

2008

Stochastic and copula models for credit derivatives

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STOCHASTIC AND COPULA MODELS FOR CREDIT DERIVATIVES

A Dissertation

Submitted to the Graduate Faculty of the
Louisiana State University and
Agricultural and Mechanical College
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

in

The Department of Mathematics

by

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August 2008

Acknowledgments

This dissertation would not have been possible without several contributions. It is a pleasure to thank my advising professor, Dr. Ambar Sengupta.

It is a pleasure also to thank the Department of Mathematics, Louisiana State University, for providing me with a pleasant working environment. A special thanks to the members of my committee, Drs. Chance, Cochran, Davidson, Perlis, and Sundar for their valuable advice.

This dissertation is dedicated to my wife, Xiaomiao Teng, my daughters Lingfei Meng and Lingxuan Meng and all my friends here: Alvaro, Wei Cai, Xuanting Cai, Cristian, Deb, Liqun Fang, Fareed, Jason, Jeremy, Julius, Lee, Ryan, Stan, Hairui Tu, Jie Wu, Dongxiang Yan, Hong Yin for their support and encouragement.

Table of Contents

Acknowledgments	ii
List of Figures	v
Abstract	vi
Chapter 1: Introduction	1
Chapter 2: A First Passage Time Estimate	11
2.1 From the Gaussian Process to Uncorrelated Brownian Motion . . .	13
2.2 Exit Time from an Orthant	14
2.3 Correlation and Some Geometric Consequences	16
2.4 Hitting Times for Processes with Drift	20
Chapter 3: Exit Times Revisited	22
3.1 Extrema of Paths	22
Chapter 4: The Gaussian Copula Model	33
4.1 Sensitivity to Correlation in the Gaussian Model	34
4.2 Sensitivity to the Threshold	38
4.3 Gamma: A Convexity Measure	41
4.4 Poisson-mix Model	42
4.5 The Large- N Limit	45
4.6 Relationship with CDO Tranche Models	46
4.7 Proxy Variables	48
4.8 Simulations and Graphs	51
Chapter 5: Significance of Certain Stochastic Integrals	54
5.1 Probability and Pricing Notions	54
5.2 Default Intensity	56
5.3 Default Intensity Integrals	57
5.4 Stochastic Integrals with Stopping Times	58
Chapter 6: Certain Stochastic Integrals with Stopping Times	60
6.1 The Vasicek Model	60
6.2 The Two-Factor Gaussian Model	65
6.3 The Multifactor Cox-Ingersoll-Ross Model	70
6.4 Stochastic Integral Representation for the Duration of a CDS	72
References	74
Appendix A: Copulas, Correlation and Girsanov's Theorem	82

A.1	Introduction to Copula	82
A.2	Correlation and Kendall's Tau	83
A.3	Girsanov's Theorem	83
	Appendix B: Code of GcBinomialSimulation.m	84
	Appendix C: Code of DerivativeOfExpLoss.m	85
	Vita	88

List of Figures

3.1	A Sample Path Of A Stochastic Process	23
4.1	Dependence of $\frac{dL_3^e}{d\rho}$ on ρ and c	52
4.2	Graph of $L = L_3^e$ against ρ and $\Phi(c)$	53

Abstract

We prove results relating to the exit time of a stochastic process from a region in N -dimensional space. We compute certain stochastic integrals involving the exit time. Taking a Gaussian copula model for the hitting time behavior, we prove several results on the sensitivity of quantities connected with the hitting times to parameters of the model, as well as the large- N behavior. We discuss the relationship of these results to certain credit derivative instruments. Relevant simulations are presented.

Chapter 1

Introduction

This dissertation forms a step towards a fuller understanding of certain hitting time questions for stochastic processes in N dimensions, especially with a view towards understanding what happens for large N . As it happens, some of these questions have arisen in connection with certain financial instruments called credit derivatives. We draw on intuition and simulation-based observations gathered in the context of these instruments, to formulate precise mathematical results and proofs. Our ultimate objectives are, however, mathematical.

- Motivation and Background

Consider a stochastic process, i.e. a random path,

$$t \mapsto X_t$$

evolving in \mathbb{R}^N . A natural and classic question in probability theory is the determination of the behavior of the first time τ when the process hits some specified set. For instance, if the process initiates at a point p_0 in a region $D \subset \mathbb{R}^N$, one may study the first time the process hits the boundary ∂D . A particular case of great simplicity is the question of when a Brownian motion

$$t \mapsto B_t$$

in *one* dimension, starting at the origin 0, reaches a point $x > 0$. It is well known (see, for instance, [74]) that this hitting time τ_x has distribution given by

$$\text{Prob} [\tau_x \leq t] = 2 \int_{x/\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} dy$$

The questions we study involve a stochastic process $t \mapsto X_t$, initiating at a point in a ‘wedge’, an unbounded subset of \mathbb{R}^N with boundary formed by hyperplanes. Ideally, one would like to know the exact joint distribution of the times at which the process hits each of the bounding hyperplanes. This task is, of course, of tremendous complexity. Even in two dimensions, the distribution of the hitting time to one of two bounding walls, is very complex (it has been studied by, among others, Rebbholz in his 1994 Berkeley PhD thesis [76]).

As it happens, and as we shall explain in detail in later chapters, hitting (or exit) time questions are of relevance to certain fundamental models of default events of bonds. The quantitative work in connection with the pricing and risk management of instruments which market default risk in portfolios has given rise to ‘phenomenological models’, called copula models, for describing the joint probability distribution of defaults of bonds in a portfolio. The usefulness of these phenomenological models suggests that, at least when the dimension N of the ambient space is large, the hitting time distributions might be approximated by large- N limits of these copula models, especially the so-called Gaussian copula. Whether this is in fact the case, remains conjectural at this time.

Our study splits into two parts, first a study of questions relating to hitting times of stochastic process, and then a study of statistical/probabilistic features of a copula model for default-time/hitting-time distributions. These features are suggested by simulation-based observations and intuitively understood phenomena used in practice. We turn now to a more detailed summary of the results we prove.

- Overview of Results

A Brownian motion $t \mapsto B(t)$ in \mathbb{R}^N is a stochastic process (thus, having random paths) which sets off at the origin 0, has continuous paths, with Gaussian incre-

ments $B_t - B_s$ (for $0 \leq s < t$) independent of the ‘past’, each component having mean 0 and variance $t - s$. Consider now a stochastic process

$$t \mapsto Y(t) = (Y_1(t), \dots, Y_N(t))$$

in \mathbb{R}^N , such that each component Y_j is a Brownian motion, but now suppose that these component are *correlated*, i.e. the correlations

$$\rho_{jk} = \text{Corr}(Y_j(t), Y_k(t))$$

are not all 0. Now consider the region

$$\{x \in \mathbb{R}^N : x_1 > -c_1, \dots, x_N > -c_N\},$$

where $c_1, \dots, c_N > 0$, which is bounded by the ‘walls’ given by:

$$j\text{-th wall} = \{x \in \mathbb{R}^N : x_j = -c_j\}.$$

In dimensions > 2 , Brownian motion is known to be ‘transient’ and escapes to infinity with probability one. Our first result provides an upper bound for the probability that the exit time is greater than t , and then we discuss several other results on the exit time and the correlation.

Next we construct a discrete approximation to the process $t \mapsto Y(t)$. We prove results showing exactly in what sense this discrete process approximates the continuous-time process. We also derive a difference equation for the probability distribution of the hitting time for the discrete process. We then indicate, informally, how this difference equation provides, as its limit, the Kolmogorov backward equation for the hitting time distribution of the continuous process.

Brownian motion is technically described through a measure, the standard *Wiener measure* μ on the space $C_0([0, \infty); \mathbb{R}^N)$ of continuous paths in \mathbb{R}^N starting at 0. As

with all measures, this measure is best understood by means of integrals

$$\int_{C_0([0,\infty);\mathbb{R}^N)} f d\mu$$

for functions f of interest. The simplest choice of such functions f are *cylinder functions*, i.e. functions of the form

$$x \mapsto f(x) = F(x(t_1), \dots, x(t_n))$$

for paths $x \in C_0([0, \infty); \mathbb{R}^N)$, time instants $0 < t_1 < \dots < t_n$, and suitable measurable functions F on \mathbb{R}^N . Another interesting standard class of functions are of the form

$$x \mapsto f(x) = e^{-\int_0^T F(x)(t) dt} g(x)(T)$$

for suitable functions F and g on the path space, and $T > 0$. Integrals of such functions are the subject the Feynman-Kac formula. Instead of a fixed time T , one may also study such integrals with a random time τ . We will take a the random exit times τ mentioned before, and obtain closed-form expressions for integrals of the type

$$\int_{C_0([0,\infty);\mathbb{R}^N)} \left[\int_0^{\tau \wedge T} e^{-\int_0^t \zeta(x,u) du} \right] d\mu(x), \quad (1.0.1)$$

where ζ may be one of several types of functions on the path space, usually specified, almost everywhere, through a stochastic differential equation. We will describe our method in Chapter 6 in the context of certain credit-derivative models.

Finally, we turn to a set of questions motivated by a phenomenological model for exit times from the region. Here we simply assume, as an *Ansatz*, that the number k of the N component paths of a process $t \mapsto Y(t)$, initiating at a point in a wedge in \mathbb{R}^N , which exit the wedge are governed by a specific ‘Gaussian copula’ law. We can formulate our results directly, without reference to the process Y . We consider jointly Gaussian variables X_1, \dots, X_N , each being standard Gaussian, with

a common positive correlation

$$\rho = \mathbb{E}[X_j X_k] > 0 \quad \text{for all } j, k \in \{1, \dots, N\},$$

specified by

$$X_i = \sqrt{\rho}Z + \sqrt{1 - \rho}\epsilon_i \quad \text{for every } i \in \{1, \dots, N\}$$

where $Z, \epsilon_1, \dots, \epsilon_N$ are independent standard Gaussians. Let $c \in \mathbb{R}$ be a ‘threshold’. We view the event $[X_j < c]$ as indicating that the j -th component of the process has exited the wedge within a fixed time horizon. Let ν be the random variable which counts the number of X_j which are below the threshold value:

$$\nu = \mathbb{1}_{[X_1 < c_1]} + \dots + \mathbb{1}_{[X_N < c_N]}. \quad (1.0.2)$$

One way to study the joint distribution of the events $[X_j < c]$ is to examine the behavior of the random variables

$$\nu_k = \min\{\nu, k\} \quad \text{for } k \in \{1, \dots, N\}, \quad (1.0.3)$$

and the expectations

$$\mathbb{E}[\nu_k].$$

Here, and always, $\mathbb{E}[Z]$ denotes the expected value of a random variable Z . When the correlation ρ increases, the distribution of ν gets heavier at both high and low values, and it is not apparent which way the expected value $\mathbb{E}[\nu_k]$ would move. We prove that, in fact, $\mathbb{E}[\nu_k]$ decreases when ρ increases.

We may also consider the ‘delta’ for $k \in \{0, 1, \dots, N\}$, given by

$$\Delta_k = \frac{\frac{\partial \mathbb{E}[\nu_k]}{\partial c}}{\frac{\partial \mathbb{E}[\nu]}{\partial c}}, \quad (1.0.4)$$

which is a normalized sensitivity of $\mathbb{E}[\nu_k]$ to changes in the threshold c .

Our main results for this ‘Gaussian copula’ model may be summarized as follows.

Theorem 1.0.1. *With notation and hypotheses as above,*

(i) *the derivative of $\mathbb{E}[\nu_k]$ with respect to the correlation parameter ρ is negative:*

$$\frac{d\mathbb{E}[\nu_k]}{d\rho} < 0$$

(ii) *There is a probability measure Δ on subsets of $\{0, 1, \dots, N\}$, such that*

$$\Delta_k = \Delta(\{0, 1, \dots, k\});$$

the delta measure Δ is given by an averaging of a certain Binomial probability distribution over a Gaussian distribution (see (4.2.6) for an explicit formula).

(iii) *The ‘Gamma’ for k , defined by*

$$\Gamma_k \stackrel{\text{def}}{=} \Delta_k \frac{\partial^2 L_N}{\partial c^2} - \frac{\partial^2 L_k}{\partial c^2}, \quad (1.0.5)$$

is positive.

(iv) *Let $Z, \epsilon_1, \epsilon_2, \dots$ be a sequence of independent standard Gaussians, and $X_i = \sqrt{\rho}Z + \sqrt{1-\rho}\epsilon_i$ for each i , with a fixed $\rho > 0$. Let $\nu^{(N)}$ be the number of j for which X_j is below c . Then the random variable $\nu^{(N)}/N$ converges almost surely to $\Phi((c - \sqrt{\rho}Z)/\sqrt{1-\rho})$, where Φ is the distribution function of the standard Gaussian.*

We also prove analogous results for Poisson distributions in place of the Binomial. These two results are inspired by observations made in quantitative finance practice.

In addition to the preceding theorems, we also present some simulations illustrating aspects of the results.

- Relationship with Credit Derivative Modeling

As noted earlier, some of our results are inspired by ideas arising from models for certain credit derivative instruments, specifically models for credit default swaps (CDS) and collateralized debt obligations (CDO).

A CDS is a credit derivative which is an agreement between two parties, A (who is buying protection) and B (who is selling protection). A pays B a premium periodically to insure the notional amount of a given defaultable bond against risk of default. If a default happens during the life of the CDS, B pays A the loss amount. Otherwise, B pays A nothing. To model the price of a CDS basically means to set up a model to find the premium A should pay B.

A CDO is a credit derivative in which the credit risk on a portfolio of defaultable assets is sold by different default levels, called *tranches*. The first tranche, 0% – 3%, is called the *equity tranche*; it is the most risky tranche. The last one, usually 30% – 100%, is called the *supersenior* tranche, and is the most secure.

The market for CDS and CDOs, which began in the mid 1990s, has grown explosively (from 7.3 trillion US dollar notional in June 2005 to 24.2 trillion in June 2007 for a certain category of CDS contracts, according to Table 19 in [9].) This, along with the current turmoil in the credit derivatives market and its ramifications to the global economy, underline the need for broader, theoretical studies of the models used in pricing and risk managing default swaps and CDOs. The present work, however, is primarily mathematical, with the objective being rigorous proofs of precisely formulated theorems. The finance context serves only as an intuitive guide, providing a qualitative guidance towards conjecturing new results relating to exit time phenomena.

The exit time of a stochastic process from the wedge described earlier, can be viewed as a simple first model of default of a bond. The process is a proxy for the value of the assets of the bond issuer, and hitting the boundary corresponds to

default of the bond. The distribution of the default time, i.e. the wedge exit time in the model, is a significant factor in the CDS premium rate for the bond.

Our results on the Gaussian copula model, and the Poisson-mix model, are connected with default behavior in a CDO. The Gaussian copula model for default behavior in a CDO associates to each CDS name i in the portfolio a standard Gaussian variable X_i ; these variables are assumed to be such that that are independent standard Gaussians $Z, \epsilon_1, \dots, \epsilon_N$ with

$$X_i = \sqrt{\rho}Z + \sqrt{1 - \rho}\epsilon_i,$$

for $i \in \{1, \dots, N\}$, for some fixed positive ρ . Name i defaults in a given time horizon if the value of X_i is below a threshold c (assumed, in this simple model, to be the same for all names). With this framework our results in Theorem 1.0.1 match what is understood through simulations and experience in actual practice (see, for instance, [70]). It may be noted that the Gaussian copula model (popularized by David Li [64]) for default behavior, though still a valuable tool in practice, has many practical deficiencies. However, from the mathematical point of view, it is a fundamental setting, with the Gaussian background measure, to prove results of elegance and simplicity.

The interaction between stochastics and financial models goes back at least to Louis Bachelier's 1900 PhD thesis using the essential ideas of Brownian motion in the context of stock price evolution. In more modern times, in 1973, Robert C. Merton and Myron S. Scholes, who were later awarded the Nobel prize, in collaboration with Fischer S. Black, developed the celebrated Black-Scholes-Merton formula to evaluate stock options, and changed pricing financial derivatives from a guessing game into solving a mathematical model. The mathematical tools they used, continuous-time stochastic calculus and stochastic differential equations, be-

came the most common language for evaluating financial instruments in industry and in academic work.

- Organization of Thesis

Chapter 2 presents our main results on the exit time of a stochastic process from a region in N -dimensional space.

In Chapter 3, we describe a discrete random walk process, as an approximation to a continuous process, and obtain a difference equation for the exit; we also discuss related results and notions.

In Chapter 4, we take a phenomenological model, mainly the Gaussian copula model, for hitting time distributions, and present our results concerning sensitivity of hitting time distribution characteristics to model parameters. We explain how these connect with ideas used in the credit-derivatives industry. We also prove a convergence result for the large- N (dimension) behavior of the model.

Chapter 5 summarizes some standard material from ‘stochastic finance’, explaining how certain integrals involving stopping times arise and how they may be interpreted in the financial context.

Chapter 6 begins with a description of certain standard models in pricing bond-related instruments. Then, in section 6.4 we describe our method for computing the integrals (1.0.1) for these models.

A few standard definitions and notions pertaining to copula and correlation are summarized in Appendix A, presented in a manner suitable for our needs.

- Brief Comments on the Bibliography

The bibliography presents primarily works which have been broadly consulted in preparing this dissertation.

The literature on exit times/first-passage-times is vast, spanning many decades. A search on mathscinet for ‘first passage time’ produces over a thousand entries. These include works in several areas of physics, biological sciences, reliability theory, and finance. The book by Oksendal [74] has been particularly useful for us.

There is also a large body of literature, with heavy current activity, relating to credit derivatives. However, very little of this is motivated mainly by the search for mathematically elegant and precise results and proofs. Indeed, a search on mathscinet for ‘credit default swaps’ and ‘collateralized debt obligations’ produces very few entries. The present dissertation should be viewed as a work of mathematics, with simulations and ideas arising in part from the finance context.

The long range goal of this line of research is the study of large- N behavior of exit times of stochastic processes in N dimensions. The ICM lecture of Williams [102], and the work of Varadhan and Williams [96], testify to depth of questions and ideas that arise in even the case of a stochastic process in a wedge.

Chapter 2

A First Passage Time Estimate

In this chapter we consider a stochastic process in \mathbb{R}^N , with continuous paths and Gaussian in nature, and study the first time this process exits from a region bounded by hyperplanes orthogonal to the coordinate vectors. We obtain an upper bound for the exit time distribution.

All through this chapter we work on a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is assumed to be richly structured enough to admit Brownian motion processes. For example, the probability space might be the space $C_0([0, \infty), \mathbb{R}^N)$, of continuous paths in \mathbb{R}^N starting at 0, equipped with Wiener measure on the completion of the σ -algebra generated by cylinder sets.

We shall use the terminology ‘first passage time’, ‘hitting time’, and ‘exit time’, interchangeably. For our purposes a distinction between these notions need not be made. In other settings, especially for processes with discontinuous paths, a distinction could be made, but that is not applicable to our discussion.

As mentioned in the Introduction, there is a large body of literature on exit times. In particular, we will use a well-known formula concerning the probability that a standard Brownian path $t \mapsto B(t)$ in \mathbb{R} , starting at the origin, reaches beyond a level $x > 0$:

$$\begin{aligned} \mathbb{P}\left[\sup_{s \in [0, t]} B(s) \geq x\right] &= \mathbb{P}\left[\sup_{s \in [0, t]} B(s) > x\right] = \mathbb{P}[\tau_x \leq t] \\ &= 2 \left(1 - \Phi(x/\sqrt{t})\right) = 2\Phi\left(-\frac{x}{\sqrt{t}}\right), \end{aligned} \tag{2.0.1}$$

where τ_x is the exit time

$$\tau_x = \inf\{t \geq 0 : B(t) \geq x\},$$

and Φ is the standard Gaussian distribution function

$$\Phi(y) = \int_{-\infty}^y (2\pi)^{-1/2} e^{-s^2/2} ds.$$

This result may be found in standard texts, such as Oksendal [74].

We will work with a Gaussian stochastic process

$$[0, 1] \rightarrow \mathbb{R}^N : (t, \omega) \mapsto Y(t; \omega) = Y_t(\omega) \in \mathbb{R}^N$$

for which each component $Y_j(\cdot)$ is a standard Brownian motion. In particular,

$$Y(0) = 0.$$

For each $j, k \in \{1, \dots, N\}$, define ρ_{jk} through

$$\rho_{jk} = \text{Corr}(Y_j(t), Y_k(t)), \tag{2.0.2}$$

and we assume that ρ_{jk} is independent of t .

Since each $Y_j(t)$ has mean 0 and variance t , it follows that

$$\mathbb{E}[Y_j(t)Y_k(t)] = t\rho_{jk}$$

Let

$$R = [\rho_{jk}],$$

the $N \times N$ matrix whose entries are the correlation terms ρ_{jk} . Note that the matrix R is symmetric.

Exit times have been studied, mostly in terms of general, abstract, results in many works. We mention Wentzell [99], Freidlin [41], Krylov [60] and Oksendal [74]. Shepp [86] studies a more specific hitting-time problem, with a parabolic boundary.

2.1 From the Gaussian Process to Uncorrelated Brownian Motion

It is a standard fact that this matrix is *non-negative-definite*; this is because

$$\sum_{j,k=1}^N z_j \rho_{jk} \bar{z}_k = \mathbb{E} \left[\left| \sum_{j=1}^N z_j Y_j(t) \right|^2 \right] \geq 0,$$

for every complex $z_1, \dots, z_N \in \mathbb{C}$. The inequality above will be an equality if and only if $\sum_{j=1}^N z_j Y_j(t)$ is 0 almost-everywhere. Thus, if we assume that $Y_1(t), \dots, Y_N(t)$ are linearly independent then the matrix R is *positive definite*.

Now, for some invertible symmetric matrix β , consider the random vector

$$W(t) = \beta^{-1} Y(t),$$

so that

$$Y(t) = \beta W(t). \tag{2.1.1}$$

We want to find the β which will make the $W(\cdot)$ a standard Brownian motion, i.e. its components (which are Gaussian) should be independent. Thus, we should have

$$\begin{aligned} \mathbb{E} [Y_j(t) Y_k(t)] &= \sum_{m,l=1}^N \mathbb{E} [\beta_{jm} W_m(t) \beta_{kl} W_l(t)] \\ &= t \sum_{1 \leq m,l \leq N} \beta_{jm} \delta_{ml} \beta_{kl} \\ &= t (\beta \beta^t)_{jk}, \end{aligned} \tag{2.1.2}$$

Thus, we should take β to be the positive definite matrix (hence, automatically, symmetric) whose square is R :

$$\beta = R^{1/2}. \tag{2.1.3}$$

Thus,

$$t \mapsto W(t)$$

is a Gaussian stochastic process whose components are independent Brownian motions. Hence $t \mapsto W(t)$ is a standard Brownian motion in \mathbb{R}^N .

2.2 Exit Time from an Orthant

Fix ‘threshold’ values

$$c_1, \dots, c_N > 0$$

and let τ_j be the exit time

$$\tau_j = \inf\{t \geq 0 : Y_j(t) \leq -c_j\} \tag{2.2.1}$$

We are interested in

$$\tau = \min\{\tau_1, \dots, \tau_N\}. \tag{2.2.2}$$

This is the first time when the process $Y(\cdot)$ has one component fall to or below the corresponding threshold level $-c_j$.

The condition

$$Y_j(t) > -c_j \quad \text{for all } j \in \{1, \dots, N\}$$

is equivalent to each component of the vector

$$\beta W(t) + c$$

being positive. This means that $\beta W(t) + c$ is in the positive orthant $(0, \infty)^N$.

Thus, τ is the first time t when $W(t) + \beta^{-1}c$ exits $\beta^{-1}((0, \infty)^N)$.

Let τ' be the first time t when $W(t) + \beta^{-1}c$ exits the half space H_v . Then

$$\tau \leq \tau' \tag{2.2.3}$$

because the process must first exit $\beta^{-1}((0, \infty)^N)$ before it can exit the half-space H_v .

Lemma 2.2.1. *The distribution of the exit time τ' is given by*

$$\text{Prob}[\tau' \leq t] = 2\Phi\left(-\frac{\langle\beta^{-1}c, v\rangle}{\sqrt{t}}\right) \quad (2.2.4)$$

In particular, τ' is finite with probability 1.

Proof. Let us write $W(t)$ as a component along the vector v and a component perpendicular to v :

$$W(t) = \langle W(t), v \rangle v + W(t) - \langle W(t), v \rangle v$$

Each of these two components is a Brownian motion. In particular,

$$t \mapsto \langle W(t), v \rangle$$

is a standard Brownian motion.

The first time $W(t) + \beta^{-1}c$ exits the half-space H_v is the first time the component $\langle W(t), v \rangle$ falls to or below the value

$$-\langle\beta^{-1}c, v\rangle.$$

So

$$\text{Prob}[\tau' \leq t] = \text{Prob}[\tau'' \leq t], \quad (2.2.5)$$

where τ'' is the first time a standard Brownian motion $t \mapsto B(t)$ hits the value $\langle\beta^{-1}c, v\rangle$.

Now (see, for example, Oksendal [74]), for any $x \geq 0$,

$$\text{Prob}\left[\sup_{0 \leq s \leq t} B_s \geq x\right] = 2\Phi\left(-\frac{x}{\sqrt{t}}\right). \quad (2.2.6)$$

Therefore,

$$\text{Prob}[\tau'' \leq t] = 2\Phi\left(-\frac{\langle\beta^{-1}c, v\rangle}{\sqrt{t}}\right) \quad (2.2.7)$$

This gives the desired result from (2.2.5). Letting $t \uparrow \infty$ shows that $\text{Prob}[\tau' < \infty]$ is 1. □

Now we have:

Proposition 2.2.1. *Let $t \mapsto Y(t)$ be a Gaussian process, each of whose components is a standard Brownian motion, and with a non-degenerate correlation matrix $R = [\rho_{jk}]$, where $\rho_{jk}t = \mathbb{E}[Y_j(t)Y_k(t)]$ for $j, k \in \{1, \dots, N\}$ and all $t > 0$. Let $c = (c_1, \dots, c_N) \in (0, \infty)^N$, and τ the exit time of $Y_t + c$ from $(0, \infty)^N$. Then:*

$$\mathbb{P}[\tau > t] \leq 2\Phi\left(\frac{\min_j c_j}{\sqrt{t}}\right) - 1. \quad (2.2.8)$$

Proof. As before, let $\beta = R^{1/2}$, and $W_t = \beta^{-1}Y_t$. The exit time τ is the first time $\beta W_t + c$ exits $(0, \infty)^N$, i.e. the first time $W_t + \beta^{-1}c$ exits $\beta^{-1}(0, \infty)^N$. Suppose n is any unit vector such that

$$\beta^{-1}((0, \infty)^N) \subset H_n = \{x \in \mathbb{R}^N : \langle x, n \rangle \geq 0\}.$$

This is equivalent to $\langle \beta^{-1}e_j, n \rangle$ being positive for each standard basis vector e_j , i.e.

$$\langle e_j, \beta^{-1}n \rangle > 0 \quad \text{for all } j \in \{1, \dots, N\}. \quad (2.2.9)$$

Then, from Lemma 2.2.1, the time τ'_n of exit of the process $t \mapsto W_t$ from H_n , satisfies

$$\mathbb{P}[\tau'_n \leq t] = 2\Phi\left(-\frac{\langle c, \beta^{-1}n \rangle}{\sqrt{t}}\right).$$

The maximum over unit vectors n for which (2.2.9) holds, occurs at $\beta^{-1}n$ equal to some e_k , and so it is for that k for which $\langle c, e_k \rangle$ is minimum. \square

2.3 Correlation and Some Geometric Consequences

We work with a Gaussian process $t \mapsto Y_t$ with Brownian components, and with correlation matrix $R = [\rho_{jk}]$ specified by

$$\mathbb{E}[Y_j(t)Y_k(t)] = \rho_{jk}t.$$

If the matrix R is positive definite and has all entries ρ_{jk} positive then the Brownian ‘factor’ process $t \mapsto W_t$ is contained in a halfspace determined by RL

Proposition 2.3.1. *If R is a positive definite matrix with all entries positive then $R^{-1/2}((0, \infty)^N)$ is contained in the half-space*

$$H_v = \{x \in \mathbb{R}^N : \langle x, v \rangle \geq 0\}$$

where v is the eigenvector of R corresponding to the largest eigenvalue. Thus, if each ρ_{jk} is positive and $c \in (0, \infty)^N$, then the process $t \mapsto W(t) + R^{-1/2}c$ lies entirely inside the half-space H_v up to time τ of exit of the process $Y + c$ from the positive orthant.

Proof. Since R is positive-definite and has all entries positive, the Perron-Frobenius theorem says that it has a unique unit eigenvector v , with all components positive, which corresponds to the largest eigenvalue λ :

$$Rv = \lambda v.$$

Since R is positive definite, λ is positive. Note also that

$$\beta = R^{1/2}$$

is a positive definite, and hence, symmetric matrix. Looking at the matrix of R relative to an orthonormal basis of eigenvectors of R , it is simply a diagonal matrix, and β is the diagonal matrix with entries given by the corresponding square-roots. In particular,

$$\beta v = \sqrt{\lambda} v$$

Consider any of the standard unit basis vectors

$$e_j = (0, 0, \dots, 0, 1, 0, \dots, 0)$$

with 1 at the j -th entry. Then

$$\langle \beta^{-1}e_j, v \rangle = \langle e_j, \beta^{-1}v \rangle = \frac{1}{\sqrt{\lambda}} \langle e_j, v \rangle > 0, \quad (2.3.1)$$

the last inequality holding because each component of the vector v is positive.

Thus

$$\beta^{-1}e_j \in H_v \stackrel{\text{def}}{=} \{x \in \mathbb{R}^N : \langle v, x \rangle > 0\}$$

Thus, β^{-1} maps each of the basis vectors into the half space H_v . Hence, it maps any positive linear combination of the e_j 's into H_v . This means that β^{-1} maps $(0, \infty)^N$ into a subset of H_v . \square

Next suppose R is positive definite and R^{-1} has all entries positive. For this case we have:

Proposition 2.3.2. *Suppose the Gaussian process $t \mapsto Y(t)$ has Brownian components, and the correlation matrix $R = [\rho_{jk}]$, where $\rho_{jkt} = \mathbb{E}[Y_j(t)Y_k(t)]$ for all $j, k \in \{1, \dots, N\}$ and $t > 0$, is such that R is invertible and $R^{-1/2}$ has all entries positive. Then*

$$W(t) + R^{-1/2}c \in (0, \infty)^N$$

for all $t \leq \tau$, where τ is the time of exit of the process $Y + c$ from the positive orthant.

Proof. Upto time τ , $R^{1/2}W(t) + c$ lies in $(0, \infty)^N$, and so $W(t) + R^{-1/2}c$ lies in $R^{-1/2}(0, \infty)^N$, and this lies inside the positive orthant if $R^{-1/2}$ has all components positive. \square

In the 2-dimensional case we can draw some conclusions concerning the expectation of the hitting time τ .

Let

$$X(t) = (Y_1(t) + c_1)(Y_2(t) + c_2)$$

Then, by Itô's lemma,

$$dX(t) = \text{martingale terms} + \rho dt$$

Consequently,

$$\mathbb{E}[X(\tau \wedge N)] = X(0) + \rho \mathbb{E}[(\tau \wedge N)] \quad (2.3.2)$$

Now at time $\tau \wedge N \leq \tau$, the process $Y + c$ is still inside $(0, \infty)^2$, and so

$$X(\tau \wedge N) \geq 0$$

Thus

$$c_1 c_2 + \rho \mathbb{E}[(\tau \wedge N)] \geq 0.$$

By monotone convergence as $N \uparrow \infty$, we have

$$c_1 c_2 \geq -\rho \mathbb{E}[\tau \mathbb{1}_{[\tau < \infty]}].$$

But we already know that $\tau < \infty$ with probability 1. So we conclude:

Proposition 2.3.3. *For the process $t \mapsto Y(t)$ in \mathbb{R}^2 if the correlation ρ is negative, then the expected hitting time $\mathbb{E}[\tau]$ is finite.*

Intuitively, if one component, say $Y_1(t)$ is very high positive (away from $-c_1$) then the negative correlation makes it likely that the other component is very low negative and so likely below the corresponding threshold ($-c_2$ for Y_2). This makes it more likely that the boundary of the region will be hit in less time than in the case of positive correlation when both components could be large simultaneously.

The case of two dimensions implies the following consequence for higher dimensions:

Proposition 2.3.4. *For the process $t \mapsto Y(t)$ in \mathbb{R}^N if the correlation between Y_j and Y_k is negative for some pair $j, k \in \{1, \dots, N\}$, then the expected hitting time $\mathbb{E}[\tau]$ is finite.*

2.4 Hitting Times for Processes with Drift

We should note that the Brownian motion we discussed above has no drift and volatility coefficient. However, by Girsanov's theorem, a variation on some of our results should still hold for Brownian motion with volatility σ and an added 'small' drift. That is, τ is still finite with probability 1 under some transformed probability measure to which Girsanov's theorem applies. (For Girsanov's theorem, see theorem A.3.1 in Appendix A.)

We focus now on a case that is more concrete. The following result is well-known, but we include a proof.

Proposition 2.4.1. *Let $t \mapsto B_t$ be standard Brownian motion in one dimension, $\mu \in \mathbb{R}$, and $a \in (0, \infty)$. Let τ be the first time $B_t + \mu t$ exits $(-\infty, a)$. Then $\tau < \infty$ with probability 1 if $\mu \geq 0$, and is equal to $e^{2a\mu}$ if $\mu < 0$.*

Proof. Let $\lambda > 0$ and $N \in \{1, 2, \dots\}$. Then

$$\begin{aligned} e^{\lambda B_{\tau \wedge N} - \lambda^2(\tau \wedge N)/2} &\leq e^{\lambda(a - \mu(\tau \wedge N)) - \lambda^2(\tau \wedge N)/2} \\ &\leq e^{a\lambda - \lambda(2\mu + \lambda)(\tau \wedge N)/2} \end{aligned} \tag{2.4.1}$$

To make this bounded, we work with λ satisfying

$$\lambda > -2\mu. \tag{2.4.2}$$

Note that, when $N \geq \tau$ then in (2.4.1), $e^{\lambda B_{\tau \wedge N} - \lambda^2(\tau \wedge N)/2}$ stabilizes at $e^{a\lambda - \lambda(2\mu + \lambda)\tau/2}$, whereas, if $N < \tau$, then it is always bounded by $e^{a\lambda - \lambda(2\mu + \lambda)N/2}$, which goes to 0.

Thus,

$$\lim_{N \rightarrow \infty} e^{\lambda B_{\tau \wedge N} - \lambda^2(\tau \wedge N)/2} = e^{a\lambda - \lambda(2\mu + \lambda)\tau/2} \mathbb{1}_{[\tau < \infty]} \tag{2.4.3}$$

From the martingale property of $t \mapsto e^{B_t - t/2}$, we have, for the bounded stopping time $\tau \wedge N$,

$$\mathbb{E} \left[e^{\lambda B_{\tau \wedge N} - \lambda^2(\tau \wedge N)/2} \right] = 1.$$

We can apply monotone convergence to the part of the expectation over $[\tau \leq N]$, and dominated convergence to the part on $[\tau > N]$, as $N \uparrow \infty$, to conclude, using (2.4.3), that

$$e^{a\lambda} \mathbb{E} \left[e^{-\lambda(2\mu+\lambda)\tau/2} \mathbb{1}_{[\tau < \infty]} \right] = 1. \quad (2.4.4)$$

If $\mu \geq 0$ we let $\lambda \downarrow 0$ and conclude that $\mathbb{E} [\mathbb{1}_{[\tau < \infty]}]$ is 1, i.e. $\tau < \infty$ with probability 1. If $\mu < 0$ then letting $\lambda \downarrow -2\mu$, and using monotone (or dominated) convergence, we see that $\mathbb{E} [\mathbb{1}_{[\tau < \infty]}]$ equals $e^{2a\mu}$. \square

The argument above provides the Laplace transform of τ :

$$\mathbb{E} [e^{-\theta\tau}] = e^{-a(\sqrt{\mu^2+2\theta}-\mu)} \quad \text{for all } \theta > 0. \quad (2.4.5)$$

(Note that the part $\tau = \infty$ disappears because the exponential term is then 0.)

This Laplace transform may be inverted. According to Krylov [60, Page 66] (by other methods),

$$\mathbb{P}[\tau > t] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^a e^{\mu y - \frac{1}{2}\mu^2 t} \left(e^{-\frac{y^2}{2t}} - e^{-\frac{(2a-y)^2}{2t}} \right) dy. \quad (2.4.6)$$

From Proposition 2.4.1 it follows that for a Gaussian process in \mathbb{R}^N , given by

$$t \mapsto Y(t) = \beta W(t) + \mu t,$$

where $\mu \in \mathbb{R}^N$, the process $Y + c$, where $c \in (0, \infty)^N$, leaves the positive orthant in finite time with probability 1 if the drift velocity vector μ such that some component of $\beta^{-1}\mu$ has the same sign as the component of $-\beta^{-1}c$.

Chapter 3

Exit Times Revisited

In this chapter we examine the exit time question for a stochastic process in \mathbb{R}^N in terms of a discrete approximation to the original process. We shall also look at the continuum case, and the Kolmogorov backward equation describing the probability distribution of the exit time. We will quote results on the solution of this equation, and also demonstrate how the equation can be transformed to a standard heat equation.

3.1 Extrema of Paths

We will work with correlated Brownian motions

$$t \mapsto Y_j(t) \in \mathbb{R}$$

for $j \in \{1, \dots, N\}$.

Fix threshold values $d_1, \dots, d_N < 0$, and let

$$\tau_j = \inf\{t \geq 0 : Y_j(t) \leq d_j\} \tag{3.1.1}$$

Since the paths of Y_j are continuous, we have the following equality of events:

$$[\tau_j \leq t] = \left[\inf_{0 \leq s \leq t} Y_j(s) \leq d_j \right]. \tag{3.1.2}$$

A simulation is shown in Figure 3.1. Ideally, one would like to determine the joint distribution of

$$(\tau_1, \tau_2, \dots, \tau_N)$$

In view of the equality (3.1.2), this is essentially equivalent to determining the behavior of the process of extrema:

$$t \mapsto \left(\inf_{0 \leq s \leq t} Y_1(s), \inf_{0 \leq s \leq t} Y_2(s), \dots, \inf_{0 \leq s \leq t} Y_N(s) \right). \tag{3.1.3}$$

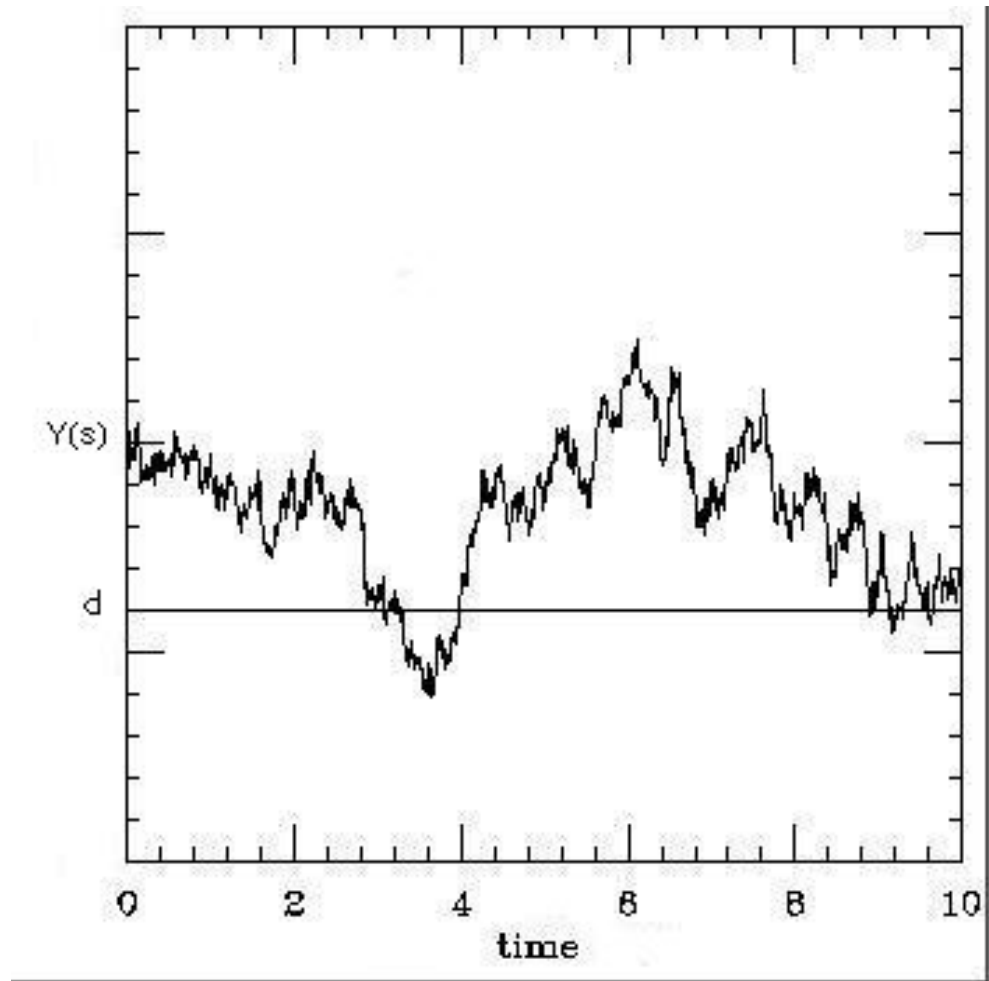


FIGURE 3.1. A Sample Path Of A Stochastic Process

We will focus mainly on the case $N = 2$.

Our objective then is to study the probability

$$P\left[\inf_{0 \leq s \leq t} Y_1(s) < d_1, \inf_{0 \leq s \leq t} Y_2(s) < d_2\right].$$

Our method will be to replace the continuous process Y with a discrete process Z , which is a random walk, which, in a limit, converges to the process Y . We first discretize time into steps of size

$$\Delta t = \delta > 0.$$

We then work with a lattice in \mathbb{R}^2 specified by

$$L_{\Delta t} = \{(m\Delta x, n\Delta y) : m, n \in \mathbb{Z}\} \quad (3.1.4)$$

where Δx and Δy are given by

$$\Delta x = \sigma_1 \sqrt{\Delta t}, \quad \text{and} \quad \Delta y = \sigma_2 \sqrt{\Delta t} \quad (3.1.5)$$

The discrete process we consider is a random walk, whose time- n position is given by

$$S_n = \sum_{j=1}^n Z_j \quad (3.1.6)$$

where Z_1, Z_2, \dots are independent identically distributed random variables, with distribution given by

$$\begin{aligned} P_{11} &:= P[Z_i = (\Delta x, \Delta y)] = \frac{1}{4} \left(1 + \rho + \frac{\mu_1 \Delta x}{\sigma_1^2} + \frac{\mu_2 \Delta y}{\sigma_2^2} \right) \\ P_{21} &:= P[Z_i = (-\Delta x, \Delta y)] = \frac{1}{4} \left(1 - \rho - \frac{\mu_1 \Delta x}{\sigma_1^2} + \frac{\mu_2 \Delta y}{\sigma_2^2} \right) \\ P_{12} &:= P[Z_i = (\Delta x, -\Delta y)] = \frac{1}{4} \left(1 - \rho + \frac{\mu_1 \Delta x}{\sigma_1^2} - \frac{\mu_2 \Delta y}{\sigma_2^2} \right) \\ P_{22} &:= P[Z_i = (-\Delta x, -\Delta y)] = \frac{1}{4} \left(1 + \rho - \frac{\mu_1 \Delta x}{\sigma_1^2} - \frac{\mu_2 \Delta y}{\sigma_2^2} \right), \end{aligned} \quad (3.1.7)$$

We assume that $\Delta t = \delta > 0$ is chosen small enough that all these transition probabilities are positive.

Lemma 3.1. Let $Y(t) = (Y_1(t), Y_2(t))$ be a two-dimensional Brownian motion with mean $(\mu_1 t, \mu_2 t)$, variance $(\sigma_1^2 t, \sigma_2^2 t)$ and correlation ρ . We use the notation and process introduced above. Then $Y(t)$ is the limit in distribution of the process

$$X_\delta(t) = \sum_{i=1}^{\lfloor t/\delta \rfloor} Z_i$$

i.e.

$$X_\delta(t) \xrightarrow{\text{dist.}} Y(t) \quad \text{as } \delta \rightarrow 0$$

Proof. Consider the characteristic function $M_{X(t)}(\theta_1, \theta_2)$ of $X(t)$ with complex θ_1 and θ_2 :

$$\begin{aligned}
M_{X(t)}(\theta_1, \theta_2) &= E[e^{(\theta_1, \theta_2)X(t)}] \\
&= E[e^{(\theta_1, \theta_2) \sum_{i=1}^{t/\Delta t} Z_i}] \\
&= \prod_{i=1}^{[t/\Delta t]} E[e^{(\theta_1, \theta_2)Z_i}] \\
&= (P_{11}e^{\theta_1\Delta x + \theta_2\Delta y} + P_{12}e^{\theta_1\Delta x - \theta_2\Delta y} \\
&\quad + P_{21}e^{-\theta_1\Delta x + \theta_2\Delta y} + P_{22}e^{-\theta_1\Delta x - \theta_2\Delta y})^{[t/\Delta t]} \\
&= (A_{11} + A_{12} + A_{21} + A_{22})^{[t/\Delta t]}.
\end{aligned}$$

where

$$\begin{aligned}
A_{11} &= P_{11}e^{\theta_1\sigma_1\sqrt{\Delta t} + \theta_2\sigma_2\sqrt{\Delta t}} \\
A_{12} &= P_{12}e^{\theta_1\sigma_1\sqrt{\Delta t} - \theta_2\sigma_2\sqrt{\Delta t}} \\
A_{21} &= P_{21}e^{-\theta_1\sigma_1\sqrt{\Delta t} + \theta_2\sigma_2\sqrt{\Delta t}} \\
A_{22} &= P_{22}e^{-\theta_1\sigma_1\sqrt{\Delta t} - \theta_2\sigma_2\sqrt{\Delta t}}
\end{aligned}$$

By using Taylor expansion on $A_{11}, A_{12}, A_{21}, A_{22}$, we have

$$\begin{aligned}
A_{11} &= \frac{1}{4}(1 + \rho + (\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2})\Delta t) \\
&\quad \times (1 + (\theta_1\sigma_1 + \theta_2\sigma_2)\sqrt{\Delta t} + \frac{1}{2}(\theta_1\sigma_1 + \theta_2\sigma_2)^2\Delta t + o(\Delta t)) \\
A_{12} &= \frac{1}{4}(1 - \rho + (\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2})\Delta t) \\
&\quad \times (1 + (\theta_1\sigma_1 - \theta_2\sigma_2)\sqrt{\Delta t} + \frac{1}{2}(\theta_1\sigma_1 - \theta_2\sigma_2)^2\Delta t + o(\Delta t)) \\
A_{21} &= \frac{1}{4}(1 - \rho - (\frac{\mu_1}{\sigma_1} - \frac{\mu_2}{\sigma_2})\Delta t) \\
&\quad \times (1 + (-\theta_1\sigma_1 + \theta_2\sigma_2)\sqrt{\Delta t} + \frac{1}{2}(\theta_1\sigma_1 - \theta_2\sigma_2)^2\Delta t + o(\Delta t)) \\
A_{22} &= \frac{1}{4}(1 + \rho - (\frac{\mu_1}{\sigma_1} + \frac{\mu_2}{\sigma_2})\Delta t) \\
&\quad \times (1 + (-\theta_1\sigma_1 - \theta_2\sigma_2)\sqrt{\Delta t} + \frac{1}{2}(\theta_1\sigma_1 + \theta_2\sigma_2)^2\Delta t + o(\Delta t)).
\end{aligned}$$

Then

$$\begin{aligned} A_{11} + A_{12} + A_{21} + A_{22} &= 1 + (\mu_1\theta_1 + \mu_2\theta_2)\Delta t \\ &\quad + \frac{1}{2}(\theta_1^2\sigma_1^2 + \theta_2^2\sigma_2^2 + 2\rho\theta_1\theta_2\sigma_1\sigma_2)\Delta t + o(\Delta t) \end{aligned}$$

and by L'Hospital's rule

$$\begin{aligned} \lim_{\Delta t \rightarrow 0} \ln M_{X(t)}(\theta_1, \theta_2) &= \lim_{\Delta t \rightarrow 0} \frac{t}{\Delta t} \ln(A_{11} + A_{12} + A_{21} + A_{22}) \\ &= (\mu_1\theta_1 + \mu_2\theta_2)t + \frac{1}{2}(\theta_1^2\sigma_1^2 + \theta_2^2\sigma_2^2 + 2\rho\theta_1\theta_2\sigma_1\sigma_2)t \\ &= \ln M_{B(t)}(\theta_1, \theta_2), \end{aligned}$$

that is

$$M_{X(t)}(\theta_1, \theta_2) \rightarrow M_{B(t)}(\theta_1, \theta_2) \text{ as } \Delta t \rightarrow 0.$$

Therefore

$$X(t) \xrightarrow{\text{dist.}} Y(t) \text{ as } \Delta t \rightarrow 0$$

□

The following result is known (see, for instance, [76]).

Lemma 3.1.1. *Let $Y(t)$ be the process defined in Lemma 3.3 and*

$$F(x_1, x_2, t) = \mathbb{P}\left[\sup_{0 \leq s \leq t} Y_1(s) \leq x_1, \sup_{0 \leq s \leq t} Y_2(s) \leq x_2\right],$$

where $0 < Y_1(0) = x_{10} < x_1$ and $0 < Y_2(0) = x_{10} < x_2$,

then $F(x_1, x_2, t)$ satisfies the following Backward Equation:

$$\frac{\partial F}{\partial t} = \frac{\sigma_1^2}{2} \frac{\partial^2 F}{\partial x_1^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 \bar{F}}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2} \frac{\partial^2 F}{\partial x_2^2} - \mu_1 \frac{\partial F}{\partial x_1} - \mu_2 \frac{\partial \bar{F}}{\partial x_2}$$

with boundary conditions:

$$F(x_1, x_2, 0) = 1, \quad F(0, x_2, t) = F(x_1, 0, t) = 0.$$

Instead of a formal proof, we present the essential argument in a manner related to simulation of the continuous processes. Our method yields a difference equation for the discrete approximation to the continuous process.

Let $\bar{F}(x_1, x_2, t)$ denote the hitting probability for the discrete process

$$t \mapsto X(t) + (x_1, x_2)$$

where (x_1, x_2) is an initial point in the lattice L_δ , with positive coordinates.

From the definition of \bar{F} , we have

$$\begin{aligned} \bar{F}(x_1, x_2, t) &= Pr[\sup_{0 \leq s \leq t} X_1(s) \leq x_1, \sup_{0 \leq s \leq t} X_2(s) \leq x_2] \\ &= P[X_1(s) \leq x_1, X_2(s) \leq x_2 \mid s \in [0, t], X_1(0) = x_{10}, X_2(0) = x_{20}] \\ &= P_{11} \cdot P[X_1(s) \leq x_1, X_2(s) \leq x_2 \mid s \in [\Delta t, t], X_1(\Delta t) = x_{10} + \Delta x \\ &\quad \text{and } X_2(\Delta t) = x_{20} + \Delta y] \\ &+ P_{21} \cdot P[X_1(s) \leq x_1, X_2(s) \leq x_2 \mid s \in [\Delta t, t], X_1(\Delta t) = x_{10} - \Delta x \\ &\quad \text{and } X_2(\Delta t) = x_{20} + \Delta y] \\ &+ P_{12} \cdot P[X_1(s) \leq x_1, X_2(s) \leq x_2 \mid s \in [\Delta t, t], X_1(\Delta t) = x_{10} + \Delta x \\ &\quad \text{and } X_2(\Delta t) = x_{20} - \Delta y] \\ &+ P_{22} \cdot P[X_1(s) \leq x_1, X_2(s) \leq x_2 \mid s \in [\Delta t, t], X_1(\Delta t) = x_{10} - \Delta x \\ &\quad \text{and } X_2(\Delta t) = x_{20} - \Delta y] \end{aligned}$$

$$\begin{aligned}
&= P_{11} \cdot Pr[X_1(s) \leq x_1 - \Delta x, X_2(s) \leq x_2 - \Delta y \mid s \in [0, t - \Delta t] | X_1(0) = x_{10} \\
&\quad \text{and } X_2(0) = x_{20}] \\
&+ P_{21} \cdot Pr[X_1(s) \leq x_1 + \Delta x, X_2(s) \leq x_2 - \Delta y \mid s \in [0, t - \Delta t] | X_1(0) = x_{10} \\
&\quad \text{and } X_2(0) = x_{20}] \\
&+ P_{12} \cdot Pr[X_1(s) \leq x_1 - \Delta x, X_2(s) \leq x_2 + \Delta y \mid s \in [0, t - \Delta t] | X_1(0) = x_{10} \\
&\quad \text{and } X_2(0) = x_{20}] \\
&+ P_{22} \cdot Pr[X_1(s) \leq x_1 + \Delta x, X_2(s) \leq x_2 + \Delta y \mid s \in [0, t - \Delta t] | X_1(0) = x_{10} \\
&\quad \text{and } X_2(0) = x_{20}] \\
&= P_{11} \cdot \bar{F}(x_1 - \Delta x, x_2 - \Delta y, t - \Delta t) + P_{21} \cdot \bar{F}(x_1 + \Delta x, x_2 - \Delta y, t - \Delta t) \\
&\quad + P_{12} \cdot \bar{F}(x_1 - \Delta x, x_2 + \Delta y, t + \Delta t) + P_{22} \cdot \bar{F}(x_1 + \Delta x, x_2 + \Delta y, t - \Delta t)
\end{aligned}$$

To summarize,

$$\begin{aligned}
\bar{F}(x_1, x_2, t) &= P_{11} \cdot \bar{F}(x_1 - \Delta x, x_2 - \Delta y, t - \Delta t) + P_{21} \cdot \bar{F}(x_1 + \Delta x, x_2 - \Delta y, t - \Delta t) \\
&\quad + P_{12} \cdot \bar{F}(x_1 - \Delta x, x_2 + \Delta y, t + \Delta t) + P_{22} \cdot \bar{F}(x_1 + \Delta x, x_2 + \Delta y, t - \Delta t)
\end{aligned} \tag{3.1.8}$$

This difference equation governs the hitting time distribution of the discrete process X .

To understand, at a formal level, the relationship with the Kolmogorov backward equation, we use Taylor expansion on the right hand side, assuming that \bar{F} arises

from a smooth enough function, defined in the continuum. Then

$$\begin{aligned}
& \bar{F}(x_1, x_2, t) \\
= & P_{11}[\bar{F} - \Delta x \frac{\partial \bar{F}}{\partial x_1} - \Delta y \frac{\partial \bar{F}}{\partial x_2} - \Delta t \frac{\partial \bar{F}}{\partial t} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 \bar{F}}{\partial x_1^2} + \frac{1}{2}(\Delta y)^2 \frac{\partial^2 \bar{F}}{\partial x_2^2} \\
& + \Delta x \Delta y \frac{\partial^2 \bar{F}}{\partial x_1 \partial x_2} + o(\Delta t^2)] \\
+ & P_{21}[\bar{F} + \Delta x \frac{\partial \bar{F}}{\partial x_1} - \Delta y \frac{\partial \bar{F}}{\partial x_2} - \Delta t \frac{\partial \bar{F}}{\partial t} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 \bar{F}}{\partial x_1^2} + \frac{1}{2}(\Delta y)^2 \frac{\partial^2 \bar{F}}{\partial x_2^2} \\
& - \Delta x \Delta y \frac{\partial^2 \bar{F}}{\partial x_1 \partial x_2} + o(\Delta t^2)] \\
+ & P_{12}[\bar{F} - \Delta x \frac{\partial \bar{F}}{\partial x_1} + \Delta y \frac{\partial \bar{F}}{\partial x_2} - \Delta t \frac{\partial \bar{F}}{\partial t} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 \bar{F}}{\partial x_1^2} + \frac{1}{2}(\Delta y)^2 \frac{\partial^2 \bar{F}}{\partial x_2^2} \\
& - \Delta x \Delta y \frac{\partial^2 \bar{F}}{\partial x_1 \partial x_2} + o(\Delta t^2)] \\
+ & P_{22}[\bar{F} + \Delta x \frac{\partial \bar{F}}{\partial x_1} + \Delta y \frac{\partial \bar{F}}{\partial x_2} - \Delta t \frac{\partial \bar{F}}{\partial t} + \frac{1}{2}(\Delta x)^2 \frac{\partial^2 \bar{F}}{\partial x_1^2} + \frac{1}{2}(\Delta y)^2 \frac{\partial^2 \bar{F}}{\partial x_2^2} \\
& + \Delta x \Delta y \frac{\partial^2 \bar{F}}{\partial x_1 \partial x_2} + o(\Delta t^2)]
\end{aligned}$$

Simplify the above equation and plug in $P_{11}, P_{21}, P_{12}, P_{22}, \Delta x$ and Δy , we have

$$\begin{aligned}
\frac{\partial \bar{F}}{\partial t} \Delta t &= \frac{\sigma_1^2}{2} \frac{\partial^2 \bar{F}}{\partial x_1^2} \Delta t + \rho \sigma_1 \sigma_2 \frac{\partial^2 \bar{F}}{\partial x_1 \partial x_2} \Delta t + \frac{\sigma_2^2}{2} \frac{\partial^2 \bar{F}}{\partial x_2^2} \Delta t \\
&\quad - \mu_1 \frac{\partial \bar{F}}{\partial x_1} \Delta t - \mu_2 \frac{\partial \bar{F}}{\partial x_2} \Delta t + o(\Delta t^2).
\end{aligned}$$

dividing both sides by Δt , we have

$$\frac{\partial \bar{F}}{\partial t} = \frac{\sigma_1^2}{2} \frac{\partial^2 \bar{F}}{\partial x_1^2} + \rho \sigma_1 \sigma_2 \frac{\partial^2 \bar{F}}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2} \frac{\partial^2 \bar{F}}{\partial x_2^2} - \mu_1 \frac{\partial \bar{F}}{\partial x_1} - \mu_2 \frac{\partial \bar{F}}{\partial x_2} + o(\Delta t).$$

and the boundary conditions come from the initial conditions:

$$0 < X_1(0) = x_{10} < x_1$$

$$0 < X_2(0) = x_{10} < x_2$$

The following theorem gives a solution of the above Kolmogorov Backward Equation.

Theorem 3.2. The Kolmogorov Backward Equation

$$\frac{\partial F}{\partial t} = \frac{\sigma_1^2}{2} \frac{\partial^2 F}{\partial x_1^2} + \rho \sigma_1 \sigma_2 \frac{\partial^2 F}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2} \frac{\partial^2 F}{\partial x_2^2} - \mu_1 \frac{\partial F}{\partial x_1} - \mu_2 \frac{\partial F}{\partial x_2} \quad (3.1.9)$$

with boundary conditions:

$$F(x_1, x_2, 0) = 1, \quad F(0, x_2, t) = F(x_1, 0, t) = 0$$

has the solution

$$F(x_1, x_2, t) = \frac{2}{\alpha' t} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta'}{\alpha'}\right) e^{-r'^2/2t} \int_0^{\alpha'} \sin\left(\frac{n\pi\theta}{\alpha'}\right) g_n(\theta) d\theta$$

where

$$\begin{aligned} g_n(\theta) &= \int_0^{\infty} r e^{-r^2/2t} \mathbb{1}_{n\pi/\alpha} \left(\frac{rr'}{t} \right) dr \\ \tan \alpha' &= -\frac{\sqrt{1-\rho^2}}{\rho} \\ \alpha &= \alpha' - \frac{\pi}{2} \\ r' &= \frac{1}{\sqrt{1-\rho^2}} \left(\frac{x_1^2}{\sigma_1^2} - \frac{2\rho x_1 x_2}{\sigma_1 \sigma_2} + \frac{x_2^2}{\sigma_2^2} \right)^{\frac{1}{2}} \\ \cos \theta &= \frac{x_1}{\sigma_1 r'} \\ \theta' &= \theta + \alpha \end{aligned}$$

Full details of a solution are worked out in e the Ph.D. thesis of Rebholz (1994) [76]. See also Caslow (1947) [22] for an approach using separation of variables. Here we shall describe how the equation can be transformed into a standard heat equation. Let

$$F(x_1, x_2, t) = e^{m_1 x_1 + m_2 x_2 + at} G(x_1, x_2, t),$$

where

$$m_1 = \frac{\mu_1\sigma_2 - \rho\mu_2\sigma_1}{(1 - \rho^2)\sigma_2\sigma_1^2} \quad m_2 = \frac{\mu_2\sigma_1 - \rho\mu_1\sigma_2}{(1 - \rho^2)\sigma_1\sigma_2^2}$$

$$a = \frac{\sigma_1^2}{2}m_1^2 + \rho\sigma_1\sigma_2m_1m_2 + \frac{\sigma_2^2}{2}m_2^2 - \mu_1m_1 - \mu_2m_2.$$

Then $F(x_1, x_2, t)$ solves

$$\frac{\partial F}{\partial t} = \frac{\sigma_1^2}{2} \frac{\partial^2 F}{\partial x_1^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 F}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2} \frac{\partial^2 F}{\partial x_2^2} - \mu_1 \frac{\partial F}{\partial x_1} - \mu_2 \frac{\partial F}{\partial x_2}$$

with boundary conditions:

$$F(x_1, x_2, 0) = 1, \quad F(0, x_2, t) = F(x_1, 0, t) = 0$$

if and only if $G(x_1, x_2, t)$ solves

$$\frac{\partial G}{\partial t} = \frac{\sigma_1^2}{2} \frac{\partial^2 G}{\partial x_1^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 G}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2} \frac{\partial^2 G}{\partial x_2^2}$$

with boundary conditions:

$$G(x_1, x_2, 0) = e^{-m_1x_1 - m_2x_2}, \quad G(0, x_2, t) = G(x_1, 0, t) = 0$$

Define two new variables ξ_1 and ξ_2 as below,

$$\xi_1 = \frac{1}{\sigma_1}x_1$$

$$\xi_2 = \frac{1}{\sqrt{1 - \rho^2}} \left(-\frac{\rho}{\sigma_1}x_1 + \frac{1}{\sigma_2}x_2 \right).$$

By the above substitution, $G(x_1, x_2, t)$ becomes $H(\xi_1, \xi_2, t)$ and the PDE

$$\frac{\partial G}{\partial t} = \frac{\sigma_1^2}{2} \frac{\partial^2 G}{\partial x_1^2} + \rho\sigma_1\sigma_2 \frac{\partial^2 G}{\partial x_1 \partial x_2} + \frac{\sigma_2^2}{2} \frac{\partial^2 G}{\partial x_2^2},$$

with boundary conditions:

$$G(x_1, x_2, 0) = e^{-m_1x_1 - m_2x_2}, \quad G(0, x_2, t) = G(x_1, 0, t) = 0,$$

becomes

$$\frac{\partial H}{\partial t} = \frac{1}{2} \frac{\partial^2 H}{\partial \xi_1^2} + \frac{1}{2} \frac{\partial^2 H}{\partial \xi_2^2}$$

with boundary conditions:

$$H(\xi_1, \xi_2, 0) = e^{-(m_1\sigma_1 + \rho m_2\sigma_2)\xi_1 - (m_2\sigma_2\sqrt{1-\rho^2})\xi_2}$$

$$H(\xi_1, \xi_2, t) = 0 \quad \text{if } \xi_1 = 0 \quad \text{and } \xi_2 > 0$$

$$H(\xi_1, \xi_2, t) = 0 \quad \text{if } \xi_2 = -\frac{\rho}{\sqrt{1-\rho^2}} \quad \text{and } \xi_2 > 0.$$

Chapter 4

The Gaussian Copula Model

In this chapter we assume, as a phenomenological model, that the exit time behavior of an underlying process is governed by a ‘Gaussian copula model.’ We will also present some analogous results for a Poisson-type model.

In more detail, we assume that there exist independent standard Gaussian variables $Z, \epsilon_1, \dots, \epsilon_N$, and parameters $\rho > 0$ and $c > 0$ such that, with

$$X_i = \sqrt{\rho}Z + \sqrt{1 - \rho}\epsilon_i, \quad (4.0.1)$$

the event that the i -th component Y_i of an underlying stochastic process exits a threshold value is given through

$$[X_i \leq c].$$

Note that we assume a common correlation

$$\mathbb{E}[X_i X_j] = \rho > 0, \text{ for all } i \neq j. \quad (4.0.2)$$

For the results of this chapter, we will draw from intuition based on credit derivative modeling of CDO instruments. To make the comparison, we should view the event $[X_i \leq c]$ as a default of a name i in a portfolio of N CDS names, within a fixed time horizon. The event that exactly k of the random variables X_i have values $\leq c$ will be called an *equity tranche*. The complementary event of having more than k such hits will be called a *senior tranche*. We view an event $[X_i \leq c]$ as a ‘loss’ of name i . We will also use terms such as ‘delta’ and ‘Gamma’, inspired by concepts in the CDO context. We also present similar results for a Poisson-type model.

Sections 4.1 to 4.5 present our results. The remainder of the chapter is largely a discussion about the relationship of the mathematical results with the CDO context.

4.1 Sensitivity to Correlation in the Gaussian Model

We use the notation

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{and} \quad \Phi(x) = \int_{-\infty}^x \phi(s) ds. \quad (4.1.1)$$

The probability that exactly j of the variables X_i are at level $\leq c$, is given by

$$p_j = \int_{\mathbb{R}} \binom{N}{j} p^j (1-p)^{N-j} \phi(x) dx \quad (4.1.2)$$

where

$$p = \mathbb{P}[X_i \leq c | Z = x] = \Phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1-\rho}}\right) \quad (4.1.3)$$

Let ν be the random variable counting the number of i for which $X_i \leq c$. A convenient way to study the joint behavior of the events $[X_i < c]$ in terms of ν , is by using the ‘cut-off’ random variables

$$\nu_k = \min\{\nu, k\} = \mathbb{1}_{[\nu=1]} + 2\mathbb{1}_{[\nu=2]} + \cdots + (k-1)\mathbb{1}_{[\nu=k-1]} + k\mathbb{1}_{[\nu \geq k]} \quad (4.1.4)$$

and the complementary variables

$$\nu_k^s = \nu - \min\{\nu, k\} = \mathbb{1}_{[\nu=k+1]} + 2\mathbb{1}_{[\nu=k+2]} + \cdots + (N-k)\mathbb{1}_{[\nu=N]} \quad (4.1.5)$$

We can now formulate our first main result for this model.

Theorem 4.1.1. *Assume that $Z, \epsilon_1, \dots, \epsilon_N$ are independent standard Gaussian variables, with $N > 1$, and let*

$$X_i = \sqrt{\rho}Z + \sqrt{1-\rho}\epsilon_i, \quad \text{for } i \in \{1, \dots, N\}$$

where $\rho \in (0, 1)$. Let $c \in \mathbb{R}$. Let ν be the random variable which counts the number of X_j with value $< c$:

$$\nu = \#\{j \in \{1, \dots, N\} : X_j < c\} \quad (4.1.6)$$

and, for $k \in \{1, \dots, N\}$,

$$\nu_k = \min\{\nu, k\} \quad (4.1.7)$$

$$\nu_k^s = \nu - \min\{\nu, k\}. \quad (4.1.8)$$

Then the expected value of ν has no dependence on ρ :

$$\frac{d\mathbb{E}[\nu]}{d\rho} = 0.$$

Moreover,

$$\frac{d\mathbb{E}[\nu_k]}{d\rho} < 0, \quad \text{and} \quad \frac{d\mathbb{E}[\nu_k^s]}{d\rho} > 0,$$

for $1 \leq k < N$.

The rest of this section is devoted to proving this result.

Since

$$\nu_k + \nu_k^s = \nu,$$

we have

$$\mathbb{E}[\nu_k] + \mathbb{E}[\nu_k^s] = \mathbb{E}[\nu]$$

Now

$$\begin{aligned} \mathbb{E}[\nu] &= \mathbb{E}\left[\sum_{j=1}^N \mathbb{1}_{[X_j < c]}\right] \\ &= N\mathbb{P}[X_1 < c] = N\Phi(c), \end{aligned}$$

which is clearly independent of ρ . Thus,

$$\frac{d\mathbb{E}[\nu_k^s]}{d\rho} = -\frac{d\mathbb{E}[\nu_k]}{d\rho}$$

So it will suffice to prove that $\frac{d\mathbb{E}[\nu_k]}{d\rho}$ is negative.

The expected value of ν_k is

$$\mathbb{E}[\nu_k] = p_1 + 2p_2 + \cdots + (k-1)p_{k-1} + k[1 - p_0 - \cdots - p_{k-1}],$$

which can be rewritten as

$$\mathbb{E}[\nu_k] = k - \sum_{j=0}^k (k-j)p_j. \quad (4.1.9)$$

From this we have

$$\begin{aligned} \frac{d\mathbb{E}[\nu_k]}{d\rho} &= - \sum_{j=0}^k (k-j) \binom{N}{j} \int_{\mathbb{R}} [jp^{j-1}(1-p)^{N-j} - (N-j)p^j(1-p)^{N-j-1}] \frac{\partial p}{\partial \rho} \phi(x) dx \\ &= \int_{\mathbb{R}} I(p) \frac{\partial p}{\partial \rho} \phi(x) dx \end{aligned}$$

where

$$I(p) \stackrel{\text{def}}{=} - \sum_{j=0}^k \binom{N}{j} (k-j) [jp^{j-1}(1-p)^{N-j} - (N-j)p^j(1-p)^{N-j-1}] \quad (4.1.10)$$

(Note that the integrand in the expression for $d\mathbb{E}[\nu_k]/d\rho$ contains an exponentially decreasing term in x^2 , which ensures that $d/d\rho$ and $\int_{\mathbb{R}} \dots dx$ can be interchanged.)

We can now compute the derivative $\partial p/\partial \rho$ from (4.1.3):

$$\begin{aligned} \frac{\partial p}{\partial \rho} &= \phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1-\rho}}\right) \frac{\sqrt{1-\rho} \left\{-\frac{1}{2\sqrt{\rho}}x\right\} - (c - \sqrt{\rho}x) \left\{-\frac{1}{2\sqrt{1-\rho}}\right\}}{1-\rho} \\ &= -\frac{(1-\rho)x - \sqrt{\rho}(c - \sqrt{\rho}x)}{2\sqrt{\rho}(1-\rho)^{3/2}} \phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1-\rho}}\right) \\ &= -\frac{x - c\sqrt{\rho}}{2\sqrt{\rho}(1-\rho)^{3/2}} \phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1-\rho}}\right). \end{aligned}$$

So

$$\begin{aligned} \frac{d\mathbb{E}[\nu_k]}{d\rho} &= - \int_{\mathbb{R}} I(p) \frac{(x - c\sqrt{\rho})}{2\sqrt{\rho}(1-\rho)^{3/2}} \phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1-\rho}}\right) \phi(x) dx \\ &\stackrel{\underbrace{\hspace{1cm}}}{y=x-c\sqrt{\rho}} = - \int_{\mathbb{R}} I(p) \frac{y}{2(1-\rho)^{3/2}\sqrt{\rho}} \phi\left(\frac{c(1-\rho) - \sqrt{\rho}y}{\sqrt{1-\rho}}\right) \phi(y + c\sqrt{\rho}) dy \\ &= - \int_{\mathbb{R}} I(p) \frac{y}{2(1-\rho)^{3/2}\sqrt{\rho}} \frac{1}{2\pi} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy \end{aligned}$$

Looking back at (4.1.3), let us write

$$p(y) = p = \Phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1 - \rho}}\right) = \Phi\left(\frac{c(1 - \rho) - \sqrt{\rho}y}{\sqrt{1 - \rho}}\right) \quad (4.1.11)$$

Note that this is clearly monotonically decreasing in y .

Returning again to the derivative $d\mathbb{E}[\nu_k]/d\rho$, we have:

$$\frac{d\mathbb{E}[\nu_k]}{d\rho} = - \int_0^\infty [I(p(y)) - I(p(-y))] \frac{y}{2(1 - \rho)^{3/2}\sqrt{\rho}} \frac{1}{2\pi} e^{-\frac{y^2}{2(1 - \rho)} - \frac{c^2}{2}} dy \quad (4.1.12)$$

As we prove below in Lemma 4.1.2, the function $I(\cdot)$ is monotonically decreasing.

Now, as noted above, for $y > 0$, we have $p(y) < p(-y)$. Hence,

$$I(p(y)) - I(p(-y)) > 0 \quad \text{for any } y > 0.$$

This implies, from (4.1.12), that

$$d\mathbb{E}[\nu_k]/d\rho < 0,$$

which is the result we had set out to prove.

We have used the following observation about $I(p)$:

Lemma 4.1.2. *Let*

$$I(p) = - \sum_{j=0}^k \binom{N}{j} (k - j) [jp^{j-1}(1 - p)^{N-j} - (N - j)p^j(1 - p)^{N-j-1}]$$

where N and k are positive integers, with $k \leq N$, and $p \in [0, 1]$. Then

$$I(p) = N - (N - k)k \binom{N}{k} \int_0^p t^{k-1}(1 - t)^{N-k-1} dt \quad (4.1.13)$$

In particular, $I(p)$ is monotonically decreasing with p , if $1 \leq k < N$.

Proof First let us rework the expression for $I(p)$:

$$\begin{aligned}
I(p) &\stackrel{\text{def}}{=} - \sum_{j=0}^k \binom{N}{j} (k-j) [jp^{j-1}(1-p)^{N-j} - (N-j)p^j(1-p)^{N-j-1}] \\
&= - \sum_{j=1}^k \binom{N}{j} (k-j)jp^{j-1}(1-p)^{N-j} \\
&\quad - \sum_{j=0}^{k-1} \binom{N}{j} (k-j)(N-j)p^j(1-p)^{N-j-1} \\
&= - \sum_{j=0}^{k-1} \left[\binom{N}{j+1} (k-j-1)(j+1) - \binom{N}{j} (k-j)(N-j) \right] p^j(1-p)^{N-j-1} \\
&= \sum_{j=0}^{k-1} \binom{N}{j} (N-j)p^j(1-p)^{N-j-1} \\
&= N(1-p)^{N-1} + \sum_{j=1}^{k-1} \binom{N}{j} (N-j)p^j(1-p)^{N-j-1}
\end{aligned}$$

Taking the derivative, we obtain

$$\begin{aligned}
I'(p) &= \sum_{j=1}^{k-1} \binom{N}{j} (N-j)jp^{j-1}(1-p)^{N-j-1} - \sum_{j=0}^{k-1} \binom{N}{j} (N-j)p^j(N-j-1)(1-p)^{N-j-2} \\
&= \sum_{j=0}^{k-2} \underbrace{\left\{ \binom{N}{j+1} (N-j-1)(j+1) - \binom{N}{j} (N-j)(N-j-1) \right\}}_0 p^j(1-p)^{N-j-2} \\
&\quad - \binom{N}{k-1} (N-k+1)(N-k)p^{k-1}(1-p)^{N-k-1}
\end{aligned}$$

Rewriting the last term, we have

$$I'(p) = -(N-k)k \binom{N}{k} p^{k-1}(1-p)^{N-k-1}$$

Integrating, and using the value N for $I(0)$, we obtain (4.1.13). \square

4.2 Sensitivity to the Threshold

We wish to study the sensitivity of the distribution of ν to changes in the threshold value c . To this end we look at

$$\frac{\partial \mathbb{E}[\nu_k]}{\partial c},$$

where

$$\nu_k = \mathbb{E} [\min\{k, \nu\}]. \quad (4.2.1)$$

The normalized form

$$\Delta_k = \frac{\frac{\partial \mathbb{E} [\nu_k]}{\partial c}}{\frac{\partial \mathbb{E} [\nu]}{\partial c}}, \quad (4.2.2)$$

is a more convenient quantity. For the denominator we observe first that

$$\mathbb{E} [\nu] = N\Phi(c)$$

and so

$$\frac{\partial \mathbb{E} [\nu]}{\partial c} = N\phi(c) = \frac{N}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}.$$

Then, from (4.1.9),

$$\mathbb{E} [\nu_k] = k - \sum_{j=0}^k (k-j)p_j, \quad (4.2.3)$$

where

$$p_j = \int_{\mathbb{R}} \binom{N}{j} p^j (1-p)^{N-j} \phi(x) dx, \quad (4.2.4)$$

and

$$p = \Phi\left(\frac{c - \sqrt{\rho}x}{\sqrt{1-\rho}}\right). \quad (4.2.5)$$

Theorem 4.2.1. *The numbers*

$$p_{\Delta_s}(k) = \int_{\mathbb{R}} \binom{N-1}{k-1} p(y)^{k-1} (1-p(y))^{N-k} \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy \quad (4.2.6)$$

for $0 \in \{1, \dots, N\}$, with $p_{\Delta_s}(0)$ being 0 by definition, specify a probability measure on $\{0, \dots, N\}$. For any $k \in \{0, 1, \dots, N\}$, we have

$$\Delta_k = \sum_{j=0}^k p_{\Delta_s}(j). \quad (4.2.7)$$

Proof. From (4.2.3) we have

$$\begin{aligned}\frac{d\mathbb{E}[\nu_k]}{dc} &= -\sum_{j=0}^k (k-j) \binom{N}{j} \int_{\mathbb{R}} [jp^{j-1}(1-p)^{N-j} - (N-j)p^j(1-p)^{N-j-1}] \frac{\partial p}{\partial c} \phi(x) dx \\ &= \int_{\mathbb{R}} I(p) \frac{\partial p}{\partial c} \phi(x) dx\end{aligned}$$

where, as seen in the proof of Lemma 4.1.2,

$$I(p) = \sum_{j=0}^{k-1} \binom{N}{j} (N-j)p^j(1-p)^{N-j-1}. \quad (4.2.8)$$

Now

$$\frac{\partial p}{\partial c} = \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{(c-\sqrt{\rho}x)^2}{2(1-\rho)}},$$

and

$$\frac{\partial p}{\partial c} \phi(x) = \frac{1}{2\pi\sqrt{1-\rho}} e^{-\frac{(x-c\sqrt{\rho})^2}{2(1-\rho)} - \frac{c^2}{2}}.$$

Setting $y = x - c\sqrt{\rho}$, we have

$$\begin{aligned}\frac{d\mathbb{E}[\nu_k]}{dc} &= \int_{\mathbb{R}} I(p) \frac{\partial p}{\partial c} \phi(x) dx \\ &= \int_{\mathbb{R}} I(p) \frac{1}{2\pi\sqrt{1-\rho}} e^{-\frac{(x-c\sqrt{\rho})^2}{2(1-\rho)} - \frac{c^2}{2}} dx \\ &= \int_{\mathbb{R}} I(p(y)) \frac{1}{2\pi\sqrt{1-\rho}} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy\end{aligned}$$

Therefore

$$\begin{aligned}\Delta_k &= \frac{\partial \mathbb{E}[\nu_k]}{\partial c} / \frac{\partial \mathbb{E}[\nu]}{\partial c} \\ &= \frac{1}{N} \int_{\mathbb{R}} I(p(y)) \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy\end{aligned} \quad (4.2.9)$$

As $I(p)$ is monotonically decreasing with p and $p(y) = \Phi\left(\frac{c(1-\rho)-\sqrt{\rho}y}{\sqrt{1-\rho}}\right)$ is monotonically increasing with c , the delta (4.2.9) decreases with increasing c .

The expression (4.2.8) shows that

$$\Delta_k = \sum_{j \in \{0,1,\dots,k\}} p_{\Delta_s}(k),$$

where

$$p_{\Delta_s}(k) = \frac{1}{N} \int_{\mathbb{R}} \binom{N}{k-1} (N-k+1) p(y)^{k-1} (1-p(y))^{N-k} \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy,$$

understood to be 0 when k is 0. This expression simplifies to (4.2.6). \square

This result confirms, for the Gaussian copula model, the generally held view that the delta with respect to index spread movements is a probability measure on the loss levels (see, for instance, [70]).

4.3 Gamma: A Convexity Measure

This section is best appreciated in the CDO terminology, which we shall use, and may be read in consultation with section 4.6 below.

Consider a portfolio with a short position on an equity tranche with losses $\leq k$ and a long position on h units of the index (entire portfolio). The expected loss of this hedged portfolio is then

$$V(h) = h\mathbb{E}[\nu] - \mathbb{E}[\nu_k].$$

The ‘convexity’ for ν_k is described through

$$\Gamma_k = \frac{\partial^2 V(h)}{\partial c^2} \Big|_{h=\Delta_k}, \tag{4.3.1}$$

i.e. it is h fixed.

Theorem 4.3.1. *For each $k \in \{0, 1, \dots, k\}$, the quantity Γ_k is positive.*

Proof. Recall that

$$\frac{d\mathbb{E}[\nu_k]}{dc} = \int_{\mathbb{R}} I(p(y)) \frac{1}{2\pi\sqrt{1-\rho}} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy$$

and

$$\frac{\partial \mathbb{E}[\nu]}{\partial c} = \frac{N}{\sqrt{2\pi}} e^{-\frac{c^2}{2}},$$

where

$$I(p) = N - (N - k)k \binom{N}{k} \int_0^p t^{k-1} (1-t)^{N-k-1} dt$$

and

$$p(y) = \Phi \left(\frac{c(1-\rho) - \sqrt{\rho}y}{\sqrt{1-\rho}} \right) \Delta_k = \frac{1}{N} \int_{\mathbb{R}} I(p(y)) \frac{1}{\sqrt{2\pi(1-\rho)}} e^{-\frac{y^2}{2(1-\rho)}} dy.$$

Then

$$\frac{\partial^2 \mathbb{E}[\nu_k]}{\partial c^2} = \frac{1}{2\pi\sqrt{1-\rho}} \int_{\mathbb{R}} \left[\frac{\partial I(p)}{\partial c} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} + (-c)I(p) e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} \right] dy,$$

and

$$\frac{\partial I(p)}{\partial c} = -(N-k)k \binom{N}{k} p^{k-1} (1-p)^{N-k-1} \cdot \frac{\sqrt{1-\rho}}{\sqrt{2\pi}} e^{-\frac{(c(1-\rho) - \sqrt{\rho}y)^2}{2(1-\rho)}}$$

On the other hand,

$$\frac{\partial^2 \mathbb{E}[\nu]}{\partial c^2} = \frac{-cN}{\sqrt{2\pi}} e^{-\frac{c^2}{2}}.$$

Therefore

$$\begin{aligned} \Gamma_k &= \Delta_k \frac{\partial^2 L_N}{\partial c^2} - \frac{\partial^2 L_k^c}{\partial c^2} \\ &= \frac{-c}{2\pi\sqrt{1-\rho}} \int_{\mathbb{R}} I(p) e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy \\ &\quad - \frac{1}{2\pi\sqrt{1-\rho}} \int_{\mathbb{R}} \left[\frac{\partial I(p)}{\partial c} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} + (-c)I(p) e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} \right] dy \\ &= -\frac{1}{2\pi\sqrt{1-\rho}} \int_{\mathbb{R}} \frac{\partial I(p)}{\partial c} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy. \end{aligned}$$

Clearly, $\frac{\partial I(p)}{\partial c} < 0$ for all $y \in \mathbb{R}$, and so $\Gamma_k > 0$. □

4.4 Poisson-mix Model

It is well known that a binomial (N, p) distribution is approximately Poisson (Np) , when N is large, p is small, and Np is fixed. This suggests study of a limiting case, with the binomials distribution replaced by a Poisson. This is the object of this section.

Under the Poisson-mix model,

$$p_j \stackrel{\text{def}}{=} \mathbb{P}[\nu = j] = \int_{\mathbb{R}} e^{-Np} \frac{(Np)^j}{j!} \phi(x) dx, \quad (4.4.1)$$

where

$$p = \mathbb{P}[X_i \leq c \mid Z = x] = \Phi \left(\frac{c - \sqrt{\rho}x}{\sqrt{1 - \rho}} \right).$$

Then we have

$$\begin{aligned} \frac{\partial \mathbb{E}[\nu_k]}{\partial \rho} &= - \sum_{j=0}^k (k-j) \int_{\mathbb{R}} e^{-Np} \left[-N \frac{(Np)^j}{j!} + N \frac{(Np)^{j-1}}{(j-1)!} \right] \frac{\partial p}{\partial \rho} \phi(x) dx \\ &= \int_{\mathbb{R}} I_1(p) \frac{\partial p}{\partial \rho} \phi(x) dx \end{aligned}$$

where

$$I_1(p) = - \sum_{j=0}^k (k-j) e^{-Np} \left(-N \frac{(Np)^j}{j!} + N \frac{(Np)^{j-1}}{(j-1)!} \right),$$

where the second term here is taken to be 0 when $j = 0$.

Lemma 4.4.1. *Let*

$$I_1(p) = - \sum_{j=0}^k (k-j) e^{-Np} \left[-N \frac{(Np)^j}{j!} + N \frac{(Np)^{j-1}}{(j-1)!} \right],$$

where N and k are positive integers, with $k \leq N$, and $p \in [0, 1]$. Then

$$I_1(p) = N - N^2 \int_0^p e^{-Nt} \frac{(Nt)^{k-1}}{(k-1)!} dt \quad (4.4.2)$$

In particular, $I_1(p)$ is monotonically decreasing with p , if $1 \leq k < N$.

Proof. First let us simplify the expression for $I_1(p)$:

$$\begin{aligned} I_1(p) &\stackrel{\text{def}}{=} - \sum_{j=0}^k (k-j) e^{-Np} \left[-N \frac{(Np)^j}{j!} + N \frac{(Np)^{j-1}}{(j-1)!} \right] \\ &= e^{-Np} \left[\sum_{j=0}^{k-1} (k-j) N \frac{(Np)^j}{j!} - \sum_{j=1}^k (k-j) N \frac{(Np)^{j-1}}{(j-1)!} \right] \\ &= e^{-Np} \left[\sum_{j=0}^{k-1} (k-j) N \frac{(Np)^j}{j!} - \sum_{j=0}^{k-1} (k-j-1) N \frac{(Np)^j}{j!} \right] \\ &= N e^{-Np} \sum_{j=0}^{k-1} \frac{(Np)^j}{j!} \end{aligned} \quad (4.4.3)$$

Taking the derivative, we obtain

$$\begin{aligned}
I_1'(p) &= -N^2 e^{-Np} \sum_{j=0}^{k-1} \frac{(Np)^j}{j!} + N^2 e^{-Np} \sum_{j=1}^{k-1} \frac{(Np)^{j-1}}{(j-1)!} \\
&= -N^2 e^{-Np} \sum_{j=0}^{k-1} \frac{(Np)^j}{j!} + N^2 e^{-Np} \sum_{j=0}^{k-2} \frac{(Np)^j}{j!} \\
&= -N^2 e^{-Np} \frac{(Np)^{k-1}}{(k-1)!}
\end{aligned}$$

Integrating, and using the value N for $I_1(0)$ by (4.4.3), we obtain equation (4.4.2). \square

The functions I_1 and I are both monotonically decreasing. By using reasoning similar to that used for the binomial case, we obtain the following result:

$$\frac{d\mathbb{E}[\nu_k]}{d\rho} < 0, \quad \text{and} \quad \frac{d\mathbb{E}[\nu_k^s]}{d\rho} > 0. \quad (4.4.4)$$

For evaluation of Δ_k , we can simply replace $I(p)$ by $I_1(p)$ and all proofs continue to be valid. Hence under the Poisson distribution, we have the same properties for Δ_k .

To evaluate Γ_k , we can replace $I(p)$ by $I_1(p)$, to obtain:

$$\begin{aligned}
\Gamma_k &= \Delta_k \frac{\partial^2 L_N}{\partial c^2} - \frac{\partial^2 L_k^c}{\partial c^2} \\
&= \frac{-c}{2\pi\sqrt{1-\rho}} \int_{\mathbb{R}} I(p) e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy \\
&\quad - \frac{1}{2\pi\sqrt{1-\rho}} \int_{\mathbb{R}} \left[\frac{\partial I(p)}{\partial c} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} + (-c) I(p) e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} \right] dy \\
&= -\frac{1}{2\pi\sqrt{1-\rho}} \int_{\mathbb{R}} \frac{\partial I(p)}{\partial c} e^{-\frac{y^2}{2(1-\rho)} - \frac{c^2}{2}} dy.
\end{aligned}$$

we only need to see the sign of $\frac{\partial I_1(p)}{\partial c}$. By Lemma 4.4.1, we have

$$\begin{aligned}
\frac{\partial I_1(p)}{\partial c} &= \frac{\partial I_1(p)}{\partial p} \frac{\partial p}{\partial c} \\
&= -N^2 e^{-Np} \frac{(Nt)^{k-1}}{(k-1)!} \cdot \frac{\partial p}{\partial c}
\end{aligned}$$

where

$$\frac{\partial p}{\partial c} = \frac{\sqrt{1-\rho}}{\sqrt{2\pi}} e^{\frac{(c(1-\rho)-\sqrt{\rho}y)^2}{2(1-\rho)}}.$$

Clearly, $\frac{\partial I_1(p)}{\partial c} < 0$. Therefore, in the Poisson approach, it is still true that $\Gamma_k > 0$ for all $k \in \{1, \dots, N\}$.

4.5 The Large- N Limit

As before, we work with the standard Gaussian copula for a portfolio of size N .

The large- N behavior has been studied in the CDO literature through simulations for various copula models. See, for example, Schönbucher [81], Andersen and Sidenius [5], and [43].

The proportion of X_i below the threshold c is

$$\bar{\nu}^{(N)} = \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{X_i \leq c}, \quad (4.5.1)$$

where we have explicitly indicated N on the left. We have then

Theorem 4.5.1. *The sequence $\bar{\nu}^{(N)}$ converges with probability 1 to the random variable $\Phi\left(\frac{c-\sqrt{\rho}Z}{\sqrt{1-\rho}}\right)$:*

$$\bar{\nu}^{(N)} \rightarrow \bar{\nu}^{(\infty)} \stackrel{\text{def}}{=} \Phi\left(\frac{c-\sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \quad \text{almost surely.}$$

Moreover,

$$\bar{\nu}^{(N)} \rightarrow \bar{\nu}^{(\infty)}$$

in L^2 .

Proof The variable $\bar{\nu}^{(N)}$ is a function of the Gaussian variable $(Z, \epsilon_1, \dots, \epsilon_N)$. For each fixed value for Z , it is the average of N independent, identically distributed (bounded) variables. So, by the law of large numbers, for each fixed value z of Z ,

$$\lim_{N \rightarrow \infty} \bar{\nu}^{(N)} = \mathbb{E}[X_i \leq c | Z = z] = \mathbb{P}[X_i \leq c | Z = z] = \Phi\left(\frac{c-\sqrt{\rho}z}{\sqrt{1-\rho}}\right)$$

almost surely in $(\epsilon_1, \dots, \epsilon_N)$. Therefore, by Fubini's theorem (which guarantees that a set with all sections of full measure is itself of full measure),

$$\lim_{N \rightarrow \infty} \bar{\nu}^{(N)} = \Phi \left(\frac{c - \sqrt{\rho} Z}{\sqrt{1 - \rho}} \right)$$

holds almost everywhere.

As for L^2 convergence, denoting $\mathbb{P}[X_i \leq c \mid Z = z]$ by $p(z)$, we have

$$\mathbb{E} \left[\left\| \frac{1}{N} \sum_{i=1}^N \mathbb{1}_{D_i} - p(Z) \right\|^2 \right] = \mathbb{E} \left[\frac{p(Z)(1 - p(Z))}{N} \right] \leq \frac{1}{N} \rightarrow 0$$

as $N \rightarrow \infty$. \square

The distribution of the limiting average loss $\bar{\nu}^{(\infty)}$ is thus

$$\mathbb{P}[\bar{\nu}^{(\infty)} \leq x] = \Phi \left(\frac{\sqrt{1 - \rho} \Phi^{-1}(x) - c}{\sqrt{\rho}} \right) \quad (4.5.2)$$

This agrees with Schönbucher [81, Eq. (23)].

4.6 Relationship with CDO Tranche Models

A (synthetic) CDO is a portfolio of credit default swaps (as explained in the Introduction), whose default risk is sliced up into tranches. A standard approach to modeling a CDO's default behavior, is to consider a proxy X_i for the firm value for each CDS name i , and declare a default if X_i fall below a threshold value c .

The Gaussian copula model for pricing CDO tranches became popular following the work of Li [64]. It is an excellent foundational model which displays qualitative characteristics observed in practice and through simulations in other models. In this model, for a homogeneous portfolio of N names, one assumes there exist $N + 1$ independent standard Gaussian variable factors

$$Z, \epsilon_1, \dots, \epsilon_N,$$

and declare that name i defaults if

$$X_i = \sqrt{\rho} Z + \sqrt{1 - \rho} \epsilon_i$$

falls below a threshold $c \in \mathbb{R}$. Here

$$\rho > 0$$

is a fixed correlation parameter.

The assumptions that the same c and same ρ operate for all the names i in the portfolio is, of course, a great simplification, for the sake of constructing a working initial model.

The threshold c controls the default probability; the default probability for name i is

$$\mathbb{P}[X_i < c] = \Phi(c),$$

where Φ is the standard Gaussian distribution function. The default probability, in turn, is related to the CDS rate, and so can be imputed from market data.

Some of the mathematical results in this chapter, for the Gaussian copula and Poisson-mix models, translate to the following in the language of CDOs:

- (i) Equity tranches are long correlation and senior tranches are short correlation;
- (ii) equity tranche deltas decrease (increase) when the index spread increases (decreases);
- (iii) tranche deltas, for index spread movements, form a probability measure on losses;
- (iv) the normalized loss in a size N portfolio converges almost surely to a random variable, of known distribution, as $N \rightarrow \infty$.

These results are supported both by intuition and simulations. If correlation rises, the probability of very few defaults increases (as well as that for many defaults), and this ought to decrease the expected loss for, at least, a low-detachment

equity tranche. It is, however, not quite clear intuitively whether this ought to work for all equity tranches. Theorem 4.1.1 establishes the result rigorously. The document [70] mentions some of these results, with justifications provided by simulations.

We proceed to further elucidate some questions concerning the proxy variables used and underlying continuous-time process. The remainder of the chapter is devoted to this objective.

4.7 Proxy Variables

A very useful procedure that underlies the idea of proxy variables is contained in the following well-known result:

Lemma 4.1. For any random variable Y , if its distribution function F_Y is strictly monotone and continuous, then $F_Y(Y)$ is uniform on $[0,1]$.

Proof. Suppose that F_Y is strictly monotone and continuous. Then its inverse function F_Y^{-1} exists, and is also strictly monotone and continuous. Let $Z = F_Y(Y)$, then its distribution function $F_Z(m)$ can be found as the following:

$$\begin{aligned} F_Z(t) &= P[F_Y(Y) \leq t] \\ &= P[Y \leq F_Y^{-1}(t)] \\ &= F_Y(F_Y^{-1}(t)) \\ &= t \end{aligned}$$

Therefore, $Z = F_Y(Y)$ is uniformly distributed on $[0,1]$. □

This idea here leads to the following very useful transformation of a stopping time τ to a standard Gaussian variable X :

Lemma 4.2. Suppose τ is a random variable with values in $[0, \infty)$, having a strictly increasing continuous distribution function F_τ . Then there is a standard Gaussian

random variable X , and a function c on $[0, \infty)$ such that

$$[\tau < t] = [X < c(t)], \quad (4.7.1)$$

for all $t \in [0, \infty)$.

Proof. We can simply take

$$X = \Phi^{-1}(F_\tau(\tau)),$$

where Φ is the standard Gaussian distribution function, and take c to be the function $\Phi^{-1} \circ F_\tau$. \square

Assume that we have N names in our portfolio, whose default behaviors are governed by N related standard Gaussian random variables, X_i , $i = 1, 2, \dots, N$. Each X_i is represented as a combination of two factors, a global factor Z and an idiosyncratic factor ϵ_i , $i = 1, 2, \dots, N$, in the following way:

$$X_i = \sqrt{\rho}Z + \sqrt{1 - \rho}\epsilon_i, \quad (4.7.2)$$

where Z and ϵ_i 's are all independent standard Gaussian random variables. Then the name i defaults by time T if X_i is below a threshold level $c_i(T)$, that is,

$$X_i \leq c_i(T) \Leftrightarrow \text{default of } i \text{ by time } T$$

To simplify our model, we assume that all names have the same threshold level c , i.e.,

$$c = c_1(T) = c_2(T) = \dots = c_N(T).$$

Usually, the default probability for any name is less than 0.5, i.e.

$$P[X_i \leq c] < 0.5,$$

which implies $c < 0$.

By some easy calculations, we have the following lemmas:

Lemma 4.3. In the Gaussian copula model (4.7.2), ρ is the correlation of names.

Proof. For distinct $i, j \in \{1, \dots, N\}$, we have

$$\begin{aligned}
\text{corr}(X_i, X_j) &= E[X_i X_j] \\
&= E[(\sqrt{\rho}Z + \sqrt{1-\rho}\epsilon_i)(\sqrt{\rho}Z + \sqrt{1-\rho}\epsilon_j)] \\
&= E[\rho Z^2 + \sqrt{\rho(1-\rho)}Z\epsilon_i + \sqrt{\rho(1-\rho)}Z\epsilon_j + (1-\rho)\epsilon_i\epsilon_j] \\
&= \rho E[Z^2] \\
&= \rho,
\end{aligned}$$

since Z and the ϵ_i are independent Gaussian random variables. \square

For the conditional probability that X_i falls below c , given the global factor Z , we have

Lemma 4.4. With notation as above,

$$\mathbb{P}[X_i \leq c \mid Z] = P[Z, \rho] = \int_{-\infty}^{\frac{c - \sqrt{\rho}Z}{\sqrt{1-\rho}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

Proof. We have

$$\begin{aligned}
P[Z, \rho] &= \mathbb{P}[X_i \leq c \mid Z] \\
&= \mathbb{P}\left[\epsilon_i \leq \frac{c - \sqrt{\rho}Z}{\sqrt{1-\rho}} \mid Z\right] \\
&= \Phi\left(\frac{c - \sqrt{\rho}Z}{\sqrt{1-\rho}}\right) \\
&= \int_{-\infty}^{\frac{c - \sqrt{\rho}Z}{\sqrt{1-\rho}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy
\end{aligned}$$

where Φ is the cumulative distribution function of standard Gaussian random variable. \square

For the distribution of ν , the number of X_i which are below the threshold c , conditional on Z , is given by

$$\mathbb{P}[X_i \leq c | Z] = \int_{-\infty}^{\infty} P[x, \rho] \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (4.7.3)$$

$$= \int_{-\infty}^{\infty} \binom{N}{k} (P[x, \rho])^k (1 - P[x, \rho])^{N-k} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx, \quad (4.7.4)$$

in the binomial case, and by

$$\mathbb{P}[\nu = k | Z] = P[k|Z] = e^{-NP[Z, \rho]} \frac{(NP[Z, \rho])^k}{k!}$$

in the Poisson case.

4.8 Simulations and Graphs

Default probability refers to $\Phi(c)$, i.e. $\mathbb{P}[X_i \leq c]$.

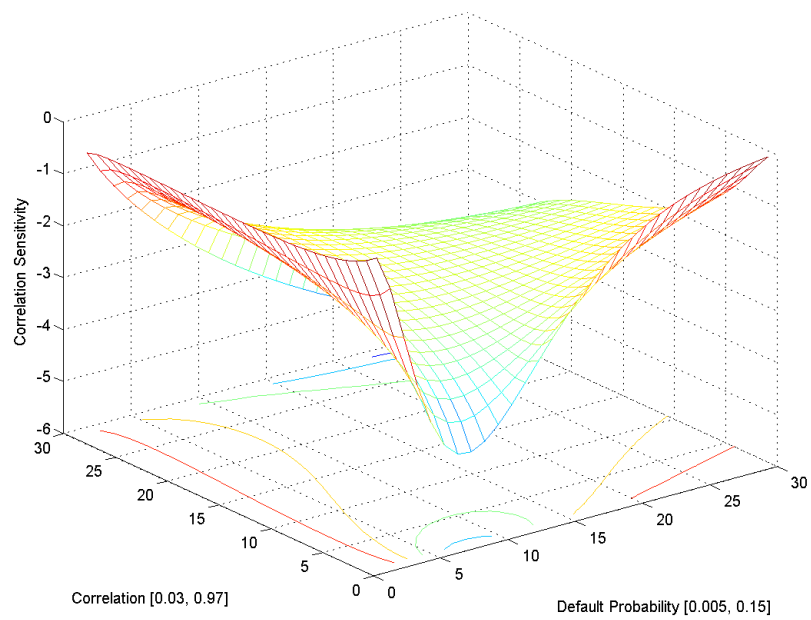


FIGURE 4.1. Dependence of $\frac{dL_3^e}{d\rho}$ on ρ and c

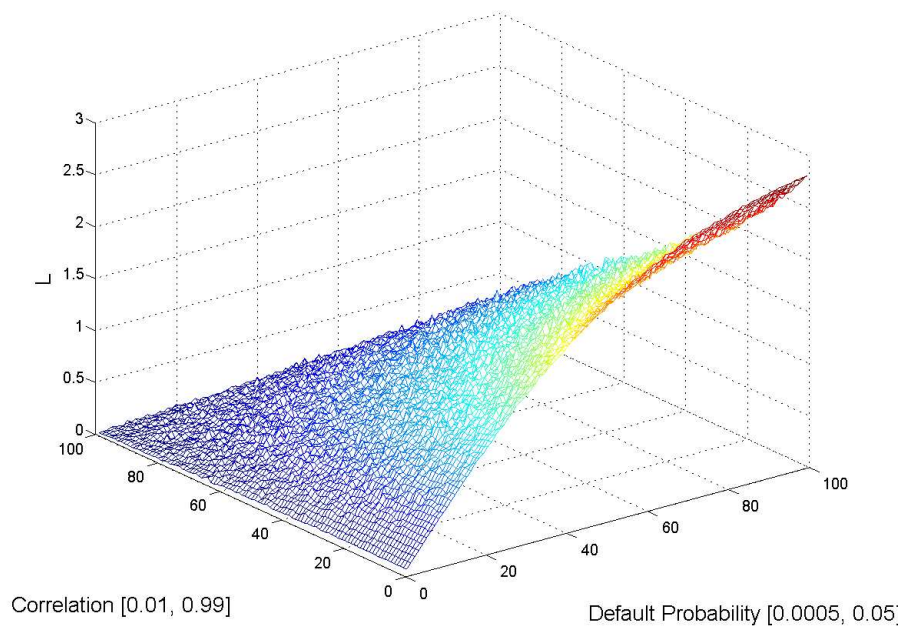


FIGURE 4.2. Graph of $L = L_3^c$ against ρ and $\Phi(c)$

Chapter 5

Significance of Certain Stochastic Integrals

The purpose of this chapter is to summarize well-known basic notions relating to pricing certain types of financial instruments. The objective is to outline how certain stochastic integrals arise from this context.

5.1 Probability and Pricing Notions

Here we summarize some standard notions on expressing prices of risky assets using probability measures. We will keep to a rather sketchy outline, looking only at a simplified abstract structure, since the topic is not central to our overall objectives.

The market price of a hypothetical asset I_A which pays off one unit of currency (or some other numeraire) if an event A happens, and nothing otherwise, is the market's estimate of the probability of the event A :

$$Q(A) = \text{price of asset } A.$$

The pricing measure arises from a market at equilibrium.

More structurally, the market is modeled by a probability space

$$(\Omega, \mathcal{F}, Q),$$

where elements of Ω are to be viewed as states or scenarios in the market, and a space of random variables

$$X : \Omega \rightarrow \mathbb{R},$$

where $X(\omega)$ is to be understood as the price of X (in some chosen fixed unit numeraire) in market state ω . In the presence of additional information, encoded in a σ -algebra $\mathcal{G} \subset \mathcal{F}$, the price is the conditional expectation

$$\mathbb{E}_Q[X|\mathcal{G}]$$

(See [84] for a framework along these lines.)

A bond is a financial instrument that yields to the issuer an agreed-upon sum, the *face value* (\$1 in our case), at a chosen *maturity date* T , and may also pay coupons prior to maturity if so agreed upon. A *default-free zero-coupon bond* is a bond that has no risk of default and pays no coupons.

Let us assume that there is a *default-free zero-coupon bond*, which pays off \$1 at time T . Under market scenario $\omega \in \Omega$, let $b(t, T; \omega)$ (we will usually suppress ω) be the price of this bond at time t . Thus, $b(t, T)$ is the value at time t of \$1 at time T . Clearly, $b(t, T) < 1$ and $b(t, T)$ is increasing with respect to t .

Consider a short moment Δt after time t , the interest rate from time t to time $t + \Delta t$ is

$$\frac{b(t + \Delta t, T) - b(t, T)}{b(t, T)\Delta t}$$

We define the force of interest $r(t)$ at time t as the limit of the average interest rate over the short moment $(t, t + \Delta t]$:

$$\begin{aligned} r(t) &\stackrel{\text{def}}{=} \lim_{\Delta t \rightarrow 0} \frac{b(t + \Delta t, T) - b(t, T)}{b(t, T)\Delta t} \\ &= \frac{1}{b(t, T)} \frac{db(t, T)}{dt} \\ &= \frac{d \log b(t, T)}{dt} \end{aligned}$$

which implies

$$\log b(T, T) - \log b(t, T) = \int_t^T r(s) ds$$

and

$$b(t, T) = e^{-\int_t^T r(s) ds}$$

since $b(T, T) = 1$.

The market price $B(t, T)$ of such bond is then

$$B(t, T) = E_Q[e^{-\int_t^T r(s) ds}].$$

5.2 Default Intensity

Now let us consider a bond which has a likelihood of defaulting. In market scenario ω , let the stopping time τ be the time-to-default, then event $[\tau \leq s]$ is the event that the bond defaults before time s for any s between the present time t and maturity T . Of course the event that the bond survives beyond time s is $[\tau > s]$.

In what follows we work with a probability space

$$(\Omega, \mathcal{F}, Q),$$

and, in the interpretation, we can view Q as the measure used for pricing instruments. Sometimes we will use the notation \mathbb{P} for Q .

We define the *default intensity* $\lambda(s)$ as the limit of the average probability of default over $(s, s + \Delta s)$, given that there is no default up to time s :

$$\begin{aligned} \lambda(s) &\stackrel{\text{def}}{=} \lim_{\Delta s \rightarrow 0} \frac{q(s < \tau \leq s + \Delta s | \tau > s)}{\Delta s} \\ &= \lim_{\Delta s \rightarrow 0} \frac{Q(\tau > s) - Q(\tau > s + \Delta s)}{Q(\tau > s)\Delta s} \\ &= -\frac{d \log Q(\tau > s)}{ds} \end{aligned}$$

Integrating both sides from t to T , we have

$$\log Q(\tau > T) - \log Q(\tau > t) = -\int_t^T \lambda(s) ds$$

and

$$Q(\tau > T) = e^{-\int_t^T \lambda(s) ds},$$

assuming that

$$Q(\tau > t) = 1.$$

Note that this is the probability of the defaultable zero-coupon bond, issued at time t , survives up to T .

Let us now build up the market price at time t of a defaultable zero-coupon bond with maturity T . At time t it is worth, as a function on Ω ,

$$e^{-\int_t^T r(s) ds} \mathbb{1}_{[\tau > T]}.$$

At time $t < T$, its price would be

$$\bar{B}(t, T) = E_Q[e^{-\int_t^T [r(s) + \lambda(s)] ds}].$$

5.3 Default Intensity Integrals

The expected values of integrals

$$B(t, T) = E_Q[e^{-\int_t^T r(s) ds}]$$

and

$$\bar{B}(t, T) = E_Q[e^{-\int_t^T [r(s) + \lambda(s)] ds}]$$

will be useful “building blocks” for our purposes.

We need to calculate out the density of the time of the first default, $Q(\tau \in (T, T + dT])$, for a specific market scenario ω :

$$\begin{aligned} Q(\tau \in (T, T + dT]) &= Q(\tau > T) - Q(\tau > T + dT) \\ &= e^{-\int_t^T \lambda(s) ds} - e^{-\int_t^{T+dT} \lambda(s) ds} \\ &= -\frac{d(e^{-\int_t^T \lambda(s) ds})}{dT} dT \\ &= \lambda(T) e^{-\int_t^T \lambda(s) ds} dT. \end{aligned}$$

Let

$$e(t, T) = E_Q \left[\lambda(T) e^{-\int_t^T \lambda(s) ds} \right]. \quad (5.3.1)$$

Then

$$\begin{aligned} e(t, T) dT &= E_Q \left[e^{-\int_t^T r(s) ds} Q(\tau \in (T, T + dT]) \right] \\ &= E_Q \left[e^{-\int_t^T r(s) ds} \lambda(T) e^{-\int_t^T \lambda(s) ds} dT \right] \\ &= E_Q \left[\lambda(T) e^{-\int_t^T [r(s) + \lambda(s)] ds} \right] dT \end{aligned}$$

To sum up, we have the following three “building blocks”:

$$B(t, T) = E_Q[e^{-\int_t^T r(s) ds}] \quad (5.3.2)$$

$$\bar{B}(t, T) = E_Q[e^{-\int_t^T [r(s)+\lambda(s)] ds}] \quad (5.3.3)$$

$$e(t, T) = E_Q \left[\lambda(T) e^{-\int_t^T [r(s)+\lambda(s)] ds} \right] \quad (5.3.4)$$

5.4 Stochastic Integrals with Stopping Times

In this section we examine certain stochastic integral expectation values, which involve a stopping time. These are motivated by an examination of credit default swaps (CDS).

Recall that a CDS is an agreement between two parties, protection buyer A and protection seller B: party A pays party B a premium periodically to insure the notional amount of a given defaultable bond against default risk. If a default happens during the life of the CDS, B pays A the loss amount. Otherwise, B pays A nothing.

Suppose that the notional amount is \$1 and τ is the *time-to-default* for a CDS contract maturing at time T . Thus, the protection seller expects to pay out

$$\begin{aligned} E \left[e^{-\int_0^\tau r(s) ds} Q(\tau \leq T) \right] &= \int_0^T E \left[e^{-\int_0^t r(s) ds} Q(\tau \in (t, t + dt]) \right] \\ &= \int_0^T e(0, t) dt \end{aligned}$$

by the third “building block” (5.3.4).

Suppose that the CDS swap rate is s_T , with premiums paid on dates t_1, t_2, \dots, t_N . Then by the second “building block” (5.3.3), the total of premiums protection buyer expects to pay out is

$$\sum_{j=1}^N s_T (t_j - t_{j-1}) \bar{B}(0, t_j).$$

The CDS spread s_T should be the price such that the expected pay outs from protection buyer and seller are equal. Therefore we have s_T in terms of the “building

blocks”

$$s_T = \frac{\int_0^T e(0, t) dt}{\sum_{j=1}^N (t_j - t_{j-1}) \bar{B}(0, t_j)}$$

In the next chapter we will see how to calculate the “building blocks” after we specify the structures on the *force of interest* $r(t)$ and the *default intensity* $\lambda(t)$.

Chapter 6

Certain Stochastic Integrals with Stopping Times

In this chapter we will evaluate certain stochastic integrals of the form

$$\mathbb{E} \left[\int_0^{T \wedge \tau} e^{-\int_0^t r(u) du} dt \right] \quad (6.0.1)$$

for specified stochastic processes

$$u \mapsto r(u)$$

and τ , to be viewed as the exit time of suitable processes, is a stopping time with specified intensity. To provide intuitive guidance and motivation we select choice for the processes r and the intensity of τ from models for default behavior of bonds.

The first three sections of this chapter summarize, in a form useful for our purposes, the essential features of certain standard models pertaining to credit default behavior. Section 6.4 is devoted to explaining our method for computing the integrals (6.0.1) for these models.

6.1 The Vasicek Model

The Vasicek model, a specific Gaussian model, is usually studied in the context of zero-coupon risk-free bonds, but the same mathematical model could be applied to the default intensity process of a risky bond. The essential idea of the Vasicek model is as following. The interest rate $r(t)$ or default intensity $\lambda(t)$ is generated by the Vasicek stochastic differential equation

$$dx(t) = (\kappa(t) - ax(t))dt + \sigma(t)dW(t),$$

the “building block” $B(t, T)$ or $Q(t, T)$ can be computed by

$$E[e^{-\int_t^T x(s) ds} | F_t] = e^{\alpha(t, T) - \beta(t, T)x(t)}.$$

We start deriving this idea by the following lemma.

Lemma 6.1. Given the following stochastic differential equation:

$$dx(t) = (\kappa(t) - ax(t))dt + \sigma(t)dW(t)$$

where a is constant, κ and σ are continuous deterministic functions, and $W(t)$ is a one-dimensional Brownian motion, we have the following solution:

$$x(t) = x(0)e^{-at} + \int_0^t e^{-a(t-s)}\kappa(s) ds + \int_0^t e^{-a(t-s)}\sigma(s) dW(s)$$

Proof. Let us consider the ‘integrating factor’ e^{at} and the process

$$y(t) = x(t)e^{at}.$$

The differential is

$$\begin{aligned} dy(t) &= d(x(t)e^{at}) \\ &= e^{at}ax(t) dt + e^{at} dx(t) \\ &= e^{at}(ax(t) dt + (\kappa(t) - ax(t))dt) + \sigma(t) dW(t) \\ &= e^{at}\kappa(t) dt + e^{at}\sigma(t) dW(t) \end{aligned}$$

Now integrating both sides of the above equation from 0 to t , we get

$$y(t) - y(0) = \int_0^t e^{as}\kappa(s) ds + \int_0^t e^{as}\sigma(s) dW(s).$$

Thus,

$$x(t)e^{\int_0^t a ds} = x(0) + \int_0^t e^{as}\kappa(s) ds + \int_0^t e^{as}\sigma(s) dW(s),$$

which implies

$$x(t) = x(0)e^{-at} + \int_0^t e^{-a(t-s)}\kappa(s) ds + \int_0^t e^{-a(t-s)}\sigma(s) dW(s).$$

□

Lemma 6.2. For a deterministic function $h(t)$, a function only of $t \in [0, \infty)$, locally square-integrable, its Itô integral $\int_0^t h(s) dW(s)$ is Gaussian with mean 0 and variance $\int_0^t h^2(s) ds$.

Proof. By the construction of the Itô integral, we know that

$$E \left[\int_0^t h(s) dW(s) \right] = 0.$$

Also by the definition of the Itô integral,

$$\int_0^t h(s) dW(s) = \lim_{n \rightarrow \infty} \int_0^t h_n(s) dW(s) \quad (6.1.1)$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n h(t_i) (W(t_{i+1}) - W(t_i)), \quad (6.1.2)$$

where the simple function $h_n(t)$ is defined by

$$h_n(t) = \sum_{i=0}^n h(t) \cdot \chi_{[t_i, t_{i+1})}(t)$$

and the above limits are in $L^2(P)$.

The differences $W(t_{i+1}) - W(t_i)$ are Gaussian $N(0, t_{i+1} - t_i)$, and are mutually independent as $W(t)$ is a Brownian motion. So the sum

$$\sum_{i=1}^n h(t_i) (W(t_{i+1}) - W(t_i)) \quad (6.1.3)$$

is also Gaussian. Therefore as the L^2 -limit of the Gaussian random variables in (6.1.3), $\int_0^t h(s) dW(s)$ is also a Gaussian random variable.

The variance of $\int_0^t h(s) dW(s)$ is, from the Itô isometry,

$$E \left[\left(\int_0^t h(s) dW(s) \right)^2 \right] = E \left[\int_0^t h(s)^2 ds \right] = \int_0^t h(s)^2 ds, \quad (6.1.4)$$

the last equation is because $h(t)$ is deterministic. \square

Theorem 6.3. Let $x(t)$ satisfy

$$dx(t) = (\kappa(t) - ax(t))dt + \sigma(t)dW(t),$$

where κ and σ are continuous deterministic functions, and let

$$\mathcal{F}_t = \sigma\{W(s) : s \leq t\}.$$

Then

$$E[e^{-\int_t^T x(s) ds} | \mathcal{F}_t] = e^{\alpha(t,T) - \beta(t,T)x(t)} \quad (6.1.5)$$

where

$$\begin{aligned} \beta(t, T) &= \frac{1}{a}(1 - e^{-a(T-t)}) \\ \alpha(t, T) &= \frac{1}{2} \int_t^T \sigma(s)^2 \beta^2(s, T) ds - \int_t^T \kappa(s) \beta(s, T) ds \end{aligned}$$

Proof. By Lemma 6.1 and 6.2, we know that the process

$$x(t) = x(0)e^{-at} + \int_0^t e^{-a(t-s)} \kappa(s) ds + \int_0^t e^{-a(t-s)} \sigma(s) dW(s)$$

is a Brownian motion with a drift

$$x(0)e^{-at} + \int_0^t e^{-a(t-s)} \kappa(s) ds.$$

The process $x(s)$ is Markov, how $x(s)$ evolves when $s \geq t$ conditional on \mathcal{F}_t depends only on the behavior of $x(t)$. So we have

$$E[e^{-\int_t^T x(s) ds} | \mathcal{F}_t] = E[e^{-\int_t^T x(s) ds} | x(t)]$$

and therefore $E[e^{-\int_t^T x(s) ds} | \mathcal{F}_t]$ is a function only of t and $x(t)$.

From now on, we denote $E[e^{-\int_t^T x(s) ds} | \mathcal{F}_t]$ by $B(t, x(t))$.

It is easy to check that if we look on $e^{-\int_0^T x(s) ds}$ as a random variable when fixing T , then

$$E[e^{-\int_0^T x(s) ds} | \mathcal{F}_t] \quad (6.1.6)$$

is a martingale with respect to t . We also have the following relationship

$$E[e^{-\int_0^T x(s) ds} | \mathcal{F}_t] = E[e^{-\int_0^t x(s) ds} e^{-\int_t^T x(s) ds} | \mathcal{F}_t] \quad (6.1.7)$$

$$= e^{-\int_0^t x(s) ds} E[e^{-\int_t^T x(s) ds} | \mathcal{F}_t] \quad (6.1.8)$$

$$= e^{-\int_0^t x(s) ds} B(t, x(t)), \quad (6.1.9)$$

which implies that $e^{-\int_0^t x(s) ds} B(t, x(t))$ is a martingale.

Then, using Itô's lemma,

$$\begin{aligned}
d\left(e^{-\int_0^t x(s) ds} B(t, x(t))\right) &= \frac{\partial e^{-\int_0^t x(s) ds} B(t, x(t))}{\partial t} dt + \frac{\partial e^{-\int_0^t x(s) ds} B(t, x(t))}{\partial x(t)} dx(t) \\
&\quad + \frac{1}{2} \frac{\partial^2 e^{-\int_0^t x(s) ds} B(t, x(t))}{\partial x(t)^2} (dx(t))^2 \\
&= e^{-\int_0^t x(s) ds} \left[-x(t) B(t, x(t)) + \frac{\partial B(t, x(t))}{\partial t} \right. \\
&\quad \left. + (\kappa(t) - ax(t)) \frac{\partial B(t, x(t))}{\partial x(t)} + \frac{1}{2} \frac{\partial^2 B(t, x(t))}{\partial x(t)^2} \sigma(t)^2 \right] dt \\
&\quad + e^{-\int_0^t x(s) ds} \sigma(t) \frac{\partial B(t, x(t))}{\partial x(t)} dW(t). \tag{6.1.10}
\end{aligned}$$

Because this is the stochastic differential of a martingale, drift part in equation (6.1.10) must be zero, i.e.

$$\frac{\partial B(t, x(t))}{\partial t} - x(t) B(t, x(t)) + (\kappa(t) - ax(t)) \frac{\partial B(t, x(t))}{\partial x(t)} + \frac{1}{2} \frac{\partial^2 B(t, x(t))}{\partial x(t)^2} \sigma(t)^2 = 0.$$

Let us try a solution of the form $B(t, x(t)) = e^{\alpha(t, T) - \beta(t, T)x(t)}$. Then the above equation becomes:

$$-x(t) + \frac{\partial \alpha(t, T)}{\partial t} - \frac{\partial \beta(t, T)}{\partial t} x(t) - (\kappa(t) - ax(t)) \beta(t, T) + \frac{1}{2} \sigma(t)^2 \beta^2(t, T) = 0$$

with initial conditions $\alpha(T, T) = \beta(T, T) = 0$ as $B(T, x(T)) = 0$ from its definition.

The above equation is true for all $x(t)$, so we obtain the following two partial differential equations:

$$\frac{\partial \beta(t, T)}{\partial t} = a\beta(t, T) - 1 \tag{6.1.11}$$

$$\frac{\partial \alpha(t, T)}{\partial t} = \kappa(t) \beta(t, T) - \frac{1}{2} \sigma(t)^2 \beta^2(t, T) \tag{6.1.12}$$

with initial conditions

$$\beta(t, T) = 0 \tag{6.1.13}$$

$$\alpha(T, T) = 0 \tag{6.1.14}$$

By solving the PDE (6.1.11) and its initial condition (6.1.13), we have

$$\beta(t, T) = \frac{1}{a}(1 - e^{-a(T-t)}).$$

By solving the PDE (6.1.12) and its initial condition (6.1.14), we have

$$\alpha(t, T) = \frac{1}{2} \int_t^T \sigma(s)^2 \beta^2(s, T) ds - \int_t^T \kappa(s) \beta(s, T) ds.$$

□

6.2 The Two-Factor Gaussian Model

In the two-factor Gaussian model, we will study two factors which mainly affect the CDS rate: interest rate of the market, $r(t)$ and default intensity of the reference, $\lambda(t)$. Financially, the interest rate and default intensity are generally correlated. We bring the ideas of the Vasicek Model from the interest rate term structure model to setup $r(t)$ and $\lambda(t)$. The following are the two-factor Gaussian Model and its assumptions:

$$dr(t) = (\kappa(t) - ar(t)) dt + \sigma(t) dW(t)$$

$$d\lambda(t) = (\bar{\kappa}(t) - \bar{a}\lambda(t)) dt + \bar{\sigma}(t) d\bar{W}(t)$$

$$dW(t)d\bar{W}(t) = \rho dt$$

where $r(t)$ is the default-free interest rate, $\lambda(t)$ is the default intensity and ρ is the correlation between their generating white noise processes, the changes of two Brownian Motions, $W(t)$ and $\bar{W}(t)$.

By Theorem 6.3 in the Vasicek model, we have the following two facts:

Fact 6.2.1. *The price of a default-free zero coupon bond at time t with payoff 1 unit at maturity T is*

$$B(t, T) = E[e^{-\int_t^T r(s) ds}] = e^{\alpha(t, T) - \beta(t, T)r(t)}$$

where

$$\begin{aligned}\beta(t, T) &= \frac{1}{a}(1 - e^{-a(T-t)}) \\ \alpha(t, T) &= \frac{1}{2} \int_t^T \sigma^2(s) \beta^2(s, T) ds - \int_t^T \beta(s, T) \kappa(s) ds\end{aligned}$$

Fact 6.2.2. *The survival probability of the reference security from t to T is*

$$E[e^{-\int_t^T \lambda(s) ds}] = e^{\bar{\alpha}(t, T) - \bar{\beta}(t, T) \lambda(t)}$$

where

$$\begin{aligned}\bar{\beta}(t, T) &= \frac{1}{\bar{a}}(1 - e^{-\bar{a}(T-t)}) \\ \bar{\alpha}(t, T) &= \frac{1}{2} \int_t^T \bar{\sigma}^2(s) \bar{\beta}^2(s, T) ds - \int_t^T \bar{\beta}(s, T) \kappa(s) ds\end{aligned}$$

Then the other “building blocks” can be calculated out by the following lemmas:

Lemma 6.4. The price of a defaultable zero coupon bond at time t with payoff 1 unit at maturity T is

$$\bar{B}(t, T) = E[e^{-\int_t^T \lambda(s) + r(s) ds}] = B(t, T) e^{\bar{\alpha}(t, T) - \bar{\beta}(t, T) \lambda(t)}$$

where

$$\begin{aligned}\bar{\beta}(t, T) &= \frac{1}{\bar{a}}(1 - e^{-\bar{a}(T-t)}) \\ \bar{\alpha}(t, T) &= \frac{1}{2} \int_t^T \bar{\sigma}^2(s) \bar{\beta}^2(s, T) ds - \int_t^T \tilde{\kappa}(s) \bar{\beta}(s, T) ds \\ \tilde{\kappa}(t) &= \bar{\kappa}(t) - \rho \beta \bar{\sigma}(t) \sigma(t)\end{aligned}$$

Proof. To evaluate

$$\bar{B}(t, T) = E_Q[e^{-\int_t^T r(s) ds} e^{-\int_t^T \lambda(s) ds}],$$

we want to change the measure Q into the new measure Q_T ,

$$Q_T(A) = \frac{\int_{\Omega} \mathbb{1}_A e^{-\int_t^T r(s) ds} dQ}{\int_{\Omega} e^{-\int_t^T r(s) ds} dQ},$$

by using *Girsanov's Theorem* to yield that

$$\bar{B}(t, T) = B(t, T)E_{Q_T}[e^{-\int_t^T \lambda(s) ds}].$$

In the Vacicek model, we know that the process

$$E[e^{-\int_t^T r(s) ds} | \mathcal{F}_u]$$

appeared in equation (6.1.6) is martingale. By relationship (6.1.9) and definition of the "building block" (5.3.2)

$$B(t, T) = E_Q[e^{-\int_t^T r(s) ds}],$$

we have

$$E[e^{-\int_t^T r(s) ds} | F_u] = e^{-\int_t^u r(s) ds} B(u, T).$$

Let

$$M(u) = \frac{e^{-\int_t^u r(s) ds} B(u, T)}{B(t, T)},$$

then clearly $M(u)$ is a martingale with expectation 1.

From Theorem 6.3 and the equations (6.1.5) and (6.1.10), we have

$$\begin{aligned} dM(u) &= \frac{1}{B(t, T)} e^{-\int_t^u r(s) ds} \sigma(u) \frac{\partial B(u, T)}{\partial r(u)} dW(u) \\ &= \frac{1}{B(t, T)} e^{-\int_t^u r(s) ds} \sigma(u) (-\beta(u, T) B(u, T)) dW(u) \\ &= -\sigma(u) \beta(u, T) M(u) dW(u), \end{aligned}$$

which implies

$$\frac{dM(u)}{M(u)} = -\sigma(u) \beta(u, T) dW(u); \quad (6.2.1)$$

Define a new Q_T by

$$\frac{dQ_T}{dQ} = M(T),$$

then by Girsanov's Theorem, the Itô process $W_{Q_T}(t)$ defined by

$$dW_{Q_T}(t) = \sigma(t) \beta(t, T) dt + dW(t) \quad (6.2.2)$$

is a Brownian Motion under the new measure Q_T . Also $\bar{B}(t, T)$ can be computed as

$$\begin{aligned}
\bar{B}(t, T) &= E_Q[M(T)e^{-\int_t^T \lambda(s) ds}] \\
&= E_Q[e^{-\int_t^T r(s) ds}]E_{Q_T}[e^{-\int_t^T \lambda(s) ds}] \\
&= B(t, T)E_{Q_T}[e^{-\int_t^T \lambda(s) ds}]
\end{aligned} \tag{6.2.3}$$

Since

$$dW(t)d\bar{W}(t) = \rho dt,$$

we have

$$\begin{aligned}
d\bar{W}(t) &= \rho dW(t) \\
&= -\rho\sigma(t)\beta(t, T) dt + \rho dW_{Q_T}(t).
\end{aligned}$$

Note that the second equal sign above is from the equation (6.2.2) of the new Brownian Motion.

Under the new measure Q_T , the default intensity becomes

$$\begin{aligned}
d\lambda(t) &= (\bar{\kappa}(t) - \bar{a}\lambda(t)) dt + \bar{\sigma}(t) d\bar{W}(t) \\
&= (\bar{\kappa}(t) - \bar{a}\lambda(t)) dt + \bar{\sigma}(t) (-\rho\sigma(t)\beta(t, T) dt + \rho dW_{Q_T}(t)) \\
&= ((\bar{\kappa}(t) - \rho\beta(t, T)\bar{\sigma}(t)\sigma(t)) - \bar{a}\lambda(t))dt + +\bar{\sigma}(t)d\bar{W}_{Q_T}(t)
\end{aligned}$$

where

$$\tilde{\kappa} = \bar{\kappa}(t) - \rho\beta(t, T)\bar{\sigma}(t)\sigma(t).$$

By theorem (6.3) again, we are able to calculate

$$E_{Q_T}[e^{\int_t^T \lambda(s) ds}]$$

under the new measure Q_T , then the result follows by the equation (6.2.3).

□

Lemma 6.5. The value at time t of a payoff of \$1 at time $T + dt$ if and only if a default happens in $[T, T + dt]$ is

$$e(t, T)dt = \bar{B}(t, T)[\lambda(t)e^{-\bar{a}(T-t)} + \int_t^T e^{-a(T-s)}\tilde{k}'(s) ds] dt$$

where

$$\tilde{k}'(t) = \bar{k}(t) - \rho\bar{\sigma}\sigma\beta - \bar{\sigma}^2\bar{\beta}.$$

Proof. Recall that

$$e(t, T)dt = E[\lambda(T)e^{-\int_t^T \lambda(s)+r(s) ds}]dt.$$

Again we want to generate a new measure \bar{Q}_T such that

$$e(t, T) = \bar{B}(t, T)E_{\bar{Q}_T}[\lambda(T)].$$

Let

$$\bar{M}(u) = \frac{e^{-\int_t^u r(s)+\lambda(s) ds}\bar{B}(u, T)}{\bar{B}(t, T)},$$

then similar to $M(u)$ in the previous lemma (6.4), $\bar{M}(u)$ is a martingale with expectation 1.

By Itô's lemma,

$$\begin{aligned} \frac{d\bar{M}(u)}{\bar{M}(u)} &= \frac{1}{\bar{M}(u)} \left[\frac{\partial\bar{M}(u)}{\partial t} dt + \frac{\partial\bar{M}(u)}{\partial r(u)} dr(u) + \frac{\partial\bar{M}(u)}{\partial\lambda(u)} d\lambda(u) \right. \\ &\quad \left. + \frac{1}{2} \frac{\partial^2\bar{M}(u)}{\partial r(u)^2} (dr(u))^2 + \frac{1}{2} \frac{\partial^2\bar{M}(u)}{\partial r(u)\partial\lambda(u)} dr(u)d\lambda(u) + \frac{1}{2} \frac{\partial^2\bar{M}(u)}{\partial\lambda(u)^2} (d\lambda(u))^2 \right] \\ &= (-\beta(u, T)\sigma(u) dW(u) - \bar{\beta}(u, T)\bar{\sigma}(u) d\bar{W}(u)) \\ &= (-\beta(u, T)\sigma(u)\rho - \bar{\beta}(u, T)\bar{\sigma}(u)) d\bar{W}(u) \end{aligned}$$

Then by Girsanov's Theorem, we have

$$d\lambda(t) = (\tilde{\kappa}'(t) - \bar{a}\lambda(t)) dt + \bar{\sigma}(t) d\bar{W}_{\bar{Q}_T}(t) \tag{6.2.4}$$

and

$$e(t, T) = \bar{B}(t, T)E_{\bar{Q}_T}[\lambda(T)] \quad (6.2.5)$$

where

$$\tilde{\kappa}' = \bar{\kappa}(t) - \rho\beta(t, T)\bar{\sigma}(t)\sigma(t) - \bar{\sigma}^2(t)\bar{\beta}(t, T).$$

Therefore the result follows by applying Theorem 6.3. \square

Putting everything together, we can work out the CDS spread

$$s_T = \frac{\int_0^T e(0, t) dt}{\sum_{j=1}^N (t_j - t_{j-1})\bar{B}(0, t_j)}$$

6.3 The Multifactor Cox-Ingersoll-Ross Model

In the multifactor *Cox-Ingersoll-Ross* (CIR) model, we will set up n independent factors which drive the interest rate $r(t)$ and the default intensity $\lambda(t)$. The model setup is the following.

The i -th factor is defined by

$$dx_i = (a_i - b_i x_i)dt + \sigma_i \sqrt{x_i} dW_i(t)$$

where $i = 1, \dots, n$ and $W_i(t)$'s are mutually independent Brownian Motions.

To make sure that each factor $x_i(t)$ is strictly positive, one assumption on the coefficients must be made:

$$a_i > \frac{1}{2}\sigma_i^2.$$

Then the interest rate and default intensity are defined as linear combinations of these n factors.

$$r(t) = \sum_{i=1}^n c_i x_i(t)$$

$$\lambda(t) = \sum_{i=1}^n \bar{c}_i x_i(t)$$

Note that although the $W_i(t)$'s are independent, $r(t)$ and $\lambda(t)$ are correlated.

The followings are the calculations of the pricing building blocks based on the multifactor CIR model. Details of the proofs can be found in Schönbucher [82] page 175-186.

Lemma 6.6. Given the following stochastic differential equation:

$$dx_i = (a_i - b_i x_i) dt + \sigma_i \sqrt{x_i} dW_i(t)$$

where a_i, b_i, σ_i are constants and $W(t)$ is a one-dimensional Brownian Motion, we have the following result:

$$E[e^{-\int_t^T c x_i(s) ds}] = H_{1i}(T - t, c) e^{-H_{2i}(T-t, c) c x_i(t)}$$

where

$$\begin{aligned} H_{1i}(T - t, c) &= \left[\frac{2\gamma_i e^{\frac{1}{2}(\gamma_i + \beta_i)(T-t)}}{(\gamma_i + \beta_i)(e^{\gamma_i(T-t)} - 1) + 2\gamma_i} \right]^{2\alpha_i/\sigma_i^2} \\ H_{2i}(T - t, c) &= \frac{2(e^{\gamma_i(T-t)} - 1)}{(\gamma_i + \beta_i)(e^{\gamma_i(T-t)} - 1) + 2\gamma_i} \\ \gamma_i &= \sqrt{\beta_i^2 + 2c\sigma_i^2} \end{aligned}$$

Lemma 6.7. The price of a defaultable zero coupon bond at time t with payoff 1 unit at maturity T is

$$\bar{B}(t, T) = \prod_{i=1}^n H_{1i}(c_i + \bar{c}_i) e^{-H_{2i}(c_i + \bar{c}_i)(c_i + \bar{c}_i) x_i(t)}$$

Lemma 6.8. The value at time t of a payoff of 1 unit at time $T + dt$ if and only if a default happens in $[T, T + dt]$ is

$$\begin{aligned} e(t, T) dt &= E[\lambda(T) e^{-\int T_t \lambda(s) + r(s) ds}] dt \\ &= \sum_{i=1}^n \bar{c}(c_i + \bar{c}_i) \left(\alpha_i H_{2i}(c_i + \bar{c}_i) + \frac{\partial H_{2i}(c_i + \bar{c}_i)}{\partial t} x_i(t) \right) \prod_{j=1}^n \bar{B}_j(t, T) dt \end{aligned}$$

The swap rate

$$\begin{aligned} s_T &= \frac{\int_0^T E[e^{-\int_t^T r(s) ds} \mathbf{1}_{\tau \in [t, t+dt]}]}{\sum_{j=1}^N (t_j - t_{j-1}) \bar{B}(0, t_j)} \\ &= \frac{\int_0^T e(0, t) dt}{\sum_{j=1}^N (t_j - t_{j-1}) \bar{B}(0, t_j)} \end{aligned}$$

can be worked out.

6.4 Stochastic Integral Representation for the Duration of a CDS

In this section we show how integrals of the form

$$E \left[\int_0^{\tau \wedge T} e^{-\int_0^t r(u) du} s_T dt \right] = s_T E \left[\int_0^{\tau \wedge T} e^{-\int_0^t r(u) du} dt \right],$$

may be computed in the models discussed in preceding sections.

The integral above arises, for instance, in measuring the *duration* of a CDS, it is the length of time over which the CDS premium would have to be paid to exactly match the expected loss payments. To simplify the case, we consider paying the rate, s_T , continuously. Then the present value of the total premium paid is

$$E \left[\int_0^{\tau \wedge T} e^{-\int_0^t r(u) du} s_T dt \right] = s_T E \left[\int_0^{\tau \wedge T} e^{-\int_0^t r(u) du} dt \right],$$

and therefore in this case,

$$\text{duration of CDS} = E \left[\int_0^{\tau \wedge T} e^{-\int_0^t r(u) du} dt \right]. \quad (6.4.1)$$

For our purposes we will take this as definition.

The following result is our essential tool for computation.

Theorem 6.9. In the two-factor Gaussian model, the duration of the CDS is

$$\begin{aligned} CDS_{\text{duration}} &= \int_0^T \bar{B}(0, t) dt \\ &= \int_0^T B(0, t) e^{\bar{\alpha} - \bar{\beta} \lambda(0)} dt \end{aligned}$$

where

$$\begin{aligned}\bar{\beta} &= \frac{1}{\bar{a}}(1 - e^{-\bar{a}t}) \\ \bar{\alpha} &= \frac{1}{2} \int_0^t \bar{\sigma}^2(s) \bar{\beta}^2 ds - \int_0^t \bar{\beta} \tilde{k}(s) ds \\ \tilde{k} &= \bar{k}(0) - \rho \beta \bar{\sigma}(0) \sigma(0)\end{aligned}$$

In the multi-factor CIR Model, the duration of CDS is

$$\begin{aligned}CDS_{duration} &= \int_0^T \bar{B}(0, t) dt \\ &= \int_0^T \prod_{i=1}^n H_{1i}(c_i + \bar{c}_i) e^{-H_{2i}(c_i + \bar{c}_i)(c_i + \bar{c}_i)x_i(0)} dt\end{aligned}\quad (6.4.2)$$

where

$$H_{1i}(T - t, c) = \left[\frac{2\gamma_i e^{\frac{1}{2}(\gamma_i + \beta_i)(T-t)}}{(\gamma_i + \beta_i)(e^{\gamma_i(T-t)} - 1) + 2\gamma_i} \right]^{2\alpha_i/\sigma_i^2}\quad (6.4.3)$$

$$H_{2i}(T - t, c) = \frac{2(e^{\gamma_i(T-t)} - 1)}{(\gamma_i + \beta_i)(e^{\gamma_i(T-t)} - 1) + 2\gamma_i}\quad (6.4.4)$$

$$\gamma_i = \sqrt{\beta_i^2 + 2c\sigma_i^2}\quad (6.4.5)$$

Proof. If τ is the default time, then

$$\begin{aligned}CDS_{duration} &= \mathbb{E} \left[\int_0^{\tau \wedge T} e^{-\int_0^t r(u) du} dt \right] \\ &= \mathbb{E} \left[\int_0^T 1_{[\tau > t]} \cdot e^{-\int_0^t r(u) du} dt \right] \\ &= \int_0^T E \left[1_{[\tau > t]} \cdot e^{-\int_0^t r(u) du} \right] dt \\ &= \int_0^T E \left[e^{-\int_0^t \lambda(u) + r(u) du} \right] dt \\ &= \int_0^T \bar{B}(0, t) dt\end{aligned}$$

This yields the desired expression. □

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Appendix A: Copulas, Correlation and Girsanov's Theorem

We summarize some standard notions and well-known results (stated here largely without proofs) related to our investigations.

A.1 Introduction to Copula

A standard reference for copulas is the book of Nelsen [73]. Details of the results of this section, and proofs, may be found in [73].

Copulas are of correlated variables, whose marginals are known. The notion of a copula can be formalized in different degrees of generality. The essential idea may be expressed as follows.

Definition A.1.1. *A function $C : [0, 1]^N \rightarrow [0, 1]$ is a copula if it is the joint distribution function of N random variables, U_1, U_2, \dots, U_N , each having uniform distribution on $[0, 1]$. Thus,*

$$C(u_1, u_2, \dots, u_N) = Q[U_1 \leq u_1, U_2 \leq u_2, \dots, U_N \leq u_N].$$

where Q is the underlying probability measure. Equivalently, a function $C : [0, 1]^N \rightarrow \mathbb{R}$ is a copula function if there is a Borel probability measure P on $[0, 1]^N$ such that

$$C(u_1, \dots, u_N) = P([0, u_1] \times \dots \times [0, u_N]),$$

for every $u_1, \dots, u_N \in [0, 1]^N$.

Copulas have properties which make them very convenient in applications. As we have seen in Lemma 4.1, any random variable X with continuous, strictly monotone distribution function can be 'converts' into the uniform variable $F_X(X)$. In view of this, one has the following observation:

Proposition A.1.1. *If $X = (X_1, X_2, \dots, X_N)$ are random variables with continuous joint distribution F_X and univariate marginal distribution functions $F_{X_1}, F_{X_2}, \dots, F_{X_N}$ which are strictly monotone and continuous, then there exists a unique copula function C such that*

$$F_X(x_1, x_2, \dots, x_N) = C(F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_N}(x_N))$$

for all $x_1, \dots, x_N \in \mathbb{R}$.

In a converse direction, there is the following result:

Proposition A.1.2. *Given N univariate marginal distribution function $F_{X_1}, F_{X_2}, \dots, F_{X_N}$ for random variables (X_1, X_2, \dots, X_N) , and any copula function C , the function defined by C on $[0, 1]^N$, the function F on \mathbb{R}^N given by*

$$F(x_1, x_2, \dots, x_N) = C(F_{X_1}(x_1), F_{X_2}(x_2), \dots, F_{X_N}(x_N)),$$

is a joint distribution function for the random variables (X_1, X_2, \dots, X_N) .

As an example, we have the copula for independent variables:

Proposition A.1.3. *If U_1, U_2, \dots, U_N are all independent, then*

$$C(u_1, u_2, \dots, u_N) = \prod_{i=1}^N u_i$$

A.2 Correlation and Kendall's Tau

The standard *correlation* ρ of two non-constant random variables X and Y is defined by

$$\rho = \frac{E[(X - E(X))(Y - E(Y))]}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

describing the dependency of X and Y .

Correlation of a different flavor is described through *Kendall's Tau* τ , defined by

$$\tau_{\text{Kendall}} = Q[(X - \tilde{X})(Y - \tilde{Y}) > 0] - Q[(X - \tilde{X})(Y - \tilde{Y}) < 0] \quad (\text{A.2.1})$$

where (\tilde{X}, \tilde{Y}) is an independent copy of (X, Y) .

The Kendall's tau of two random variables X, Y is "invariant" when they are replaced by $G(X)$ and $G(Y)$, for any monotonic function G , and so is the same if X, Y are transformed into uniform or Gaussian or any other variables. Thus τ is the right measure of correlation in the copula context. Kendall's tau is related in a simple way to the Gaussian correlation which we use,

$$\tau = \frac{2}{\pi} \arcsin \rho.$$

For a proof see [65].

A.3 Girsanov's Theorem

The following version of the Girsanov's Theorem is from Oksendal [74] page 162:

Theorem A.3.1. *Let $Y(t) \in \mathbb{R}^n$ be an Itô process of the form*

$$dY(t) = a(t, \omega)dt + dB(t); \quad t \leq T, Y_0 = 0.$$

where $T \leq \infty$ is a given constant and $B(t)$ is n -dimensional Brownian motion. Put

$$M_t = \exp\left(-\int_0^t a(s, \omega) dB_s - \frac{1}{2} \int_0^t a^2(s, \omega) ds\right); \quad 0 \leq t \leq T.$$

Assume that M_t is a martingale with respect to $F_t^{(n)}$ and P . Define the measure Q on $F_t^{(n)}$ by

$$dQ(\omega) = M_T(\omega)dP(\omega).$$

Then Q is a probability measure on $F_t^{(n)}$ and $Y(t)$ is an n -dimensional Brownian motion w.r.t. Q , for $0 \leq t \leq T$.

Appendix B: Matlab Code I

The following is the Matlab code for simulating $L = L_e^3$ against ρ and $\Phi(c)$, where $\Phi(c) = \mathbb{P}[X_i \leq c]$, taken from GcBinomialSimulation.m.

```
clear;

num_rho = 100;
num_pro = 100;
Expe = zeros(num_rho, num_pro);
for m = 1:num_rho
    for n = 1:num_pro

        delta = 100;
        globe = randn(delta,1);
        name = 100;
        rho = m/101;
        default_pro = n/2000;
        c = -sqrt(2)*erfcinv(2*default_pro);
        p = zeros(name,1);

        for k = 1:3
            c_N_k = factorial(name)/(factorial(k-1)*factorial(name-k+1));
            p(k) = 0;
            for i = 1:delta
                p_X = erfc(-(c-sqrt(rho)*globe(i))/sqrt(2*(1-rho)))/2;
                p(k) = p(k) + c_N_k * p_X^(k-1) * (1-p_X)^(name-k+1);
            end
            p(k) = p(k)/delta;
        end
        Expe(m,n) = p(2) + 2*p(3) + 3*(1-p(1)-p(2)-p(3));
    end
end
```

Appendix C: Matlab Code II

The following is the Matlab code for simulating Dependence of $\frac{dL_c^3}{d\rho}$ on ρ and c , taken from DerivativeOfExpLoss.m.

```
clear;

num_rho = 30;
num_pro = 30;
in0 = zeros(num_rho, num_pro);
in1 = in0;
in2 = in0;
exp_loss = in0;
for m = 1:num_rho
    for n = 1:num_pro

name = 100;
rho = m/30;
default_pro = n/200;

delta = 100;
globe = -3.5:7/(delta-1):3.5;
globe = globe';

for i = 2:delta
    in0(m,n) = in0(m,n) + integk0(globe(i-1),rho,default_pro)
        *(erfc(-globe(i)/sqrt(2))-erfc(-globe(i-1)/sqrt(2)))/2;
    in1(m,n) = in1(m,n) + integk1(globe(i-1),rho,default_pro)
        *(erfc(-globe(i)/sqrt(2))-erfc(-globe(i-1)/sqrt(2)))/2;
    in2(m,n) = in2(m,n) + integk2(globe(i-1),rho,default_pro)
        *(erfc(-globe(i)/sqrt(2))-erfc(-globe(i-1)/sqrt(2)))/2;
end

exp_loss(m,n) = -3*in0(m,n)-2*in1(m,n)-in2(m,n);

    end
end
meshc(exp_loss); figure(gcf)
```

```

function y = integk0(x,rho,default_pro)
N = 100;
%k = 0;

c = -sqrt(2)*erfcinv(2*default_pro);

p = erfc(-(c-sqrt(rho)*x)/sqrt(2*(1-rho)))/2;
y = (c*sqrt(rho)-x)/((1-rho)*2*sqrt(rho*(1-rho)))
    *((-N)*exp(-N*p-0.5*((c-sqrt(rho)*x)/sqrt(1-rho))^2)/sqrt(2*pi));
end

```

```

function y = integk1(x,rho,default_pro)
N = 100;
k = 1;

c = -sqrt(2)*erfcinv(2*default_pro);

p = erfc(-(c-sqrt(rho)*x)/sqrt(2*(1-rho)))/2;
y = (c*sqrt(rho)-x)/((1-rho)*2*sqrt(rho*(1-rho)))
    *((-N)*(N*p)^k/factorial(k)+(N*p)^(k-1)/factorial(k-1)*N)
    *exp(-N*p-0.5*((c-sqrt(rho)*x)/sqrt(1-rho))^2)/sqrt(2*pi);
end

```

```

function y = integk2(x,rho,default_pro)
N = 100;
k = 2;
%default_pro = .1;
c = -sqrt(2)*erfcinv(2*default_pro);
%rho = 0.5;
p = erfc(-(c-sqrt(rho)*x)/sqrt(2*(1-rho)))/2;
y = (c*sqrt(rho)-x)/((1-rho)*2*sqrt(rho*(1-rho)))
    *((-N)*(N*p)^k/factorial(k)+(N*p)^(k-1)/factorial(k-1)*N)

```

```
*exp(-N*p-0.5*((c-sqrt(rho)*x)/sqrt(1-rho))^2)/sqrt(2*pi);  
end
```

Vita

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