}^{\lambda} \lambda_j$ (i.e. the sum of the orders of the zeros of the function $u$).

As in the previous theorem, let $u$ be a finite product of informal mappings of $U$. Let $N = (\text{Range of } A_u)^\perp$. 
Then, as in the case for one conformal mapping, we claim that $H = \sum_{i=0}^{\infty} A_u^i(N)$. For again, if $f \perp A_u^i(N)$ for $i = 0, \ldots, n$, then $f$ is infinitely divisible by $A_u$ and this leads to the absurdity of $f$ having a zero of infinite order. This leads then to the following Theorem.

**Theorem 3.10** Let $u$ be a finite Blaschke product. Then $A_u$ is spatially isomorphic to a shift on $\mathbb{Z}$, where the dimension of $N = \sum$ the zeros of $u$ counted according to their multiplicity.

**Proof:** This is very easy to see in view of the fact that $A_u^i(N) = A_u^j(N)$ for all $i$ and $j$.

We now characterize the isometries in $H^\infty$ corresponding to infinite Blaschke products and to singular functions. First we prove two lemmas.

**Lemma 3.11** Let $u$ be an inner function; $h_n(z) = z^n$ and let $R = \text{range of } A_u$ and $N = R^\perp$. Then the orthogonal decomposition of $h_n$ with respect to $N \oplus R$ is given by

$$h_n = u \cdot f + (h_n - u \cdot f)$$

where $f(z) = u! \sum_{i=0}^{n} \binom{n}{i} u^{n-i}(0)z^i$.

(note: $u^i$ denotes the $i$th derivative of $u$.)

**Proof:** Since $uf$ is obviously in the range of $A_u$, it suffices to show that $h_n - uf$ is $\perp$ to the range of $A_u$; i.e. that $(ug, h_n - uf) = 0$ for all $g \in H^2(U)$. The following computation establishes this.
(u,g, h_n - uf) = (u, g, h_n) - (u, g, uf) = 

(u, g, h_n) - (g, f) = \int_T U(w)G(w)\overline{w}^n dm(w)

- \int_T G(w)\overline{F(w)}dm(w) =

(u \cdot g)^n(0) - n! \sum_{i=0}^{n} \binom{n}{i} u^{n-i}(0) \int_T G(w)w^i dm(w) =

n! \sum_{i=0}^{n} \binom{n}{i} u^{n-i}(0) g^i(0) - n! \sum_{i=0}^{n} \binom{n}{i} u^{n-i}(0) g^i(0) = 0

Lemma 3.12 No singular function or infinite Blaschke product can be the ratio of polynomials.

Proof: Suppose u is an inner function and u(z) = P(z)/Q(z) where (P, Q) = 1. Let w \in T, and suppose that Q(w) = 0. Then since (P, Q) = 1, P(w) \neq 0 and hence in some neighborhood of w, |P(z)/Q(z)| > 2.

It therefore follows that u is not inner. Hence Q has no zeros on T and therefore u is continuously extendable to T. But the only inner functions which are continuous on the boundary of the unit circle are finite Blaschke products. Hence the desired result.

Theorem 3.13 If u is an inner function having either a singular function or an infinite Blaschke product in its factorization, then the dimension of the orthogonal complement of the range of \( A_u \) is \( \infty \). It follows from this that any two such inner functions yield unitarily equivalent operators on \( H^2(U) \), namely generalized shifts on \( \sum_{i=0}^{\infty} \Theta H \) where \( \dim H = \infty \).
Proof: Let $u$ be an inner function with either a singular function or an infinite Blaschke product in its factorization. Let $e_j(z) = h_j(z) - u(z)f_j(z)$ where $h_j(z) = z^j$ and $f_j(z) = \sum_{i=0}^{\lfloor j/2 \rfloor} (i^j)u^{j-2i}(0)z^i$.

By Lemma 3.11, each $e_j$ is 1 to the range of $u$. We now claim that the $e_j$ are linearly independent.

Suppose not. Then there exists scalars $c_1, \ldots, c_n$ such that $\sum_{j=1}^{n} c_j e_j = 0$ or equivalently $0 = \sum_{j=1}^{n} c_j(h_j - uf_j)$ and hence

$$\sum_{j=1}^{n} c_j h_j = u\sum_{j=1}^{n} c_j f_j.$$ 

Therefore $u = P/Q$, where $P = \sum_{j=1}^{n} c_j f_j$ and $Q = \sum_{j=1}^{n} c_j f_j$.

But since $P$ and $Q$ are polynomials, it follows by Lemma 3.12 that $u$ is a finite Blaschke product. But this is a contradiction and hence the collection

$\{e_i\}_{i=1}^{\infty}$ is linearly independent, yielding

$$\dim (\text{Range of } A_u^1) = \infty.$$ 

The remaining part of the theorem can be demonstrated by methods similar to those used for the finite Blaschke products. Namely, we need only observe that $H^2(U) = \sum_{i=0}^{\infty} A^i_u(N)$, where $N = (\text{Range of } A_u^1)$.

We close this chapter with one more observation concerning specific operators in $H^\infty(U^1)$. More precisely, we show that $H^\infty(U)$ lacks certain operators.

**Theorem 3.14** The only compact operators in
$R^1 = \{A_f | f \in H^\infty(U^1)\}$ is the trivial one, $A_f = 0$.

**Proof:** Suppose $A_f \in R^1$ and that $A_f$ is a non-zero compact operator. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and let $B = \{h_n\}_{n=0}^{\infty} \subset H^2(U)$ where $h_n(z) = z^n$. Since $B$ is a norm bounded set and $A_f$ is a compact operator, there exists a subsequence $\{n_p\}_{p=0}^{\infty}$ of the integers such that $\{g_{p} \}_{p=0}^{\infty} = \{A_f h_{n_p}\}_{p=0}^{\infty}$ is Cauchy. Let $a_r$ be the first non-zero coefficient in the series expansion for $f$, and choose $\varepsilon < |a_r|^2$. For this value of $\varepsilon$, there exists an integer $M$ such that when $p > q > M$, $||g_p - g_q||^2 < \varepsilon$. Choose $p > q > M$ such that $j = n_p - n_q > r$. But then for this choice of $p$, $q$ we have

$$||g_p - g_q||^2 = \left| \left( \sum_{k=0}^{j} a_k z^{k+n_p} - \sum_{k=0}^{j} a_k z^{k+n_q} \right) \right|^2 = \left| \left( \sum_{k=0}^{j} a_k z^{k+n_q} \right) + \left( \sum_{k=0}^{\infty} (a_k - a_{k+j}) z^{k+n_p} \right) \right|^2,$$

$$j \sum_{k=0}^{\infty} |a_k|^2 + \sum_{k=0}^{\infty} |a_i - a_{i+j}|^2 < \varepsilon.$$

But in view of the fact that $j > r$, $\sum_{k=0}^{j} |a_k|^2$ has $|a_r|^2$ as part of its sum. This yields a contradiction since $\varepsilon$ was chosen such that $\varepsilon < |a_r|^2$. 
In this chapter, we present an alternative description for describing $H^2(U^2)$ and $H^\infty(U^2)$. In particular, we show that in some sense of the word $H^2(U^2)$ may be considered as an analytic direct integral of $H^2(U)$ over $T$, the distinguished boundary of $U$. We also show a similar result for $H^\infty(U^2)$. These results may, of course, in an appropriate manner be generalized to the higher dimensional spaces.

By the symbol $\int_T H^2(U) \, dm(w)$, the analytic direct integral of $H^2(U)$ over $T$, we shall mean the set of all vector valued functions $f$ defined almost everywhere such that:

(i) $f(w)$ is in $H^2(U)$ for almost all $w$ in $T$.

(ii) for each integer $n \geq 0$, $a_n$ is the boundary value function for an analytic function, where $f(w)(z) = \sum_{n=0}^{\infty} a_n(w) z^n$ is the series expansion for $f(w)$, whenever $f(w)$ is in $H^2(U)$.

(iii) the mapping $w \to ||f(w)||^2$ is summable on $T$ with respect to the measure $dm$.

We note that conditions (i) and (iii) actually imply that the mapping $w \to (f(w), g)$ is measurable for each $f$ in $\int_T H^2(U) \, dm(w)$ and $g$ in $H^2(U)$. For since

$$(f(w), g) = \int_T f(w)(x) \overline{g(x)} \, dm(x) = \sum_{n=0}^{\infty} \int_T a_n(w) x^n \overline{g(x)} \, dm(x)$$

and for each $n$ the mapping $w \to \int_T a_n(w) x^n \overline{g(x)} \, dm(x)$ is
evidently measurable because of the analyticity of the mapping \( w \mapsto a_n(w) \), it follows that

\[
\sum_{n=0}^{\infty} \int_{T} a_n(w)x^n \, g(x) \, dm(x) \text{ is measurable.}
\]

Of course, we equip \( \mathcal{H}(U)dm(w) \) with the obvious inner product:

\[
(f, g) = \int_{T} (f(w), g(w)) dm(w).
\]

It is easy to see that \( \mathcal{H}(U)dm(w) \) becomes a Hilbert space when equipped with this inner product and that

\[
|f|^2 = \int_{T} |f(w)|^2 dm(w)
\]

for every \( f \) in \( \mathcal{H}(U)dm(w) \).

We also define the analytic direct integral of the algebra, \( H^\infty(U) \) over \( T \). We let \( \mathcal{H}(U)dm(w) \) be the set of all vector valued functions \( B \) defined almost everywhere on \( T \) such that:

(i) \( B(w) \) is in \( H^\infty(U) \) for almost all \( w \) in \( T \).

(ii) \( w \mapsto (B(w)f, g) \) is measurable for all \( f \) and \( g \) in \( H^2(U) \).

(iii) for each integer \( n \geq 0 \), \( b_n \) is the boundary value function for an analytic function in \( U \), where \( B(w)(z) = \sum_{n=0}^{\infty} b_n(w)z^n \) is the series expansion for \( B(w) \).

(iv) the mapping \( w \mapsto ||B(w)||_\infty \) is essentially bounded.

As before, condition (ii) is actually implied by conditions (i) and (iii). We include it merely for emphasis.

If we equip \( \mathcal{H}(U)dm(w) \) with the norm
Theorem 4.1 For each \( f \) in \( \mathcal{H}^2(U) \), let \([f(w)]\) represent the boundary value function of \( f(w) \) on \( T \) for almost all \( w \) in \( T \). Then whenever \( f \) is in \( \mathcal{H}^2(U) \), \( f \) has an expansion of the form:

\[
[f(w_1)](w_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_p\, q \ w_1^p w_2^q, \quad \text{where} \quad \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} |a_p\, q|^2 < \infty \quad \text{and the series on the right converges in 2-norm.}
\]

Conversely, the equation above determines uniquely an element of \( \mathcal{H}^2(U) \). Hence it follows that \( \mathcal{H}^2(U) \) is isometric isomorphic to \( \mathcal{H}^2(U) \).

Proof: Let \( f \) be in \( \mathcal{H}^2(U) \). For almost all \( w_1 \) in \( T \), \( f(w_1) \) is in \( \mathcal{H}(U) \) and hence \( f(w_1)(w_2) = \sum_{q=0}^{\infty} a_q\, w_1^q w_2^q \) where \( \sum_{q=0}^{\infty} |a_q(w_1)|^2 < \infty \). Also, since \( a_q(w_1) \) is analytic as a function of \( w_1 \) for each \( q \) fixed we have that \( a_q \) has a series expansion

\[
a_q(w_1) = \sum_{p=0}^{\infty} a_p\, q \, w_1^p.
\]

Now

\[
f(w_1)(w_2) = \sum_{q=0}^{\infty} a_q\, w_1^q \sum_{p=0}^{\infty} a_p\, q \, w_2^p = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_p\, q \, w_1^p w_2^q.
\]
Conversely, suppose that \( \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p,q} w_1^p w_2^q \leq p o q \). We claim that the formula

\[
[f(w_1)](w_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p,q} w_1^p w_2^q
\]
determines an element, \( f \), in \( \int_T H^\infty(w) dm(w) \). In particular letting \( a_q(w_1) = \sum_{p=0}^{\infty} a_{p,q} w_1^p \) for each \( q \geq 0 \) we have that \( a_q \) is in \( L^2(T) \) since \( \sum_{p=0}^{\infty} |a_{p,q}|^2 < \infty \) for every \( q \geq 0 \). Clearly, then, for each \( q \geq 0 \) \( a_q \) is the boundary value of a function in \( H^2(U) \). Now since

\[
\int_T \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} |a_q(w_1)|^2 dm(w_1) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} |a_{p,q}|^2 < \infty
\]
we have that for almost all \( w_1 \)

\[
\sum_{q=0}^{\infty} |a_q(w_1)|^2 < \infty.
\]

It follows then that \( f(w_1) \) is in \( H^2(U) \) for almost all \( w_1 \) in \( T \). More precisely, what we have proved is that \( [f(w_1)] \) is the boundary value function of a function in \( H^2(U) \) for almost all \( w_1 \) in \( T \). The above computations also prove that the mapping \( w_1 \mapsto |f(w_1)|^2 \) is summable over \( T \). We have therefore that \( f \) is in \( \int_T H^2(U) dm(w) \).

For \( f \) in \( \int_T H^2(U) dm(w) \) we let \( Uf = g \) where

\[
g(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p,q} z_1^p z_2^q \quad (p, q \geq 0). \]

The above proof shows that \( U \) is an isometry of \( \int_T H^2(U) dm(w) \) onto \( H^2(U^2) \). We use here the following characterization of \( H^2(U^2) \).

\( H^2(U^2) \) is the set of all functions

\[
g(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p,q} z_1^p z_2^q \quad \text{where} \quad \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} |a_{p,q}|^2 < \infty.
\]
If for each \( f \) in \( f_T H^\infty(U)dm(w) \) and \( g \) in \( f_T H^2(U)dm(w) \), we let

\[
T_fg(w) = T_f(w)g(w) = fg(w)
\]

for almost all \( w \) in \( T \), then \( T_f \) is obviously a bounded linear operator on \( f_T H^2(U)dm(w) \). We let \( S^2 \) be the ring of all such operators \( T_f \) with \( f \) in \( f_T H^\infty(U)dm(w) \). We previously claimed that \( f_T H^\infty(U)dm(w) \) is a normed algebra. The following shows that this analytic direct integral actually is complete and is hence a Banach algebra. It accomplishes this by classifying \( S^2 \), the algebra of associated linear operators.

**Theorem 4.2** \( S^2 \) is unitarily equivalent to \( \mathbb{R}^2 \).

In order to facilitate the proof of Theorem 4.2, we first prove two results, one of which is interesting enough to be designated as a theorem.

**Theorem 4.3** If \( f(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p,q} z_1^p z_2^q \) is the series expansion for a function \( f \) in \( H^\infty(U^2) \) then for all nonnegative integers \( q \), the function \( f_q \) defined by

\[
f_q(z) = \sum_{p=0}^{\infty} a_{p,q} z^p
\]

is in \( H^\infty(U) \) and \( \|f_q\|_\infty \leq \|f\|_\infty \).

**Proof:** Let \( M = \|f\|_\infty \). Then by Theorem 3.3, we have that whenever \( \{b_{p,q}\} \) \( \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} |b_{p,q}|^2 < \infty \) is a double sequence such that

\[
\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} |b_{p,q}|^2 < \infty
\]

then

\[
\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p,q} b_{p,q} \leq M \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} |b_{p,q}|^2.
\]
Now let \( \{ c_i \}_{i=1}^{\infty} \) be an arbitrary sequence such that 
\[
\sum_{i=0}^{\infty} |c_i|^2 < \infty.
\]
Then if we set \( d_{i,j} \) equal to \( c_i \) whenever \( j = 0 \) and equal to 0 otherwise, we have

\[
\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |d_{i,j}|^2 = \sum_{i=0}^{\infty} |c_i|^2 < \infty.
\]

Also

\[
\sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left| \sum_{p,k} a_{p,k} c_{p-k} \right|^2 = \sum_{p=0}^{\infty} \sum_{k=0}^{\infty} \left| \sum_{m=0}^{\infty} a_{p,m} b_{p-k,k-m} \right|^2 = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left| \sum_{m=0}^{\infty} a_{p,m} b_{p-k,q-m} \right|^2 = \sum_{p=0}^{\infty} |c_p|^2
\]

for each nonnegative integer \( k \). Hence it follows by Theorem 3.2 that for each such \( k \), \( f_k(z) = \sum_{p=0}^{\infty} a_{p,k} z^p \) is the series expansion of a function in \( \mathcal{H}^\infty(U) \) and that 
\[
||f_k||_\infty \leq M = ||f||_\infty.
\]

Before proving the remaining necessary lemma, we present some conventions. In view of Theorem 4.1,

whenever \( f(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p,q} z_1^p z_2^q \) is in \( \mathcal{H}^2(U^2) \),

then \( f_{w_1}(z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p,q} w_1^p z_2^q \) is in \( \mathcal{H}^2(U) \) for almost all \( w_1 \) in \( T \). Here, the series for \( f_{w_1} \) converges in 2-norm. For \( w_1 \) in \( T \), \( 0 < r < 1 \) we let

\[
f_{rw_1}(z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p,q} (rw_1)^p z_2^q.
\]

It is easy to see that \( f_{rw_1} \) is in \( \mathcal{H}^2(U) \) whenever \( f \) is in \( \mathcal{H}^2(U^2) \); \( f_{rw_1} \) is a so-called "slice function" of \( f \).
Lemma 4.4 If $f(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p,q} z_1^p z_2^q$ represents a function in $H^2(U^2)$, then for almost all $w_1$ in $T$ there exists a sequence $\{r_n\}_{n=1}^{\infty}$ such that $0 < r_n < 1$ for $n = 1, 2, 3, \ldots$ and $f_{r_n w_1}$ converges to $f_{w_1}$ in 2-norm as $n$ converges to infinity.

Proof: Let $f$ be in $H^2(U^2)$ and $w_1$ in $T$ such that $f_{w_1}$ is in $H^2(U)$. Then since

$$f_{w_1} - f_{rw_1}(z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p,q} (1 - r^i)w_1^i z_2^q$$

we have

$$||f_{w_1} - f_{rw_1}||_2^2 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} (1 - r^i)w_1^i |z_2|^j.$$ Intergrating this expression we obtain

$$\int_T ||f_{w_1} - f_{rw_1}||_2^2 dm(w_1) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} (1 - r^i)w_1^i |z_2|^j dm(w_1) =$$

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} (1 - r^i)w_1^i |1 - r^i|^2.$$ We now claim that $\lim_{r \to 1^-} \int_T ||f_{w_1} - f_{rw_1}||_2^2 dm(w_1) = 0$ and hence it will follow that $\lim_{r \to 1^-} \int_T ||f_{w_1} - f_{rw_1}||_2^2 dm(w_1) = 0$.

To see this, let $\varepsilon > 0$ and choose $N$ such that

$$\sum_{i=N}^{\infty} \sum_{j=0}^{\infty} a_{i,j} |1 - r^i|^2 < \varepsilon/2.$$ Having chosen $N$, pick $\delta > 0$ such that

$$1 - r^i < \varepsilon/2 (\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j}^2) = \gamma$$

for $i = 1, \ldots, N$ and $1 - \delta < r < 1$. It suffices to choose $\delta$ such that $1 - \delta > N \sqrt{1 - \gamma}$ since when $1 - \delta < r$ we have
that

\[ r^N > (1 - \delta)^N > 1 - \gamma \]

and for \( i = 1, \ldots, N \)

\[ r^i > r^N > 1 - \gamma \]

or equivalently

\[ 1 - r^i < \gamma. \]

With this choice of \( \delta \), whenever \( 1 - \delta < r < 1 \) we have

\[ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |a_{i,j}|^2 |1 - r^i|^2 = \]

\[ \sum_{i=0}^{N} \sum_{j=0}^{N} |a_{i,j}|^2 (1 - r^i)^2 + \sum_{i=0}^{N} \sum_{j>N} |a_{i,j}|^2 (1 - r^i)^2 \]

or equivalently

\[ \sum_{i=0}^{N} \sum_{j=0}^{N} |a_{i,j}|^2 \gamma + \sum_{i=0}^{N} \sum_{j>N} |a_{i,j}|^2 < \]

\[ \left( \epsilon/2 \right) \left( \sum_{i=0}^{N} \sum_{j=0}^{N} |a_{i,j}|^2 \right) \left( \sum_{i=0}^{N} \sum_{j=0}^{N} |a_{i,j}|^2 \right) + \epsilon/2 = \]

\[ \epsilon/2 + \epsilon/2 = \epsilon \]

The claim is now established and we have that

\[ \lim_{r \to 1^+} \int_T |f_{r_0 w_1} - f_{w_1}|^2 \, dm(w_1) = 0. \]

Let \( \{V_n\}_{n=1}^{\infty} \) be a sequence in (0, 1) such that \( \lim_{n \to \infty} V_n = 1 \). Then

\[ \int_T |f_{r_n w_1} - f_{w_1}|^2 \, dm(w_1) \to 0 \]

and hence we may choose a subsequence of \( r_n \), say \( \{S_n\}_{n=1}^{\infty} \) such that

\[ |f_{S_n w_1} - f_{w_1}|^2 \to 0 \]

for almost all \( w_1 \) in \( T \).

We are now ready to prove Theorem 4.2.

**Proof:** Let \( U \) be as defined at the end of the proof of Theorem 4.1. We begin by showing that \( U \) is multiplicative.
To this end, we let \( f \) and \( g \) be in \( f_T H^2(U)dm(w) \) and we suppose that 
\[
[f(w_1)](w_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p,q} w_1^p w_2^q
\]
and
\[
[g(w_1)](w_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} b_{p,q} w_1^p w_2^q
\]
are their respective representations as guaranteed in Theorem 4.1. Then

\[
(Uf)(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_{p,q} z_1^p z_2^q,
\]

\[
(Ug)(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} b_{p,q} z_1^p z_2^q
\]
and

\[
(Uf)(Ug)(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (\sum_{\ell=0}^{p} \sum_{m=0}^{q} a_{\ell,m} b_{p-\ell,q-m}) z_1^p z_2^q.
\]

Since \( f * g(w_1) = f(w_1)g(w_1) \) for almost all \( w_1 \) in \( T \), it follows that 
\[
[f * g(w_1)](w_2) = [f(w_1)](w_2) \cdot [g(w_1)](w_2)
\]
\[
= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (\sum_{\ell=0}^{p} \sum_{m=0}^{q} a_{\ell,m} b_{p-\ell,q-m}) w_1^p w_2^q.
\]
We thus have

\[
[U(f * g)](z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} (\sum_{\ell=0}^{p} \sum_{m=0}^{q} a_{\ell,m} b_{p-\ell,q-m}) z_1^p z_2^q
\]
and it follows immediately that \( U(f * g) = (Uf) \cdot (Ug) \).

Now let \( F = T_f \) where \( f \) is in \( f_T H^\infty(U)dm(w) \) so that \( F \) is in \( S^2 \). Also let \( g \) be in \( f_T H^2(U)dm(w) \) and set \( h = Ug \). Setting \( b = Uf \) (note that since \( f \) is in \( f_T H^\infty(U)dm(w) \) it is also in \( f_T H^2(U)dm(w) \) we obtain

\[
UFU^{-1}h = UTfg = U f \cdot g = (Uf)(Ug) = b \cdot h;
\]
i.e. under the isometry \( U \), \( F \) maps into multiplication by \( b \) on \( H^2(U^2) \). In view of the fact that the range of \( U \) is \( H^2(U^2) \), it follows that the operator \( UFU^{-1} \) is actually multiplication by the \( H^2(U^2) \) function, \( b \).

Clearly this operator then commutes with \( R^2 \) and since \( R^2 \)
is maximal commutative (Theorem 1.1), it follows that 
b is actually in $H^\infty(U^2)$. Hence the mapping \( F \circ UFU^{-1} \) maps \( S^2 \) into \( R^2 \).

We now show that \( F \circ UFU^{-1} \) maps \( S^2 \) onto \( R^2 \). To this end, let \( b \) be in \( H^\infty(U^2) \) and set \( f = U^{-1}b \). Assuming that \( f \) is actually in \( \int_T H^\infty(U) \, dm(w) \), the following computation establishes the fact that \( UT_f U^{-1} = T_b \) and hence shows that the mapping \( F \circ UFU^{-1} \) is onto \( R^2 \).

For all \( h \) in \( H^2(U^2) \) we have

\[
UFU^{-1}h = UT_f(U^{-1}h) = U(f \cdot U^{-1}h) = (Uf) \cdot (UU^{-1}h) = b \cdot h = T_b h.
\]

We shall now show that \( f \) is in \( \int_T H^\infty(U) \, dm(w) \). Let

\[
b(z_1, z_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} b_{p,q} z_1^p z_2^q
\]

so that

\[
[f(w_1)](w_2) = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} b_{p,q} z_1^p z_2^q
\]

is the representation of an element \( f \) in \( \int_T H^2(U) \, dm(w) \). We first show that for almost all \( w_1 \) in \( T \), \( f(w_1) \) is in \( H^\infty(U) \). Using the notation of Lemma 4.4, we have that \( f(w_1) = b_{w_1} \) and hence there exist a sequence \( \{ r_n \}_{n=1}^\infty \) in \((0, 1) \) converging to 1 such that \( b_{r_n w_1} \) converges \( b_{w_1} \) in 2-norm for almost all \( w_1 \) in \( T \). It is well known that convergence in 2-norm implies uniform convergence on compacta (This is a direct application of Cauchy's Integral Formula and the Cauchy-Schwartz Inequality). But since

\[
|b_{r_n w_1}|_\infty \leq |b| \text{ for } n = 1, 2, 3, \ldots,
\]

it follows that for almost all \( w_1 \) in \( T \), \( b_{w_1} = f(w_1) \) is in \( H^\infty(U) \).
and \( \|f(w_1)\|_\infty \leq \|b\| \). We thus have conditions (i) and (iv) in the definition of \( \int_U H^\infty(U) \, dm(w) \).

It remains to be shown that for \( n = 1, 2, 3, \ldots \) \( b_n \) is the boundary value function for a function which is analytic on \( U \), where \( f(w_1)(z_2) = \sum_{n=0}^{\infty} b_n(w_1)z_2^n \) is the series expansion for \( f(w_1) \) whenever \( f(w_1) \) is in \( H^\infty(U) \). It is easy to see that \( b_n \) is the boundary value function for the mapping

\[
  z + \sum_{p=0}^{\infty} b_{p,n} z^p.
\]

This fact follows from the representation formula for \([f(w_1)](w_2)\). However, Theorem 4.3 guarantees that the above mapping is actually a bounded analytic function. This completes the proof.
BIBLIOGRAPHY


BIOGRAPHY

Willis Joseph Bourque, Jr. was born in Lafayette, Louisiana, on July 24, 1942. He attended Cathedral High School in Lafayette and graduated in May, 1959. From September, 1959, to January, 1962, he attended Louisiana State University. From June, 1962, to August, 1964, he attended the University of Southwestern Louisiana where he received a B.S.

In September, 1964, he enrolled in the graduate school of Louisiana State University where he taught as graduate assistant until August, 1968. He taught at the University of North Dakota from September, 1968, to May, 1969. He is currently teaching at the University of Southwestern Louisiana and is also a candidate for the degree of Doctor of Philosophy in Mathematics at Louisiana State University.
Candidate: Willis J. Bourque, Jr.

Major Field: Mathematics

Title of Thesis: $^\infty$ Rings of Operators

Approved:

[Signatures]

Date of Examination:

January 4, 1971