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Zeta functions of finite graphs

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Table of Contents

Acknowledgments .... **ii**

Abstract .... **iv**

Chapter 1: Introduction **1**
1.1 Definition of the Zeta Function of a Finite Graph 1
1.2 The Ihara-Hashimoto Theorem 4

Chapter 2: Properties Determined by the Zeta Function of a Finite Graph **15**
2.1 Coefficients in the Zeta Function of a Finite Graph 15
2.2 Zeta Functions of Graphs with Only One Vertex 16
2.3 Zeta Functions of Graphs with Exactly Two Vertices 18
2.4 Zeta Functions of Graphs with Exactly Three Vertices 19

Chapter 3: The Zeta Function and the Spectra of a Finite Graph **20**
3.1 Regular Graphs 22
3.2 Bipartite Graphs 26
3.3 Biregular-Bipartite Graphs 27

Chapter 4: Open Problems **35**

Bibliography **37**

Appendix: Select MAGMA Programs **38**

Vita **57**

Abstract

Ihara introduced the zeta function of a finite graph in 1966 in the context of p-adic matrix groups. The idea was generalized to all finite graphs in 1989 by Hashimoto. We will introduce the zeta function from both perspectives and show the equivalence of both forms. We will discuss several properties of finite graphs that are determined by the zeta function and show by counterexample several properties of finite graphs that are not determined by the zeta function. We will also discuss the relationship between the zeta function of a finite graph and the spectrum of a finite graph.

Chapter 1

Introduction

The zeta function of a finite graph was introduced by Ihara [Iha66] in 1966 in the context of p-adic matrix groups and was specialized to graphs in 1989 by Hashimoto [Has89]. In this chapter, we will introduce the zeta function of a finite graph. In Chapter 2, we will discuss the properties of a finite graph that are determined by its zeta function. The relationship between the spectrum of a finite graph and its zeta function is discussed in Chapter 3. We will conclude by proposing several topics and questions that require further investigation in Chapter 4. The examples provided in Chapter 2 and Chapter 3 were found using programs written in MAGMA. Several of these search programs are included in the Appendix.

1.1 Definition of the Zeta Function of a Finite Graph

A *graph* γ with vertex set V and edges set E is a finite, undirected graph and may include multiple edges, loops and multiple components. A *walk* is a sequence of vertices and edges and is denoted $(v_1, e_1, v_2, e_2, \cdots, v_{m-1}, e_{m-1}, v_m)$ where edge e_i connects vertex v_i and vertex v_{i+1}. If e_i is a loop, then a direction must also be specified along which e_i is traveled. Note that v_i and v_j may be equal for any i and j and e_i and e_j may equal for any i and j. A *cycle* is a walk in which the first and the last vertex are the same. If cycle C is unambiguous, then C may be denoted $(v_1, v_2, \cdots, v_{m-1}, v_1)$.

A cycle has *backtracking* if in the cycle a non-loop edge appears twice in immediate succession or if a loop is immediately followed by the same loop traveled in the opposite direction. A *tail* occurs in a cycle without backtracking if the first edge and the last edge in the cycle are the same non-loop edge or if the first and last edge are the same loop traveled in opposite directions. Cycle C is *primitive* if C is not the power of another cycle. In other words, cycle C cannot be obtained by repeating another cycle a finite number of times.

Cycle $C = (v_1, e_1, v_2, e_2, \cdots, v_{m-1}, e_{m-1}, v_1)$ and cycle D are *equivalent* if there exists an index i such that cycle $D = (v_i, e_i, v_{i+1}, e_{i+1}, \cdots, v_{m-1}, e_{m-1}, v_1, \cdots, v_{i-1}, e_{i-1}, v_i)$ for some vertex v_i in cycle C. Let $[C]$ denote the class of all cycles equivalent to primitive, tail-less, backtrackless cycle C, and let $\pi(\gamma)$ denote the set of all such $[C]$. Let $v([C])$ be the number of edges in representative C of equivalence class $[C]$.

Definition 1.1.1. The *Ihara zeta function* of finite graph γ is

$$Z_\gamma(u) = \prod_{[C]\in\pi(\gamma)} \left(1 - u^{v([C])}\right)^{-1}. \tag{1.1}$$

In this dissertation, we will shorten the name "Ihara zeta function of a graph" to "zeta function of a graph".

The product in Definition 1.1.1 is a formal product. For given $[C]$, the factor

$$\left(1 - u^{v([C])}\right)^{-1} = \frac{1}{1 - u^{v([C])}} = 1 + u^{v[C]} + u^{2v[C]} + \cdots$$

is a power series in u. Since γ is a finite graph, there are only a finite number of cycles with a given length. Hence, the zeta function of a finite graph is a product of power series in which each power of u appears only a finite number of times. Thus, the product in Definition 1.1.1 is a power series.

Example 1.1.2. Let γ be a n-gon graph. Then

$$Z_\gamma(u) = (1 - u^n)^{-2}$$

since there are only two non-equivalent primitive cycles without tails and backtracking, one clockwise and one counterclockwise.

The barycentric subdivision of graph γ, denoted $\gamma_{(2)}$, is created by adding a vertex to the midpoint of each edge. In other words, each edge is divided into two edges by adding one vertex per edge.

Proposition 1.1.3 ([Has89]). *Let γ be a finite graph. Then*

$$Z_\gamma(u^2) = Z_{\gamma_{(2)}}(u).$$

Proof. The proof follows directly from the definition of the zeta function of a finite graph. The equivalence classes of primitive, tail-less cycles without backtracking in graph γ have a one-to-one correspondence onto the equivalence classes of primitive, tail-less cycles without backtracking in $\gamma_{(2)}$. The bijection takes an equivalence class from graph γ where a representative of the class has $v([C])$ edges and maps it to the corresponding equivalence class in graph $\gamma_{(2)}$ where a representative has $2v([C])$ edges. Thus,

$$Z_\gamma(u^2) = \prod_{[C]\in\pi(\gamma)} \left(1 - (u^2)^{v([C])}\right)^{-1} = \prod_{[C]\in\pi(\gamma)} \left(1 - u^{2v([C])}\right)^{-1} = Z_{\gamma_{(2)}}(u).$$

■

1.2 The Ihara-Hashimoto Theorem

Let γ be a finite graph with n vertices and e edges, and let $n \times n$ matrix A be the adjacency matrix of graph γ. By definition, for $i \neq j$, element $A_{i,j}$ of matrix A is the number of edges between vertex i and vertex j and $A_{i,i}$ is twice the number of loops at vertex i. Note that $(A^n)_{i,j}$ is the number of walks of length n starting at vertex i and ending at vertex j. Define the Q-matrix of graph γ as a diagonal $n \times n$ matrix where $Q_{i,i}$ is one less than the degree of vertex i. Let I be the $n \times n$ identity matrix.

Theorem 1.2.1 ([Has89]). *The zeta function of finite graph γ can be written*

$$Z_\gamma(u) = \frac{(1-u^2)^{n-e}}{\det(I - Au + Qu^2)}.$$

We will conclude this section by proving Theorem 1.2.1 by following the proof in Stark and Terras [ST96].

Let A_m be an $n \times n$ matrix where element $(A_m)_{i,j}$ is the number of walks in graph γ of length m with no backtracking starting at vertex i and ending at vertex j. Define $A_0 = I$, and note that $A_1 = A$.

Lemma 1.2.2. *Matrix $A_2 = A^2 - (Q + I)$ and for $m \geq 3$, matrix $A_m = A_{m-1}A - A_{m-2}Q$.*

Proof. Entry $(A^2)_{i,j}$ counts the number of walks of length two from vertex i to vertex j. Entry $(A_2)_{i,j}$ counts the number of walks of length two without backtracking from vertex i to vertex j. If $i \neq j$, then there are no walks of length two from vertex i to vertex j with backtracking. So if $i \neq j$, then $(A_2)_{i,j} = (A^2)_{i,j}$. Backtracking will occur in a cycle of length two only if the cycle is of the form (i, e_1, j, e_1, i). This will occur exactly once for each edge and twice for each loop. So $(A^2)_{i,i}$ counts the degree of vertex i or $(Q_{i,i} + 1)$ more cycles than $(A_2)_{i,i}$. Thus, $A^2 = A_2 - (Q + I)$.

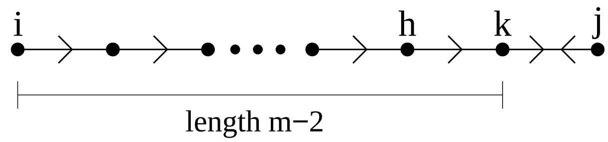

Figure 1.1: Walk $(i, \cdots, h, k, j, k)$ contains backtracking.

Now assume $m \geq 3$. Let V_k be the set of all vertices adjacent to vertex k. Note that if vertex k has a loop, then $k \in V_k$. The sum

$$\sum_{j \in V_k} (A_{m-1})_{i,j} A_{j,k}$$

computes the number of walks without backtracking of length m from vertex i to vertex k plus the number of walks with backtracking of the form depicted in Figure 1.1. Note that the walk from vertex i to vertex j through vertex k of length $m-1$ in Figure 1.1 has no backtracking, and that vertex j could be any of the $Q_{k,k}+1$ vertices adjacent to vertex k with the exception of vertex h. Thus, the number of walks of the form depicted in Figure 1.1 is $(A_{m-2})_{i,k} Q_{k,k} = (A_{m-2}Q)_{i,k}$. So $A_m = A_{m-1}A - A_{m-2}Q$ as desired. ∎

For $m \geq 1$, let t_m be the number of cycles of length m with tails and no backtracking in graph γ. Note that in this situation a tail must occur at the beginning of a cycle to avoid backtracking.

Lemma 1.2.3. *We have $t_1 = 0$ and $t_2 = 0$. For $m \geq 3$, we have $t_m = Tr[(Q-I)A_{m-2}] + t_{m-2}$.*

Proof. Since there are no walks with a tail and no backtracking of length one or two, we have $t_1 = 0$ and $t_2 = 0$.

Let V be the set of vertices in graph γ, and let V_k be the set of all vertices adjacent to vertex k. Define C_m to be the set of all cycles of length m with a tail and no backtracking.

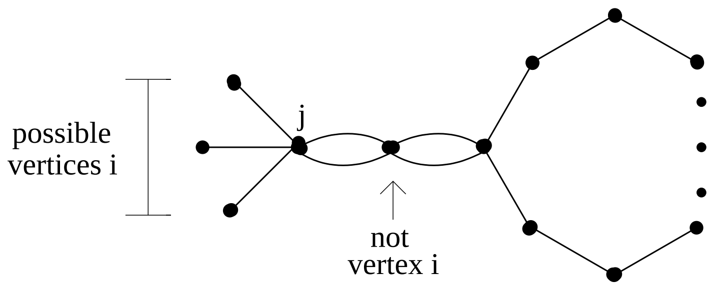

Figure 1.2: Possible vertices i if cycle D has a tail and no backtracking.

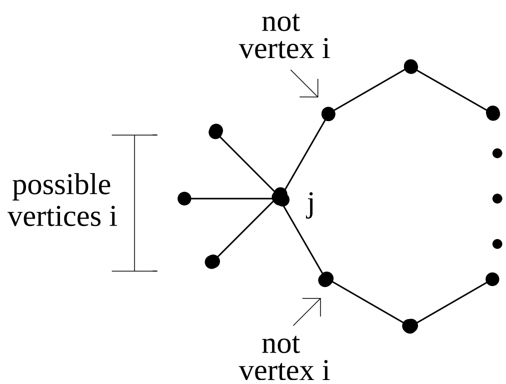

Figure 1.3: Possible vertices i if cycle D has no backtracking and no tail.

Let E_m be the set of all tail-less cycles of length m with no backtracking. Then

$$t_m = \sum_{i \in V} \text{number of elements in } C_m \text{ with initial vertex } i$$
$$= \sum_{i \in V} \sum_{j \in V_i} \text{number of elements in } C_m \text{ with initial vertex } i \text{ and second vertex } j.$$

Since a tail has to occur at the beginning of a cycle without backtracking, cycle C in C_m starting at vertex i and going directly to vertex j has the form $(i, e_1, j, e_2, \cdots, j, e_1, i)$. Let cycle $D = (j, e_2, \cdots, , e_{m-2}, j)$ be the cycle of length $m-2$ that results from removing vertex i and edge e_1 from the beginning and the end of cycle C. If cycle D does have a tail, then vertex i must be one of the $Q_{j,j}$ vertices which are adjacent to vertex j and not one step away from vertex j in cycle D. This situation is depicted in Figure 1.2. If cycle D does not have a tail, in order to avoid backtracking, vertex i must be one of the $Q_{j,j} - 1$ vertices which are adjacent to vertex j and not adjacent to vertex j in cycle D. This situation is depicted in Figure 1.3.

So,

$$
\begin{aligned}
t_m &= \sum_{i \in V} \sum_{j \in V_i} [Q_{j,j} (\text{number of elements in } C_{m-2} \text{ with initial vertex } j) \\
&\qquad + (Q_{j,j} - 1)(\text{number of elements in } E_{m-2} \text{ with initial vertex } j)] \\
&= \sum_{i \in V} \sum_{j \in V_i} [(\text{number of elements in } C_{m-2} \text{ with initial vertex } j) \\
&\qquad + (Q_{j,j} - 1)(\text{number of elements in } C_{m-2} \text{ with initial vertex } j) \\
&\qquad + (Q_{j,j} - 1)(\text{number of elements in } E_{m-2} \text{ with initial vertex } j)] \\
&= \sum_{i \in V} \sum_{j \in V_i} [(\text{number of elements in } C_{m-2} \text{ with initial vertex } j) \\
&\qquad + (Q_{j,j} - 1)(\text{number of elements in } E_{m-2} \cup C_{m-2} \text{ with initial vertex } j)] \\
&= t_{m-2} + \mathrm{Tr}((Q - I)A_{m-2})
\end{aligned}
$$

■

Lemma 1.2.4. *Let $f(u)$ be a square matrix. Then*

$$
Tr\left(-\frac{d}{du}\log\left(I - f(u)\right)\right) = Tr\left(f'(u)(I - f(u))^{-1}\right).
$$

Proof. First we will see that

$$
\frac{d}{du}(f(u))^n = \sum_{j=0}^{n-1} (f(u))^j f'(u)(f(u))^{n-j-1}.
$$

Clearly, this is true for $n = 1$. Using induction,

$$\begin{aligned}
\frac{d}{du}(f(u))^n &= \frac{d}{du}(f(u))^{n-1}f(u) \\
&= \left(\frac{d}{du}(f(u))^{n-1}\right) f(u) + (f(u))^{n-1}f'(u) \\
&= \left(\sum_{j=0}^{n-2}(f(u))^j f'(u)(f(u))^{n-1-j-1}\right) f(u) + (f(u))^{n-1}f'(u) \\
&= \sum_{j=0}^{n-2}(f(u))^j f'(u)(f(u))^{n-j-1} + (f(u))^{n-1}f'(u) \\
&= \sum_{j=0}^{n-1}(f(u))^j f'(u)(f(u))^{n-j-1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Tr}\left(\frac{d}{du}(f(u))^n\right) &= \mathrm{Tr}\left(\sum_{j=0}^{n-1}(f(u))^j f'(u)(f(u))^{n-j-1}\right) \\
&= \mathrm{Tr}\left(\sum_{j=0}^{n-1} f'(u)(f(u))^{n-1}\right) \\
&= n\mathrm{Tr}\left(f'(u)(f(u))^{n-1}\right).
\end{aligned}$$

Using $-\log(I - f(u)) = \sum_{n\geq 1}\frac{1}{n}(f(u))^n$ and $\frac{1}{I-f(u)} = \sum_{n\geq 1}(f(u))^{n-1}$, we have

$$\begin{aligned}
\mathrm{Tr}\left(-\frac{d}{du}\log(I - f(u))\right) &= \mathrm{Tr}\left(\frac{d}{du}\sum_{n\geq 1}\frac{1}{n}(f(u))^n\right) \\
&= \mathrm{Tr}\left(\sum_{n\geq 1} f'(u)(f(u))^{n-1}\right) \\
&= \mathrm{Tr}\left(f'(u)\sum_{n\geq 1}(f(u))^{n-1}\right) \\
&= \mathrm{Tr}\left(f'(u)(I - f(u))^{-1}\right).
\end{aligned}$$

■

Lemma 1.2.5. *Let I be the identity matrix, A the adjacency matrix of graph γ, and Q the Q-matrix of graph γ. Then* $\exp(\mathit{Tr}(\log(I - Au + Qu^2))) = \det(I - Au + Qu^2)$.

Proof. Let $\lambda_j(u)$ be the set of n eigenvalues of matrix $f(u) = Au - Qu^2$. Then

$$\begin{aligned}
\exp(\mathrm{Tr}(\log(I - Au + Qu^2))) &= \exp\left(\mathrm{Tr}\left(-\sum_{j\geq 1}\frac{1}{j}(f(u))^j\right)\right) \\
&= \exp\left(-\sum_{j\geq 1}\frac{1}{j}\mathrm{Tr}(f(u))^j\right) \\
&= \exp\left(-\sum_{j\geq 1}\frac{1}{j}\sum_{s=1}^{n}(\lambda_s(u))^j\right) \\
&= \exp\left(\sum_{s=1}^{n}\left(-\sum_{j\geq 1}\frac{1}{j}(\lambda_s(u))^j\right)\right) \\
&= \exp\left(\sum_{s=1}^{n}\log(1 - \lambda_s(u))\right) \\
&= \exp\left(\log\prod_{s=1}^{n}(1 - \lambda_s(u))\right) \\
&= \prod_{s=1}^{n}(1 - \lambda_s(u)) \\
&= \det(1 - Au + Qu^2).
\end{aligned}$$

■

Now, we are ready to prove Theorem 1.2.1.

Proof of Theorem 1.2.1. Using Definition 1.1.1, we have

$$\log Z_\gamma(u) = \log\left(\prod_{[C]\in\pi(\gamma)}\left(1 - u^{v([C])}\right)^{-1}\right) = -\sum_{[C]\in\pi(\gamma)}\log(1 - u^{v([C])}) = \sum_{[C]\in\pi(\gamma)}\sum_{j\geq 1}\frac{1}{j}u^{j(v([C]))}.$$

Thus,

$$\begin{aligned}\frac{d}{du}\log Z_\gamma(u) &= \sum_{[C]\in\pi(\gamma)} v([C]) \sum_{j\geq 1} u^{j(v([C]))-1} \\ &= \sum_{j\geq 1}\sum_{d\geq 1} d \sum_{\substack{[C]\in\pi(\gamma)\\ d=v([C])}} u^{j(v([C]))-1} \\ &= \sum_{j\geq 1}\sum_{d\geq 1} d \sum_{\substack{[C]\in\pi(\gamma)\\ d=v([C])}} u^{v([C^j])-1}.\end{aligned}$$

It follows that

$$u\frac{d}{du}\log Z_\gamma(u) = \sum_{j\geq 1}\sum_{d\geq 1} d \sum_{\substack{[C]\in\pi(\gamma)\\ d=v([C])}} u^{v([C^j])}.$$

Let C^* be the set of all primitive, backtrackless, tail-less cycles in graph γ. Since there are d elements in $[C]$, we have

$$\begin{aligned}u\frac{d}{du}\log Z_\gamma(u) &= \sum_{j\geq 1}\sum_{d\geq 1}\sum_{\substack{C\in C^*\\ d=v(C)}} u^{v(C^j)} \\ &= \sum_{j\geq 1}\sum_{C\in C^*} u^{v(C^j)}.\end{aligned}$$

The sums over j and cycles C combine as one sum over all cycles C with no backtracking and no tails. Let N_m be the number of cycles in graph γ of length m with no backtracking and no tails. Then

$$u\frac{d}{du}\log Z_\gamma(u) = \sum_{m\geq 1} N_m u^m. \tag{1.2}$$

Expanding $\sum_{m\geq 0}(A_m u^m)(I - Au + Qu^2)$ and simplifying using the relations from Lemma 1.2.2, we get

$$\sum_{m\geq 0}(A_m u^m)(I - Au + Qu^2) = I(1-u^2). \tag{1.3}$$

Solving for I and using

$$\frac{1}{1-u^2} = \sum_{j\geq 0} u^{2j},$$

we have

$$I = \left(\sum_{m\geq 0} A_m u^m\right)\left(\sum_{j\geq 0} u^{2j}\right)\left(I - Au + qu^2\right)$$

or

$$I = \left(\sum_{m\geq 0}\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor} A_{m-2j}u^m\right)\left(I - Au + Qu^2\right). \tag{1.4}$$

By Lemma 1.2.3, for $m \geq 2$ we have

$$\begin{aligned} t_m &= \operatorname{Tr}\left((Q-I)A_{m-2}\right) + t_{m-2} \\ &= \operatorname{Tr}\left((Q-I)A_{m-2}\right) + \operatorname{Tr}\left((Q-I)\right)A_{m-4} + t_{m-4} \\ &= \operatorname{Tr}\left((Q-I)\sum_{j=1}^{\lfloor \frac{m-1}{2}\rfloor} A_{m-2j}\right). \end{aligned} \tag{1.5}$$

We know $N_m = \operatorname{Tr}(A_m) - t_m$. Using Equation 1.5, we have

$$\begin{aligned} N_m &= \operatorname{Tr}(A_m) - \operatorname{Tr}\left((Q-I)\sum_{j=1}^{\lfloor \frac{m-1}{2}\rfloor} A_{m-2j}\right) \\ &= \operatorname{Tr}\left(A_m - (Q-I)\sum_{j=1}^{\lfloor \frac{m-1}{2}\rfloor} A_{m-2j}\right) \end{aligned} \tag{1.6}$$

for $m \geq 2$.

For $m \geq 0$, define

$$N_m^* = A_m - (Q-I)\sum_{j=1}^{\lfloor \frac{m}{2}\rfloor} A_{m-2j} = QA_m - (Q-I)\sum_{j=0}^{\lfloor \frac{m}{2}\rfloor} A_{m-2j}.$$

Then for $m \geq 1$, we have

$$\text{(1.7)} \qquad \operatorname{Tr}(N_m^*) = \begin{cases} N_m & \text{if } m \text{ is odd} \\ N_m - \operatorname{Tr}(Q - I) & \text{if } m \text{ is even.} \end{cases}$$

Using the definition of N_m^*, Equation 1.3, and Equation 1.4, we have

$$\begin{aligned} &\left(\sum_{m\geq 0} N_m^* u^m\right)(I - Au + Qu^2) \\ &\qquad = \left(Q\sum_{m\geq 0} A_m u^m - (Q - I)\sum_{m\geq 0}\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} A_{m-2j} u^m\right)(I - Au + Qu^2) \\ &\qquad = Q(1 - u^2) - (Q - I) \\ &\qquad = I - Qu^2. \end{aligned}$$

Since $N_0^* = A_0 = I$, we have

$$\begin{aligned} \left(\sum_{m\geq 1} N_m^* u^m\right)(I - Au + Qu^2) &= I - Qu^2 - (I - Au + Qu^2) \\ &= Au - 2Qu^2. \end{aligned}$$

Thus,

$$\sum_{m\geq 1} N_m^* u^m = (Au - 2Qu^2)(I - Au + Qu^2)^{-1}.$$

Using the previous equation and letting $f(u) = (Au - Qu^2)$ in Lemma 1.2.4, we have

$$\begin{aligned} \operatorname{Tr}\left(\sum_{m\geq 1} N_m^* u^m\right) &= \operatorname{Tr}\left((Au - 2Qu^2)(I - Au + Qu^2)^{-1}\right) \\ &= \operatorname{Tr}\left((-u\frac{d}{du}\log(I - Au + Qu^2)\right). \end{aligned}$$

Using Equation 1.7 and

$$\frac{1}{1-u^2} = \sum_{j\geq 0} u^{2j},$$

we have

$$\begin{aligned}
\operatorname{Tr}\left(\sum_{m\geq 1} N_m^* u^m\right) &= \sum_{m\geq 1} N_m u^m - \operatorname{Tr}\left(Q-I\right)\left(\sum_{\substack{m \text{ even}\\ m\geq 2}} u^m\right) \\
&= \sum_{m\geq 1} N_m u^m - \operatorname{Tr}(Q-I)\left(\frac{1}{1-u^2} - 1\right) \\
&= \sum_{m\geq 1} N_m u^m - \operatorname{Tr}(Q-I)\left(\frac{u^2}{1-u^2}\right).
\end{aligned}$$

Thus,

$$\operatorname{Tr}\left((-u\frac{d}{du}\log(I - Au + Qu^2)\right) = \sum_{m\geq 1} N_m u^m - \operatorname{Tr}(Q-I)\left(\frac{u^2}{1-u^2}\right).$$

Using the previous equation and Equation 1.2, we have

$$\begin{aligned}
u\frac{u}{du}\log Z_\gamma(u) &= \sum_{m\geq 1} N_m u^m \\
&= \operatorname{Tr}\left((-u\frac{d}{du}\log(I - Au + Qu^2)\right) + \operatorname{Tr}(Q-I)\left(\frac{u^2}{1-u^2}\right) \\
&= \operatorname{Tr}\left((-u\frac{d}{du}\log(I - Au + Qu^2)\right) - u\frac{u}{du}\log\left((1-u^2)^{\frac{Tr(Q-I)}{2}}\right).
\end{aligned}$$

Since both sides of this equation are zero at $u = 0$, we can integrate to get

$$-u\log Z_\gamma(u) = \operatorname{Tr}\left(\log(I - Au + Qu^2)\right) + \log\left((1-u^2)^{\frac{\operatorname{Tr}(Q-I)}{2}}\right).$$

Using Lemma 1.2.5, the theorem is proved. ■

Corollary 1.2.6. *The zeta function of a graph is the reciprocal of a polynomial.*

Proof. Let γ be a graph, and let γ^* be the graph obtained by removing any isolated or pendant vertices from graph γ. The zeta function of graph γ and the zeta function of graph

γ^* are the same since a pendant vertex or an isolated vertex does not contribute to any backtrackless, tail-less cycles. The number of edges in a graph without isolated vertices and pendant vertices is always greater than or equal to the number of vertices. Using Theorem 1.2.1, the zeta function is the reciprocal of a polynomial. ■

Since $Z_\gamma(u)$ is the reciprocal of a polynomial, we will usually consider the inverse zeta function,

$$Z_\gamma^{-1}(u) = (1-u^2)^{e-n}\det(I - Au + Qu^2), \tag{1.8}$$

which is the reciprocal of the zeta function of graph γ. Also, we will restrict our study to md_2 graphs in which every vertex has a minimal degree of at least two.

Chapter 2

Properties Determined by the Zeta Function of a Finite Graph

2.1 Coefficients in the Zeta Function of a Finite Graph

Let γ be a md_2 graph with e edges and n vertices. Let A be the adjacency matrix of graph γ and let Q be the Q-matrix. Then

$$\begin{aligned} Z_\gamma(u)^{-1} &= (1-u^2)^{e-n}\det(I - Au + Qu^2) \\ &= \left(1-u^2\right)^{e-n}\left(1 - \left(\sum_{i=1}^{n} A_{ii}\right)u + \cdots + \left(\prod_{i=1}^{n} Q_{ii}\right)u^{2n}\right) \\ &= 1 - \left(\sum_{i=1}^{n} A_{ii}\right)u + \cdots + (-1)^{e-n}\left(\prod_{i=1}^{n} Q_{ii}\right)u^{2e}. \end{aligned}$$

The degree of the zeta function of graph γ is twice the number of edges in graph γ, and the coefficient of u is negative twice the number of loops in graph γ.

However, the number of vertices is not determined by the zeta function of a graph. For example, graph γ and graph γ' depicted in Figure 2.1 have the same zeta function but a different number of vertices. However, the zeta function does determine the parity of the number of vertices. To see this, assume graph γ and graph γ' are md_2 graphs with the same zeta function. Since the coefficients on u^{2e} are equal, $\prod_{i=1}^{n} Q_{ii}$ equals $\prod_{i=1}^{n} Q'_{ii}$

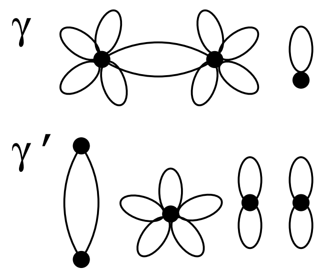

Figure 2.1: Graph γ and graph γ' have the same zeta function and a different number of vertices and components.

or differs by a negative. However, since each diagonal element in matrix Q is an positive integer, the product of the diagonal elements is always positive. So, $\prod_{i=1}^{n} Q_{ii} = \prod_{i=1}^{n} Q_{ii}'$ and $(-1)^{e-n} = (-1)^{e-n'}$. Thus, the parity of n and n' must be the same.

Figure 2.1 also shows that the zeta function of a graph does not determine the number of components in the graph. The MAGMA program used to find the graphs depicted in Figure 2.1 is given in the Appendix, see page 38.

The edge structure of a graph is also not determined by its zeta function. In Figure 2.2, graph γ has three occurrences of double edges and graph γ' has one occurrence of triple edges.

2.2 Zeta Functions of Graphs with Only One Vertex

Proposition 2.2.1. *If two graphs with only one vertex each have the same zeta function, then the graphs are isomorphic.*

Proof. An md_2 graph with only one vertex can only have loops as edges. Since two graphs with the same zeta function have the same number of loops, the graphs must be isomorphic. ∎

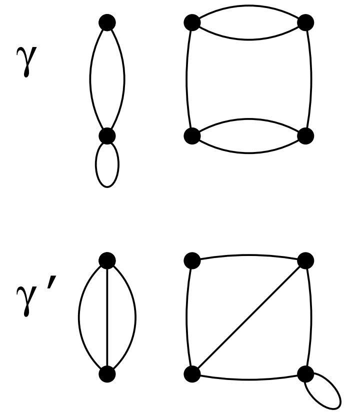

Figure 2.2: Graph γ and graph γ' have the same zeta function and different edge structure.

Proposition 2.2.2. *If md_2 graphs γ and γ' have the same zeta function and graph γ has only one vertex, then graph γ and graph γ' are isomorphic.*

Proof. Since graph γ has only one vertex, all the edges in graph γ must be loops. Since graph γ and graph γ' have the same zeta function, both graphs have the same number of edges and the same number of loops. Thus, every edge in graph γ' is also a loop. Since graph γ' has only loops, its adjacency matrix is a diagonal matrix. Let $A'_{1,1} \cdots A'_{n',n'}$ be the diagonal entries in the adjacency matrix of graph γ' where n' is the number of vertices in graph γ'. Then the adjacency matrix of graph γ is a 1×1 matrix with entry $\sum_{i=1}^{n'} A'_{i,i}$.

The coefficient on u^2 in the zeta function of graph γ is $-(e-1)+q_{1,1} = -e+1+2e-1 = e$. The coefficient on u^2 in the zeta function of graph γ' is

$$-(e-n') + \sum_{i=1}^{n'} Q_{i,i} - \sum_{i<j} A_{i,i}A_{j,j} = -e + n' + 2e - n' - \sum_{i<j} A_{i,i}A_{j,j} = e - \sum_{i<j} A_{i,i}A_{j,j}.$$

Thus, $0 = \sum_{i<j} A_{i,i}A_{j,j}$. Since $A_{i,i} > 0$ for every vertex i, the previous equation is true if and only if graph γ' has only one vertex. By Proposition 2.2.2, graph γ is isomorphic to graph γ'. ∎

2.3 Zeta Functions of Graphs with Exactly Two Vertices

Proposition 2.3.1. *If two md_2 graphs γ and γ' with exactly two vertices each have the same zeta function, then the graphs are isomorphic.*

Proof. Let

$$A = \begin{pmatrix} 2b & c \\ c & 2d \end{pmatrix} \text{ and } A' = \begin{pmatrix} 2x & y \\ y & 2z \end{pmatrix}$$

be the adjacency matrices of graph γ and graph γ' respectively where b, c, d, x, y, and z are non-negative integers. Then

$$\begin{aligned} Z_\gamma^{-1}(u) &= (1-u^2)^{b+c+d-2} \det \begin{pmatrix} 1-2bu+(2b+c-1)u^2 & -cu \\ -cu & 1-2du+(c+2d-1)u^2 \end{pmatrix} \\ &= (1-u^2)^{x+y+z-2} \det \begin{pmatrix} 1-2xu+(2x+y-1)u^2 & -yu \\ -yu & 1-2zu+(y+2z-1)u^2 \end{pmatrix} \\ &= Z_{\gamma'}^{-1}(u). \end{aligned}$$

Since graph γ and graph γ' have the same zeta function, the number of edges in each graph must be equal. So $b+c+d = x+y+z$. Thus, the determinants are equal and

$$\begin{aligned} &u^4(-2c-2b+c^2+2bc-2d+4bd+2cd+1)+u^3(-2bc-2cd-8bd+2b+2d)+ \\ &\quad u^2(-2+4bd+2d+2b-c^2+2c)+u(-2d-2b)+1 \\ &\quad = u^4(-2y-2x+y^2+2xy-2z+4xz+2yz+1)+u^3(-2xy-2yz-8xz+2x+2z) \\ &\quad + u^2(-2+4xz+2z+2x-y^2+2y)+u(-2z-2x)+1. \end{aligned}$$

The coefficients of u imply $b+d = x+z$. Since $b+c+d = x+y+z$, we have $c = y$. It follows using the coefficients on u^2 that $bd = xz$. Solving this equality for b and substituting

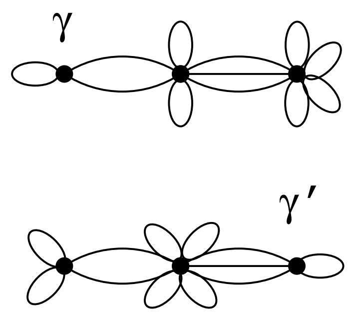

Figure 2.3: Graph γ and graph γ' have the same zeta function but are not isomorphic.

into $b + d = x + z$, we have $xz + d^2 = dx + dz$ or $x(z - d) = d(z - d)$. So either $z = d$ or $x = d$. Either way, graph γ is isomorphic to graph γ'. ■

2.4 Zeta Functions of Graphs with Exactly Three Vertices

Two graphs with the same zeta function and with exactly three vetices each are not necessarily isomorphic. The two graphs depicted in Figure 2.3 have the same zeta function but are not isomorphic. Note that this example is the smallext example with respect to the number of vertices.

Chapter 3

The Zeta Function and the Spectra of a Finite Graph

The spectrum of graph γ, denoted Spec(γ), is the set of eigenvalues with multiplicity of its adjacency matrix. In other words, two graphs have the same spectrum if their respective adjacency matrices have the same characteristic polynomial.

The zeta function of a md_2 graph does not determine the spectrum of the graph. Figure 3.1 is an example of two graphs that have the same zeta function and different spectrum. This example is the smallest example with respect to the number of edges. The MAGMA program used to find the graphs depicted in Figure 3.1 is given in the Appendix, see page 40.

The spectrum of a md_2 graph does not determine the zeta function of the graph. Figure 3.2 is an example of two graphs with the same spectrum and different zeta functions. This example is the smallest example with respect to the number of edges. Figure 3.3 is the

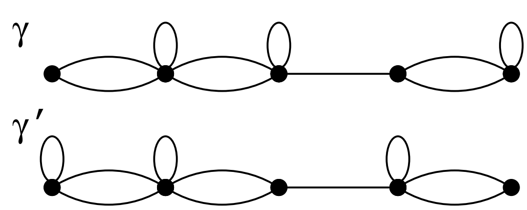

Figure 3.1: Graph γ and graph γ' have the same zeta function and different spectrum.

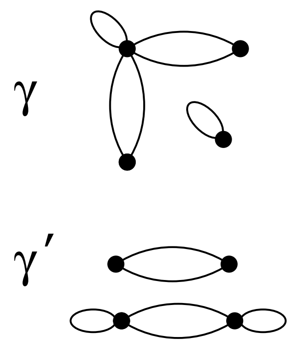

Figure 3.2: Graph γ and graph γ' have the same spectrum and different zeta functions.

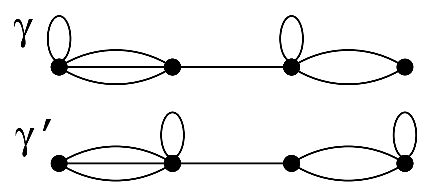

Figure 3.3: Connected graph γ and connected graph γ' have the same spectrum and different zeta functions.

smallest example with respect to the number of edges of two connected graphs with the same spectrum and different zeta functions.

If additional restrictions are placed on a md_2 graph, then there is relationship between the zeta function and the spectrum. In Section 3.1, we will show that the zeta function of a regular md_2 graph determines the spectrum of the graph and the spectrum of a regular md_2 graph determines the zeta function. We will also introduce the Riemann Hypothesis for regular graphs and discuss how the spectrum and the zeta function of a Ramanujun graph are related. In Section 3.3, we will see that the zeta function of a biregular-bipartite md_2 graph determines the graph and the spectrum of a biregular-bipartite md_2 graph determines the graph. Note that two graphs that are not both regular or both biregular-bipartite may still have the same spectrum and the same zeta function. Figure 2.2 is one such example.

3.1 Regular Graphs

A graph is *regular* if every vertex has the same degree. The largest eigenvalue of regular graph γ is $\lambda_1 = (q+1)$ where $q+1$ is the regularity of graph γ.

Lemma 3.1.1. *Let γ be a regular md_2 graph with regularity $q+1$. Then*

$$Z_\gamma^{-1}(u) = (1-u^2)^{e-n} \prod_{i=1}^{n} \left(1 - \lambda_i u + qu^2\right)$$

where $\{\lambda_i\}_{i=1}^n$ are the eigenvalues of the adjacency matrix, e is the number of edges, and n is the number of vertices.

Proof. Since $I - Au + Qu^2$ is a real symmetric matrix, there exists a matrix M such that

$$M(I - Au + Qu^2)M^{-1} = \begin{pmatrix} 1-\lambda_1 u + qu^2 & & 0 \\ & \ddots & \\ 0 & & 1-\lambda_n u + qu^2 \end{pmatrix}.$$

The determinant of this diagonal matrix equals the determinant of $I - Au + Qu^2$. So

$$Z_\gamma^{-1}(u) = (1-u^2)^{e-n} \prod_{i=1}^{n} \left(1 - \lambda_i u + qu^2\right).$$

■

Theorem 3.1.2 ([Mel01]). *If γ and $\gamma^{'}$ are regular md_2 graphs, then γ and γ' have equal zeta functions if and only if the graphs have the same spectrum.*

Proof. Let $q+1$ be the regularity of graph γ and $q^{'}+1$ be the regularity of graph $\gamma^{'}$. First assume graph γ and graph γ' have equal zeta functions where Spec(γ)=$\{\lambda_1, \ldots, \lambda_n\}$ and

Spec($\gamma^{'}$)= $\{\lambda_1', \ldots, \lambda_n'\}$. Then

$$\begin{aligned} Z_{\gamma}^{-1}(u) &= (1-u^2)^{e-n} \prod_{i=1}^{n} \left(1 - \lambda_i u + q u^2\right) \\ &= (1-u^2)^{e-n^{'}} \prod_{i=1}^{n^{'}} \left(1 - \lambda_i^{'} u + q^{'} u^2\right) \\ &= Z_{\gamma^{'}}^{-1}(u) \end{aligned}$$

where n and $n^{'}$ are the number of vertices in graph γ and graph $\gamma^{'}$ respectively and e is the number of edges.

Without loss of generality, assume $q^{'} + 1 \leq q$. Since $\lambda_1 = q + 1$, we know $\frac{1}{q}$ is a root of $Z_{\gamma}^{-1}(u)$. So $\frac{1}{q}$ must be a root of $Z_{\gamma^{'}}^{-1}(u)$. Thus, there exists an eigenvalue $\lambda_i^{'}$, namely $\lambda_i^{'} = q + \frac{q^{'}}{q}$, such that

$$1 - \lambda_i^{'} \left(\frac{1}{q}\right) + q^{'} \left(\frac{1}{q}\right)^2 = 0.$$

But,

$$\lambda_i^{'} = q + \frac{q^{'}}{q} \geq q^{'} + 1 + \frac{q^{'}}{q} > q^{'} + 1.$$

This contradicts eigenvalue $\lambda_1^{'} = q^{'} + 1$ being maximal. Thus, $q = q^{'}$.

Since graph γ and graph $\gamma^{'}$ have the same regularity and the same number of edges, graph γ and graph $\gamma^{'}$ have the same number of vertices. It follows that graph γ and graph γ' have the same number of eigenvalues. Pick the largest root of $Z_{\gamma}^{-1}(u)$ and call it α_1. Then there exists a β_i such that $\alpha_1 \beta_i = \frac{1}{q}$. This gives eigenvalue $\lambda_1 = (\alpha_1 + \beta_i)q$. Since $Z_{\gamma}^{-1}(u) = Z_{\gamma^{'}}^{-1}(u)$, the same pair of root exists for $Z_{\gamma^{'}}^{-1}(u)$. Thus $\lambda_1 = \lambda_1^{'}$. Pick the next largest root and call it α_2. This process can be repeated n times to show Spec(γ)=Spec($\gamma^{'}$).

Now assume, Spec(γ)=Spec($\gamma^{'}$)=$\{\lambda_1, \ldots, \lambda_n\}$. Since the largest eigenvalue of a regular graph is its regularity, graph γ and graph γ' have the same regularity. Since graph γ and graph $\gamma^{'}$ have the same number of vertices and the same regularity, they have the same

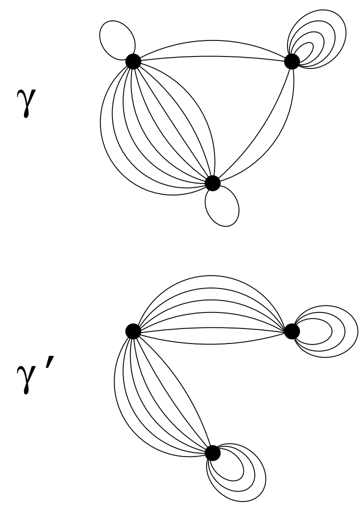

Figure 3.4: Graph γ and graph γ' are non-isomorphic, regular, and have the same zeta function.

number of edges. By Lemma 3.1.1, we have $Z_{\gamma}^{-1}(u) = Z_{\gamma'}^{-1}(u)$. ■

Corollary 3.1.3. *If γ and γ' are regular md_2 graphs with the same zeta functions or the same spectrum, then graph γ and graph γ' have the same regularity and the same number of vertices and edges.*

Figure 3.4 is the smallest example in terms of vertices of two non-isomorphic regular graphs with the same zeta function and thus the same spectrum. The MAGMA program used to find the graphs depicted in Figure 3.4 is given in the Appendix, see page 46.

In order to discuss the Riemann Hypothesis for regular graphs, we need to make a change of variable by setting $u = q^{-s}$ where graph γ has regularity $q+1$. The following proposition shows that the equality of the zeta functions of two regular graphs is not affected by the change of variable.

Proposition 3.1.4. *Let γ be a $(q+1)$-regular graph with n vertices, and let γ' be a $(q'+1)$-regular graph with n' vertices. Then $Z_{\gamma}(q^{-s}) = Z_{\gamma'}((q')^{-s})$ for all s if and only if $Z_{\gamma}(u) = Z_{\gamma'}(u)$ for all u.*

Proof. If $Z_\gamma(u) = Z_{\gamma'}(u)$, then graph γ and graph γ' have the same regularity by Corollary 3.1.3. Thus $q^{-s} = (q')^{-s}$.

Assume $Z_\gamma(q^{-s}) = Z_{\gamma'}((q')^{-s})$, and without loss of generality, assume $q' \geq q$. Let $-s = \log_q(u)$. Then

$$Z_\gamma(u) = Z_\gamma(q^{-s}) = Z_{\gamma'}\left((q')^{-s}\right) = Z_{\gamma'}\left(u^{\frac{\log q'}{\log q}}\right) = Z_{\gamma'}(u^k)$$

where $q^k = q'$. We need to show $k = 1$.

The lead coefficient of $Z_\gamma(u)$ is $\pm q^n$ and the lead coefficient of $Z_{\gamma'}(u^k)$ is $\pm(q')^{n'}$. Since q and q' are positive, $q^n = (q')^{n'}$. Substituting $q^k = q'$ and solving for n, we have $n = kn'$.

Assume graph γ has e edges and graph γ' has e' edges. Then the degree of $Z_\gamma^{-1}(u)$ is $2e$ and the degree of $Z_{\gamma'}^{-1}(u^k)$ is $2e'k$. So $e = ke'$.

Using the regularity of graph γ, we have $(q+1)n = 2e$. By substituting $e = ke'$ and $n = kn'$, we get

$$n' = \frac{2e'}{q+1}. \tag{3.1}$$

Using the regularity of graph γ, we have $(q^k+1)n' = 2e'$. Solving for n', we have

$$n' = \frac{2e'}{q^k+1}.$$

Using Equation 3.1, we can conclude that $q + 1 = q^k + 1$. Thus, $k = 1$ as desired. ∎

Definition 3.1.5 ([ST96]). Let γ be a $(q+1)$-regular graph and set $u = q^{-s}$. Then graph γ is said to satisfy the Riemann Hypothesis if and only if Re $s \in (0,1)$ and $Z_\gamma(q^{-s}) = 0$ implies Re $s = \frac{1}{2}$.

Definition 3.1.6 ([LPS88]). A $(q+1)$-regular graph is a Ramanujan graph if and only if for every eigenvalue λ_i of the adjacency matrix with the exception of $|\lambda_1| = q+1$, we have $|\lambda_i| \le 2\sqrt{q}$.

Proposition 3.1.7 ([ST96]). *Let γ be a $(q+1)$-regular graph. Then γ satisfies the Riemann Hypotheses if and only if γ is a Ramanujan graph.*

Proof. Let $\{\lambda_i\}_{i=1}^n$ be the set of eigenvalues of the adjacency matrix of graph γ. By Lemma 3.1.1, we know

$$Z_\gamma^{-1}(u) = (1-u^2)^{e-n} \prod_{i=1}^{n} \left(1 - \lambda_i u + qu^2\right).$$

Let $\{\alpha_i,\ \beta_i\}$ be the set roots of $\prod_{i=1}^n (1-\lambda_i u + qu^2)$ where $\alpha_i\beta_i = q$ and $\alpha_i + \beta_i = \lambda_i$. So

$$\alpha_i,\ \beta_i = \frac{\lambda_i \pm \sqrt{\lambda_i^2 - 4q}}{2q}.$$

For each root we can write $\alpha_i = q^{-s}$ for some complex number $-s = \sigma + i\tau$. Then $|\alpha_i| = |q^{-s}| = |q^{\sigma+i\tau}| = |q^\sigma| = \sqrt{q}$ if and only if $\sigma = \frac{1}{2}$. So $|\lambda_i| \le 2\sqrt{q}$ if and only if α_i and β_i are complex conjugates with absolute value $\sqrt{q}$. If $\lambda_i = \pm(q+1)$, then $\alpha_i, \beta_i \in \{2q, 2, -2, -2q\}$. In this situation, $|\alpha_i| = |q^s|$ implies $s = 0$ or $s = 1$.

∎

3.2 Bipartite Graphs

A *bipartite* graph is a graph in which the vertex set V can be divided into two non-empty disjoint sets U_1 and U_2 whose union equals V and where every vertex in U_i is not adjacent to any vertex in U_i. Hashimoto [Has89] remarks that the zeta function of a bipartite graph is an even polynomial. In other words, $Z_\gamma^{-1}(u) = Z_\gamma^{-1}(-u)$ for every bipartite graph γ. Thus, we will typically consider $Z_\gamma^{-1}(u^{\frac{1}{2}})$ when discussing bipartite graphs.

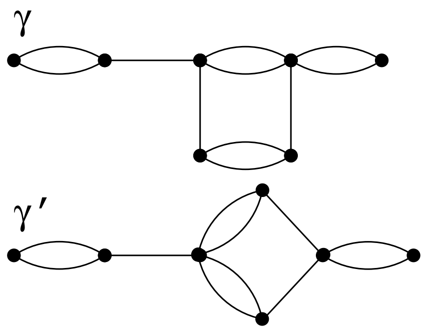

Figure 3.5: Bipartite graphs γ and γ' have the same zeta function and different spectrum.

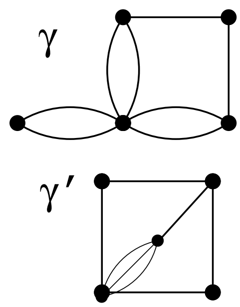

Figure 3.6: Bipartite graphs γ and γ' have the same spectrum and different zeta functions.

The zeta function and the spectrum of a bipartite graph do not determine one another as they do for regular graphs. Figure 3.5 is an example of two bipartite md_2 graphs with the same zeta function and different spectrum, and Figure 3.6 is an example of two bipartite md_2 graphs with the same spectrum and different zeta functions. The examples in Figure 3.5 and Figure 3.6 are both the smallest examples with respect to the number of edges. The MAGMA program used to find the graphs depicted in Figure 3.5 is given in the Appendix, see page 47.

3.3 Biregular-Bipartite Graphs

A *biregular-bipartite* graph is a bipartite graph where every vertex in U_1 has the same degree and every vertex in U_2 has the same degree. We will denote the number of vertices in U_1 and

U_2 as n_1 and n_2 respectively and the regularity of the vertices in U_1 and U_2 as q_1+1 and q_2+1 respectively. We also will assume without loss of generality that $n_2 \geq n_1$. With this notation, we have that the number of edges in a biregular-bipartite graph is $n_1(q_1+1) = n_2(q_2+1)$.

Hashimoto [Has89] shows the largest eigenvalue of biregular-bipartite graph γ is $\lambda_1 = \sqrt{(q_1+1)(q_2+1)}$, and this eigenvalue occurs exactly as many times as the number of components in graph γ. He also shows that a biregular-bipartite graph has spectrum of the form

$$\{\pm\lambda_1, \pm\lambda_2, \cdots, \pm\lambda_{n_1}, \underbrace{0, \cdots, 0}_{n_2-n_1}\}$$

where $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_{n_1}| \geq 0$.

Hashimoto [Has89] proves that the inverse zeta function of a connected biregular-bipartite graph γ is

$$Z_\gamma^{-1}(u^{\frac{1}{2}}) = (1-u)^{n_1q_1-n_2}(1+q_2u)^{n_2-n_1}\prod_{j=1}^{n_1}\left(1-\left(\lambda_j^2-q_1-q_2\right)u+q_1q_2u^2\right).$$

However, this expansion can be extended to graphs with multiple components since the zeta function of a non-connected graph is the product of the zeta functions of each component.

Theorem 3.3.1. *Let γ and γ' be biregular-bipartite md_2 graphs. If $Z_\gamma(u) = Z_{\gamma'}(u)$, then graph γ and graph γ' are isomorphic.*

Proof. We are given that

$$\begin{aligned} Z_\gamma^{-1}(u^{\frac{1}{2}}) &= (1-u)^{n_1q_1-n_2}(1+q_2u)^{n_2-n_1}\prod_{j=1}^{n_1}\left(1-\left(\lambda_j^2-q_1-q_2\right)u+q_1q_2u^2\right) \\ &= (1-u)^{n_1'q_1'-n_2'}(1+q_2'u)^{n_2'-n_1'}\prod_{j=1}^{n_1'}\left(1-\left({\lambda_j'}^2-q_1'-q_2'\right)u+q_1'q_2'u^2\right) \\ &= Z_{\gamma'}^{-1}(u^{\frac{1}{2}}). \end{aligned}$$

Since graph γ and graph γ' have the same number of edges,

$$n_1(q_1+1) = n_2(q_2+1) = n_1'(q_1'+1) = n_2'(q_2'+1). \tag{3.2}$$

The coefficients on u^e, where $Z_\gamma(u^{\frac{1}{2}}) = Z_{\gamma'}(u^{\frac{1}{2}})$ is a degree e polynomial, are equal so

$$q_2^{n_2} q_1^{n_1} = \left(q_2'\right)^{n_2'} \left(q_1'\right)^{n_1'}. \tag{3.3}$$

Note that $\frac{1}{q_1 q_2}$ is a root of $Z_\gamma^{-1}(u^{\frac{1}{2}})$ since $\lambda_1 = \sqrt{(q_1+1)(q_2+1)}$. So $\frac{1}{q_1 q_2}$ must be a root of $Z_{\gamma'}^{-1}(u^{\frac{1}{2}})$. If $\frac{1}{q_1 q_2} = 1$, then $q_1 = q_2 = 1$. If $\frac{1}{q_1 q_2} \neq 1$, then there exists an eigenvalue λ_j' of graph γ' such that

$$\left(\lambda_j'\right)^2 = q_1 q_2 + q_1' + q_2' + \frac{q_1' q_2'}{q_1 q_2}.$$

The largest eigenvalue of graph γ' is $\lambda_1' = \sqrt{(q_1'+1)(q_2'+1)}$. So

$$(q_1'+1)(q_2'+1) \geq q_1 q_2 + \frac{q_1' q_2'}{q_1 q_2} + q_1' + q_2'.$$

It follows that

$$q_1' q_2'(q_1 q_2 - 1) \geq q_1 q_2 (q_1 q_2 - 1).$$

Thus, either $q_1 = q_2 = 1$ or $q_1' q_2' \geq q_1 q_2$.

Similarly, since $\frac{1}{q_1' q_2'}$ is a root of $Z_{\gamma'}^{-1}(u^{\frac{1}{2}})$, we know $q_1' = q_2' = 1$ or $q_1 q_2 \geq q_1' q_2'$.

If $q_1 = q_2 = 1$ or $q_1' = q_2' = 1$, then by Equation 3.3, we have that $q_1 = q_2 = q_1' = q_2' = 1$. It follows that $n_1 = n_2 = n_1' = n_2'$ by Equation 3.2. In this situation, graph γ and graph γ' are isomorphic. If $q_1' q_2' \geq q_1 q_2$ and $q_1 q_2 \geq q_1' q_2'$, then

$$q_1 q_2 = q_1' q_2'. \tag{3.4}$$

Clearly, $Z_\gamma^{-1}(u^{\frac{1}{2}})$ has 1 as a root at least $n_1q_1 - n_2 \geq 0$ times. If $Z_\gamma^{-1}(u^{\frac{1}{2}})$ has 1 as a root more than $n_1q_1 - n_2$ times, then there exists some eigenvalue λ_j such that

$$1 - \left(\lambda_j^2 - q_1 - q_2\right)(1) + q_1q_2\,(1)^2 = 0.$$

Only the largest eigenvalue, $\lambda_1 = \sqrt{(q_1+1)(q_2+1)}$, satisfies this equation. So $Z_\gamma^{-1}(u^{\frac{1}{2}})$ has one as a root exactly $n_1q_1 - n_2 + k$ times where k is the number of components in graph γ. Similarly, $Z_{\gamma'}^{-1}(u^{\frac{1}{2}})$ has one as a root exactly $n_1^{'}q_1^{'} - n_2^{'} + k^{'}$ times where k' is the number of components in graph γ'. So, $n_1q_1 - n_2 + k = n_1^{'}q_1^{'} - n_2^{'} + k^{'}$. Without loss of generality, assume $k > k^{'}$. Then

$$n_1q_1 - n_2 < n_1^{'}q_1^{'} - n_2^{'}$$
$$n_1q_1e - n_2e < n_1^{'}q_1^{'}e - n_2^{'}e$$
$$n_1q_1(n_2q_2 + n_2) - n_2(n_1q_1 + n_1) < n_1^{'}q_1^{'}(n_2^{'}q_2^{'} + n_2^{'}) - n_2^{'}(n_1^{'}q_1^{'} + n_1^{'})$$
$$q_1q_2n_1n_2 - q_1^{'}q_2^{'}n_1^{'}n_2^{'} < n_1n_2 - n_1^{'}n_2^{'}$$
$$q_1q_2(n_1n_2 - n_1^{'}n_2^{'}) < n_1n_2 - n_1^{'}n_2^{'}.$$

So, $q_1q_2 < 1$. This is a contradiction so $k = k^{'}$ and

$$n_1q_1 - n_2 = n_1^{'}q_1^{'} - n_2^{'}. \tag{3.5}$$

From Equation 3.2, we have $n_1(q_1+1) = n_1^{'}(q_1^{'}+1)$. Subtracting Equation 3.5 from this equality, we get

$$n_1 + n_2 = n_1^{'} + n_2^{'}. \tag{3.6}$$

Thus, graph γ and graph $\gamma^{'}$ have the same number of vertices.

Assume $n_1 = n_2$. By Equation 3.3 and Equation 3.4, we have

$$\begin{aligned} (q_1 q_2)^{n_1} &= \left(q_2^{'}\right)^{n_2^{'} - n_1^{'}} \left(q_1^{'} q_2^{'}\right)^{n_1^{'}} \\ \left(q_1^{'} q_2^{'}\right)^{n_1 - n_1^{'}} &= \left(q_2^{'}\right)^{n_2^{'} - n_1^{'}} \\ \left(q_1^{'}\right)^{n_1 - n_1^{'}} &= \left(q_2^{'}\right)^{n_2^{'} - n_1} \\ \left(q_1^{'}\right)^{n_1 - n_1^{'}} &= \left(q_2^{'}\right)^{n_2^{'} - n_2}. \end{aligned}$$

Using Equation 3.6, we have either $q_1^{'} = q_2^{'}$ or $n_1 = n_1^{'}$. Either way, we can conclude from Equation 3.2 and Equation 3.4 that graph γ and graph γ' are isomorphic.

If $n_1 \neq n_2$, then $u = \frac{-1}{q_2}$ is a root of $Z_{\gamma}^{-1}(u^{\frac{1}{2}})$ and thus a root of $Z_{\gamma'}^{-1}(u^{\frac{1}{2}})$. If $q_2 = q_2^{'} \neq 0$, then using Equation 3.2 and Equation 3.4, we have that graph γ and graph γ' are isomorphic. If $q_2 \neq q_2^{'}$, then there exists an eigenvalue $\lambda_j^{'}$ such that

$$\left(\lambda_j^{'}\right)^2 = q_1 + q_2 - q_1^{'} - q_2^{'}.$$

Thus, $q_1 + q_2 - q_1^{'} - q_2^{'} \geq 0$ implies $q_1 + q_2 \geq q_1^{'} + q_2^{'}$.

Similarly, if $\frac{-1}{q_2^{'}}$ is a root of $Z_{\gamma'}^{-1}(u^{\frac{1}{2}})$, then $q_1^{'} + q_2^{'} \geq q_1 + q_2$. Thus,

$$q_1^{'} + q_2^{'} = q_1 + q_2. \tag{3.7}$$

Solving Equation 3.4 for q_1 and substituting this into Equation 3.7, we have

$$q_1^{'}(q_2^{'} - q_2) = q_2(q_2^{'} - q_2).$$

So, $q_1^{'} = q_2$ or $q_2^{'} = q_2$. Either way, by Equation 3.2, we can conclude that graph γ and graph γ' are isomorphic. ∎

The Q-spectrum of biregular-bipartite graph γ are the solutions to the modified characteristic polynomial

$$Q_\gamma(\lambda) = \det\left(\lambda I - \frac{A}{\sqrt{(q_1+1)(q_2+1)}}\right)$$

where I is the identity matrix and A is the adjacency matrix. The following discussion of the relationship between the spectrum of a graph, the Q-spectrum of a graph, and the number of edges in a biregular-bipartite graph is from Cvetković, Doob, and Sachs [CDS80].

Theorem 3.3.2. *Let γ be a biregular-bipartite graph with n vertices and let $\sigma = \sqrt{(q_1+1)(q_2+1)}$. Then*

$$Q_\gamma(\lambda) = \frac{1}{\sigma^n} P_\gamma(\sigma\lambda)$$

where $P_\gamma(\gamma)$ is the characteristic polynomial of adjacency matrix A. In other words, the spectrum of a biregular-bipartite graph determines the Q-spectrum of a biregular-bipartite graph and the Q-spectrum of a biregular-bipartite graph determines the spectrum of a biregular-bipartite .

Proof. We have that

$$Q_\gamma(\lambda) = \det\left(\lambda I - \frac{1}{\sigma}A\right) = \frac{1}{\sigma^n}\det(\lambda\sigma I - A) = \frac{1}{\sigma^n}P_\gamma(\sigma\lambda).$$

■

Theorem 3.3.3. *Let γ be a md_2 biregular-bipartite graph with n vertices and Q-spectrum $\{\lambda_1^*, \cdots, \lambda_n^*\}_Q$. Let E be the set of all edges in graph γ where edge $e_{i,j}$ connects vertex i and vertex j. Then*

$$\sum_{v=1}^{n} (\lambda_v^*)^2 = 2\sum_{e_{i,j}\in E} \frac{1}{d_i d_j}$$

where d_i is the degree of vertex i.

Corollary 3.3.4. *Let γ be a md_2 biregular-bipartite graph. If $\sigma = \sqrt{(q_1+1)(q_2+1)}$ and the Q-spectrum is $\{\lambda_1^*, \cdots, \lambda_n^*\}_Q$, then*

$$e = \frac{1}{2}\sigma^2 \sum_{v=1}^{n} (\lambda_v^*)^2$$

where e is the number of edges.

Proof. Using the notation in Theorem 3.3.3, we have that d_i equals q_1+1 or q_2+1 and d_j equals q_1+1 or q_2+1. Since γ is a biregular-bipartite graph, we have that d_i and d_j cannot both be equal to q_1+1 and both cannot be equal to q_2+1. So using Theorem 3.3.3, we have

$$\begin{aligned}\sum_{v=1}^{n} (\lambda_v^*)^2 &= 2 \sum_{e_{i,j}\in E} \frac{1}{d_i d_j} \\ &= 2\sum_{E} \frac{1}{(q_1+1)(q_2+1)} \\ &= 2\sum_{E} \frac{1}{\sigma^2} \\ &= 2\frac{e}{\sigma^2}.\end{aligned}$$

Solving for e, we have $e = \frac{1}{2}\sigma^2 \sum_{v=1}^{n} (\lambda_v^*)^2$ as desired. ∎

Using our discussion of Q-spectrum, we can now prove that the spectrum of a biregular-bipartite graph determines the graph.

Theorem 3.3.5. *Let γ and γ' be biregular-bipartite graphs. If Spec(γ) = Spec(γ'), then graph γ and graph γ' are isomorphic.*

Proof. We know that graph γ and graph γ' have the same number of vertices and the same largest eigenvalue. So we have that

$$n_1 + n_2 = n_1' + n_2' \tag{3.8}$$

and

$$(3.9)\qquad (q_1+1)(q_2+1)=(q_1^{'}+1)(q_2^{'}+1).$$

Using Theorem 3.3.2 and Corollary 3.3.4, we have that if Spec(γ)=Spec($\gamma^{'}$), then $e=e^{'}$. So,

$$(3.10)\qquad n_1(q_1+1)=n_2(q_2+1)=n_1^{'}(q_1^{'}+1)=n_2^{'}(q_2^{'}+1).$$

It follows from Equation 3.9 and Equation 3.10 that

$$n_1n_2=n_1^{'}n_2^{'}.$$

Solving this equation for n_1 and substituting it into Equation 3.8, we have

$$n_1^{'}(n_2^{'}-n_2)=n_2(n_2^{'}-n_2).$$

So, $n_1^{'}=n_2$ or $n_2^{'}=n_2$. Either way, by Equation 3.10, we can conclude that graph γ and graph γ' are isomorphic. ■

Theorem 3.3.6. *If γ and $\gamma^{'}$ are biregular-bipartite md$_2$ graphs, then graph γ and graph γ' have equal zeta functions if and only if the graphs have the same spectrum.*

Proof. The result follows from Theorem 3.3.1 and Theorem 3.3.5. ■

Chapter 4

Open Problems

In Section 2.1, we discussed several properties of the zeta function of a graph that are encoded in the coefficients of the inverse zeta function. However, we only used the coefficient of u and u^{2e} to make our arguments. For a general graph, what information is encoded in the other coefficients? More specifically, what information is encoded in the coefficients of a specialized graph like a loop-less graph?

Figure 2.1 shows that the zeta function of a graph does not determine the number of vertices or the number of components in the graph. Can additional counterexamples be found? Are there subsets of graphs besides regular graphs and biregular-bipartite graphs where the zeta function does determine the number of vertices or the number of components?

Figure 2.1 is also an example of two graphs that have the same zeta function, but do not have the same vertex degrees since graph γ' has a vertex of degree four and graph γ does not. However, both graphs have minimal degree of two and maximal degree of ten. Do two graphs with the same zeta function always have the same minimal degree and maximal degree?

In Chapter 2, we showed that if two graphs have the same zeta function and one of the two graphs is loop-less or has only one vertex, then the other graph must also have this property. Are there any other subsets of graphs beside loop-less graphs and graphs with one vertex where having one graph in the subset requires any other graph with the same zeta function to also be in the subset?

In Chapter 3, we showed a strong relationship between the zeta function of a graph and the spectrum of the graph for regular and biregular-bipartite graphs. In both cases, we assumed that the two graphs with the same zeta function or spectrum were both either regular or biregular-bipartite. Can two graphs have the same zeta function where one is regular and the other is not regular? Similarly, can two graphs have the same zeta function where one is biregular-bipartite and the other is not biregular bipartite?

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Appendix: Select MAGMA Programs

The following MAGMA code creates and stores the adjacency matrices and zeta functions of all graphs with i edges and three vertices. Then the program systematically creates a graph with i edges and five vertices and compares its zeta function to the list of the zeta functions from the graphs with three vertices. If a match occurs, the program outputs the adjacency matrix of the graph with three vertices and the adjacency matrix of the graph with five vetices. Figure 2.1 is an example of its output.

```
ZZ<u>:=PolynomialRing(Integers());
for i in [3..15] do
A2:=[**];
Zeta2:=[**];
count2:=0;
count4:=0;
for a in [0..i] do
for b in [0..i-a] do
for c in [0..i-a-b] do
for d in [0..i-a-b-c] do
for e in [0..i-a-b-c-d] do
for f in [0..i-a-b-c-d-e] do
     if a+b+c+d+e+f eq i and 2*a+b+c ge 2 and b+2*d+e ge 2
     and c+e+2*f ge 2 then
          I:=DiagonalMatrix([1,1,1]);
          Q:=DiagonalMatrix([2*a+b+c-1,b+2*d+e-1,c+e+2*f-1]);
          AA:=SymmetricMatrix([2*a,b,2*d,c,e,2*f]);
          ZZZ:=Polynomial(ZZ,(1-u^2)^(i-3)*Determinant(I-AA*u+Q*u^2*I));
          iii:=1;
          while iii le count2 do
               if Zeta2[iii] eq ZZZ then
                    iii:=count2+4;
```

```
          else
               iii:=iii+1;
          end if;
     end while;
     if iii eq count2+1 then
          Append(~A2,AA);
          Append(~Zeta2,ZZZ);
          count2:=count2+1;
     end if;
end if;
end for;
end for;
end for;
end for;
end for;
end for;
for a in [0..i] do
for b in [0..i-a] do
for c in [0..i-a-b] do
for d in [0..i-a-b-c] do
for e in [0..i-a-b-c-d] do
for f in [a..i-a-b-c-d-e] do
for g in [0..i-a-b-c-d-e-f] do
for h in [0..i-a-b-c-d-e-f-g] do
for j in [0..i-a-b-c-d-e-f-g-h] do
for k in [a..i-a-b-c-d-e-f-g-h-j] do
for l in [0..i-a-b-c-d-e-f-g-h-j-k] do
for m in [0..i-a-b-c-d-e-f-g-h-j-k-l] do
for n in [a..i-a-b-c-d-e-f-g-h-j-k-l-m] do
for p in [0..i-a-b-c-d-e-f-g-h-j-k-l-m-n] do
for q in [a..i-a-b-c-d-e-f-g-h-j-k-l-m-n-p] do
     if a+b+c+d+e+f+g+h+j+k+l+m+n+p+q eq i then
     if 2*a+b+c+d+e ge 2 then
     if b+2*f+g+h+j ge 2 then
     if c+g+2*k+l+m ge 2 then
     if d+h+l+2*n+p ge 2 then
     if e+j+m+p+2*q ge 2 then
          I:=DiagonalMatrix([1,1,1,1,1]);
          Q:=DiagonalMatrix([2*a+b+c+d+e-1,b+2*f+g+h+j-1,
          c+g+2*k+l+m-1,d+h+l+2*n+p-1,e+j+m+p+2*q-1]);
          AA:=SymmetricMatrix([2*a,b,2*f,c,g,2*k,d,h,l,2*n,e,j,m,p,2*q]);
          ZZZ:=Polynomial(ZZ,(1-u^2)^(i-5)*Determinant(I-AA*u+Q*u^2*I));
```

```
            ii:=1;
            while ii le count2 do
                 if Zeta2[ii] eq ZZZ then
                      AA;
                      print"";
                      A2[ii];
                      print"";
                      print"";
                      ii:=count2+4;
                 else
                      ii:=ii+1;
                 end if;
          end while;
     end if;
     end if;
     end if;
     end if;
     end if;
     end if;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
```

The following MAGMA code creates the adjacency matrix of each graphs with i edges and up to six vertices. It find and outputs the adjacency matrices of any two graphs that have the same zeta function and different characteristic polynomials. Figure 3.1 is an example of its output.

```
ZZ<u>:=PolynomialRing(Integers());
for i in [1..10] do
A:=[**];
Zeta:=[**];
Char:=[**];
count:=0;
for a in [0..i] do
for b in [0..i] do
for c in [0..i] do
     if a+b+c eq i and 2*a+b ge 2 and b+2*c ge 2 then
         I:=DiagonalMatrix([1,1]);
         Q:=DiagonalMatrix([2*a+b-1,b+2*c-1]);
         AA:=SymmetricMatrix([2*a,b,2*c]);
         ZZZ:=Polynomial(ZZ,(1-u^2)^(i-2)*Determinant(I-AA*u+Q*u^2*I));
         Ch:=CharacteristicPolynomial(AA);
         ii:=1;
         while ii le count do
               if Zeta[ii] eq ZZZ then
                    if Char[ii] ne Ch then
                         AA;
                         A[ii];
                    else
                         ii:=count+4;
                    end if;
               else
                     ii:=ii+1;
               end if;
          end while;
          if ii eq count+1 then
               Append(~A,AA);
               Append(~Zeta,ZZZ);
               Append(~Char,Ch);
               count:=count+1;
          end if;
     end if;
for d in [0..i] do
for e in [0..i] do
for f in [0..i] do
     if a+b+c+d+e+f eq i and 2*a+b+c ge 2 and b+2*d+e ge 2
     and c+e+2*f ge 2 then
         I:=DiagonalMatrix([1,1,1]);
         Q:=DiagonalMatrix([2*a+b+c-1,b+2*d+e-1,c+e+2*f-1]);
```

```
        AA:=SymmetricMatrix([2*a,b,2*d,c,e,2*f]);
        ZZZ:=Polynomial(ZZ,(1-u^2)^(i-3)*Determinant(I-AA*u+Q*u^2*I));
        Ch:=CharacteristicPolynomial(AA);
         ii:=1;
         while ii le count do
              if Zeta[ii] eq ZZZ then
                   if Char[ii] ne Ch then
                        AA;
                        A[ii];
                   else
                        ii:=count+4;
                   end if;
              else
                    ii:=ii+1;
              end if;
         end while;
         if ii eq count+1 then
              Append(~A,AA);
              Append(~Zeta,ZZZ);
              Append(~Char,Ch);
              count:=count1;
         end if;
     end if;
for g in [0..i] do
for h in [0..i] do
for j in [0..i] do
for k in [0..i] do
     if a+b+c+d+e+f+g+h+j+k eq i and 2*a+b+c+d ge 2 and
     b+2*e+f+g ge 2 and c+f+2*h+j ge 2 and d+g+j+2*k ge 2 then
        I:=DiagonalMatrix([1,1,1,1]);
        Q:=DiagonalMatrix([2*a+b+c+d-1,b+2*e+f+g-1,
        c+f+2*h+j-1,d+g+j+2*k-1]);
        AA:=SymmetricMatrix([2*a,b,2*e,c,f,2*h,d,g,j,2*k]);
        ZZZ:=Polynomial(ZZ,(1-u^2)^(i-4)*Determinant(I-AA*u+Q*u^2*I));
        Ch:=CharacteristicPolynomial(AA);
        ii:=1;
        while ii le count do
              if Zeta[ii] eq ZZZ then
                   if Char[ii] ne Ch then
                        AA;
                        A[ii];
                   else
```

```
                    ii:=count+4;
                end if;
            else
                 ii:=ii+1;
            end if;
        end while;
        if ii eq count+1 then
            Append(~A,AA);
            Append(~Zeta,ZZZ);
            Append(~Char,Ch);
            count:=count+1;
        end if;
    end if;
for l in [0..i] do
for m in [0..i] do
for n in [0..i] do
for p in [0..i] do
for q in [0..i] do
    if a+b+c+d+e+f+g+h+j+k+l+m+n+p+q eq i then
    if 2*a+b+c+d+e ge 2 then
    if b+2*f+g+h+j ge 2 then
    if c+g+2*k+l+m ge 2 then
    if d+h+l+2*n+p ge 2 then
    if e+j+m+p+2*q ge 2 then
        I:=DiagonalMatrix([1,1,1,1,1]);
        Q:=DiagonalMatrix([2*a+b+c+d+e-1,b+2*f+g+h+j-1,c+g+2*k+l+m-1,
        d+h+l+2*n+p-1,e+j+m+p+2*q-1]);
        AA:=SymmetricMatrix([2*a,b,2*f,c,g,2*k,d,h,l,2*n,e,j,m,p,2*q]);
        ZZZ:=Polynomial(ZZ,(1-u^2)^(i-5)*Determinant(I-AA*u+Q*u^2*I));
        Ch:=CharacteristicPolynomial(AA);
        ii:=1;
        while ii le count do
            if Zeta[ii] eq ZZZ then
                if Char[ii] ne Ch then
                    AA;
                    A[ii];
                else
                    ii:=count+4;
                end if;
            else
                 ii:=ii+1;
            end if;
```

```
          end while;
          if ii eq count+1 then
               Append(~A,AA);
               Append(~Zeta,ZZZ);
               Append(~Char,Ch);
               count:=count+1;
          end if;
     end if;
     end if;
     end if;
     end if;
     end if;
     end if;
for r in [0..i] do
for s in [0..i] do
for t in [0..i] do
for y in [0..i] do
for v in [0..i] do
for w in [0..i] do
     if a+b+c+d+e+f+g+h+j+k+l+m+n+p+q+r+s+t+y+v+w eq i then
     if 2*a+b+c+d+e+f ge 2 then
     if b+2*g+h+j+k+l ge 2 then
     if c+h+2*m+n+p+q ge 2 then
     if d+j+n+2*r+s+t ge 2 then
     if e+k+p+s+2*y+v ge 2 then
     if f+l+q+t+v+2*w ge 2 then
         I:=DiagonalMatrix([1,1,1,1,1,1]);
         Q:=DiagonalMatrix([2*a+b+c+d+e+f-1,b+2*g+h+j+k+l-1,
         c+h+2*m+n+p+q-1,d+j+n+2*r+s+t-1,e+k+p+s+2*y+v-1,f+l+q+t+v+2*w-1]);
         AA:=SymmetricMatrix([2*a,b,2*g,c,h,2*m,d,j,n,2*r,e,k,p,s,
         2*y,f,l,q,t,v,2*w]);
         ZZZ:=Polynomial(ZZ,(1-u^2)^(i-6)*Determinant(I-AA*u+Q*u^2*I));
         Ch:=CharacteristicPolynomial(AA);
         ii:=1;
         while ii le count do
               if Zeta[ii] eq ZZZ then
                    if Char[ii] ne Ch then
                         AA;
                         A[ii];
                    else
                         ii:=count+4;
                    end if;
```

```
                else
                      ii:=ii+1;
                end if;
           end while;
           if ii eq count+1 then
                Append(~A,AA);
                Append(~Zeta,ZZZ);
                Append(~Char,Ch);
                count:=count+1;
           end if;
      end if;
      end if;
      end if;
      end if;
      end if;
      end if;
      end if;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
```

The following MAGMA code finds non-isomorphic, regular graphs with three vertices that have the same zeta function. The program creates the adjacency matrix and computes the zeta function of each regular graph with i edges and three vertices. If two non-isomorphic graphs have the same zeta function, then the program outputs the adjacency matrices of the two graphs. The two graphs in Figure 3.4 are the smallest such example.

```
ZZ<u>:=PolynomialRing(Integers());
for i in [1..18] do
A:=[**];
Zeta:=[**];
count:=0;
for a in [0..i] do
for b in [0..i-a] do
for c in [0..i-a-b] do
for d in [0..i-a-b-c] do
for e in [0..i-a-b-c-d] do
for f in [0..i-a-b-c-d-e] do
     if a+b+c+d+e+f eq i and 2*a+b+c ge 2 and b+2*d+e ge 2 and c+e+2*f ge 2
     and 2*a+b+c eq b+2*d+e and b+2*d+e eq c+e+2*f then
          I:=DiagonalMatrix([1,1,1]);
          Q:=DiagonalMatrix([2*a+b+c-1,b+2*d+e-1,c+e+2*f-1]);
          AA:=SymmetricMatrix([2*a,b,2*d,c,e,2*f]);
          ZZZ:=Polynomial(ZZ,(1-u^2)^(i-3)*Determinant(I-AA*u+Q*u^2*I));
          P1:=Matrix(3,[1,0,0,0,0,1,0,1,0]);
          P2:=Matrix(3,[0,1,0,1,0,0,0,0,1]);
          P3:=Matrix(3,[0,1,0,0,0,1,1,0,0]);
          P4:=Matrix(3,[0,0,1,1,0,0,0,1,0]);
          P5:=Matrix(3,[0,0,1,0,1,0,1,0,0]);
          ii:=1;
          while ii le count do
               if Zeta[ii] eq ZZZ then
                    if A[ii] ne P1*AA*P1^(-1) and A[ii] ne P2*AA*(P2)^(-1)
                    and A[ii] ne P3*AA*(P3)^(-1) and A[ii] ne P4*AA*(P4)^(-1)
                    and A[ii] ne P5*AA*(P5)^(-1) then
                         AA;
                         print"";
                         A[ii];
                         print"";
```

```
                    print"";
                    ii:=count+4;
                else
                    ii:=ii+1;
                end if;
            else
                ii:=ii+1;
            end if;
        end while;
        if ii eq count+1 then
            Append(~A,AA);
            Append(~Zeta,ZZZ);
            count:=count+1;
        end if;
    end if;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
```

The following MAGMA code creates the adjacency matrix of each bipartite graph with i edges and up to eight vertices. It outputs the adjacency matrices of any two bipartite graphs that have the same zeta function and different characteristic polynomials. Figure 3.5 is an example of its output.

```
ZZ<u>:=PolynomialRing(Integers());
for i in [3..15] do
i;
A:=[**];
Zeta:=[**];
Char:=[**];
count:=0;
for a in [0..i] do
    if a eq i and a ge 2 then
        I:=DiagonalMatrix([1,1]);
```

```
        Q:=DiagonalMatrix([a-1,a-1]);
        AA:=SymmetricMatrix([0,a,0]);
        ZZZ:=Polynomial(ZZ,(1-u^2)^(i-2)*Determinant(I-AA*u+Q*u^2*I));
        Ch:=CharacteristicPolynomial(AA);
        ii:=1;
        while ii le count do
             if Zeta[ii] eq ZZZ then
                  if Char[ii] ne Ch then
                       print"";
                       AA;
                       A[ii];
                  else
                       ii:=count+4;
                  end if;
             else
                  ii:=ii+1;
             end if;
        end while;
             if ii eq count+1 then
                   Append(~A,AA);
                   Append(~Zeta,ZZZ);
                   Append(~Char,Ch);
                   count:=count+1;
             end if;
   end if;
for b in [0..i-a] do
for c in [0..i-a-b] do
for d in [0..i-a-b-c] do
   if a+b+c+d eq i and a+c ge 2 and b+d ge 2 and c+d ge 2 and
   a+b ge 2 and a+c eq b+d and a+b eq c+d then
        I:=DiagonalMatrix([1,1,1,1]);
        Q:=DiagonalMatrix([a+c-1,b+d-1,a+b-1,c+d-1]);
        AA:=SymmetricMatrix([0,0,0,a,b,0,c,d,0,0]);
        ZZZ:=Polynomial(ZZ,(1-u^2)^(i-4)*Determinant(I-AA*u+Q*u^2*I));
        Ch:=CharacteristicPolynomial(AA);
        ii:=1;
        while ii le count do
             if Zeta[ii] eq ZZZ then
                  if Char[ii] ne Ch then
                       print"";
                       AA;
                       A[ii];
```

```
                    else
                         ii:=count+4;
                    end if;
               else
                    ii:=ii+1;
               end if;
          end while;
               if ii eq count+1 then
                     Append(~A,AA);
                     Append(~Zeta,ZZZ);
                     Append(~Char,Ch);
                     count:=count+1;
               end if;
     end if;
for e in [0..i-a-b-c-d] do
for f in [0..i-a-b-c-d-e] do
     if a+b+c+d+e+f eq i and a+d ge 2 and b+e ge 2 and c+f ge 2
     and a+b+c ge 2 and d+e+f ge 2  and a+d eq b+e and
     a+d eq c+f and a+b+c eq d+e+f then
          I:=DiagonalMatrix([1,1,1,1,1]);
          Q:=DiagonalMatrix([a+d-1,b+e-1,c+f-1,a+b+c-1,d+e+f-1]);
          AA:=SymmetricMatrix([0,0,0,0,0,0,a,b,c,0,d,e,f,0,0]);
          ZZZ:=Polynomial(ZZ, (1-u^2)^(i-5)*Determinant(I-AA*u+Q*u^2*I));
          Ch:=CharacteristicPolynomial(AA);
          ii:=1;
          while ii le count do
               if Zeta[ii] eq ZZZ then
                    if Char[ii] ne Ch then
                         print"";
                         AA;
                         A[ii];
                    else
                         ii:=count+4;
                    end if;
               else
                    ii:=ii+1;
               end if;
          end while;
               if ii eq count+1 then
                     Append(~A,AA);
                     Append(~Zeta,ZZZ);
                     Append(~Char,Ch);
```

```
                    count:=count+1;
                end if;
     end if;
for g in [0..i-a-b-c-d-e-f] do
for h in [0..i-a-b-c-d-e-f-g] do
     if a+b+c+d+e+f+g+h eq i and a+e ge 2 and b+f ge 2 and c+g ge 2
     and d+h ge 2 and a+b+c+d ge 2 and e+f+g+h ge 2 and a+e eq c+g
     and a+e eq b+f and a+e eq d+h and a+b+c+d eq e+f+g+h then
          I:=DiagonalMatrix([1,1,1,1,1,1]);
          Q:=DiagonalMatrix([a+e-1,b+f-1,c+g-1,d+h-1, a+b+c+d-1,
          e+f+g+h-1]);
          AA:=SymmetricMatrix([0,0,0,0,0,0,0,0,0,0,a,b,c,d,0,
          e,f,g,h,0,0]);
          ZZZ:=Polynomial(ZZ, (1-u^2)^(i-6)*Determinant(I-AA*u+Q*u^2*I));
          Ch:=CharacteristicPolynomial(AA);
          ii:=1;
          while ii le count do
               if Zeta[ii] eq ZZZ then
                    if Char[ii] ne Ch then
                         print"";
                         AA;
                         A[ii];
                    else
                         ii:=count+4;
                    end if;
               else
                    ii:=ii+1;
               end if;
          end while;
               if ii eq count+1 then
                     Append(~A,AA);
                     Append(~Zeta,ZZZ);
                     Append(~Char,Ch);
                     count:=count+1;
               end if;
     end if;
for j in [0..i] do
     if a+b+c+d+e+f+g+h+j eq i and a+d+g ge 2 and b+e+h ge 2 and
     c+f+j ge 2 and a+b+c ge 2 and d+e+f ge 2 and j+g+h ge 2 and
     a+d+g eq b+e+h and a+d+g eq c+f+j and a+b+c eq d+e+f and
     a+b+c eq g+h+j then
          I:=DiagonalMatrix([1,1,1,1,1,1]);
```

```
        Q:=DiagonalMatrix([a+d+g-1,b+e+h-1,c+f+j-1,a+b+c-1,
        d+e+f-1, j+g+h-1]);
        AA:=SymmetricMatrix([0,0,0,0,0,0,a,b,c,0,d,e,f,0,0,
        g,h,j,0,0,0]);
        ZZZ:=Polynomial(ZZ,(1-u^2)^(i-6)*Determinant(I-AA*u+Q*u^2*I));
        Ch:=CharacteristicPolynomial(AA);
        ii:=1;
        while ii le count do
             if Zeta[ii] eq ZZZ then
                  if Char[ii] ne Ch then
                       print"";
                       AA;
                       A[ii];
                  else
                       ii:=count+4;
                  end if;
             else
                  ii:=ii+1;
             end if;
        end while;
             if ii eq count+1 then
                   Append(~A,AA);
                   Append(~Zeta,ZZZ);
                   Append(~Char,Ch);
                   count:=count+1;
             end if;
   end if;
for k in [0..i-a-b-c-d-e-f-g-h-j] do
   if a+b+c+d+e+f+g+h+j+k eq i and a+f ge 2 and b+g ge 2 and c+h ge 2
   and d+j ge 2 and e+k ge 2 and a+b+c+d+e ge 2 and f+g+h+j+k ge 2
   and a+f eq b+g and a+f eq c+h and a+f eq d+j and a+f eq e+k and
   a+b+c+d+e eq f+g+h+j+k then
        I:=DiagonalMatrix([1,1,1,1,1,1,1]);
        Q:=DiagonalMatrix([a+f-1,b+g-1,c+h-1,d+j-1,e+k-1, a+b+c+d+e-1,
        f+g+h+j+k-1]);
        AA:=SymmetricMatrix([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,a,b,c,d,e,0,
        f,g,h,j,k,0,0]);
        ZZZ:=Polynomial(ZZ,(1-u^2)^(i-7)*Determinant(I-AA*u+Q*u^2*I));
        Ch:=CharacteristicPolynomial(AA);
        ii:=1;
        while ii le count do
             if Zeta[ii] eq ZZZ then
```

```
                    if Char[ii] ne Ch then
                         print"";
                         AA;
                         A[ii];
                    else
                         ii:=count+4;
                    end if;
               else
                    ii:=ii+1;
               end if;
          end while;
               if ii eq count+1 then
                     Append(~A,AA);
                     Append(~Zeta,ZZZ);
                     Append(~Char,Ch);
                     count:=count+1;
               end if;
     end if;
for l in [0..i-a-b-c-d-e-f-g-h-j-k] do
for m in [0..i-a-b-c-d-e-f-g-h-j-k-l] do
     if a+b+c+d+e+f+g+h+j+k+l+m eq i and a+e+j ge 2 and b+f+k ge 2 and
     c+g+l ge 2 and d+h+m ge 2 and j+k+l+m ge 2 and a+b+c+d ge 2 and
     e+f+g+h ge 2 and a+e+j eq b+f+k and a+e+j eq c+g+l and a+e+j eq d+h+m
     and a+b+c+d eq e+f+g+h and a+b+c+d eq j+k+l+m then
          I:=DiagonalMatrix([1,1,1,1,1,1,1]);
          Q:=DiagonalMatrix([a+e+j-1,b+f+k-1,c+g+l-1,d+h+m-1,a+b+c+d-1,
          e+f+g+h-1, j+k+l+m-1]);
          AA:=SymmetricMatrix([0,0,0,0,0,0,0,0,0,0,a,b,c,d,0,
          e,f,g,h,0,0,j,k,l,m,0,0,0]);
          ZZZ:=Polynomial(ZZ, (1-u^2)^(i-7)*Determinant(I-AA*u+Q*u^2*I));
          Ch:=CharacteristicPolynomial(AA);
          ii:=1;
          while ii le count do
               if Zeta[ii] eq ZZZ then
                    if Char[ii] ne Ch then
                         print"";
                         AA;
                         A[ii];
                    else
                         ii:=count+4;
                    end if;
               else
```

```
                ii:=ii+1;
            end if;
        end while;
            if ii eq count+1 then
                Append(~A,AA);
                Append(~Zeta,ZZZ);
                Append(~Char,Ch);
                count:=count+1;
            end if;
end if;
if a+b+c+d+e+f+g+h+j+k+l+m eq i and a+g ge 2 and b+h ge 2 and
c+j ge 2 and d+k ge 2 and e+l ge 2 and f+m ge 2 and a+b+c+d+e+f ge 2
and g+h+j+k+l+m ge 2 and a+g eq b+h and a+g eq e+j and a+g eq d+k
and a+g eq e+l and a+g eq f+m and a+b+c+d+e+f eq g+h+j+k+l+m then
    I:=DiagonalMatrix([1,1,1,1,1,1,1,1]);
    Q:=DiagonalMatrix([a+g-1,b+h-1,c+j-1,d+k-1,e+l-1, f+m-1,
    a+b+c+d+e+f-1, g+h+i+j+k+l+m-1]);
    AA:=SymmetricMatrix([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
    a,b,c,d,e,f,0,g,h,j,k,l,m,0,0]);
    ZZZ:=Polynomial(ZZ,(1-u^2)^(i-8)*Determinant(I-AA*u+Q*u^2*I));
    Ch:=CharacteristicPolynomial(AA);
    ii:=1;
    while ii le count do
        if Zeta[ii] eq ZZZ then
            if Char[ii] ne Ch then
                print"";
                AA;
                A[ii];
            else
                ii:=count+4;
            end if;
        else
            ii:=ii+1;
        end if;
    end while;
        if ii eq count+1 then
            Append(~A,AA);
            Append(~Zeta,ZZZ);
            Append(~Char,Ch);
            count:=count+1;
        end if;
end if;
```

```
for n in [0..i-a-b-c-d-e-f-g-h-j-k-l-m] do
for p in [0..i-a-b-c-d-e-f-g-h-j-k-l-m-n] do
for q in [0..i-a-b-c-d-e-f-g-h-j-k-l-m-n-p] do
     if a+b+c+d+e+f+g+h+j+k+l+m+n+p+q eq i then
     if a+f+l ge 2 and b+g+m ge 2 and c+h+n ge 2  and a+b+c+d+e ge 2
     and d+j+p ge 2 and e+k+q ge 2 and f+g+h+j+k ge 2 and l+m+n+p+q ge 2
     and a+f+l eq b+g+m and a+f+l eq c+h+n and a+f+l eq d+j+p and
     a+f+l eq e+k+q and a+b+c+d+e eq f+g+h+j+k and
     a+b+c+d+e eq l+m+n+p+q then
          I:=DiagonalMatrix([1,1,1,1,1,1,1,1]);
          Q:=DiagonalMatrix([a+f+l-1,b+g+m-1,c+h+n-1,d+j+p-1,e+k+q-1,
          a+b+c+d+e-1,f+g+h+j+k-1,l+m+n+p+q-1]);
          AA:=SymmetricMatrix([0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,
          a,b,c,d,e,0,f,g,h,j,k,0,0,l,m,n,p,q,0,0,0]);
          ZZZ:=Polynomial(ZZ,(1-u^2)^(i-8)*Determinant(I-AA*u+Q*u^2*I));
          Ch:=CharacteristicPolynomial(AA);
          ii:=1;
          while ii le count do
               if Zeta[ii] eq ZZZ then
                    if Char[ii] ne Ch then
                         print"";
                         AA;
                         A[ii];
                    else
                         ii:=count+4;
                    end if;
               else
                    ii:=ii+1;
               end if;
          end while;
               if ii eq count+1 then
                     Append(~A,AA);
                     Append(~Zeta,ZZZ);
                     Append(~Char,Ch);
                     count:=count+1;
               end if;
     end if;
     end if;
for r in [0..i-a-b-c-d-e-f-g-h-j-k-l-m-n-p-q] do
     if a+b+c+d+e+f+g+h+j+k+l+m+n+p+q+r eq i then
     if a+e+j+n ge 2 and b+f+k+p ge 2 and c+g+l+q ge 2 and d+h+m+r ge 2
     and a+b+c+d ge 2 and e+f+g+h ge 2 and j+k+l+m ge 2 and
```

```
     n+p+q+r ge 2 and a+e+j+n eq b+f+k+p and a+e+j+n eq c+g+l+q and
     a+e+j+n eq d+h+m+r and a+b+c+d eq e+f+g+h and
     a+b+c+d eq e+j+k+l+m and a+b+c+d eq n+p+q+r then
          I:=DiagonalMatrix([1,1,1,1,1,1,1,1]);
          Q:=DiagonalMatrix([a+e+j+n-1,b+f+k+p-1,c+g+l+q-1,d+h+m+r-1,
          a+b+c+d-1,e+f+g+h-1,j+k+l+m-1,n+p+q+r-1]);
          AA := SymmetricMatrix([0,0,0,0,0,0,0,0,0,0,a,b,c,d,0,
          e,f,g,h,0,0,j,k,l,m,0,0,0,n,p,q,r,0,0,0,0]);
          ZZZ:=Polynomial(ZZ,(1-u^2)^(i-8)*Determinant(I-AA*u+Q*u^2*I));
          Ch:=CharacteristicPolynomial(AA);
          ii:=1;
          while ii le count do
               if Zeta[ii] eq ZZZ then
                    if Char[ii] ne Ch then
                         print"";
                         AA;
                         A[ii];
                    else
                         ii:=count+4;
                    end if;
               else
                    ii:=ii+1;
               end if;
          end while;
               if ii eq count+1 then
                     Append(~A,AA);
                     Append(~Zeta,ZZZ);
                     Append(~Char,Ch);
                     count:=count+1;
               end if;
     end if;
     end if;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
end for;
```

```
end for;
end for;
end for;
end for;
end for;
end for;
end for;
```

Vita

Debra Lynn Czarneski was born on March 19, 1979, in Green Bay, Wisconsin. She graduated in 2001, from Mount Mercy College in Cedar Rapids, Iowa, with a major in mathematics and minor in computer science. In 2003, she received a Master of Science in mathematics from Louisiana State University. Dr. Robert Perlis is currently advising her work on Zeta Functions of Graphs. Debra is a candidate for the degree of Doctor of Philosophy in mathematics, which will be awarded in August 2005. After graduation, Debra will be an assistant professor of mathematics at Simpson College in Indianola, Iowa.