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A new procedure for constructing basis vectors of $SU(3) \supset SO(3)$

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A simple and effective algebraic angular momentum projection procedure for constructing basis vectors of $SU(3) \supset SO(3) \supset SO(2)$ from the canonical $U(3) \supset U(2) \supset U(1)$ basis vectors is outlined. The expansion coefficients are components of the null-space vectors of a projection matrix with, in general, four nonzero elements in each row, where the projection matrix is derived from known matrix elements of the $U(3)$ generators in the canonical basis. The advantage of the new procedure lies in the fact that the Hill-Wheeler integral involved in the Elliott's projection operator method used previously is avoided, thereby achieving faster numerical calculations with improved accuracy. Selected analytical expressions of the expansion coefficients for the $SU(3)$ irreps $[n_{13}, n_{23}]$, or equally, $(\lambda, \mu) = (n_{13} - n_{23}, n_{23})$ with λ and μ the $SU(3)$ labels familiar from the Elliott model, are presented as examples for $n_{23} \leq 4$. Explicit formulae for evaluating $SO(3)$-reduced matrix elements of $SU(3)$ generators are derived. A general formula for evaluating the $SU(3) \supset SO(3)$ Wigner coefficients is given, which is expressed in terms of the expansion coefficients and known $U(3) \supset U(2)$ and $U(2) \supset U(1)$ Wigner coefficients. Formulae for evaluating the elementary Wigner coefficients of $SU(3) \supset SO(3)$, i. e., for the $SU(3)$ coupling $[n_{13}, n_{23}] \otimes [1, 0]$, are explicitly given with some analytical examples shown to check the validity of the results. However, the Gram-Schmidt orthonormalization is still needed in order to provide orthonormalized basis vectors.

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I. INTRODUCTION

The $SO(3)$ group embedded in $SU(3)$, i. e. the non-canonical group chain $SU(3) \supset SO(3) \supset SO(2)$, has been exploited in nuclear shell-model calculations since the pioneering work of Elliott [1, 2]. In Elliott's model, the essential rotational features of the Bohr-Mottelson collective model can be well reproduced in a shell-model framework by introducing the quadrupole-quadrupole interaction within a three-dimensional harmonic oscillator mean-field [1, 2], where the quadrupole operators are generators of $SU(3)$, while the angular momentum operators are generators of its subgroup $SO(3)$. The $SU(3)$ framework has been adopted in many studies due to its importance, as summarized in [3], and recently, in its multi-shell generalization, in the *ab initio* symmetry-adapted no-core shell model [5, 6]. $SU(3)$ and other unitary groups are also useful in quantum interferometry [4]. In both nuclear shell-model calculations and quantum interferometry, the dimension of an irreducible representation (irrep) of $SU(3)$ can be huge and may approach the classical asymptotic limits. In Elliott's work [1, 2], the basis vectors of $SU(3) \supset SO(3) \supset SO(2)$ are projected from a specific (extremal) $SU(3) \supset SU(2) \otimes U(1)$ state by using the angular momentum projection, which are often called the Elliott states. Draayer, Pursey, and Williams have made significant contributions in this direction [7, 8], in which the expansion coefficients of the basis vectors of $SU(3) \supset SO(3) \supset SO(2)$ introduced by Elliott in terms of those of the canonical group chain $U(3) \supset U(2) \supset U(1)$ have been derived. Based on these studies, the practical algorithm for calculating various coupling coefficients of $SU(3)$, including those of $SU(3) \supset SO(3)$, has been formulated [9, 10]. An optimized code for generating Clebsch-Gordan (CG) coefficients of $SU(3) \supset SO(3)$ based on vector coherent state theory [11] has been also developed [12]. The main complexities in practical calculations are two-fold. One of them lies in the fact that these calculations use a projection operator constructed by integration of the product of the rotational group element and its matrix element (Wigner's D-function) of a given angular momentum over the Euler angles. While the projection formalism can be straightforwardly implemented in computer codes, it needs to address challenges related to the accuracy and computing time for evaluating coupling coefficients of $SU(3) \supset SO(3)$, because of the use of the Hill-Wheeler integral [13] . The other difficulty is related to the fact that the Elliott states are non-orthogonal, and to calculate overlaps of the Elliott states needed in many cases is also time consuming. Similar non-orthogonal basis vectors of $SU(3) \supset SO(3) \supset SO(2)$ have been constructed by Bargmann and Moshinsky [14], and further studied by Ališauskas [15]. In addition, Sharp *et al.* have proposed the polynomial and stretched non-orthogonal bases [16, 17]. Asherova and Smirnov have used projection operators expressed as polynomials in the $SO(3)$ generators instead of integral operators in the group elements [18], for which the expressions are also complicated. The relations among different types of basis vectors of $SU(3) \supset SO(3) \supset SO(2)$ mentioned above have been detailed in [3].

Very recently, we have proposed a simple and effective angular momentum projection procedure [19] to construct the non-canonical $O(5) \supset O(3)$ basis vectors from basis vectors of $O(5) \supset O_1(3) \otimes U(1)$, in which $O(5) \supset O_1(3) \otimes U(1)$ is branching multiplicity-free, based on the group chain $U(5) \supset U(3) \otimes U(2)$. We observe that the canonical $U(3) \supset$

$U(2) \supset U(1)$ basis plays a similar role of $U(5) \supset U(3) \otimes U(2)$ used to construct the basis vectors of $O(5) \supset O(3)$. Thus, it should also be possible to construct $SU(3) \supset SO(3) \supset SO(2)$ basis vectors directly from those of $U(3) \supset U(2) \supset U(1)$ with a similar simpler algebraic formalism.

In Sec. II, the canonical and non-canonical bases of $SU(3)$ are briefly reviewed. In Sec. III, based on the results shown in Sec. II, the basis vectors of $SU(3) \supset SO(3) \supset SO(2)$ are expanded in terms of those of $U(3) \supset U(2) \supset U(1)$, from which a four-term relation among the expansion coefficients is explicitly derived. Analytical expressions of the expansion coefficients for the $SU(3)$ irreps $[n_{13}, n_{23}]$ with $n_{23} \leq 4$ are presented. In Sec. IV, explicit formulae for evaluating $SO(3)$-reduced matrix elements of $SU(3)$ generators are derived. In Sec. V, a general formula for evaluating the $SU(3) \supset SO(3)$ Wigner coefficients is given, which is expressed in terms of the expansion coefficients and known $U(3) \supset U(2)$ and $U(2) \supset U(1)$ Wigner coefficients. Formulae for evaluating the elementary Wigner coefficients of $SU(3) \supset SO(3)$, i. e., for the $SU(3)$ coupling $[n_{13}, n_{23}] \otimes [1, 0]$, are explicitly given with some analytical examples shown to check the validity of the results.

II. CANONICAL AND NON-CANONICAL BASES OF $SU(3)$

The generators of $U(N)$ can be denoted by $\{E_{ij}\}$ $(1 \leq i, j \leq N)$ satisfying the following commutation and Hermitian conjugation relations:

$$[E_{ij},\ E_{lk}] = \delta_{jl} E_{ik} - \delta_{ik} E_{lj}, \tag{1}$$

$$(E_{ij})^{\dagger} = E_{ji}. \tag{2}$$

There is an obvious subgroup $U(N-1)$ of $U(N)$ generated by $\{E_{ij}\}$ $(1 \leq i, j \leq N-1)$. Thus, one gets the canonical chain of $U(N)$ with $U(N) \supset U(N-1) \supset \cdots \supset U(2) \supset U(1)$, for which the basis vectors were constructed firstly by Gel'fand and Zetlin [20], and then discussed by Moshinsky, Beidenharn and Louck, and many others in various ways [21–23]. The reduction $U(N) \downarrow U(N-1)$ for any $N \geq 2$ is multiplicity-free. By removing the first order algebraic invariant (Casimir operator), $C_1(U(N)) = \sum_{i=1}^{N} E_{ii}$, which is obviously commutative with all generators $\{E_{ij}\}$ $(1 \leq i, j \leq N)$ of $U(N)$, the remaining $N^2 - 1$ generators generate $SU(N)$.

Let $[\nu_1, \nu_2, \cdots, \nu_N]$, where $\nu_1, \nu_2, \cdots, \nu_N$ are positive integers obeying $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_N$, be an irrep of $U(N)$. It is well known that an irrep $[\nu_1 + m, \nu_2 + m, \cdots, \nu_N + m]$, where $m \geq -\nu_N$ is an integer, and $[\nu_1, \nu_2, \cdots, \nu_N]$ have the same dimension and the representation matrices of any element of $U(N)$ for these two irreps differ only by an overall phase factor [24]. For $SU(N)$ case, the irreps $[\nu_1, \nu_2, \cdots, \nu_N]$ and $[\nu_1 + m, \nu_2 + m, \cdots, \nu_N + m]$ are equivalent because the determinant of a representation matrix of any element in $SU(N)$ should be 1, which requires the phase factor being 1. A general discussion on the above equivalence may be found in [24]. Therefore, for the $SU(3)$ case, an irrep can be denoted by $[n'_{13}, n'_{23}, n'_{33}] \equiv [n_{13} = n'_{13} - n'_{33}, n_{23} = n'_{23} - n'_{33}, 0]$, where $[n'_{13}, n'_{23}, n'_{33}]$ is used to label the corresponding irrep of $U(3)$, where n_{i3} are zero or integers obeying the betweenness condition $n_{13} \geq n_{23} \geq 0$. Incidentally, for the (λ, μ) labels of an $SU(3)$ irrep used in the Elliott model, the relation is $(\lambda, \mu) = (n_{13} - n_{23}, n_{23}) = (n'_{13} - n'_{23}, n'_{23} - n'_{33})$. Therefore, the irrep denoted as $[n_{13},\ n_{23}]$ of $SU(3)$ is also the irrep $[n_{13},\ n_{23}, 0]$ of $U(3)$. The general (canonical) basis vectors of $U(3) \supset U(2) \supset U(1)$ may be denoted by [20]

$$\left| \begin{array}{c} [n_{13}, n_{23}, n_{33}] \\ [n_{12},\ n_{22}] \\ n_{11} \end{array} \right\rangle, \tag{3}$$

with the betweenness conditions:

$$\begin{array}{c} n_{13} \geq n_{12} \geq n_{23} \geq n_{22} \geq n_{33}, \\ n_{12} \geq n_{11} \geq n_{22}. \end{array} \tag{4}$$

The matrix representations of $U(N)$ in the canonical basis are well-known [20]. For example, nonzero matrix elements of E_{13} and E_{32} in the canonical basis $U(3) \supset U(2) \supset U(1)$ can be expressed explicitly as

$$\left\langle \begin{array}{c} [n_{13}, n_{23}, n_{33}] \\ [n_{12}+1, n_{22}] \\ n_{11}+1 \end{array} \right| E_{13} \left| \begin{array}{c} [n_{13}, n_{23}, n_{33}] \\ [n_{12}, n_{22}] \\ n_{11} \end{array} \right\rangle = \left[\frac{(n_{11} - n_{22} + 1)(n_{13} - n_{12})(n_{12} - n_{23} + 1)(n_{12} - n_{33} + 2)}{(n_{12} - n_{22} + 1)(n_{12} - n_{22} + 2)} \right]^{\frac{1}{2}}, \tag{5}$$

$$\left\langle \begin{array}{c} [n_{13}, n_{23}, n_{33}] \\ [n_{12}, n_{22}+1] \\ n_{11}+1 \end{array} \right| E_{13} \left| \begin{array}{c} [n_{13}, n_{23}, n_{33}] \\ [n_{12}, n_{22}] \\ n_{11} \end{array} \right\rangle = -\left[\frac{(n_{12} - n_{11})(n_{13} - n_{22} + 1)(n_{23} - n_{22})(n_{22} - n_{33} + 1)}{(n_{12} - n_{22} + 1)(n_{12} - n_{22})} \right]^{\frac{1}{2}}, \tag{6}$$

$$\left\langle \begin{array}{c} [n_{13}, n_{23}, n_{33}] \\ [n_{12}-1, n_{22}] \\ n_{11} \end{array} \right| E_{32} \left| \begin{array}{c} [n_{13}, n_{23}, n_{33}] \\ [n_{12}, n_{22}] \\ n_{11} \end{array} \right\rangle = \left[\frac{(n_{12}-n_{11})(n_{13}-n_{12}+1)(n_{12}-n_{23})(n_{12}-n_{33}+1)}{(n_{12}-n_{22})(n_{12}-n_{22}+1)} \right]^{\frac{1}{2}}, \quad (7)$$

$$\left\langle \begin{array}{c} [n_{13}, n_{23}, n_{13}] \\ [n_{12}, n_{22}-1] \\ n_{11} \end{array} \right| E_{32} \left| \begin{array}{c} [n_{13}, n_{23}, n_{33}] \\ [n_{12}, n_{22}] \\ n_{11} \end{array} \right\rangle = \left[\frac{(n_{11}-n_{22}+1)(n_{13}-n_{22}+2)(n_{23}-n_{22}+1)(n_{22}-n_{33})}{(n_{12}-n_{22}+2)(n_{12}-n_{22}+1)} \right]^{\frac{1}{2}}. \quad (8)$$

For E_{12}, a nonzero matrix element only depends on the sub-irrep of $U(2) \supset U(1)$ and is given by

$$\left\langle \begin{array}{c} [n_{13}, n_{23}, n_{33}] \\ [n_{12}, n_{22}] \\ n_{11}+1 \end{array} \right| E_{12} \left| \begin{array}{c} [n_{13}, n_{23}, n_{33}] \\ [n_{12}, n_{22}] \\ n_{11} \end{array} \right\rangle = [(n_{11}-n_{22}+1)(n_{12}-n_{11})]^{\frac{1}{2}} \quad (9)$$

because $\{E_{12},\ E_{21},\ E_{11},\ E_{22}\}$ are generators of the subgroup $U(2)$.

It can be observed that the matrix elements of the generators $\{E_{ij}\}$ $(1 \le i, j \le 3)$ shown in (5)-(9) are all given in functions of two-number differences among the six quantum numbers n_{ij} $(1 \le i \le j \le 3)$ satisfying the betweenness conditions (4). One can check that there is an exact correspondence between the basis vectors of $U(3) \supset U(2) \supset U(1)$:

$$\left| \begin{array}{c} [n_{13}, n_{23}, n_{33}] \\ [n_{12},\ n_{22}] \\ n_{11} \end{array} \right\rangle = \left| \begin{array}{c} [n_{13}-n_{33},\ n_{23}-n_{33},\ 0] \\ [n_{12}-n_{33},\ n_{22}-n_{33}] \\ n_{11}-n_{33} \end{array} \right\rangle, \quad (10)$$

under which the matrix representations of $U(3)$ are the same. Therefore, in general, the $SU(3)$ irrep $[n_{13} = n'_{13} - n'_{33}, n_{23} = n'_{23} - n'_{33}] \equiv [n'_{13}, n'_{23}, n'_{33}]$. However, the right-hand-side of (10) should be considered, when any $SU(3)$ basis vector is expended in terms of the canonical $U(3) \supset U(2) \supset U(1)$ basis vectors. This correspondence will be helpful when one considers the Kronecker product of two or more irreps of $SU(3)$ expanded in the canonical chain, which will be used in Sec. V.

After a linear transformation, the generators of $SU(3)$ can also be expressed in its non-canonical basis, i. e. in the $SU(3) \supset SO(3)$ basis, with generators given by

$$\begin{aligned} L_0 &= E_{11} - E_{22},\ L_+ = \sqrt{2}(E_{13} + E_{32}),\quad L_- = (L_+)^\dagger = \sqrt{2}(E_{31} + E_{23}), \\ Q_2 &= E_{12},\ Q_1 = \sqrt{\tfrac{1}{2}}(E_{32} - E_{13}),\quad Q_0 = \sqrt{\tfrac{1}{6}}(E_{11} + E_{22} - 2E_{33}), \\ Q_{-1} &= -(Q_1)^\dagger = \sqrt{\tfrac{1}{2}}(E_{31} - E_{23}),\quad Q_{-2} = (Q_2)^\dagger = E_{21}, \end{aligned} \quad (11)$$

where $\{L_+, L_-, L_0\}$ are generators of the subgroup $SO(3)$, which may be identified as the angular momentum operators satisfying the usual commutation relations:

$$[L_0,\ L_\pm] = \pm L_\pm,\ [L_+,\ L_-] = 2L_0, \quad (12)$$

and Q_μ $(\mu = 2, 1, \ldots, -2)$ are quadrupole (moment) operators realized in the Elliott model for nuclei. One can verify that the second-order invariant (Casimir operator) of $SU(3)$ may be expressed as

$$\begin{aligned} C_2(SU(3)) &= \textstyle\sum_\mu (-1)^\mu Q_\mu Q_{-\mu} + \frac{1}{2}\left(\frac{1}{2}(L_+L_- + L_-L_+) + L_0^2\right) \\ &= \mathbf{Q}\cdot\mathbf{Q} + \tfrac{1}{2}\mathbf{L}\cdot\mathbf{L} = \textstyle\sum_{1\le i,j\le 3} E_{ij}E_{ji} - \frac{1}{3}\left(\sum_{i=1}^3 E_{ii}\right)^2, \end{aligned} \quad (13)$$

where $C_2(U(3)) = \sum_{1\le i,j\le 3} E_{ij}E_{ji}$ is the second-order Casimir operator of $U(3)$. Under the basis vector (3) with $n_{33} = 0$, the eigenvalue of $C_1(U(3))$ and that of $C_2(U(3))$ are given by

$$\begin{aligned} \langle C_1(U(3))\rangle &= \left\langle \textstyle\sum_{i=1}^3 E_{ii} \right\rangle = n_{13} + n_{23}, \\ \langle C_2(U(3))\rangle &= \left\langle \textstyle\sum_{1\le i,j\le 3} E_{ij}E_{ji} \right\rangle = n_{13}(n_{13}+2) + n_{23}^2. \end{aligned} \quad (14)$$

According to the labeling convention in nuclear physics introduced by Elliott [1, 2], the $SU(3)$ irrep $[n_{13}, n_{23}]$ is also labeled by (λ, μ) with $\lambda = n_{13} - n_{23}$ and $\mu = n_{23}$. Then, the eigenvalue of $C_2(SU(3))$ defined in (13) under the basis vector (3) is given by

$$\langle C_2(SU(3))\rangle = \langle C_2(U(3))\rangle - \tfrac{1}{3}\langle C_1(U(3))\rangle = \tfrac{2}{3}(\lambda^2 + \mu^2 + \lambda\mu + 3\lambda + 3\mu). \quad (15)$$

III. BASIS VECTORS OF $SU(3) \supset SO(3)$

The basis vector (3) is also an eigenstate of L_0 with eigenvalue $M = 2n_{11} - n_{12} - n_{22}$. For a given irrep $[n_{13}, n_{23}, 0]$ of $U(3)$ [or $SU(3)$], all possible basis vectors of $U(3) \supset U(2) \supset U(1)$ shown in (3) restricted by the betweenness conditions (4) form a complete set. Therefore, the basis vectors of $SU(3) \supset SO(3) \supset SO(2)$ can be expanded in terms of the basis vectors of $U(3) \supset U(2) \supset U(1)$ with the restriction on the $SO(2)$ quantum number $M = 2n_{11} - n_{12} - n_{22}$. The possible basis vectors of $U(3) \supset U(2) \supset U(1)$ spanning the subspace with $M = 2n_{11} - n_{12} - n_{22} \geq 0$ can be illustrated in the weight projection diagram for the example of the $SU(3)$ irrep [4, 2], shown in Fig. 1, where n_{11} is the quantum number of the $U(1)$ generator E_{11}, and $n_{12} + n_{22}$ is the eigenvalue of the $U(2)$ invariant $E_{11} + E_{22}$. In Fig. 1, the degeneracy equals exactly to the number of integer partitions of a fixed number $n_{12} + n_{22}$ restricted by the betweenness conditions: $n_{23} \leq n_{12} \leq n_{13}$ and $0 \leq n_{22} \leq n_{23}$, from which the number of basis vectors involved in the projection with fixed M can easily be counted.

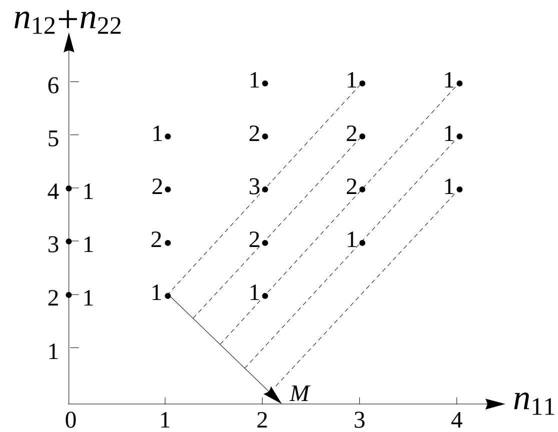

FIG. 1: The weight projection diagram for $SU(3) \supset SO(3) \supset SO(2)$ for the irrep $[n_{13}, n_{23}] = [4,\ 2]$, where the solid dots are the corresponding $U(3)$ weights labeled in the canonical chain by $2 \leq n_{12} + n_{22} \leq 6$ and $0 \leq n_{11} \leq 4$, with the corresponding degeneracy (shown by the number near the dots). The weights connected by the the dashed lines are involved in the projection with fixed M for $M = 0, \ldots, 4$ indicated in the order from the top left to the bottom right by the crossing points of the dashed lines with the M axis.

Thus, in order to find all basis vectors of $U(3) \supset U(2) \supset U(1)$ with fixed M in the irrep $[n_{13}, n_{23}]$ of $SU(3)$, it suffices to consider all possible irreps $[n_{12}, n_{22}]$ of $U(2)$ embedded in the canonical chain satisfying the condition (4) for this case. In constructing the basis vectors of $SU(3) \supset SO(3) \supset SO(2)$ for the irrep $[n_{13}, n_{23}]$ of $SU(3)$, there is a freedom to choose a specific basis vector of $SU(3) \supset SO(3) \supset SO(2)$ with the angular momentum quantum number L and the quantum number of the third component of the angular momentum M. Practically, it is convenient to choose the highest or the lowest weight state of $SO(3)$ with $M = L$ or $M = -L$. Let

$$\left| \begin{array}{l} [n_{13}, n_{23}] \\ \zeta\ L,\ \ M \end{array} \right\rangle \tag{16}$$

with $M = L$ or $M = -L$ be the basis vector of $SU(3) \supset SO(3) \supset SO(2)$ for the highest or the lowest weight state of $SO(3)$ for this case, where ζ is the multiplicity label needed in the reduction $[n_{13}, n_{23}] \downarrow L$. (16) should satisfy

$$L_{\pm} \left| \begin{array}{c} [n_{13}, n_{23}] \\ \zeta\ L,\ \ M = \pm L \end{array} \right\rangle = 0. \tag{17}$$

The condition (17) is also the constraint useful in determining the expansion coefficients of the basis vector (16) expanded in terms of the basis vectors of $U(3) \supset U(2) \supset U(1)$, which will be used in what follows. In the following, we only construct the highest weight state of $SO(3)$ with $M = L$. Once the basis vector (16) for the highest weight state of $SO(3)$ with $M = L$ is known, the basis vector of $SU(3) \supset SO(3) \supset SO(2)$ for any M can be expressed in the standard way as

$$\left| \begin{array}{c} [n_{13}, n_{23}] \\ \zeta\ L,\ \ M \end{array} \right\rangle = \sqrt{\frac{(L+M)!}{(2L)!(L-M)!}}\ (L_{-})^{L-M} \left| \begin{array}{c} [n_{13}, n_{23}] \\ \zeta\ L,\ \ M = L \end{array} \right\rangle, \tag{18}$$

where $L \geq 0$ should be satisfied.

According to the restriction $M = 2n_{11} - n_{12} - n_{22}$ and the betweenness conditions shown in (4), we find that all possible basis vectors within the $U(3)$ irrep $[n_{13}, n_{23}, 0]$ and with $M = k \geq 0$ are given as follows:

$$\left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t, t] \\ k+q \end{array} \right\rangle, \tag{19}$$

where

$$\begin{array}{c} 0 \leq k \leq n_{13}, \quad 0 \leq t \leq n_{23}, \\ \mathrm{Max}\left[t, \mathrm{IntM}[\frac{1}{2}(t-k+n_{23})]\right] \leq q \leq \mathrm{Int}[\frac{1}{2}(n_{13}-k+t)], \end{array} \tag{20}$$

in which $\mathrm{Int}[x]$ is the integer part of x, and $\mathrm{IntM}[x]$ is the largest integer closest to x defined by

$$\mathrm{IntM}[x] = \begin{cases} \mathrm{Int}[x]+1 & \text{if} \quad x - \mathrm{Int}[x] > 0, \\ \mathrm{Int}[x] & \text{if} \quad x - \mathrm{Int}[x] = 0. \end{cases} \tag{21}$$

The basis vectors of $SU(3) \supset SO(3) \supset SO(2)$, for $L = M$, the $SO(3)$ highest-weight state, may be expanded in terms of (19) as

$$\left| \begin{array}{c} [n_{13}, n_{23}] \\ \zeta \ L = M = k \end{array} \right\rangle = \sum_{t=0}^{n_{23}} \sum_{q=\mathrm{Max}\left[t, \mathrm{IntM}[\frac{1}{2}(t-k+n_{23})]\right]}^{\mathrm{Int}[\frac{1}{2}(n_{13}-k+t)]} c_{qt}^{(\zeta)}([n_{13}, n_{23}, 0], L) \left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t, t] \\ k+q \end{array} \right\rangle, \tag{22}$$

where $\{c_{qt}^{(\zeta)} \equiv c_{qt}^{(\zeta)}([n_{13}, n_{23}, 0], L)\}$ are the expansion coefficients, which must satisfy

$$\sqrt{\frac{1}{2}} L_{+} \left| \begin{array}{c} [n_{13}, n_{23}] \\ \zeta \ L = M = k \end{array} \right\rangle = (E_{13} + E_{32}) \left| \begin{array}{c} [n_{13}, n_{23}] \\ \zeta \ L = M = k \end{array} \right\rangle = 0. \tag{23}$$

The $U(3)$ generators $\{E_{13}, E_{23}\}$ are rank-1 irreducible tensor operators $\{T_{\mu}^{[1,0]}\}$ ($\mu = 1$ or 0) of $U(2)$ with $E_{13} = T_{1}^{[1,0]}$ and $E_{23} = T_{0}^{[1,0]}$. The action of E_{13} or E_{32} onto the basis vector of $U(3) \supset U(2) \supset U(1)$ shown in (19) useful for (23) can be summarized as follows:

$$\begin{aligned} E_{13} \left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t, t] \\ k+q \end{array} \right\rangle = & \left\langle \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t+1, t] \\ k+q+1 \end{array} \right| E_{13} \left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t, t] \\ k+q \end{array} \right\rangle \left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t+1, t] \\ k+q+1 \end{array} \right\rangle + \\ & \left\langle \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t, t+1] \\ k+q+1 \end{array} \right| E_{13} \left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t, t] \\ k+q \end{array} \right\rangle \left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t, t+1] \\ k+q+1 \end{array} \right\rangle, \end{aligned} \tag{24}$$

$$\begin{aligned} E_{32} \left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t, t] \\ k+q \end{array} \right\rangle = & \left\langle \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t-1, t] \\ k+q \end{array} \right| E_{32} \left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t, t] \\ k+q \end{array} \right\rangle \left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t-1, t] \\ k+q \end{array} \right\rangle + \\ & \left\langle \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t, t-1] \\ k+q \end{array} \right| E_{32} \left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t, t] \\ k+q \end{array} \right\rangle \left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t, t-1] \\ k+q \end{array} \right\rangle. \end{aligned} \tag{25}$$

By using (24) and (25), and the explicit matrix elements shown in (5)-(8), together with (22), Eq. (23) can be written as

$$\begin{aligned} \sqrt{\tfrac{1}{2}} L_{+} \left| \begin{array}{c} [n_{13}, n_{23}] \\ \zeta \ L = M = k \end{array} \right\rangle = \textstyle\sum_{t,q} \Bigg\{ & c_{q,t}^{(\zeta)} \left[\tfrac{(k+q-t+1)(n_{13}-k-2q+t)(k+2q-t-n_{23}+1)(k+2q-t+2)}{(k+2q-2t+1)(k+2q-2t+2)} \right]^{\frac{1}{2}} + \\ & c_{q+1,t}^{(\zeta)} \left[\tfrac{(q-t+1)(n_{13}-k-2q+t-1)(k+2q-t-n_{23}+2)(k+2q-t+3)}{(k+2q-2t+2)(k+2q-2t+3)} \right]^{\frac{1}{2}} + \\ & c_{q+1,t+1}^{(\zeta)} \left[\tfrac{(k+q-t+1)(n_{13}-t+1)(n_{23}-t)(t+1)}{(k+2q-2t+1)(k+2q-2t+2)} \right]^{\frac{1}{2}} - \\ & c_{q,t-1}^{(\zeta)} \left[\tfrac{(q-t+1)(n_{13}-t+2)(n_{23}-t+1)t}{(k+2q-2t+2)(k+2q-2t+3)} \right]^{\frac{1}{2}} \Bigg\} \left| \begin{array}{c} [n_{13}, n_{23}, 0] \\ [k+2q-t+1, t] \\ k+q+1 \end{array} \right\rangle = 0, \end{aligned} \tag{26}$$

which, thus, leads to the following four-term relation to determine the expansion coefficients $\{c^{(\zeta)}_{q,t}\}$:

$$c^{(\zeta)}_{q,t}\left[\frac{(k+q-t+1)(n_{13}-k-2q+t)(k+2q-t-n_{23}+1)(k+2q-t+2)}{k+2q-2t+1}\right]^{\frac{1}{2}}+c^{(\zeta)}_{q+1,t}\left[\frac{(q-t+1)(n_{13}-k-2q+t-1)(k+2q-t-n_{23}+2)(k+2q-t+3)}{k+2q-2t+3}\right]^{\frac{1}{2}}+$$
$$c^{(\zeta)}_{q+1,t+1}\left[\frac{(k+q-t+1)(n_{13}-t+1)(n_{23}-t)(t+1)}{k+2q-2t+1}\right]^{\frac{1}{2}}-c^{(\zeta)}_{q,t-1}\left[\frac{(q-t+1)(n_{13}-t+2)(n_{23}-t+1)t}{k+2q-2t+3}\right]^{\frac{1}{2}}=0. \tag{27}$$

Similar to the projection procedure for $O(5)\supset O(3)$ shown in [19], one can construct a matrix equation of (27) with

$$\mathbf{P}([n_{13},n_{23}],k)\mathbf{c}^{(\zeta)}=\Lambda\mathbf{c}^{(\zeta)}, \tag{28}$$

where $\mathbf{c}^{(\zeta)}\equiv\mathbf{c}^{(\zeta)}([n_{13},n_{23}],k)$, for which the transpose is arranged as $\left(\mathbf{c}^{(\zeta)}\right)^{\mathrm{T}}=(c^{(\zeta)}_{0,0},c^{(\zeta)}_{1,0},c^{(\zeta)}_{2,0},\dots,c^{(\zeta)}_{1,1},c^{(\zeta)}_{2,1},\dots)$. Possible nonzero components of $\mathbf{c}^{(\zeta)}$ for some specific cases are shown in Table I. Entries of the angular momentum projection matrix $\mathbf{P}([n_{13},n_{23}],k)$ can easily be read out from Eq. (27). The components of the eigenvector $\mathbf{c}^{(\zeta)}$ corresponding to $\Lambda=0$ provide the expansion coefficients $\{c^{(\zeta)}_{q,t}\}$ of (22). Once the matrix $\mathbf{P}([n_{13},n_{23}],k)$ is constructed, it can be verified that the number of $\Lambda=0$ solutions of Eq. (28) for sufficiently large n_{13} equals exactly to the number of rows of $\mathbf{P}([n_{13},n_{23}],k)$ with all entries zero. However, some entries of $\mathbf{P}([n_{13},n_{23}],k)$ will be zero or become complex for some specific values of n_{13} and n_{23}. In such cases, a nonzero solution of $\{c^{(\zeta)}_{q,t}([n_{13},n_{23}],k)\}$ does not exist, which will be examined for specific cases separately in the following. Actually, the eigenvectors $\mathbf{c}^{(\zeta)}([n_{13},n_{23}],k)$ belong to the null space of $\mathbf{P}([n_{13},n_{23}],k)$. Since there are many ways currently available to find null-space vectors of a matrix, to find solutions of Eq. (20) with $\Lambda=0$ becomes practically easy. Furthermore, $\left(\mathbf{c}^{(\zeta')}([n_{13},n_{23}],k)\right)^{\mathrm{T}}\cdot\mathbf{c}^{(\zeta)}([n_{13},n_{23}],k)\neq0$ when the multiplicity is greater than 1 mainly because the projection matrix $\mathbf{P}([n_{13},n_{23}],k)$ is nonsymmetric. Therefore, the $SU(3)\supset SO(3)\supset SO(2)$ basis vectors (22) constructed from the expansion coefficients obtained according to (27) are also non-orthogonal with respect to the multiplicity label ζ. The Gram-Schmidt process may be adopted in order to construct orthonormalized basis vectors of $SU(3)\supset SO(3)\supset SO(2)$. Thus, the multiplicity of $L=k$ for the given irrep $[n_{13},n_{23}]$ is given by the number of linearly independent null space vectors of $\mathbf{P}([n_{13},n_{23}],k)$.

It is known that computing time and memory requirements needed to numerically solve the null-space problem (28) depend mainly on the number of terms $d(k)$, with $k=L$, needed in the expansion (22), which equals to the number of columns of $\mathbf{P}([n_{13},n_{23}],k)$. Generally, it would take CPU time on the order of $\mathrm{O}(d^3)$ with a unit inversely proportional to the CPU frequency, while the memory size should depend on rows×columns of $\mathbf{P}$. Since $\mathbf{P}$ is a sparse matrix with only four nonzero elements in each row, the memory size increases with d linearly.

For a given irrep $[n_{13},n_{23}]$ of $SU(3)$, the number of terms $d(k)$ with $k=L$ needed in the expansion (22) increases with the increasing of n_{23} and $p=n_{13}-L$, where $\mathrm{Min}[p]=0$ and $\mathrm{Max}[p]=n_{13}$ are always satisfied in the reduction $[n_{13},n_{23}]\downarrow L=n_{13}-p$, of which the multiplicity is determined by the number of independent solutions of (28). Since $n_{13}\geq n_{23}$ is always satisfied, to estimate the asymptotic behavior of $d(L)$ with $L=n_{13}-p$ for $p=0,1,2,\dots,n_{13}$ with sufficiently large n_{13}, namely for the situation with large n_{23} and $n_{13}-L$ cases , which yields the p-dependence of $d(L)$ in the large n_{13} and n_{23} limit, the dimensions $d(L)$ as functions of p for several n_{23} values up to the large n_{23} limit are plotted and shown in Fig. 2. It is clearly seen in Fig. 2 that $d(L)$ can always be fitted by a polynomial, for which the order slightly increases with the increasing of n_{23}. However, the order of the polynomial becomes fixed in the large n_{23} limit, from which the p-dependence of $d(L)$ can be estimated. It is found that a best-fit analysis in the large n_{23} limit yields $d(L)|_{n_{23}\to\infty}\approx\mathrm{Int}[1+p+\frac{1}{4}p^2]$ with $L=n_{13}-p$, which shows that $d(L)$ in the large n_{23} limit increases with p quadratically.

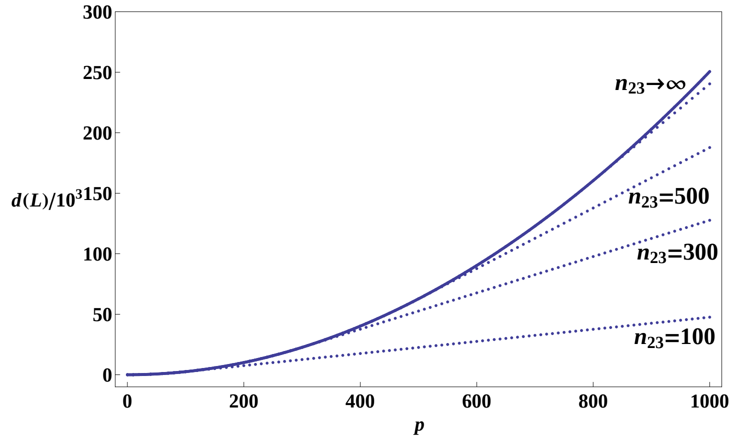

FIG. 2: The number of terms, $d(L)$, needed in the expansion (22) as a function of p for several n_{23} values, including that in the large n_{23} limit (the solid line), with a sufficiently large n_{13} value, where $L = n_{13} - p$.

When $n_{23} = 0$, only $t = 0$ is allowed. There are only two terms involved in (27) for this case with

$$c_{q,0}\,[(k+q+1)(n_{13}-k-2q)]^{\frac{1}{2}} + c_{q+1,0}\,[(q+1)(n_{13}-k-2q-1)]^{\frac{1}{2}} = 0, \tag{29}$$

from which one obtains

$$c_{q,0} = c_{0,0}(-1)^q \left[\frac{(k+q)!(n_{13}-k)!!(n_{13}-k-2q-1)!!}{k!q!(n_{13}-k-2q)!!(n_{13}-k-1)!!}\right]^{\frac{1}{2}}. \tag{30}$$

For a symmetric irrep $[n_{13}, 0]$ of $SU(3)$, the basis vectors of $U(3) \supset U(2) \supset U(1)$ used in the expansion (22) can be realized in terms of three boson creation operators $\{a_1^\dagger, a_2^\dagger, a_3^\dagger\}$, which carry the angular momentum $l = 1$ with quantum number of its third component $\nu = 1, -1$, and 0, respectively, as

$$\left|\begin{matrix}[n_{13},0,0]\\ [k+2q,0]\\ k+q\end{matrix}\right\rangle = \frac{a_1^{\dagger k+q} a_2^{\dagger q} a_3^{\dagger n_{13}-k-2q}}{[(k+q)!q!(n_{13}-k-2q)!]^{\frac{1}{2}}}|0\rangle = \left[\frac{k!}{(k+q)!q!(n_{13}-k-2q)!}\right]^{\frac{1}{2}} (a_1^\dagger a_2^\dagger)^q (a_3^{\dagger 2})^{\mathrm{Int}[\frac{n_{13}-k}{2}]-q} \left|\begin{matrix}[k,0,0]\\ [k,0]\\ k\end{matrix}\right\rangle, \tag{31}$$

where $|0\rangle$ is the boson vacuum state, and

$$\left|\begin{matrix}[k,0,0]\\ [k,0]\\ k\end{matrix}\right\rangle = \frac{1}{\sqrt{k!}} a_1^{\dagger k}|0\rangle. \tag{32}$$

Substituting (30) and (31) into (22), we get

$$\left|\begin{matrix}[n_{13},0]\\ L=M=k\end{matrix}\right\rangle = \frac{c_{0,0}}{(\mathrm{Int}[\frac{1}{2}(n_{13}-k)])!}\left[\frac{(n_{13}-k)!!}{(n_{13}-k-1)!!}\right]^{\frac{1}{2}} (\frac{1}{2}a_3^{\dagger 2} - a_1^\dagger a_2^\dagger)^{\mathrm{Int}[\frac{1}{2}(n_{13}-k)]} \left|\begin{matrix}[k,0,0]\\ [k,0]\\ k\end{matrix}\right\rangle, \tag{33}$$

which, up to a normalization constant, is consistent with the result of the symmetric irrep $[n_{13}, 0]$ of $SU(3)$ in the $SU(3) \supset SO(3)$ basis shown previously [3], where $\frac{1}{2}a_3^{\dagger 2} - a_1^\dagger a_2^\dagger$ is the boson pairing operator with angular momentum zero.

TABLE I: Allowed $(q,\ t)$ combinations in the basis vectors (22) of $SU(3) \supset SO(3) \supset SO(2)$ for the irrep $[n_{13}, n_{23}]$ with $1 \leq n_{23} \leq 4$ and a sufficiently large n_{13} value, and $L = k = n_{13} - p$ for $p = 0, 1, \ldots, 4$, expanded in terms of those of $U(3) \supset U(2) \supset U(1)$ with the corresponding multiplicity Multi$([n_{13}, n_{23}], k)$, where $d(k)$ is the total number of terms needed in the expansion for a given p.

n_{23}	k	(q,t)	$d(k)$	Multi$([n_{13},n_{23}],k)$
1	n_{13}	$(0,0)$	1	1
	$n_{13}-1$	$(0,0),(1,1)$	2	1
	$n_{13}-2$	$(0,0),(1,0),(1,1)$	3	1
	$n_{13}-3$	$(0,0),(1,0),(1,1),(2,1)$	4	1
	$n_{13}-4$	$(0,0),(1,0),(2,0),(1,1),(2,1)$	5	1
2	n_{13}	$(0,0)$	1	1
	$n_{13}-1$	$(0,0),(1,1)$	2	1
	$n_{13}-2$	$(0,0),(1,0),(1,1),(2,2)$	4	2
	$n_{13}-3$	$(0,0),(1,0),(1,1),(2,1),(2,2)$	5	1
	$n_{13}-4$	$(0,0),(1,0),(2,0),(1,1),(2,1),(2,2),(3,2)$	7	2
3	n_{13}	$(0,0)$	1	1
	$n_{13}-1$	$(0,0),(1,1)$	2	1
	$n_{13}-2$	$(0,0),(1,0),(1,1),(2,2)$	4	2
	$n_{13}-3$	$(0,0),(1,0),(1,1),(2,1),(2,2),(3,3)$	6	2
	$n_{13}-4$	$(0,0),(1,0),(2,0),(1,1),(2,1),(2,2),(3,2),(3,3)$	8	2
4	n_{13}	$(0,0)$	1	1
	$n_{13}-1$	$(0,0),(1,1)$	2	1
	$n_{13}-2$	$(0,0),(1,0),(1,1),(2,2)$	4	2
	$n_{13}-3$	$(0,0),(1,0),(1,1),(2,1),(2,2),(3,3)$	6	2
	$n_{13}-4$	$(0,0),(1,0),(2,0),(1,1),(2,1),(2,2),(3,2),(3,3),(4,4)$	9	3

Hence, we only consider cases with $n_{23} \geq 1$ in what follows. For given $L = k = 0,\ 1,\ 2, \ldots, n_{13}$ of $SO(3)$, the number of solutions, Multi$([n_{13}, n_{23}], k)$, of Eq. (28) with $\zeta = 1,\ 2, \ldots,$ Multi$([n_{13}, n_{23}], k)$ equals exactly to the multiplicity in the reduction $SU(3) \downarrow SO(3)$ for the $SU(3)$ irrep $[n_{13}, n_{23}]$, which may be calculated by the formula [7]

$$\mathrm{Multi}([n_{13},n_{23}],k) = \mathrm{IntP}[\frac{1}{2}(n_{13}-k)] - \mathrm{IntP}[\frac{1}{2}(n_{13}-n_{23}+1-k)] - \mathrm{IntP}[\frac{1}{2}(n_{23}+1-k)] + 1, \tag{34}$$

where

$$\mathrm{IntP}[x] = \begin{cases} \mathrm{Int}[x] & \text{if } \ x \geq 0, \\ 0 & \text{if } \ x < 0, \end{cases} \tag{35}$$

derived according to Elliott's reduction rules [1]:

$$K = \mathrm{Min}[n_{13}-n_{23}, n_{23}], \quad \mathrm{Min}[n_{13}-n_{23}, n_{23}] - 2, \quad \ldots, \quad 0 \ \text{ or } \ 1,$$

$$L = \begin{cases} K,\ K+1,\ K+2,\ \ldots,\ K + \mathrm{Max}[n_{13}-n_{23}, n_{23}] & \text{for } K > 0, \\ \mathrm{Max}[n_{13}-n_{23}, n_{23}],\ \mathrm{Max}[n_{13}-n_{23}, n_{23}] - 2,\ \ldots, 0 \ \text{ or } \ 1 & \text{for } K = 0. \end{cases} \tag{36}$$

Multi$([n_{13}, n_{23}], k)$ with $k \geq n_{13} - 4$ for the $SU(3)$ irrep $[n_{13}, n_{23}]$ $(1 \leq n_{23} \leq 4)$ is also shown in the last column of Table I.

In the following, we list some $\mathbf{P}([n_{13}, n_{23}], k = L)$ matrices and the corresponding expansion coefficients $\{c^{(\zeta)}_{q,t}\}$. There is always a freedom in choosing the global phase. In our calculation, we always set $c^{(\zeta)}_{0,0} > 0$, while the relative phase is completely determined by the eigen-equation (28). Since $\mathbf{c}^{(\zeta)}([n_{13}, n_{23}], k)$ is non-orthogonal with respect to ζ when Multi$([n_{13}, n_{23}], k) > 1$, the corresponding orthonormalized null-space vectors of $\mathbf{P}$ will be denoted by $\tilde{\mathbf{c}}^{(\zeta)}$, while ζ will be omitted if Multi$([n_{13}, n_{23}], k) = 1$.

For any irrep of $SU(3)$ labeled by $[n_{13}, n_{23}]$, the largest two angular momentum states with $L = n_{13}$ and $n_{13} - 1$ are always multiplicity-free. It can be verified that $\mathbf{P}([n_{13}, n_{23}], n_{13}) = 0$ with $\tilde{c}_{0,0}([n_{13}, n_{23}], n_{13}) = 1$, which is trivial corresponding to one unique highest-weight state of $SU(3) \supset SO(3)$ with $k = n_{13}$. When $k = n_{13} - 1$,

$$\mathbf{P}([n_{13}, n_{23}], n_{13} - 1) = \begin{pmatrix} \sqrt{n_{13} - n_{23}}, & \sqrt{n_{23}} \end{pmatrix} \tag{37}$$

with $(\mathbf{c}([n_{13}, n_{23}], n_{13} - 1))^{\mathrm{T}} = (c_{0,0}, c_{1,1})$. Since there is one row with all entries zero, the multiplicity of $k = n_{13} - 1$ is $\mathrm{Multi}([n_{13}, n_{23}], n_{13} - 1) = 1$ for $n_{13} > n_{23}$. The normalized expansion coefficients are $\tilde{c}_{0,0}([n_{13}, n_{23}], n_{13} - 1) = \sqrt{\frac{n_{23}}{n_{13}}}$, $\tilde{c}_{1,1}([n_{13}, n_{23}], n_{13} - 1) = -\sqrt{\frac{n_{13} - n_{23}}{n_{13}}}$ for $n_{13} > n_{23}$. It can be observed that all entries of the projection matrix $\mathbf{P}$ should be real numbers depending on n_{13}, n_{23}, and k. If some entries of $\mathbf{P}$ become complex for some specific values of n_{13}, n_{23}, and k, the corresponding entries in $\mathbf{c}$ must be zero. For example, $\tilde{c}_{00}$ should be zero when $n_{23} > n_{13}$, at which the first entry in (37) is complex, though $n_{23} > n_{13}$ obviously violates the betweenness conditions of $SU(3)$. For all possible values of n_{13}, n_{23}, and k, the boundary condition (20) is causal in determining allowed (q, t) values. For example, only $(q, t) = (1, 1)$ is allowed when $n_{13} = n_{23}$, which results in $c_{0,0} = c_{1,1} = 0$ consistent with the branching rule of $SU(3) \supset SO(3)$, namely, $L = n_{13} - 1$ dose not occur in this case.

For $n_{23} = 1$ and $k = n_{13} - 2$,

$$\mathbf{P}([n_{13}, 1], n_{13} - 2) = \begin{pmatrix} \sqrt{2n_{13}(n_{13} - 2)}, & \sqrt{n_{13} - 1}, & \sqrt{n_{13} + 1} \\ 0, & -1, & \sqrt{(n_{13} - 1)(n_{13} + 1)} \end{pmatrix}, \tag{38}$$

where and in the following, rows with all entries zero in the matrix $\mathbf{P}([n_{13}, n_{23}], k)$ are omitted. Since there is one row with all entries zero in (38) when $n_{13} > 2$, the multiplicity of $L = n_{13} - 2$ is $\mathrm{Multi}([n_{13}, 1], n_{13} - 2) = 1$ for $n_{13} > 2$. The normalized nonzero expansion coefficients corresponding to $\Lambda = 0$ are

$$\tilde{c}_{0,0} = \sqrt{\frac{n_{13}+1}{(2n_{13}-1)(n_{13}-1)}}, \quad \tilde{c}_{1,0} = -\sqrt{\frac{2(n_{13}+1)(n_{13}-2)}{(2n_{13}-1)n_{13}}}, \quad \tilde{c}_{1,1} = -\sqrt{\frac{2(n_{13}-2)}{(2n_{13}-1)(n_{13}-1)n_{13}}}$$

for $n_{13} > 2$.

For $n_{23} = 1$ and $k = n_{13} - 3$,

$$\mathbf{P}([n_{13}, 1], n_{13} - 3) = \begin{pmatrix} \sqrt{3(n_{13} - 1)(n_{13} - 3)}, & \sqrt{2(n_{13} - 2)}, & \sqrt{n_{13} + 1}, & 0 \\ 0, & -\sqrt{n_{13} + 1}, & \sqrt{2(n_{13} - 2)n_{13}^2}, & \sqrt{(n_{13} - 1)(n_{13} + 1)} \\ 0, & \sqrt{(n_{13}^2 - 1)(n_{13} - 1)}, & 0, & \sqrt{(n_{13} - 1)(n_{13} + 1)} \end{pmatrix}. \tag{39}$$

Since there is one row with all entries zero in (39) when $n_{13} > 3$, the multiplicity of $k = n_{13} - 3$ is $\mathrm{Multi}([n_{13}, 1], n_{13} - 3) = 1$ for $n_{13} > 3$. The normalized nonzero expansion coefficients corresponding to $\Lambda = 0$ are

$$\tilde{c}_{0,0} = \sqrt{\frac{3}{(2n_{13}-3)(n_{13}-2)}}, \quad \tilde{c}_{1,0} = -\sqrt{\frac{2(n_{13}-3)}{(2n_{13}-3)(n_{13}-1)}}, \tilde{c}_{1,1} = -\sqrt{\frac{(n_{13}-3)(n_{13}+1)}{(2n_{13}-3)(n_{13}-1)(n_{13}-2)}}, \tilde{c}_{2,1} = \sqrt{\frac{2(n_{13}-3)}{2n_{13}-3}}$$

for $n_{13} > 3$.

For $n_{23} = 1$ and $k = n_{13} - 4$,

$$\mathbf{P}([n_{13}, 1], n_{13} - 4) = \begin{pmatrix} \sqrt{4(n_{13} - 2)(n_{13} - 4)}, & \sqrt{3(n_{13} - 3)}, & 0, & \sqrt{n_{13} + 1}, & 0 \\ 0, & -\sqrt{n_{13} + 1}, & 0, & \sqrt{3(n_{13} - 3)(n_{13} - 1)^2}, & \sqrt{2n_{13}(n_{13} - 2)} \\ 0, & \sqrt{2n_{13}(n_{13} - 2)^2}, & \sqrt{2(n_{13} - 1)^2}, & 0, & \sqrt{(n_{13} - 2)(n_{13} + 1)} \\ 0, & 0, & -\sqrt{2}, & 0, & \sqrt{(n_{13} - 2)(n_{13} + 1)} \end{pmatrix}. \tag{40}$$

Since there is also one row with all entries zero in (40) when $n_{13} > 4$, the multiplicity of $L = n_{13} - 4$ is $\mathrm{Multi}([n_{13}, 1], n_{13} - 4) = 1$ for $n_{13} > 4$. The normalized nonzero expansion coefficients corresponding to $\Lambda = 0$ are

$$\tilde{c}_{0,0} = \sqrt{\frac{3(n_{13}+1)}{(2n_{13}-3)(2n_{13}-5)(n_{13}-3)}}, \quad \tilde{c}_{1,0} = -\sqrt{\frac{4(n_{13}-4)(n_{13}+1)}{(2n_{13}-3)(2n_{13}-5)(n_{13}-2)}}, \quad \tilde{c}_{2,0} = \sqrt{\frac{4(n_{13}-4)(n_{13}-2)(n_{13}+1)}{(2n_{13}-3)(2n_{13}-5)n_{13}}},$$

$$\tilde{c}_{1,1} = -\sqrt{\frac{12(n_{13}-4)}{(2n_{13}-3)(2n_{13}-5)(n_{13}-2)(n_{13}-3)}}, \quad \tilde{c}_{2,1} = \sqrt{\frac{8(n_{13}-4)}{(2n_{13}-3)(2n_{13}-5)n_{13}}}$$

for $n_{13} > 4$. In general, a given angular momentum L occurs only once in the reduction $SU(3) \downarrow SO(3)$ for the irrep $[n_{13}, 1]$.

For $n_{23} = 2$ and $k = n_{13} - 2$,

$$\mathbf{P}([n_{13}, 2], n_{13} - 2) = \begin{pmatrix} \sqrt{2n_{13}(n_{13} - 3)}, & \sqrt{n_{13} - 2}, & \sqrt{2(n_{13} + 1)}, & 0 \\ 0, & -\sqrt{2}, & \sqrt{(n_{13} - 2)(n_{13} + 1)}, & \sqrt{2n_{13}} \end{pmatrix}. \tag{41}$$

Since there are two rows with all entries zero in (41) when $n_{13} > 3$, the multiplicity of $L = n_{13} - 2$ is Multi$([n_{13}, 2], n_{13} - 2) = 2$ for $n_{13} > 3$. After the Gram-Schmidt orthogonalization, the two sets of normalized nonzero expansion coefficients corresponding to $\Lambda = 0$ are

$$\tilde{c}^{(1)}_{0,0} = \sqrt{\tfrac{n_{13}-2}{2n_{13}^2-3n_{13}-8}}, \quad \tilde{c}^{(1)}_{1,0} = -\sqrt{\tfrac{2(n_{13}-3)n_{13}}{2n_{13}^2-3n_{13}-8}}, \quad \tilde{c}^{(1)}_{1,1} = 0, \quad \tilde{c}^{(1)}_{2,2} = -\sqrt{\tfrac{2(n_{13}-3)}{2n_{13}^2-3n_{13}-8}}$$

$$\tilde{c}^{(2)}_{0,0} = \sqrt{\tfrac{18(n_{13}-3)(n_{13}+1)}{(2n_{13}-1)(n_{13}-1)(2n_{13}^2-3n_{13}-8)}}, \quad \tilde{c}^{(2)}_{1,0} = -\frac{2(n_{13}-4)\sqrt{(n_{13}-2)(n_{13}+1)}}{\sqrt{(n_{13}-1)n_{13}(2n_{13}-1)(2n_{13}^2-3n_{13}-8)}},$$

$$\tilde{c}^{(2)}_{1,1} = -\sqrt{\tfrac{2(2n_{13}^2-3n_{13}-8)}{(n_{13}-1)n_{13}(2n_{13}-1)}}, \quad \tilde{c}^{(2)}_{2,2} = \sqrt{\tfrac{(2n-5)^2(n_{13}+1)(n_{13}-2)}{(n_{13}-1)(2n_{13}-1)(2n_{13}^2-3n_{13}-8)}}$$

for $n_{13} > 3$.

For $n_{23} = 2$ and $k = n_{13} - 3$,

$$\mathbf{P}([n_{13}, 2], n_{13} - 3) = \begin{pmatrix} \sqrt{3(n_{13} - 1)(n_{13} - 4)}, & \sqrt{2(n_{13} - 3)}, & \sqrt{2(n_{13} + 1)}, & 0, & 0 \\ 0, & -\sqrt{2(n_{13} + 1)}, & \sqrt{2(n_{13} - 3)n_{13}^2}, & \sqrt{(n_{13} - 2)(n_{13} + 1)}, & \sqrt{2n_{13}^2} \\ 0, & \sqrt{n_{13} - 2}, & 0, & \sqrt{2}, & 0 \\ 0, & 0, & 0, & -\sqrt{2}, & \sqrt{(n_{13} - 2)(n_{13} + 1)} \end{pmatrix}. \tag{42}$$

Since there is one row with all entries zero in (42) when $n_{13} > 4$, the multiplicity of $k = n_{13} - 3$ is Multi$([n_{13}, 2], n_{13} - 3) = 1$ for $n_{13} > 4$. The normalized nonzero expansion coefficients corresponding to $\Lambda = 0$ are

$$\tilde{c}_{0,0} = \sqrt{\tfrac{6(n_{13}+1)}{(2n_{13}-3)(n_{13}-1)(n_{13}-2)}}, \quad \tilde{c}_{1,0} = -\sqrt{\tfrac{4(n_{13}+1)(n_{13}-3)(n_{13}-4)}{(2n_{13}-3)(n_{13}-1)^2(n_{13}-2)}}, \quad \tilde{c}_{1,1} = -\sqrt{\tfrac{(n_{13}-4)(n_{13}+3)^2}{(2n_{13}-3)(n_{13}-1)^2(n_{13}-2)}},$$

$$\tilde{c}_{2,1} = \sqrt{\tfrac{2(n_{13}-3)(n_{13}-4)(n_{13}+1)}{(2n_{13}-3)(n_{13}-1)^2}}, \quad \tilde{c}_{2,2} = \sqrt{\tfrac{4(n_{13}-3)(n_{13}-4)}{(2n_{13}-3)(n_{13}-1)^2(n_{13}-2)}},$$

for $n_{13} > 4$.

For $n_{23} = 2$ and $k = n_{13} - 4$,

$$\mathbf{P}([n_{13}, 2], n_{13} - 4) = \begin{pmatrix} \sqrt{4(n_{13} - 5)}, & \sqrt{\tfrac{3(n_{13}-4)}{n_{13}-2}}, & 0, & \sqrt{\tfrac{2(n_{13}+1)}{n_{13}-2}}, & 0, & 0, & 0 \\ 0, & -\sqrt{\tfrac{2(n_{13}+1)}{(n_{13}-1)^2}}, & 0, & \sqrt{3(n_{13} - 4)}, & \sqrt{\tfrac{2(n_{13}-3)n_{13}}{(n_{13}-1)^2}}, & \sqrt{\tfrac{2n_{13}}{n_{13}-1}}, & 0 \\ 0, & \sqrt{\tfrac{n_{13}(n_{13}-3)}{n_{13}-1}}, & 1, & 0, & \sqrt{\tfrac{(n_{13}+1)}{n_{13}-1}}, & 0, & 0 \\ 0, & 0, & -2, & 0, & \sqrt{\tfrac{(n_{13}-2)^2(n_{13}+1)}{n_{13}-1}}, & 0, & \sqrt{\tfrac{2n_{13}(n_{13}-2)}{n_{13}-1}} \\ 0, & 0, & 0, & 0, & -\sqrt{\tfrac{2}{n_{13}-1}}, & \sqrt{2(n_{13} - 3)}, & \sqrt{\tfrac{(n_{13}+1)(n_{13}-2)}{(n_{13}-1)n_{13}}} \end{pmatrix}. \tag{43}$$

Since there are two rows with all entries zero in (43) when $n_{13} > 5$, the multiplicity of $L = n_{13} - 4$ is Multi$([n_{13}, 2], n_{13} - 4) = 2$ for $n_{13} > 5$. The two sets of nonzero expansion coefficients corresponding to $\Lambda = 0$ are

$$c^{(1)}_{0,0} = \frac{3n_{13}^4-32n_{13}^3+75n_{13}^2-6n_{13}-56}{\sqrt{96(n_{13}-5)(n_{13}-4)(n_{13}-3)n_{13}^2(n_{13}-1)^2}}, \quad c^{(1)}_{1,0} = -\frac{(n_{13}-2)(n_{13}^2-5n_{13}-4)}{\sqrt{8(n_{13}-3)(n_{13}-2)n_{13}^2}},$$

$$c^{(1)}_{2,0} = \sqrt{\frac{(n_{13}-2)(n_{13}^2-3n_{13}-2)^2}{8(n_{13}-1)n_{13}}}, \quad c^{(1)}_{1,1} = \sqrt{\frac{(n_{13}+1)(n_{13}-2)(n_{13}^2-n_{13}+4)^2}{12(n_{13}-3)(n_{13}-4)n_{13}^2(n_{13}-1)^2}},$$

$$c^{(1)}_{2,1} = -\sqrt{\frac{(n_{13}-2)(n_{13}+1)}{2n_{13}}}, \quad c^{(1)}_{2,2} = 0, \quad c^{(1)}_{3,2} = -1;$$

$$c^{(2)}_{0,0} = \sqrt{\frac{(3n_{13}^2-23n_{13}+28)^2(n_{13}-2)(n_{13}+1)}{48(n_{13}-5)(n_{13}-4)n_{13}(n_{13}-1)}}, \quad c^{(2)}_{1,0} = -\sqrt{\frac{(n_{13}-4)^2(n_{13}+1)(n_{13}-1)}{4n_{13}}},$$

$$c^{(2)}_{2,0} = \tfrac{1}{2}(n_{13} - 2)\sqrt{(n_{13} - 3)(n_{13} + 1)}, \quad c^{(2)}_{1,1} = \frac{n_{13}^2-n_{13}+4}{\sqrt{6(n_{13}-4)n_{13}(n_{13}-1)}},$$

$$c^{(2)}_{2,1} = -\sqrt{(n_{13} - 3)(n_{13} - 1)}, \quad c^{(2)}_{2,2} = -1, \quad c^{(2)}_{3,2} = 0$$

for $n_{13} > 5$, which are not normalized and non-orthogonal because the orthonormalized expressions are much too complicated. For such cases, the orthogonalization could be preformed numerically, which is a straightforward task.

For $n_{23} = 3$ and $k = n_{13} - 2$,

$$\mathbf{P}([n_{13}, 3], n_{13} - 2) = \begin{pmatrix} \sqrt{2n_{13}(n_{13}-4)}, & \sqrt{n_{13}-3}, & \sqrt{3(n_{13}+1)}, & 0 \\ 0, & -\sqrt{3}, & \sqrt{(n_{13}-3)(n_{13}+1)}, & \sqrt{4n_{13}} \end{pmatrix}. \tag{44}$$

Since there are two rows with all entries zero in (44) when $n_{13} > 4$, the multiplicity of $L = n_{13} - 2$ is $\text{Multi}([n_{13}, 3], n_{13} - 2) = 2$ for $n_{13} > 4$. After the Gram-Schmidt orthogonalization, the two sets of normalized nonzero expansion coefficients corresponding to $\Lambda = 0$ are

$$\tilde{c}^{(1)}_{0,0} = \sqrt{\frac{2(n_{13}-3)}{4n_{13}^2-11n_{13}-18}}, \quad \tilde{c}^{(1)}_{1,0} = -\sqrt{\frac{4(n_{13}-4)n_{13}}{4n_{13}^2-11n_{13}-18}}, \quad \tilde{c}^{(1)}_{1,1} = 0, \quad \tilde{c}^{(1)}_{2,2} = -\sqrt{\frac{3(n_{13}-4)}{4n_{13}^2-11n_{13}-18}}$$

$$\tilde{c}^{(2)}_{0,0} = \sqrt{\frac{75(n_{13}-4)(n_{13}+1)}{(2n_{13}-1)(n_{13}-1)(4n_{13}^2-11n_{13}-18)}}, \quad \tilde{c}^{(2)}_{1,0} = -\frac{(n-6)\sqrt{6(n_{13}-3)(n_{13}+1)}}{\sqrt{(n_{13}-1)n_{13}(2n_{13}-1)(4n_{13}^2-11n_{13}-18)}},$$

$$\tilde{c}^{(2)}_{1,1} = -\sqrt{\frac{2(4n_{13}^2-11n_{13}-18)}{(n_{13}-1)n_{13}(2n_{13}-1)}}, \quad \tilde{c}^{(2)}_{1,2} = \sqrt{\frac{2(n_{13}+1)(n_{13}-3)(2n_{13}-7)^2}{(n_{13}-1)(2n_{13}-1)(4n_{13}^2-11n_{13}-18)}}$$

for $n_{13} > 4$.

For $n_{23} = 3$ and $k = n_{13} - 3$,

$$\mathbf{P}([n_{13}, 3], n_{13} - 3) =$$

$$\begin{pmatrix} \sqrt{3(n_{13}-1)(n_{13}-5)}, & \sqrt{2(n_{13}-4)}, & \sqrt{3(n_{13}+1)}, & 0, & 0, & 0 \\ 0, & -\sqrt{3(n_{13}+1)}, & \sqrt{2(n_{13}-4)n_{13}^2}, & \sqrt{(n_{13}-3)(n_{13}+1)}, & \sqrt{2n_{13}^2}, & 0 \\ 0, & \sqrt{n_{13}-3}, & 0, & \sqrt{3}, & 0, & 0 \\ 0, & 0, & 0, & -2, & \sqrt{(n_{13}-3)(n_{13}+1)}, & \sqrt{3(n_{13}-1)} \end{pmatrix}. \tag{45}$$

Since there are two rows with all entries zero in (45) when $n_{13} > 5$, the multiplicity of $k = n_{13} - 3$ is $\text{Multi}([n_{13}, 3], n_{13} - 3) = 2$ for $n_{13} > 5$. After the Gram-Schmidt orthogonalization, the two sets of normalized nonzero expansion coefficients corresponding to $\Lambda = 0$ are

$$\tilde{c}^{(1)}_{0,0} = \sqrt{\frac{3(3n_{13}-7)^2}{(6n_{13}^3-37n_{13}^2+16n_{13}+159)(n_{13}-2)}}, \quad \tilde{c}^{(1)}_{1,0} = -\sqrt{\frac{18(n_{13}-1)(n_{13}-4)(n_{13}-5)}{(n_{13}-2)(6n_{13}^3-37n_{13}^2+16n_{13}+159)}},$$

$$\tilde{c}^{(1)}_{1,1} = -\sqrt{\frac{3(n_{13}+1)(n_{13}-1)(n_{13}-5)}{(n_{13}-2)(6n_{13}^3-37n_{13}^2+16n_{13}+159)}}, \quad \tilde{c}^{(1)}_{2,1} = \sqrt{\frac{6(n_{13}-3)(n_{13}-4)(n_{13}-5)(n_{13}-1)}{(n_{13}-2)(6n_{13}^3-37n_{13}^2+16n_{13}+159)}},$$

$$\tilde{c}^{(1)}_{2,2} = 0, \ \tilde{c}^{(1)}_{3,3} = \sqrt{\frac{8(n_{13}-3)(n_{13}-4)(n_{13}-5)}{(n_{13}-2)(6n_{13}^3-37n_{13}^2+16n_{13}+159)}},$$

$$\tilde{c}^{(2)}_{0,0} = \sqrt{\frac{288(n_{13}+1)(n_{13}-4)(n_{13}-5)}{(n_{13}-1)(n_{13}-2)(2n_{13}-3)(6n_{13}^3-37n_{13}^2+16n_{13}+159)}}, \quad \tilde{c}^{(2)}_{1,0} = -\sqrt{\frac{12(n_{13}^2-23n_{13}+57)^2(n_{13}+1)}{(n_{13}-1)^2(n_{13}-2)(2n_{13}-3)(6n_{13}^3-37n_{13}^2+16n_{13}+159)}},$$

$$\tilde{c}^{(2)}_{1,1} = -\sqrt{\frac{8(n_{13}-4)(4n_{13}^2-13n_{13}-27)^2}{(n_{13}-1)^2(n_{13}-2)(2n_{13}-3)(6n_{13}^3-37n_{13}^2+16n_{13}+159)}}, \quad \tilde{c}^{(2)}_{2,1} = \sqrt{\frac{4(n_{13}-3)(n_{13}+1)(2n_{13}^2-23n_{13}-57)^2}{(n_{13}-1)^2(n_{13}-2)(2n_{13}-3)(6n_{13}^3-37n_{13}^2+16n_{13}+159)}},$$

$$\tilde{c}^{(2)}_{2,2} = \sqrt{\frac{6n_{13}^3-37n_{13}^2+16n_{13}+159}{(n_{13}-1)^2(n_{13}-2)(2n_{13}-3)}}, \quad \tilde{c}^{(2)}_{3,3} = -\sqrt{\frac{3(n_{13}-3)(n_{13}+1)(2n_{13}^2-13n_{13}+23)^2}{(n_{13}-1)(n_{13}-2)(2n_{13}-3)(6n_{13}^3-37n_{13}^2+16n_{13}+159)}}$$

For $n_{23} = 3$ and $k = n_{13} - 4$,

$$\mathbf{P}([n_{13}, 3], n_{13} - 4) =$$

$$\begin{pmatrix} \sqrt{4(n_{13}-6)}, & \sqrt{\frac{3(n_{13}-5)}{n_{13}-2}}, & 0, & \sqrt{\frac{3(n_{13}+1)}{n_{13}-2}}, & 0, & 0, & 0, & 0 \\ 0, & -\sqrt{\frac{3(n_{13}+1)}{(n_{13}-1)^2}}, & 0, & \sqrt{3(n_{13}-5)}, & \sqrt{\frac{2(n_{13}-4)n_{13}}{(n_{13}-1)^2}}, & \sqrt{\frac{4n_{13}}{n_{13}-1}}, & 0, & 0 \\ 0, & \sqrt{\frac{2(n_{13}-4)n_{13}}{(n_{13}-1)}}, & \sqrt{\frac{2(n_{13}-3)}{n_{13}-2}}, & 0, & \sqrt{\frac{3(n_{13}+1)}{n_{13}-1}}, & 0, & 0, & 0 \\ 0, & 0, & -\sqrt{\frac{6}{n_{13}-2}}, & 0, & \sqrt{\frac{(n_{13}-3)(n_{13}+1)}{n_{13}-1}}, & 0, & \sqrt{\frac{4n_{13}}{n_{13}-1}}, & 0 \\ 0, & 0, & 0, & 0, & -\sqrt{\frac{4}{(n_{13}+1)(n_{13}-1)}}, & \sqrt{\frac{2(n_{13}-4)}{n_{13}+1}}, & \sqrt{\frac{n_{13}-3}{(n_{13}-1)n_{13}}}, & \sqrt{\frac{3(n_{13}-1)}{(n_{13}+1)n_{13}}} \\ 0, & 0, & 0, & 0, & 0, & 0, & -\sqrt{\frac{3}{n_{13}+1}}, & \sqrt{n_{13}-3} \end{pmatrix}. \tag{46}$$

Since there are two rows with all entries zero in (46) when $n_{13} > 6$, the multiplicity of $L = n_{13} - 4$ is $\text{Multi}([n_{13}, 3], n_{13} - 4) = 2$ for $n_{13} > 6$. The two sets of nonzero expansion coefficients corresponding to $\Lambda = 0$ are

$$c_{0,0}^{(1)} = \frac{\sqrt{(n_{13}+1)(n_{13}-2)}(n_{13}^2+5n_{13}-30)}{\sqrt{48(n_{13}-6)(n_{13}-5)(n_{13}-4)}}, \quad c_{1,0}^{(1)} = -\frac{\sqrt{n_{13}+1}(n_{13}^2+5n_{13}-18)}{6\sqrt{2(n_{13}-4)}}, \quad c_{2,0}^{(1)} = \sqrt{\frac{(n_{13}-2)(n_{13}-3)(n_{13}-1)(n_{13}+1)(n_{13}+6)^2}{72n_{13}}},$$

$$c_{1,1}^{(1)} = -\frac{n_{13}^2+n_{13}-10}{\sqrt{8(n_{13}-4)(n_{13}-5)}}, \quad c_{2,1}^{(1)} = -\frac{(n_{13}-2)(n_{13}+3)}{2\sqrt{n_{13}}}, \quad c_{2,2}^{(1)} = 0, \quad c_{3,2}^{(1)} = \sqrt{\frac{(n_{13}+1)(n_{13}-3)}{3}}, \quad c_{3,3}^{(1)} = 1;$$

$$c_{0,0}^{(2)} = \sqrt{\frac{(n_{13}+1)n_{13}(n_{13}-1)(n_{13}-2)}{8(n_{13}-5)(n_{13}-6)}}, \quad c_{1,0}^{(2)} = -\sqrt{\frac{(n_{13}^2-1)n_{13}}{12}}, \quad c_{2,0}^{(2)} = \frac{1}{\sqrt{12}}\sqrt{(n_{13}-3)(n_{13}-4)(n_{13}-2)(n_{13}+1)},$$

$$c_{1,1}^{(2)} = -\sqrt{\frac{3(n_{13}-1)n_{13}}{4(n_{13}-5)}}, \quad c_{2,1}^{(2)} = \frac{1}{\sqrt{2}}\sqrt{(n_{13}-4)(n_{13}-1)}, \quad c_{2,2}^{(2)} = 1, \quad c_{3,2}^{(2)} = 0, \quad c_{3,3}^{(1)} = 0$$

for $n_{13} > 5$, which are not normalized and non-orthogonal.

For $n_{23} = 4$ and $k = n_{13} - 2$,

$$\mathbf{P}([n_{13}, 4], n_{13} - 2) = \begin{pmatrix} \sqrt{2n_{13}(n_{13}-5)}, & \sqrt{n_{13}-4}, & \sqrt{4(n_{13}+1)}, & 0 \\ 0, & -2, & \sqrt{(n_{13}-4)(n_{13}+1)}, & \sqrt{6n_{13}} \end{pmatrix}. \tag{47}$$

Since there are two rows with all entries zero in (47) when $n_{13} > 5$, the multiplicity of $L = n_{13}-2$ is Multi$([n_{13}, 4], n_{13} - 2) = 2$ for $n_{13} > 5$. After the Gram-Schmidt orthogonalization, the two sets of normalized nonzero expansion coefficients corresponding to $\Lambda = 0$ are

$$\tilde{c}_{0,0}^{(1)} = \sqrt{\frac{3(n_{13}-4)}{6n_{13}^2 - 23n_{13} - 32}}, \quad \tilde{c}_{1,0}^{(1)} = -\sqrt{\frac{6(n_{13}-5)n_{13}}{6n_{13}^2 - 23n_{13} - 32}}, \quad \tilde{c}_{1,1}^{(1)} = 0, \quad \tilde{c}_{2,2}^{(1)} = -\sqrt{\frac{4(n_{13}-5)}{6n_{13}^2 - 23n_{13} - 32}}$$

$$\tilde{c}_{0,0}^{(2)} = \sqrt{\frac{196(n_{13}-5)(n_{13}+1)}{(2n_{13}-1)(n_{13}-1)(6n_{13}^2 - 23n_{13} - 32)}}, \quad \tilde{c}_{1,0}^{(2)} = -\frac{2(n-8)\sqrt{2(n_{13}-4)(n_{13}+1)}}{\sqrt{(n_{13}-1)n_{13}(2n_{13}-1)(6n_{13}^2 - 23n_{13} - 32)}},$$

$$\tilde{c}_{1,1}^{(2)} = -\sqrt{\frac{2(6n_{13}^2 - 23n_{13} - 32)}{(n_{13}-1)n_{13}(2n_{13}-1)}}, \quad \tilde{c}_{1,2}^{(2)} = \sqrt{\frac{3(n_{13}+1)(n_{13}-4)(2n_{13}-9)^2}{(n_{13}-1)(2n_{13}-1)(6n_{13}^2 - 23n_{13} - 32)}}$$

for $n_{13} > 5$.

For $n_{23} = 4$ and $k = n_{13} - 3$,

$$\mathbf{P}([n_{13}, 4], n_{13} - 3) = \begin{pmatrix} \sqrt{3(n_{13}-1)(n_{13}-6)}, & \sqrt{2(n_{13}-5)}, & \sqrt{4(n_{13}+1)}, & 0, & 0, & 0 \\ 0, & -\sqrt{4(n_{13}+1)}, & \sqrt{2(n_{13}-5)n_{13}^2}, & \sqrt{(n_{13}-4)(n_{13}+1)}, & \sqrt{6n_{13}^2}, & 0 \\ 0, & \sqrt{n_{13}-4}, & 0, & 2, & 0, & 0 \\ 0, & 0, & 0, & -\sqrt{6}, & \sqrt{(n_{13}-4)(n_{13}+1)}, & \sqrt{6(n_{13}-1)} \end{pmatrix}. \tag{48}$$

Since there are two rows with all entries zero in (48) when $n_{13} > 6$, the multiplicity of $k = n_{13}-3$ is Multi$([n_{13}, 4], n_{13} - 3) = 2$ for $n_{13} > 6$. After the Gram-Schmidt orthogonalization, the two sets of normalized nonzero expansion coefficients corresponding to $\Lambda = 0$ are

$$\tilde{c}_{0,0}^{(1)} = \sqrt{\frac{12(n_{13}-3)^2}{(2n_{13}^3-17n_{13}^2+24n_{13}+63)(n_{13}-2)}}, \quad \tilde{c}_{1,0}^{(1)} = -\sqrt{\frac{8(n_{13}-1)(n_{13}-5)(n_{13}-6)}{(n_{13}-2)(2n_{13}^3-17n_{13}^2+24n_{13}+63)}},$$

$$\tilde{c}_{1,1}^{(1)} = -\sqrt{\frac{(n_{13}+1)(n_{13}-1)(n_{13}-6)}{(n_{13}-2)(2n_{13}^3-17n_{13}^2+24n_{13}+63)}}, \quad \tilde{c}_{2,1}^{(1)} = \sqrt{\frac{2(n_{13}-4)(n_{13}-5)(n_{13}-6)(n_{13}-1)}{(n_{13}-2)(2n_{13}^3-17n_{13}^2+24n_{13}+63)}},$$

$$\tilde{c}_{2,2}^{(1)} = 0, \quad \tilde{c}_{3,3}^{(1)} = \sqrt{\frac{2(n_{13}-4)(n_{13}-5)(n_{13}-6)}{(n_{13}-2)(2n_{13}^3-17n_{13}^2+24n_{13}+63)}},$$

$$\tilde{c}_{0,0}^{(2)} = \sqrt{\frac{216(n_{13}+1)(n_{13}-5)(n_{13}-6)}{(n_{13}-1)(n_{13}-2)(2n_{13}-3)(2n_{13}^3-17n_{13}^2+24n_{13}+63)}}, \quad \tilde{c}_{1,0}^{(2)} = -\sqrt{\frac{16(n_{13}^2-16n_{13}+51)^2(n_{13}+1)}{(n_{13}-1)^2(n_{13}-2)(2n_{13}-3)(2n_{13}^3-17n_{13}^2+24n_{13}+63)}},$$

$$\tilde{c}_{1,1}^{(2)} = -\sqrt{\frac{2(n_{13}-5)(7n_{13}^2-31n_{13}-48)^2}{(n_{13}-1)^2(n_{13}-2)(2n_{13}-3)(2n_{13}^3-17n_{13}^2+24n_{13}+63)}}, \quad \tilde{c}_{2,1}^{(2)} = \sqrt{\frac{4(n_{13}-4)(n_{13}+1)(n_{13}^2-16n_{13}-51)^2}{(n_{13}-1)^2(n_{13}-2)(2n_{13}-3)(2n_{13}^3-17n_{13}^2+24n_{13}+63)}},$$

$$\tilde{c}_{2,2}^{(2)} = \sqrt{\frac{6(2n_{13}^3-17n_{13}^2+24n_{13}+63)}{(n_{13}-1)^2(n_{13}-2)(2n_{13}-3)}}, \quad \tilde{c}_{3,3}^{(2)} = -\sqrt{\frac{(n_{13}-4)(n_{13}+1)(2n_{13}^2-17n_{13}+39)^2}{(n_{13}-1)(n_{13}-2)(2n_{13}-3)(2n_{13}^3-17n_{13}^2+24n_{13}+63)}}.$$

For $n_{23}=4$ and $k=n_{13}-4$,

$$\mathbf{P}([n_{13},4],n_{13}-4)=$$
$$\begin{pmatrix}
\sqrt{4(n_{13}-7)}, & \sqrt{\frac{3(n_{13}-6)}{n_{13}-2}}, & 0, & \sqrt{\frac{4(n_{13}+1)}{n_{13}-2}}, & 0, & 0, & 0, & 0, & 0 \\
0, & -\sqrt{\frac{4(n_{13}+1)}{(n_{13}-1)^2}}, & 0, & \sqrt{3(n_{13}-6)}, & \sqrt{\frac{2(n_{13}-5)n_{13}}{(n_{13}-1)^2}}, & \sqrt{\frac{6n_{13}}{n_{13}-1}}, & 0, & 0, & 0 \\
0, & \sqrt{\frac{2(n_{13}-5)n_{13}}{(n_{13}-1)}}, & \sqrt{\frac{2(n_{13}-4)}{n_{13}-2}}, & 0, & \sqrt{\frac{4(n_{13}+1)}{n_{13}-1}}, & 0, & 0, & 0, & 0 \\
0, & 0, & -\sqrt{\frac{8}{n_{13}-2}}, & 0, & \sqrt{\frac{(n_{13}-4)(n_{13}+1)}{n_{13}-1}}, & 0, & \sqrt{\frac{6n_{13}}{n_{13}-1}}, & 0, & 0 \\
0, & 0, & 0, & 0, & -\sqrt{\frac{6}{(n_{13}+1)(n_{13}-1)}}, & \sqrt{\frac{2(n_{13}-5)}{n_{13}+1}}, & \sqrt{\frac{n_{13}-4}{(n_{13}-1)n_{13}}}, & \sqrt{\frac{6(n_{13}-1)}{(n_{13}+1)n_{13}}}, & 0 \\
0, & 0, & 0, & 0, & 0, & 0, & -\sqrt{\frac{6}{n_{13}+1}}, & \sqrt{n_{13}-4}, & \sqrt{\frac{4(n_{13}-2)}{n_{13}+1}}
\end{pmatrix}. \quad (49)$$

Since there are three rows with all entries zero in (49) when $n_{13}>7$, the multiplicity of $L=n_{13}-4$ is $\mathrm{Multi}([n_{13},4],n_{13}-4)=3$ for $n_{13}>7$. The two sets of nonzero expansion coefficients corresponding to $\Lambda=0$ are

$$c^{(1)}_{0,0}=\frac{\sqrt{n_{13}-4}(n_{13}^2+5n_{13}-38)}{\sqrt{48(n_{13}-7)(n_{13}-6)(n_{13}-5)}},\quad c^{(1)}_{1,0}=-\frac{\sqrt{(n_{13}-4)(n_{13}-2)}(n_{13}+7)}{6\sqrt{2(n_{13}-5)}},$$
$$c^{(1)}_{2,0}=\sqrt{\frac{(n_{13}-2)^2(n_{13}+4)^2(n_{13}-1)}{72n_{13}}},\quad c^{(1)}_{1,1}=\sqrt{\frac{(n_{13}-2)(n_{13}-4)(n_{13}+1)}{6(n_{13}-6)(n_{13}-5)}},$$
$$c^{(1)}_{2,1}=\sqrt{\frac{(n_{13}-4)(n_{13}-2)(n_{13}+1)}{9n_{13}}},\quad c^{(1)}_{2,2}=0,\quad c^{(1)}_{3,2}=\sqrt{\frac{2(n_{13}-2)}{3}},\quad c^{(1)}_{3,3}=0,\ \ c^{(1)}_{4,4}=1;$$

$$c^{(2)}_{0,0}=\frac{\sqrt{(n_{13}+1)(n_{13}-2)}(n_{13}^2+9n_{13}-58)}{\sqrt{192(n_{13}-5)(n_{13}-6)(n_{13}-7)}},\quad c^{(2)}_{1,0}=-\frac{\sqrt{(n_{13}+1)}(n_{13}^2+9n_{13}-34)}{12\sqrt{2(n_{13}-5)}},$$
$$c^{(2)}_{2,0}=\frac{n_{13}+10}{12\sqrt{2n_{13}}}\sqrt{(n_{13}-1)(n_{13}-2)(n_{13}-4)(n_{13}+1)},\quad c^{(2)}_{1,1}=-\frac{n_{13}^2+3n_{13}-22}{\sqrt{24(n_{13}-5)(n_{13}-6)}},$$
$$c^{(2)}_{2,1}=\frac{1}{6\sqrt{n_{13}}}\sqrt{(n_{13}-2)(n_{13}+5)},\qquad c^{(2)}_{2,2}=0,\quad c^{(2)}_{3,2}=\frac{1}{\sqrt{6}}\sqrt{(n_{13}-4)(n_{13}+1)},\quad c^{(2)}_{3,3}=1\quad c^{(2)}_{4,4}=0;$$

$$c^{(3)}_{0,0}=\sqrt{\frac{(n_{13}+1)n_{13}(n_{13}-1)(n_{13}^2-7n_{13}+10)}{16(n_{13}-5)(n_{13}-6)(n_{13}-7)}},\quad c^{(3)}_{1,0}=-\frac{\sqrt{(n_{13}+1)(n_{13}-1)n_{13}}}{2\sqrt{6}},$$
$$c^{(3)}_{2,0}=\frac{1}{2\sqrt{6}}\sqrt{(n_{13}-2)(n_{13}-4)(n_{13}-5)(n_{13}+1)},\quad c^{(3)}_{1,1}=-\sqrt{\frac{(n_{13}-1)n_{13}(n_{13}^2-7n_{13}+10)}{2(n_{13}-5)(n_{13}-6)(n_{13}-2)}},$$
$$c^{(3)}_{2,1}=\frac{1}{3}\sqrt{(n_{13}-1)(n_{13}+5)},\quad c^{(3)}_{2,2}=1,\quad c^{(3)}_{3,2}=0,\quad c^{(3)}_{3,3}=0,\quad c^{(3)}_{4,4}=0$$

for $n_{13}>7$, which are not normalized and non-orthogonal. From these examples, one can check that the multiplicity $\mathrm{Multi}([n_{13},n_{23}],L)$ in the reduction $SU(3)\downarrow SO(3)$ for the irrep $[n_{13},\ n_{23}]\downarrow L$ determined by Eq. (28) is indeed consistent with the formula (34). The advantage of the projection (28) lies in the fact that the Hill-Wheeler integral involved in the projection operator method used previously [1] can be avoided, and the null-space vectors of the projection matrix $\mathbf{P}$ can now be obtained easily, e. g., by using the built-in function NullSpace[$\mathbf{P}$] in Wolfram Mathematica. Then, the Gram-Schmidt orthogonalization of the eigenvectors $\{\mathbf{c}^{(\zeta)}\}$ of $\mathbf{P}$ should be carried out, e. g., by using the built-in function Orthogonalize[$\mathbf{c}^{(1)},\dots,\mathbf{c}^{(\mathrm{Multi})}$] in Wolfram Mathematica if $\mathrm{Multi}\equiv\mathrm{Multi}([n_{13},n_{23}],L)>1$.

IV. MATRIX REPRESENTATIONS OF $SU(3)\supset SO(3)$

Once the orthonormalized expansion coefficients $\tilde{\mathbf{c}}^{(\zeta)}$ are obtained according to the angular momentum projection method shown in the previous section, one can easily calculate matrix elements of $SU(3)$ generators $\{Q_\mu,\ L_\mu\}$ given in (11). Since matrix elements of the angular momentum operators $\{L_\mu\}$ are well-known, which are irrelevant to the irreps of $SU(3)$ involved, only formulae for the matrix elements of Q_μ in the $SU(3)\supset SO(3)\supset SO(2)$ basis will be provided.

In the $SU(3)\supset SO(3)\supset SO(2)$ basis, Q_μ are rank-2 irreducible tensor operators of $SO(3)$ and using the Wigner-Eckart theorem for matrix elements for the $SU(3)\supset SO(3)\supset SO(2)$ case, we have

$$\left\langle \begin{matrix}[n_{13},n_{23}]\\ \zeta'\ L'\ M'\end{matrix}\right| Q_\mu \left|\begin{matrix}[n_{13},n_{23}]\\ \zeta\ L\ M\end{matrix}\right\rangle=\langle LM,2\mu|L'\ M'\rangle\left\langle \begin{matrix}[n_{13},n_{23}]\\ \zeta'\ L'\end{matrix}\right\| Q \left\|\begin{matrix}[n_{13},n_{23}]\\ \zeta\ L\end{matrix}\right\rangle, \quad (50)$$

where $\langle LM, 2\mu | L' \ M' \rangle$ is the CG coefficient of $SO(3)$, and $\left\langle \begin{matrix} [n_{13}, n_{23}] \\ \zeta' \ L' \end{matrix} \middle\| Q \middle\| \begin{matrix} [n_{13}, n_{23}] \\ \zeta \ L \end{matrix} \right\rangle$ is the $SO(3)$-reduced matrix element. In the calculation, we ensure that L' always exists in the $SO(3)$ coupling $L \otimes 2$ and that $M' = M + \mu$ is always satisfied. By using (22) and the expressions of Q_μ in terms of $\{E_{ij}\}$ shown in (11), the left-hand-side of (50) can be expressed in terms of the expansion coefficients $\tilde{\mathbf{c}}^{(\zeta)}$ and the matrix elements of $\{E_{ij}\}$ in the canonical basis of $U(3)$ given in (5)-(9). In the following, we list nonzero $SO(3)$-reduced matrix elements of Q derived in this way:

$$\left\langle \begin{matrix} [n_{13}, n_{23}] \\ \zeta' \ L+2 \end{matrix} \middle\| Q \middle\| \begin{matrix} [n_{13}, n_{23}] \\ \zeta \ L \end{matrix} \right\rangle = \sum_{q,t} \tilde{c}^{(\zeta')}_{q-1,t}(L+2) \tilde{c}^{(\zeta)}_{q,t}(L) \sqrt{(L+q-t+1)(q-t)}, \tag{51}$$

$$\begin{aligned} \left\langle \begin{matrix} [n_{13}, n_{23}] \\ \zeta' \ L+1 \end{matrix} \middle\| Q \middle\| \begin{matrix} [n_{13}, n_{23}] \\ \zeta \ L \end{matrix} \right\rangle = & \sqrt{\tfrac{L+2}{2L}} \textstyle\sum_{q,t} \tilde{c}^{(\zeta')}_{q-1,t}(L+1) \tilde{c}^{(\zeta)}_{q,t}(L) \sqrt{\tfrac{(q-t)(n_{13}-L-2q+t+1)(L+2q-t-n_{23})(L+2q-t+1)}{(L+2q-2t)(L+2q-2t+1)}} + \\ & \sqrt{\tfrac{L+2}{2L}} \textstyle\sum_{q,t} \tilde{c}^{(\zeta')}_{q-1,t-1}(L+1) \tilde{c}^{(\zeta)}_{q,t}(L) \sqrt{\tfrac{(L+q-t+1)(n_{13}-t+2)(n_{23}-t+1)t}{(L+2q-2t+1)(L+2q-2t+2)}} + \\ & \sqrt{\tfrac{L+2}{2L}} \textstyle\sum_{q,t} \tilde{c}^{(\zeta')}_{q,t+1}(L+1) \tilde{c}^{(\zeta)}_{q,t}(L) \sqrt{\tfrac{(q-t)(n_{13}-t+1)(n_{23}-t)(t+1)}{(L+2q-2t)(L+2q-2t+1)}} - \\ & \sqrt{\tfrac{L+2}{2L}} \textstyle\sum_{q,t} \tilde{c}^{(\zeta')}_{q,t}(L+1) \tilde{c}^{(\zeta)}_{q,t}(L) \sqrt{\tfrac{(L+q-t+1)(n_{13}-L-2q+t)(L+2q-t-n_{23}+1)(L+2q-t+2)}{(L+2q-2t+1)(L+2q-2t+2)}} \end{aligned} \tag{52}$$

for $L \neq 0$, and

$$\left\langle \begin{matrix} [n_{13}, n_{23}] \\ \zeta' \ L \end{matrix} \middle\| Q \middle\| \begin{matrix} [n_{13}, n_{23}] \\ \zeta \ L \end{matrix} \right\rangle = \sqrt{\frac{(L+1)(2L+3)}{6L(2L-1)}} \left(6 \sum_{q,t} q \ \tilde{c}^{(\zeta')}_{q,t}(L) \tilde{c}^{(\zeta)}_{q,t}(L) + \delta_{\zeta'\zeta}(3L - 2n_{13} - 2n_{23}) \right) \tag{53}$$

for $L \neq 0$. When $L = 0$,

$$\left\langle \begin{matrix} [n_{13}, n_{23}] \\ \zeta' \ L' \ \mu \end{matrix} \middle| Q_\mu \middle| \begin{matrix} [n_{13}, n_{23}] \\ \zeta \ 0 \ 0 \end{matrix} \right\rangle = 0 \tag{54}$$

for any μ with $L' = 0$ or 1. By using (51)-(53), other non-zero reduced matrix elements of Q can be obtained by the conjugation relation:

$$\left\langle \begin{matrix} [n_{13}, n_{23}] \\ \zeta \ L \end{matrix} \middle\| Q \middle\| \begin{matrix} [n_{13}, n_{23}] \\ \zeta' \ L' \end{matrix} \right\rangle = (-1)^{L-L'} \sqrt{\frac{2L'+1}{2L+1}} \left\langle \begin{matrix} [n_{13}, n_{23}] \\ \zeta' \ L' \end{matrix} \middle\| Q \middle\| \begin{matrix} [n_{13}, n_{23}] \\ \zeta \ L \end{matrix} \right\rangle. \tag{55}$$

The matrix representations of $SU(3) \supset SO(3) \supset SO(2)$ are thus obtained completely. Table II shows the $SO(3)$-reduced matrix elements of Q for $[n_{13}, n_{23}] = [n, 1]$ and some specific k values, which are derived according to (51)-(53).

TABLE II: Some $SO(3)$-reduced matrix elements $\left\langle \begin{matrix} [n,1] \\ L_1 \end{matrix} \middle\| Q \middle\| \begin{matrix} [n,1] \\ L \end{matrix} \right\rangle$, where the entries with "–" can be obtained from the corresponding upper part entries shown in the table by using the conjugation relation (55).

L_1	$L=n$	$L=n-1$	$L=n-2$	$L=n-3$	$L=n-4$
n	$\sqrt{\frac{(n+1)(2n+3)(n-2)^2}{6n(2n-1)}}$	$-\sqrt{\frac{2(n+1)}{n}}$	$-\sqrt{\frac{2(n+1)(n-2)}{2n-1}}$	0	0
$n-1$	–	$\sqrt{\frac{(2n+1)(n+3)^2(n-2)^2}{6n(n-1)(2n-3)}}$	$-\sqrt{\frac{4(n+1)}{(2n-1)(n-1)}}$	$-\sqrt{\frac{2n(n-3)}{2n-3}}$	0
$n-2$	–	–	$\frac{2n^3-7n^2+n-8}{\sqrt{6(n-1)(n-2)(2n-1)(2n-5)}}$	$-\sqrt{\frac{2(n+1)(2n-1)}{(n-2)(2n-3)}}$	$-\sqrt{\frac{4(n-1)(n-4)(2n-1)}{(2n-3)(2n-5)}}$

Alternatively, by using the Racah factorization lemma, the $SO(3)$-reduced matrix elements of Q can also be expressed as

$$\left\langle \begin{matrix} [n_{13}, n_{23}] \\ \zeta' \ L' \end{matrix} \right\| Q \left\| \begin{matrix} [n_{13}, n_{23}] \\ \zeta \ L \end{matrix} \right\rangle = \langle [n_{13}, n_{23}], \rho = 1 \| Q \| [n_{13}, n_{23}] \rangle \left\langle \begin{matrix} [n_{13}, n_{23}] & [2,1] \\ \zeta \ L & 2 \end{matrix} \middle| \begin{matrix} [n_{13}, n_{23}], \ \rho = 1 \\ \zeta' \ L' \end{matrix} \right\rangle, \tag{56}$$

where $\langle [n_{13}, n_{23}], \rho = 1 \| Q \| [n_{13}, n_{23}] \rangle$ is the SU(3)-reduced matrix element given by

$$\langle [n_{13}, n_{23}], \rho = 1 \| Q \| [n_{13}, n_{23}] \rangle = \sqrt{\langle C_2(SU(3)) \rangle}, \tag{57}$$

in which $\langle C_2(SU(3)) \rangle$ is given by (15), and $\left\langle \begin{matrix} [n_{13}, n_{23}] & [2,1] \\ \zeta \ L & 2 \end{matrix} \middle| \begin{matrix} [n_{13}, n_{23}], \ \rho = 1 \\ \zeta' \ L' \end{matrix} \right\rangle$ is the Wigner coefficient of $SU(3) \supset SO(3)$, where ρ is the outer-multiplicity label needed in the coupling $[n_{13}, n_{23}] \otimes [2,1] \downarrow [n_{13}, n_{23}]$. For example, one may check against the results shown in Table II derived according to (51)-(53) that the Wigner coefficients of $SU(3) \supset SO(3)$ obtained according to (56) with the SU(3)-reduced matrix element given in (57) are consistent with the results listed in [25].

V. WIGNER COEFFICIENTS OF $SU(3) \supset SO(3)$

Since the basis vectors of $SU(3) \supset SO(3) \supset SO(2)$ are expanded in terms of those of $U(3) \supset U(2) \supset U(1)$, the correspondence (10) will be useful when one considers the Kronecker product $[n_{13}, n_{23}, 0] \otimes [n'_{13}, n'_{23}, 0] \downarrow [n''_{13}, n''_{23}, n''_{33}]$, in which $[n''_{13}, n''_{23}, n''_{33}]$ may contain three-rowed irreps with $n''_{33} \neq 0$. Using the correspondence (10), we have

$$\left| \begin{matrix} [n_{13} - n_{33}, n_{23} - n_{33}] \\ \zeta \ L = M = k \end{matrix} \right\rangle = \sum_{t,q} \tilde{c}^{(\zeta)}_{q - n_{33}, t - n_{33}}([n_{13} - n_{33}, n_{23} - n_{33}, 0], L) \left| \begin{matrix} [n_{13}, n_{23}, n_{33}] \\ [k + 2q - t, t] \\ k + q \end{matrix} \right\rangle \tag{58}$$

because the basis vectors of $SU(3) \supset SO(3) \supset SO(2)$ are always expanded in terms of those of $U(3) \supset U(2) \supset U(1)$, which will be used in the following, while the expansion coefficients $\tilde{\mathbf{c}}^{(\zeta)}$ are derived from $n_{33} = 0$ cases only.

Since the $SO(3)$ highest-weight state (22) is known, a general $SU(3) \supset SO(3) \supset SO(2)$ basis vector can be expressed as

$$\left| \begin{matrix} [n_{13}, n_{23}] \\ \zeta \ L, M \end{matrix} \right\rangle = \sqrt{\tfrac{(L+M)!}{(2L)!(L-M)!}} \, (L_-)^{L-M} \left| \begin{matrix} [n_{13}, n_{23}] \\ \zeta \ L, L \end{matrix} \right\rangle =$$

$$\sqrt{\tfrac{(L+M)! 2^{L-M}}{(2L)!(L-M)!}} \sum\nolimits_{q,t} \tilde{c}^{(\zeta)}_{qt}([n_{13}, n_{23}, 0], L) \, (E_{31} + E_{23})^{L-M} \left| \begin{matrix} [n_{13}, n_{23}, 0] \\ [k + 2q - t, t] \\ k + q \end{matrix} \right\rangle. \tag{59}$$

There are many ways to derive Wigner coefficients of $SU(3) \supset SO(3)$ when the expansion coefficients $\mathbf{c}^{(\zeta)}$ are known. For example, by using the explicit expressions of the matrix elements of E_{31} and E_{23} given in (5)-(8), (59) can be expressed as

$$\left| \begin{matrix} [n_{13}, n_{23}] \\ \zeta \ L, L - \mu \end{matrix} \right\rangle = \sqrt{\frac{2^\mu (2L - \mu)!}{(2L)! \mu!}} \sum_{q,t} A^\mu_{qt}([n_{13}, n_{23}, 0], L, \zeta) \left| \begin{matrix} [n_{13}, n_{23}, 0] \\ [L - \mu + 2q - t, t] \\ L - \mu + q \end{matrix} \right\rangle, \tag{60}$$

where the expansion coefficients $A^\mu_{q,t}([n_{13}, n_{23}, 0], L, \zeta)$ can be obtained by consecutively applying the generators $E_{31} + E_{23}$ on to the highest-weight state in (59) with

$$A^0_{q,t}([n_{13}, n_{23}, 0], L, \zeta) = \tilde{c}^{(\zeta)}_{q,t}([n_{13}, n_{23}, 0], L),$$

$$\begin{aligned} A^{\mu+1}_{qt}([n_{13}, n_{23}, 0], L, \zeta) = {} & A^\mu_{q,t}([n_{13}, n_{23}, 0], L, \zeta) \alpha_1([n_{13}, n_{23}, 0], L, \mu, q, t) + \\ & A^\mu_{q,t+1}([n_{13}, n_{23}, 0], L, \zeta) \alpha_2([n_{13}, n_{23}, 0], L, \mu, q, t) + \\ & A^\mu_{q-1,t}([n_{13}, n_{23}, 0], L, \zeta) \beta_1([n_{13}, n_{23}, 0], L, \mu, q, t) + A^\mu_{q-1,t-1}([n_{13}, n_{23}, 0], L, \zeta) \beta_2([n_{13}, n_{23}, 0], L, \mu, q, t), \end{aligned} \tag{61}$$

in which

$$\alpha_1([n_{13},n_{23},0],L,\mu,q,t)=\sqrt{\tfrac{(L+q-t-\mu)(1+L+2q-t-\mu)(L-n_{23}+2q-t-\mu)(1-L+n_{13}-2q+t+\mu)}{(L+2q-2t-\mu)(1+L+2q-2t-\mu)}},$$
$$\alpha_2([n_{13},n_{23},0],L,\mu,q,t)=-\sqrt{\tfrac{(1+n_{13}-t)(n_{23}-t)(q-t)(1+t)}{(-1+L+2q-2t-\mu)(L+2q-2t-\mu)}},$$
$$\beta_1([n_{13},n_{23},0],L,\mu,q,t)=\sqrt{\tfrac{(q-t)(L+2q-t-\mu)(2-L+n_{13}-2q+t+\mu)(-1+L-n_{23}+2q-t-\mu)}{(-1+L+2q-2t-\mu)(L+2q-2t-\mu)}},$$
$$\beta_2([n_{13},n_{23},0],L,\mu,q,t)=\sqrt{\tfrac{(2+n_{13}-t)t(-1-n_{23}+t)(-L-q+t+\mu)}{(-1-L-2q+2t+\mu)(-L-2q+2t+\mu)}}. \tag{62}$$

Similar to the basis vectors, any irreducible tensor operators of $SU(3)$ in the $SU(3)\supset SO(3)$ basis can be expanded in terms of those in the $U(3)\supset U(2)\supset U(1)$ basis with

$$T^{[n'_{13},n'_{23}]}_{\zeta'\ L',L'}=\sum_{t_2q_2}\tilde{c}^{(\zeta)}_{q_2,t_2}([n'_{13},n'_{23},0],L')T^{[n'_{13},n'_{23},0]}_{[L'+2q_2-t_2,t_2],L'+q_2}. \tag{63}$$

As long as the $SU(3)$-reduced matrix element $\left\langle[n''_{13}-n''_{33},n''_{23}-n''_{33}],\rho\|T^{[n'_{13},n'_{23}]}\|[n_{13},n_{23}]\right\rangle$ is nonzero, which is also the $U(3)$-reduced matrix element $\left\langle[n''_{13},n''_{23},n''_{23}],\rho\|T^{[n'_{13},n'_{23},0]}\|[n_{13},n_{23},0]\right\rangle$, using the factorization lemma similar to that used for (56) and the expansion formulae shown in (22) and (60), one can express the related Wigner coefficient of $SU(3)\supset SO(3)$ in terms those of $U(3)\supset U(2)\supset U(1)$ from matrix elements of $\mathbf{T}^{[n'_{13},n'_{23}]}$ given in (63) with

$$\left\langle\begin{matrix}[n_{13},n_{23}] & [n'_{13},n'_{23}]\\ \zeta\ L & \zeta'\ L'\end{matrix}\middle|\begin{matrix}[n''_{13}-n''_{33},n''_{23}-n''_{33}],\rho\\ \zeta''\ L''\end{matrix}\right\rangle=\langle L\ L-\mu,L'\ L'|L''\ L''\rangle^{-1}\sqrt{\tfrac{(2L-\mu)!}{(2L)!\mu!}}\times$$
$$\sum_{q_it_i}A^{\mu}_{q_1,t_1}([n_{13},n_{23},0],L,\zeta)\tilde{c}^{(\zeta)}_{q_2,t_2}([n'_{13},n'_{23},0],L')\tilde{c}^{(\zeta')}_{q_3-n''_{33},t_3-n''_{33}}([n''_{13}-n''_{33},n''_{23}-n''_{33},0],L'')\times$$
$$\left\langle\begin{matrix}[n_{13},n_{23},0] & [n'_{13},n'_{23},0]\\ [\mu+2q_1-t_1,t_1] & [L'+2q_2-t_2,t_2]\end{matrix}\middle|\begin{matrix}[n''_{13},n''_{23},n''_{33}],\rho\\ [L''+2q_3-t_3,t_3]\end{matrix}\right\rangle\times$$
$$\left\langle\begin{matrix}[\mu+2q_1-t_1,t_1] & [L'+2q_2-t_2,t_2]\\ \mu+q_1 & L'+q_2\end{matrix}\middle|\begin{matrix}[L''+2q_3-t_3,t_3]\\ L''+q_3\end{matrix}\right\rangle \tag{64}$$

for $\mu=0,\ 1,\ \ldots,\ L$, where $L''=L'+L-\mu$ is assumed, $\langle L\ M,L'\ M'|L''\ M''\rangle$ is the CG coefficient of $SO(3)$, and $\left\langle\begin{matrix}[n_{13},n_{23},0] & [n'_{13},n'_{23},0]\\ [n_{12},n_{22}] & [n'_{12},n'_{22}]\end{matrix}\middle|\begin{matrix}[n''_{13},n''_{23},n''_{33}],\rho\\ [n''_{12},n''_{22}]\end{matrix}\right\rangle$ and $\left\langle\begin{matrix}[n_{12},n_{22}] & [n'_{12},n'_{22}]\\ n_{11} & n'_{11}\end{matrix}\middle|\begin{matrix}[n''_{12},n''_{22}]\\ n''_{11}\end{matrix}\right\rangle$ are the Wigner coefficient of $U(3)\supset U(2)$ and that of $U(2)\supset U(1)$, respectively, in which ρ is the outer-multiplicity label needed in the $SU(3)$ coupling $[n_{13},n_{23}]\otimes[n'_{13},n'_{23}]\downarrow[n''_{13},n''_{23},n''_{33}]$. The $U(3)\supset U(2)$ Wigner coefficients with outer-multiplicity needed in (64) have been available numerically [10, 12], while the Wigner coefficients of $U(2)\supset U(1)$ can be expressed as the CG coefficients of $SU(2)$ with

$$\left\langle\begin{matrix}[n_{12},n_{22}] & [n'_{12},n'_{22}]\\ n_{11} & n'_{11}\end{matrix}\middle|\begin{matrix}[n''_{12},n''_{22}]\\ n''_{11}\end{matrix}\right\rangle=$$
$$\left\langle\tfrac{1}{2}(n_{12}-n_{22}),\ n_{11}-\tfrac{1}{2}(n_{12}+n_{22});\ \ \tfrac{1}{2}(n'_{12}-n'_{22}),\ n'_{11}-\tfrac{1}{2}(n'_{12}+n'_{22})\mid\tfrac{1}{2}(n''_{12}-n''_{22}),\ n''_{11}-\tfrac{1}{2}(n''_{12}+n''_{22})\right\rangle. \tag{65}$$

When the rank of the irreducible tensor $\mathbf{T}^{[n'_{13},n'_{23}]}$ is low, one can avoid the recursive process for the expansion coefficients shown in (61). For example, for the fundamental representation of $SU(3)$, using the L_- operator similar to (59), we can introduce the corresponding rank-1 irreducible tensor operators of $SU(3)$ with

$$T^{[1,0]}_{L=1,\ M=1}=T^{[1,0,0]}_{[1,0],\ 1},\quad T^{[1,0]}_{L=1,\ M=0}=T^{[1,0,0]}_{[0,0],\ 0},\quad T^{[1,0]}_{L=1,\ M=-1}=T^{[1,0,0]}_{[1,0],\ 0}, \tag{66}$$

where the right-hand-side is expressed as the irreducible tensor operator of $U(3)\supset U(2)\supset U(1)$. As long as the $SU(3)$-reduced matrix element, $\langle[n'_{13}-n'_{33},n'_{23}-n'_{33}]\|T^{[1,0]}\|[n_{13},n_{23}]\rangle$ is nonzero, using the factorization lemma similar to that used for (64) and the expansion shown in (22), one can express elementary Wigner coefficients of

$SU(3) \supset SO(3)$ in terms of those of $U(3) \supset U(2) \supset U(1)$ from matrix elements of $\mathbf{T}^{[1,0]}$ given in (66) with

$$\left\langle \begin{array}{cc} [n_{13},n_{23}] & [1,0] \\ \zeta\ L & 1 \end{array} \middle| \begin{array}{c} [n'_{13}-n'_{33},n'_{23}-n'_{33}] \\ \zeta'\ L+1 \end{array} \right\rangle = \sum_{qt} \tilde{c}^{(\zeta')}_{q-n'_{33},t-n'_{33}}([n'_{13},n'_{23},n'_{33}],L+1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23},0],L)\times$$
$$\left\langle \begin{array}{cc} [n_{13},n_{23},0] & [1,0,0] \\ [L+2q-t,t] & [1,0] \end{array} \middle| \begin{array}{c} [n'_{13},n'_{23},n'_{33}] \\ [L+2q-t+1,t] \end{array} \right\rangle \left\langle \begin{array}{cc} [L+2q-t,t] & [1,0] \\ L+q & 1 \end{array} \middle| \begin{array}{c} [L+2q-t+1,t] \\ L+q+1 \end{array} \right\rangle +$$
$$\sum_{qt} \tilde{c}^{(\zeta')}_{q-n'_{33},t+1-n'_{33}}([n'_{13},n'_{23},n'_{33}],L+1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23},0],L)\times$$
$$\left\langle \begin{array}{cc} [n_{13},n_{23},0] & [1,0,0] \\ [L+2q-t,t] & [1,0] \end{array} \middle| \begin{array}{c} [n'_{13},n'_{23},n'_{33}] \\ [L+2q-t,t+1] \end{array} \right\rangle \left\langle \begin{array}{cc} [L+2q-t,t] & [1,0] \\ L+q & 1 \end{array} \middle| \begin{array}{c} [L+2q-t,t+1] \\ L+q+1 \end{array} \right\rangle, \tag{67}$$

$$\left\langle \begin{array}{cc} [n_{13},n_{23}] & [1,0] \\ \zeta\ L & 1 \end{array} \middle| \begin{array}{c} [n'_{13}-n'_{33},n'_{23}-n'_{33}] \\ \zeta'\ L \end{array} \right\rangle = \sqrt{\tfrac{L+1}{L}} \sum_{qt} \tilde{c}^{(\zeta')}_{q-n'_{33},t-n'_{33}}([n'_{13},n'_{23},n'_{33}],L+1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23},0],L)\times$$
$$\left\langle \begin{array}{cc} [n_{13},n_{23},0] & [1,0,0] \\ [L+2q-t,t] & [0,0] \end{array} \middle| \begin{array}{c} [n'_{13},n'_{23},n'_{33}] \\ [L+2q-t,t] \end{array} \right\rangle \tag{68}$$

for $L \geq 1$, and

$$\left\langle \begin{array}{cc} [n_{13},n_{23}] & [1,0] \\ \zeta\ L & 1 \end{array} \middle| \begin{array}{c} [n'_{13}-n'_{33},n'_{23}-n'_{33}] \\ \zeta'\ L-1 \end{array} \right\rangle = \sqrt{\tfrac{2L+1}{2L-1}} \sum_{qt} \tilde{c}^{(\zeta')}_{q+1-n'_{33},t+1-n'_{33}}([n'_{13},n'_{23},n'_{33}],L-1)\times$$
$$\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23},0],L)\left\langle \begin{array}{cc} [n_{13},n_{23},0] & [1,0,0] \\ [L+2q-t,t] & [1,0] \end{array} \middle| \begin{array}{c} [n'_{13},n'_{23},n'_{33}] \\ [L+2q-t+1,t] \end{array} \right\rangle \left\langle \begin{array}{cc} [L+2q-t,t] & [1,0] \\ L+q & 0 \end{array} \middle| \begin{array}{c} [L+2q-t+1,t] \\ L+q \end{array} \right\rangle +$$
$$\sqrt{\tfrac{2L+1}{2L-1}} \sum_{qt} \tilde{c}^{(\zeta')}_{q+1-n'_{33},t+1-n'_{33}}([n'_{13},n'_{23},n'_{33}],L-1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23},0],L)\times$$
$$\left\langle \begin{array}{cc} [n_{13},n_{23},0] & [1,0,0] \\ [L+2q-t,t] & [1,0] \end{array} \middle| \begin{array}{c} [n'_{13},n'_{23},n'_{33}] \\ [L+2q-t,t+1] \end{array} \right\rangle \left\langle \begin{array}{cc} [L+2q-t,t] & [1,0] \\ L+q & 0 \end{array} \middle| \begin{array}{c} [L+2q-t,t+1] \\ L+q \end{array} \right\rangle \tag{69}$$

for $L \geq 1$.

TABLE III: Elementary $U(3) \supset U(2)$ Wigner coefficients $\left\langle \begin{array}{cc} [n_{13},n_{23},0] & [1,0,0] \\ [n'_{12},n'_{22}] & [\tau,0] \end{array} \middle| \begin{array}{c} [n'_{13},n'_{23},n'_{33}] \\ [n_{12},n_{22}] \end{array} \right\rangle$.

$[n'_{12},n'_{22}]$,	τ	$[n'_{13},n'_{23},n'_{33}]=[n_{13}+1,n_{23},0]$	$[n'_{13},n'_{23},n'_{33}]=[n_{13},n_{23}+1,0]$	$[n'_{13},n'_{23},n'_{33}]=[n_{13},n_{23},1]$
$[n_{12}-1,n_{22}]$,	1	$\sqrt{\frac{(n_{12}-n_{23})(n_{12}+1)(n_{13}-n_{22}+2)}{(n_{12}-n_{22}+1)(n_{13}-n_{23}+1)(n_{13}+2)}}$	$-\sqrt{\frac{(n_{13}-n_{12}+1)(n_{12}+1)(n_{23}-n_{22}+1)}{(n_{12}-n_{22}+1)(n_{13}-n_{23}+1)(n_{23}+1)}}$	$-\sqrt{\frac{(n_{13}-n_{12}+1)(n_{12}-n_{23})n_{22}}{(n_{12}-n_{22}+1)(n_{13}+2)(n_{23}+1)}}$
$[n_{12},n_{22}-1]$,	1	$\sqrt{\frac{(n_{23}-n_{22}+1)n_{22}(n_{13}-n_{12}+1)}{(n_{12}-n_{22}+1)(n_{13}-n_{23}+1)(n_{13}+2)}}$	$\sqrt{\frac{(n_{13}-n_{22}+2)(n_{12}-n_{23})n_{22}}{(n_{12}-n_{22}+1)(n_{13}-n_{23}+1)(n_{23}+1)}}$	$-\sqrt{\frac{(n_{13}-n_{22}+2)(n_{23}-n_{22}+1)(n_{12}+1)}{(n_{12}-n_{22}+1)(n_{13}+2)(n_{23}+1)}}$
$[n_{12},n_{22}]$,	0	$\sqrt{\frac{(n_{13}-n_{12}+1)(n_{13}-n_{22}+2)}{(n_{13}-n_{23}+1)(n_{13}+2)}}$	$\sqrt{\frac{(n_{12}-n_{23})(n_{23}-n_{22}+1)}{(n_{13}-n_{23}+1)(n_{23}+1)}}$	$\sqrt{\frac{n_{22}(n_{12}+1)}{(n_{13}+2)(n_{23}+1)}}$

By using the elementary $U(3) \supset U(2)$ Wigner coefficients shown in Table III and $SU(2)$ CG coefficients (65), Eqs. (67)-(69) can further be expressed as

$$\left\langle \begin{array}{cc} [n_{13},n_{23}] & [1,0] \\ \zeta\ L & 1 \end{array} \middle| \begin{array}{c} [n_{13}+1,n_{23}] \\ \zeta'\ L+1 \end{array} \right\rangle = \sum_{qt} \tilde{c}^{(\zeta')}_{q,t}([n_{13}+1,n_{23}],L+1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L)\times$$
$$\sqrt{\tfrac{(L+2q-t-n_{23}+1)(L+2q-t+2)(n_{13}-t+2)(L+q-t+1)}{(n_{13}-n_{23}+1)(n_{13}+2)(L+2q-2t+2)(L+2q-2t+1)}} - \sum_{qt} \tilde{c}^{(\zeta')}_{q,t+1}([n_{13}+1,n_{23}],L+1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L)\times$$
$$\sqrt{\tfrac{(n_{13}-L-2q+t+1)(n_{23}-t)(t+1)(q-t)}{(n_{13}-n_{23}+1)(n_{13}+2)(L+2q-2t)(L+2q-2t+1)}}, \tag{70}$$

$$\left\langle \begin{array}{cc} [n_{13},n_{23}] & [1,0] \\ \zeta\ L & 1 \end{array} \middle| \begin{array}{c} [n_{13}+1,n_{23}] \\ \zeta'\ L \end{array} \right\rangle = \sum_{qt} \tilde{c}^{(\zeta')}_{q,t}([n_{13}+1,n_{23}],L)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L) \times \sqrt{\tfrac{(n_{13}-L-2q+t+1)(n_{13}-t+2)(L+1))}{(n_{13}-n_{23}+1)(n_{13}+2)L}} \tag{71}$$

for $L \geq 1$,

$$\left\langle \begin{array}{cc} [n_{13},n_{23}] & [1,0] \\ \zeta\ L & 1 \end{array} \middle| \begin{array}{c} [n_{13}+1,n_{23}] \\ \zeta'\ L-1 \end{array} \right\rangle = \sum_{qt} \tilde{c}^{(\zeta')}_{q+1,t}([n_{13}+1,n_{23}],L-1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L) \times \sqrt{\tfrac{(L+2q-t-n_{23}+1)(L+2q-t+2)(n_{13}-t+2)(q-t+1)(2L+1)}{(n_{13}-n_{23}+1)(n_{13}+2)(L+2q-2t+2)(L+2q-2t+1)(2L-1)}} + \sum_{qt} \tilde{c}^{(\zeta')}_{q+1,t+1}([n_{13}+1,n_{23}],L-1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L) \times \sqrt{\tfrac{(n_{13}-L-2q+t+1)(n_{23}-t)(t+1)(L+q-t)(2L+1)}{(n_{13}-n_{23}+1)(n_{13}+2)(L+2q-2t)(L+2q-2t+1)(2L-1)}} \tag{72}$$

for $L \geq 1$,

$$\left\langle \begin{array}{cc} [n_{13},n_{23}] & [1,0] \\ \zeta\ L & 1 \end{array} \middle| \begin{array}{c} [n_{13},n_{23}+1] \\ \zeta'\ L+1 \end{array} \right\rangle = -\sum_{qt} \tilde{c}^{(\zeta')}_{q,t}([n_{13},n_{23}+1],L+1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L) \times \sqrt{\tfrac{(n_{13}-L-2q+t)(L+2q-t+2)(n_{23}-t+1)(L+q-t+1)}{(n_{13}-n_{23}+1)(n_{23}+1)(L+2q-2t+2)(L+2q-2t+1)}} - \sum_{qt} \tilde{c}^{(\zeta')}_{q,t+1}([n_{13},n_{23}+1],L+1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L) \times \sqrt{\tfrac{(n_{13}-t+1)(L+2q-n_{23}-t)(t+1)(q-t)}{(n_{13}-n_{23}+1)(n_{23}+1)(L+2q-2t)(L+2q-2t+1)}}, \tag{73}$$

$$\left\langle \begin{array}{cc} [n_{13},n_{23}] & [1,0] \\ \zeta\ L & 1 \end{array} \middle| \begin{array}{c} [n_{13},n_{23}+1] \\ \zeta'\ L \end{array} \right\rangle = \sum_{qt} \tilde{c}^{(\zeta')}_{q,t}([n_{13},n_{23}+1],L)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L) \times \sqrt{\tfrac{(L+2q-t-n_{23})(n_{23}-t+1)(L+1))}{(n_{13}-n_{23}+1)(n_{23}+1)L}} \tag{74}$$

for $L \geq 1$,

$$\left\langle \begin{array}{cc} [n_{13},n_{23}] & [1,0] \\ \zeta\ L & 1 \end{array} \middle| \begin{array}{c} [n_{13},n_{23}+1] \\ \zeta'\ L-1 \end{array} \right\rangle = -\sum_{qt} \tilde{c}^{(\zeta')}_{q+1,t}([n_{13},n_{23}+1],L-1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L) \times \sqrt{\tfrac{(n_{13}-L-2q+t)(n_{23}-t+1)(L+2q-t+2)(L-q+t)(2L+1)}{(n_{13}-n_{23}+1)(n_{23}+1)(L+2q-2t+2)(L+2q-2t+1)(2L-1)}} + \sum_{qt} \tilde{c}^{(\zeta')}_{q+1,t+1}([n_{13},n_{23}+1],L-1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L) \times \sqrt{\tfrac{(n_{13}-t+1)(L+2q-t-n_{23})(t+1)(L+q-t)(2L+1)}{(n_{13}-n_{23}+1)(n_{23}+1)(L+2q-2t)(L+2q-2t+1)(2L-1)}} \tag{75}$$

for $L \geq 1$,

$$\left\langle \begin{array}{cc} [n_{13},n_{23}] & [1,0] \\ \zeta\ L & 1 \end{array} \middle| \begin{array}{c} [n_{13}-1,n_{23}-1] \\ \zeta'\ L+1 \end{array} \right\rangle = -\sum_{qt} \tilde{c}^{(\zeta')}_{q-1,t-1}([n_{13}-1,n_{23}-1],L+1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L) \times \sqrt{\tfrac{(n_{13}-L-2q+t)(L+2q-t-n_{23}+1)t(L+q-t+1)}{(n_{13}+2)(n_{23}+1)(L+2q-2t+2)(L+2q-2t+1)}} + \sum_{qt} \tilde{c}^{(\zeta')}_{q-1,t}([n_{13}-1,n_{23}-1],L+1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L) \times \sqrt{\tfrac{(n_{13}-t+1)(n_{23}-t)(L+2q-2t+1)(q-t)}{(n_{13}+2)(n_{23}+1)(L+2q-2t)(L+2q-2t+1)}}, \tag{76}$$

$$\left\langle \begin{array}{cc} [n_{13},n_{23}] & [1,0] \\ \zeta\ L & 1 \end{array} \middle| \begin{array}{c} [n_{13}-1,n_{23}-1] \\ \zeta'\ L \end{array} \right\rangle = \sum_{qt} \tilde{c}^{(\zeta')}_{q-1,t-1}([n_{13}-1,n_{23}-1],L)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L) \times \sqrt{\tfrac{t(L+2q-t+1)(L+1))}{(n_{13}+2)(n_{23}+1)L}} \tag{77}$$

for $L\geq 1$, and

$$\left\langle \begin{array}{cc} [n_{13},n_{23}] & [1,0] \\ \zeta\ L & 1 \end{array} \right| \left. \begin{array}{c} [n_{13}-1,n_{23}-1] \\ \zeta'\ L-1 \end{array} \right\rangle = -\sum_{qt} \tilde{c}^{(\zeta')}_{q,t-1}([n_{13}-1,n_{23}-1],L-1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L)\times$$
$$\sqrt{\tfrac{(n_{13}-L-2q+t)(L+2q-t-n_{23}+1)t(q-t+1)(2L+1)}{(n_{13}+2)(n_{23}+1)(L+2q-2t+2)(L+2q-2t+1)(2L-1)}} - \sum_{qt} \tilde{c}^{(\zeta')}_{q,t}([n_{13}-1,n_{23}-1],L-1)\tilde{c}^{(\zeta)}_{q,t}([n_{13},n_{23}],L)\times$$
$$\sqrt{\tfrac{(n_{13}-t+2)(n_{23}-t)(L+2q-2t+1)(L+q-t)(2L+1)}{(n_{13}+2)(n_{23}+1)(L+2q-2t)(L+2q-2t+1)(2L-1)}} \tag{78}$$

for $L\geq 1$.

By using (70)-(78), some elementary $SU(3)\supset SO(3)$ Wigner coefficients can be evaluated analytically, for which some examples are shown in Tables IV-V, which are used to check the validity of the results. Similarly, one can also use (67)-(69) to get other elementary $SU(3)\supset SO(3)$ Wigner coefficients with $SU(3)\downarrow SO(3)$ branching multiplicity when the orthonormalized expansion coefficients $\mathbf{c}^{(\zeta)}$ are obtained by solving the angular momentum matrix projection (28).

TABLE IV: Elementary $SU(3)\supset SO(3)$ Wigner coefficients $\left\langle \begin{array}{cc} [n_{13},n_{23}] & [1,0] \\ L & 1 \end{array} \right| \left. \begin{array}{c} [n'_{13},n'_{23}] \\ n_{13} \end{array} \right\rangle$.

L	$[n'_{13},n'_{23}]=[n_{13}+1,n_{23}]$	$[n'_{13},n'_{23}]=[n_{13},n_{23}+1]$
n_{13}	$\sqrt{\frac{n_{23}}{n_{13}(n_{13}-n_{23}+1)}}$	$\sqrt{\frac{(n_{13}+1)(n_{13}-n_{23})}{n_{13}(n_{13}-n_{23}+1)}}$
$n_{13}-1$	$\sqrt{\frac{(n_{13}+1)(n_{13}-n_{23})}{n_{13}(n_{13}-n_{23}+1)}}$	$-\sqrt{\frac{n_{23}}{n_{13}(n_{13}-n_{23}+1)}}$

TABLE V: Elementary $SU(3)\supset SO(3)$ Wigner coefficients $\left\langle \begin{array}{cc} [n_{13},1] & [1,0] \\ L & 1 \end{array} \right| \left. \begin{array}{c} [n'_{13},n'_{23}] \\ n_{13}-1 \end{array} \right\rangle$.

L	$[n'_{13},n'_{23}]=[n_{13}+1,1]$	$[n'_{13},n'_{23}]=[n_{13},2]$	$[n'_{13},n'_{23}]=[n_{13}-1,0]$
n_{13}	$-\sqrt{\frac{2(n_{13}+1)^2(n_{13}-1)}{n_{13}^2(n_{13}+2)(2n_{23}-1)}}$	$-\sqrt{\frac{(n_{13}-2)(n_{13}-1)(2n_{13}+1)}{2n_{13}^2(2n_{13}-1)}}$	$-\sqrt{\frac{(n_{13}+1)(2n_{13}+1)}{2(n_{13}+2)(2n_{13}-1)}}$
$n_{13}-1$	$\sqrt{\frac{2(2n_{13}+1)}{n_{13}^2(n_{13}-1)(n_{13}+2)}}$	$\sqrt{\frac{(n_{13}+1)^2(n_{13}-2)}{2n_{13}^2(n_{13}-1)}}$	$-\sqrt{\frac{n_{13}+1}{2(n_{13}+2)}}$
$n_{13}-2$	$\sqrt{\frac{(n_{13}-2)(n_{13}+1)(2n_{13}+1)}{(n_{13}+2)(n_{13}-1)(2n_{13}-1)}}$	$-\sqrt{\frac{(n_{13}+1)}{(n_{13}-1)(2n_{13}-1)}}$	$-\sqrt{\frac{n_{13}-2}{(n_{13}+2)(2n_{13}-1)}}$

VI. SUMMARY

In this paper, a simple and effective angular momentum projection to construct basis vectors of $SU(3) \supset SO(3) \supset SO(2)$ from the canonical $U(3) \supset U(2) \supset U(1)$ basis vectors is outlined. We show that the expansion coefficients can be obtained as components of the null-space vectors of a projection matrix, for which, in general, there are only four nonzero elements in each row. There are currently available well-optimized algorithms for computing the null-space vectors of a matrix. Hence, to evaluate the expansion coefficients for the $SU(3) \supset SO(3) \supset SO(2)$ basis in terms of the basis of the canonical chain becomes more practical than Elliott's projection operator method. In the new projection scheme, the Hill-Wheeler integral involved in Elliott's projection operator method is avoided. Thus, the new projection method provides a fast and accurate numerical algorithm, since tedious high-precision numerical processing for the Hill-Wheeler integral is needed in the previous Elliott projection operator method. Some analytical expressions of the expansion coefficients for the $SU(3)$ irreps $[n_{13}, n_{23}]$ with $n_{23} \leq 4$ are presented as examples. Explicit formulae for evaluating $SO(3)$-reduced matrix elements of $SU(3)$ generators are derived, which are expressed in terms of the expansion coefficients. A general formula for evaluating the $SU(3) \supset SO(3)$ Wigner coefficients is given, which is expressed in terms of the expansion coefficients and known $U(3) \supset U(2)$ and $U(2) \supset U(1)$ Wigner coefficients. Formulae for evaluating the elementary Wigner coefficients of $SU(3) \supset SO(3)$ are explicitly given with some analytical examples shown to check the validity of results. Since the expansion coefficients are the components of null-space vectors of the projection matrix, there is always arbitrariness in choosing these vectors [19, 26]. Therefore, the null-space vectors are not orthogonal in general. The Gram-Schmidt orthonormalization is still needed in order to obtain orthonormalized basis vectors. It will be our next work to compile a code for calculating $SO(3)$-reduced matrix elements of the $SU(3)$ generators and coupling coefficients of $SU(3) \supset SO(3)$ according to the new projection method proposed in this paper. We can then check the runtime and compare with other existing codes using the Elliott's projection operator method, from which the efficiency of the new method can then be actually revealed.

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