Operators on Hilbert Space.

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ABSTRACT

Partial isometries without non-trivial reducing spaces, conjugate homogenous linear operators, and candidates for an operator without non-trivial invariant subspaces are discussed. Attention is restricted to separable Hilbert spaces.

In the first chapter, a partial isometry having no non-trivial reducing space, with its null-space and that of its adjoint both of dimension one is shown to be anti-unitarily equivalent to its adjoint. The notion of an orthogonal chain of vectors with respect to a partial isometry is introduced and investigated. Examples are given based on the idea of an orthonormal basis skewed with respect to a fixed basis. The implications of an operator being anti-unitarily equivalent to its adjoint are clarified, especially in the case where the anti-unitary operator giving the equivalence is a conjugation.

In Chapter II, a structure theorem for Hermitian conjugate operators and for normal conjugate operators is given. The maximal ideal space theory is employed.

Chapter III introduces the concept of a completely normal operator and provides four candidates for an operator without a non-trivial invariant subspace, based on the polar decomposition of an operator. Pairs are given of a unitary and a Hermitian operator, both completely normal and without common non-trivial invariant subspaces.
PREREQUISITE MATERIAL AND CONVENTIONS

Much of the contents of this section is standard. It is included for the sake of completeness.

\( \mathbb{R} \) denotes the field of real numbers and \( \mathbb{C} \) the field of complex numbers. If \( \lambda \in \mathbb{C} \), then \( \overline{\lambda} \) denotes its complex conjugate. \( \mathcal{H} \) denotes a separable Hilbert space over \( \mathbb{C} \). If \( \xi \) and \( \eta \) are vectors or elements of \( \mathcal{H} \), then \( (\xi | \eta) \) denotes their inner product. The norm of \( \xi \) is written \( \|\xi\| \). If \( \mathcal{K} \) is a Hilbert space, then we denote by \( \dim \mathcal{K} \) the dimension of \( \mathcal{K} \), the cardinality of a complete orthonormal system of vectors in \( \mathcal{K} \). In this paper the dimension of all Hilbert spaces considered is less than or equal to the cardinality of the integers. A linear manifold of \( \mathcal{K} \) is a non-void subset of \( \mathcal{K} \) closed with respect to vector addition and scalar multiplication. A subspace is a linear manifold closed in the topology given by the norm in \( \mathcal{K} \).

If \( \{\xi_a\}_{a \in A} \) is a set of vectors in \( \mathcal{K} \), then \( \mathcal{J}[\xi_a]_{a \in A} \) denotes the smallest linear manifold containing this set or the span of this set; \( \mathcal{J}[\xi]_{a \in A} \) denotes the smallest subspace containing this set or its closed span. A line above a subset of \( \mathcal{K} \) denotes its closure in the norm topology unless otherwise stated.

If \( \mathcal{M} \) is a linear manifold, then \( \mathcal{M}^\perp \) is the subspace of
all vectors orthogonal to \( m \); \( m^\perp = \{ \xi \in \mathcal{H} \mid (\xi | \eta) = 0 \ \forall \ \eta \in m \} \). If \( \{ \mathcal{K}_a \}_{a \in A} \) is a pairwise orthogonal family of subspaces of \( \mathcal{H} \), then \( \bigoplus_{a \in A} \mathcal{K}_a \) denotes the smallest subspace containing each subspace in the family. If \( m \) and \( \mathcal{N} \) are linear manifolds, then \( m + \mathcal{N} \) is the smallest linear manifold containing both \( m \) and \( \mathcal{N} \); \( m + \mathcal{N} = \{ \xi + \eta \mid \xi \in m, \eta \in \mathcal{N} \} \). We use the symbol \( \perp \) to denote the orthogonality of a vector with another vector, of a vector to a linear manifold, and of one linear manifold to another. Thus, \( \xi \perp \eta \) is equivalent to \( (\xi | \eta) = 0 \); \( \xi \perp \mathcal{K} \) is equivalent to \( (\xi | \eta) = 0 \ \forall \ \eta \in \mathcal{K} \); \( \mathcal{L} \perp \mathcal{K} \) is equivalent to \( (\xi | \eta) = 0 \ \forall \ \xi \in \mathcal{L}, \forall \ \eta \in \mathcal{K} \). If \( \mathcal{K} \) is a linear manifold of \( \mathcal{H} \), then \( \mathcal{N} \cap \mathcal{K} = \{ \xi \in \mathcal{H} \mid \xi \perp \mathcal{K} \} \).

We assume familiarity with the notion of a bounded operator on \( \mathcal{H} \), or more precisely, with the notion of a complex homogenous, linear, bounded operator. Since we discuss only bounded operators, we refer to them simply as operators. If \( A \) is an operator, we let \( \| A \| \) denote the operator norm of \( A \); we let \( A^* \) denote the adjoint of \( A \), so \( (A\xi | \eta) = (\xi | A^*\eta) \ \forall \ \xi, \eta \in \mathcal{H} \). If \( A \) is an operator, then we let \( \mathcal{R}_A = \{ A\xi \mid \xi \in \mathcal{H} \} \), the range of \( A \). We let \( \mathcal{N}_A = \{ \eta \mid A\eta = 0 \} \), the null-space of \( A \). If \( A \) is an operator, then \( \mathcal{R}^\perp_A = \mathcal{N}_{A^*} \) or \( \mathcal{R}^\perp_A = \mathcal{N}^\perp_{A^*} \). An operator \( P \) is a projection or a projection operator if and only if \( P = P^2 \) and \( P = P^* \). \( \mathcal{R}_P \) is a subspace; given a subspace \( \mathcal{K} \) there is a projection whose range is \( \mathcal{K} \). An operator \( H \) is Hermitian if and only if \( H = H^* \). An operator \( N \) is normal if and only if \( N^*N = NN^* \).
An operator $K$ is an isometry or an isometric operator if and only if $\|K\xi\| = \|\xi\|$ $\forall \xi \in \mathcal{H}$. An operator $U$ is unitary if and only if it is isometric and $\mathcal{R}_U = \mathcal{H}$. If $U$ is an operator, then $U$ being unitary is equivalent to $U^*U$ and $UU^*$ both being the identity operator on $\mathcal{H}$. A partial isometry on $\mathcal{H}$ is an operator $S$ such that for some subspace $\mathcal{K}$ of $\mathcal{H}$, we have $\|S\xi\| = \|\xi\|$ $\forall \xi \in \mathcal{K}$, and $S\eta = 0$ for $\eta \perp \mathcal{K}$. $\mathcal{K}$ is denoted by $\mathcal{B}_S$. The following statements concerning an operator $S$ are equivalent: 1) $S$ is a partial isometry; 2) $S^*$ is a partial isometry; 3) $S^*S$ is a projection; 4) $SS^*$ is a projection. Furthermore, $S^*S$ is the projection onto $\mathcal{B}_S = \mathcal{R}_{S^*}$; $SS^*$ is the projection onto $\mathcal{B}_{S^*} = \mathcal{R}_S$.

We employ the usual concepts of invertibility, spectrum, eigenvector or proper vector, eigenvalue, approximate eigenvalue, and approximate point spectrum of an operator. We denote the spectrum of an operator $A$ by $\sigma(A)$ and its approximate point spectrum by $\pi(A)$.

If $\mathcal{K}$ is a subset of $\mathcal{H}$ and $A$ is an operator on $\mathcal{H}$, then $A\mathcal{K} = \{A\xi \mid \xi \in \mathcal{K}\}$. If $\mathcal{A}$ is a set of operators and $\eta_0 \in \mathcal{H}$, then $A\eta_0 = \{A\eta_0 \mid A \in \mathcal{A}\}$. If $\mathcal{A}$ is a ring of operators on a Hilbert space $\mathcal{K}$, $\xi_0 \in \mathcal{K}$, and the smallest subspace containing $\mathcal{A}\xi_0$ is $\mathcal{K}$, then we say $\xi_0$ is a cyclic vector for $\mathcal{A}$. If $A$ is an operator on $\mathcal{K}$ and $\zeta_0 \in \mathcal{K}$, then we say $\zeta_0$ is cyclic for $A$ (for $A$ and $A^*$) if and only if $\zeta_0$ is cyclic for the algebraic ring generated by $A$ (by $A$ and $A^*$).
A linear manifold $\mathcal{L}$ is invariant for an operator $A$ if and only if $A\mathcal{L} \subseteq \mathcal{L}$; $\mathcal{L}$ reduces $A$ if and only if $A\mathcal{L} \subseteq \mathcal{L}$ and $A^*\mathcal{L} \subseteq \mathcal{L}$. A linear manifold is invariant for a set of operators if and only if it is invariant under each operator in the set. A linear manifold reduces a set of operators if and only if it reduces each operator in the set. If a linear manifold $\mathcal{M}$ is invariant for an operator $A$, then the subspace $\mathcal{M}$ is also invariant under $A$.

Now we turn for a moment to less common material. A conjugate homogenous, linear, bounded operator, or conjugate operator, is a mapping $B$ of $\mathcal{V}$ into itself such that $B$ is continuous in the norm topology and $B(\alpha \xi + \beta \eta) = \overline{\alpha} B\xi + \overline{\beta} B\eta$ $\forall \alpha, \beta \in \mathbb{C}$, $\forall \xi, \eta \in \mathcal{V}$. The norm of $B$ is defined as for operators and is denoted $\| B \|$. If $B$ is a conjugate operator on $\mathcal{V}$, then there is a conjugate operator $B^*$, the adjoint of $B$, satisfying $(B\xi | \eta) = (B^*\eta | \xi)$$\forall \xi, \eta \in \mathcal{V}$. The existence of the adjoint of a conjugate operator is proven using the Riesz Representation theorem for bounded linear functionals on $\mathcal{V}$; the proof is analogous to that for the existence of the adjoint of an operator. See p. 38-40, [1].

If $B$ is a conjugate operator, then $B^{**} = B$ and $(\lambda B)^* = \lambda B^*$ $\forall \lambda \in \mathbb{C}$. If $B$ and $C$ are conjugate operators, then $B + C$ is a conjugate operator and $BC$ is an operator. We have $(B + C)^* = B^* + C^*$. The adjoint of the operator $BC$ is the product of the adjoints of the conjugate operators $B$ and $C$.
in the opposite order; thus \((BC)^* = C^*B^*\). If \(A\) is an operator
and \(B\) is a conjugate operator, then \(AB\) and \(BA\) are conjugate
operators; \((AB)^* = B^*A^*\) and \((BA)^* = A^*B^*\). We use the symbol
\(^*\) above an operator or above a conjugate operator to denote
its adjoint without confusion. We do not add a non-zero
operator and a non-zero conjugate operator.

If \(B\) is a conjugate operator, then we define \(R_B\) and \(T_B\)
as for operators. A conjugate operator \(H\) is Hermitian if and
only if \(H = H^*\). A conjugate operator \(N\) is normal if and only
if \(N^*N = NN^*\). An anti-unitary operator is a conjugate opera-
tor \(U\) such that \(\| U\xi \| = \| \xi \| \forall \xi \in \mathcal{H}\) and \(R_U = \mathcal{H}\). A con-
jugate operator \(U\) is anti-unitary if and only if \(U^*U =UU^* = \)
1, the identity operator on \(\mathcal{H}\).

We employ the terms ring and symmetric ring as in [3].
The maximal ideal space of a commutative Banach ring with
identity is a compact Hausdorff space in the weakest topology
in which the Gelfand transforms of ring elements are con-
tinuous; p.197, [3]. We use \(\mathcal{B}(\mathcal{H})\) to denote the ring of all
operators on \(\mathcal{H}\). A norm-closed commutative symmetric subring
of \(\mathcal{B}(\mathcal{H})\) with identity is isometrically isomorphic to the ring
\(C(\mathcal{M})\) of all continuous functions on the maximal ideal space
\(\mathcal{M}\) of the subring. We use \(\hat{A}(m)\) to denote the continuous func-
tion on \(\mathcal{M}\) which is the image or Gelfand transform of an
operator \(A\) in the subring. \((\hat{A}^*)(m) = \hat{A}(m) \forall m \in \mathcal{M}. \| A \| = \)
\(\sup_{m \in \mathcal{M}} |\hat{A}(m)|\). We refer to p. 230-232, [3].
Since we are restricting our attention to separable Hilbert spaces, each maximal commutative symmetric subring of $\mathcal{B}(\mathcal{U})$ has a cyclic vector by a maximality argument. If $\mathcal{A}$ is a maximal commutative symmetric ring of operators on $\mathcal{U}$, $\xi_0$ is a cyclic vector for $\mathcal{A}$, and $M$ is the maximal ideal space of $\mathcal{A}$, then $\mathcal{A}$ is isometrically isomorphic to $C(M)$, $\xi_0$ corresponds to a regular Borel measure $\mu$ on $M$ with support equal to $M$, and there is an isometric mapping of $\mathcal{U}$ onto $L^2(M, \mu)$. Moreover, if we denote the image of $\xi \in \mathcal{U}$ by $\xi(m) \in L^2(M, \mu)$, then we have $(A\xi)(m) = \hat{A}(m) \xi(m)$ for $A \in \mathcal{A}$, $\forall \xi \in \mathcal{U}$; p. 247, [3].

We presuppose knowledge of the weak and strong topologies for $\mathcal{B}(\mathcal{U})$. Multiplication with one factor fixed is continuous in either of the two topologies. The transition from $A$ to $A^*$ is continuous in the weak topology. The weak and the strong closures of symmetric subrings of $\mathcal{B}(\mathcal{U})$ coincide. A weakly closed symmetric subring of $\mathcal{B}(\mathcal{U})$ is generated by its projection operators, i.e., the minimal subring of $\mathcal{B}(\mathcal{U})$ containing all the projection operators of the given subring is the subring itself; p. 441-449, [3]. If $\mathcal{A}$ is a weakly closed symmetric commutative subring of $\mathcal{B}(\mathcal{U})$ with a cyclic vector, then $\mathcal{A}$ is a maximal commutative subring; we refer to Corollary 1.1 of [8]. A direct proof using the maximal ideal space theory without decomposition theory is possible.

We use the definitions of measure-preserving transformation, ergodic transformation, and measure algebra as in [2].
If $M$ is the maximal ideal space of a weakly closed commutative symmetric subring of $\mathcal{B}(\mathcal{N})$, then the closure of each open set in $M$ is open; p.31, [5]. The measure $\mu$ corresponding to a cyclic vector for a maximal such subring is a regular Borel measure. For each measurable set $S$, there exists a unique clopen set $U$ for which $\mu \left( (S-U) \cup (U-S) \right) = 0$; p.48, [5]. Thus the clopen sets form a complete set of representatives for the equivalence classes which comprise the measure algebra $(M,\mu)$. 
CHAPTER I

PARTIAL ISOMETRIES

Our study of partial isometries having no non-trivial reducing spaces began as an attempt to generalize the following result of von Neumann:

Let $\mathcal{H}$ be a Hilbert space of dim $> 1$. Let $S$ be an isometry on $\mathcal{H}$ such that $S$ has no non-trivial reducing space. Then $\mathcal{H}$ is of dimension one. If $\mathcal{H}$ is of countably infinite dimension.

Outline of proof: If $\mathcal{R}_S = \{0\}$, then $S$ is unitary. By the spectral theorem for normal operators and the fact that dim $\mathcal{H} > 1$, we have that $S$ has non-trivial reducing spaces in contradiction of the hypothesis. Hence $\mathcal{R}_S \neq \{0\}$.

We fix $\xi_0 \in \mathcal{H}$ such that $\|\xi_0\| = 1$. Then for $0 \leq j < i < \infty$, we have $(S^i \xi_0 | S^j \xi_0) = (S^* S^j \xi_0 | \xi_0) = (S^{i-j} \xi_0 | \xi_0) = 0$ since $\xi_0 \in \mathcal{R}_S$. So $(S^i \xi_0 | S^j \xi_0) = \delta_{i,j}$ for $0 \leq i, j < \infty$. But $\mathcal{F}\{S^i \xi_0\}_{i=0}^\infty$ reduces $S$. In fact, if $\sum_{i=0}^\infty a_i S^i \xi_0$ is an element of this closed span, then we have that $S^* (\sum_{i=0}^\infty a_i S^i \xi_0) = \sum_{i=0}^\infty a_{i+1} S^i \xi_0$. So $\mathcal{F}\{S^i \xi_0\}_{i=0}^\infty = \mathcal{H}$.

We now give two examples of partial isometries having no non-trivial reducing spaces. Both partial isometries are defined on countably infinite dimensional Hilbert space. To facilitate the construction of these and of further examples,
we will fix a method for determining an orthonormal basis for separable *"skewed"* with respect to a given basis \( \{e_i\}_{i=1}^{\infty} \).

We proceed as follows. Given the orthonormal basis \( \{e_i\}_{i=1}^{\infty} \), we define a second set of vectors \( \{\xi_i\}_{i=1}^{\infty} \). Let \( \xi_1 = (\sqrt{2})^{-1}(e_1 + e_2) \); for \( n \geq 2 \), let \( \xi_n = (\sqrt{2})^{-n}e_1 - \sum_{i=2}^{n}(\sqrt{2})^{i-n-2}e_i + (\sqrt{2})^{-1}e_{n+1} \). First we verify that \( \|\xi_i\| = 1 \), \( i = 1, 2, \ldots \).

Clearly \( \|\xi_1\| = 1 \). For \( n \geq 2 \), \( \|\xi_n\|^2 = \|(\sqrt{2})^{-n}e_1\|^2 + \sum_{i=2}^{n}\|(\sqrt{2})^{i-n-2}e_i\|^2 + \|(\sqrt{2})^{-1}e_{n+1}\|^2 \) by the parallelogram law.

\[
\|\xi_n\| = 2^{-n} + \sum_{i=2}^{n}2^{i-n-2} + 2^{-1} = 1. \text{ So } \|\xi_n\| = 1 \text{ for } n \geq 1.
\]

Now we show \( (\xi_n | \xi_m) = 0 \) for \( 1 \leq m < n < \infty \). Suppose \( n \geq 2 \).

By definition, \( \xi_n = (\sqrt{2})^{-n}e_1 - (\sqrt{2})^{-n}e_2 + \eta_n \), where \( \eta_n \in \mathcal{F}(e_i)_{i=3}^{n+1} \). \( (\eta_n | (\sqrt{2})^{-1}[e_1+e_2]) = (\eta_n | \xi_1) = 0 \). Also, \( (\sqrt{2})^{-n}(e_1-e_2) \) is orthogonal to \( (\sqrt{2})^{-1}(e_1+e_2) \). So \( \xi_1 \perp \xi_n \), \( n \geq 2 \). For \( n > m \geq 2 \), we have \( (\xi_n | \xi_m) = 0 \).

\[
(\sqrt{2})^{-n}e_1 - \sum_{i=2}^{n}(\sqrt{2})^{i-n-2}e_i + (\sqrt{2})^{-1}e_{n+1}|(\sqrt{2})^{-m}e_1 - \sum_{i=2}^{m}(\sqrt{2})^{i-m-2}e_i + (\sqrt{2})^{-1}e_{m+1}) = 0
\]

So for \( n > m \geq 2 \), we have \( (\xi_n | \xi_m) = 0 \). Hence \( \{\xi_i\}_{i=1}^{\infty} \) is an orthonormal set.

It remains to show that \( \mathcal{F}(e_i)_{i=1}^{\infty} = \mathcal{F}(\xi_i)_{i=1}^{\infty} \). \( (e_1 | \xi_n) = 0 \), \( n \geq 1 \). So \( \sum_{n=1}^{\infty}|(e_1 | \xi_n)|^2 = \sum_{n=1}^{\infty}(\sqrt{2})^{-n} = 1 \).

By Parseval's identity, we have \( e_1 \in \mathcal{F}(\xi_i)_{i=1}^{\infty} \). Similarly, for
m > 1, \((e_m|g_n) = 0\) for \(n < m - 1\); \((e_m|g_{m-1}) = [\sqrt{2}]^{-1}\); and 
\((e_m|g_{m-1+k}) = -[\sqrt{2}]^{-k-1}\) for \(k \geq 1\). Hence we have 
\(\sum_{n=1}^{\infty} (e_m|g_n)^2 = \sum_{n=m}^{\infty} (e_m|g_n)^2 = 2^{-1}\) and 
\(\sum_{n=m}^{\infty} 2^{m-2-n} = 1\). So 
\(e_m \in \mathcal{J}(g_1)_{i=1}^\infty\). Clearly \(\mathcal{J}(g_1)_{i=1}^\infty \subseteq \mathcal{J}(e_1)_{i=1}^\infty\). Thus we have 
\(\mathcal{J}(g_1)_{i=1}^\infty = \mathcal{J}(e_1)_{i=1}^\infty\).

**Definition 1.1:** If \(\{e_1\}_{i=1}^\infty\) is an orthonormal basis for \(V\), 
then \(\{g_1\}_{i=1}^\infty\), as constructed above, is also. We will refer 
to \(\{g_1\}_{i=1}^\infty\) as the **skewed basis** for \(V\) with respect to the basis \(\{e_1\}_{i=1}^\infty\).

While the above computations facilitate proof, they 
seem to obscure the idea involved in building a basis 
skewed with respect to a given basis. A more intuitive 
explanation is in order. Choose \(\eta_1 = [\sqrt{2}]^{-1}[e_1+e_2]\). Then 
take a vector of norm one in \(\mathcal{J}(e_1, e_2) \oplus \mathcal{J}(\eta_1)\), calling 
it \(\eta_1\). Let \(\eta_2 = [\sqrt{2}]^{-1}[\eta_1 + e_3]\). Again take a vector of 
norm one in \(\mathcal{J}(e_1, e_2, e_3) \oplus \mathcal{J}(\eta_1, \eta_2)\), calling it \(\eta_2\). Let 
\(\eta_3 = [\sqrt{2}]^{-1}[\eta_2 + e_4]\). And so on. This process will yield 
a similar basis \(\{\eta_1\}_{i=1}^\infty\) also "skewed" with respect to \(\{e_1\}_{i=1}^\infty\). 
In the first explanation, we fix \(\eta_1 = [\sqrt{2}]^{-1}[e_1-e_2]\); but 
any multiple of this vector by a complex number of modulus 
one would suffice.

**Example 1.2** The first example is relatively simple and makes 
no use of a skewed basis. Consider \(V\) with a "double-ended" 
orthonormal basis \(\{e_1\}_{i=-\infty}^\infty\). The partial isometry \(L_1\) is defined 
as follows:
\[ L_1(e_j) = e_{j+1} \text{ for } -\infty < j < 0 \text{ and } 1 < j < \infty \]
\[ L_1(e_0) = [\sqrt{2}]^{-1}[e_1 + e_2] \]
\[ L_1(e_1) = 0 \]

So \( \eta_{L_1} = \mathcal{J}(e_1) \), \( \mathcal{K}_{L_1}^\perp = \eta_{L_1}^* = \mathcal{J}([\sqrt{2}]^{-1}[e_1 - e_2]) \). Since \( L_1^*L_1 = 1 \)
on \( \mathcal{K}_{L_1} \), we have
\[ L_1^*(e_{j+1}) = e_j \text{ for } -\infty < j < 0 \text{ and } 1 < j < \infty \]

We claim that \( L_1 \) has no non-trivial reducing space. We employ two steps. First we will show that \( e_1 \) is cyclic for \( L_1 \) and \( L_1^* \). Secondly we will show that each non-zero reducing space for \( L_1 \) contains \( e_1 \). Hence \( L_1 \) will be proven to have no non-trivial reducing space.

Let \( \mathcal{K}_1 \) be a reducing space for \( L_1 \) containing \( e_1 \). \( L_1^*(e_1) = (e_1| [\sqrt{2}]^{-1}[e_1 + e_2])e_0 + (e_1| [\sqrt{2}]^{-1}[e_1 - e_2])e_0 = [\sqrt{2}]^{-1}e_0 \).
\[ L_1^*(e_1) = [\sqrt{2}]^{-1}e_{-n+1} \text{ for } n \geq 1. \]
So \( \mathcal{J}(e_1)_{i=\infty}^1 \subset \mathcal{K}_1 \).
\[ L_1(e_0) = [\sqrt{2}]^{-1}[e_1 + e_2], \quad L_1([\sqrt{2}]^{-1}[e_1 + e_2]) = ([\sqrt{2}]^{-1}[e_1 + e_2]|e_1)e_0 + ([\sqrt{2}]^{-1}[e_1 + e_2]|e_2)e_3 = [\sqrt{2}]^{-1}e_3 \]
\[ L_1([\sqrt{2}]^{-1}[e_1 + e_2] = [\sqrt{2}]^{-1}e_{n+2} \text{ for } n \geq 1. \]
Lastly. \( L_1^*L_1([\sqrt{2}]^{-1}[e_1 + e_2] = [\sqrt{2}]^{-1}e_2 \), since \( L_1^*L_1 \) is the projection onto \( \mathcal{K}_{L_1} \). So \( \mathcal{J}(e_1)_{i=2}^\infty \subset \mathcal{K}_1 \). Hence \( \mathcal{K} = \mathcal{K}_1 \), and the first step is completed.

Now let \( \mathcal{K}_1^\perp \) be a non-zero reducing space for \( L_1 \). We must show \( e_1 \in \mathcal{K}_1^\perp \). Pick \( \zeta \in \mathcal{K}_1^\perp, \zeta \neq 0 \). Let \( \zeta = \sum a_i^\infty e_i \).
\( \zeta \neq 0 \) implies \( a_m \neq 0 \) for some \( m \); we will assume \( m \geq 1 \). The case where \( m < 1 \) is treated in a similar fashion. If \( m = 1 \), then \( \zeta - L_1^*L_1\zeta = (1 - L_1^*L_1)\zeta = \sum a_i^\infty e_i - \sum a_i^1 e_i = -a_1 e_1 \).
If \( m > 1 \), let \( p \) be the least integer of those \( m > 1 \) such that \( a_m \neq 0 \). Thus \( \zeta = \sum_{i=1}^{\infty} a_i e_i + \sum_{i=p}^{\infty} a_i e_i \). \( L_1^{*p-2}(\zeta) = \sum_{i=p}^{\infty} a_i e_i - \varepsilon - 2 + \sum_{i=1}^{\infty} a_i e_i \). Now \( a_{2+p-2} = a_p \). So

\[
(1-L_1 L_1^*) L_1^{*p-2} \zeta = (L_1^{*p-2} \delta \varepsilon/2 - 1)[e_1 - e_2] + \varepsilon/2 - 1[e_1 - e_2] = -\varepsilon/2 a_p[e_1 - e_2].
\]

So \( 1/2 - 1[e_1 - e_2] \in K_1 \). But \( (1-L_1 L_1^*) \varepsilon/2 - 1[e_1 - e_2] = 1/2 - 1[e_1] \). We then have \( e_1 \in K_1 \) and \( K_1' = \mathcal{K} \).

We have shown that \( L_1 \) is a partial isometry with no non-trivial reducing space, \( \dim \eta_{L_1} = \dim \eta_{L_1^*} = 1 \). Also, we note that \( \eta_{L_1} = \mathcal{J}(e_1) \) is not cyclic for \( L_1^* \). In fact, \( e_2 L_1 L_1^* e_1 \) for \( n \geq 0 \). Similarly, \( \eta_{L_1^*} \) is not cyclic for \( L_1^* \).

**Example 1.3** Our second example makes use of a skewed basis.

Let \( \mathcal{K} \) be a Hilbert space with orthonormal basis \( \{e_i\}_{i=0}^\infty \).

With respect to the orthonormal set \( \{e_i\}_{i=1}^\infty \), let \( \{\xi_i\}_{i=1}^\infty \) be the skewed basis. We recall that \( \zeta_i = 1/2 - 1[e_1 + e_2] \), \( \xi_n = 1/2 - 1[e_1 + e_2] + 1/2 - 1[e_{n+1}] \) for \( n \geq 2 \), and \( \mathcal{J}(\xi_i)_{i=1}^\infty = \mathcal{J}(e_i)_{i=1}^\infty \). We define \( L_2 \) as follows:

\[
L_2(e_0) = \xi_1
\]

\[
L_2(e_i) = \xi_i \quad \text{for} \quad i \geq 2
\]

So \( \eta_{L_2} = \mathcal{J}(e_1) \) and \( \mathcal{R}_{L_2} = \mathcal{J}(\xi_i)_{i=1}^\infty \). Hence \( \mathcal{R}_{L_2}^\perp = \eta_{L_2^*} = \mathcal{J}(e_0) \).

To show that \( L_2 \) has no non-trivial reducing space, we proceed in two steps. First, we show that \( e_0 \) is cyclic for \( L_2 \).

Second, we show that a non-zero reducing space for \( L_2 \) contains \( e_0 \).

Let \( \mathcal{K}_2 \) be a subspace invariant under \( L_2 \) and containing \( e_0 \). If \( \xi_i \in \mathcal{K}_2 \) for \( 1 \leq i < \infty \), then we have that \( \mathcal{K}_2 = \mathcal{K} \) since
\[ \mathcal{J}(e_0) \oplus \mathcal{J}(e_1) = \mathcal{K}^2. \] Obviously \( \xi_1 = \xi_2 = \mathcal{L}_2(\xi_0) \in \mathcal{K}_2. \) Also, \( \mathcal{L}_2(\xi_1) = \mathcal{L}_2([\sqrt{2}]^{-1}[e_1+e_2]) = \mathcal{L}_2([\sqrt{2}]^{-1}e_2) = [\sqrt{2}]^{-1}\xi_2. \) Now suppose for some \( j \geq 2 \) we have \( \xi_j \not\in \mathcal{K}_2. \) Let \( q \) be the least integer of those \( j \) for which \( \xi_j \not\in \mathcal{K}_2. \) Thus \( \mathcal{J}(\xi_i)_{i=1}^{q-1} \subseteq \mathcal{K}_2 \) but \( \xi_q \not\in \mathcal{K}_2. \)

Since \( q \geq 3 \), we have \( q-1 \geq 2 \) and

\[ \xi_{q-1} = [\sqrt{2}]^{-q-1}[e_1] - \sum_{i=2}^{q-1} [\sqrt{2}]^{-i-q-1}[e_1] + [\sqrt{2}]^{-1}[e_q] \]

\[ \mathcal{L}_2(\xi_{q-1}) = \mathcal{L}_2([\sqrt{2}]^{-q-1}[e_1]) - \sum_{i=2}^{q-1} [\sqrt{2}]^{-i-q-1}\mathcal{L}_2(e_1) + [\sqrt{2}]^{-1}\mathcal{L}_2(e_q) = 0 - \sum_{i=2}^{q-1} [\sqrt{2}]^{-i-q-1}\xi_1 + [\sqrt{2}]^{-1}\xi_q. \]

Now \( \mathcal{L}_2(\xi_{q-1}) \in \mathcal{K}_2 \). But \( \mathcal{J}(\xi_i)_{i=1}^{q-1} \subseteq \mathcal{K}_2 \) by choice of \( q \). So \( \xi_q \in \mathcal{K}_2 \) and we reach a contradiction. Hence \( \mathcal{J}(\xi_i)_{i=1}^{\infty} \subseteq \mathcal{K}_2 \) and \( \mathcal{K}_2 = \mathcal{K}. \)

Secondly, let \( \mathcal{K}_2^1 \) be a non-zero reducing subspace for \( \mathcal{L}_2 \). We must show \( e_o \in \mathcal{K}_2^1 \). Let \( \zeta \in \mathcal{K}_2^1, \zeta \neq 0, \zeta = a_o e_o + \sum_{i=1}^{\infty} a_i \xi_i \). If \( a_o \neq 0 \), then \( (1 - \mathcal{L}_2^sL_2^s)\zeta = a_o e_o \neq 0 \). Otherwise, we note that \( (L_2^s\xi_i | \xi_{i-1}) = (e_i | \xi_{i-1}) = [\sqrt{2}]^{-1} \neq 0 \) for \( i \geq 2 \).

Also, \( (L_2^s\xi_1 | \xi_k) = 0 = (e_1 | \xi_k) \) for \( 1 \leq k < i-1 \). Hence if \( \zeta = \sum_{i=1}^{\infty} a_i \xi_i \) with \( a_s \neq 0 \), we have \( (L_2^s| e_1) = [\sqrt{2}]^{-s}a_s \) and \( (L_2^s| e_o) = [\sqrt{2}]^{-s}a_s \). Thus \( (1 - \mathcal{L}_2^sL_2^s)\zeta = [\sqrt{2}]^{-s}a_s e_o \). In any case, we have shown that \( e_o \in \mathcal{K}_2^1 \). So \( \mathcal{K}_2^1 = \mathcal{K} \), and we see that \( \mathcal{L}_2 \) has no non-trivial reducing space.

Also, \( \dim \mathcal{N}_2 = 1 = \dim \mathcal{N}_2^* \); and \( e_o \) is cyclic for \( \mathcal{L}_2 \). As a result of lemma 1.4.1 to be proven later, we know that \( \mathcal{N}_2^* \) being cyclic for \( \mathcal{L}_2 \) implies \( \mathcal{N}_2^* \) is cyclic for \( \mathcal{L}_2^* \).

**Theorem 1.4** Let \( U \) be a partial isometry such that \( \dim \mathcal{N}_U = \dim \mathcal{N}_U^* = 1 \). Let \( \mathcal{N}_U = \mathcal{J}(e_o) \); let \( \mathcal{N}_U^* = \mathcal{J}(e_o^*). \) Then for each
pair \( P(x), Q(x) \) of polynomials over the complex numbers, we have 
\[
(P(U^*)e_o | Q(U^*)e_o) = (\overline{P(U)}\eta_o | \overline{Q(U)}\eta_o) = (\overline{Q(U)}\eta_o | \overline{P(U)}\eta_o)
\]
where \( P(x) \) denotes the polynomial obtained from \( P(x) \) by conjugation of coefficients.

Proof: We recall that \( UU^*, U^*U, 1-UU^*, 1-U^*U \) are the projections onto \( \xi_{U^*}, \xi_U, \eta_{U^*}, \) and \( \eta_U \), respectively. \( (e_o | e_o) = 1 = (\eta_o | \eta_o) \). So the assertion is true for constant polynomials.

We suppose that for \( 0 \leq i, j \leq n-1 \) it is true that 
\[
(U^1e_o | U^j e_o) = (U^1_\eta_o | U^j_\eta_o). \]
We will show that this equality holds for \( 0 \leq i, j \leq n \). If \( i = j = 0 \), then the equality is clear. If \( i = 0 \) and \( j \neq 0 \) or if \( i \neq 0 \) and \( j = 0 \), then both sides of the equality are zero, since \( \mathcal{R}_{U^*} \mathcal{1}_{\eta_U} \) and \( \mathcal{R}_U \mathcal{1}_{\eta_{U^*}} \). Thus we can suppose that \( 0 < i, j < n \). 

\[
(U^1e_o | U^j e_o) = (UU^*U^1e_o | U^j e_o) = (U^1e_o | U^j e_o) - ([I-UU^*]U^1e_o | U^j e_o) = (U^1e_o | U^j e_o) - (U^1e_o | U^j e_o)\]

By symmetry in the above computations, it is clear that 
\[
(U^1_\eta_o | U^j_\eta_o) = (U^1_\eta_o | U^j_\eta_o) - (U^1_\eta_o | e_o) \cdot (e_o | U^j_\eta_o). \]

Clearly, 
\[
(U^1_\eta_o | e_o) \cdot (e_o | U^j_\eta_o) = (\eta_o | U^1_\eta_o) \cdot (\eta_o | U^j_\eta_o) = (U^1_\eta_o | U^j_\eta_o). \]

So \( (U^1e_o | U^j e_o) = (U^1_\eta_o | U^j_\eta_o) \) for \( 0 \leq i, j < \infty \) by induction.

Now let \( P(x) = \sum_{i=0}^{n} a_i x^i \); let \( Q(x) = \sum_{j=0}^{m} b_j x^j \). 

\[
(P(U^*)e_o | Q(U^*)e_o) = \sum_{i,j} a_i b_j (U^i e_o | U^j e_o) = \sum_{i,j} a_i b_j (U^i_\eta_o | U^j_\eta_o) = \sum_{i,j} a_i b_j (\eta_o | \eta_o)
\]
Lemma 1.41. Let $U$ be a partial isometry on $\mathcal{K}$ such that $U$ has no non-trivial reducing space and $\dim \eta_U = \dim \eta_{U^*} = 1$. Let $\eta_U = \mathcal{J}(e_0)$; let $\eta_{U^*} = \mathcal{J}(\eta_0)$. Then $\mathcal{J}[U^i\eta_0]_{i=0}^\infty = \mathcal{K}$ is equivalent to $\mathcal{J}[U^{*i}e_0]_{i=0}^\infty = \mathcal{K}$.

Proof: By the symmetry in the following argument, it will be clear that it is enough to show that $\mathcal{J}[U^i\eta_0]_{i=0}^\infty = \mathcal{K}$ implies that $\mathcal{J}[U^{*i}e_0]_{i=0}^\infty = \mathcal{K}$. We assume that $\mathcal{J}[U^i\eta_0]_{i=0}^\infty = \mathcal{K}$.

Thus there exists a sequence of polynomials over the complex numbers, $\{P_i(x)\}_{i=1}^\infty$, such that $\|P_i(U)\eta_0 - e_0\|^2 < 2^{i-1}$. By theorem 1.4 we know that $\|P_i(U)\eta_0\|^2 = \|P_i(U^*)e_0\|^2$. So

\[ \|P_i(U)\eta_0 - e_0\|^2 = \|P_i(U)\eta_0\|^2 + \|e_0\|^2 - (P_i(U)\eta_0|e_0) - (e_0|P_i(U)\eta_0) = \|P_i(U^*)e_0\|^2 + \|\eta_0\|^2 - (\eta_0|P_i(U^*)e_0) - (\overline{P_i(U^*)}e_0|\eta_0) = \|\overline{P_i(U^*)}e_0\|^2 - \|\eta_0\|^2 \leq 2^{-i}. \]

Hence $\eta_0 \in \mathcal{J}[U^{*i}e_0]_{i=0}^\infty$. Let $\mathcal{K} = \mathcal{J}[U^{*i}e_0]_{i=0}^\infty$. Clearly $\mathcal{K}$ is invariant under $U^*$. Since $\mathcal{R}_{U^*} \perp \eta_U$, we can write $\mathcal{K} = \mathcal{J}(e_0) \oplus U^*\mathcal{K}$. But $\eta_0 \in \mathcal{K}$ implies $U^*$ is a partial isometry when $U^*$ is restricted to $\mathcal{K}$. So the domain of $U^*|\mathcal{K}$ is $\mathcal{K} \ominus \mathcal{J}(\eta_0) = \mathcal{K} \ominus \mathcal{J}(\eta_0)$. So we have $\mathcal{K} = \mathcal{J}(e_0) \oplus U^*(\mathcal{K} \ominus \mathcal{J}(\eta_0))$. If $\xi \in \mathcal{K}$, then $\xi = ae_0 + U\xi_1$ where $a \in \mathcal{C}$ and $\xi_1 \in \mathcal{K} \ominus \mathcal{J}(\eta_0)$. So $UG = U(ae_0) + UU\xi_1 = 0 + UU\xi_1 = \xi_1$. So $\mathcal{K}$ is invariant under $U$ as well as under $U^*$. Therefore $\mathcal{K} = \mathcal{K}$. As noted in the beginning of the proof, we can interchange the roles of $U$ and $U^*$ along with those of $e_0$ and $\eta_0$ in the above argument. Thus the lemma is proven.
We now introduce a definition of a chain of vectors with respect to a partial isometry $U$. Von Neumann's theorem on isometries without non-trivial reducing spaces means that each such isometry generates a chain whose span is dense in the Hilbert space.

Definition 1.5 A chain with respect to a partial isometry $U$ is a sequence of vectors of the form $\{U^i \xi_0\}_{i=0}^{\infty}$ with the requirement that $(U^i \xi_0 | U^j \xi_0) = \delta_{ij}$, $0 \leq i, j < \infty$.

Theorem 1.6 Let $U$ be a partial isometry on $\mathcal{H}$ such that $U$ has no non-trivial reducing space. Then the following statements are equivalent:

(a) $\mathcal{J}(U^* \xi_0) \neq \emptyset$;

(b) There exists $\xi_0 \in \mathcal{H}$ such that $\| U^i \xi_0 \| = 1$ for $0 \leq i < \infty$;

(c) There is a chain with respect to $U$, a $U$-chain.

Proof:

(a) implies (b). Since $\mathcal{J}(U^* \xi_0) \neq \emptyset$, we can pick $\xi_0$ in this subspace such that $\| \xi_0 \| = 1$. Now $\mathcal{J}(U^* \xi_0) \neq \emptyset$ is invariant with respect to $U$ and is orthogonal to $\xi_0$. So $U$ restricted to this subspace is an isometry. Hence (b) follows.

(b) implies (c). Let $\mathcal{K} = \mathcal{J}(U^i \xi_0)_{i=0}^{\infty}$. $\mathcal{K}$ is invariant with respect to $U$. Also, $U$ is an isometry when restricted to $\mathcal{K}$.

Let $\mathcal{G} = \mathcal{K} \oplus U\mathcal{K}$. We suppose $\mathcal{G} = \{0\}$. Then $\mathcal{K} = U\mathcal{K}$ or $U^* \mathcal{K} = U^* U \mathcal{K} = \mathcal{K}$ since $\mathcal{K} \perp \xi_0$ and $U^* U = I$ on $\mathcal{J}_0$. $\mathcal{K}$ is thus a reducing space, contradicting the hypothesis about $U$. So $\mathcal{G} \neq \{0\}$. Let $\xi_1 \in \mathcal{G}$ such that $\| \xi_1 \| = 1$. For $i > j$, $(U^i \xi_1 | U^j \xi_1) =$
\[(U^*JU^i|\xi_1) = (U^i-J|\xi_1) = 0 \text{ since } \xi_1 \in \chi \otimes U\chi. \text{ And }\]
\[
\eta_{U\perp} \chi \text{ implies that } (U^i|\xi_1) = 1 \text{ for } 0 \leq i < \infty. \text{ Thus } \]
\[
(U^i|\xi_1)(U^j|\xi_1) = \delta_{ij} \text{ for } 0 \leq i, j < \infty. \]
(c) implies (a). Let \( \xi_1 \) be such that \((U^i|\xi_1)(U^j|\xi_1) = \delta_{ij} \text{ for } 0 \leq i, j < \infty. \) So \( U^i|\xi_1 \nabla \eta_U \text{ for } 0 \leq i < \infty. \) So \( \eta_{U\perp} \nabla \{U^i|\xi_1\}_i=0.\)
This closed span is invariant with respect to \( U. \) So we have \( \chi \otimes \{U^i|\xi_1\}_i=0 \) is invariant with respect to \( U^* \) and contains \( \eta_U. \) Thus \( \{U^i|\xi_1\}_i=0 \subseteq \chi \otimes \{U^i|\xi_1\}_i=0 \neq \chi. \)

Lemma 1.61 Suppose \( U \) is a partial isometry on \( \chi, \chi \) is a subspace of \( \chi, \) and \( UX \subseteq \chi. \) Then \( \chi \) is a reducing subspace for \( U \) if and only if \( \chi \otimes UX \subseteq \eta_{U^*} \) and \( \eta_U = (\eta_U \cap \chi) \otimes (\eta_U \otimes \chi) = (\eta_U \cap \chi) \otimes (\eta_U \otimes \chi). \)

Proof: We suppose \( \chi \) reduces \( U. \) Then \( \chi \) and \( \chi \perp \) are invariant with respect to \( 1 - U^*U, \) the projection onto \( \eta_U. \) Thus \( \eta_U = (1-U^*U)\chi = (1-U^*U)(\chi \otimes \chi^\perp) = ((1-U^*U)\chi)(1-U^*U)\chi,) = (\eta_U \cap \chi) \otimes (\eta_U \otimes \chi^\perp). \) Similarly, since \( UX \subseteq \chi \) and \( UX \perp \chi \perp \subseteq \chi \perp, \) we have that \( \chi \otimes UX \subseteq (\chi \otimes UX) \otimes (\chi \otimes UX) = \chi \otimes (UX \otimes UX) = \chi \otimes \eta_{U^*}. \)

Now we suppose that \( \chi \) is an invariant subspace for \( U \) such that \( \chi \otimes UX \subseteq \eta_{U^*} \) and \( \eta_U = (\eta_U \cap \chi) \otimes (\eta_U \otimes \chi). \) We must show \( U^*\chi \subseteq \chi. \) \( \chi \otimes \eta_U = \chi \otimes [(\eta_U \cap \chi) \otimes (\eta_U \otimes \chi)] = \chi \otimes (\eta_U \cap \chi) \text{ since } \chi \otimes (\eta_U \otimes \chi) = \{0\}. \) Thus \( \chi \cap \beta_U = \chi \otimes (\eta_U \cap \chi) \text{ or } \chi = (\chi \cap \beta_U) \oplus (\chi \cap \eta_U). \) Now \( UX = U(\chi \cap \beta_U). \) \( \chi \otimes UX \subseteq \eta_{U^*} \text{ implies that } \chi = UX \oplus (\chi \otimes UX) = U(\chi \cap \beta_U) \oplus (\chi \cap \eta_U). \) Thus \( U^*\chi = U^*[U(\chi \cap \beta_U) \oplus (\chi \cap \eta_U)] = U^*U(\chi \cap \beta_U) + U^*(\chi \cap \eta_U) = \)
Theorem 1.7 Let $U$ be a partial isometry on $\mathcal{V}$ such that $U$ has no non-trivial reducing space and $\dim \mathcal{H}_U = \dim \mathcal{H}_{U^*} = 1$. Let $\mathcal{H}_U = \mathcal{J}(e_o)$ and $\mathcal{H}_{U^*} = \mathcal{J}(\eta_o)$. Then there exists a unique anti-unitary operator $S$ on $\mathcal{V}$ satisfying the conditions $SUS^{-1} = U^*$ and $S\eta_o = e_o$. Also we have $S^2\eta_o = \eta_o$. [We will see later that under these conditions $S^2\eta_o = \eta_o$ implies that $S = e$.]

Proof: Since $S^{-1} = S^*$, we see that $SUS^* = U^*$ is equivalent to $S^*US^* = U$. This is in turn equivalent to $S^*US = U^*$ or $S^{-1}US = U^*$. Let $\mathcal{J}(U^i\eta_o)_{i=0}^{\infty} = \mathcal{L}$. We consider two cases: $\mathcal{L} = \mathcal{V}$ and $\mathcal{L} \neq \mathcal{V}$.

First we suppose that $\mathcal{L} = \mathcal{V}$. By Lemma 1.4 $\mathcal{J}(U^i e_o)_{i=0}^{\infty} = \mathcal{V}$. We define $S\eta_o = e_o$. Thus $\eta_o = S^{-1}e_o$. Now if $SUS^{-1} = U^*$, then $SU^NS^{-1} = U^N$ for $N \geq 0$. If $S$ is conjugate homogenous and linear and if $P(x)$ is a polynomial over $\mathcal{C}$, then we have $SP(U)S^{-1} = \overline{P(U^*)}$. Thus if it is possible to define $S$ meeting the requirements of the theorem, we see that $SP(U)S^{-1}e_o = \overline{P(U^*)}e_o$ or $S[P(U)\eta_o] = \overline{P(U^*)}e_o$ on $\mathcal{J}(U^i\eta_o)_{i=0}^{\infty}$. By Theorem 1.4, for $\xi_1, \xi_2 \in \mathcal{L}$, we have $||S_{\xi_1}|| = ||\xi_1||$; so $S$ is well-defined. By the same theorem $(S_{\xi_1}|S_{\xi_2}) = (\xi_1|\xi_2)$. $S$ is conjugate homogenous and linear. We extend the domain of definition of $S$ to $\mathcal{J}(U^i\eta_o)_{i=0}^{\infty} = \mathcal{L} = \mathcal{V}$. On this closed span we clearly have $SUS^{-1} = U^*$. $SUS^{-1}$ is a bounded linear operator which equals $U^*$ on a dense linear manifold of $\mathcal{V}$ and hence equals $U^*$ on the
entire space. The range of $S$ includes $\mathcal{F}(U^*e_0)_{i=0}^\infty$ and hence is $\mathcal{H}$. Therefore, $S$ is anti-unitary; the theorem is proven in the first case.

Now we suppose $\mathcal{L} \neq \mathcal{H}$. By lemma 1.41 $\mathcal{F}(U^*e_0)_{i=0}^\infty \neq \mathcal{H}$.

Let $\mathcal{M} = \mathcal{F}(U^*e_0)_{i=0}^\infty$. If $e_0 \in \mathcal{L}$, then lemma 1.61 implies that $\mathcal{L}$ reduces $U$. So $e_0 \neq \mathcal{L}$. Let $P$ and $Q$ be the projections onto $\mathcal{L}$ and $\mathcal{M}$ respectively. Let $\mathcal{L}' = \mathcal{L} \oplus \mathcal{J}(e_0)$. Let $\xi_0 = [\|e_0 - Pe_0\|^2]^{-1}[e_0 - Pe_0]$, $\mathcal{L}' \oplus \mathcal{L} = \mathcal{J}(\xi_0)$. $\mathcal{L} = \mathcal{J}(\eta_0) \oplus \mathcal{U}\mathcal{L}$, since $R_{\mathcal{U}}U^*e_0$. $\mathcal{U}\mathcal{L}' = \mathcal{U}\mathcal{L}$. $\mathcal{L}' = \mathcal{L} \oplus \mathcal{J}(\xi_0) = \mathcal{J}(\eta_0) \oplus \mathcal{U}\mathcal{L}' \oplus \mathcal{J}(\xi_0)$. Since $\eta_0 \subset \mathcal{L}'$, we can write $\mathcal{L}' = \mathcal{J}(\eta_0) \oplus \mathcal{J}(\xi_0)$. With these facts we will now show that $\mathcal{H} = \mathcal{L} \oplus \mathcal{F}(U^*\xi_0)_{i=0}^\infty$. $U\xi_0 \in \mathcal{U}\mathcal{L}' = \mathcal{U}\mathcal{L} \subset \mathcal{L}$. If $\xi \in \mathcal{L}'$ and $\xi \perp \xi_0$, then we have $\xi = a\eta_0 + U\xi_2$, where $a \in \mathcal{C}$ and $\xi_2 \in \mathcal{L}' \cap \mathcal{N}_{\mathcal{U}}$. Hence $U^*\xi = aU^*\eta_0 + U^*U\xi_2 = \xi_2 \in \mathcal{L}'$. So $U^*\xi_0 \perp \mathcal{L}'$.

$\xi_0 \perp \mathcal{L}$ implies $\xi_0 \perp U^n\mathcal{L}$ for $n \geq 1$. $U^n\mathcal{L} = U^n\mathcal{L}'$. So $U^*\xi_0 \perp \mathcal{L}'$ for $n \geq 1$. Since $\eta_0 \subset \mathcal{L}$, we see that $(U^*\xi_0 | \xi_0) = 0$ and the norm of $U^*\xi_0$ is equal to one. More generally, $(U^*\xi_0 | U^*\xi_0) = \delta_{ij}$ for $0 \leq i, j \leq \infty$. Let $\zeta \in \mathcal{L} \oplus \mathcal{F}(U^*\xi_0)_{i=0}^\infty$, $\zeta = \xi_1 + \sum_{i=0}^\infty a_i U^*\xi_0$ where $\xi_1 \in \mathcal{L}$. $U\zeta = U\xi_1 + a_0 U\xi_0 + \sum_{i=0}^\infty a_i U^{i+1} \xi_0$. But we noted earlier $\xi_1 \in \mathcal{L}$ implies $U\xi_1 \in \mathcal{L}$ and $U\xi_0 \in \mathcal{U}\mathcal{L}' \subset \mathcal{L}$. Thus $U\zeta \in \mathcal{L} \oplus \mathcal{F}(U^*\xi_0)_{i=0}^\infty$. $U^*\xi = U^*\xi_1 + \sum_{i=0}^\infty a_i U^{i-1} U^*\xi_0$. We noted above that $\xi_1 \in \mathcal{L}$ implies that $U^*\xi_1 \in \mathcal{L} \oplus \mathcal{J}(\xi_0)$. Thus $U^*\xi \in \mathcal{L} \oplus \mathcal{F}(U^*\xi_0)_{i=0}^\infty$, and so this space reduces $U$. Hence $\mathcal{L} \oplus \mathcal{F}(U^*\xi_0)_{i=0}^\infty = \mathcal{H}$.

Similarly, if we set $\zeta_0 = \|\eta_0 - Q\eta_0\|^{-1}[\eta_0 - Q\eta_0]$, then we have $(U^i\zeta_0 | U^j\zeta_0) = \delta_{ij}$ for $0 \leq i, j < \infty$ and likewise
By the proof of lemma 1.41, for a polynomial $R(x)$ we have $\|R(U)e_0 - \eta_0\|^2 = \|\overline{R(U^*)}e_0 - \eta_0\|^2$.

So if $(R_i(x))_{i=1}^{\infty}$ is a sequence of polynomials such that $R_i(U)e_0$ converges (norm) to $P\eta_0$ as $i$ goes to infinity, then $R_i(U^*)e_0$ converges to $Q\eta_0$. Here we use the elementary fact that the projection of a vector $\eta_1$ onto a subspace $\mathcal{J}$ is that vector $\eta_2$ of $\mathcal{J}$ such that $\inf_{\eta\in\mathcal{J}}\|\eta-\eta_1\|$ is attained. Thus $S(P\eta_0) = Q\eta_0$, where we define $S$ as in the first case.

In this case, $S$ is a conjugate homogeneous linear isometry mapping $\mathcal{Z}$ onto $\mathcal{M}$; $S$ must be extended to meet the requirements of the theorem. For $\xi \in \mathcal{M}$ we have $SU^{-1}\xi = U^*\xi$, just as in case one.

Now we enlarge the domain of definition of $S$. As we shall see, this extension depends only on our definition of $S(\xi_0)$ or of $S(e_0 - P\eta_0)$. First we show that $S(e_0 - P\eta_0)$ can be defined in at most one way to satisfy the theorem.

From the equation $SU^{-1} = U^*$ we see that $S^{-1}H_{U^*} = H_U$ or $H_{U^*} = S^*H_U$. Since $S$ is norm-preserving or isometric, we see that $S\eta_0 = a\eta_0$, where $|a| = 1$, $a \in \mathbb{C}$. Considering that $S\eta_0 = Q\eta_0$, we obtain that $0 = (e_0 - P\eta_0|P\eta_0) = (S\eta_0 - S\eta_0|S\eta_0) = (a\eta_0 - Q\eta_0|Q\eta_0) = a(\eta_0|Q\eta_0) - (Q\eta_0|Q\eta_0) = |a - 1|(Q\eta_0|Q\eta_0)$.

So $a = 1$ if $(Q\eta_0|Q\eta_0) \neq 0$; but if $\eta_0 \perp \mathcal{M}$, then $\mathcal{M}$ reduces $U$ by another application of lemma 1.61. So $a = 1$ and $S\eta_0 = \eta_0$.

Therefore if $S$ satisfies the condition of the theorem, we have $S(e_0 - P\eta_0) = \eta_0 - Q\eta_0$ or $S\xi_0 = \xi_0$. 
Again SUS\(^{-1}\) = U* implies that SU\(^n\)S\(^{-1}\) = U*\(^n\), n ≥ 1;
SU\(^n\)S\(^{-1}\)\(\xi\)_o = U*\(^n\)\(\xi\)_o or SU\(^n\)\(\xi\)_o = U*\(^n\)\(\xi\)_o. By this observation we
must define S(\(\Sigma\_{i=1}^m b_i U^i \xi\)_o) = \(\Sigma\_{i=1}^m b_i U^i \xi\)_o. Since \((U^i \xi\)_o|U^j \xi\)_o) = δ\(_{i,j}\) for 0 ≤ i, j < ∞, we see that \((\xi_3|\xi_4) = \langle S\xi_3|S\xi_4\rangle\) for \(\xi_3, \xi_4 \in \mathcal{J}(U^i \xi\)_o\)\(^\infty\). By linearity and continuity we
extend the domain of definition of S from \(\mathcal{L} \oplus \mathcal{J}(U^i \xi\)_o\)\(^\infty\) to \(\mathcal{L} \oplus \mathcal{J}(U^i \xi\)_o\)\(^\infty\). It is clear that SUS\(^{-1}\)\(\xi\) = U*\(\xi\) for \(\xi\) ∈ 
\(\mathcal{J}(U^i \xi\)_o\)\(^\infty\) and hence for \(\xi\) in the closure of this span. So
SUS\(^{-1}\) = U* on \(\mathcal{M} \oplus \mathcal{J}(U^i \xi\)_o\)\(^\infty\) = \(\mathcal{H}\). In summary, S\(\mathcal{L}\) = \(\mathcal{M}\);
S(\(\mathcal{J}(U^i \xi\)_o\)\(^\infty\)) = \(\mathcal{J}(U^i \xi\)_o\)\(^\infty\); S is anti-unitary; S\(\eta\)_o = \(\eta\)_o;
S\(^2\)\(\eta\)_o = S\(\eta\)_o = \(\eta\)_o; and SUS\(^{-1}\) = U*. So the theorem is proven
in either case. We note that S\(^2\)\(\eta\)_o = \(\eta\)_o is a consequence of the
conditions SUS\(^{-1}\) = U* and S\(\eta\)_o = \(\eta\)_o.

Lemma 1.71 Suppose A is a linear operator with no non-trivial
reducing space. Let S be unitary or anti-unitary such that
SAS\(^*\) = A\(^*\). Then S\(^2\) = c \cdot 1 for some c ∈ \(\mathbb{C}\).

Proof: SAS\(^*\) = A\(^*\). Taking adjoints, we have SA*S\(^*\) = A or
S*A = A\(^*\) = SAS\(^*\). So S\(^2\)A = A\(^2\) and S\(^2\)A\(^*\) = A*S\(^2\). Since A
has no non-trivial reducing space, the Double Commutant
Theorem implies that S\(^2\) is a scalar multiple of the identity;
see p.448, [3].

Lemma 1.72 If S is anti-unitary such that S\(^2\) = c \cdot 1 where
c ∈ \(\mathbb{C}\), then c = 1 or c = -1.

Proof: S\(^2\) = c \cdot 1 implies that S\(^2\)S\(^*\) = cS\(^*\) or S = cS\(^*\). Taking
adjoints, we have S\(^*\) = cS. So S = cS\(^*\) = \(\bar{c}\)S\(^*\). Thus c - \(\bar{c}\) = 0.
Since |c| = 1 and c is a real number, we have c = 1 or c = -1.
Corollary 1.73 If \( S \) is an anti-unitary operator satisfying the conditions of theorem 1.7, then \( S^2 = 1 \).

Proof: Lemma 1.71 implies that \( S^2 \) is a multiple of the identity. Lemma 1.72 implies that \( S^2 = 1 \) or \( S^2 = -1 \). \( S^2 \eta_0 = \eta_0 \) implies that \( S^2 = 1 \).

Theorem 1.80 Let \( U \) be a partial isometry on \( \mathcal{H} \) such that \( U \) has no non-trivial reducing space and \( \dim \eta_U = 1 \leq \dim \eta_{U^*} < \infty \). Let \( \zeta_0 \in \mathcal{H} \) such that \( \| U^i \xi_0 \| = 1 \) for \( 0 \leq i < \infty \). Then there exists \( \xi_0 \in \mathcal{H} \) such that \( \| U^i \xi_0 \| = 1 \) for \( 0 \leq i < \infty \).

Proof: Let \( \eta_U = \mathcal{J}(e_o) \). If \( \dim \eta_{U^*} = 1 \), then the theorem is true by lemma 1.41 and theorem 1.6. We proceed by induction, assuming the theorem true for \( \dim \eta_{U^*} = n-1 \) and proving it to hold in case \( \dim \eta_{U^*} = n \). Let \( \eta_{U^*} \) have the orthonormal basis \( \{ \eta_i \}_{i=1}^n \), where \( \eta_i \perp e_o \) for \( 2 \leq i \leq n \). Let \( \mathcal{K} = \mathcal{J}(U^i \eta_1)_{i=0}^\infty \) since \( \mathcal{R}_U \perp \eta_{U^*} \). If \( e_o \in \mathcal{K} \) or if \( e_o \in \mathcal{K} \perp \), then \( \mathcal{K} \) is a reducing space by lemma 1.61. Since \( \| U^i \zeta_0 \| = 1 \) for \( 0 \leq i < \infty \), we have \( U^i \zeta_0 \perp \eta_{U^*} \) or \( (U^i \zeta_0 | \eta_1) = 0 = (\zeta_0 | U^i \eta_1) \) for \( 0 \leq i < \infty \). So \( \zeta_0 \in \mathcal{K} \perp \). Let \( P \) be the projection onto \( \mathcal{K} \), and let \( \theta_0 = ||e_o - Pe_o||^{-1}[e_o - Pe_o] \). Let \( \mathcal{L} = \mathcal{K} + \mathcal{J}(e_o) \). \( \mathcal{L} = \mathcal{K} \oplus \mathcal{J}(\theta_0) \). \( \mathcal{K} = \mathcal{J}(\eta_1) \oplus \mathcal{U} \mathcal{K} \). \( \mathcal{U} \mathcal{K} = \mathcal{U} \mathcal{L} \). Since \( \mathcal{H} \subset \mathcal{L} \), we have \( \mathcal{U} \mathcal{L} = \mathcal{U}(\mathcal{L} \cap \mathcal{B}_U) \). Thus \( \mathcal{L} = \mathcal{J}(\eta_1) \oplus \mathcal{U}(\mathcal{L} \cap \mathcal{B}_U) \oplus \mathcal{J}(\theta_0) \). \( \mathcal{U} \mathcal{L} \perp \theta_0 \) implies that \( \mathcal{L} \perp U^* \theta_0 \). \( U^* \mathcal{K} = U^*(\mathcal{L} \mathcal{J}(\theta_0)) = U^*\mathcal{J}(\eta_1) \oplus \mathcal{U}(\mathcal{L} \cap \mathcal{B}_U) \) = \( \mathcal{L} \cap \mathcal{B}_U \); or \( U^* \mathcal{K} = \mathcal{L} \cap \mathcal{B}_U \). So \( U^* \mathcal{K} \perp \mathcal{L} \) or \( \mathcal{L} \perp U^* \mathcal{L} \). On \( \mathcal{K} \perp \), we define \( \tilde{U} \) a partial isometry as follows: \( \tilde{U} \theta_0 = 0 \), \( \tilde{U} \xi = U \xi \) for \( \xi \in (\mathcal{K} \mathcal{J}(\theta_0)) = \mathcal{L} \). Since \( \mathcal{H} \subset \mathcal{L} \), we see that \( \eta_\tilde{U} = \mathcal{J}(\theta_0) \) and
We claim that \( \bar{U}^* = U^* \) on \( X^\perp \), and that \( \bar{U} \) has no non-trivial reducing space as an operator on \( X^\perp \). \( X^\perp = \mathcal{J}(\theta_o) \oplus \mathcal{L}^\perp \).

We recall \( U \in \mathcal{L} \) or \( U^* \in \mathcal{L}^\perp \). But \( U^* \mathcal{L} \not\subseteq \mathcal{L}^\perp \). So \( U^* \mathcal{K} = U^*(\mathcal{J}(\theta_o) \oplus \mathcal{L}^\perp) \subseteq \mathcal{L}^\perp \). Now let \( \xi_1, \xi_2 \in \mathcal{K}^\perp \). We write \( \xi_2 = a \theta_o + \xi_2' \), where \( \xi_2' \in \mathcal{L}^\perp \). Then \( (U^* \xi_1 | \xi_2) = (\xi_1 | U\theta_o + \xi_2') = (\xi_1 | \bar{U}(a \theta_o + \xi_2')) = (\xi_1 | \bar{U} \xi_2') = (\xi_1 | U \xi_2') = (U^* \xi_1 | \xi_2') = (U^* a \theta_o + U^* \xi_2') = (U^* a \theta_o + U^* \xi_2') \). So \( \bar{U}^* = U^* \) on \( \mathcal{K}^\perp \). Now we suppose \( \bar{U} \) has a non-trivial reducing space \( \mathfrak{m} \) contained in but not equal to \( \mathcal{K}^\perp \).

If \( \bar{U} \bar{U} \mathfrak{m} = \mathfrak{m} \), we have \( \theta_o \perp \mathfrak{m} \). On \( \mathcal{K}^\perp \oplus \mathcal{J}(\theta_o) \), \( U = \bar{U} \) and \( U^* = \bar{U}^* \). Thus \( \mathfrak{m} \) reduces \( U \), a contradiction. If \( \bar{U} \bar{U} \mathfrak{m} \) is properly contained in \( \mathfrak{m} \), then we have \( \mathfrak{m} \oplus \bar{U} \bar{U} \mathfrak{m} = \mathcal{J}(\theta_o) \), since \( \bar{U} \bar{U} \mathfrak{m} \) is the projection onto \( \mathfrak{m} \). In this case we claim \( \mathcal{K} \oplus \mathfrak{m} \) is a non-trivial reducing space for \( U \). Clearly \( \mathcal{K} \oplus \mathfrak{m} \neq \mathcal{K} \). \( \mathcal{K} \oplus \mathfrak{m} = \mathcal{K} \oplus \mathcal{J}(\theta_o) \oplus (\mathfrak{m} \oplus \mathcal{J}(\theta_o)) \). \( \mathfrak{m} \subset \mathcal{K} \); \( U^* \subset \mathcal{K} \oplus \mathcal{J}(\theta_o) = \mathcal{L} \); also \( U(\mathfrak{m} \oplus \mathcal{J}(\theta_o)) = \bar{U}(\mathfrak{m} \oplus \mathcal{J}(\theta_o)) \subset \mathfrak{m} \). \( U^* \mathfrak{m} = \bar{U}^* \mathfrak{m} \subset \mathfrak{m} \). Finally \( \theta_o \in U \mathcal{L} \subseteq \mathcal{K} \). So \( \mathcal{L} \oplus \mathfrak{m} \) reduces \( U \), a contradiction. Therefore \( \bar{U} \) has no non-trivial reducing space. \( \eta_{\bar{U}} = \mathcal{J}(\theta_o) \); \( \eta_{\bar{U}^*} = \mathcal{J}(\eta_{\bar{U}}) = \eta_{\bar{U}} \eta_{\bar{U}^*} = \mathcal{J}(\eta_{\bar{U}}) \).

Since \( \bar{U}^* = U^* \) on \( \mathcal{K}^\perp \) and \( \mathcal{J}(\eta_{\bar{U}}) \subset \mathcal{K}^\perp \). So \( \dim \eta_{\bar{U}} = 1 \) and \( \dim \eta_{\bar{U}^*} = n-1 \). Also, \( \zeta_o \in \mathcal{K}^\perp \) and \( |\bar{U}^* \zeta_o| = |U^* \zeta_o| = 1 \) for \( 0 \leq i < \infty \). By the induction hypothesis we have that there exists \( \xi_o \in \mathcal{K}^\perp \) such that \( |\bar{U}^i \xi_o| = 1 \) for \( 0 \leq i < \infty \). Thus \( \bar{U}^i \zeta_o \) for \( 0 \leq i < \infty \). So \( |U^i \zeta_o| = 1 \) for \( 0 \leq i < \infty \). |
and there is a $L^*_3$-chain. We define $L_3$ as follows on $\mathcal{F}\{e_i\}_{i=-\infty}^\infty = \mathcal{H}$:

$$L_3e_1 = e_{i+1} \text{ for } -\infty < i < 1$$
$$L_3e_2 = [\sqrt{2}]^{-1}[e_3+e_4]$$
$$L_3e_{2n+1} = e_{2n+3} \text{ for } 0 < n < \infty$$
$$L_3e_4 = e_6; \quad L_3e_6 = [\sqrt{2}]^{-1}[e_8+e_{10}]; \quad L_3e_8 = 0$$
$$L_3e_{2n} = e_{2n+2} \text{ for } 5 < n < \infty.$$  

So $\eta_{L_3} = \mathcal{F}\{e_8\}; \quad \eta_{L^*_3} = \mathcal{F}\{[\sqrt{2}]^{-1}[e_3-e_4], [\sqrt{2}]^{-1}[e_8-e_{10}]\}$. To show that $L_3$ has no non-trivial reducing space, we prove that $e_8$ is cyclic for $L_3$ and $L^*_3$. Then we prove any non-zero reducing space for $L_3$ must contain $e_8$. The proof is so similar to that in earlier examples that it is not repeated here.

We observe that $[U^*([\sqrt{2}]^{-1}[e_3+e_4])]_{i=0}^\infty$ is a $L^*_3$-chain. $[U^1e_{10}]_{i=0}^\infty$ and $[U^1e_3]_{i=0}^\infty$ are both $L_3$-chains; the existence of at least one $L_3$-chain is proven by theorems 1.80 and 1.6.

We do not know if there is a partial isometry $U$ satisfying the conditions of theorem 1.80 with the added requirement that $\dim \eta_{U^*} \geq 3$.

Now we make an estimate which relates the number of pairwise orthogonal $U^*$-chains to $\dim \eta_U$, where $U$ is a partial isometry having no non-trivial reducing space.

**Lemma 1.85** Let $\mathcal{K}$ be a subspace of the Hilbert space $\mathcal{H}$, and let $\mathcal{L}$ be a $n$-dimensional subspace of $\mathcal{K}$. Then we have that $\dim \{(\mathcal{K}+\mathcal{L}) \ominus \mathcal{K}\} \leq n$.

**Proof:** Let $\xi \in \mathcal{L}$, $\zeta \in \mathcal{K}$. Then $\xi + \zeta \perp \mathcal{K}$ is equivalent to $-P\xi = P\zeta$ where $P$ is the projection operator onto $\mathcal{K}$. $\zeta \in \mathcal{K}$ is
equivalent to $P\xi = \zeta$. So $\xi + \zeta \perp \mathcal{K}$ is equivalent to $\zeta = -P\xi$.

Let $\{\xi_i\}_{i=1}^{\infty}$ be a linear basis for $\mathcal{L}$. Then $\{\xi_i - P\xi_i\}_{i=1}^{n}$ is a spanning set for $(\mathcal{L} + \mathcal{K}) \theta \mathcal{K}$. Thus $\dim((\mathcal{L} + \mathcal{K}) \theta \mathcal{K}) \leq n$.

**Theorem 1.90** Let $U$ be a partial isometry on $\mathcal{H}$ such that $U$ has no non-trivial reducing space and $1 \leq \dim \mathcal{H}_U < \infty$. Then there is a set $D$ of vectors in $\mathcal{H}$ such that:

1) Cardinality of $D \leq \dim \mathcal{H}_U$. If $D$ is empty, we write $\text{card } D = 0$.

2) $(U^k \xi_i | U^m \xi_j) = \delta_{ij} \delta_{km}$ for $0 \leq k, m < \infty$ and $\xi_i, \xi_j \in D$ where $\xi_i \neq \xi_j$ for $i \neq j$.

3) If $\xi_0 \in \mathcal{H}$ such that $\|U^k \xi_0\| = 1$ for $0 \leq k < \infty$, then $\text{card } D$.

**Proof:** Let $\mathcal{K} = \mathcal{H}(U^* \mathcal{H}_U)_{i=0}^{\infty}$. If $\xi_0$ is such that $U^k \xi_0 \perp \mathcal{H}_U$ for $0 \leq k < \infty$, then $\xi_0 \perp \mathcal{K}$. $\mathcal{K} = \mathcal{H}_U \theta U \mathcal{K}$. $\mathcal{K}$ is invariant under $U$. Let $\mathcal{L} = \mathcal{K} + U \mathcal{K}$. $U \mathcal{L} = U \mathcal{K} \subset \mathcal{K}$. So $\mathcal{K} = \mathcal{H}_U \theta U \mathcal{L}$.

Let $\mathcal{G} = ((\mathcal{H}_U \theta U \mathcal{L}) + \mathcal{H}_U \theta U \mathcal{L}) \theta (\mathcal{H}_U \theta U \mathcal{L}) = \mathcal{L} \theta \mathcal{K}$. By lemma 1.85, we have $\dim \mathcal{G} \leq \dim \mathcal{H}_U$. $\mathcal{G} \perp \mathcal{K}$ implies $\mathcal{G} \perp U \mathcal{H}_U$ or $\mathcal{G} \subset B_{U^*}$. $U \mathcal{L} \subset \mathcal{K}$ implies $U \mathcal{L} \perp \mathcal{G}$ or $\mathcal{L} \perp U \mathcal{G}$. In fact, $U^{t+} \mathcal{G} \perp \mathcal{L}$ for $t \geq 0$ since $\mathcal{L}$ is invariant under $U^*$. If $\mathcal{G} \neq \{0\}$, then let $D = \{\xi_i\}_{i=1}^{\infty}$, an orthonormal basis for $\mathcal{G}$. We see that condition 1) is satisfied. We suppose that $(U^k \xi_i | U^m \xi_j) \neq 0$. We can assume $k \geq m$. $(U^k \xi_i | U^m \xi_j) \neq 0$. But $k - m > 0$ implies that $U^k \xi_i \perp \mathcal{L}$. So $k - m = 0$. $(\xi_i | \xi_j) \neq 0$ implies that $i = j$. So 2) is satisfied. $\mathcal{K} \theta (\mathcal{S} \theta \mathcal{F}(U^j \xi_i)_{j=0}^{\infty})$ is invariant under $U$ and $U^*$, as in earlier arguments. In the case $\mathcal{H}_U \subset \mathcal{K}$ or $\mathcal{G} = \{0\}$, we have $\mathcal{K} = \mathcal{H}$ by lemma 1.61. In this situation
where $D$ is the empty set, we interpret $\sum_{i=1}^{s} \mathcal{F}[U^g_j]_{i=0} = \{0\}$. So in any case $\sum_{i=1}^{s} \mathcal{F}[U^g_j]_{i=0} = K$. Hence $3$ holds.

By theorem 1.80 we know that a partial isometry $U$ satisfying $\dim \mathcal{H}_U = 1 = \dim \mathcal{H}_U^*$, having a $U^*$-chain, and without non-trivial reducing spaces must have a $U$-chain. A natural question is whether the $U$-chain or a $U^*$-chain must be orthogonal to a $U^*$-chain. The following example is best compared with example 1.2, $L_1$.

**Example 1.91** The following example will be constructed in several steps. We first construct a partial isometry in the following way. Let $\mathcal{K}$ be a Hilbert space with orthonormal basis $\{e_i\}_{i=0}^{\infty}$. Let $\{\xi_i\}_{i=1}^{\infty}$ be the skewed basis with respect to the orthonormal set $\{e_1, e_2\} \cup \{e_{2i+1}\}_{i=2}^{\infty}$. Similarly, let $\{\theta_i\}_{i=1}^{\infty}$ be the skewed basis with respect to the orthonormal set $\{e_3, e_4\} \cup \{e_{2i}\}_{i=3}^{\infty}$. That is, $\xi_1 = [\sqrt{2}]^{-1}[e_1 + e_2]$; $\xi_2 = [\sqrt{2}]^{-2}[e_1 - e_2] + [\sqrt{2}]^{-1}e_5$; ... Also, $\theta_1 = [\sqrt{2}]^{-1}[e_3 + e_4]$; $\theta_2 = [\sqrt{2}]^{-2}[e_3 - e_4] + [\sqrt{2}]^{-1}e_6$; ... We define $U^*$ as follows:

- $U^*e_0 = \xi_1$; $U^*e_1 = \theta_1$; $U^*e_3 = \xi_2$; $U^*e_5 = \theta_2$
- $U^*e_{2n} = \xi_n$ for $n \geq 3$; $U^*e_{2n+1} = \theta_n$ for $n \geq 3$

So $\mathcal{H}_{U^*} = \mathcal{J}[e_2, e_4]$; $\mathcal{H}_U = \mathcal{R}_{U^*} = \mathcal{J}[e_0]$. It is clear that $e_0$ is cyclic for $U^*$ and that each non-zero reducing space for $U$ must contain $\mathcal{H}_U$. So $U$ has no non-trivial reducing space. Given $\theta \in \mathcal{K}$ with $\|\theta\| = 1$, we can easily obtain an integer $k$ such that $\|U^k\theta\| < 1$. Loosely said, $U$ moves each vector toward $\mathcal{H}_U$. So there is no $U$-chain. By theorem 1.80, there is no $U^*$-chain.
Now we proceed to the next stage of the construction.

Let $X = \mathcal{J}[U_\mathcal{V}^{1\mathcal{V}}e_{\mathcal{V}}]_{i=0}^{\infty}$. Since $K \perp \eta_{U*}$, we have $e_{\mathcal{V}} \perp X$. By lemma 1.61, if $e_{\mathcal{V}} \in X$ or if $e_{\mathcal{V}} \perp X$, then $X$ reduces $U$. Note that $(U^2e_{\mathcal{V}}|e_{\mathcal{V}}) \neq 0$. Let $Z = X + \mathcal{J}e_{\mathcal{V}} = X \oplus \mathcal{J}e_{\mathcal{V}}$, where $P$ is the projection onto $X$.

Now let $V$ denote the restriction of $U$ to $Z$. Since $\eta_{U} \subset Z$, we have that $V$ is a partial isometry on $Z$. $\eta_{V} = \mathcal{J}e_{\mathcal{V}}$. $\eta_{V*} = \mathcal{L} \oplus \mathcal{V} \mathcal{L} = \mathcal{L} \oplus \mathcal{U} \mathcal{L} = \mathcal{J}e_{\mathcal{V}}, e_{\mathcal{V}}, e_{\mathcal{V}}$. We claim that $V$ has no non-trivial reducing space. A reducing space for $V$ must contain $e_{\mathcal{V}}$, or else its orthogonal complement must; for $\eta_{V}$ is one-dimensional. So let $Z'$ be a reducing space for $V$ containing $e_{\mathcal{V}}$. Then $Z' \perp$ is invariant under $V$ and hence under $U$. But this contradicts the fact that there is no $U$-chain, after an application of theorem 1.6. We also claim that there is no $V$-chain. This is clear, since a $V$-chain would be a $U$-chain. Therefore, by theorem 1.80 there is no $V*$-chain.

Now we arrive at the last step of the construction. Let $\xi_{\mathcal{V}} = \| e_{\mathcal{V}} - Pe_{\mathcal{V}} \|^{-1}[e_{\mathcal{V}} - Pe_{\mathcal{V}}]$. Let $\{\eta_{i}\}_{i=1}^{\infty}$ be an orthonormal set of vectors orthogonal to $X$ and hence orthogonal to $Z$. We define our operator $L_{Z}$ on $Z \oplus \mathcal{J}[\eta_{i}]_{i=1}^{\infty}$ as follows:

The restrictions of $L_{Z}$, $V$, and $U$ to $Z$ are the same.

$L_{Z}\eta_{1} = \xi_{\mathcal{V}}; L_{Z}\eta_{1} = \eta_{i-1}$ for $i \geq 2$.

So $\eta_{L_{Z}} = \mathcal{J}e_{\mathcal{V}}$; $\eta_{L_{Z}} = \mathcal{J}e_{\mathcal{V}}$. By an argument analogous to the one just employed to demonstrate that $V$ has no non-trivial reducing space, we see that $L_{Z}$ has no non-trivial reducing space.
We have a $L_{11}$-chain, namely $(L_{11}^{i}z_{0})_{1=i=0}^{∞}$. So by theorems 1.80 and 1.6, there is a $L_{4}$-chain. If $ζ$ is a vector in a $L_{4}$-chain, then $(ζ|η_{k}) ≠ 0$ for a co-final set of integers $(i_k)$. That is, if $(ζ|η_{m}) = 0$ for all $m$ greater than some fixed integer $M$, then $(L_{4}^{m}ζ)_{m=M+1}^{∞}$ is a $L_{4}$-chain contained in $L$. But $L_{4}$ = $V$ on $L$, yielding a $V$-chain and a contradiction. Thus we find that no $L_{4}$-chain is orthogonal to the $L_{11}$-chain $(L_{11}^{i}z_{0})_{1=i=0}^{∞}$. By theorem 1.90 $(L_{11}^{i}z_{0})_{1=i=0}^{∞}$ or rather the span of this $L_{11}$-chain contains each $L_{4}$-chain, since $J(L_{11}^{i}η_{L_{11}})_{1=i=0}^{∞} = K = J(U^{i}e_{4})_{1=i=0}^{∞}$. In other words, the span of the $L_{11}$-chain which contains the span of every other $L_{11}$-chain need not be orthogonal to the span of the corresponding $L_{4}$-chain or to the span of any $L_{4}$-chain.

We have only partially answered the question we posed before presenting the above example. It is not clear whether a given $L_{4}$-chain must be orthogonal to at least one $L_{11}$-chain. In example 1.91, the construction of the partial isometry $U$ in the first stage can be used to build partial isometries such that the difference in dimension of the null space of the operator with that of the null space of its adjoint is a given integer. In fact, the operator $U$ on $V$ in 1.91 might be called a $U^*$-chain of defect two, since $e_{4}$ is cyclic for $U^*$ and dim $η_{U^*} = 2$.

Other examples may be constructed by minor modification of those examples given. For a partial isometry $U$, one might
ask what significance the orthogonality of $\mathcal{H}_U$ and $\mathcal{H}_{U*}$ might have. If $\mathcal{H}_U \perp \mathcal{H}_{U*}$, then $U^*U$ commutes with $UU^*$. In example 1.2 we see that $\mathcal{H}_{L_1}$ is not orthogonal to $\mathcal{H}_{L_1^*}$. We now define a partial isometry $S_1$ which is a "finite-dimensional" modification of $L_1$ such that $\mathcal{H}_{S_1} \perp \mathcal{H}_{S_1^*}$. Let $\{e_i\}_{i=-\infty}^{\infty} \cup \{\eta_0, \eta_1\}$ be an orthonormal basis in $\mathcal{V}$. Let $S_1 e_j = e_{j+1}$ for $-\infty < j < 0$ and $1 < j < \infty$. Let $S_1 e_0 = [\sqrt{2}]^{-1}[e_1 + e_2]$; let $S_1 \eta_0 = \eta_1$; let $S_1 \eta_1 = [\sqrt{2}]^{-1}[e_1 - e_2]$. So $\mathcal{H}_{S_1} = \mathcal{J}(e_1)$ and $\mathcal{H}_{S_1^*} = \mathcal{J}(\eta_1)$. Thus we see that at least in simple cases the orthogonality of null spaces seems to be of no consequence.

We conclude this section with three more examples. First, we exhibit $L_5$ having no non-trivial reducing space and $\dim \mathcal{H}_{L_5} = \dim \mathcal{H}_{L_5^*} = \infty$. Secondly, we have $L_6$ having again no non-trivial reducing space and $\dim \mathcal{H}_{L_6} = 1$, $\dim \mathcal{H}_{L_6^*} = \infty$. So the finite dimensionality of the null space of a partial isometry together with the hypothesis of no non-trivial reducing space need not imply the finite dimensionality of the null space of its adjoint operator. Our final example is $L_7$; $L_7$ satisfies the hypothesis of theorem 1.7 with the exception that $\dim \mathcal{H}_{L_7} = \dim \mathcal{H}_{L_7^*} = 2$. However, $L_7$ is not anti-unitarily equivalent to its adjoint.

**Example 1.92** Let $\mathcal{V}$ be a Hilbert space with orthonormal basis $\{e_i\}_{i=1}^{\infty}$. Let $\eta_1 = e_1$; let $[\sqrt{2}]^{-1}[e_{2i-1} - e_{2i+1}] = \eta_{2i}$ for $1 \leq i \leq \infty$; let $[\sqrt{2}]^{-1}[e_{2i} + e_{2i+1}]$ for $1 \leq i < \infty$. Let $L_5$ be the partial isometry with domain space $\mathcal{J}(\eta_{2i+1})_{i=0}^{\infty}$, with range
space \( \mathcal{F}(e_{2i})_{i=1}^{\infty} \), and defined as follows: \( L_5 e_{2i+1} = e_{2i+1} \) for \( 0 \leq i < \infty \). So \( \eta_{L_5} = \mathcal{F}(e_{2i})_{i=1}^{\infty} \); \( \eta_{L_5^*} = \mathcal{F}(e_{2i+1})_{i=0}^{\infty} \). It is clear that \( e_1 \) is cyclic for \( L_5 \). Loosely said, \( L_5^* \) moves vectors toward \( \eta_1 = e_1 \). So a reducing space for \( L_5 \) contains a vector of the form \( \sum_{n=0}^{\infty} a_{2n+1} e_{2n+1} \) with \( a_1 \neq 0 \). If we designate such a vector as \( \xi_0 \), then we have \( \xi_0 - 2 \sum_{i=0}^{5} L_5 L_5^* \xi_0 = a_1 e_1 \). So \( L_5 \) has no non-trivial reducing space, \( \text{dim } \eta_{L_5} = \text{dim } \eta_{L_5^*} = \infty \).

**Example 1.93** Let \( \mathcal{H} \) be a Hilbert space with the orthonormal basis \( \{e_i\}_{i=1}^{\infty} \cup \{(\eta_k)_{k=1}^{\infty}\}_{j=1}^{\infty} \). Let \( \theta = \sum_{i=1}^{\infty} [\sqrt{2}]^{-i} e_1 \). Let \( L_6 = \mathcal{F}(\theta) \) be defined as follows:

\[
\begin{align*}
L_6\theta &= 0 \\
L_6 e_j &= e_j \quad \text{for } 1 \leq j < \infty. \\
L_6 e_k &= \eta_{k+1} \quad \text{for all } (j,k) \text{ such that } j \neq k, 0 < j,k < \infty. \\
\end{align*}
\]

Let \( \xi_i = \| e_i - (e_i | \theta) \theta \|^{-1} [e_i - (e_i | \theta) \theta] \) for \( 1 \leq i < \infty \).

Let \( \{\xi_i\}_{i=1}^{\infty} \) be the Gramm orthonormalization of \( \{\xi_i\}_{i=1}^{\infty} \).

\[
L_6 \xi_i = \eta_{i+1} \quad \text{for } 1 \leq i < \infty.
\]

So \( \eta_{L_6} = \mathcal{F}(\theta); \) \( \eta_{L_6^*} = \mathcal{F}(\eta_1)_{j=1}^{\infty} \). We note that \( (\xi_i | e_j) \neq 0 \) for \( 1 \leq i,j < \infty \). We will sketch a proof that \( L_6 \) has no non-trivial reducing space. It is enough to show that a non-zero reducing space for \( L_6 \) must contain \( \mathcal{F}(\eta_1)_{j=1}^{\infty} = \eta_{L_6^*} \), since \( \eta_{L_6^*} \) is cyclic for \( L_6 \). It can be shown that if \( \xi \in \mathcal{H}, \xi \neq 0 \), then there is a polynomial \( P(x,y) \) such that \( P(L_6,L_6^*) \xi \neq 0 \), \( P(L_6,L_6^*) \xi \neq \mathcal{F}(\theta) \), and \( (P(L_6,L_6^*) \xi | \theta) \neq 0 \). If we let \( \xi_1 = P(L_6,L_6^*) \xi \), then we can write \( \xi_1 - L_6^* L_6 \xi_1 = \sum_{i=m}^{\infty} a_i e_i \) where \( a_m \neq 0 \). But \( L_6^m (\sum_{i=m}^{\infty} a_i e_i) = L_6^m (\sum_{i=m}^{\infty} a_i e_i) = a_m \). Now
\[ L_m = e \cdot L^*_m = c \cdot e \] with \( c \) a non-zero complex number.

\[ \eta^j_m = \sum_{i=1}^{\infty} b_i e_i, \quad b_i \neq 0 \text{ for } 1 \leq i < \infty. \] So by a process similar to the one used in obtaining \( \eta^m_1 \), we can now obtain \( \eta^j_1 \) for \( 1 \leq j < \infty \). Thus \( L_0 \) has no non-trivial reducing space.

Example 1.94 Let \( \mathcal{X} \) be a Hilbert space with orthonormal basis \( \{e_i\}_{i=-\infty}^{\infty} \). Let \( \{\theta_i\}_{i=-\infty}^{-5} \) be the skewed basis with respect to the orthonormal set \( \{e_i\}_{i=-\infty}^{-5} \). Let \( \{\eta_i\}_{i=4}^{\infty} \) be the skewed basis with respect to the orthonormal set \( \{e_i\}_{i=4}^{\infty} \). Let \( L_7 \) be defined as follows:

\[ L_7 e_1 = e_1 \text{ for } -\infty < i < -5; \quad L_7 e_{-5} = e_{-4}; \quad L_7 e_{i} = e_{i+1} \text{ for } -4 \leq i < -1; \quad L_7 e_{0} = [\sqrt{2}]^{-1}[e_1 + e_2]; \quad L_7 e_1 = 0; \quad L_7 e_2 = e_3; \]

\[ L_7 e_3 = \eta_4; \quad L_7 e_4 = 0; \quad L_7 e_1 = \eta_1 \text{ for } 5 \leq i < \infty. \]

We sketch a proof that \( L_7 \) has no non-trivial reducing space. We suppose that \( K \), a non-zero subspace, reduces \( L_7 \). If we choose a non-zero vector in \( K \), then either the vector is not orthogonal to \( \eta^j_7 \) or there is a polynomial \( P(x, y) \) with non-commuting variables such that \( P(L_7, L^*_7) \) applied to the vector is not orthogonal to \( \eta^j_7 \). So \( K \) contains a non-zero vector \( \xi \) of the form \( ae_1 + be_4 \). If \( a = 0 \), then \( (L_7 L^*_7 \xi | e_1) \neq 0 \). So we can assume \( a \neq 0 \). We project \( \xi \) onto \( \eta^j_7 \), obtaining a non-zero multiple of \([\sqrt{2}]^{-1}[e_1 - e_2]\). We project this vector back onto \( \eta^j_7 \) to obtain a non-zero multiple of \( e_1 \). But \( e_1 \) is cyclic for \( L_7 \) and \( L^*_7 \). Hence \( K = \mathcal{X} \).

We suppose \( S \) is anti-unitary such that \( SL_7 S^* = L^*_7 \). As before, we have \( S^* L_7 S = L^*_7 \), \( L_7 = S^* L^*_7 S \), and \( L^*_7 = S^* L^*_7 S \) for
\( n \geq 0 \). Since \( S^* L^*_7 S = L^*_7 \), we have \( S \mathcal{L}_{L^*_7} \subseteq \mathcal{L}_{L^*_7} \). We let \( S_{-5} = \zeta \in \mathcal{L}_{L^*_7} \), \( \| \zeta \| = 1 \). \( \| L^*_7 e_{-5} \| = 1 \) for \( 0 \leq k \leq 6 \). \( \| L^*_7 m \zeta \| < 1 \) for some \( m \), \( 1 \leq m \leq 3 \), by direct verification using the fact that \( \zeta \in \mathcal{L}_{L^*_7} \). But \( L^*_7 e_{-5} = S^* L^*_7 \zeta \), \( 1 \leq m \leq 3 \). So we have a contradiction. Hence \( L^*_7 \) is not anti-unitarily equivalent to \( L^*_7 \).

We now consider a few consequences of an operator's being anti-unitarily equivalent to its own adjoint. First we note that every normal operator is anti-unitarily equivalent to its own adjoint. In fact, if \( N \) is a normal operator on \( \mathcal{K} \), \( \mathcal{A} \) is a maximal commutative symmetric ring of operators containing \( N \), and \( \xi_o \) is a cyclic vector for \( \mathcal{A} \), then we have \( \mathcal{K} \) is isomorphic to \( L^2(M, \mathcal{K}) \) and \( \mathcal{A} \) is isometric-isomorphic to \( C(M) \), where \( M \) is the maximal ideal space of \( \mathcal{A} \). Thus if \( U \) is the conjugate operator which maps the pre-image of \( \xi(m) \) to the pre-image of \( \xi(m) \) for \( \xi(m) \in L^2(M, \mathcal{K}) \), then it is clear that \( U N U^* = N^* \), since \( N(m) = \hat{N}(m) \). In view of the fact that the class of operators anti-unitarily equivalent to their adjoints includes the normal operators, one might expect that operators in this class have some properties in common with normal operators.

**Lemma 1.95** Let \( A \) be a linear operator anti-unitarily equivalent to \( A^* \). Then the spectrum of \( A \) is the approximate point spectrum of \( A \).

**Proof:** Let \( U \) be an anti-unitary operator such that \( U A U^* = A^* \).
The approximate point spectrum of an operator is contained in its spectrum. We suppose \( \lambda \) is a complex number which is not in the approximate point spectrum of \( A \). Then there is \( \alpha > 0 \) such that \( \| (A-\lambda I)\xi \| \geq \alpha \| \xi \| \) for \( \xi \in \mathcal{H} \). We claim that \( \mathcal{H}_{A-\lambda I} \) is dense in \( \mathcal{H} \). If \( (A-\lambda I)\xi = 0 \) for all \( \xi \in \mathcal{H} \), then \( (A^*-\bar{\lambda} I)\zeta = 0 \). Now \( \| (A^*-\bar{\lambda} I)\zeta \| = \| (U_{1}AU^* - U_{1}U^*)\zeta \| = \| (A-\lambda I)U^*\zeta \| = 0 \). So \( U^*\zeta = 0 = \zeta \). Hence if \( \lambda \) is not an approximate eigenvalue, then \( (A-\lambda I)^{-1} \) exists.

**Lemma 1.96** Let \( \sigma(A) \) denote the spectrum of \( A \), where \( A \) is a linear operator. If there is an anti-unitary operator \( V \) such that \( VAV^* = A^* \), then \( \sigma(A) = \overline{\sigma(A^*)} \) and \( A*A \) is unitarily equivalent to \( AA^* \).

**Proof:** \( VAV^* = A^* \); \( V(A-\bar{\lambda} I)V^* = A^*-\lambda I \). So the existence of \( (A-\bar{\lambda} I)^{-1} \) is equivalent to the existence of \( (A^*-\lambda I)^{-1} \). Hence \( \sigma(A) = \overline{\sigma(A^*)} \).

\( VAV^* = A^* \) implies that \( VA^*V^* = A \). \( AA^* = VA^*AV^* \). As mentioned earlier, we can find \( U_{1} \) an anti-unitary operator such that \( U_{1}^* = U_{1} \) and \( U_{1}AA^*U_{1}^* = AA^* \). So \( AA^* = U_{1}AA^*U_{1}^* = U_{1}VA^*AV^*U_{1}^* = (U_{1}V)A^*A(U_{1}V)^* \). Since \( U_{1}V \) is the product of anti-unitary operators and thus unitary, the lemma holds.

We are unable to state necessary and sufficient conditions for an operator to be anti-unitarily equivalent to its adjoint. If \( A \) is an operator which is anti-unitarily equivalent to \( A^* \) by the anti-unitary \( U \), then \( (cA), c \in \mathcal{C} \), is anti-unitarily equivalent to \( (cA)^* \) by the same anti-unitary \( U \). This need not hold for unitary equivalence. We do not know whether the
anti-unitary operator $U$ giving the anti-unitary equivalence of an operator $A$ to its adjoint $A^*$ can always be taken so that $U = U^*$, or, what is the same, so that $U^2 = 1$. Each anti-unitary operator $U$ such that $U = U^*$ is given by conjugation of Fourier coefficients with respect to an orthonormal basis; there exists an orthonormal basis $\{e_i\}_{i=1}^\infty$ such that for $\sum_{i=1}^\infty a_i e_i \in \mathcal{H}$, we have $U(\sum_{i=1}^\infty a_i e_i) = \sum_{i=1}^\infty \overline{a}_i e_i$. Such an anti-unitary operator is called a conjugation; p.357-360, [9].

**Lemma 1.97** Let $A$ be a linear operator on $\mathcal{H}$; then $A$ is anti-unitarily equivalent to $A^*$ by a conjugation if and only if with respect to some basis $\{e_i\}_{i=1}^\infty$ the matrix $(a_{ij})$ of $A$ satisfies $a_{ij} = a_{ji}$ for $1 \leq i, j < \infty$.

**Proof:** We suppose $UAU^* = A^*$, with $U$ anti-unitary and $U = U^*$. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis for $\mathcal{H}$ with respect to which $U$ is conjugation of Fourier coefficients. $UAe_j = U^{\infty}_{i=1} a_{ij} e_i = \sum_{i=1}^\infty \overline{a}_{ij} e_i$. So the matrix of $UAU^*$ is $(\overline{a}_{ij})$. But the matrix of $A^*$ is $(\overline{a}_{ji})$. Thus $a_{ij} = a_{ji}$ for $1 \leq i, j < \infty$.

We suppose now that the matrix $(a_{ij})$ of the operator $A$ with respect to the basis $\{e_i\}_{i=1}^\infty$ is such that $a_{ij} = a_{ji}$ for $1 \leq i, j < \infty$. Let $U$ be the anti-unitary operator given by conjugation of Fourier coefficients with respect to $\{e_i\}_{i=1}^\infty$. By reversing the above computations we see that $UAU^* = A^*$. |

If $H$ is a Hermitian operator, then with respect to some basis $H$ has the matrix $(h_{ij})$ where $h_{ij} = h_{ji}$. Since the matrix of $H^* = H$ is $(\overline{h}_{ij})$, we see that $h_{ij}$ is a real number.
for $1 \leq i, j < \infty$. Thus lemma 1.97 implies that a Hermitian operator has a real matrix with respect to some basis. If $N$ is a normal operator, then $N$ also satisfies the hypothesis of lemma 1.97. So there is a basis with respect to which the matrix $(b_{ij})$ of $N$ satisfies $b_{ij} = b_{ji}$. When we write out what it means for the matrices $(b_{ij})$ and $(\overline{b}_{ij})$ to commute, we have $\sum_k b_{ik} \overline{b}_{jk} = \sum_k \overline{b}_{ki} b_{kj} = \sum_k \overline{b}_{ik} b_{jk}$. So the inner product of each row of the matrix $(b_{ij})$ of $N$ with each other row is a real number. Also, it is clear by reversing the above argument that each such matrix yields a normal operator.
CHAPTER II
NORMAL CONJUGATE OPERATORS

We now obtain a structure theorem on Hermitian conjugate operators. In Stone's pioneering book on transformations on Hilbert space, there is a structure theorem for a conjugation; p. 357-360,[9]. A conjugation \( J \) is an anti-unitary operator such that \( J^2 = 1 \). It is easy to see that \( J^2 = 1 \) is equivalent to \( J = J^* \). A conjugation \( J \) is given by conjugation of Fourier coefficients with respect to a suitable orthonormal basis depending on \( J \); or \( J \) can be represented as conjugation of functions on a suitable \( L^2 \)-space. Stone's book contains interesting material on real linear operators, linear operators which commute with a conjugation operator.

**Theorem 2.1** Let \( H \) be a Hermitian conjugate operator on \( \mathcal{H} \). Then there exists a symmetric commutative ring of operators \( \mathcal{A} \) such that \( K \in \mathcal{A}, K = K^* \) implies that \( KH = HK \); and \( \mathcal{A} \) is maximal with respect to set inclusion in the class of all commutative symmetric subrings of \( \mathcal{B}(\mathcal{H}) \).

**Proof:** Let \( \mathcal{A}' \) be a symmetric commutative ring maximal with respect to set inclusion in the class \( \mathcal{B} \) of symmetric commutative rings such that \( \mathcal{A}' \subseteq \mathcal{B}, K \in \mathcal{A}' \), \( K = K^* \) imply that \( KH = HK \) and \( H^2 \in \mathcal{A} \). The ring generated by \( H^2 \) is an element of \( \mathcal{B} \). Let \( \mathcal{A} \) be a maximal commutative symmetric ring containing \( \mathcal{A}' \). We will show that \( \mathcal{A}' = \mathcal{A} \). We have that \( \mathcal{H} \) is isomorphic
to $L^2(M,\mu)$, where $M$ is the maximal ideal space of $\mathfrak{a}$ and $\mu$ is regular Borel measure corresponding to a fixed vector $\xi_0$ which is cyclic for $\mathfrak{a}$. By construction, the mapping of $\xi$ to $\xi(m)$ from $\mathfrak{a}$ to $L^2(M,\mu)$ has the property that $\xi_0(m) = 1$ for all $m \in M$. Hence forward we identify $\mathfrak{a}$ with $L^2(M,\mu)$. We define $U(\xi(m)) = \xi(m)$. Clearly $U$ is anti-unitary, $U^2 = 1$ or $U = U^*$. Since a Hermitian operator $K$ in $\mathfrak{a}$ has a real-valued Gelfand transform $\hat{K}(m)$ in $C(M)$, we have that $UK = KU$ for $K \in \mathfrak{a}$, $K = K^*$. We claim $UH \in \mathfrak{a}'$. $UH^2 = H^2U$ since $H^2 \in \mathfrak{a}'$ and $H^2$ is Hermitian. $(UH)^* = H^*U^* = HU$. So $UH(UH)^* = UH^2U = H^2U^2 = H^2$; and $(UH)^*(UH) = HU^2H = H^2$. Thus $UH$ is a normal operator. We must show that the Hermitian parts of $UH$ commute with $H$.

$$UH = 2^{-1}[UH+HU] + i[2i]^{-1}[UH-HU]$$
$$2^{-1}[UH+HU]H = 2^{-1}[UH^2+HUH] = 2^{-1}[H^2U+HUH]$$
$$H(2^{-1}[UH+HU]) = 2^{-1}[H^2U+HUH]$$
So $H$ commutes with $2^{-1}[UH+HU]$.

Now, $H[2i]^{-1}[UH-HU] = -(2i)^{-1}[HUH-H^2U] = [2i]^{-1}[H^2U-HUH]$
Thus $2^{-1}[UH+HU]$ and $[2i]^{-1}[UH-HU]$ commute with $H$. Now we show that $UH$ commutes with $\mathfrak{a}'$. Let $A \in \mathfrak{a}'$; let $A = K_1 + iK_2$ where $K_1$ and $K_2$ are Hermitian elements of $\mathfrak{a}'$. Since $\mathfrak{a}' \subset \mathfrak{a}$, we see $UK_i = K_iU$ for $i = 1, 2$. $UHA(K_1+iK_2) = U(HK_1-iHK_2) = U(K_1-iK_2)H = (K_1+iK_2)UH = AUH$. So $UH$ commutes with $\mathfrak{a}'$. Hence, if we ad-join $UH$ and $(UH)^*$ to $\mathfrak{a}'$, then we obtain a ring in the class $\mathbb{B}$ from which $\mathfrak{a}'$ was chosen. Since $\mathfrak{a}'$ is maximal in $\mathbb{B}$, we have $UH \in \mathfrak{a}'$. 


We now show that $\mathcal{A}'$ is weakly closed. If $\{a_n\}$ is a net of operators converging in the weak topology for $\mathcal{B}(\mathcal{V})$ to an operator $A$, then $\{a_n^*\}$ converges weakly to $A^*$. Thus $2^{-1}[a_n + a_n^*]$ converges weakly to $2^{-1}[A + A^*]$, and $\{[2n]^{-1}[a_n - a_n^*]\}$ converges weakly to $[2n]^{-1}[A - A^*]$. Also, if $\{a_n\}$ is a net weakly converging to $A$, then $\{a_n A\}$ converges weakly to $A H$ and $\{H a_n\}$ converges weakly to $H A$. If $K$ is in the weak closure of $\mathcal{A}'$, then $K$ commutes with each operator in $\mathcal{A}'$. The Hermitian parts of $K$ are weakly approximated by Hermitian elements of $\mathcal{A}'$ which commute with $H$. Thus, by the above remarks on weak convergence, the Hermitian parts of $K$ commute with $H$. So $\mathcal{A}'$ is weakly closed.

Now we show that $\xi_0$ is a cyclic vector for $\mathcal{A}'$. Let $\mathcal{A}'[\xi_0] = \mathcal{V}_1$. Let $P$ be the projection onto $\mathcal{V}_1$. Clearly $P$ commutes with $\mathcal{A}'$. Since $\xi_0(m) = 1$ for all $m \in M$, we have $U\xi_0 = \xi_0$.

Sums of the form $\sum_{i=1}^{n} a_i A_i \xi_0$ with $A_i \in \mathcal{A}'$ and $A_i = A_i^*$ are dense in $\mathcal{V}_1$. We recall that $U A_i = A_i U$. Thus $U(\sum_{i=1}^{n} a_i A_i \xi_0) = \sum_{i=1}^{n} a_i U A_i \xi_0 = \sum_{i=1}^{n} a_i A_i \xi_0 = U \sum_{i=1}^{n} a_i A_i \xi_0 = \sum_{i=1}^{n} a_i A_i \xi_0$. So $\mathcal{V}_1$ is invariant under $U$. $U H \in \mathcal{A}'$ implies that $U H \mathcal{V}_1 \subset \mathcal{V}_1$. Now $H = U(UH)$. So $H \mathcal{V}_1 \subset \mathcal{V}_1$ and $HP = PHP$. Taking adjoints, $P^* H^* = P^* H^* P^*$ or $PH = PHP$. So $PH = HP$.

Again by the maximality of $\mathcal{A}'$ in the class $\mathcal{B}$, we have $P \in \mathcal{A}'$. So $P \in \mathcal{A}$. Now $1 \in \mathcal{A}'$. By definition of $P$, $(1-P)\xi_0 = 0$ or $[1 - P(m)]\xi_0(m) = 0$ for almost all $m \in M$. Since $P(m)$ is continuous and $\xi_0(m) = 1$ for all $m \in M$, we have $P(m) = 1$ for all $m \in M$. So $P = 1$ and $\xi_0$ is cyclic for $\mathcal{A}'$. Since $\mathcal{A}'$ is weakly closed, symmetric, and with a cyclic vector, we have that $\mathcal{A}'$ is maximal. So $\mathcal{A} = \mathcal{A}'$. 
Corollary 2.11  Let $H$ be a Hermitian conjugate operator and let $\mathcal{A}$ be the maximal commutative symmetric ring of theorem 2.1. Let $\xi_0$ be cyclic for $\mathcal{A}$, and let $U$ be conjugation of functions in $L^2(M,\mu)$ where $M$ is the maximal ideal space of $\mathcal{A}$ and $\mu$ is the regular Borel measure corresponding to $\xi_0$. Then $UH$ and $HU$ are in $\mathcal{A}$, and $H$ can be represented on $L^2(M,\mu)$ by $H\xi(m) = (HU)(m)\xi(m)$ for $\xi \in \mathcal{K}$.

Proof: From theorem 2.1, we have $UH$ and hence $HU$ are in $\mathcal{A}' = \mathcal{A}$. $H = U(UH)$. So $(H\xi)(m) = (UH\xi)(m) = U(UH)(m) = (UH)m\xi(m)$ $= (UH)^*(m)\xi(m) = (HU)(m)\xi(m)$. |

We note that if $U$ is a unitary or anti-unitary operator, then $U^2 = 1$ is equivalent to $U = U^*$. Now we turn to study conjugate normal operators. The following lemma is analogous to the familiar one concerning operators.

Lemma 2.2  Let $N$ be a conjugate operator. Then $|| N^*\theta || = || N\theta ||$ for all $\theta \in \mathcal{K}$ is equivalent to $N$ being normal.

Proof: Since $N^*N$ and $NN^*$ are linear operators, we have the usual polar decomposition of the inner product for $(N^*N\xi|\eta)$ and for $(NN^*\xi|\eta)$. Hence $N^*N = NN^*$ is equivalent to $(N^*N\theta|\theta) = (NN^*\theta|\theta)$ for all $\theta \in \mathcal{K}$. But this last equation is equivalent to $|| N\theta ||^2 = || N^*\theta ||$. |

We have found no fruitful definition of spectrum for a conjugate operator. However, the next lemma gives a hint to the structure of a normal conjugate operator by noting a property of the spectrum of its square.
Lemma 2.3 If $N$ is a normal conjugate operator, then the spectrum of $N^2$ is closed under conjugation; that is, $\sigma(N^2) = \sigma(N^2)$.

Proof: $N^2$ is a normal operator, since $N^*N = NN^*$ implies $N^2N^* = N^{*2}N^2$. Let $c \in \sigma(N^2)$. We know the spectrum of the normal operator $N^2$ is its approximate point spectrum. Let $\alpha > 0$. There exists non-zero $\xi \in \mathcal{H}$ such that $\|N^2\xi - c\xi\| < \alpha \|\xi\|$. Then $\|N^2(N\xi) - \overline{c}(N\xi)\| < \|N\|\alpha\|\xi\|$. So if $N\xi \neq 0$, then $\overline{c}$ is in the approximate point spectrum of $N^2$. But if $N\xi = 0$, then $N^2\xi = 0$. In case $c \neq 0$, by choice of $\alpha < |c|$ we have $N^2\xi \neq 0$ and hence $N\xi \neq 0$. If $c = 0$, then $\overline{c} \in \sigma(N^2)$.

Before continuing a development of the structure of normal conjugate operators, we mention an easily obtained but gross result relating normal operators and normal conjugate operators. Let $N$ be a normal (conjugate normal) operator. Then there is a conjugation $U$ such that $UN$ is conjugate normal (normal) and $U$ commutes with $N^*N$. This is obtained by taking a conjugation $U$ which commutes with $N^*N$.

In view of the fact that some of the computations to follow are rather long, we give two examples to which the reader can refer the machinery of our structure theorem, 2.8.

Example 2.51 Let $\mathcal{H} = \mathcal{F}\{e_i\}_{i=1}^{\infty}$ where $e_i \perp e_j$ for $i \neq j$ and $\|e_i\| = 1$ for $1 \leq i < \infty$. We define $U$ as follows:

$Ue_{2i} = -e_{2i-1}$ for $1 \leq i < \infty$

$Ue_{2i-1} = e_{2i}$ for $1 \leq i < \infty$
We extend $U$ by conjugate linearity; 
\[ U(\sum_{i=1}^{\infty} a_i e_i) = -\sum_{i=1}^{\infty} \bar{a}_2 i e_{2i-1} + \sum_{i=1}^{\infty} \bar{a}_{2i-1} e_{2i}. \]
So $U^2 = -1$ or $U = -U^*$. 

**Example 2.52** Let $H$ be a Hilbert space with orthonormal basis 
\[ \{f_n\}_{n=-\infty}^{\infty}. \]
Let $S(f_n) = f_{n+1}$ for $-\infty < n < \infty$, and extend by conjugate linearity so that 
\[ S(\sum_{n=-\infty}^{\infty} b_n f_n) = \sum_{n=-\infty}^{\infty} b_n f_{n+1}. \]
We note that if we set $R f_n = f_{n+1}$ and extend by linearity, we obtain the familiar unitary operator, the bilateral shift, which can be represented by multiplication by $e^{i\theta}$ on $L^2([0,2\pi])$ since 
\[ e^{i\theta} e^{i\theta} = e^{i(n+1)\theta} \text{ and } \mathcal{F}[e^{i\theta}]_{n=-\infty}^{\infty} = L^2([0,2\pi]). \]
The results of theorem 2.8 are easily accessible in the case of $S$. To represent $S$, we let $f_n = e^{i\theta}$ for $-\infty < n < \infty$ and $\theta \in [0,2\pi]$; $U$ denotes the conjugation of functions in $L^2([0,2\pi])$; finally, we let $T$ be the mapping of $[0,2\pi]$ onto itself given by $T(\theta) = -\theta \mod 2\pi$. In the usual fashion we identify $0$ with $2\pi$ and $[0,2\pi]$ with the unit circle in the complex plane. Then we see 
\[ S(\sum_{n=-\infty}^{\infty} b_n e^{i\theta}) = \sum_{n=-\infty}^{\infty} b_{n-1} e^{i\theta} = e^{i\theta} \sum_{n=-\infty}^{\infty} b_n e^{i\theta} = R(\sum_{n=-\infty}^{\infty} b_n e^{i\theta}) = RU(\sum_{n=-\infty}^{\infty} b_n e^{i\theta}) = RU(\sum_{n=-\infty}^{\infty} b_n e^{i\theta}) = \mathcal{F}[e^{i\theta}]_{n=-\infty}^{\infty} = L^2([0,2\pi]). \]
So if $f(\theta) \in L^2([0,2\pi])$, we can write $(Sf)(\theta) = e^{i\theta} \mathcal{F}[T(\theta)]$. $S$ is given by a rotation $T$, a conjugation $U$, and a multiplication $R$ on $L^2([0,2\pi])$. We note that $T^2$ is the identity map on $[0,2\pi]$. Our structure theorem, 2.8, is simply a generalization from this example.

**Lemma 2.6** Let $N$ be a normal conjugate operator on $H$; let $L = \{f|Nf = 0\}$. On $K = \mathcal{K} \oplus L$, we have that $N = U\sqrt{(N^*N)}$, where $U$ is an anti-unitary operator which commutes with $N$ and with $N^*$. Also, $\mathcal{K}$ reduces $N$. 


Proof: By lemma 2.2, \| N \xi \| = \| N^* \xi \| \text{ for all } \xi \in \mathcal{X}. \text{ So } \mathcal{L} \text{ is the null-space of } N^*. \text{ If } N^*N \xi = 0 \text{ for some } \xi \in \mathcal{X}, \text{ then } 0 = (N^*N \xi, \xi) = (N \xi, N \xi); \text{ so } N \xi = 0. \text{ Clearly } N \xi = 0 \text{ implies that } N^*N \xi = 0. \text{ So } \mathcal{L} \text{ is the null-space of } N^*N \text{ and, by a similar argument, of } NN^*. \text{ Since } \sqrt{(N^*N)} \text{ can be approximated in the norm by polynomials in } N^*N \text{ with real coefficients, and since } N \text{ and } N^* \text{ both commute with } N^*N, \text{ we have that } \sqrt{(N^*N)} \text{ commutes with } N \text{ and with } N^*. \text{ From this point on, we consider only the subspace } \mathcal{K}, \text{ as is clearly sufficient.}

If } \eta \in \mathcal{X} \text{ such that } (N^*N \xi, \eta) = 0 \text{ for all } \xi \in \mathcal{X}, \text{ then } (\xi, N^*N \eta) = 0 \text{ for all } \xi \in \mathcal{X}. \text{ So } N^*N \eta = 0. \text{ So } \eta = 0 \text{ since } \eta \in \mathcal{L}. \text{ Similarly, the ranges of } N \text{ and of } N^* \text{ are dense in } \mathcal{X}. \text{ We define } U(\sqrt{(N^*N)} \xi) = N \xi \text{ for all } \xi \in \mathcal{X}. \text{ Since the range of } N^*N \text{ is dense in } \mathcal{X}, \text{ so is the range of } \sqrt{(N^*N)}. \text{ } \| \sqrt{(N^*N)} \xi \|^2 = (\sqrt{(N^*N)} \xi, \sqrt{(N^*N)} \xi) = (N^*N \xi, \xi) = (N \xi, N \xi) = \| N \xi \|^2. \text{ So } U \text{ is isometric and well-defined. For } c \text{ in the complex numbers, we have } U(c \sqrt{(N^*N)} \xi) = U(\sqrt{(N^*N)} c \xi) = N(c \xi) = \overline{c} N \xi. \text{ So } U \text{ is conjugate homogenous. Since the range of } N \text{ is dense in } \mathcal{X}, \text{ we can extend } U \text{ to an anti-unitary operator by conjugate linearity and continuity.}

We now need to verify that } U \text{ commutes with } N \text{ and with } N^*. \text{ We have that } N \text{ and } N^* \text{ commute with } \sqrt{(N^*N)}. \text{ } (UN)\sqrt{(N^*N)} \xi = U(N^*N)(N \xi) = N(U \sqrt{(N^*N)} \xi) = (UN) \sqrt{(N^*N)} \xi. \text{ So } UN = NU \text{ on a dense linear manifold of } \mathcal{X} \text{ and hence on all of } \mathcal{X}. \text{ Similarly, } UN^* = N^*U. \text{ }

The above construction is analogous to the usual polar decomposition for operators; p. 284, [3].
Lemma 2.7 Let \( U \) be an anti-unitary operator on \( \mathcal{U} \). Let \( \mathcal{A} \) be a weakly closed commutative symmetric ring with identity satisfying \( \mathcal{U} \mathcal{A} \mathcal{U}^* = \mathcal{A} \) and \( U^2 \in \mathcal{A} \). Let \( \mathcal{P} \) denote the set of projection operators of \( \mathcal{A} \). Then \( \mathcal{P} = \mathcal{P}_0 \oplus \mathcal{P}_1 \oplus \mathcal{P}_2 \), where

1) \( P \in \mathcal{P}_0 \) implies that \( UPU^* = P \)
2) \( \mathcal{P}_2 = U\mathcal{P}_1 U^* \) or \( \mathcal{P}_2 = \{ UPU^* | P_1 \in \mathcal{P}_1 \} \)
3) \( \mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2 \) all have maximum elements in the order on projections such that \( T_1 \leq T_2 \) is equivalent to \( T_1 T_2 = T_2 T_1 = T_1 \). Moreover, each \( \mathcal{P}_1 \) is the set of all projections in \( \mathcal{A} \) less than or equal to the maximum projection in \( \mathcal{P}_1 \), for \( 0 \leq i \leq 2 \).

Note: This lemma is so algebraically simple that it can probably be found proven more generally elsewhere.

Proof: Let \( \pi(P) = UPU^* \). Since \( \mathcal{U} \mathcal{A} \mathcal{U}^* = \mathcal{A} \), we have \( \pi(\mathcal{P}) \subseteq \mathcal{P} \).

\[
\pi^2(P) = \pi(UPU^*) = U^2 PU^* U^* U^2 = PU^2 U^* U^2 = P \text{ since } U^2 \in \mathcal{A}.
\]

We consider subsets \( \mathcal{P} \) of \( \mathcal{P} \) satisfying \( P_1, P_2 \in \mathcal{P} \) implies \( \pi(P_1) \perp \pi(P_2) \). The zero projection forms such a set. The union of an increasing tower of such sets is again such a set. By the Hausdorff Maximal Property (H.M.P.), there is a maximal set \( \mathcal{P}_1 \) in the class of all \( \mathcal{P} \). The supremum of an increasing tower of projections in \( \mathcal{P}_1 \) is again a projection in \( \mathcal{P}_1 \), since \( \mathcal{A} \) is weakly closed. Let \( Q_1 \) be a maximal projection in \( \mathcal{P}_1 \) again using H.M.P.. We claim that \( Q_1 \) is the maximum or largest element in \( \mathcal{P}_1 \). Let \( P \in \mathcal{P}_1 \). \( P-Q_1 P \) is in \( \mathcal{P}_1 \) since \( (P-Q_1 P) \leq P \) and \( P \in \mathcal{P}_1 \), \( P' \in \mathcal{P}_1 \) imply \( P' \leq P \). But \( Q_1 \oplus (P-Q_1 P) \geq Q_1 \), and \( Q_1 \oplus (P-Q_1 P) \) is in \( \mathcal{P}_1 \). So
\( P - Q_1 P = 0 \) or \( P \leq Q_1 \). Thus \( Q_1 \) is the maximum or largest element of \( \Theta_1 \). \( \pi \) preserves products, direct sums, and hence ordering on \( \Theta \). \( P \in \Theta, P \leq Q_1 \) imply that \( \pi(P) \leq \pi(Q_1) \); \( R \in \Theta, R \perp \pi(Q_1) \) imply that \( R \perp \pi(P) \). So \( P \in \Theta, P \leq Q_1 \) imply that \( P \in \Theta_1 \).

\( \pi(Q_1) \) is the maximum element of \( \pi(\Theta_1) \). We set \( \pi(\Theta_1) = \Theta_2 \); we set \( \pi(Q_1) = Q_2 \). If \( P \in \Theta, P \leq Q_2 = \pi(Q_1) \), then \( \pi(P) \leq \pi^2(Q_1) = Q_1 \). So \( \pi(P) \in \Theta_1 \) and \( \pi^2(P) = P \in \pi(\Theta_1) = \Theta_2 \).

Now we let \( 1 - (Q_1 \Theta Q_2) = Q_0 \). We suppose that \( P \in \Theta, P \leq Q_0 \). \( P \perp Q_1, P \perp Q_2 \) imply that \( \pi(P) \perp Q_1, \pi(P) \perp Q_2 \) since \( \pi(Q_1) = Q_2 \) and \( \pi(Q_2) = \pi^2(Q_1) = Q_1 \). In order to show that \( \pi(P) = P \) for \( P \leq Q_0, P \in \Theta \), we consider \( \pi(P) - \pi(P)P \).

\[
\pi[\pi(P) - \pi(P)P] = P - \pi(P)P. \quad \text{But} \quad [P - \pi(P)P][\pi(P) - \pi(P)P] = P - \pi(P)P + [\pi(P)P]^2 = 0
\]

So \( \pi[\pi(P) - \pi(P)P] \perp [\pi(P) - \pi(P)P] \). Also, \( \pi(P) - \pi(P)P \) and \( P - \pi(P)P \) are orthogonal to \( Q_1 \) and \( Q_2 \) since both \( P \) and \( \pi(P) \) are. By the maximality of \( \Theta_1 \) and of \( Q_1 \), we have that \( \pi(P) - \pi(P)P = 0 = P - \pi(P)P \). So \( P = \pi(P) \). In short, \( P \in \Theta, P \leq Q_0 \) imply that \( \pi(P) = P \). Thus we let \( \Theta_0 = \{P \in \Theta \mid P \leq Q_0\} \).

If \( P \in \Theta \), then \( P = Q_0 P \oplus Q_1 P \oplus Q_2 P; Q_0 P \in \Theta_0, Q_1 P \in \Theta_1, Q_2 P \in \Theta_2 \).

We are now in a position to prove a representation theorem for normal conjugate operators. As is often the case, the examples, such as 2.51 and 2.52, are analyzed first and the techniques of proof in the general case stumbled upon later.
Theorem 2.8 Let $N$ be a normal conjugate operator on $\mathcal{H}$. Then there is a maximal commutative symmetric ring $\mathcal{A}$ such that on $M$, the maximal ideal space of $\mathcal{A}$, we have $(N\xi)(m) = g(m)\xi(T(m))$ for $\xi(m) \in L^2(M,\mu)$ isomorphic to $\mathcal{H}$, where $T$ is a measure-preserving transformation of $M$ with respect to the measure $\mu$ determined by some cyclic vector $\xi_0$. Also, we have $|g(m)| = |g(T(m))|$ almost everywhere with respect to $\mu$. Moreover, the transformation $T$ is such that $T^2$ is the identity map on $M$; $g(m)$ is a bounded measurable function. Every such mapping of $\mathcal{H}$ into itself is a normal conjugate operator.

Proof: We assume $\|N\| < 1$. Let $\mathcal{L} = \mathcal{H}_N$. By lemma 2.6, we have $\mathcal{L} = \mathcal{H}_N = \mathcal{H}_{N^*} = \mathcal{H}_{N^*N} = \mathcal{H}_{NN^*}$. Let $\mathcal{K} = \mathcal{H} \otimes \mathcal{L}$. All further remarks pertain to the restriction of $N$ to $\mathcal{K}$. Let $\mathcal{N} = U\sqrt{N^*N}$, as in lemma 2.6. $U$ commutes with $N$ and $N^*$. Using the spectral representation theorem for the normal operator $\mathcal{N}^2$, we write $\mathcal{N}^2 = \int_\mathcal{C} \lambda \, d\mathcal{P}_\lambda$, where $\mathcal{C}$ is the complex plane. Let $\mathcal{S}$ be the subset of the complex plane excluding the real axis; $\mathcal{S} = \{ x + iy | y \neq 0 \}$. Let $\mathcal{K}_1 = (\int_\mathcal{S} \lambda \, d\mathcal{P}_\lambda) \mathcal{K}$. Let $\mathcal{K}_3 = (\int_{\mathcal{R}} \lambda \, d\mathcal{P}_\lambda) \mathcal{K}$, where $\mathcal{R}$ is the real line. $\mathcal{K}_1$ and $\mathcal{K}_3$ reduce $\mathcal{N}^2$, and $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_3$ by the spectral theorem. We claim that $\mathcal{K}_1$ and $\mathcal{K}_3$ each reduce the conjugate operator $N$. $\mathcal{N}^2 = \int_\mathcal{C} \lambda \, d\mathcal{P}_\lambda = \mathcal{U} \mathcal{N}^2 \mathcal{U}^* = \int_\mathcal{C} \lambda \, d\mathcal{P}_\lambda$. By the uniqueness of spectral measure, cf. p. 71, [1], we have that $P(\Delta) = U P(\Delta) U^*$ for $\Delta$ a Borel subset of $\mathcal{C}$. If $\Delta'$ is a subset of $\mathcal{R}$, then we have that $U P(\Delta') U^* = P(\Delta')$. Let $Q_1 = \int_{\text{Im } \lambda < 0} \lambda \, d\mathcal{P}_\lambda = P(\Delta_1)$, where $\Delta_1 = \{ \lambda \in \mathcal{C} | \text{Im } \lambda < 0 \}$. Let $Q_2 = \int_{\text{Re } \lambda < 0} \lambda \, d\mathcal{P}_\lambda = \int_{\text{Im } \lambda > 0} \lambda \, d\mathcal{P}_\lambda$. The proof continues with these definitions.
\[ \int_{\lambda > 0} \text{Im} \, \lambda > 0 \, \text{Id} \, \lambda = P(\Delta_2), \text{ where } \Delta_2 = \{ \lambda \in \mathbb{C} | \text{Im} \, \lambda > 0 \}. \text{ Thus } UQ_1U^* = Q_2. \text{ Also, } \kappa_1 = Q_1(\kappa_1) \oplus Q_2(\kappa_1). \text{ Our plan is to deal with } \kappa_1 \text{ and } \kappa_3 \text{ separately but analogously.} \]

First, we consider \( \kappa_1. \) \( \kappa_1 = Q_1(\kappa_1) \oplus Q_2(\kappa_1), \) where each summand is a reducing space for \( N^2. \) Since \( U \) commutes with \( N \) and \( N^* \), it follows that \( U^2 \) commutes with \( N^2 \) and \( N^*^2 = (N^2)^*; \)

\( Q_1(\kappa_1) \) and \( Q_2(\kappa_1) \) both reduce \( U^2 \) since they are both spectral subspaces of \( N^2. \) Let \( \mathcal{B}_1 \) be the weakly closed symmetric commutative ring generated by \( N^2 \) and \( U^2. \sqrt{(N^*N)} = \sqrt{(N^*^2N^2)} \in \mathcal{B}_1. \)

Since linear combinations of products of \( U^2, U^*^2, N^2, \) and \( N^*^2 \) are weakly dense in \( \mathcal{B}_1, \) we have \( U\mathcal{B}_1U^* = \mathcal{B}_1. \) Now we will show that we can extend \( \mathcal{B}_1 \) to a maximal commutative symmetric overring \( \mathcal{A}_1 \) on \( \kappa_1 \) satisfying \( U\mathcal{A}_1U^* = \mathcal{A}_1. \)

Since \( Q_1 \) commutes with both \( U^2 \) and \( N^2, \) we have that \( \mathcal{B}_1[Q_1(\kappa_1)] \subset Q_1(\kappa_1). \) We pick \( \zeta_1 \) of norm one in \( Q_1(\kappa_1). \) If \( \overline{\mathcal{B}_1\zeta_1} \neq Q_1(\kappa_1), \) then we pick \( \zeta_2 \) of norm one in the subspace \( \{Q_1(\kappa_1)\} \oplus \overline{\mathcal{B}_1\zeta_1}. \) Using the symmetry of the ring \( \mathcal{B}_1, \) it is easy to see that \( \overline{\mathcal{B}_1\zeta_1} \perp \overline{\mathcal{B}_1\zeta_2}. \) Proceeding in this fashion and employing the Hausdorff Maximality Principle, we can write \( Q_1(\kappa_1) = \sum_{i=1}^{\infty} \oplus \overline{\mathcal{B}_1\zeta_i} \) where \( \zeta_i \in Q_1(\kappa_1) \) for \( 1 \leq i < \infty. \)

Since \( UQ_1U^* = Q_2, \) we have \( UQ_1U^*(\kappa_1) = Q_2(\kappa_1). \) Since \( U^*(\kappa_1) = \kappa_1, \) it is true that \( UQ_1(\kappa_1) = Q_2(\kappa_1). \) Let \( R_i \) be the projection operator with range \( \overline{\mathcal{B}_1\zeta_i} \) for \( 1 \leq i < \infty. \) Then \( R_i \) commutes with \( \mathcal{B}_1 \) since \( R_i(\kappa_1) \) reduces \( \mathcal{B}_1, \) for \( 1 \leq i < \infty. \) We claim that \( UR_iU^* \) is the projection onto \( \overline{\mathcal{B}_1\zeta_i} \) for \( 1 \leq i < \infty. \) Let \( \mathcal{B} \in \mathcal{B}_1. \) Let \( \mathcal{B} = UB'U^* \), where \( B' \in \mathcal{B}_1. \) Then we have:
\[ UR_i U^* B U^* = UR_i U^* B U^* U^* j = UR_i B U^* j = \begin{cases} 0 \text{ if } i \neq j \\ UR_i B U^* j \text{ if } i = j \end{cases} \]

But \( UR_i B U^* j = UB j = U^2 B j = BU j \). So \( UR_i U^* \) is the projection onto \( \{ \theta_1 U^* \} \) for \( 1 \leq i < \infty \), and \( Q_2(\chi_1) = \sum_{i=1}^{\infty} \Omega(\theta_1 U^* \chi_1) \).

We adjoin \( \{ R_i \}_{i=1}^{\infty} \) and \( \{ UR_i U^* \}_{i=1}^{\infty} \) to the restriction of the ring \( B_1 \) to \( \chi_1 \). The weak closure of this new ring is called \( \sigma_1 \). Clearly \( U \sigma_1 U^* = \sigma_1 \). \( \sigma_1 \) has \( \left( \sum_{i=1}^{\infty} 2^{-i} \nu \right) \) as a cyclic vector. So \( \sigma_1 \) is a maximal commutative symmetric ring on \( \chi_1 \). The restriction of \( \sqrt{N^*N} \) to \( \chi_1 \) is in \( \sigma_1 \); the restriction of \( U^2 \) to \( \chi_1 \) is in \( \sigma_1 \) by definition of \( B_1 \).

Now we consider \( \chi_3 = (\int_R^1 \text{id} P_1) \chi \). \( \chi_3 \) reduces \( U \) and \( N^2 \) since \( \chi_1 \) does. Hence \( \chi_3 \) reduces \( N = U \sqrt{N^*N} \). The restriction of \( N^2 \) to \( \chi_3 \) is Hermitian, since it has real spectrum. The following remarks apply to \( \chi_3 \) and restrictions of operators or of conjugate operators to \( \chi_3 \).

\( N = U \sqrt{N^*N} \). \( N^2 = U^2 N^* N. N^* = \sqrt{N^*N} U^* = U^* \sqrt{N^*N} \).

\( N^* = U^* 2 N^* N. N^* \chi_3 \) is dense in \( \chi_3 \), since \( N^* N \chi \) is dense in \( \chi \) and \( \chi_3 \) reduces \( N^* N \). Thus \( N^2 = N^* \chi = U^2 \chi N^* \chi = U^* \chi^2 \).

In implies that \( U^2 = U^* \). So \( U^2 \) is Hermitian on \( \chi_3 \). Let \( U^2 = P_1 - P_2 \), where \( P_1 \oplus P_2 \) is the projection onto \( \chi_3 \). \( U^2 - U^2 = P_1 - P_2 = P_1 \). So \( P_1 \) and \( P_2 \) commute with \( N \) and with \( U \).

\( P_1(\chi_3) \) and \( P_2(\chi_3) \) reduce \( N \) and \( U. U^2 = 1 \) on \( P_1(\chi_3) \) and hence \( U = U^* \) on \( P_1(\chi_3) \). Thus \( N = N^* \) on \( P_1(\chi_3) \). \( U^2 = -1 \) or \( U = -U^* \) on \( P_2(\chi_3) \). So \( N = -N^* \) on \( P_2(\chi_3) \).
On $P_1(\mathcal{K}_3)$, we have $N = N^*$. By theorem 2.1, there is a maximal commutative symmetric ring $\mathcal{A}_2$ containing $N^2$ on $P_1(\mathcal{K}_3)$. For $H \in \mathcal{A}_2$, $H$ Hermitian, we have $HU = UH$ or $NH =HN$ on $P_1(\mathcal{K}_3)$.

On $P_2(\mathcal{K}_3)$ the situation is slightly more difficult. We note that $N = -N^*$ on $P_2(\mathcal{K}_3)$. For $\xi \in P_2(\mathcal{K}_3)$, $(N^2|\xi) = (N^*\xi|\xi) = -(N\xi|\xi)$. So $(N\xi|\xi) = 0$. $(N^2N|\xi) = (N^NN\xi|\xi) = (N^*N\xi|N\xi) = [-1]^n(N^NN\xi|N\xi) = 0$. So if $P(x)$ is a polynomial, we have $(P(N^2)|\xi|N\xi) = 0$. We recall that $\mathcal{B}_1$ is the weakly closed symmetric commutative ring generated by $U^2$ and $N^2$. We have $[\mathcal{B}_1\xi] \perp N\xi$ for $\xi \in P_2(\mathcal{K}_3)$. Now $\mathcal{N}(N\xi) \in \mathcal{B}_1$, and $N = \mathcal{N}(N\xi) U$. So $[\mathcal{B}_1\xi] \perp U\xi$. Now we are in a position to construct an overring of the ring $\mathcal{B}_1$ restricted to $P_2(\mathcal{K}_3)$ as we did with $\mathcal{B}_1$ restricted to $\mathcal{K}_1$.

We fix $\xi_1$ of norm one in $P_2(\mathcal{K}_3)$. Let $S_1$ be the projection onto $[\mathcal{B}_1\xi_1]$. Since $\mathcal{B}_1$ is symmetric and $U\xi_1 \perp [\mathcal{B}_1\xi_1]$, we have $[\mathcal{B}_1U\xi_1] \perp [\mathcal{B}_1\xi_1]$. We claim $US_1U^*$ is the projection onto $[\mathcal{B}_1U\xi_1]$. $[\mathcal{B}_1U\xi_1] = [U\mathcal{B}_1\xi_1] = U[\mathcal{B}_1\xi_1]$. If $\xi \perp [\mathcal{B}_1U\xi_1]$, then $\xi \perp U[\mathcal{B}_1\xi_1]$ or $U\xi \perp [\mathcal{B}_1\xi_1]$. So $S_1U\xi = 0 = US_1U\xi$. If, on the other hand, $\xi \in [\mathcal{B}_1U\xi_1] = U[\mathcal{B}_1\xi_1]$, then $U\xi \in [\mathcal{B}_1\xi_1]$. $S_1U\xi = U\xi$. So $US_1U\xi = \xi$.

Now we choose $\xi_2$ of norm one in $[(\mathcal{B}_1\xi_1) \circ (\mathcal{B}_1U\xi_1)]$, where we take the orthogonal complement in $P_2(\mathcal{K}_3)$. Since $(\mathcal{B}_1\xi_1) \circ (\mathcal{B}_1U\xi_1)$ reduces the ring $\mathcal{B}_1$, $N$, and $U$, we can repeat the above argument to obtain $[\mathcal{B}_1\xi_2] \perp [\mathcal{B}_1U\xi_2]$ and $S_2,US_2U^*$ as the projections onto these subspaces. By a maximality argument, we can write $P_2(\mathcal{K}_3) = \bigoplus_{i=1}^\infty [\mathcal{B}_1\xi_1] \circ \bigoplus_{i=1}^\infty [\mathcal{B}_1U\xi_1]$ where the norm
of $\xi_1$ is one for $1 \leq i < \infty$. We let $S_1$ be the projection onto $[\xi_1 S_1]$ for $1 \leq i < \infty$. Then $US_1U^*$ is the projection onto $[\xi_1 S_1]$ for $1 \leq i < \infty$. We adjoin $\{S_1\}_{i=1}^{\infty}$ and $\{US_1U^*\}_{i=1}^{\infty}$ to the restriction of the ring $\mathcal{B}_1$ to $P_2(\mathcal{K}_3)$. We denote the weak closure of this symmetric commutative overring by $\mathcal{A}_3$. Then $\sum_{i=1}^{\infty} 2^{-i} S_1 + \sum_{i=1}^{\infty} 2^{-i} U S_1$ is a cyclic vector for $\mathcal{A}_3$. So $\mathcal{A}_3$ is a maximal commutative symmetric ring on $P_2(\mathcal{K}_3)$. $U \mathcal{A}_3 U^* = \mathcal{A}_3$ by construction.

We have $\mathcal{A}_1$ on $\mathcal{K}_1$, $\mathcal{A}_2$ on $P_1(\mathcal{K}_3)$, and $\mathcal{A}_3$ on $P_2(\mathcal{K}_3)$, each ring maximal commutative symmetric on its subspace. Let $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$. $\mathcal{A}$ is a maximal commutative symmetric ring on $\mathcal{K}$. Since $\mathcal{K}_1$, $P_1(\mathcal{K}_3)$, and $P_2(\mathcal{K}_3)$ each reduce $U$, and $U \mathcal{A}_1 U^* = \mathcal{A}_1$ for $1 \leq i \leq 3$, we have $U \mathcal{A} U^* = \mathcal{A}$. Also, $U^2$ and $\mathcal{A}(N \times N)$ are both in $\mathcal{A}$.

We are now ready to apply lemma 2.7 to $\mathcal{A}$ and $\Theta$, the set of projection operators of $\mathcal{A}$. We can write $\Theta = \Theta_0 \oplus \Theta_1 \oplus \Theta_2$, where $P \in \Theta_0$ implies $U P U^* = P$, $P \in \Theta_1$ implies $U P U^* \in \Theta_2$, and $P \in \Theta_2$ implies $U P U^* \in \Theta_1$. Let $S_0$, $S_1$, and $S_2$ be the maximum projections of $\Theta_0$, $\Theta_1$, and $\Theta_2$ respectively. We choose $\Theta_0$ of norm $[\sqrt{3}]^{-1}$ and cyclic for $S_0 \mathcal{A} S_0$ on $S_0 \mathcal{K}$. We choose $\Theta_1$ of norm $[\sqrt{3}]^{-1}$ and cyclic for $S_1 \mathcal{A} S_1$ on $S_1 \mathcal{K}$. Now $\| U \Theta_1 \| = [\sqrt{3}]^{-1}$.

Since $US_1 U^* = S_2$, we have $US_1 \mathcal{K} = S_2 \mathcal{K}$ and $US_1 = S_2 U$. We find:

$$S_2 \mathcal{A} S_2 U \Theta_1 = S_2 \mathcal{A} S_1 \Theta_1 = S_2 U \Theta_1 = S_2 (U \mathcal{A} U^*) U \Theta_1 = S_2 U \Theta_1 = S_2 U S_1 \mathcal{K} = S_2 \mathcal{K}.$$  

So $U \Theta_1$ is cyclic for $S_2 \mathcal{A} S_2$ on $S_2 \mathcal{K}$. For $P \in \Theta_1$, it holds that $(P \Theta_1 | \Theta_1) = (U P \Theta_1 | U \Theta_1) = (U P U^* (U \Theta_1) | U \Theta_1)$. Let $\xi_0 = \Theta_0 \oplus \Theta_1 \oplus \Theta_2$. Let $M$ be the maximal ideal space of $\mathcal{A}$, and let $\mu$ be the regular
Borel measure corresponding to the cyclic vector $\xi_0$. We let $T$ denote the map of clopen sets of $M$ which corresponds to the mapping of $P$ to $UPU^*$ for $P \in \Theta$; if $\hat{P}(m) = \chi_{\Delta}(m)$ and $(UPU^*)(m) = \chi_{\Delta'}(m)$ where $\Delta$ and $\Delta'$ are clopen sets in $M$, we set $T(\Delta) = \Delta'$. By construction of the cyclic vector $\xi_0$, namely, since $(P\Theta_1|\Theta_1) = (UPU^*(U\Theta_1)|U\Theta_1)$ for $P \in \Theta$, we have that $\mu(\Delta) = \mu(T[\Delta])$ for $\Delta \subseteq M$, $\Delta$ clopen. Since the clopen sets of $M$ form a complete set of representatives for the measure algebra of $(M,\mu)$, we see that $T$ induces an automorphism of the measure algebra. Moreover, we can define $T$ in a pointwise fashion in the following way. If $m \in M$, then $UmU^* \in M$. If $m$ is a maximal ideal and $P \in \Theta$, then $P \in m$ if and only if $UPU^* \in UmU^*$. Or $\hat{P}(m) = 0$ if and only if $(UPU^*)(UmU^*) = 0$. So if we define $T(m) = UmU^*$, then the pointwise map agrees with the set map $T$; and hence the same notation is justified.

We have $\xi_0(m) = 1$ for all $m \in M$. Let $f(m) = (U\xi_0)(m)$. Now $U = (UPU^*)U$, so $U(\chi_{\Delta}) = \chi_{T(\Delta)}U$ for $\Delta$ clopen, or $U(\chi_{\Delta})(m)\xi_0(m) = (\chi_{T(\Delta)})(m)(U\xi_0)(m)$. Since $U$ is anti-unitary, we have that $\|U(\chi_{\Delta})(m)\|^2 = \mu(\Delta) = \int_{T(\Delta)}|f(m)|^2\mu(m) = \mu(T[\Delta])$. So $|f(m)| = 1$ almost everywhere. If $\sum_{i=1}^na_i\chi_{\Delta_i} \in L^2(M,\mu)$, we have:

$$U(\sum_{i=1}^na_i\chi_{\Delta_i}) = \sum_{i=1}^n\overline{a_i}U(\chi_{\Delta_i}) = \sum_{i=1}^n\overline{a_i}f|_{T(\Delta_i)}(m) = f(m)\sum_{i=1}^n\overline{a_i}\chi_{T(\Delta_i)}.$$  

By continuity and linearity, we have $(U\xi)(m) = f(m)\xi(T[m])$ for $\xi(m) \in L^2(M,\mu)$. Now $N = U\sqrt{(N^*N)} = \sqrt{(N^*N)}U$. So $(N\xi)(m) =$
Let \( g(m) = (\sqrt{N^*N})(m) f(m) \). Then

\[
(N\xi)(m) = \int_M g(m) \xi(T[m]) \eta(m) d\mu(m) = \int_M g(T[m]) \xi(m) \eta(T[m]) d\mu(m) = (N*\xi)(\eta)
\]

for all \( \xi, \eta \in K \).

Thus \( (N*S)(m) = g(T[m]) \xi(T[m]) \).

Since \( N^*N\xi = NN^*\xi \), we have \( g(T[m])g(T[m])\xi(m) = g(m)g(m)\xi(m) \).

So \( |g(T[m])| = |g(m)| \) almost everywhere with respect to \( \mu \).

For completeness' sake, we could adjoin to \( \mathcal{A} \) any maximal commutative symmetric ring on \( \mathcal{L} = \mathcal{L}_N \), defining \( f \) to be the zero function and \( T \) to be the identity map on the maximal ideal space of the adjoined ring.
CHAPTER III
THE INVARIANT SUBSPACE PROBLEM

We now consider the well known question of whether each bounded operator on a countably infinite dimensional Hilbert space has a non-trivial invariant subspace. We suppose that the answer is no and attempt to find candidates for an example of an operator without a non-trivial invariant subspace.

Let \( S \) be an operator on \( \mathcal{H} \) without a non-trivial invariant subspace. Let \( S = UA \), where \( UA \) is the polar decomposition of \( S \). We claim that in this case \( U \) is unitary. Since \( \eta_S \) is invariant under \( S \), we have \( \eta_S = \{0\} \). Since \( \eta_A = \eta(S^*S) \subset \eta_S \), we have \( \eta_A = \{0\} \). Since \( \mathcal{R}_U = \mathcal{R}_A \), we have \( \mathcal{R}_U = \mathcal{H} \). Since \( \mathcal{R}_S \) is invariant under \( S \), it holds that \( \mathcal{R}_S = \mathcal{H} = \mathcal{R}_U \). Thus \( U \) is a partial isometry with domain space \( \mathcal{H} \) and range space \( \mathcal{H} \). Hence \( U \) is unitary.

If \( S \) has no non-trivial invariant subspace, then \( U \) and \( A \) have no common non-trivial invariant subspace. In this chapter we give four examples of operators \( \{S_1\}_{i=1}^4 \) having dense range and zero null-space such that \( S_1 \) has polar decomposition \( U_1A_1 \) where \( U_1 \) and \( A_1 \) have no common non-trivial invariant subspace. At least one of the examples to be given here, the first, is well known.

To construct these examples, we note the following facts. If \( A \) is a Hermitian operator on \( \mathcal{H} \) such that \( A \) generates a maximal commutative symmetric ring \( \sigma(A) \) in the weak operator
topology, then every invariant subspace for $A$ corresponds to a projection in $\mathcal{A}(A)$. For if $\mathcal{X}$ is invariant under $A$ and hence under $A^* = A$, then $P_{\mathcal{X}} A = AP_{\mathcal{X}}$, where $P_{\mathcal{X}}$ is the projection onto $\mathcal{X}$. Thus $P_{\mathcal{X}}$ is in $\mathcal{A}(A)$ inasmuch as $\mathcal{A}(A)$ is generated by $A$.

If $U$ is a unitary operator such that there is a sequence $\{n_p\}_{p=1}^{\infty}$ of positive integers for which $U^{n_p}$ converges to $U^*$ in the strong topology and such that $U$ and $U^*$ generate a maximal commutative symmetric ring $\mathcal{A}(U)$ in the weak topology, then again every subspace $\mathcal{Z}$ invariant under $U$ corresponds to a projection in $\mathcal{A}(U)$. The same conclusion holds if we replace $U^{n_p}$ by a net of polynomials in $U$ or by a net of continuous or Borel functions of $U$ which converges strongly to $U^*$.

Thus some operators have an easily described set of invariant subspaces. We are led to the following definition.

**Definition 3.0** An operator $W$ on $\mathcal{H}$ is said to be completely normal if and only if each invariant subspace for $W$ is a reducing subspace for $W$.

Clearly an operator $W$ is completely normal if and only if $W^*$ is completely normal. Every Hermitian operator is completely normal. We have no characterization of which normal operators are completely normal. We believe that there exist non-normal completely normal operators, but we have no examples known to be such. An operator without a non-trivial invariant subspace would be an example.

**Lemma 3.1** A unitary operator $U$ is completely normal if and
only if for each subspace \( \mathcal{K} \) such that \( U\mathcal{K} \subseteq \mathcal{K} \), it is true that \( U\mathcal{K} = \mathcal{K} \).

Proof: We suppose that \( U \) is completely normal; let \( \mathcal{K} \) be such that \( U\mathcal{K} \subseteq \mathcal{K} \). If \( \xi \in \mathcal{K} \Theta U\mathcal{K} \), then \( \xi \perp U\mathcal{K} \) implies that \( U\xi \perp \mathcal{K} \).

Thus \( \xi = 0 \) or \( U\mathcal{K} = \mathcal{K} \).

Now we suppose that \( U \) is such that \( U\mathcal{K} \subseteq \mathcal{K} \) implies that \( U\mathcal{K} = \mathcal{K} \). Let \( \mathcal{K}_1 \) be invariant under \( U \). Then \( U\mathcal{K}_1 = \mathcal{K}_1, U^*U\mathcal{K}_1 = U^*\mathcal{K}_1 \), and \( \mathcal{K}_1 = U^*\mathcal{K}_1 \). So \( \mathcal{K}_1 \) reduces \( U \), and \( U \) is completely normal.

Lemma 3.2 If \( N \) is a normal, completely normal operator such that \( N \) and \( N^* \) generate a maximal commutative symmetric ring \( \mathcal{A}(N) \) in the weak topology, then the projection onto each invariant subspace of \( N \) is an element of \( \mathcal{A}(N) \).

Proof: Let \( N\mathcal{K} \subseteq \mathcal{K} \); let \( P \) be the projection onto \( \mathcal{K} \). \( N\mathcal{K} \subseteq \mathcal{K} \) implies that \( PNP = NP \). Since \( N \) is completely normal, it holds that \( NP = PN, N^*P = PN^* \), and \( P \in \mathcal{A}(N) \) by the maximality of \( \mathcal{A}(N) \).

We now define some completely normal unitary operators which will be used in constructing the candidates for an example of an operator without an invariant subspace.

Consider the measure space \([0, 2\pi], \mu\), where \( \mu \) is Lebesgue measure. If \( \theta \in \mathbb{R} \), let \( \tilde{\theta} \) be such that \( 0 \leq \tilde{\theta} < 2\pi \), and \( \tilde{\theta} \) is congruent to \( \theta \) modulo \( 2\pi \). For \( \theta \in [0, 2\pi] \), let \( T(\theta) = (\tilde{\theta} - 1) \).

\( T \) is a measure-preserving ergodic transformation; p. 25-30, [2]. \( T^{-1}(\theta) = (\tilde{\theta} + 1) \). \( (T^n(0))_{n=0}^{\infty} \) is dense in \([0, 2\pi]\) in the usual
interval topology. Translation of functions is continuous in $L^2([0,2\pi])$, $\int_0^{2\pi} |f(\theta)-f(\theta+a)|^2 d\theta$ goes to zero as $a$ goes to zero. In the first example, we will define $U_1 f(\theta) = f(T[\theta])$ for $f \in L^2([0,2\pi])$. $U_1$ is a unitary operator; $U_1^* f(\theta) = f(T^{-1}[\theta])$.

Since $\{T^n(0)\}_{n=0}^{\infty}$ is dense in $[0,2\pi]$, we can find a sequence of integers $\{k_p\}_{p=1}^{\infty}$ such that $|T^{k_p}(0) - T^{-1}(0)|$ goes to zero as $p$ goes to infinity. Thus $U_1^{k_p}$ converges strongly to $U_1^*$, and $U_1$ is completely normal.

Our final completely normal unitary operator depends on analytic function theory rather than ergodic theory. We claim that for $\alpha$ such that $0 < \alpha < 2\pi$, the span of $\{e^{in\theta}\}_{n=1}^{\infty}$ is dense in the continuous functions on $[0,\alpha]$ under the supremum norm. The proof depends on theorems 3.9 and 3.6 and corollary 3.3.1 of [6]. If the span of $\{e^{in\theta}\}_{n=1}^{\infty}$ is not dense in the continuous functions on $[0,\alpha]$, then there is a regular Borel measure given by $\varphi$, a normalized function of bounded variation on $[0,\alpha]$ such that $\int_0^\alpha e^{in\theta} d\varphi = 0$ for $n \geq 1$.

Let $k = \varphi(0^+) - \varphi(0)$. We define $\psi$ as follows:

$$
\psi(\theta) = \begin{cases} 
\varphi(\theta) - 2^{-1}k & \text{for } 0 < \theta < \alpha \\
2^{-1}[\varphi(\alpha) - \psi(\alpha^-)] - 2^{-1}k & \text{for } \theta = \alpha \\
\varphi(\alpha) - 2^{-1}k & \text{for } \alpha < \theta < 2\pi \\
\varphi(0) & \text{for } \theta = 0, \text{ and } \varphi(\alpha) \text{ for } \theta = 2\pi
\end{cases}
$$

Thus $\psi$ is a normalized function of bounded variation such that $\int_0^{2\pi} e^{in\theta} d\psi = 0$ for $n \geq 1$. By theorem 3.9 of [6], $\psi$ is absolutely continuous and $\psi'(\theta) = \lim_{r \to 1} g(re^{in\theta})$ almost everywhere, with $g$ analytic and beschränktartige by theorem 3.6.
$\psi'(\theta) = 0$ for $0 < \theta < 2\pi$. So by corollary 3.3.1 of [6], $g = 0$.

Thus $\psi = 0$, and $\varphi = 0$. So $\{e^{in\theta}\}_{n=1}^{\infty}$ are dense in the continuous functions on $[0,a]$ for $0 < a < 2\pi$.

Hence the span of $\{e^{i\alpha n\theta}\}_{n=1}^{\infty}$ is dense in the continuous functions on $[0,2\pi]$ in the supremum norm for $0 < \alpha < 2\pi$. We fix $\alpha$ between $0$ and $2\pi$. For $f \in L^2([0,2\pi])$, we define $U_3f(\theta) = e^{i\alpha \theta}f(\theta)$. $U_3$ is unitary. Since polynomials in $e^{i\alpha \theta}$ can be found to approximate $e^{-i\alpha \theta}$ in the supremum norm, we know that these polynomials in $U_3$ approximate $U_3^*$ in the operator norm. So $U_3$ is completely normal.

**Example 1** As we said earlier, we define $U_1f(\theta) = f(\theta-1) = f'(T[\theta])$ for $f \in L^2([0,2\pi])$. Let $A_1f(\theta) = \theta f(\theta)$. By the Weierstrass approximation theorem, the norm closure of the ring generated by $A_1$ contains all operators given by multiplications by continuous functions which vanish at zero.

Hence the weakly closed ring generated by $A_1$ is $L^\infty([0,2\pi])$, a maximal commutative symmetric ring; p.352, IV,[3]. By lemma 3.2, each invariant subspace of $A_1$ corresponds to a Borel subset $E$ of $[0,2\pi]$ and is the subspace of square-summable functions which are supported on $E$. Since $T$ corresponding to $U_1$ is ergodic on $[0,2\pi]$, the only subsets $F$ of $[0,2\pi]$ such that $T^{-1}(F) = F$ are such that $\mu(F)$ is equal to $0$ or to $2\pi$, where $\mu$ denotes Lebesgue measure. So $U_1$ and $A_1$ have only trivial common invariant subspaces.

We give a second proof of this fact, using that $U_1$ is
completely normal. Since \( U_1 e^{in\theta} = e^{inT(\theta)} = e^{in(\theta-1)} = e^{-in}e^{in\theta} \), \( U_1 \) has a complete orthonormal set of eigenvectors with discrete eigenvalues. Thus if \( P_n \) is the projection onto \( \mathcal{H}[e^{in\theta}] \) for \(-\infty < n < \infty\), we have \( P_n \) is in the weakly closed ring generated by \( U_1 \) and \( U_1^* \). So by lemma 3.2, each invariant subspace for \( U_1 \) is of the form \( \mathcal{H}[e^{in\theta}]_{n \in J} \), where \( J \) is a subset of the integers. Since \( |e^{in\theta}| = 1 \) for \( \theta \in [0,2\pi] \) and \(-\infty < n < \infty\), our previous characterization of the invariant subspaces for \( A_1 \) shows that \( U_1 \) and \( A_1 \) have no common non-trivial invariant subspace. We note that \( A_1 \) has no eigenvectors and that \( U_1 \) has a complete set of eigenvectors. We set \( S_1 = U_1 A_1 \). We refer the reader to [4] for a more complete discussion of this type of operator.

**Example 2** In this example both \( U_2 \) and \( A_2 \) have a complete set of eigenvectors. Let \( \mathcal{K} = \mathcal{H}[e^{in\theta}]_{n=0}^\infty = H^2([0,2\pi]) \). Let \( \{\eta_j\}_{j=0}^\infty \) be the skewed basis with respect to \( \{e^{in\theta}\}_{n=0}^\infty \); this construction was given in the beginning of the chapter on partial isometries. We claim that a subspace \( \mathcal{K} \) of the form \( \mathcal{K} = \mathcal{H}[e^{in\theta}]_{n \in A} = \mathcal{H}[\eta_j]_{j \in B} \) where \( A \) and \( B \) are subsets of the integers, is either \([0]\) or \( \mathcal{K} \). If \( \mathcal{K} \neq [0] \), we note that \( (e^0|\eta_j) \neq 0 \) for each \( j \in B \); for \( (e^0|\eta_j) \neq 0 \) for \( 0 \leq j < \infty \) by construction of the skewed basis. Since \( \mathcal{K} \) is of the form \( \mathcal{H}[e^{in\theta}]_{n \in A} \), we have that \( e^0 \in \mathcal{K} \) or \( e^0 \perp \mathcal{K} \). If \( B \) is non-empty, we have \( e^0 \in \mathcal{K} \). Since \( \mathcal{K} \) is of the form \( \mathcal{H}[\eta_j]_{j \in B} \), we have that \( \eta_j \in \mathcal{K} \) or \( \eta_j \perp \mathcal{K} \) for \( 0 \leq j < \infty \). Since \( e^0 \in \mathcal{K} \) and \( (e^0|\eta_j) \neq 0 \)
for $0 \leq j < \infty$, we have $\eta_j \in \mathcal{K}$. So $\mathcal{K} = \mathcal{X}$.

We let $U_2$ be the restriction of $U_1$ to $H^2([0, 2\pi]) = \mathcal{F}\{e^{in\theta}\}_{n=-\infty}^\infty$, a subspace of $\mathcal{F}\{e^{in\theta}\}_{n=-\infty}^\infty$ which reduces $U_1$. [For an alternate definition of $H^2([0, 2\pi])$, see [6].] $U_2$ is completely normal. Each invariant subspace for $U_2$ is of the form $\mathcal{F}\{e^{in\theta}\}_{n \in A}$, where $A$ is a subset of the non-negative integers, by lemma 3.2.

We let $A_2(\sum_{j=0}^{\infty} a_j \eta_j) = \sum_{j=0}^{\infty} a_j [1-2^{-j-1}]\eta_j$. Or, if $Q_j$ is the projection onto $\mathcal{F}\{\eta_j\}$, then $A_2 = \sum_{j=0}^{\infty} [1-2^{-j-1}]Q_j$. Each invariant subspace of $A_2$ is of the form $\mathcal{F}\{\eta_j\}_{j \in B}$, where $B$ is a subset of the non-negative integers, by lemma 3.2.

Thus $U_2$ and $A_2$ have only the trivial subspaces as common invariant subspaces. We let $S_2 = U_2A_2$. The property of the skewed basis used in this example is the motivation for the choice of the description "skewed."

**Example 3** In this example, $U_3$ has no eigenvectors and $A_3$ has a complete set of eigenvectors. As before, we fix $\alpha$ such that $0 < \alpha < 2\pi$ and define $U_3f(\theta) = e^{i\alpha\theta}f(\theta)$ for $f \in L^2([0, 2\pi])$.

If $R_n$ is the projection onto $\mathcal{F}\{e^{in\theta}\}$, we define $A_3 = \sum_{n=0}^{\infty} (1-3^n)R_n + \sum_{n=1}^{\infty} (1-2^{-n})R_n$. Since each invariant subspace of $U_3$ corresponds to a Borel subset of $[0, 2\pi]$ and each invariant subspace of $A_3$ is of the form $\mathcal{F}\{e^{in\theta}\}_{n \in D}$, where $D$ is a subset of the integers, we see that $U_3$ and $A_3$ have only the trivial subspaces as common invariant subspaces. We set $S_3 = U_3A_3$.

**Example 4** In our final example both $U_4$ and $A_4$ have no eigen-
vectors. In fact, the Hilbert space is \( L^2([0, 2\pi]) \), and \( U_4 = U_3 \). We set \( V\text{e}^{i\theta} = e^i(n+1)\theta \) or \( Vf(\theta) = e^{i\theta}f(\theta) \) for \( f \in L^2([0, 2\pi]) \). We recall that \( U_1f(\theta) = f(T[\theta]) = f(\theta-1) \). \( U_1Vf(\theta) \) 
\[ U_1e^{i\theta}f(\theta) = e^{i(\theta-1)}f(\theta-1) = e^{-i}e^{i\theta}f(\theta-1) = e^{-i}VU_1f(\theta) \] 
So \( U_1V = e^{-i}VU_1 \). Moreover, any proper non-zero subspace of 
\( L^2([0, 2\pi]) \) invariant under \( U_4 = U_3 \) is not invariant under 
\( U_1V_1 \), since \( T \) is ergodic.

For \( e^{i0\theta} = e^0 \), we have that \( \{V^j e^0\}_{j=-\infty}^\infty \) is an orthonormal basis for \( L^2([0, 2\pi]) \). Now \( U_1^j e^0 = e^0 \) for \( -\infty < j < \infty \). Thus \( U_1V = e^{-i}VU_1 \) implies that \( (U_1V)^j e^0 = c_j V^j e^0 \), where \( |c_j| = 1 \) for 
\( -\infty < j < \infty \). So \( \{(U_1V)^j e^0\}_{j=-\infty}^\infty \) is an orthonormal basis for \( \mathcal{V} = L^2([0, 2\pi]) \); and \( U_1V \) is the bilateral shift with respect to this basis. So we represent \( U_1V \) as a multiplication by \( e^{i\theta} \) on \( L^2([2\pi, 4\pi]) \). \( U_1Vg(\theta) = e^{i\theta}g(\theta) \) for \( g(\theta) \in L^2([2\pi, 4\pi]) \).

We set \( A_4g(\theta) = (\theta-2\pi)g(\theta) \) for \( g(\theta) \in L^2([2\pi, 4\pi]) \). The invariant subspaces for \( A_4 \) correspond to Borel subsets of 
\( [2\pi, 4\pi] \) by lemma 3.2. Each such subspace is invariant under 
\( U_1V \), since \( U_1V \) is represented as a multiplication on \( L^2([2\pi, 4\pi]) \). But no non-trivial invariant subspace for \( U_4 \) is invariant under \( U_1V \) since \( T \) is ergodic, as mentioned earlier. So \( U_4 \) and \( A_4 \) have no non-trivial invariant subspace in common.
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