Dimer models for knot polynomials

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DIMER MODELS FOR KNOT POLYNOMIALS

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Agricultural and Mechanical College
in partial fulfillment of the
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in
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by
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It takes a village to raise a child.

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Abstract

A dimer model consists of all perfect matchings on a (bipartite) weighted signed graph, where the product of the signed weights of each perfect matching is summed to obtain an invariant. In this paper, the construction of such a graph from a knot diagram is given to obtain the Alexander polynomial. This is further extended to a more complicated graph to obtain the twisted Alexander polynomial, which involved “twisting” by a representation. The space of all representations of a given knot complement into the general linear group of a fixed size can be described by the same graph.

This work also produces a bipartite weighted signed graph to obtain the Jones polynomial for the infinite class of pretzel knots as well as for some other constructions. This is a corollary to a stronger result that calculates the activity words for the spanning trees of the Tait graph associated to a pretzel knot diagram, and this has several other applications, as well, including the Tutte polynomial and the spanning tree model of reduced Khovanov homology.
Chapter 1

Introduction

A *dimer* in chemistry terms is a polymer with exactly two atoms and one bond. In graph theory, this would look like an edge with two distinct vertices; in tiling problems common in combinatorics, this would look like a domino; these formulations have been much studied in their fields. A *dimer covering* is thus a perfect matching of a graph or a domino tiling of a gameboard on a grid.

A *dimer model* for some invariant is a weighting and a signing of the edges of a graph such that

\[
\sum_m \prod_{e \in m} \mu(e)
\]

gives this invariant after summing over all perfect matchings \(m\) and all edges in the perfect matching.

The main work in this paper provides the weighting and the signing \(\mu(e)\) of edges of a graph obtained from a knot to compute three knot invariants: the Alexander polynomial, the twisted Alexander polynomial, and the Jones polynomial.

The twisted Alexander polynomial is not nearly as well-understood as the classical
Alexander polynomial. Work has been done to translate classical properties to the twisted version; however, these vary over different representations. The new dimer model introduced here gives a large graph associated with a knot diagram on which the twisted Alexander polynomial can be computed over any representation into $GL_p(R)$ for fixed $p$.

The Jones polynomial has been formulated in terms of the spanning trees of a graph obtained from the knot diagram, and the Matrix Tree Theorem associates spanning trees to non-zero terms in a matrix determinant expansion. However, due to computational complexity work by Jaeger, Vertigan, and Welsh [JVW90], the Jones polynomial of an arbitrary knot cannot be computed in polynomial time. This work introduces a dimer model for polynomial computation for the infinite class of pretzel knots. Furthermore, the unweighted underlying graph in this model is the same as the one in the dimer model for the Alexander polynomial.

The organization of this work is as follows.

Sufficient background on knots, graph, and dimers appears in Chapter 2.

A background in Knot Theory is given in Section 2.1; further general background on the subject can be found in [Lic97]. In particular here, some notation is established for the class of pretzel knots that will be considered in Section 3.3.

The background in Graph Theory provided in Section 2.2 defines some common operations on graphs, some important graphs, and some important matrices that will be used in Chapter 3. Further general background on the subject can be found in [Bol98]. The notion of activity on spanning trees is introduced here, as well as Lemma 2.2.14 which is used to determine the activity of pretzel knots.

The graphs, spanning trees, and matrices are brought together in a discussion of Dimers in Section 2.3, which features some tricks that are implemented in Chapter 3. In particular,
the building up of Proposition 2.3.8 provides the framework for the main results in Section 3.3.

The main results on dimer models for knot polynomials appear in Chapter 3 in three sections: Theorem 3.1.5 for the Alexander polynomial, Theorem 3.2.3 for the twisted Alexander polynomial with Proposition 3.2.4 to consider any representation, and Theorem 3.3.7 for activity words that gives Corollary 3.3.5 for the Jones polynomial.

Finally some applications and further directions appear in Chapter 4, including a potential new way of computing reduced Khovanov homology.
Chapter 2

Terminology and Constructions

2.1 Knot Theory

A knot $K$ is an embedding of the circle $S^1$ into a three-dimensional manifold. In all the discussion below, and for that matter, in the usual context, this manifold is $\mathbb{R}^3$ or its one-point compactification $S^3$, which does not matter since the knot can be moved sufficiently far from this compactification point. The embedding is usually smooth, although knots can also be composed of piece-wise linear arcs.

A link $L$ with $\ell$ components is an embedding of $\ell$ disjoint copies of $S^1$ as above. Much of the discussion below can be extended to links, but in general only knots are be considered unless otherwise specified.

Each of these closed one-manifolds can be oriented in either direction. See an orientation of the trefoil knot in Figure 2.1.

One project in Knot Theory is to be able to distinguish a knot from another or discern that they are indeed the same. Two knots are considered topologically equivalent if there is
an ambient isotopy taking one to the other.

The traction in this field has come from the discovery of a large number of knot invariants, or algebraic objects algorithmically assigned to a knot that are invariant over any of the topologically equivalent embeddings. The knot polynomials discussed in Chapter 3 are such objects in the Laurent polynomials.

However, knots are accessible as purely combinatorial data, and so one need not think of them in terms of the (differential) topology of three-space.

2.1.1 A Combinatorial Perspective

A knot diagram $D$ is the projection of a knot onto the plane along with under- and over-crossing information, so long as each crossing is transverse and there are no triple points.

Due to a powerful result by Reidemeister in 1926 (see for example [Lic97]), two diagrams represent the same knot if there is a transformation from one to the other by isotopies of arcs along with a sequence of moves, now called Reidemeister moves, which are local configurations of one, two, and three crossings.

These represent local configurations, and outside of these neighborhoods the diagrams must remain the same. Consider these subdiagrams themselves, called tangles, as configurations of $n$ arcs whose $2n$ endpoints lie along an outer circle. Sometimes additional closed loops are considered within these tangles, and sometimes these arcs and closed loops are oriented.
For $n = 2$ with no interior closed loops in the unoriented case, three configurations are shown in Figure 2.2.

The unoriented crossing appears with its two smoothings; the first is called the $A$-smoothing, and the second is called the $B$-smoothing. In practice, the first smoothing is distinguished by rotating the over-strand of the crossing counterclockwise towards the understrand, sweeping out the region of the $A$-smoothing.

When one of these tangles in a diagram is replaced by any of the three, denote the link diagrams by $L_x$, $L_0$, and $L_\infty$, respectively. It is important to consider link diagrams and not just knot diagrams because replacing a crossing by either of the smoothings (and vice versa) can change the number of components of the link.

For $n = 2$ with no interior closed loops in the oriented case, three configurations are shown in Figure 2.3.

When one of these tangles in a diagram is replaced by any of the three, denote the link diagrams by $L_+$, $L_-$, and $L_0$, respectively. It is again important to consider link diagrams and not just knot diagrams because replacing a positive or negative crossing by the oriented
smoothing (and vice versa) can change the number of components of the link. Even though
the notation $L_0$ is used for both the oriented and unoriented cases, it should be clear by
context which is being considered.

One can consider how a given link invariant changes over these three configurations. In
some cases, this can be expressed by a *Skein relation* as in Chapter 3.

One can stack unoriented crossings vertically or horizontally and remain a tangle with
$n = 2$ and no interior closed loops. Each stack is called a *twist region*, and these can be
joined up together to form what is called a *rational tangle*. Tangles with $n = 2$ can be closed
in two ways to obtain knots and links; the closure of a rational tangle is a *rational knot or
link*.

There is an infinite subclass of unoriented $n = 2$ tangles where unoriented crossings
appear stacked in $k$ parallel columns. The column is considered positive if the unoriented
crossing above is stacked vertically and is considered negative if the unoriented crossing is
stacked horizontally and then righted into a column.

The *pretzel knot* $P = P(n_1, n_2, \ldots , n_k)$ is the closure of a tangle described above with $k$
columns, each of $|n_i| \in \mathbb{N}$ crossings where the sign of $n_i$ determines whether the which way
the crossings are stacked. For example see Figure 2.4 where in this case all the crossings are
positive, that is, each $n_i > 0$.

Some twist regions may be unnecessary overall, and so it can simplify a diagram by
removing them ahead of time. One way of doing so is to identify certain crossings at which
one can untwist.

A crossing is *nugatory* if a circle can be drawn in the plane meeting the diagram only at
the crossing. Nugatory crossings can be added or removed by the first Reidemeister move.
Since one might be interested in accounting for this, the following definition will be useful.

The *writhe* of an oriented diagram is the difference of the number of positive crossings and the number of negative crossings. Alternatively, it is the sum over all crossings of the evaluation $+1$ for positive crossings and $-1$ for negative crossings as in Figure 2.5.

![Positive and negative crossings](image)

Figure 2.5: Positive and negative crossings contribute $+1$ and $-1$ to the writhe.

### 2.2 Graph Theory

A *graph* $G = (V, E)$ is an ordered pair of disjoint sets such that $E \subset V \times V$ is (multi-)subset of unordered pairs of $V$. In the discussion below, $G$ is always a finite graph, that is, $V$ and $E$ are finite. The elements $v$ of $V$ are called *vertices* and the elements $e = v_1v_2$ of $E$ are called *edges*. If an edge $vv = e \in E$ is named by the same vertex twice, it is called a *loop*, and if there are several edges named by the same two vertices $v_1v_2$, these are called *parallel*
edges or multi-edges. A graph without loops and parallel edges can be called simple and a graph with these can be called a multi-graph. A graph whose edges are oriented is called a directed graph or digraph; its edges are called directed edges.

A graph is an abstract object, but if it can be embedded in the plane without its edges crossing one another, it is called planar. This plane embedding is called a plane graph, and here one can see further combinatorial data: the regions of the plane minus the graph. These 2-cells, which again can be defined combinatorially, are called faces, and these are each homeomorphic to a disk with the exception of the one that borders the one-point compactification at infinity, called the outer face, infinite face, or universal face. Viewed with this one-point compactification, this graph can be embedded on the sphere, a genus-zero surface such that every face \( f \in F \) is homeomorphic to a disk.

By Euler’s formula, the numbers of vertices, edges, and faces of a connected plane graph are related by the formula

\[ |V| - |E| + |F| = 2. \]

The dual \( G^* \) of a plane graph \( G \) is the graph whose vertex set \( V^* = F \) is the set of faces of \( G \), whose faces \( F^* = V \), and whose edges are in bijective correspondence with the original edges. Thus by Euler’s formula, the dual of a plane graph is also a plane graph. The explicit embedding of the dual can be obtained by taking the barycenters of each of the faces as the new vertices and having each dual edge cross each original edge exactly once.

In general, one can embed any graph on a(n orientable,) closed surface such that there are no crossings and such that each face is indeed homeomorphic to a disk. Thus the surface of interest is minimal genus.

A graph embedded on such a surface in such a way is called a dessin d’enfant (literally,
a children’s drawing), but it goes by several other names as well, including fat graph, ribbon graph, and combinatorial map.

2.2.1 Operations on Graphs

Given an edge $e$ in a graph $G = (V, E)$, the operation of deletion produces the graph $G\setminus e$ whose vertex set is the same and whose edge set is $E - \{e\}$. A subgraph $H$ of a graph $G$ is any graph obtained from $G$ by a sequence of deletions.

Given an edge $e = v_1v_2$ in a graph $G = (V, E)$, the operation of contraction produces the graph $G/e$ whose vertex set is $V - \{v_1, v_2\} \cup v$ where the new vertex $v$ replaces both $v_1$ and $v_2$ and whose edge set is $E - \{e\}$ where other occurrences of $v_1$ and $v_2$ are replaced by the new vertex $v$. A minor $H'$ of a graph $G$ is any graph obtained from a subgraph $H$ of $G$ by a sequence of contractions.

Specifically, these operations do exactly what they claim: deleting an edge $e$ from the graph; or contracting the edge and identifying the two end points. Furthermore, these operations are dual: the deletion of an edge in $G$ (and then taking its dual) and the contraction of the corresponding edge in the dual $G^*$ (and then taking its dual) both result in the same pair of graphs $G$ and $G^*$.

The dual of a loop is a coloop or bridge. A tree is a graph with no cycles: every edge is a bridge. Note that it has a single face, and so its dual is a single vertex with some number of loops.

A spanning tree is a one-component subgraph that has no cycles and whose vertex set is the same as the original graph. A rooted spanning tree in a directed graph is a spanning tree together with a choice of a fixed vertex or root, from which all edges are directed outward.
The following property is quite remarkable.

**Property 2.2.1.** Given a plane graph $G$ and a spanning tree $T \subset E$, the complementary edges $T^c = E - T$ form a spanning tree of the dual graph $G^*$.

The *complete graph* $K_n$ is the simple graph on $n$ vertices farthest away from being planar (for $n \geq 5$, at least) because every vertex is adjacent to every other one. A graph is *bipartite* if its vertex set can be written $V = V_1 \sqcup V_2$ as the disjoint union of two sets such that each edge has one end in $V_1$ and the other in $V_2$. A bipartite graph with $|V_1| = |V_2|$ will be called *balanced* below. The *complete bipartite graph* $K_{n_1,n_2}$ is the simple graph that includes every possible edge from the $n_1$ vertices of $V_1$ to the $n_2$ vertices of $V_2$.

The condition on edges in a bipartite graph makes it more natural to consider the following concept.

**Definition 2.2.2.** A *matching* of a graph is a subgraph where every vertex of the original graph is incident with at most one edge of the subgraph. A *perfect matching* is a subgraph where every vertex is incident with exactly one of the edges. One says that this subgraph covers the original vertex set, and each vertex is incident with no more than one of the edges of the perfect matching.

On the subject of treating individual edges, there are two more dual operations which will be discussed below.

A *subdivision* of an edge $v_1v_2$ results in two edges $v_1v_3$ and $v_3v_2$; specifically, a new vertex $v_3$ is added at the barycenter of the edge, splitting it into two edges.

In terms of the dual graph, this is equivalent to the *doubling* of an edge, that is, including an additional edge between the same two vertices.
The set of edges between the same two vertices will be referred to as a parallel edge class. The dual of this parallel edge edge will be non-standardly referred to as a path, even though this term usually represents something far more general in graph theory.

### 2.2.2 Graphs Arising from Knots

A signed graph is a graph whose edges are assigned either a + or a − sign.

Given an oriented knot $K$ and a diagram $D$ with $n$ crossings, checkerboard-color the regions of the diagram. Construct the signed Tait graph by taking the vertex set to be the set of black regions and the signed edge set to be the set of crossings of the diagram along with sign information as in Figure 2.6.

![positive negative](image)

Figure 2.6: Crossings determine the sign of the edges in the signed Tait graph.

Note that there are actually two graphs here: one for the black regions and its dual for the white regions. Each of these graphs has $n$ edges and is a plane graph, so the notion of duality makes sense. For example, the projection of the trefoil in Figure 2.1 has as its Tait graphs the triangle $K_3$ and its dual, a single parallel edge class of three edges between two vertices.

**Definition 2.2.3.** Given a knot $K$ and a diagram $D$, the projection itself, with its over- and under-crossing information removed, is a 4-valent graph, which shall be called the projection graph.

For a knot diagram with $n$ crossings, the projection graph is a plane graph with $n$ four-
valent vertices corresponding to the original crossings, $2n$ edges, and thus $n + 2$ faces by Euler’s formula. See for example Figure 2.7 for the projection graph of the trefoil given by the knot diagram in Figure 2.1.

![Figure 2.7: A projection graph of the trefoil.](image)

The main graph of interest in the discussion below is the following new graph associated to a knot diagram. A similar notion can be found in [HV08].

**Definition 2.2.4.** The **overlaid Tait graph** is the signed bipartite graph obtained from a projection graph as follows. The first vertex set of the new graph is the vertex set of the projection graph, and the second vertex set is the set of barycenters of all the faces of the projection graph. Two vertices are adjacent in the new graph if the associated vertex and face in the projection graph are incident with the same edge.

Note: The signs of the edges do not arise from the signs of the original Tait graphs but may be assigned somewhat arbitrarily according to a Kasteleyn weighting. For more on this, see Section 2.3 below.

As the name suggests, the overlaid Tait graph can be understood as the bipartite graph whose first vertex set is the set of intersection points of the edges and whose second vertex set is the union of the vertex sets of both the original Tait graph and its dual. The edges of this graph are the half-edges of both of the Tait graphs.
For a knot diagram with $n$ crossings, the overlaid Tait graph is a plane graph with $n$ four-valent vertices in the first vertex set corresponding to the original crossings of the diagram, $n + 2$ vertices in the second vertex set corresponding to the faces of the projection graph, a total of $4n$ edges understood as four around each of the crossings, and thus $2n$ faces by Euler’s formula.

**Property 2.2.5.** Each edge in the projection graph corresponds to a square face in the overlaid Tait graph.

*Proof.* Each edge in the projection graph is incident with exactly two vertices and exactly two faces, which correspond to four vertices in the overlaid Tait graph, as in Figure 2.8.

In order to discuss perfect matchings on this bipartite graph, the size of the two vertex sets must be equal. Thus the graph below will be of more interest.

**Definition 2.2.6.** The balanced overlaid Tait graph can be obtained from the overlaid Tait graph by deleting two vertices from the larger vertex set that lie on the same square face along with all the edges incident with them.

For the sake of simplicity of construction, one can create the balanced overlaid Tait graph directly from the projection graph by neglecting two adjacent faces, that is, by neglecting the two faces incident with a single edge of the projection graph.
Furthermore, the vertex corresponding to the universal face is an obvious candidate for deletion. See for example the balanced overlaid Tait graph associated with the diagram of the trefoil in Figure 2.1 with the universal face and the upper face deleted in Figure 2.9.

Figure 2.9: A balanced overlaid Tait graph of the trefoil.

There are two matrices that can be associated with a (simple) graph that help to encode the combinatorial data. Two other matrices will be constructed from these to be used below.

**Definition 2.2.7.** The incidence matrix has its rows labelled by the edges of the graph and its columns labelled by the vertices of the graph. The $ij$-th entry is 0 if the $i$-th edge is not incident with the $j$-th vertex; if it is incident, the entry is 1 for an unweighted, unsigned graph or can be replaced by the weight and sign of the edge in the weighted, signed version.

This matrix can be very far from square since the number of edges of a simple graph can be as many as $\binom{|V|}{2}$ in a simple graph. However, if one unites the incidence matrix of a plane graph with the incidence matrix of the dual graph, the matrix becomes $|E| \times (|V| + |F|)$, which is $|E| \times (|E| + 2)$ by Euler’s formula. When columns corresponding to one vertex and to one face are removed thus creating a square, call this the squared incidence matrix.

**Definition 2.2.8.** The adjacency matrix has both its rows and its columns labelled by the
vertices of the graph. The $ij$-th entry is again 0 if the $i$-th vertex is not adjacent to the $j$-th vertex; if it is adjacent, the entry is again 1 for an unweighted, unsigned simple graph or can be replaced by the weight and sign of the edge that connects them in the weighted, signed version.

The adjacency matrix of a bipartite graph has the property that upon rearranging the labels, the matrix can be presented in block form

$$\begin{pmatrix}
0 & M \\
M^T & 0
\end{pmatrix}$$

with some submatrix $M$ and its transpose $M^T$ along with zero blocks. Clearly, the whole adjacency matrix is redundant, and only the submatrix $M$ need be considered. Let this submatrix be called the *bipartite adjacency submatrix* associated to the graph.

Note that in a balanced bipartite graph, each block or submatrix is square at half the size of the original matrix.

**Proposition 2.2.9.** Choose two omitted faces in the projection graph associated with a knot diagram. Then the squared incidence matrix of the (unweighted, unsigned) Tait graph associated with the knot diagram is in fact the bipartite adjacency submatrix of the (unweighted, unsigned) balanced overlaid Tait graph associated with the knot diagram.

*Proof.* By construction, the rows of both matrices represent the original crossings of the knot diagram, and the columns of both matrices represent the faces of the projection graph associated to the knot diagram. Before weighting and signing the edges, the zero and non-zero entries of the matrix are the same by incidence/adjacency. 

\[\square\]
Furthermore, one can obtain the balanced overlaid Tait graph directly given the bipartite adjacency submatrix.

In order to introduce a relatively obscure matrix operation, it will be useful to take a particular vantage point of a more common one:

Recall that the determinant of a matrix $M = (m_{ij})$ can be defined by

$$
det(M) = \sum_{\sigma} \prod_i (-1)^{\text{sign}(\sigma)} m_{i\sigma(i)}$$

summing over all permutations $\sigma$ in the symmetric group. Thus the terms in the determinant expansion must be considered with sign. However, it may be useful to consider these without sign, as in the following definition.

**Definition 2.2.10.** The *permanent* or *unsigned determinant* of a matrix $M = (m_{ij})$ is

$$perm(M) = \sum_{\sigma} \prod_i m_{i\sigma(i)}$$

summing over all permutations $\sigma$ in the symmetric group.

### 2.2.3 Activity on Spanning Trees

In the way that one seeks to distinguish knots by knot invariants, one can also distinguish graphs by way of graph invariants. Several popular graph polynomials can be viewed as specializations of a single one:

**Theorem 2.2.11.** *(Tutte)* For a graph $G$ and an edge $e$, the Tutte polynomial is the unique
graph polynomial satisfying:

\[
T(G; x, y) = \begin{cases} 
  xT(G \setminus e; x, y) & \text{if } e \text{ is a bridge,} \\
  yT(G/e; x, y) & \text{if } e \text{ is a loop,} \\
  T(G \setminus e; x, y) + T(G/e; x, y) & \text{if } e \text{ is neither a bridge nor a loop.}
\end{cases}
\]

When applied repeatedly, this deletion-contraction formula can be used to reduce a graph to its most basic components: bridges and loops. It is important to note, then, that this inherently orders the edges in some arbitrary way.

A deletion-contraction model like the one above can be exploited to decompose a graph in other ways, as well. The number of spanning trees of a graph can be expressed as the sum of the number of spanning trees in the deletion graph and the number of spanning trees in the contraction graph for any given edge.

**Definition 2.2.12.** To a spanning tree \( T \) of a signed graph whose \( n \) edges are ordered, Tutte assigns an activity word of length \( n \) in the alphabet \( L, D, \ell, d, \overline{L}, \overline{D}, \overline{\ell}, \overline{d} \). A positive edge \( e \in T \) is internally active (or live, \( L \)) if it is the lowest numbered edge that reconnects \( T \setminus e \); otherwise it is internally inactive (or dead, \( D \)). A positive edge \( e \notin T \) is externally active (or live, \( \ell \)) if it is the lowest numbered edge in the unique cycle \( T \cup e \); otherwise it is externally inactive (or dead, \( d \)). Negative edges \( \overline{L}, \overline{D}, \overline{\ell}, \overline{d} \) are defined similarly.

For an unsigned graph \( G \), let \( i(T) \) and \( e(T) \) count the numbers of internally and externally active letters, respectively, in the activity word associated to a tree \( T \).
Theorem 2.2.13. (Tutte) For a graph $G$, the Tutte polynomial can be written

$$T(G; x, y) = \sum_T x^{e(T)} y^{\varepsilon(T)}$$

as the sum over all spanning trees $T$ of the graph $G$.

The following lemma and its other versions will be used to determine the activity words of spanning trees on certain subgraphs of signed graphs whose edges are ordered. The first one is for positive path subgraphs, that is, on subgraphs whose vertices are all two-valent in both the graph and the subgraph except for two single-valent vertices in the subgraph which can be multi-valent in the graph. Another way to view this is by the repeated subdivision of an edge.

Lemma 2.2.14. (Classification of activity on paths) Suppose that a path $P_{k+1}$ of $k + 1$ positive edges indexed sequentially by $i, \ldots, i + k$ for some $k > 0$ belongs to a graph $G$ such that all interior vertices of the path have degree exactly two in $G$. Then when determining the activity word for $G$ given a spanning tree $T$, the portion of the activity word associated to the path $P_{k+1}$ must be one of the following:

1. $L \ldots L$ or $D \ldots D$ when all edges of $P_{k+1}$ are included in $T$, or

2. $L \ldots LdD \ldots D$ omitting only the $(i + j)$-th edge for $0 \leq j \leq k$, or

3. $\ell D \ldots D$ when $i = 1$ and a few other exceptional cases.

Proof. First suppose that the path $P_{k+1}$ is contained in the spanning tree $T$; then each of its edges are labelled by $L$ or $D$. Supposing the $i$-th edge is labelled $L$, this is the least-indexed edge to reconnect the severed tree $T - \{i\}$, and so no edge indexed less than $i + 1$ can
reconnect the severed tree $T - \{i + 1\}$. Iterate $j$ times to get the path labelled by $L \ldots L$.

Supposing the $i$-th edge is labelled $D$, there is an edge indexed less than $i$ that reconnects the severed tree $T - \{i\}$, and so this edge also reconnects the severed tree $T - \{i + 1\}$. Iterate $j$ times to get the path labelled by $D \ldots D$. Take note that only these two possibilities arise here, as this face is used below.

Now suppose that the edge indexed by $i + j$ of the path $P_{k+1}$ is not contained in the spanning tree $T$; then it must be labelled by either $\ell$ or $d$. The cycle contained in $T \cup \{i + j\}$ must also contain the edge indexed by $i + j - 1$, so the $(i + j)$-th edge must be labelled by $d$ unless $j = 0$. Note that no other edge may be omitted without disconnecting the tree, so the rest of the edges of $P_{k+1}$ must be labelled by $L$ or $D$. Then by the argument above, each path $i, \ldots, i + j - 1$ and $i + j + 1, \ldots, i + k$ must be one of $L \ldots L$ or $D \ldots D$.

The string before the omitted edge $i + j$ must contain an $L$ (and therefore be $L \ldots L$ by the argument above) because only the edges $i + j - 1$ and $i + j$ reconnect the severed tree $T - \{i + j - 1\}$. The string following the omitted edge $i + j$ must contain a $D$ (and therefore be $D \ldots D$ by the argument above) because only the edges $i + j$ and $i + j + 1$ reconnect the severed tree $T - \{i + j + 1\}$.

The negative case is similar, with each activity letter replaced by its negative counterpart. The dual version and its negative case work, as well, and the proofs are similar.

**Lemma 2.2.15.** *(Classification of activity on parallel edge classes)* Suppose a $k + 1$-parallel edge class $P_{k+1}$ of $k + 1$ positive edges indexed sequentially by $i, \ldots, i + k$ for some $k > 0$ belongs to a graph $G$. Then when determining the activity word for $G$ given a spanning tree $T$, the portion of the activity word associated to the parallel edge class $P_{k+1}$ must be one of the following:
1. either \(\ell \ldots \ell\) or \(d \ldots d\) when no edges of \(P_{k+1}\) are included in \(T\), or

2. \(\ell \ldots \ell D d \ldots d\) when only the \((i + j)^{th}\) edge for \(0 \leq j \leq k\) is included in \(T\), or

3. \(L d \ldots d\) when \(i = 1\) and a few other exceptional cases.

### 2.3 Dimer Model

A dimer in chemistry terms is a polymer with exactly two atoms and one bond. In graph theory, this would look like an edge with two distinct vertices; in tiling problems common in combinatorics, this would look like a domino; these formulations have been much studied in their fields. A dimer covering is thus a perfect matching of a graph or a domino tiling of a gameboard on a grid.

A dimer model for some invariant is a weighting and a signing of the edges of a graph such that

\[
\sum_m \prod_{e \in m} \mu(e)
\]

gives this invariant after summing over all perfect matchings \(m\) and all edges in the perfect matching. The main work in this paper provides the weighting and the signing \(\mu(e)\) of edges to obtain knot polynomials.

**Proposition 2.3.1.** The terms in the permanent expansion of a bipartite adjacency submatrix associated with a(n unsigned) balanced bipartite graph give the complete list of perfect matchings of the graph.

**Proof.** Each term in the permanent expansion is a permutation \(\sigma\) matching each vertex \(i\) in the first vertex set to a vertex \(\sigma(i)\) in the second vertex set. \(\square\)
2.3.1 Kasteleyn Weighting

This of course raises the question of providing signs so that the determinant can be recovered. The following accomplishes this.

**Definition 2.3.2.** A *Kasteleyn weighting* of a plane bipartite graph is a choice of sign for each edge such that the number of negatives around a particular face is

- odd if the face has length 0 mod 4 or
- even if the face has length 2 mod 4.

Since only bipartite graphs are considered here, there can be no odd cycles, and thus these are the only cases.

**Lemma 2.3.3.** If an edge is deleted from a graph with a Kasteleyn weighting, the resulting graph still has a Kasteleyn weighting.

**Proof.** Suppose the edge is incident with two faces of length $f_1$ and $f_2$. Then if the edge is deleted, the new face that replaces both has length $f_1 + f_2 - 2$. Since the edge was counted twice, the number of negatives in this new face changes by 0 or 2 (an even number) compared with the sum of the number of negatives in $f_1$ and $f_2$. According to the four cases listed in Table 2.1, the Kasteleyn weighting rule is preserved. □

Kauffman’s signing convention (appearing [Kau06] in the future republication of [Kau83]) described in Figure 2.10 provides a convenient trick to distribute signs to the edges of the balanced overlaid Tait graph coming from a knot diagram.

**Proposition 2.3.4.** Kauffman’s trick provides a Kasteleyn weighting for the balanced overlaid Tait graph coming from an oriented knot diagram.
Table 2.1: The four cases for deleting an edge in a Kasteleyn weighting.

<table>
<thead>
<tr>
<th>$f_1 \mod 4$</th>
<th># negs</th>
<th>$f_2 \mod 4$</th>
<th># negs</th>
<th>$f_1 + f_2 - 2 \mod 4$</th>
<th># negs</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 odd</td>
<td>0 odd</td>
<td>2 even</td>
<td>0 odd</td>
<td>2 even</td>
<td>0 odd</td>
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<td>2 even</td>
</tr>
</tbody>
</table>

Figure 2.10: Kauffman’s trick to yield a Kasteleyn weighting.

Proof. By Property 2.2.5, each of the faces in the overlaid Tait graph is a square. Two vertices from the same set incident with a single face are deleted to get the balanced overlaid Tait graph, creating a single larger universal face.

Given a square face as in Figure 2.11 (repeated from above), the assigning of a negative edge according to Figure 2.10 affects exactly one of the northwest and southwest sides of the square. Thus exactly one edge of every square face is negatively signed.

Thus Kauffman’s trick provides a Kasteleyn weighting for the overlaid Tait graph as-

Figure 2.11: A square face of the balanced overlaid Tait graph.
Figure 2.12: Kuperberg’s tripling trick to reduce crossings per edge.

associated with a knot diagram, and by Lemma 2.3.3, the edge deletions that result in the balanced overlaid Tait graph do not affect this weighting.

Because a Kasteleyn weighting may not exist for a bipartite graph that is not planar, Kuperberg [Kup98] alters the graph in such a way as to preserve the total weighted count of perfect matchings. The configuration of a single edge crossing another single edge is replaced by a butterfly configuration as in Figure 2.13. To ensure that an edge is not crossed by more than a single edge, the tripling operation, or two successive subdivisions, depicted in Figure 2.12 preserves the bipartite property while segmenting off an edge to be turned into a butterfly.

**Proposition 2.3.5.** Before the assigning of a Kasteleyn weighting of signs to the edges, the local contribution of an edge weighted by $w$ to all the perfect matchings in a dimer model is equal to the local contribution of edges weighted by 1, $-1$, and $w$ after Kuperberg’s tripling operation on a graph.

**Proof.** Observe that if the edge weighted by $w$ prior to a tripling is indeed included in the perfect matching, then afterward the perfect matching must include the two edges weighted by 1 and by $w$, not altering the product. If the edge is omitted from the perfect matching, then the new edge weighted by $-1$ must be included. Also note that there are no other local configurations to be considered in a perfect matching on the whole graph.
At the level of the weighted bipartite adjacency submatrix, this looks like the following

\[
\begin{pmatrix} M & c \\ r & w \end{pmatrix} \rightarrow \begin{pmatrix} M & 0 & c \\ r & 1 & 0 \\ 0 & -1 & w \end{pmatrix},
\]

with some untouched submatrix given by \( M \) and some not-necessarily-non-zero row \( r \) and column \( c \).

The Kasteleyn weighting signs of these new edges are positive except for the two edges with negative weights, who receive additional negative signs, thus making all the terms positive in the determinant expansion. \( \square \)

**Proposition 2.3.6.** The local contribution of two crossing edges weighted by \( v \) and \( w \) to all the perfect matchings in a dimer model is equal to the local contribution of edges weighted according to Figure 2.13 after Kuperberg’s butterfly replacement operation on a graph.

**Proof.** Now if the two edges weighted by \( v \) and \( w \) are indeed included in a perfect matching, then afterward there are three local configurations that work: the three vertical edges weighted \(-1, 1, \) and \(-vw\); the first vertical edge \(-1 \) and the horizontal \( v \) and \( w \); and the horizontal \( 1 \) and \( 1 \) with the vertical \(-vw\). Thus the result after the operation is \(-vw\). If only \( v \) is included, then afterward the only configuration is \( 1 \) and \( v \); if only \( w \) is included
initially, then afterward the only configuration is 1 and \( w \). If neither of the two edges is included initially, the only configuration is the central vertical 1. Also note that there are no other local configurations to be considered in a perfect matching on the whole graph.

At the level of the signed and weighted bipartite adjacency submatrix, this looks like the following

\[
\begin{pmatrix}
M & c_1 & c_2 \\
-1 & 1 & 0 \\
0 & 1 & w \\
0 & v & -vw
\end{pmatrix},
\]

with some untouched submatrix given by \( M \) and some not-necessarily-non-zero rows \( r_i \) and columns \( c_i \).

The Kasteleyn weighting signs of these new edges are positive except for the two edges with negative weights, who receive additional negative signs, thus making all the terms positive in the determinant expansion.

It is important to note that the implementation of Kuperberg’s tricks, while solving the problem of a Kasteleyn weighting, may significantly alter the original balanced overlaid Tait graph obtained from a knot diagram. Specifically, these operations add at least one vertex to the set of vertices associated with crossings, so Kauffman’s trick can only be used towards a Kasteleyn weighting before the Kuperberg operations.

### 2.3.2 Dimers and Spanning Trees

**Proposition 2.3.7.** Suppose a balanced bipartite graph is a plane graph that is given a Kasteleyn weighting. Then the terms in the determinant expansion of a bipartite adjacency
submatrix associated with the graph give the complete list of perfect matchings of the signed graph, up to an overall sign.

Proof. From Proposition 2.3.1 above, only the sign of each term in the determinant expansion needs to be checked against the signs of the edges in the perfect matchings. It is enough to demonstrate the sign difference between any two terms and the sign difference between the two corresponding perfect matchings are indeed the same.

Suppose two permutations that do not give zero terms differ by exactly one transposition. This holds if and only if there are four non-zero terms arranged as corners of a rectangle in the matrix. This holds if and only if there are two vertices from each of the two vertex sets incident with both of the vertices in the other two set, that is, if and only if there is a square face in the graph.

Since the face has an odd number of signs by the Kasteleyn weighting, the two perfect matchings, which differ only on the opposite sides of this square, must have opposite signs.

Any two permutations differ in some number of transpositions, so this can be extended to all terms.

The dimer model is indeed a useful way to look at knot diagrams, especially in light of the correspondence between perfect matchings of the balanced overlaid Tait graph and rooted spanning trees of the Tait graph. There is some ambiguity in the previous spanning tree model.

First there is the question of which of the two Tait graphs is easier to consider. Secondly each edge of the graph actually represents two directed edges that coincide, and so when an edge is chosen, there is a question of which direction is being considered. This direction comes both from the selection of the other edges in the spanning tree and also from the choice of
the omitted vertex in the Tait graph. There were two omitted faces in the projection graph, and one corresponds to a vertex in the Tait graph and the other to a vertex in the dual of the Tait graph. The omitted vertex is then considered the root, and all edges of each spanning tree are directed away from it.

**Proposition 2.3.8.** Given a knot diagram and the choice of two omitted faces in the projection graph associated with the knot diagram, there is a bijection between perfect matchings of the balanced overlaid Tait graph associated with the knot diagram and rooted spanning trees of the Tait graph associated with the knot diagram.

**Proof.** By Proposition 2.3.7, there is a bijection between perfect matchings of the balanced overlaid Tait graph associated with the knot diagram and the terms in the permanent expansion of the bipartite adjacency submatrix. By Proposition 2.2.9, the bipartite adjacency submatrix is the squared incidence matrix of the Tait graph associated with the knot diagram. Then it is enough to show that the non-zero terms of the permanent expansion of the squared incidence matrix give the complete list of rooted spanning trees of the Tait graph associated with the knot diagram.

A non-zero term in the permanent expansion of the squared incidence matrix is a bijection between edges of the Tait graph and the set composed of all but one of the vertices and all but one of the faces. This partitions the edge set into a collection of edges $T$ in the Tait graph and the complement $T^c = E - T$ in the dual graph.

By Property 2.2.1, $T$ is a spanning tree if and only if $T^c$ is a spanning tree in the dual graph. To check that $T$ is indeed a spanning tree and that $T^c$ is indeed a spanning tree in the dual, first note that by construction both of these span all except the omitted vertices.

If there is a cycle $C \subset T$, then since there are an equal number of vertices and edges, the
omitted vertex cannot be part of the cycle. Thus it must be on one side, say the outside, of $C$. Since $C$ partitions the faces into two non-empty sets, the omitted face must be on one side of $C$. By construction the omitted vertex must be on the omitted face, so these must be on the same side: the outside. Then so that $T^c$ can span, and since there must be an equal number of dual vertices and edges on the inside of $C$, there must be a cycle in the dual on the inside. Then so that $T$ can span, there must be a cycle in the original graph on the inside of that cycle in the dual. Repeating this process yields an infinite graph, which is a contradiction.

Since $T$ does not have any cycles, it must be incident with $|T| + 1$ vertices, and thus it is incident with the omitted vertex as well. The dual tree $T^c$ is similarly incident with the omitted dual vertex.

On the other hand, a rooted spanning tree $T$ and its dual rooted spanning tree $T^c$ yield a matching by associating each edge with the vertex it is directed toward in the graph and the dual graph.

To a crossing in the knot diagram, there are exactly four configurations of edges that can be incident with the associated vertex in the overlaid Tait graph, and there are exactly four configurations of directed edges that can be associated to it in the Tait graph. It is easy to see the relationship between these, as in Figure 2.14.
Figure 2.14: The correspondence between Tait graph and overlaid Tait graph edges.
Chapter 3

Knot Polynomials

In this chapter, a dimer model is presented for the Alexander polynomial, the twisted Alexander polynomial, and the Jones polynomial. The main results are Theorem 3.1.5 for the Alexander polynomial, Theorem 3.2.3 for the twisted Alexander polynomial, and Theorem 3.3.7 for activity words that gives Corollary 3.3.5 for the Jones polynomial.

3.1 Alexander Polynomial

The first knot polynomial, due to Alexander in 1923, can be defined in terms of the infinite cyclic cover $X$ of a knot complement $S^3 \setminus K$. One obtains this covering by gluing together a countably infinite number of copies of the Seifert surface of the knot; let $t$ be the translation to the next copy. Then the first homology group $H_1(X; \mathbb{Z})$ is a module over the integral Laurent polynomials $\mathbb{Z}[t^\pm]$; this is called the Alexander module. A presentation matrix for this finitely presentable module is called the Alexander matrix. The ideal generated by all full-rank minors of this matrix is called the Alexander ideal, which is always non-zero and principal.
Definition 3.1.1. The Alexander polynomial $\Delta_K(t)$ of a knot $K$ is the generator of the Alexander ideal up to a unit, that is, up to some $\pm t^k$.

There are several easily constructed presentations for the Alexander module; these can be defined combinatorially based on a knot diagram $D$ with $n$ crossings.

First consider the Wirtinger presentation of the fundamental group $\pi_1(X; \mathbb{Z})$ of the knot complement. The $n$ generators are the loops around each of the arcs $x_j$, that is, around the connected components of the drawing of the diagram, following along the knot from one undercrossing to the next undercrossing. The $n$ relations $r_i = x_i x_k^\pm - x_k^\pm x_j$ come from the $\pm$ crossings with $x_k$ coming from the overstrand.

Construct the Fox calculus Jacobian matrix $M_{Wirt} = (m_{ij})$ whose entries $m_{ij}$ are the Fox calculus derivatives $\partial r_i / \partial x_j$. For the row corresponding to the relation $r_i = x_i x_k^\pm - x_k^\pm x_j$, the non-zero terms are $1$, $-x_k^\pm$, and either $x_i - 1$ or $-x_k^{-1}(x_i - 1)$ in columns $i$, $j$, and $k$, respectively.

In order to get the first homology $H_1(X) = \pi_1 / [\pi_1, \pi_1]$ from this presentation, abelianize the entries by sending each $x_j \mapsto t$, a generator of $\langle t \rangle \cong \mathbb{Z} \cong \pi_1(X; \mathbb{Z})$. Notice that this corresponds to the meridian of the knot. Then the non-zero terms of each row are $1$, $-t^\pm$, and $t^\pm - 1$.

Because one of the relations is redundant, the presentation matrix $M_{Wirt}$ is $n - 1 \times n$, and thus to get a full-rank minor, only one column need be deleted.

Alternate Definition 3.1.2. The Alexander polynomial $\Delta_K(t)$ of a knot $K$ can be defined as the determinant of a full-rank minor of the Alexander matrix $M_{Wirt}$ given by the abelianization of the Wirtinger presentation matrix.

In order to discuss the Dehn presentation of the fundamental group $\pi_1(X; \mathbb{Z})$ of the knot
complement, consider the 4-valent projection graph $G'$ obtained from the knot diagram $D$ by ignoring over- and under-crossing information. The $n$ crossings of $D$ become the $n$ vertices of the graph $G'$, and because there are $2n$ edges and the graph is a plane diagram, there must be $n + 2$ faces by Euler's formula.

The $n + 1$ generators of the Dehn presentation are loops through each of the faces $f_j$ of $G'$ returning through the outer face, which then corresponds to the identity. The $n$ relations $v_i = f_{j_1}f_{j_2}^{-1} - f_{j_3}f_{j_4}^{-1}$ correspond to the vertices of $G'$, that is, the crossings of the original diagram, and are given by the information at each crossing as in Figure 3.1.

![Figure 3.1: Dehn generators.](image)

Construct the Fox calculus Jacobian matrix $M_{Dehn} = (m_{ij})$ whose entries $m_{ij}$ are the Fox calculus derivatives $\partial v_i / \partial f_j$. For the row corresponding to the relation $v_i = f_{j_1}f_{j_2}^{-1} - f_{j_3}f_{j_4}^{-1}$, the four non-zero terms are 1, $-f_{j_1}f_{j_2}^{-1}$, $-1$, and $f_{j_3}f_{j_4}^{-1}$ in columns $j_1$, $j_2$, $j_3$, and $j_4$, respectively.

In order to get the first homology $H_1(X) = \pi_1/[\pi_1, \pi_1]$ from this presentation, abelianize the entries by sending each $f_{j_1}f_{j_2}^{-1} \mapsto t$, a generator of $\langle t \rangle \cong \mathbb{Z} \cong \pi_1(X; \mathbb{Z})$. Notice that this again corresponds to the meridian of the knot since $f_{j_1}f_{j_2}^{-1} = x_k$ above. Then the non-zero terms of each row are 1, $-t$, $-1$, and $t$.

The presentation matrix $M_{Dehn}$ is $n \times n + 1$, and thus to get a full-rank minor, only one column need be deleted. This column must be associated with a face adjacent to the universal face.
Alternate Definition 3.1.3. The Alexander polynomial $\Delta_K(t)$ of a knot $K$ can be defined as the determinant of a full-rank minor of the Alexander matrix $M_{Dehn}$ given by the abelianization of the Dehn presentation matrix.

One can construct this matrix combinatorially without having to depend on all the algebra in the background. See Figure 3.2.

Another formulation of the polynomial, also found by Conway in 1969, was defined recursively on the number of crossings of the knot diagram by means of a Skein relation.

Alternate Definition 3.1.4. The Alexander polynomial $\Delta_K(t)$ of a knot $K$ can be defined by

1. Normalization: $\Delta_U(t) = 1$

2. Skein relation: $\Delta_{L_+}(t) - \Delta_{L_-}(t) = (t^{1/2} - t^{-1/2})\Delta_{L_0}(t)$

where $U$ is the unknot, which can be depicted as a simple closed curve.

Observe that the half-powers of $t$ come into play when the number of components changes by one, and that the Alexander polynomial of a link lives in $\mathbb{Z}[t^{\pm 1/2}]$.

3.1.1 Dimer Construction for the Alexander Polynomial

The main result for this polynomial is the following:
Theorem 3.1.5. (Dasbach [CDR10]) Let $G_\Delta$ be a balanced overlaid Tait graph associated with a knot diagram $D$ for a knot $K$. Then

$$
\sum_m \prod_{e \in m} \mu(e) = \Delta_K(t),
$$

the Alexander polynomial of $K$ up to sign and a power of $t$, after summing over all perfect matchings $m$ and all edges in the perfect matching.

The proof relies on the construction of a signed, weighted bipartite graph whose associated matrix is indeed the Alexander matrix.

Proof. For a knot $K$ and diagram $D$ with $n$ crossings, construct the balanced overlaid Tait graph where the two omitted regions in the projection graph are the universal face and some other face adjacent to it. This graph, together with a weighting coming from Figure 3.2 and a signing, is the one named $G_\Delta$ in the theorem.

The bipartite adjacency submatrix $M_\Delta$ associated with this balanced overlaid Tait graph has rows associated with the crossings of the knot diagram and has columns associated with the faces of the projection graph.

The signing of the edges in the graph comes from a Kasteleyn weighting; for convenience Kauffman’s trick can be used. The weighting of the edges follows from Figure 3.3 and also gives the appropriate entry of the bipartite adjacency submatrix. Together these give $\mu(e)$, the signed weighting on the edges.

By Alternate Definition 3.1.3, the determinant of this matrix is the Alexander polynomial up to a signed power of $t$. By Proposition 2.3.7, this determinant also gives the perfect matchings of the graph $G_\Delta$. □
3.1.2 Example

Example 3.1.6. Given the diagram for the left-handed trefoil knot from Figure 2.1, here again is the balanced overlaid Tait graph as seen in Figure 2.9. The weighting (and signing) as described above have been labelled in Figure 3.4 below.

![Figure 3.4: A balanced overlaid Tait graph for the trefoil knot.](image)

The three perfect matchings in this graph are: the three vertical lines labelled by \((-t\), 1, and 1, respectively; the first vertical line and two horizontal lines labelled by \((-t\), 1, and \((-t\), respectively; and the first two horizontal lines followed by the third vertical line labelled by 1, 1, and 1, respectively.

Summing together the products of these weights for each perfect matching, one obtains 
\[-t + t^2 + 1\], which is indeed the Alexander polynomial of the trefoil.
3.2 Twisted Alexander Polynomial

For a representation $\rho$ taking a finitely presented group (like the fundamental group of a knot complement) to the general linear group $GL_p(R)$ of some Noetherian unique factorization domain $R$, Wada defined an invariant $W = \frac{D}{f} \in R[t^{\pm 1}]$ called the twisted Alexander polynomial [Wad94]. As in the case of the classical Alexander polynomial, both the numerator and denominator are defined by taking the determinant of a presentation matrix for a certain module.

The numerator $D$ of $W$ was defined by X.S. Lin using regular Seifert surfaces [Lin01]. In his work Lin refers to $D$ as the twisted Alexander polynomial. To avoid ambiguity $W$ will be called the Wada invariant while $D$ will be called the Lin polynomial $L$. The latter is the focus of this work.

To be clear, both of these polynomials depend on the choice of the finitely presented group and the representation $\rho$, but for brevity’s sake, these indices will be neglected.

One relatively fast way to produce a representation is by using a $p$-coloring of the arcs of the knot diagram by $[p] = \{1, \ldots, p\}$ for some odd prime, where at each crossing the average of the two under-arcs is equal to the over-arc modulo $p$, and then the finitely presented group is the dihedral group with $2p$ elements.

Beginning with the original $n - 1 \times n$ Wirtinger presentation matrix $M_{\text{Wirt}}$ as defined above, replace each entry by a $p \times p$ block by sending 1 to the identity matrix and sending $t$ to $tX_i$, a matrix related to the over-crossing arc $x_i$. Let $\tilde{M}_{\text{Wirt}}$ be this enhanced matrix.

Here $X_i$ is the matrix assigned to the over-crossing arc at this particular crossing, and a different matrix can be assigned to each crossing. In the case of a representation coming from a coloring, take $X_i$ to be the composition of a reflection and some number of rotations,
where that number is the color assigned to the over-crossing strand. Although it is nice when $X_i$ is a permutation matrix, more complicated representations can be incorporated.

**Alternate Definition 3.2.1.** The *Lin polynomial* $L_K(t)$ of a knot $K$ can be defined as the determinant of a full-rank minor of an enhanced Alexander matrix $\widetilde{M}_{Wirt}$ given by the abelianization of an enhanced Wirtinger presentation matrix.

However, much more traction can be gained by starting with the original $n \times n + 1$ Dehn presentation matrix and repeating the steps above to enhance it using the rule in Figure 3.5.

**Alternate Definition 3.2.2.** The *Lin polynomial* $L_K(t)$ of a knot $K$ can be defined as the determinant of a full-rank minor of the enhanced Alexander matrix $\widetilde{M}_{Dehn}$ given by the abelianization of the enhanced Dehn presentation matrix.

### 3.2.1 Dimer Construction for the Twisted Alexander Polynomial

The main result for this polynomial is the following:

**Theorem 3.2.3.** Let $\widetilde{G}_L$ be an enhanced graph obtained from a balanced overlaid Tait graph $G_{\Delta}$ associated with a knot diagram $D$ for a knot $K$. Then

$$\sum_m \prod_{e \in m} \mu(e) = L_K(t),$$

38
the Lin polynomial of $K$ up to sign and a power of $t$, after summing over all perfect matchings $m$ and all edges in the perfect matching.

As above, the proof relies on the construction of a signed, weighted bipartite graph whose associated matrix is indeed the Lin polynomial. This new graph $\tilde{G}_L$ can be obtained from the graph $G_\Delta$ in the previous section by replacing each vertex with $p$ copies, stacked in levels for convenience of ordering, and by replacing each edge with copies based on the entries of the associated block. There will either by $p$ parallel edges, taking each vertex to one on the same level, for the identity block, or there will be some twisting, as assigned in Figure 3.6.

For the case of a representation coming from a $p$-coloring, the matrix is a permutation $\sigma$, and thus the twisting occurs by sending a vertex at level $i$ to level $\sigma(i)$ as in Figure 3.7. However, the most general matrix can be associated with a complete bipartite graph $K_{p,p}$ between these levelled vertices.

**Proposition 3.2.4.** The graph obtained from the balanced overlaid Tait graph by replacing each vertex with $p$ copies and each edge with the complete bipartite graph $K_{p,p}$ can be
used to encode any representation into $GL_p(R)$, the general linear group of order $p$ of some Noetherian unique factorization domain $R$, by way of the enhanced Alexander matrix.

Proof. A representation sending generators of the fundamental group of a knot complement into $GL_p(R)$ replaces each linear combination of generators in the entries of the full-rank minor of the Alexander matrix with a $p \times p$ block in the enhanced matrix. This enhanced matrix describes the unweighted graph obtained from the balanced overlaid Tait as above. The weighting on the edges can be determined from each $p \times p$ block, with corresponding edges deleted for zero entries in the block.

In the more general case but also possibly for any representation, this new enhanced graph $\tilde{G}_L$ may not be planar. Since this may prevent the existence of a Kasteleyn weighting, Kuperberg’s tripling and butterfly tricks can be used without changing the determinant of the matrix by more than a sign.

Proof. For a knot $K$ and diagram $D$ with $n$ crossings, construct the matrix $M_\Delta$, which is a full rank minor of the Dehn presentation matrix $M_{Dehn}$ of the fundamental group of the knot complement.

Let $\tilde{M}_{Lin}$ be the matrix obtained from $M_\Delta$ by replacing each entry $m_{ij}$ with a $p \times p$ block as follows: replace $\pm 1$ with $\pm Id$, the $p \times p$ identity matrix, and in each original row $v_i$ replace $\pm t$ with $\pm tX_i$ for some matrix $X_i$ coming from a representation as described above as in Figure 3.6.

The matrix $\tilde{M}_{Lin}$, which is a full rank minor of $\tilde{M}_{Dehn}$, can serve as the adjacency matrix of a weighted, signed bipartite graph whose first vertex set can be associated with $p$ copies of each of the crossings of the original diagram and whose second vertex set can be associated
with $p$ copies of each of the faces of the projection graph. For two adjacent vertices in the original balanced overlaid Tait graph, the $2p$ vertices arranged in levels from two distinct sets derive their adjacencies from either the identity block taking the $k$-th level to the $k$-th level or the matrix coming from the representation.

The weighting of the edges follows from Figure 3.6 and also gives the appropriate entry of the bipartite adjacency submatrix. A Kasteleyn weighting cannot be assigned yet because the graph may not be planar. Use Kuperberg’s tripling and butterfly replacements to obtain a plane graph $\tilde{G}_L$ with more vertices and edges than the original. Since it is a plane graph, any Kasteleyn weighting will do. Together these give $\mu(e)$, the signed weighting on the edges.

By Alternate Definition 3.2.2, the determinant of the original enhanced matrix $\tilde{M}_{Dehn}$ gives the Alexander polynomial up to a signed power of $t$. By Propositions 2.3.5 and 2.3.6, the new matrix has the same determinant. By Proposition 2.3.7, this determinant also gives the perfect matchings of the new graph $\tilde{G}_L$.

### 3.2.2 Example

**Example 3.2.5.** Given the diagram for the left-handed trefoil knot from Figure 2.1, take the representation coming from a coloring of the trefoil by $X_i = \tau \alpha C(x_i)$, where $C(x_i)$ is the color of the $x_i$ arc and $\tau$ and $\alpha$ are the reflection and rotation generators, respectively, of the dihedral group of order $2 \cdot 3$. Then in the general linear group,

$$\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau \alpha = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \tau \alpha^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The balanced overlaid Tait graph for the Alexander polynomial in Figure 3.4 is enhanced
to create the balanced overlaid Tait graph for the twisted Alexander polynomial in Figure 3.8.

Observe that in this case the graph is actually planar, even though this is not a plane graph. See a plane graph version together with the weighting above and signing coming from some arbitrary Kasteleyn weighting in Figure 3.9.

Summing together the products of these weights for each of the eight perfect matchings, one obtains \( t^6 - t^5 - t^4 + 2t^3 - t^2 - t + 1 \), which is indeed the Lin polynomial of the trefoil given the representation coming from a coloring.
3.3 Jones Polynomial

Although Vaughn Jones formulated his polynomial in algebraic terms in 1983, much work has been done to make this combinatorially accessible. The Jones polynomial which lies in $\mathbb{Z}[t^{\pm 1/2}]$ can be defined by the Kauffman bracket along with a correction term due to the writhe of the knot diagram.

The bracket polynomial is defined for an unoriented diagram because it uses the smoothings of Figure 3.10 (repeated from above) which cannot have a consistent orientation. Recall that when one of these tangles in a diagram is replaced by any of the three, denote the link diagrams by $L_\times$, $L_0$, and $L_\infty$. It is important to consider link diagrams and not just knot diagrams because replacing a crossing by either of the smoothings (and vice versa) can change the number of components of the link.

![Figure 3.10: An unoriented crossing and the two smoothings.](image)

Recall that the first smoothing is called the $A$-smoothing, and the second is called the $B$-smoothing. In practice, the first smoothing is distinguished by rotating the over-strand of the crossing counterclockwise towards the under-strand, sweeping out the region of the $A$-smoothing.

**Definition 3.3.1.** The Kauffman bracket polynomial $\langle L \rangle$ of a link $L$ can be defined by

1. Normalization: $\langle U \rangle = 1$
2. Stabilization: $\langle L \sqcup U \rangle = (-A^2 + A^{-2})\langle L \rangle$
3. Smoothing relation: \( \langle L^\infty \rangle = A\langle L_0 \rangle + A^{-1}\langle L_\infty \rangle \)

where \( U \) is the unknot, which can be depicted as a simple closed curve.

Even though the notation \( L_0 \) is used for both the oriented and unoriented cases, it should be clear by context which is being considered.

**Definition 3.3.2.** The *Jones polynomial* \( V_K(t) \) of a knot \( K \) given a diagram \( D \) can be defined via the Kauffman bracket polynomial by

\[
V_K(t) = (-A^3)^{-w(D)}\langle K \rangle,
\]

where \( w(D) \) is the writhe of the (oriented) diagram, and the substitution \( A = t^{-1/4} \) is used.

Alternately, the Jones polynomial can be defined by an oriented Skein relation similar to the one for the Alexander polynomial.

**Alternate Definition 3.3.3.** The *Jones polynomial* \( V_K(t) \) of a knot \( K \) can be defined by

1. Normalization: \( V_U(t) = 1 \)

2. Stabilization: \( V_{L\sqcup U}(t) = (t^{1/2} + t^{-1/2})V_L(t) \)

3. Skein relation: \( t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} + t^{-1/2})V_{L_0}(t) \)

where \( U \) is the unknot, which can be depicted as a simple closed curve.

The unoriented smoothings can be understood as either deletions or contactions on the underlying Tait graph associated with the knot diagram. That is, if the region swept out has a vertex associated with it, then the edge associated to the crossing is contracted, and
Table 3.1: Polynomial activity evaluations.

<table>
<thead>
<tr>
<th>Activity letter</th>
<th>L</th>
<th>D</th>
<th>ℓ</th>
<th>d</th>
<th>L̅</th>
<th>D̅</th>
<th>ℓ̅</th>
<th>d̅</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tutte polynomial evaluation</td>
<td>x</td>
<td>1</td>
<td>y</td>
<td>1</td>
<td>--</td>
<td>--</td>
<td>--</td>
<td>--</td>
</tr>
<tr>
<td>Jones polynomial evaluation</td>
<td>$-A^{-3}$</td>
<td>A</td>
<td>$-A^3$</td>
<td>$A^{-1}$</td>
<td>$-A^3$</td>
<td>$A^{-1}$</td>
<td>$-A^{-3}$</td>
<td>A</td>
</tr>
</tbody>
</table>

if two regions on the sides have vertices associated with them, then the edge associated to the crossing is deleted.

Thus by the deletion-contraction formula for the Tutte polynomial, it should be of no surprise that the Jones polynomial can be obtained as a specialization of the Tutte polynomial.

**Alternate Definition 3.3.4.** As in [Thi87], the *Jones polynomial* $V_K(t)$ of a knot $K$ can be defined as the evaluation given in Table 3.1 of the activity words associated with the spanning trees of the signed Tait graph associated with the diagram.

Furthermore, the Jones polynomial is the graded euler characteristic of a bigraded knot homology theory called Khovanov homology [Kho03]. The reduced Khovanov homology version has a spanning tree model developed independently by [CK09] and [Weh08]. Here the generators are spanning trees of the underlying Tait graph of the knot diagram, but the first authors give the bigradings from activity words. In this case, although the first-order differential can be determined directly by activity, the higher-order differentials are maps through other smoothings unrelated to spanning trees.

See Chapter 4 for more on this subject.
3.3.1 Dimer Construction for the Jones Polynomial

The main result of this section, Theorem 3.3.7, is in some ways much more general than in previous sections, although the cases for which it holds may seem far more restricted. It is important to note that this exact construction cannot work for general knots, as this would result in a polynomial time calculation of the Jones polynomial, which cannot hold in general.

However, there are several smaller classes of Tait graphs for which this can work; this paper will focus on those graphs obtained from the infinite class of pretzel knots with an arbitrary number \( k \) of twist region columns. Here is a corollary to the more powerful theorem:

**Corollary 3.3.5.** Given a pretzel knot \( P = P(n_1, n_2, \ldots, n_k) \), consider the usual diagram \( D \) with the \( n = n_1 + \ldots + n_k \) crossings labelled from left to right and downward on the first column and then upward on the remaining columns. Let \( G_V \) be a balanced overlaid Tait graph for the diagram \( D \) with the two omitted faces of the projection graph corresponding to the universal face and the upper deck supported by the columns as in Figure 3.11.

Then

\[
\sum_m \prod_{e \in m} \mu(e) = V_P(t),
\]

the Jones polynomial of \( P \) up to sign, after summing over all perfect matchings \( m \) and all edges in the perfect matching.

For further applications using these activity words, see Section 4.2.

**Proof.** Use Theorem 3.3.7 and see the evaluations for the activity letters given in Table 3.1.
These activity words are assigned to the spanning trees via a special weighting on the edges of a balanced overlaid Tait graph.

Recall that in a balanced overlaid Tait graph the second vertex set is the union of the vertex sets of both the original Tait graph and its dual. Thus this graph is more technically a tripartite graph whose three vertex sets correspond to the (ordered) edges, (all but one of the) vertices, and (all but one of the) faces of the specified Tait graph.

**Definition 3.3.6.** The *activity weighting* on an edge of a balanced overlaid Tait graph associated to a knot diagram whose $n$ crossings are ordered is determined by three distinctions: positive or negative, internal or external, and live or dead.

The activity weighting for an edge incident with a vertex from the first set is positive or negative if the edge corresponding to the crossing in the chosen Tait graph is signed positive or negative, respectively, according to Figure 2.6.

The activity weighting for an edge is internal or external depending on whether it is incident with a vertex in the second or third set, respectively, according to the sets above. Note that every edge is incident with a vertex in the first set, so this is a partition.
The activity weighting for an edge is live or dead depending on whether or not it connects the lowest-ordered vertex from the first set to the other vertex with which it is incident. That is, given a vertex in the second or third set, label all of the edges around it as dead except for the one incident with the lowest-ordered vertex in the first set, and this last edge is labelled live.

These three choices determine the activity letter assigned as the weighting of the edge.

At the level of the bipartite adjacency submatrix, this activity weighting amounts to the following rules. Ordered rows associated to the original ordered crossings (vertices of the first set) contain only positive or only negative letters following the sign of the original crossing in the specific Tait graph considered. Columns associated to the vertices of the second set are internal, and columns associated to the vertices of the third set are external. The first non-zero entry in a column is live; the rest are dead.

For the diagram of the \((n_1, n_2, \ldots, n_k)\)-pretzel knot in Figure 3.11, which has all positive edges as depicted, see as an example the following (unsigned) bipartite adjacency submatrix:
Observe that there are \( k \) blocks, where the \( i \)-th block is \( n_i \times (n_i - 1) \), followed by some 
\( 1 + (k - 1) = k \) columns. The first column has non-zero entries in the first position of 
each block, except for the first block which has a non-zero entry in the last position. The 
remaining columns have non-zero entries in two consecutive blocks.

Some care must be taken to order these activity weights correctly when assembling them 
into activity words. The ordering of the edges in a perfect matching comes from the ordering 
of the original crossings, that is, by the ordering of the first vertex set.

**Theorem 3.3.7.** Given a pretzel knot \( P = P(n_1, n_2, \ldots, n_k) \), consider the usual diagram \( D \) 
with the \( n = n_1 + \ldots + n_k \) crossings labelled from left to right and downward on the first 
column and then upward on the remaining columns. Let \( G_A \) be a balanced overlaid Tait graph 
for the diagram \( D \) with the two omitted faces of the projection graph corresponding to the 
universal face and the upper deck supported by the columns as in Figure 3.11.
Then obtaining a partial ordering of edges by the ordering of the original crossings,

\[
\sum_m \prod_{e \in m} \mu(e)
\]

gives the complete list of activity words associated to spanning trees of the Tait graph associated with the diagram of \( P \), after summing over all perfect matchings \( m \) and all edges in the perfect matching.

**Proof.** The signing of edges here can be any Kasteleyn weighting; for convenience Kauffman’s trick can be used. The activity weighting described above is used.

By Proposition 2.3.8, there is a bijection between the perfect matchings of the balanced overlaid Tait graph and the rooted spanning trees of the Tait graph associated to a knot diagram. It is enough to show that the activity weighting of the perfect matching gives the activity word of the associated spanning tree.

Since there are \( n = n_1 + n_2 + \ldots + n_k \) edges and \( 1 + (n_1 - 1) + (n_2 - 1) + \ldots + (n_k - 1) + 1 = n - k + 2 \) vertices in the Tait graph, each spanning tree \( T \) omits exactly \( k - 1 \) edges. These \( k - 1 \) edges must come from distinct columns in order for \( T \) to be acyclic and connected.

The activity word associated to \( T \) can be decomposed into the activity words of the \( k \) columns, which are called paths in this paper, and considered according to Lemma 2.2.14.

Suppose that an edge from each column except for the first is omitted. Then by the lemma, the activity word of the first column will be \( L \ldots L \), and the activity words of the remaining columns will range from \( dD \ldots D \) to \( L \ldots LdD \ldots D \) to \( L \ldots Ld \).

Suppose that an edge from each column except for the \( i \)-th is omitted. Then by the lemma, the activity words of the first \( i-1 \) columns will range from \( \ell D \ldots D \) to \( L \ldots LdD \ldots D \).
to $L \ldots Ld$, the activity word of the $i$-th column will be $D \ldots D$, and the activity words of the remaining columns will range from $dD \ldots D$ to $L \ldots LdD \ldots D$ to $L \ldots Ld$.

As in the argument above, consider the location of the pivots in the last $k - 1$ columns. Note that if they both belong to the same $i$-th block, there must be a zero pivot in the $n_i \times (n_i - 1)$ block earlier, and so this choice does not contribute to the permanent expansion.

When a pivot in each of the last $k - 1$ columns is chosen, it forces the pivots of the corresponding blocks, with $L$’s chosen above the pivot row and $D$’s below in each block. This then forces the pivot in the first of the last $k$ columns, which in turn forces the pivots in the remaining block to be uniform.

This gives the activity words described above.

This is not the only class for which the activity weighting works. Here is a technique to begin with a knot diagram that works and extend it outside of the class of pretzel knots.

**Proposition 3.3.8.** Consider an edge in the Tait graph that is incident with the omitted vertex and the omitted face; order this last amongst the $n$ edges. Then if the activity weighting on a balanced overlaid Tait graph provides a dimer model for the knot diagram associated with the Tait graph, this can be extended to the balanced overlaid Tait graph associated with a Tait graph graph that subdivides or doubles the edge.

Note that for the $(n_1, n_2, \ldots, n_k)$-pretzel knot, if $n_k > 1$ and the $n$-th edge is doubled, the end result is not a pretzel knot or link.

**Proof.** Consider the squared incidence matrix of the original knot diagram. The row corresponding to a non-loop, non-bridge $n$-th edge, which is incident with both the omitted vertex and the omitted face, has only two non-zero entries.
If the edge is neither a bridge nor a loop, these entries are $D$ and $d$. After subdividing this edge, the matrix gets a new row corresponding to the $n + 1$-st edge and a new column corresponding to the new vertex. The entries in this column are zero except for an $L$ and a $D$ in the $n$-th and $n + 1$-st rows, respectively, and the last new entry is another $d$ in the $n + 1$-st row below the first $d$ in the $n$-th row mentioned above.

Configurations in the determinant expansion for the final two terms in the new matrix have only three options: $DD$ and $dD$, which preserve all of the first $n$ choices of pivots, and $Ld$, where the first $n - 1$ choices are preserved and the $d$ of the $n$-th row gets replaced by the $d$ in the $n + 1$-st row. These are exactly the three possibilities for the activity words associated to the spanning trees by Lemma 2.2.14.

The dual case of doubling works similarly.

The first Reidemeister move adds either bridges or loops to the Tait graph. These are easily handled by the following:

**Proposition 3.3.9.** If the activity weighting on a balanced overlaid Tait graph associated to a knot diagram provides a dimer model, this can be extended to one whose Tait graph is the same as before together with an additional bridge or loop incident with both the omitted vertex and the omitted face.

*Proof.* If the $n + 1$-st edge is a bridge or a loop, this amounts only to adding the terms $L$ or $\ell$, respectively, to the end of the activity words, and this appears in the expansion because of columns with only a single non-zero entry. 

\[\square\]
### 3.3.2 Examples

**Example 3.3.10.** Given the $(-2, 3, 3)$-pretzel knot (also known as $8_{19}$) in Figure 3.12, the squared incidence matrix is the following:

$$
\begin{pmatrix}
    L & \bar{L} & \bar{L} & \bar{L} & \bar{L} & \bar{L} & \bar{L} & \bar{L} \\
    D & D & L & D & L & D & L & D \\
    L & D & L & D & L & D & L & D \\
    D & L & D & L & D & L & D & D \\
    L & D & L & D & L & D & D & D \\
    D & L & D & D & L & D & D & D \\
    L & D & L & D & L & D & D & D \\
    D & L & D & D & L & D & D & D \\
\end{pmatrix}
$$

The permanent of this unsigned matrix is $-A^{-32} + A^{-20} + A^{-12} = -t^8 + t^5 + t^3$, which is indeed the Jones polynomial of $8_{19}$.

Alternately, a Kasteleyn weighting can be given using Kauffman’s trick, and then the determinant can be taken of the signed matrix.

**Example 3.3.11.** Given the $(-2, 3, 7)$-pretzel knot, which is useful in the construction of three manifolds, the squared incidence matrix is the following:
The permanent of this matrix is
\[-A^{-40} + A^{-36} - A^{-32} + A^{-16} + A^{-8} = -t^{10} + t^9 - t^8 + t^2.\]

Alternately, a Kasteleyn weighting can be given using Kauffman’s trick, and then the determinant can be taken of the signed matrix.
Chapter 4

Further Directions

4.1 Consideration of a Twisted Skein Relation

There is no known Skein relation for the twisted Alexander polynomial; however, it seems unlikely that the diagrams $L_+, L_-$, and $L_0$ defined above would always have consistent representations. In fact, dealing with a representation coming from a coloring, two differently-colored strands in $L_0$ cannot be replaced by three differently-colored strands for the positive and negative crossings in $L_+$ and $L_-$, respectively.

Yet the following shows that two differently-colored strands in $L_0$ can be replaced by a twist region of $p$ positive or negative crossings that are properly colored. Perhaps these configurations could be called $L_+^p$ and $L_-^p$, respectively.

Property 4.1.1. The smoothing in Figure 4.1 (repeated from above) whose strands are colored by $\alpha \neq \beta$ can be replaced by a $p$ or $-p$ twist region with exactly $p$ new arcs added, each distinctly colored.
Proof. Supposing $\beta \equiv \gamma \alpha$, the new arcs must be labelled (in order):

$$\{\alpha \equiv (1 - 0 \gamma) \alpha, (2 - \gamma) \alpha, (3 - 2 \gamma) \alpha, \ldots, (p - (p - 1) \gamma) \alpha \equiv \gamma \alpha\}.$$ 

In fact, in the negative case the crossings appear in opposite order.

Supposing that for some $a$ and $b$, it holds that $a - (a - 1) \gamma \equiv b - (b - 1) \gamma$, then $(a - b) - (a - b) \gamma \equiv 0$, which implies that $(a - b)(1 - \gamma) \equiv 0$. This in turn implies either $a = b$ or $1 \equiv \gamma$. Since $\alpha$ and $\beta$ are distinct, $a$ and $b$ must be the same, and the list of colors above is distinct mod $p$. $\square$

The usual Skein relation depicts twist regions of 1, -1, and 0 crossings, and appropriately for the Alexander polynomial, only $p = 1$ color is considered.

**Question 4.1.2.** Can a Skein relation for the Twisted Alexander polynomial exist comparing $p$, $-p$, and 0 twists, which would preserve the coloring of the two strands in the smoothing?

### 4.2 More Corollaries to Theorem 3.3.7

The Jones polynomial is an evaluation of the Tutte polynomial, which can also be defined in terms of activity. Thus a dimer model for the Tutte polynomial of the Tait graph of a pretzel knot can be obtained in this way.

**Corollary 4.2.1.** Given the Tait graph $G$ of an all positive pretzel knot $P_+ = P(n_1, n_2 \ldots, n_k)$
Table 4.1: Reduced Khovanov homology activity evaluations.

<table>
<thead>
<tr>
<th>Activity letter</th>
<th>$L$</th>
<th>$D$</th>
<th>$\ell$</th>
<th>$d$</th>
<th>$\overline{L}$</th>
<th>$\overline{D}$</th>
<th>$\overline{\ell}$</th>
<th>$\overline{d}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>chain complex of reduced Khovanov homology evaluation</td>
<td>$uv$</td>
<td>$v$</td>
<td>$u^{-1}$</td>
<td>1</td>
<td>$u^{-1}$</td>
<td>1</td>
<td>$u$</td>
<td>1</td>
</tr>
</tbody>
</table>

(where $n_i > 0$) in its usual diagram $D$ with the $n = n_1 + \ldots + n_k$ crossings labelled from left to right and downward on the first column and then upward on the remaining columns.

Let $G_T$ be a balanced overlaid Tait graph for the diagram $D$ with the two omitted faces of the projection graph corresponding to the universal face and the upper deck supported by the columns as in Figure 3.11.

Then

$$\sum_m \prod_{e \in m} \mu(e) = T(G; x, y)$$

the Tutte polynomial of $G$, after summing over all perfect matchings $m$ and all edges in the perfect matching.

Proof. Use Theorem 3.3.7 and see the evaluations for the activity letters given in Table 3.1.

The Jones polynomial is also the graded euler characteristic of Khovanov homology, which can be generated by spanning trees of the Tait graph associated to a knot diagram as in [CK09] and [Weh08]. The first authors define two gradings that come from activity letters; these are presented in Table 4.1.

Corollary 4.2.2. Given a pretzel knot $P = P(n_1, n_2, \ldots, n_k)$, consider the usual diagram $D$ with the $n = n_1 + \ldots + n_k$ crossings labelled from left to right and downward on the first
column and then upward on the remaining columns. Let $G_{Kh}$ be a balanced overlaid Tait graph for the diagram $D$ with the two omitted faces of the projection graph corresponding to the universal face and the upper deck supported by the columns as in Figure 3.11.

Then

$$\sum_m \prod_{e \in m} \mu(e)$$
gives the two-variable polynomial $\tilde{C}Kh_P(t)$ for the reduced Khovanov chain complex of $P$ up to sign, after summing over all perfect matchings $m$ and all edges in the perfect matching.

**Proof.** Use Theorem 3.3.7 and see the evaluations for the activity letters given in Table 4.1. \qed

In order to get reduced Khovanov homology from the chain complex, the differential is needed. There are differently ordered differential maps given in [CK09]. The first is the easiest to see in the bipartite adjacency submatrix.

**Corollary 4.2.3.** The first order differential for the spanning tree model of reduced Khovanov homology corresponds to very particular $2 \times 2$ blocks of this matrix: two rows and two columns who meet at four nonzero terms in each of the following configurations.

$$\begin{pmatrix} L & d \\ \bar{D} & \bar{d} \end{pmatrix}, \begin{pmatrix} \bar{d} & \bar{L} \\ d & D \end{pmatrix}, \begin{pmatrix} \bar{\ell} & \bar{D} \\ d & D \end{pmatrix}, \begin{pmatrix} D & \ell \\ \bar{D} & \bar{d} \end{pmatrix}$$

Perhaps collections of edges that do not give spanning trees can be used to produce the higher-order differentials through the context of the squared incidence matrix. This would lead to a positive answer to the following question.
Question 4.2.4. Can the reduced Khovanov homology of pretzel knots can be computed via the squared incidence matrix alone?
Bibliography


Vita

Moshe Cohen was born in 1982 in Tarrytown, New York. He received his Bachelor of Science in mathematics from Binghamton University (State University of New York) in May 2004 and arrived in Baton Rouge, Louisiana, for his graduate studies that fall. He received his Master of Science in mathematics from Louisiana State University in May 2006 and his Doctor of Philosophy in mathematics from Louisiana State University in August 2010. He accepted a research post-doctoral position at Bar-Ilan University in Ramat Gan, Israel, to work under Mina Teicher.