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## Dissipative Lipschitz dynamics

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DISSIPATIVE LIPSCHITZ DYNAMICS

A Dissertation

Submitted to the Graduate Faculty of the  
Louisiana State University and  
Agricultural and Mechanical College  
in partial fulfillment of the  
requirements for the degree of  
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in

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by

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*Dedicated to my son Christian  
and the memory of my grandfather  
Asdrubal*

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# Notation

$\mathbb{R}^n$	Euclidean $n$ -dimensional space
$\ \cdot\ $	Euclidean norm
$\langle \cdot, \cdot \rangle$	Euclidean inner product
$B_n$	The closed unit ball in $\mathbb{R}^n$
$\text{co } E$	The convex hull of the set $E$
$d_S(x)$	The distance from $x$ to the set $S$
$\text{proj}_S(x)$	The set of all the points in $S$ that realize the distance between $x$ and $S$
$N_S^P(x)$	The proximal normal cone of $S$ at $x$
$N_L^P(x)$	The limiting normal cone of $S$ at $x$
$\partial_P f(x)$	The proximal subdifferential of $f$ at $x$
$T_S^B(x)$	The Bouligand cone of $S$ at $x$
$T_S^C(x)$	The Clarke cone of $S$ at $x$
$C^\circ$	The polar of the set $C$
$H$ (resp. $h$ )	The upper (resp. lower) Hamiltonian



# Abstract

In this dissertation we study two related important issues in control theory: invariance of dynamical systems and Hamilton-Jacobi theory associated with optimal control theory. Given a control system modeled as a differential inclusion  $\dot{x}(t) \in F(t, x(t))$  and a closed set  $S$ , we provide necessary and sufficient conditions for the strong invariance property of the pair  $(S, F)$  when the right-hand side  $F$  satisfies a dissipative Lipschitz (DL) condition. We show that when  $F$  is almost upper semicontinuous and satisfies the (DL) property, these conditions can be expressed in terms of approximate Hamilton-Jacobi inequalities, subsuming in particular the classic infinitesimal characterization of strongly invariant Lipschitz dynamics. In the important case when the multifunction  $F$  is the sum of a maximal dissipative and a Lipschitz multifunction, the approximate inequalities turn into an exact mixed-type inequality that involves the lower and upper Hamiltonian of the dissipative and the Lipschitz piece respectively. We then extend this Hamiltonian characterization to nonautonomous systems by assuming a potentially discontinuous differential inclusion whose right-hand side is the sum of an almost upper semicontinuous dissipative and an almost lower semicontinuous dissipative Lipschitz multifunction. Finally, a Hamilton-Jacobi theory is developed for the minimal time problem of a system with possibly discontinuous monotone Lipschitz dynamics. This is achieved by showing the minimal time function  $T_S(\cdot)$  associated to an upper semicontinuous and a monotone Lipschitz data is characterized as the unique proximal semi-solution to an approximate Hamilton-Jacobi equation.

**Key words:** differential inclusion, strong invariance, minimal time function, Hamilton-Jacobi inequality.

# Introduction

Nonlinear/non-smooth dynamical systems theory arise from those physical phenomena that involve rigid bodies that stick to each other, and whose contact phase is interrupted when one body “slips” with respect to another. In addition to this frictional behavior, impacts among different parts of the system can also be observed.

From a mathematical point of view, dynamical systems of this type are difficult objects to handle due to the potentially discontinuous or non-differentiable right-hand side that arises in their resulting models. One simple, but important example in mechanical engineering exhibiting such a “slip-stick” motion is the pendulum with *dry* or *Coulomb friction* (see [21, 22, 37]). The pendulum consists of a mass being attached to a spring, and the mass moves in a straight tube due to a sinusoidal force, and has contact to the wall of the tube. Depending on the size of the dry friction between the mass and the wall, and the magnitude of the force applied to the mass, the mass moves up or down, or it sticks to the wall. Problems of this type have been modeled with the help of a *dissipative differential Inclusion*

$$\dot{x}(t) \in D(x(t)), \quad x(0) = x_0,$$

where the term dissipative refers as to the “decreasing” nature of the dynamics, which is in general explained by the relationship between the state and velocities of the system. More precisely, the multifunction  $D(\cdot)$  is called dissipative if  $\langle x - y, u - v \rangle \leq 0$  for all  $x$  and  $y$ , and all  $u \in D(x)$  and  $v \in D(y)$ .

The class of dissipative differential inclusions also contains the class of the “gradient inclusions”

$$\dot{x}(t) \in -\partial_P V(x(t)), \quad x(0) = x_0,$$

where the *proximal subdifferential of  $V$  at  $x$* ,  $\partial_P V(x)$ , is an object designed to extend the notion of differentiability to lower semicontinuous functions  $V$  (see chapter 1 for definitions and properties).

A natural generalization of the dissipative concept for multifunctions is the following *dissipative Lipschitz* (DL) condition which was first introduced by T. Donchev in [24] under the name of *one-sided Lipschitz* (OSL) property: A multifunction  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is dissipative Lipschitz if for every bounded set  $\Omega \subset \mathbb{R}^n$  there exists a constant  $k_\Omega$  such that for  $x, y \in \Omega$  and  $u \in F(x)$  there is  $v \in F(y)$  satisfying

$$\langle u - v, x - y \rangle \leq k_\Omega \|x - y\|^2.$$

Obviously the dissipative Lipschitz property reduces to a quadratic quasi-monotone condition if the multifunction is singleton-valued, but of course is more general.

It is well known that the *Lipschitz property* plays a fundamental role in establishing many results in optimal control theory. However, it is also known that there exists a generous number of important non-Lipschitz dynamics, and therefore their understanding and interpretation do not fall under the scope of the standard theory. The dissipative Lipschitz dynamical systems lie in this last category, as is corroborated by considering the following prototype of discontinuous dissipative multifunctions

$$D(x) = \begin{cases} 1 & \text{if } x < 0 \\ \{-1, 1\} & \text{if } x = 0 \\ -1 & \text{if } x > 0. \end{cases} \quad (1)$$

The multifunction  $D$  is used to explain the dry friction force acting on the mass-spring system mentioned at the beginning. For such reasons, the investigation of

(DL) dynamical systems in optimal control theory seems to be a promising vein that we attempt to explore diligently.

The main goal pursued in this dissertation is the establishment of necessary and sufficient conditions for the strong invariance property of a control system modeled as a differential inclusion

$$\begin{aligned} \dot{x}(t) &\in F(t, x(t)) \quad \text{a.e., } t \in [t_0, t_1), \\ x(t_0) &= x_0, \end{aligned} \tag{2}$$

where the initial condition  $x_0$  lies in a closed set  $S$ , which due to the invariance property is the recipient of the trajectories  $x(\cdot)$  of (2) (see below for the definitions of the invariant properties). The novelty of the pursued conditions is based on the fact that the multifunction  $F$  satisfies weaker assumptions than heretofore imposed. In fact, in addition to the dissipative Lipschitz condition, only a semicontinuity property will be assumed on the multifunction  $F$ , which contrasts with classical Lipschitz assumptions found in the literature on the subject [12, 15, 16, 17, 31, 38]. Afterward, and under similar assumptions on the data, in the last part of this work we present local versions of these invariance results, which leads us to an application in Hamilton-Jacobi Theory.

We briefly recall the main concepts used in this work. Let  $I = [0, \infty)$ ,  $S \subseteq \mathbb{R}^n$  be closed, and  $F$  as above. The pair  $(S, F)$  is *weakly invariant* (called “viable” in [2, 3]) if for each interval  $[t_0, t_1) \subset I$  and  $x_0 \in S$ , there exists a solution  $x(\cdot)$  of (2) satisfying  $x(t) \in S$  for all  $t \in [t_0, t_1)$ . Similarly,  $(S, F)$  is *strongly invariant* provided for each interval  $[t_0, t_1) \subset I$  and  $x_0 \in S$ , every solution  $x(\cdot)$  of (2) satisfies  $x(t) \in S$  for all  $t \in [t_0, t_1)$ .

Topics in invariance theory have provided the foundation for considerable current research in control theory and optimization [1, 2, 16, 26, 31, 32, 37]. Apparently the first invariance result was by Nagumo [40] in the context of differential equations,

that is, for a singleton-valued map  $F(x) = \{f(x)\}$  with  $f(\cdot)$  continuous. It was shown that weak invariance of  $(S, F)$  is equivalent to the *tangential-type* condition  $f(x) \in T_S^B(x)$  for all  $x \in S$ , where

$$T_S^B(x) := \left\{ v : \liminf_{h \downarrow 0} \frac{d_S(x + hv)}{h} = 0 \right\}$$

is the *contingent* or *Bouligand cone* of  $S$  at  $x$  ( $d_S(x) := \min\{\|x - s\| : s \in S\}$  is the distance function to  $S$ ). Of course if  $f(\cdot)$  is locally Lipschitz (or more generally, admits unique trajectories), then the notions of weak and strong invariance coincide. Brezis [10] rediscovered the tangential characterization by assuming  $f(\cdot)$  is locally Lipschitz. Bony [8] introduced the *proximal normal cone*

$$N_S^P(x) := \left\{ \zeta : \exists \sigma > 0 \text{ such that } \langle \zeta, x' - x \rangle \leq \sigma \|x' - x\|^2 \forall x' \in S \right\}$$

and proved that the *normal-type* condition  $\langle f(x), \zeta \rangle \leq 0$  for all  $x \in S$  and  $\zeta \in N_S^P(x)$  is also a characterization of invariance (again, with  $f(\cdot)$  locally Lipschitz). The proximal normal cone has since become a major instrument in non-smooth analysis; see [16]. Hartman [33] proved the equivalence of weak invariance and the tangential condition under a continuity assumption, and independently, Crandall [19] included the equivalence with the normal condition. Redheffer and Walter [43] made extensions to any real or complex inner product space and replaced a regularity assumption on  $f(\cdot)$  by a (possibly discontinuous) quasi-monotone condition, under which solutions to the ODE are nonetheless unique. They showed the tangential-type condition was sufficient for weak invariance and implies the normal-type one. We already mentioned that the dissipative Lipschitz condition used in this work reduces to the quasi-monotone condition if  $F$  is singleton-valued, but of course is more general. All of the previously cited work is within the framework of ordinary differential equations, however, the notions

of weak and strong invariance are nonetheless distinct unless the solutions of the dynamic equation are unique. Hence the cited characterizations are really of weak invariance and only include strong invariance if the two notions coincide.

Throughout this work, we consider general  $F : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with the standard assumptions (SH) of nonemptiness, convex-compact valuedness, linear growth, and almost upper semicontinuity. These assumptions have shown to be enough to guarantee the existence of at least one solution to (2) (cf. [3, 9, 21]).

Weak invariance for differential inclusions has been more extensively studied than strong invariance, and we first review its history. In the finite dimensional and autonomous ( $F$  independent of  $t$ ) case, the tangential condition

$$F(x) \cap T_S^B(x) \neq \emptyset, \quad \forall x \in S \quad (3)$$

was shown by Haddad [34] to characterize weakly invariant systems. Ledyev [39] proved a similar statement when  $F$  has measurable  $t$ -dependence, and Frankowska, Plaskacz, and Rzezuchowski [31] proved a nonautonomous tangential version where the containing set  $S$  could also depend “absolutely continuously” on  $t$ . Rapaport and Vinter [41] showed that the  $t$ -dependence of  $S$  could be weakened on intervals where the  $t$ -dependence of  $F$  is strengthened. A normal-type criterion was first proven by Veliov [50], and says that weak invariance is equivalent to

$$\min_{v \in F(x)} \langle v, \zeta \rangle \leq 0 \quad (4)$$

for every  $x \in S$  and  $\zeta \in N_S^P(x)$ . A unified treatment of the autonomous results is contained in [16], while nonautonomous versions of this treatment can be found in the works by Frankowska and Plaskacz [32], and Donchev [26].

The following simple example in dimension one shows that strong invariance is not necessary for either (3) or (4) if the data only satisfies (SH) with upper semicontinuity in the state variable: let  $F(x) = \{\text{sgn}(x)\sqrt{|x|}\}$  and  $S = \{0\}$ . Clearly

(3) and (4) both hold, but there are trajectories that begin at and leave the origin. A Lipschitz property of  $F$  typically has been assumed for the purpose of characterizing strong invariance. Clarke [13] proved the first truly strong invariance theorem (that is, when strong and weak invariance do not coincide) by considering autonomous Lipschitz multifunctions, and proved that strong invariance is equivalent to the velocity set  $F(x)$  being contained in the *Clarke tangent cone* at each point  $x \in S$ . Tangential characterizations of strong invariance are also included in Frankowska, Plaskacz, and Rzezuchowski [31] with measurable time-dependent inclusions and time varying data. The strong normal condition is that

$$\max_{v \in F(x)} \langle v, \zeta \rangle \leq 0 \tag{5}$$

holds for all  $x \in S$  and  $\zeta \in N_S^P(x)$ . Krastanov [37] showed the equivalence of strong invariance to (5) under the same assumptions as Clarke, which in light of subsequent advances in nonsmooth analysis, can readily be seen as equivalent to Clarke's tangential condition independent of invariance (quick sketch of proof: the Clarke normal and tangent cones are polars, and the Clarke normal cone is the closed convex hull of the limits of proximal normals). Clarke, Ledyaev, and Radulescu [15] have also made extensions to Hilbert spaces with appropriate modifications; we will address infinite dimensional versions of our results in a future work.

The first non-Lipschitz state-dependent characterization of strong invariance appeared recently in [26], where *almost continuity* plus the dissipative Lipschitz property was assumed. In this dissertation we will significantly extend all the results contained in the latter works by replacing both the Lipschitz and continuity assumptions by the (DL) and a semicontinuity assumptions on  $F$ , and carrying out further extensions as well. It should be noted that under assumptions (SH) and the (DL) property, criterion (5) is no longer necessary for strong invariance, which can

be easily seen by considering the pair  $(S, F)$  with  $S = \{0\}$  and  $F = \text{co } D$  (see (1)). This simple example shows that a replacement condition for strong invariance is needed when  $F$  is upper semicontinuous and dissipative Lipschitz. The new necessary and sufficient conditions have been published by the author in joint work with T. Donchev and P. Wolenski [29] and have been expressed in terms of approximate Hamiltonian inequalities. They will be presented in chapter 4 in Theorem 4.3.

By strengthening the  $t$ -dependence of the multifunction  $F$  with uniform continuity and assuming only upper semicontinuity in the state variable, and the (DL) condition, we prove that one of the approximate Hamiltonian inequalities turns into a characterization of strong invariance (see Corollary 4.4). Furthermore, assuming  $\mathcal{L} \times \mathcal{B}$ -measurability, upper semicontinuity in  $x$ , and the (DL) property for  $F(\cdot, \cdot)$ , we handle the case when the constraint set  $S$  is absolutely continuous time-dependent. Under this last setting, we are able to provide a sufficient condition for the strong invariance of  $(S, F)$  that is similar to the one obtained in case of a time-independent constraint set (see Theorem 4.7). Using the special character of dissipative multifunctions, we study the special case of an autonomous system with control parameters that enjoys a decomposition of the form  $F = D + G$ , with  $D$  upper semicontinuous and dissipative, and  $G$  being Lipschitz. The criterion for strong invariance then takes the particular mixed-type Hamiltonian form

$$h_D(x, \zeta) + H_G(x, \zeta) \leq 0, \quad \text{for all } x \in E, \text{ and all } \zeta \in N_E^P(x) \quad (6)$$

(see definitions of  $h_F$  and  $H_F$  for a multifunction  $F$  in chapter 1). An extension of criterion (6) to a completely discontinuous nonautonomous case is also obtained in Theorem 3.2, when  $D$  is “almost upper semicontinuous” and dissipative, and  $G$  is “almost lower semicontinuous” and (DL). In particular, by removing the dissipative



piece  $D$  in this nonautonomous case, we will see in chapter 3 that

$$H_G(t, x, \zeta) \leq 0, \quad \text{for all } x \in E, \text{ all } \zeta \in N_S^P(x), \text{ and a.e., } t \in I \quad (7)$$

characterizes the strong invariance property for almost lower semicontinuous (DL) dynamics.

It is interesting to note that our results under the dissipative Lipschitz assumption are only of normal-type, and that a condition involving the Clarke tangent cone is still not known.

The last part of this dissertation concerns the problem of developing a Hamilton-Jacobi (HJ) theory under a discontinuous *monotone Lipschitz* (ML) approach, where the term “monotone” is associated to conjugates of dissipative dynamics [21, 22, 29]. For this purpose the minimal time function  $T_S(\cdot)$  associated to an upper semicontinuous and monotone Lipschitz multifunction is considered, and we show this function is characterized as the unique proximal semi-solution to an approximate Hamilton-Jacobi equation.

Due to the need of accommodating more general types of solutions to HJ equations, non-classical approaches have become well-established, the most noteworthy being the so-called viscosity theory. Apparently the first subdifferential definition of generalized solution was given in the context of Lipschitz functions and appeared in 1977 [35]. By stressing the relevance of invariance, Subbotin [46, 47] introduced the minimax approach with Dini derivatives in the context of differential games. Subsequently, the method of viscosity solutions was presented by Crandall and Lions [20], which is strongly linked to classical PDE theory. The proximal solutions to HJ equations were investigated by Clarke and Ledyeu [14] where various concepts were unified, and Clarke and Vinter [18] constructed verification functions to generate solutions that are not necessarily unique. Also remarkable is the subdif-

ferential characterization provided by Barron and Jensen [7], and Frankowska [30] for certain cases, where lower semicontinuous viscosity solutions were considered the first time. See also Clarke et al. [17].

The minimal time function  $T_S(\cdot)$  has also received significant attention in HJ theory. Under local controllability assumptions, the continuity property of  $T_S(\cdot)$  plays an important role in the viscosity approach presented in [4, 5, 23, 44]. On the other hand, and under the same viscosity approach, it is shown in the works of Bardi and Staicu [6], and Soravia [45] that controllability hypotheses are not required if more restrictive conditions are imposed on the target set  $S$ . Assuming the Lipschitz property of the data, Wolenski and Zhuang showed [52] the minimal time function  $T_S(\cdot)$  is the unique lower semicontinuous solution to an HJ equation. The results exposed in this dissertation on the minimal time function were proved by the author in joint work with T. Donchev and P. Wolenski [29] and they are tailored according to the ideas presented in [52], where the role of invariance is emphasized (see also [30]). This last is done by extending the local versions of the invariant results contained in [52] to the more general dissipative Lipschitz framework (see Theorem 5.8 and Theorem 5.10).

The organization of this dissertation is as follows.

Chapter 1 contains the technical background on nonsmooth analysis and differential inclusions theory that we shall use throughout the thesis. The chapter presents a miscellany of classical statements and definitions related to proximal analysis, existence theory, and properties of trajectories of discontinuous right-hand sides. A brief comparison between the Lipschitz and the dissipative Lipschitz property is also given in the language of Hamiltonians.

Chapters 2, 3, and 4 are completely devoted to the strong invariance issue. The case of perturbed dissipative systems is approached under two different sets of

assumptions. By constructing suitable feedback selections, in chapter 2 we first characterize the trajectories of an autonomous system whose right-hand side is an upper semicontinuous and dissipative Lipschitz multifunction. Such a characterization is then applied to systems of the form  $F = D + G$  when  $D$  is upper semicontinuous and dissipative, and  $G$  is Lipschitz. A criterion for strong invariance is provided in terms of the lower and upper Hamiltonian of the dissipative and Lipschitz piece respectively. An extension of this last result is obtained in chapter 3 when the multifunction  $F = D + G$  is nonautonomous, the  $D$  component is almost upper semicontinuous and dissipative, and  $G$  is almost lower semicontinuous and dissipative Lipschitz. Chapter 4 treats the case of a general nonautonomous multifunction that satisfies the dissipative Lipschitz requirement with almost upper semicontinuity. A characterization of strong invariance is first given in a form of an invariance principle, which leads to more practical criteria in terms of approximate Hamiltonian inequalities.

Finally, in chapter 5 we study a minimal time control problem of an autonomous dynamical system. We establish local versions of the invariant characterizations obtained in chapter 4 to develop a Hamilton-Jacobi theory that applies to upper semicontinuous monotone Lipschitz dynamics.

# Chapter 1

## Preliminaries

The theoretical background required in this thesis is based on some nonsmooth analysis constructs and existence theory for differential inclusions. We summarize these prerequisites in this chapter. Since [3, 16, 21, 36, 48, 49] contain detailed proofs for the theorems, corollaries, and propositions stated in this chapter, our exposition will be rather brief. Occasionally we will sketch a proof for which a modification is necessary to make it apply under our setting.

### 1.1 Proximal Analysis

#### 1.1.1 Cones

We start by considering a nonempty set  $S \subset \mathbb{R}^n$  and recalling the *distance function*  $d_S(\cdot)$  associated to the set  $S$  is the nonnegative valued map defined by

$$d_S(x) := \inf\{\|x - s\| : s \in S\}, \quad \text{for all } x \in \mathbb{R}^n.$$

It is well known that when  $S$  is closed, for each  $x \in \mathbb{R}^n$  the existence of a point  $s \in S$  such that  $d_S(x) = \|x - s\|$  is guaranteed. When a point  $s \in S$  enjoys this last property, we say that  $s$  is a *closest point* or a *projection* of  $x$  onto  $S$ . We denote the set of such projections by  $\text{proj}_S(x)$ .

Let us assume that  $S$  is closed. For a given  $x \in \mathbb{R}^n$ , the vector  $x - s$  with  $s \in \text{proj}_S(x)$  is called a *proximal normal direction* to  $S$  at  $s$ ; any vector of the form  $\zeta = t(x - s)$ , with  $t \geq 0$ , is called a *proximal normal* to  $S$  at  $s$ , and the set

$$N_S^P(s) := \{t(x - s) : t \geq 0, \text{ and } s \in \text{proj}(x) \text{ for some } x \in \mathbb{R}^n\} \quad (1.1)$$

is called the *proximal normal cone* to  $S$  at  $s$ . It is well known (see Proposition 1.15(a) of [16]) that the following *proximal normal inequality* is a characterization

for the elements in  $N_S^P(s)$ :  $\zeta \in N_S^P(s)$  if and only if there exists  $\sigma = \sigma(\zeta, s) \geq 0$  such that

$$\langle \zeta, s' - s \rangle \leq \sigma \|s' - s\|^2 \quad \text{for all } s' \in S. \quad (1.2)$$

More even, for each  $\varepsilon > 0$  and  $s \in S$ , we have  $\zeta \in N_S^P(s)$  if and only if there exists  $\sigma(\zeta, s) > 0$  so that (1.2) holds for all  $s' \in S \cap (s + \varepsilon B)$ , that is,  $N_S^P(s) = N_{(S \cap (s + \varepsilon B))}^P(s)$  for all  $\varepsilon > 0$  (this is item (b) in Proposition 1.15 of [16]). The natural closure operation to apply to  $N_S^P(x)$  gives rise to an element from which we will benefit in the future. The *limiting normal cone* to  $S$  at  $x \in S$  is the set

$$N_S^L(x) := \left\{ \lim_{i \rightarrow \infty} \zeta_i : \zeta_i \in N_S^P(x_i), x_i \xrightarrow{S} x \right\}, \quad (1.3)$$

where  $x_i \xrightarrow{S} x$  signifies that  $x_i \rightarrow x$ , and  $x_i \in S$  for all  $i$ . One of the motivations for defining the limiting normal cone is that  $N_S^P(x)$  is potentially trivial, i.e.,  $N_S^P(x) = 0$  for “many”  $x$ ; in pointwise considerations it is  $N_S^L(x)$  that may incorporate normality conditions. Other important constructions in our analysis are the following *contingent* or *Bouligand cone* of  $S$  at  $x$

$$T_S^B(x) := \left\{ v : \liminf_{h \downarrow 0} \frac{d_S(x + hv)}{h} = 0 \right\}, \quad (1.4)$$

and the *Clarke tangent cone* to  $S$  at  $x \in S$

$$T_S^C(x) := \left\{ v : \limsup_{y \rightarrow x, t \downarrow 0} \frac{d_S(y + tv) - d_S(y)}{t} = 0 \right\}. \quad (1.5)$$

The appeal of the second cone is that it is always convex. However, the price to pay for this nice property is that the Clarke cone may often be reduced to the trivial cone  $\{0\}$ . Actually, the inclusion  $T_S^C(x) \subseteq T_S^B(x)$  holds in general for all  $x \in S$ , and when  $S$  is a convex set, the Bouligand and Clarke cones coincide. Another important observation about the Clarke tangent cone is the following characterization (see [16], page 85)

$$T_S^C(x) = [N_S^L(x)]^\circ := \{v : \langle v, p \rangle \leq 0 \ \forall p \in N_S^L(x)\}, \quad \text{for all } x \in S, \quad (1.6)$$

where  $[N_S^L(x)]^\circ$  is called *the polar* of  $N_S^L(x)$ . Finally, and under the same polar operation we have also the inclusion

$$T_S^B(x) \subseteq [N_S^P(x)]^\circ, \quad (1.7)$$

which holds for all  $x \in S$  (see page 91 of [16]).

### 1.1.2 Proximal Subgradients

We now briefly review another concept that will be of importance in chapter 5, namely, the proximal subgradients. We start by considering a function that may allow the value  $+\infty$  at a given point, that is, a function  $\theta : \mathbb{R}^n \rightarrow (-\infty, \infty]$  with *effective domain* and *epigraph* defined respectively by

$$\text{dom } \theta := \{x \in \mathbb{R}^n : \theta(x) < \infty\},$$

and

$$\text{epi } \theta := \{(x, r) \in \text{dom } \theta \times \mathbb{R} : r \geq \theta(x)\}.$$

Let  $U \subseteq \mathbb{R}^n$  be open. The function  $\theta$  is *lower semicontinuous* at  $x \in U$  provided that

$$\theta(x) \leq \liminf_{y \rightarrow x} \theta(y).$$

In case  $\theta$  is lower semicontinuous at every point  $x$  of  $U$ , we simply say that  $\theta$  is lower semicontinuous on  $U$ . Complementary to lower semicontinuity we say that  $g$  is *upper semicontinuous* at  $x$  if  $-g$  is lower semicontinuous at  $x$ . In order to avoid pathological cases, we will restrict our attention to those lower semicontinuous functions for which  $\text{dom } \theta \cap U \neq \emptyset$ , with  $\theta$  being lower semicontinuous on  $U$ .

A vector  $\xi \in \mathbb{R}^n$  is a *proximal subgradient* of  $\theta$  at  $x \in \text{dom } \theta$  provided  $(\xi, -1) \in N_{\text{epi } \theta}^P(x, \theta(x))$ . Notice that when  $\theta(\cdot)$  is lower semicontinuous, the set  $\text{epi } \theta$  is always a closed subset of  $\mathbb{R}^{n+1}$ . The set (which could be empty) of all proximal

subgradients of  $\theta(\cdot)$  at  $x$  is denoted by  $\partial_P\theta(x)$ , and it is referred as the *proximal subdifferential* of  $\theta$  at  $x$ . If  $x \notin \text{dom}\theta$ , then  $\partial_P\theta(x) = \emptyset$ . Now let  $(\xi, -\lambda) \in \mathbb{R}^n \times \mathbb{R}$  belongs to  $N_{\text{epi}\theta}^P(x, r)$  for some  $(x, r) \in \text{epi}\theta$ . It necessarily follows that  $\lambda \geq 0$ , that  $r = \theta(x)$  if  $\lambda > 0$ , and that  $\lambda = 0$  if  $r > \theta(x)$ . In the last case  $(\xi, 0) \in N_{\text{epi}\theta}^P(x, \theta(x))$ .

The following characterization is the so-called *proximal subgradient inequality*, and it is perhaps the most widely used description of these proximal elements:  $\zeta \in \partial_P\theta(x)$  if and only if there exist positive numbers  $\sigma$  and  $\eta$  such that

$$\theta(y) \geq \theta(x) + \langle \zeta, y - x \rangle - \sigma \|y - x\|^2 \quad \forall y \in x + \eta B_n.$$

Note that since a cone is involved in the definition of  $\partial_P\theta(x)$ , if  $\alpha > 0$  and  $(\xi, -\alpha) \in N_{\text{epi}\theta}^P(x, \theta(x))$ , then  $\xi/\alpha \in \partial_P\theta(x)$ .

We close this subsection by stating a result due to Rockafellar. This is referred to as the *Horizontal Approximation Theorem*, and expresses how horizontal proximal normals to epigraphs can be approximated by nonhorizontal ones, which then correspond to subgradients (see page 67 of [16]).

**Theorem 1.1. (Rockafellar)** *Let  $\theta$  be a lower semicontinuous function on  $\mathbb{R}^n$ , and  $(\xi, 0) \in N_{\text{epi}\theta}^P(x, \theta(x))$ . Then for every  $\varepsilon > 0$  there exists  $y \in x + \varepsilon B_n$  and  $(\zeta, -\lambda) \in N_{\text{epi}\theta}^P(y, \theta(y))$ , with  $\lambda > 0$ , such that*

$$|\theta(y) - \theta(x)| < \varepsilon, \quad \|(\xi, 0) - (\zeta, -\lambda)\| < \varepsilon.$$

□

## 1.2 Differential Inclusions

Throughout our work we shall consider multifunctions  $F$  mapping elements of  $I \times \mathbb{R}^n$  to the subsets of  $\mathbb{R}^n$ , where  $I = [0, \infty)$ . The control systems of our interest

are those that can be modelled as a *differential inclusion* with initial condition

$$\begin{aligned} \dot{x}(t) &\in F(t, x(t)) \quad \text{a.e., } t \in [a, b) \\ x(a) &= x_0, \end{aligned} \tag{1.8}$$

where  $[a, b) \subseteq I$  is a given interval, and  $x_0 \in \mathbb{R}^n$ . We call a *solution* (or *trajectory*) of the differential inclusion (1.8) to be any absolutely continuous map  $x(\cdot)$  whose derivative  $\dot{x}(t)$  belongs to  $F(t, x(t))$  for almost all  $t \in [a, b)$  and satisfying  $x(a) = x_0$ . The following miscellany provides us with a background on multifunctions that we will call upon in the future chapters.

Let  $A \subseteq \mathbb{R}^m$ . A multifunction  $E : A \rightrightarrows \mathbb{R}^n$  is called *upper semicontinuous* at  $x \in A$  if for every  $\epsilon > 0$  there is  $\delta > 0$  (depending on  $x$  and  $\epsilon$ ) such that  $E((x + \delta B_m) \cap A) \subseteq E(x) + \epsilon B_n$ . The multifunction  $G : A \rightrightarrows \mathbb{R}^n$  is *lower semicontinuous* at  $x \in A$  if for every  $\epsilon > 0$  there exists  $\delta > 0$  (depending on  $x$  and  $\epsilon$ ) for which  $G(x) \subseteq G(y) + \epsilon B_n$ , for all  $y \in (x + \delta B_m) \cap A$ . The multifunction  $E$  (respectively  $G$ ) is upper semicontinuous (lower semicontinuous) in  $A$  provided it is upper semicontinuous (lower semicontinuous) at each  $x \in A$ . Alternatively to the previous semicontinuity properties, we have the following measurability approach which is prominently used in the literature: A multifunction  $E : A \rightrightarrows \mathbb{R}^n$  is called *measurable* provided the set

$$E^{-1}(V) := \{x \in A : E(x) \cap V \neq \emptyset\} \tag{1.9}$$

is Lebesgue measurable for every open (or closed) set  $V$  of  $\mathbb{R}^n$ . Some basic operations with measurable multifunctions preserve the measurability as in the single-valued case. We record one of them in the next proposition for future use (see Propositions 3.3 and 3.4(a) in [21]).

**Proposition 1.2.** *Let  $E, K : [a, b) \rightrightarrows \mathbb{R}^n$  be measurable multifunctions with compact values such that  $E(t) \cap K(t) \neq \emptyset$  on  $[a, b)$ . Then  $E \cap K$  is measurable.  $\square$*



### 1.2.1 Standing Hypotheses

Unless it is otherwise indicated, we consider general  $F : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  with the following standing hypotheses (SH) in force throughout this work.

- For each  $(t, x) \in I \times \mathbb{R}^n$ ,  $F(t, x) \subset \mathbb{R}^n$  is nonempty, convex, and compact;
- There is a locally integrable function  $c(\cdot) : I \rightarrow \mathbb{R}$  so that  $\|v\| \leq c(t)(1 + \|x\|)$  for almost all  $t \in I$  and all  $(x, v) \in \text{gr } F(t, \cdot)$ ;
- $F(\cdot, \cdot)$  is almost upper semicontinuous on  $I \times \mathbb{R}^n$ ; that is, for every compact interval  $\tilde{I} \subset I$  and each  $\varepsilon > 0$ , there exists a closed set  $\mathcal{N}_\varepsilon \subseteq \tilde{I}$  with Lebesgue measure  $\mu(\tilde{I} \setminus \mathcal{N}_\varepsilon) < \varepsilon$ , and so that the restriction of  $F$  to  $\mathcal{N}_\varepsilon \times \mathbb{R}^n$  is upper semicontinuous.

As in the qualitative study of differential equations, the previous linear growth condition (second standing hypothesis above) is also used to establish boundedness of solutions for differential inclusions. This last goal is mostly achieved by application of the well-known *Gronwall's inequality*, which we provide here in a general version that is convenient for our purposes.

**Proposition 1.3. (*Gronwall's lemma*)** *Let  $y(\cdot)$  be a nonnegative absolutely continuous function defined on  $[a, b] \subset I$  satisfying*

$$\dot{y}(t) \leq k(t)y(t) + c(t), \quad \text{a.e., } t \in [a, b]$$

*for some nonnegative integrable functions  $k(\cdot)$  and  $c(\cdot)$  defined on  $[a, b]$ . Then, for all  $t \in [a, b]$  the following inequality holds*

$$y(t) \leq e^{\int_a^t k(s) ds} \left[ y(a) + \int_a^t c(s) ds \right].$$

□

For general compact valued multifunctions  $F : I \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ , the almost upper semicontinuity property (see (SH)) is also referred in the literature as the *weakened Scorza-Dragoni property* (see [48]), which turns out to be equivalent to the graph of the multifunction  $F$  being almost closed if we add convexity to the values of  $F$  and the linear growth condition (first and second assumptions above in (SH)): for every compact interval  $\tilde{I} \subset I$  and each  $\varepsilon > 0$ , there exists a closed set  $\mathcal{N}_\varepsilon \subseteq \tilde{I}$  with Lebesgue measure  $\mu(\tilde{I} \setminus \mathcal{N}_\varepsilon) < \varepsilon$ , such that  $\text{gr } F(\cdot, \cdot) := \{(t, x, v) : v \in F(t, x)\}$  is closed in  $\mathcal{N}_\varepsilon \times \mathbb{R}^m \times \mathbb{R}^n$ . The existence of solutions to (1.8) under (SH) is well known; it has been established in the following result taken from [21] (see also [9] for a treatment on Banach spaces).

**Theorem 1.4.** *Let  $K : I \rightrightarrows \mathbb{R}^n$  be an upper semicontinuous multifunction with closed values,  $M = \text{gr } K$ , and  $F : M \rightrightarrows \mathbb{R}^n$  satisfies (SH) (with closed valuedness required instead of compactness) on  $M$ , and for some  $\mathcal{N} \subseteq I$ , with  $\mu(\mathcal{N}) = 0$  the conditions*

$$\begin{cases} \{1\} \times F(t, x) \cap T_M^B(t, x) \neq \emptyset & \text{for } t \in I \setminus \mathcal{N}, x \in K(t) \\ (\{1\} \times \mathbb{R}^n) \cap T_M^B(t, x) \neq \emptyset & \text{for } t \in \mathcal{N}, x \in K(t) \end{cases}$$

*are satisfied. Then (1.8) has a solution for each  $x_0 \in K(a)$ .  $\square$*

The next proposition collects some properties of almost semicontinuous and measurable multifunctions. We point out that item (a) and its lower semicontinuous version (see statement below) are contained in Lemma 2.3.11 of [48], while item (b) can be found in page 24 of [21].

**Proposition 1.5.** *Let  $F : [a, b] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  and  $G : [a, b] \rightrightarrows \mathbb{R}^n$  be multifunctions such that  $F$  satisfies (SH) and  $G$  is measurable with closed values. Let  $u : [a, b] \rightarrow \mathbb{R}^n$  be continuous. Then the following assertions hold:*

(a) *The multifunction  $F(\cdot, u(\cdot))$  is measurable.*

(b) The multifunction  $G$  has the *Lusin property*, that is, there exists a collection  $\{J_r\}_{r=1}^{\infty}$  of pairwise disjoint compact sets such that  $\bigcup_{r=1}^{\infty} J_r \subset [a, b]$  has full measure in  $[a, b]$ , and the restriction of  $G(\cdot)$  to each  $J_r$  is continuous.

Moreover, assertion (a) also holds if we replace the almost upper semicontinuity assumption in (SH) by almost lower semicontinuity (see definition below).  $\square$

Selection procedures arise frequently in existence theory, and lower semicontinuity is perhaps the most common structural hypothesis assumed on multifunctions to guarantee regularity properties of the selections built upon it. The following result, which belongs to the folklore of set-valued theory, corroborates the last assertion in its most general setting (see [3] for its proof).

**Theorem 1.6. (Michael)** *Let  $X$  be a metric space,  $Y$  a Banach space, and  $x_0 \in X$ . Let  $G : X \rightrightarrows Y$  be lower semicontinuous with closed and convex values. Then for every  $y_0 \in G(x_0)$  there exists  $g : X \rightarrow Y$ , a continuous selection for  $G$ , such that  $g(x_0) = y_0$ .*  $\square$

A “more affordable” setting for existence of selectors is perhaps the main feature of the celebrated *Measurable Selection Theorem* which we state next (see page 111 of [11]).

**Theorem 1.7. (Castaing-Valadier)** *Let  $E : A \rightrightarrows \mathbb{R}^n$  be measurable, closed, and nonempty on  $A$ . Then there exists a measurable function  $g : A \rightarrow \mathbb{R}^m$  such that  $g(x) \in E(x)$  for all  $x \in A$ .*  $\square$

For nonautonomous systems the following time-dependent version of the lower semicontinuity property is also of natural interest, and we will have in fact opportunity to invoke it when establishing some of our results in chapter 3. The compact valued multifunction  $G : I \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  is called *almost lower semicontinuous* (or it is said to possess the *weak Scorza-Draconi property*, according to [48]) if for every

compact interval  $\tilde{I} \subset I$  and each  $\varepsilon > 0$ , there exists a closed set  $\mathcal{N}_\varepsilon \subseteq \tilde{I}$  with Lebesgue measure  $\mu(\tilde{I} \setminus \mathcal{N}_\varepsilon) < \varepsilon$ , and so that the restriction of  $G$  to  $\mathcal{N}_\varepsilon \times \mathbb{R}^m$  is lower semicontinuous. Due to the necessity of using continuous selections in chapter 3, the following single-valued version of the almost semicontinuity property is also considered: A function  $g : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}$  is said *almost lower semicontinuous* if for every  $\varepsilon > 0$  there exists a compact  $I_\varepsilon \subset [a, b]$ , with Lebesgue measure  $\mu([a, b] \setminus I_\varepsilon) < \varepsilon$ , and such that  $g$  restricted to  $I_\varepsilon \times \mathbb{R}^n$  is lower semicontinuous. Although we will rarely state it, we say  $g$  is *almost upper semicontinuous* if naturally  $-g$  is almost lower semicontinuous. Finally, a function  $f : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}$  is *almost continuous* if it is both almost lower and almost upper semicontinuous.

In the study of almost semicontinuity properties the following notion of points of density plays an important role (see page 274 of [36]): Let  $\mathcal{M}$  be a Lebesgue measurable subset of  $\mathbb{R}$ . A point  $t \in \mathbb{R}$  is called a *point of density* of  $\mathcal{M}$  if

$$\lim_{h \downarrow 0} \frac{\mu(\mathcal{M} \cap [t - h, t + h])}{2h} = 1. \quad (1.10)$$

The set of all points of density  $\tilde{\mathcal{M}}$  of  $\mathcal{M}$  has full measure in  $\mathcal{M}$ , that is  $\mu(\mathcal{M} \setminus \tilde{\mathcal{M}}) = 0$ . Moreover, the equality  $\text{cl}(\tilde{\mathcal{M}}) = \text{cl}(\tilde{\mathcal{M}} \setminus \mathcal{V})$  holds for any set of null measure  $\mathcal{V} \subset \mathbb{R}$ , and when  $\mathcal{M}$  is closed it follows that  $\tilde{\mathcal{M}} \subset \mathcal{M}$ .

The following statement is a particular case of a result due to Tolstonogov [49] (see also Theorems 2.3.2 and 2.3.3 of [48], and Remark 2.3.4 of the same reference). It represents a characterization of the previous almost semicontinuity properties in terms of joint measurability. As usual,  $\mathcal{L} \times \mathcal{B}$  denotes the smallest  $\sigma$ -field containing the products of Lebesgue measurable subsets of  $I$  and Borel measurable subsets of  $\mathbb{R}^n$ .

**Theorem 1.8.** *Assume  $E : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is compact valued, and upper (lower) semicontinuous in  $x$  for all  $t \in I$ . Then  $E$  is almost upper (respectively almost*

lower) semicontinuous if and only if  $E$  is  $\mathcal{L} \times \mathcal{B}$  measurable. In particular, the sufficiency holds if the upper (lower) semicontinuity property in the state variable is satisfied for almost all  $t \in I$ .  $\square$

### 1.3 Hamiltonians

Both the *maximized* and *minimized Hamiltonians* play crucial roles in our analysis, and recall they are defined respectively by

$$H_F(t, x, \zeta) := \sup\{\langle v, \zeta \rangle : v \in F(t, x)\}, \quad (1.11)$$

$$h_F(t, x, \zeta) := \inf\{\langle v, \zeta \rangle : v \in F(t, x)\}. \quad (1.12)$$

If the system (1.8) is autonomous, the  $t$  will be dropped as an independent variable, and if it is clear which multifunction is being considered, the subscript  $F$  will also be omitted. Among the properties of the Hamiltonians, there are some of them which we need to recall due to their prominent use in this work. We list them in the following proposition whose proof is left to the reader as an exercise (see page 188 in [16] for an autonomous version of some of these properties).

**Proposition 1.9.** *Assume  $F$  and  $G$  satisfy (SH). Then the following properties hold:*

- (a) *For all  $(t, x, p)$  we have  $h_F(t, x, p) = -H_F(t, x, -p)$ .*
- (b)  *$h_F$  is almost lower semicontinuous in  $(t, x, p)$ , concave, and continuous in  $p$ .*
- (c)  *$H_F$  is almost upper semicontinuous in  $(t, x, p)$ , convex, and continuous in  $p$ .*
- (d) *If  $\tau : I \rightarrow \mathbb{R}$  and  $u : I \rightarrow \mathbb{R}^n$  are measurable, then  $h_F(\tau(\cdot), u(\cdot), p)$  is measurable.*
- (e)  *$h_{F+G} = h_F + h_G$  and  $H_{F+G} = H_F + H_G$ .*

*Moreover, if  $F$  is almost continuous then  $h_F$  and  $H_F$  are also almost continuous.*  $\square$

### 1.3.1 The Main Structural Hypotheses

The next definitions are related to the structural behavior of multifunctions. They will be of importance not only when designing selection feedbacks for  $F$  in chapter 2, but also in future chapters, when we cover the lack of continuous selections by means of a subsidiary invariance principle. The first definition is classic, and has been of relevance in control theory. A multifunction  $F$  is *Lipschitz* if there exists a locally integrable  $k(\cdot)$  defined on  $I$  such that for all  $x, y \in \mathbb{R}^n$ , the following inclusion holds

$$F(t, y) \subseteq F(t, x) + k(t)\|y - x\|B_n, \quad \text{a.e., } t \in I. \quad (1.13)$$

By adding convexity to the values of the multifunction  $F$ , it is a simple exercise to see that the previous condition is equivalent to the upper Hamiltonian of  $F$  being Lipschitz: There is a locally integrable function  $k(\cdot)$  defined on  $I$  such that

$$|H_F(t, x, \zeta) - H_F(t, y, \zeta)| \leq k(t)\|\zeta\| \|x - y\| \quad (1.14)$$

for all  $x, y \in \mathbb{R}^n$ , all  $\zeta \in \mathbb{R}^n$ , and almost all  $t \in I$ . In order to establish the main results of the present work we will assume the *dissipative Lipschitz* (DL) condition, which first appeared in [24] under the name of *one-sided Lipschitz*: there exists a locally integrable function  $k(\cdot)$  on  $I$  such that for all  $x, y \in \mathbb{R}^n$ , and  $u \in F(t, x)$  there is  $v \in F(t, y)$  satisfying

$$\langle x - y, u - v \rangle \leq k(t)\|x - y\|^2, \quad (1.15)$$

for almost all  $t \in I$ . In Hamiltonian terms, and again assuming the convex-valuedness of  $F$ , the (DL) property is equivalent to: There exists a locally integrable function  $k(\cdot)$  on  $I$  such that

$$H_F(t, x, x - y) - H_F(t, y, x - y) \leq k(t) \|x - y\|^2 \quad (1.16)$$

for all  $x, y \in \mathbb{R}^n$  and almost all  $t \in I$ . It is clear that (1.13) implies (1.15), and the latter is seen to be strictly weaker since the multifunction (defined on  $\mathbb{R}$ )

$$F_1(x) := \begin{cases} -1 & \text{if } x > 0 \\ [-1, 1] & \text{if } x = 0 \\ 1 & \text{if } x < 0, \end{cases} \quad (1.17)$$

satisfies the dissipative Lipschitz condition but not the Lipschitz one. Another simple example is  $F_2(x) = \{-\operatorname{sgn}(x)\sqrt{|x|}\}$ . Notice that both multifunctions  $F_1$  and  $F_2$  satisfy (SH) and (1.16) with  $k(t) \equiv 0$ . The dissipative Lipschitz notion arises as a generalization of the concept of *Dissipativity*, which is associated to phenomena involving rigid bodies that stick to each other, and whose contacts are interrupted by slip phases, like a pendulum with *dry* or *Coulomb friction* (see [1, 21, 22]). A multifunction  $D$  is *dissipative* if

$$\langle u - v, x - y \rangle \leq 0, \quad \text{for all } (x, u), (y, v) \in \operatorname{gr} D, \text{ a.e., } t \in I. \quad (1.18)$$

In the language of Hamiltonians, the latter condition is equivalent to

$$H_D(t, y, y - x) - h_D(t, x, y - x) \leq 0, \quad (1.19)$$

for all  $(x, u), (y, v) \in \operatorname{gr} D$  and almost all  $t \in I$ . Conjugates of dissipative and dissipative Lipschitz multifunctions also join the cast in our work, and they are simply defined by changing the polarity in (1.18) and (1.15): A multifunction  $M : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is *monotone* (*monotone Lipschitz*) if the multifunction  $-M$  is dissipative (respectively dissipative Lipschitz).

## 1.4 Compactness of Trajectories

The following technical result is an almost upper semicontinuous version of theorem 3.5.24 of [16]. It concerns to a multifunction  $E : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  satisfying (SH)

for which the linear growth property has been replaced with the following global boundedness condition: there exists  $M > 0$  such that

$$\sup\{\|v\| : v \in E(t, x)\} \leq M \quad \text{for all } x \in \mathbb{R}^n, \text{ and a.e. } t \in I.$$

**Theorem 1.10. (A weak sequential compactness theorem)** *Assume  $I = [a, b]$  and let  $\{v_i\}$  be a sequence in  $L_n^2(I)$  such that*

$$v_i(t) \in E(t, u_i(t)) + r_i(t)B_n \quad \text{a.e., } t \in I, \quad (1.20)$$

where the sequence of measurable functions  $u_i(\cdot)$  converges a.e. to  $u_0(t)$ , and the nonnegative measurable functions  $\{r_i\}$  converge to 0 in  $L_1^2(I)$ . Then there exists a subsequence  $\{v_{i_j}\}$  of  $\{v_i\}$  which converges weakly in  $L_n^2(I)$  to a limit  $v_0(\cdot)$  that satisfies

$$v_0(t) \in E(t, u_0(t)) \quad \text{a.e., } t \in I.$$

**Proof.** The sequence  $\{r_i\}$  converges to 0 in  $L_1^2(I)$ , and then it is bounded in  $L_1^2(I)$ . Using the global boundedness of  $E$  and condition (1.20) we obtain

$$\|v_i(t)\|^2 \leq (M + |r_i(t)|)^2 \quad \text{a.e., } t \in I,$$

which readily implies the boundedness of the sequence  $\{v_i(\cdot)\}$  in  $L_n^2(I)$ . We invoke weak compactness in  $L_n^2(I)$  to guarantee the existence of a subsequence  $\{v_{i_j}\}$  of  $\{v_i\}$  that converges weakly in  $L_n^2(I)$  to some  $v_0(\cdot)$ . To obtain the result, we need to show that actually  $v_0(t) \in E(t, u_0(t))$  a.e.,  $t \in I$ . In fact, for each  $i$  and almost all  $t \in I$  condition (1.20) implies the existence of  $w_i(t) \in E(t, u_i(t))$  and  $z_i(t) \in B$  such that  $v_i(t) = w_i(t) - r_i(t)z_i(t)$ . Since  $E(t, u_i(t))$  is convex, for each  $p \in \mathbb{R}^n$  the separation theorem leads to  $h(t, u_i(t), p) \leq \langle w_i(t), p \rangle$  for all  $i$ , and a.e.  $t \in I$ . Therefore,

$$h(t, u_i(t), p) \leq \langle v_i(t), p \rangle + r_i(t)\langle z_i(t), p \rangle \leq \langle v_i(t), p \rangle + r_i(t)\|p\|, \quad \text{a.e., } t \in I.$$



Since the multifunction  $E$  is almost upper semicontinuous, the function  $(t, u) \mapsto h(t, u, p)$  is almost lower semicontinuous for each  $p \in \mathbb{R}^n$ , and hence the function  $t \mapsto h(t, u_i(t), p)$  turns out to be measurable for each  $p \in \mathbb{R}^n$  (see proposition 1.9 (c)). Now, let  $\varepsilon_k > 0$ ,  $\varepsilon_k \downarrow 0$ , and fix  $k$ . The almost lower semicontinuity of  $h(\cdot, \cdot, p)$  implies the existence of a closed set  $\mathcal{I}_k \subseteq I$  with Lebesgue measure  $\mu(I \setminus \mathcal{I}_k) < \varepsilon_k$ , and so that the restriction of  $h(\cdot, \cdot, p)$  to  $\mathcal{I}_k \times \mathbb{R}^n$  is lower semicontinuous. Let  $A$  be any measurable subset of  $I$ . We have

$$\int_{A \cap \mathcal{I}_k} \{\langle v_{i_j}(t), p \rangle + \|p\| r_{i_j}(t) - h(t, u_{i_j}(t), p)\} dt \geq 0, \quad (1.21)$$

since the integrand is nonnegative almost everywhere. As  $j \rightarrow \infty$ , we know

$$\int_{A \cap \mathcal{I}_k} \langle v_{i_j}(t), p \rangle dt \rightarrow \int_{A \cap \mathcal{I}_k} \langle v_0(t), p \rangle dt$$

by weak convergence, and of course  $\int_{A \cap \mathcal{I}_k} r_{i_j}(t) dt \rightarrow 0$ . We recall that Fatou's lemma yields

$$\int_{A \cap \mathcal{I}_k} \liminf_{j \rightarrow \infty} h(t, u_{i_j}(t), p) dt \leq \liminf_{j \rightarrow \infty} \int_{A \cap \mathcal{I}_k} h(t, u_{i_j}(t), p) dt. \quad (1.22)$$

The previous statements lead to the following relations

$$\begin{aligned} \int_{A \cap \mathcal{I}_k} \langle v_0(t), p \rangle dt & - \int_{A \cap \mathcal{I}_k} \liminf_{j \rightarrow \infty} h(t, u_{i_j}(t), p) dt \\ & \geq \int_{A \cap \mathcal{I}_k} \langle v_0(t), p \rangle dt - \liminf_{j \rightarrow \infty} \int_{A \cap \mathcal{I}_k} h(t, u_{i_j}(t), p) dt \\ & \geq \int_{A \cap \mathcal{I}_k} \langle v_0(t), p \rangle dt - \limsup_{j \rightarrow \infty} \int_{A \cap \mathcal{I}_k} h(t, u_{i_j}(t), p) dt \\ & = \int_{A \cap \mathcal{I}_k} \langle v_0(t), p \rangle dt + \liminf_{j \rightarrow \infty} \left( - \int_{A \cap \mathcal{I}_k} h(t, u_{i_j}(t), p) dt \right) \\ & = \liminf_{j \rightarrow \infty} \left( \int_{A \cap \mathcal{I}_k} \langle v_{i_j}(t), p \rangle dt + \|p\| \int_{A \cap \mathcal{I}_k} r_{i_j}(t) dt \right. \\ & \quad \left. - \int_{A \cap \mathcal{I}_k} h(t, u_{i_j}(t), p) dt \right) \geq 0. \end{aligned} \quad (1.23)$$

Let  $\tilde{\mathcal{I}}_k$  denote the set of density points of  $\mathcal{I}_k$ . The lower semicontinuity of  $h(\cdot, \cdot, p)$  on  $\mathcal{I}_k \times \mathbb{R}^k$  implies that for  $t \in A \cap \tilde{\mathcal{I}}_k$  we have

$$h(t, u_0(t), p) \leq \liminf_{\tau \rightarrow t, y \rightarrow u_0(t)} h(\tau, y, p) \leq \liminf_{j \rightarrow \infty} h(t, u_{i_j}(t), p),$$

where the middle liminf in the previous inequalities is taken all over the sequences  $(\tau, y) \rightarrow (t, u_0(t))$  in  $\mathcal{I}_k \times \mathbb{R}^n$ . Using the previous inequalities in (1.23), and recalling that  $\mu(\tilde{\mathcal{I}}_k) = \mu(\mathcal{I}_k)$ , we obtain

$$\int_{A \cap \mathcal{I}_k} \{\langle v_0(t), p \rangle - h(t, u_0(t), p)\} dt \geq 0.$$

Since  $A$  is arbitrary in  $I$ , it follows that

$$\langle v_0(t), p \rangle \geq h(t, u_0(t), p) \quad \text{a.e., } t \in \mathcal{I}_k.$$

The fact of  $\varepsilon_k > 0$  being arbitrary implies the previous condition holds for a.e.  $t \in \bigcup_{k \geq 1} \mathcal{I}_k$ , and hence it holds for a.e.  $t \in I$  since  $\mu(\bigcup_{k \geq 1} \mathcal{I}_k) = \mu(I)$ . Now let  $\{p_i\}$  be a countable dense subset of  $\mathbb{R}^n$ . Then for each  $i$  there is an exceptional set  $\Omega_i \in I$  with null measure such that

$$\langle v_0(t), p_i \rangle \geq h(t, u_0(t), p_i) \quad \text{for all } t \in I \setminus \Omega_i. \quad (1.24)$$

Let's define  $\Omega := \bigcup_i \Omega_i$ . It is obvious that  $\Omega$  has null measure. Therefore, for any  $t \in I \setminus \Omega$  the continuity of  $h(t, u_0(t), \cdot)$  and the density of  $\{p_i\}$  imply (1.24) holds for all  $p \in \mathbb{R}^n$ , which is equivalent (again, by the separation theorem) to

$$v_0(t) \in E(t, u_0(t)) \quad \text{for all } t \in I \setminus \Omega,$$

and hence the proof is complete.  $\square$

**Corollary 1.11.** *Assume  $F$  satisfies (SH). Let  $\{x_i\}$  be a sequence of arcs on  $I$  such that the set  $\{x_i(a)\}$  is bounded, and satisfying*

$$\dot{x}_i(t) \in F(t, x_i(t) + y_i(t)) + r_i(t)B \quad \text{a.e., } t \in I,$$

where  $\{y_i\}$  and  $\{r_i\}$  are sequences of measurable functions on  $I$  such that  $y_i$  converges to 0 in  $L_n^2(I)$  and  $r_i \geq 0$  converges to 0 in  $L_1^2(I)$ . Then there exists a subsequence of  $\{x_i\}$ , which converges uniformly to an arc  $x$  that is a trajectory of  $F$ , and whose derivatives converge weakly to  $\dot{x}$ .

**Proof.** The proof is identical to that of Theorem 4.1.11 of [16], the only difference being the application of Theorem 1.10 (above) instead of Theorem 3.5.24 of [16].

□

We end this chapter by recalling a basic concept in control theory regarding trajectories. The *reachable set* from  $(a, x_0)$  at time  $a + T$ , is defined as

$$\mathcal{R}_F^{(T)}(a, x_0) := \{x(a + T) : x(\cdot) \text{ solves (1.8)}\},$$

and the notation  $\mathcal{R}_G^{[a, a+T]}(a, x_0)$  signifies the set of all points reachable from  $x_0$  at a time between  $a$  and  $a + T$ . Some of the elementary properties of the reachable set are given in the next lemma, which appeared in [51].

**Proposition 1.12.** *For each compact set  $K \subset \mathbb{R}^n$ , and  $\varepsilon > 0$ , and each  $a \in I$ , there exists  $T > 0$  such that  $[a, a + T] \subset I$ , and for which*

$$\mathcal{R}_F^{[a, a+T]}(a, K) := \bigcup_{x \in K} \mathcal{R}_F^{[a, a+T]}(a, x) \subset K + \varepsilon B.$$

Moreover, if  $x(\cdot)$  is a trajectory of  $E$  on  $[a, a + T]$  with initial point  $x(a) = x_0$  and  $T < \infty$ , and satisfies

$$\liminf_{t \uparrow T} \|x(a + t)\| < \infty,$$

then the limit of  $x(a + t)$  exists as  $t \uparrow T$ .

□

## Chapter 2

# Strong Invariance: A Feedback Selection Approach

In this chapter we present a criterion for strong invariance for a non-Lipschitz autonomous system that enjoys a decomposition of particular interest. The referred system is modeled as the sum of a dissipative and a Lipschitz set-valued map, with the dissipative piece being only upper semicontinuous. We discuss in more details the results contained in joint work [42] with P. Wolenski, which are mainly based on the construction of trajectories via feedback selections. By using the special character of dissipative multifunctions, we show how the strong invariance property can be characterized in terms of a mixed-type Hamiltonian inequality. Our general reference for this chapter is the book of Clarke *et al.* “*Nonsmooth analysis and control theory*” [16] from which the main ideas were conceived.

### 2.1 The Setting

Throughout this chapter we will consider autonomous systems with control parameters of the form

$$\begin{aligned} \dot{x}(t) &\in F(x(t)) \quad \text{a.e., } t \in I \\ x(0) &= x_0, \end{aligned} \tag{2.1}$$

where  $I = [0, \infty)$ , and the multifunction  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is a given set-valued map that enjoys a decomposition of the form  $F = D + G$ , where the corresponding pieces  $D$  and  $G$  satisfy the following properties:

- $D(\cdot)$  and  $G(\cdot)$  have nonempty, compact, and convex values;
- $D(\cdot)$  is dissipative and upper semicontinuous on  $\mathbb{R}^n$ ;
- $G(\cdot)$  is Lipschitz.

The initial condition  $x_0$  is a point taken from a nonempty closed set  $S \subseteq \mathbb{R}^n$  which is also of significant importance in our discussion. We also assume the following linear growth condition is satisfied on  $F$ : There exists a positive constant  $c \in \mathbb{R}$  such that

$$\sup\{\|v\| : v \in F(x)\} \leq c(1 + \|x\|), \text{ for all } x \in \mathbb{R}^n.$$

It is clear that the multifunction  $F = D + G$  satisfies (SH) with the almost upper semicontinuity property replaced by simple upper semicontinuity, and hence the existence of solutions to (2.1) is guaranteed according to Theorem 1.4. As it was mentioned in the introduction, we will drop the time variable  $t$  out of the upper and lower Hamiltonians, and therefore these are now respectively defined as

$$H(x, p) = \sup\{\langle p, v \rangle : v \in F(x)\}, \quad (2.2)$$

and

$$h(x, p) = \inf\{\langle p, v \rangle : v \in F(x)\}. \quad (2.3)$$

## 2.2 Euler Solutions

Suppose that we are interested in studying the flow of the Cauchy initial-value problem

$$\dot{x}(t) = f(t, x(t)), \quad x(a) = x_0, \quad (2.4)$$

where  $f : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a given function, and we want to determine whether the resulting trajectory  $x(t)$  approaches the given closed set  $S \subseteq \mathbb{R}^n$ . A natural way of testing if this is the case, is to choose for a given  $t \in [a, b]$  a point  $s \in \text{proj}_S(x(t))$ , and check if the product  $\langle f(t, x(t)), x(t) - s \rangle$  is negative. If the last holds,  $\dot{x}(t)$  will “point toward  $S$ ”. The previous technique is known as *proximal aiming* and it has proven to be very useful in consolidating Hamiltonian conditions for weak invariance. We will take advantage of it in the same way as in [16] by means of the

Euler approximations approach, which we now proceed to recall. Let us consider again the Cauchy problem (2.4), and a not necessarily uniform partition

$$\pi := \{\tau_0, \tau_1, \dots, \tau_{N-1}, \tau_N\}$$

of the interval  $[a, b]$ , where  $\tau_0 = a$  and  $\tau_N = b$ . Suppose that an “approximate” trajectory  $x_{j-1}(\cdot)$  has been defined on  $[\tau_{j-1}, \tau_j]$ . We define  $x_j(\cdot)$  in the following way: we first set the  $j$ -th node to be  $x_j = x_{j-1}(\tau_j)$  and choose

$$\begin{aligned} x_j(t) &:= x_j + \int_{\tau_j}^t f(\tau_j, x_j) dr \\ &= x_j + f(\tau_j, x_j)(t - \tau_j) \end{aligned} \tag{2.5}$$

for all  $t \in [\tau_j, \tau_{j+1}]$ . Under the previous scheme, we obtain the following piecewise defined *Euler polygonal arc*  $x^\pi(\cdot)$  on  $[a, b]$ :  $x^\pi(t) := x_j(t)$  if  $t \in [\tau_j, \tau_{j+1}]$ . Now we consider the diameter of the partition  $\pi$

$$\mathcal{D}(\pi) := \max\{\tau_j - \tau_{j-1} : 1 \leq j \leq N\}.$$

An *Euler solution* to the Cauchy problem (2.4) is any absolutely continuous arc that is the uniform limit of Euler polygonal arcs  $x_{\pi_i}$  that correspond as above to some sequence of partitions  $\pi_i$  with the property  $\mathcal{D}(\pi_i) \rightarrow 0$  as  $i \rightarrow \infty$ . Regardless of the regularity of the function  $f$  that defines (2.4), the existence of one of such Euler solutions is guaranteed by the following result (see page 183 of [16] for proof).

**Theorem 2.1.** *Suppose the function  $f : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfies the linear growth condition*

$$\|f(t, x)\| \leq c(1 + \|x\|), \text{ for all } x \in \mathbb{R}^n, \text{ a.e., } t \in [a, b],$$

*and is otherwise arbitrary. Then:*

- (a) *At least one Euler solution  $x(\cdot)$  to the Cauchy problem (2.4) exists on  $[a, b]$ , and any Euler solution is Lipschitz.*

(b) Any Euler solution for (2.4) satisfies

$$\|x(t) - x(a)\| \leq (t - a)e^{\gamma(t-a)}(c + \gamma\|x(a)\|), \quad a \leq t \leq b.$$

□

The next result confirms the heuristic observation made earlier about Euler solutions. It is given here in the case of a function  $f$  that satisfies a linear growth condition. However, the proof can be easily extended to the case when  $f$  is bounded on  $[a, b] \times \Omega$  for any bounded set  $\Omega \subset \mathbb{R}^n$ .

**Proposition 2.2. (*Proximal aiming*)** *Suppose the function  $f : [a, b] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfies the linear growth condition*

$$\|f(t, x(t))\| \leq c(1 + \|x\|), \text{ for all } x \in \mathbb{R}^n, \text{ a.e., } t \in [a, b],$$

and let  $x(\cdot)$  be an Euler arc for  $f$  on  $[a, b]$ . Let  $\Omega$  be an open set containing  $x(t)$  for all  $t \in [a, b]$ , and having the property that there is a continuous function  $\theta(\cdot, \cdot)$  such that for each  $(t, z) \in [a, b] \times \Omega$  there exists  $s \in \text{pro}_S(z)$  satisfying

$$\langle f(t, z), z - s \rangle \leq \theta(t, z)d_S(z).$$

Then,

$$\frac{d}{dt}d_S(x(t)) \leq \theta(t, x(t)), \text{ a.e.}$$

on any interval on which  $d_S(x(t)) > 0$ , or on any interval in which  $\theta(t, x(t)) \geq 0$ .

**Proof.** Let  $x(\cdot)$  be an Euler arc on  $[a, b]$  for  $f$ , and let  $x_\pi$  be an element in the sequence of polygonal arcs that converges uniformly to  $x(\cdot)$ . Let  $t \in [a, b]$ ; then there is  $i$ , with  $0 \leq i < N_\pi$ , such that  $\tau_i \leq t < \tau_{i+1}$ . Let us define  $\bar{x}_\pi := x_\pi|_{[t, b]}$ . Notice that the partition for  $\bar{x}_\pi$  is  $[t, \tau_{i+1}], [\tau_{i+1}, \tau_{i+2}], \dots, [\tau_{N_\pi-1}, \tau_{N_\pi}]$ . Moreover, since  $x(t) \in \Omega$  for all  $t \in [a, b]$ , we can assume without loss of generality that

$\bar{x}_\pi(r) \in \Omega$  for all  $r \in [t, b]$  due to the uniform convergence of  $x_\pi$ . According to the hypothesis, for the nodes  $x_i = \bar{x}_\pi(t)$ ,  $x_j = \bar{x}_\pi(\tau_j)$ ,  $i < j \leq N_\pi$  there exist  $s_j \in \text{pro}_S(x_j)$  that satisfy  $\langle f(\tau_j, x_j), x_j - s_j \rangle \leq \theta(\tau_j, x_j) d_S(x_j)$ , for all  $j$  such that  $i \leq j \leq N_\pi$ . Let  $m$  be the uniform bound for  $\|\dot{x}_\pi\|_\infty$  obtained in the proof of Theorem 2.1 (see again page 183 of [16]). Then we can estimate

$$\begin{aligned}
d_S^2(x_{i+1}) &\leq \|x_{i+1} - s_i\|^2 \\
&= \|x_{i+1} - x_i\|^2 + \|x_i - s_i\|^2 + 2\langle x_{i+1} - x_i, x_i - s_i \rangle \\
&\leq m^2(\tau_{i+1} - t)^2 + d_S^2(x_i) + 2\left\langle \int_t^{\tau_{i+1}} \dot{\bar{x}}_\pi(r) dr, x_i - s_i \right\rangle \\
&= m^2(\tau_{i+1} - t)^2 + d_S^2(x_i) + 2 \int_t^{\tau_{i+1}} \langle \dot{\bar{x}}_\pi(r), x_i - s_i \rangle dr \\
&= m^2(\tau_{i+1} - t)^2 + d_S^2(x_i) + 2 \int_t^{\tau_{i+1}} \langle f(t, \bar{x}_\pi(t)), x_i - s_i \rangle dr \\
&\leq m^2(\tau_{i+1} - t)^2 + d_S^2(x_i) + 2 \int_t^{\tau_{i+1}} \theta(t, \bar{x}_\pi(t)) d_S(\bar{x}_\pi(t)) dr,
\end{aligned}$$

where we have used the linearity of the inner product, the definition of  $\bar{x}_\pi$ , and the hypothesis on  $\Omega$  and  $f$  respectively. The same estimate can be obtained at each of the remaining nodes:

$$d_S^2(x_j) \leq m^2(\tau_j - \tau_{j-1})^2 + d_S^2(x_{j-1}) + 2 \int_{\tau_{j-1}}^{\tau_j} \theta(\tau_{j-1}, \bar{x}_\pi(\tau_{j-1})) d_S(\bar{x}_\pi(\tau_{j-1})) dr$$

for  $i \leq j \leq N_\pi$ . The previous sets of inequalities yields

$$\begin{aligned}
d_S^2(x_j) &\leq m^2(\tau_j - \tau_{j-1})^2 + d_S^2(x_{j-1}) + 2 \int_{\tau_{j-1}}^{\tau_j} \theta(\tau_{j-1}, \bar{x}_\pi(\tau_{j-1})) d_S(\bar{x}_\pi(\tau_{j-1})) dr \\
&\leq m^2(\tau_j - \tau_{j-1})^2 + k^2(\tau_{j-1} - \tau_{j-2})^2 + d_S^2(x_{j-2}) \\
&\quad + 2 \int_{\tau_{j-2}}^{\tau_{j-1}} \theta(\tau_{j-2}, \bar{x}_\pi(\tau_{j-2})) d_S(\bar{x}_\pi(\tau_{j-2})) dr \\
&\quad + 2 \int_{\tau_{j-1}}^{\tau_j} \theta(\tau_{j-1}, \bar{x}_\pi(\tau_{j-1})) d_S(\bar{x}_\pi(\tau_{j-1})) dr,
\end{aligned}$$



which then implies

$$\begin{aligned}
d_S^2(x_j) &\leq m^2 \sum_{l=1}^{j-1} (\tau_{l+1} - \tau_l)^2 + d_S^2(x_i) + 2 \sum_{l=1}^{j-1} \int_{\tau_l}^{\tau_{l+1}} \theta(\tau_l, \bar{x}_\pi(\tau_l)) d_S(\bar{x}_\pi(\tau_l)) dr \\
&\leq m^2 \mathcal{D}(\pi) \sum_{l=1}^{j-1} (\tau_{l+1} - \tau_l) + d_S^2(x_i) + 2 \sum_{l=1}^{j-1} \int_{\tau_l}^{\tau_{l+1}} \theta(\tau_l, \bar{x}_\pi(\tau_l)) d_S(\bar{x}_\pi(\tau_l)) dr \\
&= m^2 \mathcal{D}(\pi)(b - t) + d_S^2(x_i) + 2 \sum_{l=1}^{j-1} \int_{\tau_l}^{\tau_{l+1}} \theta(\tau_l, \bar{x}_\pi(\tau_l)) d_S(\bar{x}_\pi(\tau_l)) dr \quad (2.6)
\end{aligned}$$

for  $i \leq j \leq N_\pi$ . Now let us consider the sequence  $\bar{x}_{\pi_i}$  of polygonal arcs that are the restriction on  $[t, b]$  of the original sequence  $x_{\pi_i}$ . Notice that  $\bar{x}_{\pi_i}$  converges uniformly to  $x$  on  $[t, b]$  and  $\mathcal{D}(\pi_i) \rightarrow 0$  as  $i \rightarrow \infty$ . Moreover, (2.6) holds for every node, the constant  $m$  is the same for every  $\bar{x}_{\pi_i}$ , and  $d_S(\cdot)$  is a continuous function. Taking now the limit when  $j \rightarrow \infty$  in (2.6) and considering all the previous properties, we have that for each  $\tau$ , with  $a \leq t < \tau$ , the following inequality holds

$$d_S^2(x(\tau)) \leq d_S^2(x(t)) + 2 \int_t^\tau \theta(r, x(r)) d_S(x(r)) dr. \quad (2.7)$$

For the second half of the proof we define  $g(\tau) := \int_t^\tau \theta(r, x(r)) d_S(x(r)) dr$  on  $[t, b]$ .

For this function the fundamental theorem of calculus yields

$$\frac{d}{d\tau} g(\tau) = \theta(\tau, x(\tau)) d_S(x(\tau)) \quad \text{for all } \tau \in [t, b]. \quad (2.8)$$

On the other hand, if we integrate between  $\tau$  and  $\tau + h$  in (2.7), we obtain

$$\begin{aligned}
\frac{d}{d\tau} g(\tau) &= \lim_{h \rightarrow 0} \frac{\int_\tau^{\tau+h} \theta(r, x(r)) d_S(x(r)) dr}{h} \\
&\geq \frac{1}{2} \lim_{h \rightarrow 0} \frac{d_S^2(x(\tau + h)) - d_S^2(x(\tau))}{h} \\
&= d_S(x(\tau)) \frac{d}{d\tau} d_S(x(\tau)), \quad \text{a.e. } \tau \in [t, b]. \quad (2.9)
\end{aligned}$$

Comparing (2.8) with (2.9) we get

$$\theta(\tau, x(\tau)) d_S(x(\tau)) \geq d_S(x(\tau)) \frac{d}{d\tau} d_S(x(\tau)), \quad (2.10)$$

almost everywhere in  $[t, b]$ . Since  $t$  was taken arbitrarily in  $[a, b]$ , it follows that the previous inequality holds almost everywhere in  $[a, b]$ . If  $I$  is a subinterval of  $[a, b]$  in which  $d_S(x(\tau)) > 0$ , then this term can be canceled from both sides of (2.10). Whence  $\frac{d}{d\tau}d_S(x(\tau)) \leq \theta(\tau, x(\tau))$  a.e.  $\tau \in I$ . If  $\theta(\tau, x(\tau)) \geq 0$  and  $d_S(x(\tau)) = 0$ , then  $\frac{d}{d\tau}d_S(x(\tau)) = 0 = \frac{d}{d\tau}0 \leq \theta(\tau, x(\tau))$ , and then the proof is complete.  $\square$

We close this section with an autonomous version of Corollary 4.1.12 of [16], which is one of the main applications of the weak sequential compactness result established in chapter 1 (see Corollary 1.10). It confirms that, under (SH), trajectories can be calculated as Euler solutions for selections of  $F$ . We provide the proof for completeness.

**Proposition 2.3.** *Assume  $F$  is an autonomous multifunction satisfying (SH). Let  $f$  be any selection of  $F$ , and let  $x(\cdot)$  be an Euler solution to (2.4). Then  $x(\cdot)$  is a trajectory of  $F$  on  $[a, b]$ .*

**Proof.** Let  $f$  be an arbitrary selection of  $F$  and let  $x(\cdot)$  be any Euler solution to the associated Cauchy problem (2.4). Let also  $x_{\pi_i}(\cdot)$  be a sequence of Euler polygonal arcs with corresponding partitions  $\pi_i := \{\tau_0^i, \tau_1^i, \dots, \tau_{N-1}^i, \tau_N^i\}$ , and whose uniform limit is  $x(\cdot)$ . Let us fix  $i$  and let  $t \in (a, b)$  satisfying  $\tau_{j-1}^i < t < \tau_j^i$  for some  $j$ . We set  $x_{\pi_i}(\tau_j^i) := x_j^i$  and define  $y_i(t) := x_j^i - x_{\pi_i}(t) = x_{\pi_i}(\tau_j^i) - x_{\pi_i}(t)$ . Therefore, we can write

$$\dot{x}_{\pi_i}(t) = f(\tau_j^i, x_j^i) \in F(x_j^i) = F(x_{\pi_i}(t) + y_i(t)).$$

Notice that

$$\|y_i(t)\|_\infty \leq m \sup_{t \in [a, b]} |\tau_j^i - t| \leq m \mathcal{D}(\pi_i),$$

since the functions  $x_{\pi_i}(\cdot)$  are Lipschitz with the same Lipschitz rank  $m$  (see proof of Theorem 2.1 in page 183 of [16]). It follows that  $y_i(\cdot)$  is measurable, and since

$\mathcal{D}(\pi_i) \rightarrow 0$ , the sequence  $y_i(\cdot)$  converges to 0 uniformly. The fact that  $x(\cdot)$  is a trajectory of  $F$  follows now from application of Corollary 1.11.  $\square$

## 2.3 Invariance

This section is devoted to the main result of the chapter: a characterization of strongly invariant systems for the particular type of multifunction described in section 2.1. By using the character of the dissipativity property we are able to extend the ideas given in chapter 4 of [16] to produce aiming feedback selections that work under our setting. We first recall the notion of invariance and briefly mention some aspects on the evolution of our problem. The pair  $(S, F)$  is called *strongly invariant* provided that whenever  $x_0 \in S$ , every trajectory of the differential inclusion (2.1) also lies in  $S$  for all time. Strong invariance and its cousin *weak invariance* (which is the property that *at least one* trajectory lies in  $S$ ) play important roles in Hamilton-Jacobi theory because they are characterized by Hamilton-Jacobi inequalities. Strong invariance is characterized by

$$H(x, \zeta) \leq 0 \quad \forall x \in S, \zeta \in N_S^P(x), \quad (2.11)$$

where  $N_S^P(x)$  denotes the proximal normal cone of  $S$  at  $x \in S$ . The usual characterizations of strong invariance ([16], Theorem 4.3.8) require the data to be Lipschitz with respect to the Hausdorff metric, a property that under convex valuedness is equivalent to the existence of a constant  $k > 0$  (for technical simplicity, we use global formulations here) so that

$$|H(x, p) - H(y, p)| \leq k|p| |x - y| \quad \forall x, y, p \in \mathbb{R}^n. \quad (2.12)$$

A recent result by T. Donchev [26] characterizes strong invariance under the weaker assumptions of continuity plus the dissipative Lipschitz (DL) condition (a nonautonomous version is also included in [26]). The (global) definition of (DL) (cf.

(1.16)) for convex valued multifunctions is there exists  $k \geq 0$  so that

$$H(x, x - y) - H(y, x - y) \leq k |x - y|^2 \quad \forall x, y \in \mathbb{R}^n. \quad (2.13)$$

It is easy to show (2.12) implies (2.13), but the converse is not true in general, as can be seen by letting  $F(x) = \{-\text{sgn}(x)\sqrt{|x|}\}$ . This example, with  $S = \{0\}$ , is covered by the strong invariance result in [26] but not by the one in [16]. In the case of discontinuous  $F$ , condition (1.15) generalizes the notion of dissipativity; a multifunction  $D$  is dissipative if  $\langle u - v, x - y \rangle \leq 0$  for all  $(x, u)$  and  $(y, v)$  in the graph of  $D$ , which is stronger than the case of (DL) with  $k = 0$ . Dissipative multifunctions have been studied by several authors (e.g. see [3] for properties and [3, 22] for physical applications). Under the convexity assumption the dissipativity condition of the multimap  $D$  is equivalent to:

$$H(x, x - y) \leq h(y, x - y), \quad \text{for all } x, y \in \mathbb{R}^n. \quad (2.14)$$

The conclusion of the main result of this chapter (Theorem 2.7 below) will hold for multifunctions of the form  $F = D + G$ , where  $D$  is dissipative and  $G$  is Lipschitz. This setting does not cover the most general case of (DL) multimaps, since not every multifunction satisfying (1.16) has this form. However, the understanding of the invariance property for these particular systems gave us some ideas that were useful in the quest for a strong invariant Hamiltonian criterion for general (DL) multimaps. We now recall a traditional weak invariant criterion that we shall invoke in a moment (see Theorem 4.2.10 of [16]).

**Theorem 2.4.** *Assume  $F$  is any autonomous multifunction satisfying (SH). Then the following assertions are equivalent:*

- (a)  $F(x) \cap T_S^B(x) \neq \emptyset$  for all  $x \in S$ .
- (b)  $F(x) \cap \text{co}T_S^B(x) \neq \emptyset$  for all  $x \in S$ .

(c)  $h(x, \zeta) \leq 0$  for all  $x \in S$ , and all  $\zeta \in N_S^P(x)$ .

(d) The system  $(S, F)$  is weakly invariant.

□

The next result extends Theorem 4.3.4 of [16] to autonomous (DL) multifunctions satisfying (SH).

**Theorem 2.5.** *Suppose  $(S, F)$  is weakly invariant. Then there exists an autonomous feedback selection  $f$  of  $F$  for which any Euler solution  $x(\cdot)$  of  $\dot{x}(t) = f(x(t))$  with  $x(0) \in S$  satisfies  $x(t) \in S$  for all  $t \geq 0$ .*

**Proof.** The proof is a modification of that given in [16], the difference being that the (DL) assumption (2.13) is invoked instead of the Lipschitz one (2.12). In fact, for each  $x \in \mathbb{R}^n$ , let  $s_x \in \text{proj}_S(x)$ , and define  $\tilde{f}(x) := v_x$  where  $v_x \in F(s_x)$  minimizes  $\langle v, x - s_x \rangle$  over  $v \in F(s_x)$ . Next we choose  $u_x \in F(x)$  so that

$$\langle u_x - \tilde{f}(x), x - s_x \rangle \leq k \|x - s_x\|^2,$$

which exists by the (DL) property (see (1.15)) with  $y = s_x$ . We define our selection as  $f(x) := u_x$ . Let now  $x_0 \in S$  and  $x(\cdot)$  be an Euler solution on  $[a, b]$  from  $x_0$  and generated by  $f$ . We will show that  $x(t) \in S$  for all  $t \in [a, b]$ . For every  $x \in \mathbb{R}^n$  by definition of proximal normal cone we have  $x - s_x \in N_S^P(s_x)$ . Since  $(S, F)$  is weakly invariant, Theorem 2.4 implies  $h(x, x - s_x) \leq 0$ . Using this last we calculate

$$\begin{aligned} \langle f(x), x - s \rangle &= \langle \tilde{f}(x), x - s_x \rangle + \langle f(x) - \tilde{f}(x), x - s_x \rangle \\ &= h(x, x - s_x) + \langle f(x) - \tilde{f}(x), x - s_x \rangle \\ &\leq \langle u_x - \tilde{f}(x), x - s_x \rangle \\ &\leq k \|x - s_x\|^2 \\ &= k \|x - s_x\| d_S(x), \end{aligned}$$

for all  $x \in \mathbb{R}^n$ , which is an aiming condition with  $\theta(t, x) := k\|x - s_x\|$ . Therefore, application of Proposition 2.2 implies that for  $x(\cdot)$  the following estimate holds

$$\frac{d}{dt}d_S(x(t)) \leq k d_S(x(t)), \text{ a.e., } t \in [a, b].$$

Using Gronwall's inequality we obtain

$$d_S(x(t)) \leq k d_S(x(a))e^{k(t-a)} = 0$$

since  $x(a) = x_0 \in S$ . Thus,  $x(t) \in S$  for all  $t \in [a, b]$ .

**Corollary 2.6.** *Euler solutions obtained from selections of  $F$  coincide with the absolutely continuous trajectories of  $(DI)$ .*

**Proof.** The sufficiency has been shown in Proposition 2.3. In order to prove the necessity, let  $\bar{x}$  be a trajectory of  $F$  and define  $\tilde{S} := \{(t, \bar{x}(t)) : t \geq a\}$ . It is a simple exercise to show that  $\tilde{S}$  is a closed subset of  $\mathbb{R}^{n+1}$ . Let us consider the augmented multifunction

$$(\{1\} \times F)(t, x) := \{1\} \times F(x), \quad \text{for all } (t, x) \in [a, \infty) \times \mathbb{R}^n.$$

Obviously  $\{1\} \times F$  satisfies (SH) with the almost upper semicontinuity replaced by joint upper semicontinuity, and since  $F$  is (DL) with constant  $k$  it follows that  $\{1\} \times F$  also satisfies the (DL) property with the same constant  $k$ . Moreover, the system  $(\tilde{S}, \{1\} \times F)$  is weakly invariant. Therefore, by Theorem 2.5 there exists a selection of  $\{1\} \times F$  under which  $\tilde{S}$  is invariant. This selection is necessarily of the form  $\{1\} \times f$ , where  $f$  is a selection of  $F$ . If  $\tilde{x}$  is an Euler solution to  $\dot{x}(t) = f(t, x(t))$ , with  $x(a) = \bar{x}(a)$ , then  $(t, \tilde{x}(t))$  is also an Euler solution to  $(\dot{y}, \dot{x})(t) = (1, f(t, x(t)))$ , with  $(y, x)(a) = (a, \bar{x}(a))$ . Therefore, the same Theorem 2.5 guarantees that  $(t, \tilde{x}(t))$  is an invariant trajectory, that is,  $(t, \tilde{x}(t)) \in G$ , for all  $t \in [a, \infty)$ , from which it necessarily follows that  $\tilde{x}(t) = \bar{x}(t)$  for all  $t \in [a, \infty)$ , and hence  $\bar{x}$  is the unique Euler solution. □

We now proceed to establish a characterization for strong invariance for multifunctions of the form  $F = D + G$ , where  $D$  is dissipative and  $G$  is Lipschitz. Such  $F$  satisfy (2.13) with constant  $k$  equal to the Lipschitz constant for  $G$ , and Corollary 2.6 is applicable. We continue to use the Hamiltonians  $H$  and  $h$  as defined in (2.2) and (2.3), but also will use  $H_D$ ,  $H_G$ ,  $h_D$ , and  $h_G$  that are the upper and the lower Hamiltonians of the summands  $D$  and  $G$ , respectively. We will also make use of Proposition 1.9(d) which establishes  $H(x, p) = H_D(x, p) + H_G(x, p)$  and  $h(x, p) = h_D(x, p) + h_G(x, p)$  for all  $(x, p)$ .

The infinitesimal characterization (2.11) for strong invariance in the Lipschitz case is no longer necessary for discontinuous  $F$ , as seen by the following simple example. Let

$$F(x) = \begin{cases} \left\{ -\frac{x}{|x|} \right\} & \text{if } x \neq 0 \\ [-1, 1] & \text{if } x = 0, \end{cases} \quad (2.15)$$

then  $(\{0\}, F)$  is strongly invariant but (2.11) fails. The example demonstrates that a replacement for (2.11) is required for discontinuous  $F$ . The following result provides such a condition for the special case  $F = D + G$ .

**Theorem 2.7. (*The main result*)** *The system  $(S, D + G)$  is strongly invariant if and only if*

$$h_D(x, \zeta) + H_G(x, \zeta) \leq 0 \quad (2.16)$$

for all  $x \in S$  and all  $\zeta \in N_S^P(x)$ .

**Proof.** Suppose  $(S, D + G)$  is strongly invariant,  $x_0 \in S$  and  $\zeta \in N_S^P(x_0)$ . Let  $v \in G(x_0)$  be such that  $H_G(x_0, \zeta) = \langle v, \zeta \rangle$ , and define  $g(x) := \{\text{proj}_{G(x)}(v)\}$ , whose values are singletons that constitute a continuous selection of  $G$  (see page 196 of [16]). Therefore, the multifunction  $D + g$  is upper semicontinuous, and satisfies the rest of the standing hypotheses. Since  $(D + g)(x) := D(x) + \{g(x)\} \subseteq (D + G)(x)$ ,

the strong invariance of  $(S, D + G)$  implies the weak invariance of  $(S, D + g)$ , and thus Theorem 2.4 yields  $h_D(x_0, \zeta) + h_g(x_0, \zeta) \leq 0$ , which proves the necessity since  $h_g(x_0, \zeta) = \langle v, \zeta \rangle = H_G(x_0, \zeta)$ .

To prove sufficiency, let  $x(\cdot)$  be a trajectory of  $D + G$  with  $x(0) = x_0 \in S$ . By Corollary 2.6, there exists a selection  $f$  of  $D + G$  such that  $x(\cdot)$  is an Euler solution to (2.1). Let  $x \in \mathbb{R}^n$  and  $s \in \text{proj}_S(x)$ . By dissipativity,

$$H_D(x, x - s) \leq h_D(s, x - s), \quad (2.17)$$

and the Lipschitz condition (2.12) implies

$$H_G(x, x - s) \leq H_G(s, x - s) + k\|x - s\|^2. \quad (2.18)$$

Since  $f$  is a selection of  $F = D + G$ , (2.17) and (2.18) amount to

$$\begin{aligned} \langle f(x), x - s \rangle &\leq H(x, x - s) \\ &= H_D(x, x - s) + H_G(x, x - s) \\ &\leq h_D(s, x - s) + H_G(s, x - s) \\ &\quad + k\|x - s\|^2. \end{aligned}$$

Using (2.16), the inequality  $\langle f(x), x - s \rangle \leq k\|x - s\|^2$  is obtained, which is the crucial aiming condition obtained on Theorem 2.5. The rest of the proof proceeds identically as in the mentioned Theorem.  $\square$



# Chapter 3

## Nonautonomous Perturbed Dissipative Systems

It is well known that the study of nonautonomous systems with control parameters is of natural interest in control theory. Among the different reasonable settings for time dependent systems, the almost semicontinuous approach provides a more general alternative than joint measurability and semicontinuity in the state variable, as the reader can corroborate in Theorem 1.8. By following joint work [27] with T. Donchev and P. Wolenski, the author shows how the criterion obtained in the previous chapter can be extended to data that is time dependent and satisfies more general assumptions on the  $G$  component of the multimap. We shall deal with a discontinuous right hand side that is the sum of an almost upper semicontinuous dissipative and an almost lower semicontinuous (DL) multifunction. The new nonautonomous characterization for strong invariance not only resembles the one obtained in chapter 1, but also vindicates that a classical Hamiltonian condition is still in effect for almost lower semicontinuity dynamics.

### 3.1 Assumptions and Some Precedents

We start by considering the control system modeled as the differential inclusion

$$\begin{aligned} \dot{x}(t) &\in F(t, x(t)) \quad \text{a.e., } t \in I \\ x(t_0) &= x_0, \end{aligned} \tag{3.1}$$

where the given multifunction  $F : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  has compact convex values and  $I = [0, \infty)$ . The focus in this chapter is on the case where the multifunction  $F$  has the form

$$F(t, x) = D(t, x) + G(t, x), \tag{3.2}$$

where  $D$  is upper semicontinuous and dissipative in  $x$  and  $G$  is lower semicontinuous and dissipative Lipschitz; precise assumptions will be given below. As it was mentioned in the previous paragraph, our goal is to extend the main result that appears in the previous chapter by considering the nonautonomous system (3.1) with a more general class of multimaps  $G$ .

To proceed with the discussion, we need to recall the time dependent version of the main concepts of the theory of invariant systems, also called “viability theory” by some authors (see [2, 3]). For a multifunction  $F : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  the pair  $(S, F)$  is said to be *strongly invariant* when for every  $x_0 \in S$ , and  $\tau > 0$ , every solution  $x(\cdot)$  of the differential inclusion  $\dot{x}(t) \in F(t, x(t))$ , with initial condition  $x(\tau) = x_0$  satisfies  $x(t) \in S$  for all  $t > \tau$ . If for every  $x_0 \in S$  and  $\tau > 0$  one of such solutions  $x(\cdot)$  exists and satisfies  $x(t) \in S$  for all  $t > \tau$ , the pair  $(S, F)$  is then called *weakly invariant* or *viable*.

There are two main approaches for characterizing the notions of invariance, one involving tangential conditions and the other using normal cones and Hamilton-Jacobi inequalities. Data assumptions for the characterization of weak invariance are very general, whereas for strong invariance, simple examples show that additional assumptions are needed. A Lipschitz assumption has usually been invoked for this purpose in the literature [16, 31]. Clarke [16] showed that strong invariance is equivalent to

$$F(x) \subseteq T_S^C(x) \quad \forall x \in S, \quad (3.3)$$

where  $T_S^C(x)$  is the Clarke tangent cone. Krastanov [37] gave an infinitesimal characterization of normal-type, by showing strong invariance is equivalent to

$$H_F(x, \zeta) \leq 0 \quad \forall x \in S, \quad \forall \zeta \in N_S^P(x), \quad (3.4)$$

see [15] for a Hilbert space version. These results assume the data  $F$  is autonomous and Lipschitz. Donchev [25] was the first to extend these characterizations beyond the autonomous Lipschitz case to almost continuous dissipative Lipschitz multifunctions. Ríos and Wolenski [42] proved an autonomous normal-type characterization that allows for a discontinuous component, which has been detailed in chapter 1. The main result of this chapter shows that strong invariance can be characterized under more general assumptions, and is given in the normal framework via Hamilton-Jacobi inequalities.

The Hamiltonian is prominently featured in our results, but the assumptions are most easily formulated in these terms as well. Recall that the nonautonomous minimized and maximized Hamiltonians associated to the multifunction  $F$  are given, respectively, by

$$H_F(t, x, p) = \sup\{\langle p, v \rangle : v \in F(t, x)\} \quad (3.5)$$

and

$$h_F(t, x, p) = \inf\{\langle p, v \rangle : v \in F(t, x)\}. \quad (3.6)$$

We also recall the dissipative Lipschitz condition as a modification of the Lipschitz one:  $F$  is dissipative Lipschitz if there is a locally integrable function  $k : I \rightarrow \mathbb{R}^n$  such that

$$H_F(t, x, x - y) - H_F(t, y, x - y) \leq k(t)\|x - y\|^2, \quad \forall x, y \in \mathbb{R}^n, \text{ a.e., } t \in I. \quad (3.7)$$

The dissipative Lipschitz condition was introduced in [26]; see also [25]. It is obvious that (3.7) is weaker than (1.14), and is strictly weaker, since for example in dimension one,  $F(x) = \sqrt[3]{-x}$  satisfies (3.7) but not (1.14). To establish our result the maps  $D(\cdot, \cdot)$  and  $G(\cdot, \cdot)$  will be endowed with the following properties:

- The multifunction  $F = D + G$  satisfies the linear growth condition: There is a locally integrable function  $c(\cdot)$  on  $I$  so that  $\|v\| \leq c(t)(1 + \|x\|)$  for almost all  $t \in I$  and all  $(x, v) \in \text{gr } F(t, \cdot)$ ;
- $D(\cdot, \cdot)$  and  $G(\cdot, \cdot)$  have nonempty, compact, and convex values;
- $D(\cdot, \cdot)$  is almost upper semicontinuous and dissipative;
- $G(\cdot, \cdot)$  is almost lower semicontinuous and dissipative Lipschitz.

The dissipative requirement is that

$$H_F(t, x, x - y) - h_F(t, y, x - y) \leq 0 \quad \forall x, y \in \mathbb{R}^n, \text{ a.e.}, t \in I. \quad (3.8)$$

Note that, under the previous hypotheses the multimap  $F = D + G$  satisfies (SH), and is (DL) with constant  $k$  equal to the (DL) constant for  $G$ . However,  $F$  is neither necessarily almost upper semicontinuous nor almost lower semicontinuous.

## 3.2 Weak Invariance

This section is devoted to the following weak invariant Theorem, which is the time-independent constraint version of a result proved in [26]. We will have opportunity to invoke it in the next section and next chapter as well.

**Theorem 3.1.** *Let  $\tilde{I} \subseteq I$  be a subinterval. Suppose the multifunction  $G : \tilde{I} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  satisfies (SH) on  $\tilde{I} \times \mathbb{R}^n$ , and  $S \subset \mathbb{R}^n$  is closed. Then  $(S, G)$  is weakly invariant if and only if there exists a null set  $A \subset \tilde{I}$  such that*

$$h_G(t, x, \zeta) \leq 0, \quad (3.9)$$

*for all  $t \in \tilde{I} \setminus A$ ,  $x \in S$ , and  $\zeta \in N_S^P(x)$ . In this case, (3.9) holds at all points of density of a certain countable family of pairwise disjoint closed sets  $I_r \subset \tilde{I}$  ( $k=1, 2, \dots$ ) for which the restriction of  $G(\cdot, \cdot)$  to every  $I_r \times \mathbb{R}^n$  is upper semicontinuous.*

**Proof.** To prove the necessity we proceed as in page 29 of [9]. In fact, the almost upper semicontinuity of  $G$  implies the existence of a countable family of pairwise disjoint closed sets  $I_r \subset \tilde{I}$  such that the restriction of  $G(\cdot, \cdot)$  to every  $I_r \times \mathbb{R}^n$  is upper semicontinuous, and the union  $\bigcup_{r \geq 1} I_r$  has full measure in  $\tilde{I}$  (see also page 24 of [21]). Let  $\tilde{I}_r$  denote the points of density of  $I_r$ , and define  $A := \tilde{I} \setminus \left( \bigcup_{r \geq 1} \tilde{I}_r \right)$  which has measure zero in  $\tilde{I}$  since  $\mu(\tilde{I}_r) = \mu(I_r)$ . Let  $t_0 \in \tilde{I}_r$  for some  $r$ ,  $x_0 \in S$ , and  $x(\cdot)$  an invariant solution to (4.1) on  $[t_0, t_1) \subset \tilde{I}$ . Let also  $\{t_j\}$  be a sequence in  $[t_0, t_1) \cap I_r$  such that  $t_j \rightarrow t_0$ , and set  $h_j = |t_j - t_0|$ . The absolute continuity of  $x(\cdot)$ , the compact-convex valuedness of  $G$ , and upper semicontinuity of  $G$  restricted to  $I_r \times \mathbb{R}^n$  imply the existence of a subsequence  $\{t_{j_r}\}$ , which we relabel as  $\{t_j\}$ , such that

$$v := \lim_{j \rightarrow \infty} \frac{x(t_j) - x(t_0)}{h_j} = \lim_{j \rightarrow \infty} \frac{1}{h_j} \int_{t_0}^{t_j} \dot{x}(s) ds \in G(t_0, x_0).$$

By definition of Bouligand cone (1.4) it follows that  $v \in T_S^B(x_0)$ , and this yields to  $G(t_0, x_0) \cap T_S^B(x_0) \neq \emptyset$ . The necessity of (3.9) now follows from the inclusion

$$T_S^B(x) \subset [N_S^P(x)]^\circ := \left\{ w : \langle w, v \rangle \leq 0, \forall v \in T_S^B(x) \right\}, \quad (3.10)$$

which holds for all  $x \in S$  (see (1.7)).

For the converse, let  $x_0 \in S$  and  $[t_0, t_1) \subset I$  be given. With the help of Gronwall's inequality we can assume without loss of generality that  $G$  is globally bounded (see page 52 of [21] for details):

$$\|G(t, x)\| := \max\{\|v\| : v \in G(t, x)\} \leq 1.$$

Let also  $\pi$  be an arbitrary partition  $t_0 = \tau_0 < \tau_1 \cdots < \tau_N = t_1$ , and suppose that an approximate trajectory  $x_{j-1}(\cdot)$  has been defined on  $[\tau_{j-1}, \tau_j]$ . To define  $x_j(\cdot)$  on the next subinterval we proceed in the following way: we set  $x_j = x_{j-1}(\tau_j)$  and choose

$s_j \in \text{proj}_S(x_j)$ . Let us consider on  $[\tau_j, \tau_{j+1}]$  the multimap  $t \mapsto G(t, s_j)$ , which is measurable according to Proposition 1.5(b), and has compact values. The Lusin property of measurable multifunctions (see Proposition 1.5(a)) implies that for each  $\varepsilon > 0$  there is  $I_\varepsilon \subset [\tau_j, \tau_{j+1}]$ , with  $\mu([\tau_j, \tau_{j+1}] \setminus I_\varepsilon) < \varepsilon$ , and such that  $G(\cdot, s_j)$  restricted to  $I_\varepsilon$  is continuous. It turns out that the restriction of  $h_G(\cdot, s_j, x_j - s_j)$  on  $I_\varepsilon$  is also continuous. Accordingly, the multifunction

$$t \mapsto \{v \in G(t, s_j) : h_G(t, s_j, x_j - s_j) = \langle v, x_j - s_j \rangle\}, \quad (3.11)$$

has closed graph on  $I_\varepsilon$ , and therefore it is measurable on the same compact  $I_\varepsilon$ . Due to the arbitrariness of  $\varepsilon > 0$ , we obtain that (3.11) is measurable on  $I$ . The closed valuedness of (3.11) is also clear. By Theorem 1.7 the existence of a measurable selection  $g_j(t) \in G(t, s_j)$  is guaranteed, which according to (3.9), must satisfy  $\langle g_j(t), x_j - s_j \rangle \leq 0$  a.e.  $t \in [\tau_j, \tau_{j+1}]$ . Next, we define

$$x_j(t) := x_{j-1}(\tau_j) + \int_{\tau_j}^t g_j(r) dr \quad (3.12)$$

for all  $t \in [\tau_j, \tau_{j+1}]$ . Under the previous scheme, we obtain the following piecewise defined approximate trajectory  $x^\pi(\cdot)$  on  $[t_0, t_1]$ :  $x^\pi(t) := x_j(t)$  if  $t \in [\tau_j, \tau_{j+1}]$ . By Following the argument shown in Proposition 2.2 we calculate for each  $0 \leq j \leq N - 1$ :

$$\begin{aligned} d_S^2(x_{j+1}) &\leq \|x_{j+1} - s_j\|^2 \\ &= \|x_j - s_j\|^2 + \|x_{j+1} - x_j\|^2 + 2\langle x_{j+1} - x_j, x_j - s_j \rangle \\ &\leq d_S^2(x_j) + (\tau_{j+1} - \tau_j)^2 + 2 \int_{\tau_j}^{\tau_{j+1}} \langle g_j(t), x_j - s_j \rangle dt \\ &\leq d_S^2(x_j) + (\tau_{j+1} - \tau_j)^2. \end{aligned}$$

From the previous it's easy to see that for  $k = 1, 2, \dots, N$  the following estimates hold

$$d_S^2(x_k) \leq d_S^2(x_0) + \sum_{j=1}^k (\tau_j - \tau_{j-1})^2 \leq \mathcal{D}(\pi)(t_1 - t_0),$$

where  $\mathcal{D}(\pi) = \max_{1 \leq j \leq N}(\tau_j - \tau_{j-1})$ . Taking a sequence of partitions  $\pi_i$ , with  $\mathcal{D}(\pi_i) \rightarrow 0$ , and using the compactness of the respective approximate trajectories  $\{x^{\pi_i}\}$  (Corollary 1.11), we obtain an invariant solution to (4.1), from which the sufficiency of (3.9) follows.  $\square$

### 3.3 Main Result

In this section we establish a criterion for strong invariance that replaces condition (3.4) for the special case of a (DL) multifunction  $F = D + G$  under the assumptions given above. We remark that since  $F$  is not necessarily almost semicontinuous, the existence of solutions to (4.1) is not assured by the usual theory. However, in our case, existence follows since there exists an almost continuous (Caratheodory) selection  $g(\cdot, \cdot)$  of  $G(\cdot, \cdot)$  (see proof of Theorem 3.2 below). Then  $D + g$  is almost upper semicontinuous and has a solution according to Theorem 1.4. Moreover,  $(D + g)(x) := D(x) + \{g(x)\} \subseteq D(x) + G(x) =: (D + G)(x)$ , and so obviously any trajectory of  $D + g$  is a trajectory of  $D + G$ .

The following result is the main contribution of this chapter.

**Theorem 3.2.** *The system  $(S, D + G)$  is strongly invariant if and only if there exists a set  $\mathcal{I} \subset I$  of full measure such that*

$$h_D(t, x, \xi) + H_G(t, x, \xi) \leq 0, \quad \forall (t, x) \in \mathcal{I} \times S, \quad \forall \zeta \in N_S^P(x). \quad (3.13)$$

**Proof.** Suppose the system  $(S, D + G)$  is strongly invariant. Without loss of generality we can assume that there is collection of pairwise disjoint compact sets  $\{J_r\}_{r=1}^\infty$  such that  $\bigcup_{r=1}^\infty J_r \subset I$  has full measure, and the restrictions of  $D(\cdot, \cdot)$  and  $G(\cdot, \cdot)$  on  $J_r \times \mathbb{R}^n$  are upper semicontinuous and lower semicontinuous respectively for all  $r$ . Let  $I_r \subset J_r$  be the set consisting of all points of density of  $J_r$ , and consider  $\mathcal{I} = \bigcup_{r=1}^\infty I_r$ . Now fix  $t_0 \in \mathcal{I}$ ,  $x_0 \in S$ , and  $\zeta \in N_S^P(x_0)$ , and choose  $v \in G(t_0, x_0)$  so

that

$$H_G(t_0, x_0, \zeta) = \langle v, \zeta \rangle.$$

Let  $j$  be the integer for which  $t_0 \in I_j$ . For each integer  $r$  Michael's Theorem 1.6 assures the existence of a selection  $g_r$  of  $G$  that is continuous on  $J_r \times \mathbb{R}^n$ , and such that  $g_j(t_0, x_0) = v$ . Let us define  $g(t, x) := g_r(t, x)$  if  $(t, x) \in I_r \times \mathbb{R}^n$ . It follows that  $g : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  is measurable on  $I$  for all fixed  $x$  and continuous in  $x$  for almost all  $t \in I$ . Therefore, the multifunction  $D + g$  defined by  $(D + g)(t, x) := \{d + g(t, x) : d \in D(t, x)\}$  is almost upper semicontinuous, and its restriction to every  $J_r \times \mathbb{R}^n$  is upper semicontinuous. The assumption of strong invariance of  $D + G$  readily implies that  $D + g$  is weakly invariant, and therefore Theorem 3.1 and Proposition 1.9 yield

$$h_D(t_0, x_0, \zeta) + h_g(t_0, x_0, \zeta) \leq 0, \quad (3.14)$$

where  $h_g(t, x, \zeta) := \langle g(t, x), \zeta \rangle$ . By the choice of  $v \in G(t_0, x_0)$ , one has  $h_g(t_0, x_0, \zeta) = H_G(t_0, x_0, \zeta)$ , and so (3.13) follows immediately from (3.14). This proves the necessity.

For the sufficiency, assume that (3.13) holds, and let  $[t_0, t_1] \subseteq I$  and  $x(\cdot)$  be a trajectory of (3.1) with  $x(t_0) = x_0 \in S$ . By Proposition 1.5 the multifunctions  $\{\dot{x}(t) - d : d \in D(t, x(t))\}$  and  $G(t, x(t))$  are both measurable in  $t$  on  $[t_0, t_1]$ , and their intersection is nonempty for almost all  $t \in [t_0, t_1]$  since  $x(\cdot)$  is a solution of (3.1). Therefore, Proposition 1.2 implies the multifunction

$$E(t) := \left[ \{\dot{x}(t) - d : d \in D(t, x(t))\} \cap G(t, x(t)) \right]$$

is measurable. Consequently, Theorem 1.7 guarantees the existence of a measurable selection  $v(t) \in E(t)$  a.e.,  $t \in [t_0, t_1]$ . For  $\delta > 0$ , consider the multifunction

$$G_\delta(t, x) := \{g \in G(t, x) : \langle g - v(t), x - x(t) \rangle < k(t)|x - x(t)|^2 + \delta\}, \quad (3.15)$$



where  $k(\cdot)$  is the dissipative Lipschitz rank from the assumption on  $G$ , and define  $\tilde{G}_\delta(t, x) := \text{cl } G_\delta(t, x)$  for all  $x \in \mathbb{R}^n$ , and almost all  $t \in [t_0, t_1]$ . Since  $G$  is assumed to be dissipative Lipschitz and  $v(t) \in G(t, x(t))$  it follows that  $\tilde{G}_\delta(t, x)$  is nonempty. Moreover,  $\tilde{G}_\delta(t, x)$  is compact since  $\text{cl } G_\delta(t, x) \subseteq G(t, x)$  and  $G(t, x)$  is compact. The convexity of  $G_\delta$  simply follows from the linearity of the inner product (see proof of Proposition 4.2 where the argument is given in detail). Therefore,  $\text{cl } G_\delta(t, x)$  is clearly convex. We now show how the almost lower semicontinuity of  $G$  implies that of  $\tilde{G}_\delta(\cdot, \cdot)$ . In fact; the almost lower semicontinuity of  $G$  and the Lusin property of the measurable functions  $v(\cdot)$  and  $k(\cdot)$  implies the existence of a collection of pairwise disjoint compact sets  $J_r \subset [t_0, t_1]$  with  $\bigcup_{r=1}^\infty J_r$  having full measure in  $[t_0, t_1]$ , and such that the restrictions of  $G$ ,  $v$  and  $k$  on each  $J_r \times \mathbb{R}^n$  are respectively lower semicontinuous and continuous. Let  $I_r \subset J_r$  the set consisting of all points of density of  $J_r$ , and fix  $(t, x) \in I_r \times \mathbb{R}^n$  for some  $r$ . Let us also consider a sequence  $(t_i, x_i) \in J_r \times \mathbb{R}^n$  such that  $(t_i, x_i) \rightarrow (t, x)$  and  $g \in G_\delta(t, x)$ . By definition of  $G_\delta(t, x)$  there exists  $\gamma > 0$  for which

$$\langle g - v(t), x - x(t) \rangle \leq k(t) \|x - x(t)\|^2 + \delta - \gamma. \quad (3.16)$$

Now the lower semicontinuity of  $G$  on  $J_r \times \mathbb{R}^n$  implies the existence of  $g_i \in G(t_i, x_i)$  satisfying  $g_i \rightarrow g$ . Due the continuity of the inner product and norm, it is possible then to find  $i$  large enough for which the following inequalities hold

$$\langle g_i - v(t_i), x_i - x(t_i) \rangle < \langle g - v(t), x - x(t) \rangle + \frac{\gamma}{4}, \quad (3.17)$$

$$k(t) \|x - x(t)\|^2 < k(t_i) \|x_i - x(t_i)\|^2 + \frac{\gamma}{4}. \quad (3.18)$$

Therefore, by connecting inequalities (3.16), (3.17), and (3.18) we have that for  $i$  large enough

$$\begin{aligned}
\langle g_i - v(t_i), x_i - x(t_i) \rangle &< \langle g - v(t), x - x(t) \rangle + \frac{\gamma}{4} \\
&\leq k(t)\|x - x(t)\|^2 + \delta - \gamma + \frac{\gamma}{4} \\
&< k(t_i)\|x_i - x(t_i)\|^2 + \delta - \frac{3}{4}\gamma + \frac{\gamma}{4} \\
&< k(t_i)\|x_i - x(t_i)\|^2 + \delta,
\end{aligned}$$

which implies  $g_i \in G_\delta(t_i, x_i)$  for  $i$  large enough. We have shown that the restriction of  $G_\delta$  to  $J_r \times \mathbb{R}^n$  is lower semicontinuous. Since it is clear that the closure of compact valued lower semicontinuous multifunctions is again lower semicontinuous, it follows that  $\tilde{G}_\delta = \text{cl } G_\delta$  is almost lower semicontinuous. We now define the set

$$\tilde{I} = \bigcup_{r=1}^{\infty} (I_r \cap \mathcal{I}).$$

Note that  $\tilde{I} \subset \mathcal{I}$  has full measure in  $[t_0, t_1)$ , and that without loss of generality we can also assume the restriction of  $D$  to each  $J_r \times \mathbb{R}^n$  to be upper semicontinuous. Using again Michael's Theorem as in the proof of the necessity part we find a selection  $g$  of  $\tilde{G}_\delta$  whose restriction to every  $J_r \times \mathbb{R}^n$  is continuous. The fact that  $g(\cdot, \cdot) \in \tilde{G}_\delta(\cdot, \cdot) \subset G(\cdot, \cdot)$  and assumption (3.13) imply that for all  $t \in \tilde{I}$ , all  $x \in S$ , and all  $\zeta \in N_S^P(x)$ , we have

$$\begin{aligned}
h_D(t, x, \zeta) + h_g(t, x, \zeta) &\leq h_D(t, x, \zeta) + H_{\tilde{G}_\delta}(t, x, \zeta) \\
&\leq h_D(t, x, \zeta) + H_G(t, x, \zeta) \leq 0.
\end{aligned}$$

This implies the weak invariance of  $(S, D + g)$  (see Theorem 3.1), and so there exists an absolutely continuous arc  $y_\delta(\cdot)$  satisfying

$$\dot{y}_\delta(t) - g(t, y_\delta(t)) =: d(t) \in D(t, y_\delta(t)), \quad y_\delta(t_0) = x_0$$

and  $y_\delta(t) \in S$  for all  $t \in [t_0, t_1)$ . Recall that for almost all  $t \in [t_0, t_1)$ ,

$$\begin{aligned} v(t) &\in G(t, x(t)), \\ \dot{x}(t) - v(t) &\in D(t, x(t)), \text{ and} \\ g(t, y_\delta(t)) &\in G_\delta(t, y_\delta(t)). \end{aligned}$$

By definition (3.15) and the dissipativity of  $D$ , one has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |y_\delta(t) - x(t)|^2 &= \langle \dot{y}_\delta(t) - \dot{x}(t), y_\delta(t) - x(t) \rangle \\ &= \langle g(t, y_\delta(t)) - v(t), y_\delta(t) - x(t) \rangle \\ &+ \langle d(t) - (\dot{x}(t) - v(t)), y_\delta(t) - x(t) \rangle \\ &\leq k(t) |y_\delta(t) - x(t)|^2 + \delta. \end{aligned}$$

Gronwall's lemma (Proposition 1.3) implies  $|y_\delta(t) - x(t)|^2 \leq 2\delta e^{2 \int_{t_0}^t |k(s)| ds} (t - t_0)$ . Since  $\delta > 0$  is arbitrary, it follows that  $y_\delta(t) \rightarrow x(t)$  as  $\delta \downarrow 0$  for all  $t \in [t_0, t_1)$ . Since  $y_\delta(t) \in S$  for all  $t \in [t_0, t_1)$  and  $S$  is closed, we conclude  $x(t) \in S$ . Hence  $(S, D + G)$  is strongly invariant as claimed.  $\square$

The mixed-type Hamiltonian condition (3.13) reveals an intrinsic property of trajectories of dissipative and dissipative Lipschitz maps: the invariance property is preserved under the sum of these systems if the pieces are separately invariant. In fact, we establish this in the following corollary.

**Corollary 3.3.** *Under assumptions of Theorem 3.2, if  $(S, D)$  and  $(S, G)$  are strongly invariant, then  $(S, D + G)$  is strongly invariant.*

**Proof.** By setting first  $D(\cdot, \cdot) \equiv 0$  and then  $G(\cdot, \cdot) \equiv 0$ , we obtain from Theorem 2.5 that there is a set of full measure  $\mathcal{I} \subset I$  (common for both  $G$  and  $D$ ) such that  $H_G(t, x, \zeta) \leq 0$  and  $h_D(t, x, \zeta) \leq 0$ , for all  $t \in \mathcal{I}$ ,  $x \in S$  and  $\zeta \in N_S^P(x)$ . Adding these two inequalities, and again applying the criterion (3.13), the result follows immediately.  $\square$

The isolation of the pieces used in the preceding corollary shows how the condition (3.13) subsumes the Hamiltonian criterion for strong invariance given in theorem 2 of [25] for the  $G$  piece of  $F$ . Actually, we have the following extension of that result when the state constraints is a closed set  $S$ , and the Hilbert space is  $\mathbb{R}^n$ .

**Theorem 3.4.** *There exists a set of full measure  $\tilde{I} \subset I$  such that the following conditions are equivalent:*

- (a)  $G(t, x) \subseteq T_S^C(x) \quad \forall (t, x) \in \tilde{I} \times S$ .
- (b)  $G(t, x) \subseteq T_S^B(x) \quad \forall (t, x) \in \tilde{I} \times S$ .
- (c)  $G(t, x) \subseteq \text{co}T_S^B(x) \quad \forall (t, x) \in \tilde{I} \times S$ .
- (d)  $H_G(t, x, \zeta) \leq 0 \quad \forall (t, x) \in \tilde{I} \times S, \forall \zeta \in N_S^P(x)$ .
- (e)  $(G, S)$  is strongly invariant.

**Proof.** The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are tautologies (see section 1.1 for details). The equivalence between (d) and (e) is given in Theorem 3.2 by setting  $D \equiv 0$ . In order to close the chain of implications, only (e) $\Rightarrow$ (a) needs to be shown. To do so, we follow the same argument given in [16, 25] which uses the well known characterization of the Clarke's tangent cone:  $T_S^C(x) = [N_S^L(x)]^\circ$  (see (1.6)). Let  $\{J_r\}_{r=1}^\infty$  be a collection of pairwise disjoint compact sets such that  $\bigcup_{r=1}^\infty J_r \subset I$  has full measure in  $I$ , and the restriction of  $G(\cdot, \cdot)$  to each  $J_r \times \mathbb{R}^n$  is lower semicontinuous. Let  $(t, x) \in (\tilde{I} \cap I_r) \times S$ , where again  $I_r \subseteq J_r$  denotes the points of density of  $J_r$ ,  $\tilde{I}$  is as in (d), and  $v \in G(t, x)$ . For  $\zeta \in N_S^L(x)$  there exists  $\zeta_i \in N_S^P(x_i)$  with  $x_i \rightarrow x$ ,  $x_i \in S$ , and  $\zeta_i \rightarrow \zeta$ . Let also  $t_i \in J_r \cap \tilde{I}$  be such that  $t_i \rightarrow t$ . The existence of the sequence  $t_i$  is guaranteed by the fact that  $t \in I_r \subset \text{cl}(I_r) = \text{cl}(I_r \setminus (I \setminus \tilde{I}))$  (see note in chapter 1 about properties of points of density). By the lower semicontinuity of  $G(\cdot, \cdot)$  on  $J_r \times \mathbb{R}^n$ , there are  $v_i \in G(t_i, x_i)$

such that  $v_i \rightarrow v$ . But then, condition (d) leads to  $\langle v_i, \zeta_i \rangle \leq H_G(t_i, x_i, \zeta_i) \leq 0$ , for  $i = 1, 2, \dots$ . Therefore,  $\langle v, \zeta \rangle = \lim_{i \rightarrow \infty} \langle v_i, \zeta_i \rangle \leq 0$ . Since  $\zeta \in N_S^L(x)$  is arbitrary, it follows that  $v \in [N_S^L(x)]^\circ$ .  $\square$

## Chapter 4

# Strong Invariance under Almost Upper Semicontinuity

The existence of continuous selections is perhaps one of the most attractive features of lower semicontinuous dynamics. It is one of their key ingredients to determine existence of trajectories, and also facilitates conditions under which infinitesimal criteria for invariance can be established. This last in particular was appreciated in the previous chapter when we discarded the dissipative component  $D$  from the nonautonomous multifunction  $F = D + G$  (see Theorem 3.4). In this chapter we consider again the strong invariance issue for a pair  $(S, F)$  whose multifunction  $F$ , in addition to the dissipative Lipschitz condition, also satisfies the almost upper semicontinuity property. Due to the lack of continuous selections for upper semicontinuous dynamics, an obstacle immediately arises: the optimal velocities for the upper Hamiltonian at points  $x \in S$  do not provide enough information to characterize strong invariance. In order to tackle this difficulty, we make use of a subsidiary invariance principle which characterizes the strong invariance property in terms of the weak invariance of some multifunctions  $G(\cdot, \cdot) \subseteq F(\cdot, \cdot)$ . A time-dependent version of an infinitesimal characterization of weak invariance is then applied to the auxiliary pairs  $(G, S)$ , which yields to limiting Hamiltonian inequalities that are sufficient and necessary conditions for the strong invariance of  $(S, F)$ . The discussion, which follows joint work [28] with T. Donchev and P. Wolenski, also incorporates the extension of the sufficient condition for strong invariance to the case of time-dependent constraints.

## 4.1 The Setting

In this chapter we shall deal with a system with control parameters that can be modelled as a differential inclusion

$$\begin{aligned}\dot{x}(t) &\in F(t, x(t)) \quad \text{a.e., } t \in [t_0, t_1), \\ x(t_0) &= x_0.\end{aligned}\tag{4.1}$$

Here the given data is a multifunction  $F : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  endowed with the standing hypotheses (SH)

- For each  $(t, x) \in I \times \mathbb{R}^n$ ,  $F(t, x) \subset \mathbb{R}^n$  is nonempty, convex, and compact;
- $F(\cdot, \cdot)$  is almost upper semicontinuous on  $I \times \mathbb{R}^n$ ;
- There is a locally integrable function  $c(\cdot) : I \rightarrow \mathbb{R}$  so that  $\|v\| \leq c(t)(1 + \|x\|)$  for almost all  $t \in I$  and all  $(x, v) \in \text{gr } F(t, \cdot)$ .

As in the previous chapters  $I = [0, \infty)$  and  $[t_0, t_1) \subset I$ . Recall that a solution (or trajectory)  $x(\cdot)$  to (4.1) is an absolutely continuous map whose derivative  $\dot{x}(t)$  belongs to  $F(t, x(t))$  for almost all  $t \in [t_0, t_1)$ . As it was pointed out before, our goal is to obtain necessary and sufficient conditions for strong invariance under weaker assumptions on  $F$  than heretofore imposed. In particular, the criterion that we pursue subsumes the mixed-type Hamiltonian characterization provided in chapter one, and also extends all the Hamiltonian-type results known in the literature, even for the nonautonomous case.

Recall that under the previous (SH) the inclusion (4.1) admits at least one solution for each  $[t_0, t_1) \subseteq I$  and  $x_0 \in \mathbb{R}^n$  (see Theorem 1.4).

## 4.2 The Main Structural Hypothesis

Since the pioneering work by Clarke [13] in the multivalued framework, a Lipschitz property (1.13) of the multifunction  $F$  has been typically assumed for the purpose

of characterizing strong invariance. The characterizations found in the literature are of two types: Tangential and Hamiltonian. The Clarke's cone condition (3.3) provided in [13, 16] for autonomous data, and a measurable time-dependent criterion involving the Bouligand cone, which appeared in [31], are perhaps the most pedestrian contributions in the tangential setting. On the other hand, and under the same assumptions as Clarke [13], Krastanov [37] showed the strong invariance property is equivalent to the upper Hamiltonian condition

$$H_F(x, \zeta) \leq 0 \tag{4.2}$$

holding for all  $x \in S$  and  $\zeta \in N_S^P(x)$ . Clarke, Ledyaev, and Radulescu [15] have made extensions to Hilbert spaces with appropriate modifications, also under the Lipschitz approach.

The first non-Lipschitz characterization of strong invariance appeared recently in [26], where almost continuity plus the dissipative Lipschitz property (1.15) was assumed. In this last work the author showed the following time-dependent Hamiltonian condition, among others, is a characterization for strong invariance of  $(S, F)$  (see Theorem 2 of the mentioned reference): there exists a set of null measure  $B \subset I$  such that

$$H(t, x, \zeta) \leq 0, \quad \text{for all } (t, x) \in (I \setminus B) \times S \text{ and all } \zeta \in N_S^P(x). \tag{4.3}$$

We emphasize that the (DL) condition (1.15) is much weaker than the Lipschitz assumption (1.13), as the reader can appreciate by considering again the multifunction (1.17). Moreover, the multifunction  $F$  in the mentioned example is upper semicontinuous and satisfies the (DL) requirement with constant  $k = 0$  (actually, it is dissipative). However, for  $S := \{0\}$  the system  $(S, F)$  is strongly invariant and condition (4.3) fails. This illustrates the necessity of finding a replacement



Hamiltonian condition for strong invariance for general (DL) dynamics under assumptions (SH). We have accomplished the task of establishing such a criterion under the Hamiltonian framework. Nevertheless, the quest for an equivalent condition involving the Clarke's tangent cone is still ongoing.

### 4.3 Main Result

The next result is a new characterization for strong invariance that applies to systems satisfying (1.15). It establishes that the strong invariance property is equivalent to the weak invariance of all the subsystems satisfying the standing assumptions. To make this precise, we need to introduce the following definition.

**Definition 4.1.** *We say that  $G$  is a submultifunction of  $F$  if there exists a subinterval  $\tilde{I} \subseteq I$  for which  $G : \tilde{I} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  satisfies (SH) on  $\tilde{I} \times \mathbb{R}^n$ , and  $G(t, x) \subset F(t, x)$  for all  $x \in \mathbb{R}^n$  and almost all  $t \in \tilde{I}$ .*

We remark that in the following proposition, nonautonomous submultifunctions must come under consideration even when  $F$  is autonomous, in which case one has  $G(t, x) \subseteq F(x)$  for all  $t \in \tilde{I}$ .

**Proposition 4.2. (Invariance principle)** *Let us assume that (SH) and (1.15) hold. The system  $(S, F)$  is strongly invariant if and only if for every submultifunction  $G$  of  $F$  that is defined on  $\tilde{I} \times \mathbb{R}^n$ , there exists a null set  $A_G \subset \tilde{I}$  such that  $h_G(t, x, \zeta) \leq 0 \forall \zeta \in N_S^P(x), \forall x \in S$ , and  $\forall t \in \tilde{I} \setminus A_G$ .*

**Proof.** We first consider the “if” direction. Let  $[t_0, t_1) \subset I$ ,  $x_0 \in S$ , and  $x(\cdot)$  be a solution to (4.1). Let us consider the following multifunction

$$G(t, x) := \{v \in F(t, x) : \langle \dot{x}(t) - v, x(t) - x \rangle \leq k(t) \|x(t) - x\|^2\},$$

which is defined on  $\tilde{I} \times \mathbb{R}^n$ , with  $\tilde{I} := [t_0, t_1)$ . We first show that  $G$  inherits the properties (SH) from  $F$  on  $\tilde{I} \times \mathbb{R}^n$ . In fact,

- Let  $t \in \tilde{I}$  be such that  $\dot{x}(t) \in F(t, x(t))$  and let  $x \in \mathbb{R}^n$ . Due to the (DL) condition (1.15) there exists  $v \in F(t, x)$  such that  $\langle \dot{x}(t) - v, x(t) - x \rangle \leq k(t)\|x(t) - x\|^2$ , which implies  $v \in G(t, x)$  according to the definition of  $G$ . This proves the nonemptiness of  $G$ .
- The multifunction  $G$  automatically inherits the linear growth condition from  $F$  since  $G(t, x) \subset F(t, x)$  for all  $(t, x)$  in the domain of  $G$ .
- Let  $\lambda \in (0, 1)$  and  $v, w \in G(t, x)$ . By definition of  $G$  we have that the following inequalities hold:

$$\langle \dot{x}(t) - v, x(t) - x \rangle \leq k(t)\|x(t) - x\|^2, \quad (4.4)$$

and

$$\langle \dot{x}(t) - w, x(t) - x \rangle \leq k(t)\|x(t) - x\|^2. \quad (4.5)$$

Notice that  $\lambda v + (1 - \lambda)w \in F(t, x)$  due to the convexity assumption on  $F(t, x)$ . Multiplying inequalities (4.4) and (4.5) by  $\lambda$  and  $1 - \lambda$  respectively and adding the resulting inequalities yield to

$$\begin{aligned} \langle \dot{x}(t) - (\lambda v + (1 - \lambda)w), x(t) - x \rangle &= \langle \lambda(\dot{x}(t) - v), x(t) - x \rangle \\ &\quad + \langle (1 - \lambda)(\dot{x}(t) - w), x(t) - x \rangle \\ &\leq \lambda k(t)\|x(t) - x\|^2 \\ &\quad + (1 - \lambda)k(t)\|x(t) - x\|^2 \\ &= \lambda k(t)\|x(t) - x\|^2, \end{aligned}$$

which implies  $\lambda v + (1 - \lambda)w \in G(t, x)$ . This proves that  $G(t, x)$  is convex.

- The compactness of  $G(t, x)$  follows from the continuity of  $\langle \cdot, \cdot \rangle$ : Let  $v_i \in G(t, x)$  be a sequence. Therefore,  $\langle \dot{x}(t) - v_i, x(t) - x \rangle \leq k(t)\|x(t) - x\|^2$  for

all  $i$ . Since  $v_i \in F(t, x)$  and  $F(t, x)$  is compact, there exists a subsequence  $v_{i_j}$  of  $v_i$  such that  $v_{i_j} \rightarrow v$ , for some  $v \in F(t, x)$ . Therefore,

$$\langle \dot{x}(t) - v, x(t) - x \rangle = \lim_{j \rightarrow \infty} \langle \dot{x}(t) - v_{i_j}, x(t) - x \rangle \leq k(t) \|x(t) - x\|^2,$$

which means that  $v \in G(t, x)$ . Since  $v_{i_j} \in G(t, x)$ , it follows that  $G(t, x)$  is compact.

- Finally we show that  $G$  is indeed almost upper semicontinuous on  $\tilde{I} \times \mathbb{R}^n$ . For this purpose, let  $\varepsilon > 0$ . The almost upper semicontinuity of  $F$  and the Lusin property of the measurable functions  $k(\cdot)$  and  $\dot{x}(\cdot)$  imply the existence of a compact  $I_\varepsilon \subseteq \tilde{I}$ , common for  $G$ ,  $k$ , and  $\dot{x}$ , with  $\mu(\tilde{I} \setminus I_\varepsilon) < \varepsilon$ , and such that the restrictions of  $F$ ,  $k$ , and  $\dot{x}$  on  $I_\varepsilon \times \mathbb{R}^n$  are upper semicontinuous and continuous respectively. Let  $J_\varepsilon \subseteq I_\varepsilon$  denote the set of points of density of  $I_\varepsilon$  and let  $(t, x) \in J_\varepsilon \times \mathbb{R}^n$ . Let  $(t_i, x_i, v_i) \in \text{gr } G$ , with  $(t_i, x_i) \in I_\varepsilon \times \mathbb{R}^n$  and  $(t_i, x_i, v_i) \rightarrow (t, x, v)$ . Then we have  $v_i \in G(t_i, x_i)$  for all  $i$ . Notice that  $v_i \in F(t_i, x_i)$  for all  $i$ , and since  $\text{gr } F$  is closed in  $I_\varepsilon \times \mathbb{R}^{2n}$  it follows that  $v \in F(t, x)$ . Moreover, the definition of  $G(t_i, x_i)$  implies the following sequence of inequalities

$$\langle \dot{x}(t_i) - v_i, x(t_i) - x_i \rangle \leq k(t_i) \|x(t_i) - x_i\|^2, \quad i = 1, 2, \dots$$

Taking limit when  $i \rightarrow \infty$  in the previous inequalities we obtain

$$\langle \dot{x}(t) - v, x(t) - x \rangle \leq k(t) \|x(t) - x\|^2,$$

which now amounts to  $v \in G(t, x)$ . This proves  $\text{gr } G$  is closed in  $I_\varepsilon \times \mathbb{R}^{2n}$ , which is equivalent to the upper semicontinuity of  $G$  on  $I_\varepsilon \times \mathbb{R}^{2n}$ .

We have shown that  $G$  is actually a submultifunction of  $F$ . By hypothesis, there exists a null set  $A_G \subset \tilde{I}$  such that  $h_G(t, x, \zeta) \leq 0 \quad \forall \zeta \in N_S^P(x), \forall x \in S$ , and

$\forall t \in \tilde{I} \setminus A_G$ . This last implies, according to Theorem 3.1, that the pair  $(S, G)$  is weakly invariant, and therefore there exists a solution  $y(\cdot)$  to

$$\dot{y}(t) \in G(t, y(t)) \quad \text{a.e., } t \in [t_0, t_1], \quad y(t_0) = x_0 \quad (4.6)$$

such that  $y(t) \in S$  for all  $t \in [t_0, t_1]$ . From the definition of  $G$  we must have

$$\langle \dot{x}(t) - \dot{y}(t), x(t) - y(t) \rangle \leq k(t) \|x(t) - y(t)\|^2 \quad \text{a.e., } t \in \tilde{I},$$

which immediately implies

$$\frac{d}{dt} \|x(t) - y(t)\|^2 \leq 2k(t) \|x(t) - y(t)\|^2 \quad \text{a.e., } t \in \tilde{I}.$$

It follows from Gronwall's inequality (Proposition 1.3) that

$$\|x(t) - y(t)\|^2 \leq e^{2 \int_{t_0}^t k(s) ds} \left[ \|x(t_0) - y(t_0)\|^2 \right] = 0$$

for all  $t \in [t_0, t_1]$ . Therefore,  $x(t) = y(t) \in S$  for all  $t \in \tilde{I}$ . Hence the system  $(S, F)$  is strongly invariant.

To prove the converse, let  $G$  be a submultifunction of  $F$  defined on  $\tilde{I} \times \mathbb{R}^n$ . By Theorem 1.4 there exist trajectories of  $G$  which must of course also be trajectories of  $F$ , and since  $F$  is strongly invariant, such trajectories belong to  $S$ . Hence  $(S, G)$  is weakly invariant, and therefore by Theorem 3.1, there exists a null set  $A_G \subset \tilde{I}$  such that  $h_G(t, x, \zeta) \leq 0$  for all  $\zeta \in N_S^P(x)$ ,  $x \in S$ , and  $t \in \tilde{I} \setminus A_G$ .  $\square$

The following will be derived from Proposition 4.2. It provides necessary and sufficient conditions for strong invariance in terms of the original data. We shall use the following notation: For a given nonzero vector  $\zeta \in \mathbb{R}^n$ ,  $y \rightarrow_\zeta x$  denotes the limit of  $y$  approaching  $x$  along the vector  $\zeta$ ; in other words,  $y \rightarrow_\zeta x$  if and only if  $y \rightarrow x$  and  $\frac{y-x}{\|y-x\|} \rightarrow \frac{\zeta}{\|\zeta\|}$ .

**Theorem 4.3.** *Under assumptions (SH) and (1.15), the following assertions hold:*

i) *If there is a null set  $A \subset I$  such that*

$$\limsup_{y \rightarrow \zeta x, \tau \rightarrow t} H_F(\tau, y, \zeta) \leq 0, \quad (4.7)$$

*for all  $t \in I \setminus A$ , all  $x \in S$ , and all  $\zeta \in N_S^P(x)$ , then the system  $(S, F)$  is strongly invariant.*

ii) *If the system  $(S, F)$  is strongly invariant, then there is a null set  $D \subset I$  such that*

$$\liminf_{y \rightarrow \zeta x, \tau \rightarrow t} H_F(\tau, y, \zeta) \leq 0, \quad (4.8)$$

*for all  $t \in I \setminus D$ , all  $x \in S$ , and all  $\zeta \in N_S^P(x)$ .*

**Proof.** Assume (4.7) holds, and that  $A \subset I$  is the null set satisfying (4.7). We will show any submultifunction  $G$  of  $F$  is weakly invariant, and so strong invariance of  $(S, F)$  will follow directly from Proposition 4.2.

Let  $G : \tilde{I} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  be a submultifunction of  $F$ . As it was pointed out before, the almost upper semicontinuity of  $G$  on  $\tilde{I} \times \mathbb{R}^n$  implies the existence of a countable family of pairwise disjoint closed sets  $J_r \subset \tilde{I}$  such that the restriction of  $G(\cdot, \cdot)$  on  $J_r \times \mathbb{R}^n$  is upper semicontinuous for all  $r$ , and the set  $\bigcup_{r \geq 1} J_r$  has full measure in  $\tilde{I}$ . Let  $\tilde{J}_r$  denote the points of density of  $J_r$ , and define  $\tilde{A} := \tilde{I} \setminus \left( \bigcup_{r \geq 1} \tilde{J}_r \right)$  which has null measure in  $\tilde{I}$ . Define

$$A_G := \left( A \cap \tilde{I} \right) \cup \tilde{A}$$

which also has measure zero in  $\tilde{I}$ . Now fix  $t \in \tilde{I} \setminus A_G$ ,  $x \in S$  and  $\zeta \in N_S^P(x)$ . For some  $r$  we have  $t \in \tilde{J}_r \setminus \left( A \cap \tilde{I} \right)$ . By taking the  $\liminf$  all over the sequences  $(\rho, z) \rightarrow (t, x)$  in  $J_r \times \mathbb{R}^n$ , the lower semicontinuity of  $h_G(\cdot, \cdot, \zeta)$  yields

$$h_G(t, x, \zeta) \leq \liminf_{z \rightarrow x, \rho \rightarrow t} h_G(\rho, z, \zeta),$$

and obviously, all over the special sequences  $y \rightarrow_{\zeta} x$ , and the sequences  $\lambda \rightarrow t$  in  $J_r \setminus (A \cap \tilde{I})$ , we have

$$\liminf_{z \rightarrow x, \rho \rightarrow t} h_G(\rho, z, \zeta) \leq \liminf_{y \rightarrow_{\zeta} x, \lambda \rightarrow t} h_G(\lambda, y, \zeta).$$

If we consider now, as in (4.7), all possible sequences  $\tau \rightarrow t$  in  $I \setminus A$ , and the fact  $h_G \leq H_F$ , it is then readily seen that

$$\liminf_{y \rightarrow_{\zeta} x, \lambda \rightarrow t} h_G(\lambda, y, \zeta) \leq \limsup_{y \rightarrow_{\zeta} x, \tau \rightarrow t} H_F(\tau, y, \zeta).$$

The last three inequalities and condition (4.7) imply  $h_G(t, x, \zeta) \leq 0$ , and thus the sufficiency of (4.7) for strong invariance now follows directly from Proposition 4.2.

Conversely, assume that the system (S,F) is strongly invariant. Since  $F$  is almost upper semicontinuous and  $k(\cdot)$  is measurable (i.e.  $k(\cdot)$  is Lusin), there exists a countable family of pairwise disjoint closed sets  $I_r \subset I$  (common for both  $F$  and  $k$ ) such that the restrictions of  $F(\cdot, \cdot)$  and  $k(\cdot)$  on each  $I_r \times \mathbb{R}^n$  are upper semicontinuous and continuous respectively, and the union  $\bigcup_{r \geq 1} I_r$  has full measure. In the same fashion as before, let  $\tilde{I}_r$  denote the points of density of  $I_r$ , and define  $D := I \setminus \left( \bigcup_{r \geq 1} \tilde{I}_r \right)$  which has measure zero. We claim that condition (4.8) holds for each  $t \in I \setminus D = \bigcup_{r \geq 1} \tilde{I}_r$ . To see this, let  $\bar{t} \in \tilde{I}_r$  for some  $r$ , and also let  $x \in S$ ,  $\zeta \in N_S^P(x)$ , and a sequence  $y_i \rightarrow_{\zeta} x$  be given. For each  $i$ , we consider the multifunction

$$t \mapsto \{v \in F(t, y_i) : H_F(t, y_i, \zeta) = \langle v, \zeta \rangle\}. \quad (4.9)$$

Notice that by Proposition 1.9(a) we have  $H_F(t, y_i, \zeta) = -h_F(t, y_i, \zeta)$ . Therefore, the extension to the whole interval  $I$  of the argument given in the proof of Theorem 3.1 for the multifunction (3.11) guarantees the multifunction in (4.9) is measurable with compact values, and thus admits a measurable selection  $v_i(\cdot)$ . Now

define  $G_i : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  by

$$G_i(t, y) := \{w \in F(t, y) : \langle v_i(t) - w, y_i - y \rangle \leq k(t)\|y_i - y\|^2\}.$$

For almost all  $t \in I$ , each  $G_i(t, y)$  is nonempty by assumption (1.15). Furthermore, since  $F(\cdot, \cdot)$  is well-defined on each  $I_r \times \mathbb{R}^n$  we can extend  $v_i(\cdot)$ , if it is necessary, to each  $\tilde{I}_r$  such that now  $H_F(t, y_i, \zeta) = \langle v_i(t), \zeta \rangle$  holds for all  $t \in \bigcup_{r \geq 1} \tilde{I}_r$ . Therefore, we can assume that  $G_i(\cdot, \cdot)$  is also well-defined on  $\bigcup_{r \geq 1} (\tilde{I}_r \times \mathbb{R}^n)$ . By proceeding in similar way as in Proposition 4.2 it can be shown that every  $G_i(\cdot, \cdot)$  satisfies (SH). Proposition 4.2 then implies the existence of a null set  $A_i \subset I$  for which

$$h_{G_i}(t, x, \zeta) \leq 0 \quad \forall t \in I \setminus A_i. \quad (4.10)$$

Let  $\hat{A} := \bigcup_i A_i$ , which also has null measure. It was noted in chapter 1 that

$$\text{cl}(\tilde{I}_r) = \text{cl}\{\tilde{I}_r \setminus \hat{A}\},$$

and therefore (since  $\bar{t} \in \tilde{I}_r$ ), there exists a sequence  $\{t_j\} \subseteq \tilde{I}_r \setminus \hat{A}$  such that  $t_j \rightarrow \bar{t}$ . It follows from (4.10) that  $h_{G_i}(t_j, x, \zeta) \leq 0$  for all  $j \geq 1$  and all  $i \geq 1$ . In particular,

$$h_{G_i}(t_i, x, \zeta) \leq 0 \quad \text{for all } i \geq 1. \quad (4.11)$$

Now let  $w_i \in G_i(t_i, x)$  satisfy  $\langle w_i, \zeta \rangle = h_{G_i}(t_i, x, \zeta)$ . Due to the compactness of  $F(\bar{t}, x)$  the sequences  $v_i(t_i)$  and  $w_i$  are bounded (since  $(t_i, y_i) \rightarrow (\bar{t}, x)$  and  $F$  is upper semicontinuous at  $(\bar{t}, x)$ ). Also,  $k(t_i) \rightarrow k(\bar{t})$  due to the continuity of  $k(\cdot)$  on  $I_r$ . Rearranging terms from the definition of  $G_i(t_i, x)$  yields

$$\left\langle v_i(t_i), \frac{y_i - x}{\|y_i - x\|} \right\rangle \leq k(t_i)\|y_i - x\| + \left\langle w_i, \frac{y_i - x}{\|y_i - x\|} \right\rangle. \quad (4.12)$$

The following is justified by the properties of  $v_i(t_i)$ , since  $y_i \rightarrow_\zeta x$  and  $t_i \rightarrow \bar{t}$ , by (4.12), the properties of  $w_i$ , and by (4.11), respectively.

$$\begin{aligned}
\liminf_{i \rightarrow \infty} H_F(t_i, y_i, \zeta) &= \liminf_{i \rightarrow \infty} \langle v_i(t_i), \zeta \rangle \\
&= \liminf_{i \rightarrow \infty} \|\zeta\| \left\langle v_i(t_i), \frac{y_i - x}{\|y_i - x\|} \right\rangle \\
&\leq \liminf_{i \rightarrow \infty} \|\zeta\| \left\langle w_i, \frac{y_i - x}{\|y_i - x\|} \right\rangle \\
&= \liminf_{i \rightarrow \infty} \langle w_i, \zeta \rangle \\
&= \liminf_{i \rightarrow \infty} h_{G_i}(t_i, x, \zeta) \leq 0.
\end{aligned}$$

Hence condition (4.8) is satisfied with the liminf taken all over the sequences  $y_i \rightarrow_\zeta x$  and all  $t_i \rightarrow \bar{t}$ . However,  $\bar{t}$  does not depend on the particular sequences, and so (4.8) holds as stated.  $\square$

Notice that when the system (4.1) is autonomous, the limsup in (4.7), and the liminf in (4.8) can be replaced by a limsup taken only over  $y \rightarrow_\zeta x$ . The following characterization establishes this under some generality.

**Corollary 4.4.** *Suppose  $F$  satisfies (SH) and (1.15), but the almost upper semi-continuity requirement is strengthened by assuming  $F(\cdot, x)$  is uniformly continuous with modulus of continuity independent of  $x$  on bounded sets, and  $F(t, \cdot)$  is upper semicontinuous for all  $t \in I$ . Then the system  $(S, F)$  is strongly invariant if and only if*

$$\limsup_{y \rightarrow_\zeta x} H_F(t, y, \zeta) \leq 0, \tag{4.13}$$

for all  $t \in I$ , all  $x \in S$ , and all  $\zeta \in N_S^P(x)$ .  $\square$

**Remark 4.5.** *Taking the liminf and limsup over  $y \rightarrow_\zeta x$  in the previous results can be replaced by the a priori weaker conditions of taking these limits over  $\delta \rightarrow 0^+$  and with  $y = x + \delta\zeta$  without changing the necessity or the sufficiency of strong invariance.*



## 4.4 Time Dependent Constraints.

We now suppose the constraint set  $S$  depends on  $t$ , and is a multifunction  $S : I \rightrightarrows \mathbb{R}^n$  with nonempty closed values. Weak (resp. strong) invariance concepts are naturally extended to this fully nonautonomous case by requiring the existence of at least one (resp. all) solution(s) of (4.1) to satisfy

$$x(t) \in S(t) \quad \forall t \in [t_0, t_1]. \quad (4.14)$$

We follow [32] by assuming the constraint multifunction is *absolutely continuous*, which means that for every  $[t_0, t_1] \subset I$ , every compact  $C \subset \mathbb{R}^n$ , and  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any sequence  $\{(t_i, s_i)\}_{i=1}^m$  of pairwise disjoint subintervals of  $[t_0, t_1]$ , one has that  $\sum_{i=1}^m (s_i - t_i) < \delta$  implies

$$\sum_{i=1}^m \max \left[ \text{ex} \left( S(t_i) \cap C, S(s_i) \right), \text{ex} \left( S(s_i) \cap C, S(t_i) \right) \right] < \varepsilon,$$

where  $\text{ex}(S_1, S_2) := \inf \{ \alpha > 0 : S_1 \subset S_2 + \alpha B_n \}$  denotes the excess function of a compact  $S_1$  from a closed set  $S_2$ .

The following statement (Theorem 4.6) is a time-dependent constraint version of Theorem 3.1 and is proved in [32]. In contrast with Theorem 3.1, the authors in [32] use a bigger cone to characterize weak invariance in terms of Hamilton-Jacobi inequalities. This cone is referred as the “subnormal cone” of  $S$  at  $x$ , and it turns out to be the polar of the contingent cone of  $S$  at  $x$ :  $N_S^\circ(x) := [T_S^B(x)]^\circ$  (recall (1.6) and (1.7)). The almost upper semicontinuity requirement in (SH) is also replaced in [32] by the following stronger assumptions (SSH) (see Theorem 1.8), which we now also adopt:

- $F(\cdot, \cdot)$  is  $\mathcal{L} \times \mathcal{B}$  (Lebesgue-Borel)-measurable;
- $F(t, \cdot)$  is upper semicontinuous on  $\mathbb{R}^n$  for almost all  $t \in I$ ; that is, the graph  $\text{gr } F(t, \cdot) := \{(x, v) : v \in F(t, x)\}$  is closed in  $\mathbb{R}^n \times \mathbb{R}^n$  for almost all  $t \in I$ .

**Theorem 4.6.** *Let  $\tilde{I} \subseteq I$  be a subinterval, and suppose the multifunction  $G : \tilde{I} \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  satisfies (SH) on  $\tilde{I} \times \mathbb{R}^n$  with the almost upper semicontinuity assumption replaced by (SSH). Assume  $S : \tilde{I} \rightrightarrows \mathbb{R}^n$  is absolutely continuous with closed values. Then the following conditions are equivalent:*

(i) *the system  $(S, G)$  is weakly invariant.*

(ii) *there exists a null set  $A \subset \tilde{I}$  such that*

$$-p + H_G(t, x, -\zeta) \geq 0, \quad (4.15)$$

*for all  $t \in \tilde{I} \setminus A$ , all  $x \in S(t)$ , and all  $(p, \zeta) \in N_{\text{gr}S}^\circ(t, x)$ .*

(iii) *there is a null set  $D \subset \tilde{I}$  such that*

$$(\{1\} \times G(t, x)) \cap \overline{\text{co}}(T_{\text{gr}S}(t, x)) \neq \emptyset, \quad (4.16)$$

*for all  $t \in \tilde{I} \setminus D$ , and all  $x \in S(t)$ . □*

The following sufficient condition extends the normal-type result contained in Theorem 2 of [26]. Under the new set of hypotheses (SSH), we require the submultifunctions  $G(\cdot, \cdot) \subset F(\cdot, \cdot)$  to satisfy the conditions of Theorem 4.6.

**Theorem 4.7.** *In addition to (SH) and the reinforce conditions (SSH) imposed on  $(S, F)$ , suppose  $F$  also satisfies (1.15). If there is a null set  $A \subset I$  such that*

$$p + \limsup_{y \rightarrow_\zeta x, \tau \rightarrow t} H_F(\tau, y, \zeta) \leq 0 \quad (4.17)$$

*holds for all  $t \in I \setminus A$ , all  $x \in S(t)$ , and for all  $(p, \zeta) \in N_{\text{gr}S}^\circ(t, x)$ , then the system  $(S, F)$  is strongly invariant.*

**Proof.** The proof follows the same argument used in the proof of Theorem 4.3. Indeed, we need to make use of the time-dependent constraint version of Proposition 4.2 (invariance principle), whose proof will be built-in here. In fact, let  $G$  be

any submultifunction of  $F$  defined on  $\tilde{I} \times \mathbb{R}^n$ , with  $\tilde{I} \subseteq I$  a subinterval, and  $A \subset I$  be the null set such that (4.17) holds. Let again  $\{J_r\}_{r \geq 1}$  be a family of pairwise disjoint closed subsets of  $\tilde{I}$  such that  $G(\cdot, \cdot)$  is upper semicontinuous on  $J_r \times \mathbb{R}^n$  for all  $r$ , the set  $\bigcup_{r \geq 1} J_r \subset I$  has full measure, and  $\tilde{J}_r$  denotes the points of density of  $J_r$ . Consider the set with null measure  $\tilde{A} := \tilde{I} \setminus \left( \bigcup_{k \geq 1} \tilde{J}_k \right)$ . Keeping the same labels and meaning of the sequences involved in the proof of part (i) in Theorem 4.3 we have that, for each  $t \in \tilde{I} \setminus (A \cup \tilde{A})$ ,  $x \in S(t)$ , and  $(p, \zeta) \in N_{grS}^\circ(t, x)$ , the following estimates hold

$$\begin{aligned} p + h_G(t, x, \zeta) &\leq p + \liminf_{z \rightarrow x, \rho \rightarrow t} h_G(\rho, z, \zeta) \\ &\leq p + \liminf_{y \rightarrow_\zeta x, \lambda \rightarrow t} h_G(\lambda, y, \zeta) \\ &\leq p + \limsup_{y \rightarrow_\zeta x, \tau \rightarrow t} H_F(\tau, y, \zeta) \leq 0. \end{aligned}$$

The Hamiltonian identity  $H_G(t, x, -\zeta) = -h_G(t, x, \zeta)$  lets us apply Theorem 4.6 to guarantee the weak invariance of the subsystem  $(S, G)$ . In particular, and in the same spirit of Proposition 4.2, given an interval  $[t_0, t_1] \subset I$ , an initial condition  $x_0 \in S(t_0)$ , and a solution  $y(\cdot)$  to (4.1), it is now routine to show the auxiliary multifunction

$$\tilde{G}(t, x) := \{w \in F(t, x) : \langle \dot{y}(t) - w, y(t) - x \rangle \leq k(t) \|y(t) - x\|^2\},$$

satisfies the assumptions of Theorem 4.6 on  $[t_0, t_1] \times \mathbb{R}^n$ , and therefore  $(S, \tilde{G})$  is weakly invariant. Let then  $x(\cdot)$  be a solution to the problem

$$\begin{aligned} \dot{x}(t) &\in \tilde{G}(t, x(t)) \quad \text{a.e., } t \in [t_0, t_1], \\ x(t_0) &= x_0 \in S(t_0), \quad x(t) \in S(t), \forall t \in [t_0, t_1]. \end{aligned} \tag{4.18}$$

According to the definition of  $\tilde{G}$  the solution  $x(\cdot)$  must satisfy

$$\frac{d}{dt} \|x(t) - y(t)\|^2 \leq 2k(t) \|x(t) - y(t)\|^2 \quad \text{a.e., } t \in [t_0, t_1],$$

which by Gronwall's inequality again implies  $y(t) = x(t) \in S(t)$  for all  $t \in [t_0, t_1)$ . Hence, the system  $(S(\cdot), F)$  is strongly invariant. This establishes the sufficiency of (4.17).  $\square$

**Remark 4.8.** *Due to Theorem 4.6, and the invariance principle included in the proof of Theorem 4.7, the following tangential condition can be automatically added as a criterion for strong invariance:*

For every submultifunction  $G$  of  $F$  defined on  $\tilde{I} \times \mathbb{R}^n$ , there exists a null set  $A_G \subset \tilde{I}$  such that  $(\{1\} \times G(t, x)) \cap \overline{\text{co}}(T_{\text{gr}S}(t, x)) \neq \emptyset$ , for all  $t \in \tilde{I} \setminus A_G$ , and all  $x \in S(t)$ .

# Chapter 5

## Hamilton-Jacobi Theory

The last part of this thesis is devoted to an application of invariance results to Hamilton-Jacobi theory. Given a closed target set  $S$ , we consider the minimal time function  $T_S(\cdot)$  of an autonomous monotone Lipschitz dynamic and prove that  $T_S$  is the unique proximal semi-solution to an approximate Hamilton-Jacobi equation. Inspired by the ideas of Wolenski and Zhuang [52], we are able to establish local versions of the invariant results that were proved in chapter 4 for upper semi-continuous dynamics. These local versions are then applied to particular invariant state-augmented systems, resulting in an characterization of the minimal time function in terms of approximate Hamiltonian inequalities, which involve an analytic boundary condition. When the system is modeled as the sum of a monotone and a monotone Lipschitz set-valued map, we show how the approximate inequalities turn into exact ones, thanks to the special character of the monotonicity property. The exposition will be based on the results contained in joint work with Donchev and Wolenski [29], but of course will be more expanded in details.

### 5.1 Assumptions

Throughout this chapter we consider a minimal time control problem which consists of a closed target set  $S$  and a system modeled as the autonomous differential inclusion

$$\begin{aligned}\dot{x}(t) &\in F(x(t)) \quad \text{a.e., } t \in I \\ x(0) &= x,\end{aligned}\tag{5.1}$$

where  $I := [0, \infty)$ , and  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is a multifunction satisfying the following standing hypotheses (SH).

- For each  $x \in \mathbb{R}^n$ ,  $F(x) \subset \mathbb{R}^n$  is nonempty, convex, and compact;
- $F(\cdot)$  is upper semicontinuous on  $\mathbb{R}^n$ ; that is, that the graph  $\text{gr } F(\cdot) := \{(x, v) : v \in F(x)\}$  is closed in  $\mathbb{R}^n \times \mathbb{R}^n$ ;
- There exist a constant  $c$  so that  $\sup\{\|v\| : v \in F(x)\} \leq c(1 + \|x\|)$  for all  $x \in \mathbb{R}^n$ .

The goal in the minimal time control problem is to steer an initial point  $x$  to the target set along a trajectory of the system in minimal time. The minimal time value is denoted by  $T_S(x)$ , and the function  $x \rightarrow T_S(x)$  is called the minimal time function (see below for a more formal definition). Our goal in this chapter is to show that, under the monotone Lipschitz assumption on the multifunction  $F$ , the minimal time function is the unique proximal semi-solution of an approximate Hamilton-Jacobi equation satisfying an approximate boundary condition. The word “approximate” means that limiting Hamiltonian inequalities will replace the exact Hamilton-Jacobi equation that is obtained when a Lipschitz property is satisfied by the dynamics (see [52]). In the same spirit of [52], local versions of the Hamiltonian characterizations of invariance will be pertinent. For such a reason, we will present the proof of the invariant results that were obtained in the last chapter with convenient modifications as to hold under the local requirements.

In the previous chapter we used the language of submultifunctions to access some of the invariance properties that were hidden due to the lack of continuous selections. In the present discussion we will benefit in the same way, by updating the definition of submultifunction to make it fit the local invariance setting, which we will now proceed to do.

**Definition 5.1.** We will say that  $G : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is a submultifunction of  $F$  if it satisfies  $G(t, x) \subset F(x)$  for all  $x \in \mathbb{R}^n$  and almost all  $t \in I$ , and the following nonautonomous hypotheses (NSH)

- For each  $(t, x) \in I \times \mathbb{R}^n$ ,  $G(t, x) \subset \mathbb{R}^n$  is nonempty, convex, and compact;
- $G(\cdot, \cdot)$  is almost upper semicontinuous on  $I \times \mathbb{R}^n$ ; that is, for every compact interval  $\tilde{I} \subset I$  and each  $\varepsilon > 0$ , there exists a closed set  $\mathcal{N}_\varepsilon \subseteq \tilde{I}$  with Lebesgue measure  $\mu(\tilde{I} \setminus \mathcal{N}_\varepsilon) < \varepsilon$ , and so that the graph  $\text{gr } G(\cdot, \cdot) := \{(t, x, v) : v \in G(t, x)\}$  is closed in  $\mathcal{N}_\varepsilon \times \mathbb{R}^n \times \mathbb{R}^n$ .

Notice that the linear growth condition

$$\sup\{\|v\| : v \in G(t, x)\} \leq c(1 + \|x\|) \quad \forall x \in \mathbb{R}^n, \text{ and a.e., } t \in I$$

is obviously inherited from  $F$  due to the definition of submultifunction. Let now  $\tau \in I$  and consider the following auxiliary control system

$$\begin{aligned} \dot{x}(t) &\in G(t, x(t)) \quad \text{a.e., } t \in [\tau, \tau + T) \\ x(\tau) &= x, \end{aligned} \tag{5.2}$$

where  $G$  is a submultifunction of  $F$  and  $T > 0$ . It is clear that any trajectory  $x(\cdot)$  of (5.2) is also a trajectory of (5.1) on the interval  $[\tau, \tau + T)$ . With this in mind, we introduce the following time-dependent version of the terminology given in [52], and point out some implications on the submultifunction issue.

**Definition 5.2.** Suppose  $G : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is a submultifunction of  $F$ , and let  $U \subset \mathbb{R}^n$  be open. Let  $x(\cdot)$  be a trajectory of  $G$  that is defined on  $[\tau, \tau + T) \subseteq I$  with  $x(\tau) = x \in U$ . Then  $T$  is an escape time from  $U$ , in which case we write  $T := \text{Esc}(x(\cdot); U)$ , provided at least one of the following conditions holds:

- (a)  $T = \infty$  and  $x(t) \in U$  for all  $t \geq \tau$ ,

- (b)  $x(t) \in U$  for all  $t \in [\tau, \tau + T)$  and  $\|x(t)\| \rightarrow \infty$  as  $t \uparrow \tau + T$ , or  
(c)  $T < \infty$ ,  $x(t) \in U$  for all  $t \in [\tau, \tau + T)$ , and  $d_{U^c}(x(t)) \rightarrow 0$  as  $t \uparrow \tau + T$ .

The main observation about the previous definition is that under (SH), any trajectory of  $G$  can be extended to a trajectory that has an escape time. We record this in the next result.

**Proposition 5.3.** *Let  $x(\cdot)$  be a trajectory of  $G$  defined on  $[\tau, \tau + T)$  with  $T > 0$ . If  $T$  is not an escape time from  $U$ , then  $x(\cdot)$  can be extended to a trajectory  $\tilde{x}(\cdot)$  defined on a strictly larger interval  $[\tau, \tau + \tilde{T})$  (i.e.  $\tilde{T} > T$ ), and in which  $\tilde{T} = \text{Esc}(\tilde{x}(\cdot); U)$ .*

**Proof.** The argument for this proof naturally extends that of Proposition 2.5 of [52] and depends only on Proposition 1.12, which is a result regarding the trajectories of (5.2).  $\square$

We now refresh another definition given in [52]. Suppose  $U \subseteq \mathbb{R}^n$  is open,  $\tau \geq 0$ , and  $x \in U$ . The set of all trajectories of  $G$  originating from  $x$  at time  $\tau$  that remain in  $U$  over a maximal interval is denoted by  $\Upsilon_{(G,U)}(\tau, x)$ . That is,  $\Upsilon_{(G,U)}(\tau, x)$  consists of those trajectories  $x(\cdot)$  of  $G$  defined on a half-open interval  $[\tau, \tau + T)$  with  $x(\tau) = x$  and for which  $\text{Esc}(x(\cdot); U) = T$ . By Proposition 5.3, the set  $\Upsilon_{(G,U)}(\tau, x)$  is nonempty for each  $(\tau, x)$ .

**Remark 5.4.** *Due to the time-independence of  $F$ , in the case  $G = F$  we can suppose without loss of generality that  $\tau = 0$  for any trajectory  $x(\cdot)$  of  $G$  with initial point  $x(\tau) = x \in U$ , and therefore we write  $\Upsilon_{(F,U)}(x)$  instead of  $\Upsilon_{(F,U)}(0, x)$ .*

## 5.2 The Minimal Time Function

Suppose that  $S \subset \mathbb{R}^n$  is a closed set. The minimal time function  $T_S(\cdot) : \mathbb{R}^n \rightarrow [0, \infty]$  is defined as

$$T_S(x) := \inf\{T : \exists x(\cdot) \text{ satisfying (5.1) and } x(T) \in S\}. \quad (5.3)$$



If no trajectory of  $F$  originating from  $x$  can reach  $S$  in finite time, then the above infimum is taken over the empty set, and hence  $T_S(x) = \infty$  in this case, which is the usual convention. If  $x \in S$ , then  $T(x) = 0$  by definition, which is consistent with the above definition if we allow trajectories to be defined on the degenerate interval  $[\tau, \tau]$ .

The following result provides perhaps the most remarkable properties of the minimal time function that we need to use along the chapter. We recommend the reader to consult its proof given in detail in proposition 2.6 of [52].

**Proposition 5.5.** *Assume  $F$  satisfies (SH). If  $x \in S^c \cap \text{dom } T_S$ , then there exists  $x(\cdot) \in \Upsilon_{(F, S^c)}(x)$  with  $\text{Esc}(x(\cdot), S^c) = T_S(x)$  and  $x(T_S(x)) \in S$  (that is, the infimum in (5.3) is attained). Furthermore,  $T_S(\cdot)$  is lower semicontinuous on  $\mathbb{R}^n$ .  $\square$*

### 5.3 Invariance

We shall apply invariance results to objects obtained through modifying the given data thus these concepts are introduced in terms other than  $S$  and  $F$ . Moreover, since the notion of submultifunction is a key object to establish invariance criteria to be used in the next section, we present our definitions based on this language, which is consistent with the particular case  $G = F$ . Moreover, we require our notions to be local, which contrasts with the global setting used in chapter 4.

Suppose  $E \subset \mathbb{R}^n$  is nonempty,  $U \subset \mathbb{R}^n$  is open. Let  $G : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  be a submultifunction of  $F$ .

- We say that  $(E, G)$  is weakly invariant in  $U$  provided that for all  $(\tau, x) \in I \times (E \cap U)$ , there exists a solution  $x(\cdot)$  to (5.2) whose extension  $\tilde{x}(\cdot) \in \Upsilon_{(G, U)}(\tau, x)$  satisfies  $\tilde{x}(t) \in E$  for all  $t \in [\tau, \tau + \text{Esc}(\tilde{x}(\cdot); U))$ .

- We say  $(E, G)$  is strongly invariant in  $U$  provided that for every  $(\tau, x) \in I \times (E \cap U)$ , every solution  $x(\cdot)$  to (5.2) has the property that its extension  $\tilde{x}(\cdot) \in \Upsilon_{(G,U)}(\tau, x)$  satisfies  $\tilde{x}(t) \in E$  for all  $t \in [\tau, \tau + Esc(\tilde{x}(\cdot); U))$ .

We now close this section with two autonomous results that appeared in [52]. These results relate the previous invariance concepts to the minimal time problem by means of a comparison between  $T_S(\cdot)$  and certain lower semicontinuous function  $\theta(\cdot)$ . The proof of the first result (Proposition 5.6 below) can be consulted in Proposition 3.1 of [16], while the proof for Proposition 5.7 is provided here for completeness. Recall that the closed set  $S$  is given. For a multifunction  $\Gamma : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ , we consider the augmentation  $-\Gamma \times \{1\} : I \times \mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^{n+1}$  defined as

$$(-\Gamma \times \{1\})(t, (x, r)) := \{(-v, 1) : v \in \Gamma(t, x)\} \subset \mathbb{R}^{n+1}.$$

A similar notation is in effect for the multifunction  $\Gamma \times \{-1\}$ .

**Proposition 5.6.** *Suppose  $F$  satisfies (SH), and let  $E := \text{epi } T_S$ . Then  $(E, F \times \{-1\})$  is weakly invariant in  $U := S^c \times \mathbb{R}$ . Even more,  $(E, -F \times \{1\})$  is strongly invariant in  $\mathbb{R}^{n+1}$ .  $\square$*

**Proposition 5.7.** *Suppose  $F$  satisfies (SH), and  $\theta : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is lower semicontinuous and satisfies  $\theta(s) = 0$  for all  $s \in S$ . Let  $E := \text{epi } \theta$  and  $U := S^c \times \mathbb{R}$ .*

- (a) *If  $(E, F \times \{-1\})$  is weakly invariant in  $U$  and  $\theta(\cdot)$  is bounded below on  $\mathbb{R}^n$ , then  $\theta(x) \geq T_S(x)$  for all  $x \in \mathbb{R}^n$ .*
- (b) *If  $(E, -F \times \{1\})$  is strongly invariant in  $\mathbb{R}^{n+1}$ , then  $\theta(x) \leq T_S(x)$  for all  $x \in \mathbb{R}^n$ .*

**Proof.** We first proceed to show that (a) holds. Let  $x \in \mathbb{R}^n$ . By definition of  $T_S$  we see that the conclusion is clear if  $x \in S$  or  $\theta(x) = \infty$ . Therefore, we can assume that  $x \in S^c \cap \text{dom } \theta$ . By the assumption of weak invariance, there exists a trajectory

$z(\cdot) \in \Upsilon_{(F \times \{-1\}, U)}(x, \theta(x))$  satisfying  $z(t) \in E$  for  $t \in [0, Esc(z(\cdot); U)]$ . Note that necessarily  $z(t) = (x(t), \theta(x) - t)$ , where  $x(\cdot) \in \Upsilon_{(F, S^c)}(x)$ . By the definition of  $U$  we have

$$T := Esc(z(\cdot), U) = Esc(x(\cdot), S^c). \quad (5.4)$$

Notice that the definition of  $E$  and the fact that  $z(t) \in E$  for  $t \in [0, Esc(z(\cdot); U)]$  imply

$$\theta(x(t)) \leq \theta(x) - t \text{ for all } t \in [0, T]. \quad (5.5)$$

Since  $\theta$  is bounded below, it follows from (5.5) that  $T < \infty$ . By the linear growth condition on  $F$  and (5.4) we obtain from a standard argument that  $\inf_{t \in [0, T]} \|x(t)\| < \infty$ . The application of Proposition 1.12 then leads to  $\lim_{t \uparrow T} x(t) = y \in S$ . Let us set  $x(T) = y$ . Moreover, observe that by the lower semicontinuity of  $\theta(\cdot)$  we have

$$\theta(x(T)) \leq \liminf_{t \uparrow T} \theta(x(t)) \leq \liminf_{t \uparrow T} (\theta(x) - t) = \theta(x) - T, \quad (5.6)$$

and the boundary condition on  $\theta$  says that  $\theta(x(T)) = 0$ . Hence from (5.6) we have  $\theta(x) \geq T$ . Finally, the definition of  $T_S$  as an infimum yields to  $T \geq T_S(x)$ , and we conclude that  $\theta(x) \geq T \geq T_S(x)$ , which is (a). Now we prove the complementary inequality proposed in (b). In fact, let again  $x \in \mathbb{R}^n$ . If  $T_S(x) = \infty$  or  $x \in S$  then there is nothing to show. Therefore, we can assume  $x \in S^c \cap \text{dom } T_S$ . Let  $\eta > 0$ . By definition of the minimal time function there exists  $x(\cdot) \in \Upsilon_{F, S^c}(x)$  with  $Esc(x(\cdot); S^c) =: T < T_S(x) + \eta$  and  $x(T) \in S$ . Let us define  $z(t) := (x(T - t), t)$  which is obviously a trajectory of  $-F \times \{1\}$  originating from  $(x(T), 0) \in E$  (notice that indeed  $\theta(x(T)) = 0$ ). By the strong invariance assumption the trajectory  $z(\cdot)$  remains in  $E$ , and thus by definition of  $E$  we must have

$$t \geq \theta(x(T - t)) \text{ for all } t \in [0, T].$$

In particular, by setting  $t = T$  in the previous inequality we obtain

$$\theta(x) = \theta(x(0)) \leq T < T_S(x) + \eta.$$

Since  $\eta > 0$  was taken arbitrarily, it follows that (b) holds.  $\square$

## 5.4 Hamiltonians Inequalities

In this section we present new local Hamiltonian criteria for invariance. For this purpose, we recall one more time a definition of great use in flow invariance theory and optimization. The *minimized Hamiltonian* associated to the multifunction  $G : I \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  is the single-valued function  $h_G : I \times \mathbb{R}^{m+n} \rightarrow \mathbb{R}$  defined as

$$h_G(t, x, \zeta) = \inf\{\langle v, \zeta \rangle : v \in G(t, x)\}.$$

We have already pointed out how classic strong invariance theorems for multifunctions have required the satisfaction of a Lipschitz hypothesis, and we remarked this last property can be stated in terms of upper Hamiltonians if we assume the convex valuedness of  $F$ . An alternative version of the Lipschitz definition for multifunctions is given in terms of the lower Hamiltonian by simply considering Proposition 1.9(a). Since we attempt to keep the presentation similar to the one provided in [52], it is pertinent at this point to refresh the main structural hypotheses imposed on the multifunction by using the “minimum” approach: The multifunction  $F$  is locally Lipschitz if for any bounded  $C \subset \mathbb{R}^n$  there exists a constant  $k_C$  such that

$$|h_F(x, \zeta) - h_F(y, \zeta)| \leq k_C \|\zeta\| \|x - y\| \quad (5.7)$$

for all  $x, y \in C$ , all  $\zeta \in \mathbb{R}^n$ . Given a closed set  $E \subseteq \mathbb{R}^n$  and  $U \subseteq \mathbb{R}^n$  open, the local Hamiltonian criterion in Theorem 3.1(b) of [52] establishes that under the Lipschitz property, the system  $(E, F)$  is strongly invariant in  $U$  if and only if

$$h_F(x, -\zeta) \geq 0, \quad \text{for all } x \in E \cap U \text{ and all } \zeta \in N_E^P(x). \quad (5.8)$$

The strong invariance results of this chapter will hold under the conjugate of the following global *monotone Lipschitz* (ML) condition: there exists a constant  $m$  for which

$$h_F(x, x - y) - h_F(y, x - y) \geq m \|x - y\|^2 \quad (5.9)$$

for all  $x, y \in \mathbb{R}^n$ . It is clear that the global (5.7) implies (5.9) with  $m = -k$  ( $= -k_{\mathbb{R}^n}$ ). However, condition (5.9) is strictly weaker than (5.7) as it can be appreciated in the following simple example, which is the negative version of (2.15): let  $F : \mathbb{R} \rightrightarrows \mathbb{R}$  defined by

$$F_1(x) := \begin{cases} 1 & \text{if } x > 0 \\ [-1, 1] & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

The multifunction  $F_1$  satisfies (5.9) but not (5.7). Another simple example is  $F_2(x) = \{\text{sgn}(x)\sqrt{|x|}\}$ . By changing the polarity in the previous definition, we recover the now familiar dissipative Lipschitz (DL) property: for each bounded  $C \subset \mathbb{R}^n$  there exists a constant  $k_C$  such that

$$h_F(x, x - y) - h_F(y, x - y) \leq k_C \|x - y\|^2 \quad (5.10)$$

for all  $x, y \in C$ . It is clear that  $F$  is monotone Lipschitz if and only if its conjugate  $-F$  is dissipative Lipschitz. Notice that  $-F_1$  also satisfies (SH), and for  $E = \{0\}$  and  $U = \mathbb{R}$  the system  $(E, -F_1)$  is strongly invariant in  $U$ . Nevertheless, condition (5.8) fails since  $h_{(-F_1)}(0, -\zeta) < 0$  for any  $\zeta \in N_E^P(0) \cap (0, \infty)$ .

The next weak invariant result for submultifunctions is the local version of Theorem 4.2. Since its proof proceeds almost identically as in the mentioned Theorem, most of the details will be omitted, but we will emphasize those points where both proofs differ.

**Theorem 5.8.** *Let  $F$  satisfy (SH), and suppose  $G$  is a submultifunction of  $F$ , and  $E \subset \mathbb{R}^n$  is closed. Then  $(E, G)$  is weakly invariant in  $U$  if and only if there exists a null set  $A \subset I$  such that*

$$h_G(t, x, \zeta) \leq 0, \quad (5.11)$$

for all  $t \in I \setminus A$ ,  $x \in E \cap U$ , and  $\zeta \in N_E^P(x)$ .

**Proof.** The proof of the necessity follows without any obstacle the same argument given in Proposition 4.2, and we only need to point out that the inclusion  $T_E^B(x) \subset [N_E^P(x)]^\circ$  is in particular satisfied for all  $x \in E \cap U$ , which implies that (5.11) holds as stated.

For the converse, let  $x_0 \in E \cap U$  and  $(\tau, x_0) \in I \times (E \cap U)$  be given. Let  $\varepsilon > 0$  be such that  $S := \{x_0\} + \varepsilon B_n \subset U$ . According to Proposition 1.12 there is  $T > 0$  such that any trajectory  $x(\cdot)$  defined on  $[\tau, \tau + T]$ , with  $x(\tau) = x_0$ , is totally contained in  $S$ . Let  $\{\pi\}$  be a sequence of partitions (not necessarily uniform) of the interval  $[\tau, \tau + T]$  satisfying  $\mathcal{D}(\pi) \rightarrow 0$ . Assume that a typical element  $\pi$  of the sequence has nodes  $\tau = t_0 < t_1 \cdots < t_N = \tau + T$ . As in Proposition 4.2 (see also [16]) it is possible to construct a sequence of approximate trajectories  $\{x^\pi(\cdot)\}$  on  $[\tau, \tau + T]$ , which are piecewise defined as follows: for every  $j = 1, 2, \dots$  and every  $t \in [t_j, t_{j+1}]$

$$\begin{aligned} x^\pi(t) &:= x_j(t), \\ x_j(t) &:= x_{j-1}(t_j) + \int_{t_j}^t g_j(r) dr, \\ g_j(t) &\in G(t, s_j). \end{aligned}$$

In the previous  $s_j \in \text{proj}_{(E \cap S)}(x_{j-1}(t_j))$  and  $g_j(\cdot)$  is a measurable selection taken from the auxiliary multifunction

$$t \mapsto \{v \in G(t, s_j) : h_G(t, s_j, x_j - s_j) = \langle v, x_j - s_j \rangle\},$$

which is measurable and closed-valued in  $[t_j, t_{j+1}]$ , and all the properties above holding for  $j = 1, 2, \dots$ . Using Corollary 1.11 we obtain a convergent subsequence of  $\{x^\pi(\cdot)\}$ , whose limit is a solution  $x(\cdot)$  to (5.2) defined on  $[\tau, \tau + T]$ , and with range contained in  $E \cap S \subseteq E \cap U$ . We now can choose a trajectory  $\tilde{x}(\cdot)$  on a half-open interval  $[\tau, \tau + \tilde{T})$ , with maximal range that remains in  $E \cap U$ , and we must then have  $\tilde{T} = Esc(x(\cdot); U)$ , from which the sufficiency of (5.11) follows.  $\square$

The next result is the local-autonomous version of the characterization for strong invariance given in Proposition 4.2, and which applies to systems satisfying (5.10). The proof is very similar to that of Proposition 4.2. However, some considerations were made to preserve the same argument under the new definition of submultifunctions.

**Proposition 5.9. (*Local invariance principle*)** *Let  $F$  satisfy (SH) and (5.10). The system  $(E, F)$  is strongly invariant in  $U$  if and only if for every submultifunction  $G$  of  $F$ , there exists a null set  $A_G \subset I$  such that  $h_G(t, x, \zeta) \leq 0 \forall \zeta \in N_E^P(x)$ ,  $\forall x \in E \cap U$ , and  $\forall t \in I \setminus A_G$ .*

**Proof.** We first consider the “if” direction. Let  $x_0 \in E$ , and a trajectory  $x(\cdot) \in \Upsilon_{(F,U)}(x_0)$ . Let  $T := Esc(x(\cdot); U)$ , and consider the multifunction  $G : [0, T) \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  defined as

$$G(t, x) := \{v \in F(x) : \langle \dot{x}(t) - v, x(t) - x \rangle \leq k \|x(t) - x\|^2\}. \quad (5.12)$$

In Proposition 4.2 we gave the detailed argument to show that  $G$  indeed inherits the properties (SH) from  $F$  on the interval  $[0, T)$ . We need now to distinguish two cases:

$T = \infty$ : It follows then that  $G$  is a submultifunction of  $F$ , and by hypothesis, there exists a null set  $A_G \subset I$  such that  $h_G(t, x, \zeta) \leq 0 \forall \zeta \in N_E^P(x)$ ,  $\forall x \in E \cap U$ ,

and  $\forall t \in I \setminus A_G$ . This last implies, according to Theorem 5.8, that the pair  $(E, G)$  is weakly invariant in  $U$ , and therefore there exists a trajectory  $y(\cdot)$  with invariant extension  $\tilde{y}(\cdot) \in \Upsilon_{(F,U)}(x_0)$  defined on  $[0, Esc(\tilde{y}(\cdot); U))$ . Let  $s := Esc(\tilde{y}(\cdot); U)$ . We claim that necessarily  $s = \infty$ . In fact, let us assume that  $s < \infty$ . From the definition of  $G$  we must have

$$\langle \dot{x}(t) - \dot{\tilde{y}}(t), x(t) - \tilde{y}(t) \rangle \leq k \|x(t) - \tilde{y}(t)\|^2 \quad \text{a.e., } t \in [0, s),$$

which immediately implies

$$\frac{d}{dt} \|x(t) - y(t)\|^2 \leq 2k \|x(t) - y(t)\|^2 \quad \text{a.e., } t \in [0, s).$$

It follows from Gronwall's inequality that  $x(t) = \tilde{y}(t)$  for all  $t \in [0, s)$ . By the continuity of  $x(\cdot)$  we have  $\lim_{t \uparrow s} \tilde{y}(t) = \lim_{t \uparrow s} x(t) = x(s)$ , from which it is easy to extend  $\tilde{y}(\cdot)$  to the whole interval  $[0, Esc(x(\cdot); U))$  by defining

$$z(t) := \begin{cases} \tilde{y}(t) & \text{if } t \in [0, s) \\ x(t) & \text{if } t \in [s, \infty), \end{cases}$$

and this contradicts the maximality of the range of  $\tilde{y}(\cdot)$  in  $U$ . Therefore, it must be  $s = \infty$ , and this implies  $x(t) \in E$  for all  $t \in I$ .

$T < \infty$ : Let  $0 < \delta < T$ , and set  $t_k = k(T - \delta)$  for  $k = 0, 1, 2, \dots$ . We next extend the multifunction  $G$  given in (5.12) over  $[0, \infty) = I$ : for each  $x \in \mathbb{R}^n$  we define

$$G_\delta(t, x) := \begin{cases} G(t, x) & \text{a.e. } t \in [0, t_1) \\ \text{co}(G(t_1, x) \cup G(t_0, x)) & t = t_k, \text{ and } k = 1, 2, \dots \\ G(t - t_k, x) & \text{a.e. } t \in (t_k, t_{k+1}), k = 1, 2, \dots \end{cases} \quad (5.13)$$

That  $G_\delta(t, x)$  is nonempty, convex, and compact follows immediately from the definition (5.13) and the fact that the convex hull of compact sets is compact



and obviously convex. To show the linear growth condition is satisfied by  $G_\delta$  it is enough to consider the case when  $v \in G_\delta(t_k, x)$  for  $k = 1, 2, \dots$ . By definition of  $G_\delta(t_k, x)$  there exist  $\lambda_i \in [0, 1]$ , and  $v_i \in G(t_1, x) \cup G(t_0, x)$  with  $\sum_{i=1}^{n-1} \lambda_i = 1$ , and such that  $v = \sum_{i=1}^{n-1} \lambda_i v_i$ . Therefore,

$$\begin{aligned} \|v\| &= \left\| \sum_{i=1}^{n-1} \lambda_i v_i \right\| \leq \sum_{i=1}^{n-1} \lambda_i \|v_i\| \\ &\leq \sum_{i=1}^{n-1} \lambda_i c (1 + \|x\|) \\ &= c (1 + \|x\|), \end{aligned}$$

which implies the linear growth condition due to the arbitrariness of  $v$  in  $G_\delta(t_k, x)$ . To guarantee that  $G_\delta$  is a submultifunction of  $F$  it only remains to be shown that  $G_\delta(\cdot, \cdot)$  is indeed almost upper semicontinuous on  $I$ . For this purpose let  $[a, b] \subset I$ . The desired property obviously holds if  $[a, b] \subset [t_k, t_{k+1})$  for some  $k \geq 0$  due to almost upper semicontinuity of the piece  $G(\cdot - t_k, \cdot)$  on the interval  $[t_k, t_{k+1})$ . Let us assume that  $[a, b] \subset [t_k, t_{k+2})$  for some  $k$ , and additionally  $t_k \leq a < t_{k+1} < b < t_{k+2}$ . Since  $G(\cdot - t_k, \cdot)$  and  $G(\cdot - t_{k+1}, \cdot)$  are almost upper semicontinuous respectively on  $[t_k, t_{k+1}]$  and  $[t_{k+1}, t_{k+2}]$ , for  $\varepsilon > 0$  there exist compact sets  $I_\varepsilon^1 \subset [a, t_{k+1}]$  and  $I_\varepsilon^2 \subset [t_{k+1}, b]$ , with  $\mu([a, t_{k+1}] \setminus I_\varepsilon^1) < \varepsilon/2$ ,  $\mu([t_{k+1}, b] \setminus I_\varepsilon^2) < \varepsilon/2$ , and having the property that the restriction of  $G(\cdot - t_k, \cdot)$  and  $G(\cdot - t_{k+1}, \cdot)$  on  $I_\varepsilon^1 \times \mathbb{R}^n$  and  $I_\varepsilon^2 \times \mathbb{R}^n$  respectively are upper semicontinuous. It is clear that  $\mu([a, b] \setminus (I_\varepsilon^1 \cup I_\varepsilon^2)) < \varepsilon$  and  $I_\varepsilon^1 \cup I_\varepsilon^2$  is compact in  $[a, b]$ . Let  $(\tau, x) \in J_\varepsilon \times \mathbb{R}^n$  where  $J_\varepsilon$  denotes the set of density points of  $I_\varepsilon^1 \cup I_\varepsilon^2$ , and let  $(\tau_i, x_i) \rightarrow (\tau, x)$ , with  $\tau_i \in I_\varepsilon^1 \cup I_\varepsilon^2$ . The only interesting situation is when  $\tau = t_{k+1}$ , since otherwise the sequence  $\tau_i$  finally lies either in  $[a, t_{k+1}]$  or in  $[t_{k+1}, b]$ , and therefore there is  $i$  large enough such that

$$\text{either } G_\delta(\tau_i, x_i) = G(\tau_i - t_k, x_i) \subseteq G(\tau - t_k, x) + \varepsilon B = G_\delta(\tau, x) + \varepsilon B,$$

$$\text{or } G_\delta(\tau_i, x_i) = G(\tau_i - t_{k+1}, x_i) \subseteq G(\tau - t_{k+1}, x) + \varepsilon B = G_\delta(\tau, x) + \varepsilon B.$$

Therefore we can assume  $\tau = t_{k+1}$ . If for  $i$  large enough we have  $\tau_i \in [a, t_{k+1}]$ , then

$$G_\delta(\tau_i, x_i) = G(\tau_i - t_k, x_i) \subseteq G(t_1, x) + \varepsilon B \subset G_\delta(t_{k+1}, x) + \varepsilon B.$$

Similarly if  $\tau_i \in [t_{k+1}, b]$  for  $i$  large enough. Let us now consider the disjoint subsequences  $\tau_{i_j}$  and  $\tau_{i_r}$  of  $\tau_i$  such that  $\tau_{i_j} \in [a, t_{k+1}]$ , and  $\tau_{i_r} \in [t_{k+1}, b]$ . For  $\varepsilon > 0$  there exist  $j$  and  $r$  large enough such that for  $i \geq \max\{i_j, i_r\}$  we have

$$G_\delta(\tau_i, x_i) = \begin{cases} G(\tau_{i_j} - t_k, x_{i_j}) \subset G(t_1, x) + \varepsilon B \subset G_\delta(t_{k+1}, x) + \varepsilon B & \text{if } \tau_i = \tau_{i_j}, \\ G(\tau_{i_r} - t_{k+1}, x_{i_r}) \subset G(t_0, x) + \varepsilon B \subset G_\delta(t_{k+1}, x) + \varepsilon B & \text{if } \tau_i = \tau_{i_r}. \end{cases}$$

The previous study implies the restriction of  $G_\delta$  to  $(I_\varepsilon^1 \cup I_\varepsilon^2) \times \mathbb{R}^n$  is upper semicontinuous. Therefore, the multifunction  $G_\delta$  is almost upper semicontinuous and in particular satisfies (NSH). Moreover, due to the time-independence of  $F$  it follows that  $G_\delta(t, x) \subset F(x)$  a.e.  $t \in I$ , and all  $x \in \mathbb{R}^n$ , that is,  $G_\delta$  is a submultifunction of  $F$ . Therefore, the hypothesis guarantees the existence of a trajectory  $y_\delta(\cdot)$  of  $G_\delta$ , whose extension  $\tilde{y}(\cdot) \in \Upsilon_{(G,U)}(0, x_0)$  satisfies  $\tilde{y}(t) \in E$  for all  $t \in [0, Esc(\tilde{y}(\cdot); U))$ . Due to the maximality of the range in  $U$ , we see as in the case  $T = \infty$  that  $t_1 \leq Esc(\tilde{y}(\cdot); U)$ , and again the definition of  $G_\delta$  and Gronwall's inequality yield to  $x(t) = \tilde{y}_\delta(t) \in E$  on  $[0, t_1)$ . We have seen that  $x(t) \in E \cap U$  for all  $t \in [0, T - \delta)$  and all  $0 < \delta < T$ , which implies  $x(t) \in E \cap U$  for all  $t \in [0, T)$ . Hence the system  $(E, F)$  is strongly invariant in  $U$ .

To prove the converse, let  $G$  be a submultifunction of  $F$ , and let  $(\tau, x_0) \in I \times (E \cap U)$ . Let  $\varepsilon > 0$  be such  $S := \{x_0\} + \varepsilon B_n \subset U$ . The usual theory (see Theorem 1.4 or Corollary 1.11) guarantees that, under (SH), there exists a trajectory  $x(\cdot)$  of  $G$ , which according to Proposition 1.12 can be chosen so that  $x(t) \in S$  for all  $t \in [\tau, \tau + T]$ , and some  $T > 0$ . Due to Proposition 5.3 such a trajectory can be extended (if necessary) to a trajectory  $\tilde{x}(\cdot) \in \Upsilon_{(G,U)}(\tau, x_0)$ . Since  $F$  is autonomous, it is clear that (if  $\tau > 0$ )  $\tilde{y}(t) \equiv \tilde{x}(t + \tau) \in \Upsilon_{(F,U)}(x_0)$ , and since  $(E, F)$  is strongly

invariant in  $U$ , the range of  $\tilde{y}(\cdot)$  belongs to  $E \cap U$ , and so does the range of  $\tilde{x}(\cdot)$ . Hence  $(E, G)$  is weakly invariant in  $U$ , and therefore by Theorem 5.8 there exists a null set  $A_G \subset I$  such that  $h_G(t, x, \zeta) \leq 0$  for all  $\zeta \in N_E^P(x)$ ,  $x \in E \cap U$ , and  $t \in I \setminus A_G$ .  $\square$

The following is an adapted version of Corollary 5 of [28], from which we will benefit in its autonomous version. It represents a replacement condition for the Hamiltonian criterion given in Theorem 3.1(b) of [52]. As it was pointed out in Theorem 4.3, we will make use of the following notation: For a given nonzero vector  $\zeta \in \mathbb{R}^n$ ,  $y \rightarrow_\zeta x$  signifies the limit of  $y$  approaching  $x$  along the vector  $\zeta$ ; that is,  $y \rightarrow_\zeta x$  if and only if  $y \rightarrow x$  and  $\frac{y-x}{\|y-x\|} \rightarrow \frac{\zeta}{\|\zeta\|}$ .

**Theorem 5.10.** *Suppose  $F$  satisfies (SH) and the dissipative Lipschitz condition (5.10). Then the system  $(E, F)$  is strongly invariant in  $U$  if and only if*

$$\liminf_{y \rightarrow_\zeta x} h_F(y, -\zeta) \geq 0, \quad (5.14)$$

for all  $x \in E \cap U$ , and all  $\zeta \in N_E^P(x)$ .

**Proof.** We first show that for any submultifunction  $G$  of  $F$  the approximate Hamiltonian condition (5.14) forces (5.11) to hold, and so strong invariance of  $(E, F)$  will follow directly from Proposition 5.9.

Let  $G : I \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  be a submultifunction of  $F$ . Let  $\tilde{I} \subset I$  be a compact interval and  $\varepsilon > 0$ . The almost upper semicontinuity of  $G$  implies the existence of a closed set  $I_\varepsilon \subset \tilde{I}$  with Lebesgue measure  $\mu(\tilde{I} \setminus I_\varepsilon) \leq \varepsilon$ , and such that the restriction of  $G(\cdot, \cdot)$  on  $I_\varepsilon \times \mathbb{R}^n$  is upper semicontinuous. Let again  $J_\varepsilon$  denote the points of density of  $I_\varepsilon$ . Now fix  $t \in J_\varepsilon$ ,  $x \in E \cap U$ , and  $\zeta \in N_E^P(x)$ . By taking the  $\liminf$  all over the sequences  $(\tau, z) \rightarrow (t, x)$  in  $I_\varepsilon \times \mathbb{R}^n$ , the lower semicontinuity of  $h_G(\cdot, \cdot, \zeta)$  yields

$$-h_G(t, x, \zeta) \geq - \liminf_{z \rightarrow x, \tau \rightarrow t} h_G(\tau, z, \zeta). \quad (5.15)$$

It can be shown that

$$-\liminf_{z \rightarrow x, \tau \rightarrow t} h_G(\tau, z, \zeta) = \limsup_{z \rightarrow x, \tau \rightarrow t} -h_G(\tau, z, \zeta),$$

and since  $-h_G \geq -H_F$ , and  $-H_F(\cdot, \zeta) = h_F(\cdot, -\zeta)$  we obtain

$$\begin{aligned} \limsup_{z \rightarrow x, \tau \rightarrow t} -h_G(\tau, z, \zeta) &\geq \limsup_{z \rightarrow x, \tau \rightarrow t} h_F(z, -\zeta) \\ &= \limsup_{z \rightarrow x} h_F(z, -\zeta), \end{aligned} \quad (5.16)$$

and obviously, all over the special sequences  $y \rightarrow_\zeta x$ , we have

$$\limsup_{z \rightarrow x} h_F(z, -\zeta) \geq \liminf_{y \rightarrow_\zeta x} h_F(y, -\zeta). \quad (5.17)$$

Inequalities (5.15), (5.16), and (5.17), and condition (5.14) imply  $-h_G(t, x, \zeta) \geq 0$ , and therefore the Hamiltonian inequality (5.11) holds in  $\tilde{I}$  off of the exceptional set  $\tilde{A}_G := \tilde{I} \setminus \bigcup_{\varepsilon > 0} J_\varepsilon$ , which has null measure. By covering  $I$  with a countable union of disjoint compact intervals, and repeating the previous argument on each of such intervals, we see the sufficiency of (5.14) for strong invariance follows from Proposition 5.9 .

Conversely, assume that the system  $(E, F)$  is strongly invariant in  $U$ . Let  $x \in E \cap U$ ,  $\zeta \in N_E^P(x)$ , and a sequence  $y_i \rightarrow_\zeta x$  be given. For each  $i$ , let  $v_i \in F(y_i)$  be such that  $h_F(y_i, -\zeta) = \langle v_i, -\zeta \rangle$ . Now define  $G_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  by

$$G_i(y) := \{w \in F(y) : \langle v_i - w, y_i - y \rangle \leq k \|y_i - y\|^2\}.$$

For all  $y \in \mathbb{R}^n$ , each  $G_i(y)$  is nonempty by assumption (5.10), and again from the argument provided in Proposition 4.2 it readily follows that every  $G_i(\cdot)$  inherits from  $F$  the properties given in (SH). According to Proposition 5.9, we must have

$$h_{G_i}(x, \zeta) \leq 0. \quad (5.18)$$

Now choose  $w_i \in G_i(x)$  such that  $\langle w_i, \zeta \rangle = h_{G_i}(x, \zeta)$ . Due to the compactness of  $F(x)$ , the sequences  $v_i$  and  $w_i$  are bounded (since  $y_i \rightarrow x$  and  $F$  is upper semicontinuous at  $x$ ). Rearranging terms from the definition of  $G_i(x)$  yields

$$\left\langle v_i, -\frac{y_i - x}{\|y_i - x\|} \right\rangle \geq -k\|y_i - x\| + \left\langle w_i, -\frac{y_i - x}{\|y_i - x\|} \right\rangle. \quad (5.19)$$

The following is justified by the properties of  $v_i$ , since  $y_i \rightarrow_\zeta x$ , by (5.19), and the properties of  $w_i$ .

$$\begin{aligned} \liminf_{i \rightarrow \infty} h_F(y_i, -\zeta) &= \liminf_{i \rightarrow \infty} \langle v_i, -\zeta \rangle \\ &= \liminf_{i \rightarrow \infty} \|\zeta\| \left\langle v_i, -\frac{y_i - x}{\|y_i - x\|} \right\rangle \\ &\geq \liminf_{i \rightarrow \infty} \|\zeta\| \left\langle w_i, -\frac{y_i - x}{\|y_i - x\|} \right\rangle \\ &= \liminf_{i \rightarrow \infty} \langle w_i, -\zeta \rangle \\ &= \liminf_{i \rightarrow \infty} -h_{G_i}(x, \zeta) \geq 0. \end{aligned}$$

Hence condition (5.14) is satisfied with the liminf taken all over the sequences  $y_i \rightarrow_\zeta x$ , and so (5.14) holds as stated.  $\square$

**Remark 5.11.** *Taking the liminf over  $y \rightarrow_\zeta x$  in the previous result can be replaced by the a priori weaker condition of taking this limit over  $\delta \rightarrow 0^+$  and with  $y = x + \delta\zeta$  without changing the equivalence with strong invariance.*

We next interpret these results in terms of state-augmented data and epigraphs of lower semicontinuous functions. The following proposition is the analogue of Proposition 3.3 in [52].

**Proposition 5.12.** *Suppose  $F$  satisfies (SH),  $\theta : \mathbb{R}^n \rightarrow (-\infty, \infty]$  is lower semicontinuous, and  $E = \text{epi } \theta$ .*

(a)  *$(E, F \times \{-1\})$  is weakly invariant in  $S^c \times \mathbb{R}$  if and only if*

$$1 + h_F(x, \xi) \leq 0,$$

for all  $x \notin S$ , and all  $\xi \in \partial_P \theta(x)$ .

(b) If additionally  $F$  satisfies (5.9), then  $(E, -F \times \{1\})$  is strongly invariant in  $\mathbb{R}^{n+1}$  if and only if

$$1 + \liminf_{y \rightarrow_\xi x} h_F(y, \xi) \geq 0,$$

for all  $x \in \mathbb{R}^n$ , and all  $\xi \in \partial_P \theta(x)$ .

**Proof.** The proof of (a) is Proposition 3.3(a) of [52]. To prove (b) let  $(y, \xi) \in \mathbb{R}^{2n}$ ,  $r \in \mathbb{R}$ , and  $\rho < 0$  and note that

$$\begin{aligned} h_{(-F \times \{1\})}((y, r), -(\xi, \rho)) &= \inf_{v \in -F(y)} \langle (v, 1), (-\xi, -\rho) \rangle \\ &= -\rho + \inf_{v \in F(y)} \langle v, \xi \rangle \\ &= -\rho + h_F(y, \xi). \end{aligned} \tag{5.20}$$

( $\Rightarrow$ ) We first show the “only if” direction. Suppose  $x \in \mathbb{R}^n$  and  $\xi \in \partial_P \theta(x)$ . Therefore  $\zeta = (\xi, -1) \in N_E^P(x, \theta(x))$ . The following is a consequence of Remark 5.11, using  $\rho = -1$  in (5.20), and Theorem 5.10.

$$\begin{aligned} 1 + \liminf_{y \rightarrow_\xi x} h_F(y, \xi) &\geq 1 + \liminf_{(y,r) \rightarrow_{(\xi,-1)} (x,\theta(x))} h_F(y, \xi) \\ &= \liminf_{(y,r) \rightarrow_\zeta (x,\theta(x))} h_{(-F \times \{1\})}((y, r), -\zeta) \\ &\geq 0. \end{aligned} \tag{5.21}$$

( $\Leftarrow$ ) To prove the converse, we will show that any submultifunction  $\tilde{G}$  of  $-F \times \{1\}$  is weakly invariant in  $\mathbb{R}^{n+1}$ , and therefore the result will follow from the Invariance principle (Proposition 5.9). In fact, let  $\tilde{G}(\cdot, \cdot, \cdot) : I \times \mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^{n+1}$  be a submultifunction of  $-F \times \{1\}$ . Therefore, it follows that  $\tilde{G}(t, x, r) = -G(t, x, r) \times \{1\}$  for some multifunction  $G : I \times \mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^n$ . Notice that due to the dimension of the domain of  $\tilde{G}$  we cannot guarantee the  $r$ -independence of the multifunction  $G$ , and therefore it does not necessarily follow that  $G$  is a submultifunction of  $F$ . However,

it is clear that  $G$  does satisfy (NSH) with  $n$  replaced by  $n + 1$ . Now let  $\tilde{I} \subset I$  be a compact interval,  $(x, r) \in E$ , and  $(\xi, \rho) \in N_E^P(x, r)$ . The nature of epigraphs yields  $\rho \leq 0$ . We proceed now to distinguish two cases:

$\rho < 0$ : It follows that  $r = \theta(x)$ . The almost upper semicontinuity of  $G$  implies the existence of a closed set  $I_\varepsilon \subset \tilde{I}$  with Lebesgue measure  $\mu(\tilde{I} \setminus I_\varepsilon) \leq \varepsilon$ , and such that the restriction of  $G(\cdot, \cdot, \cdot)$  on  $I_\varepsilon \times \mathbb{R}^{n+1}$  is upper semicontinuous. Let again  $J_\varepsilon$  denote the points of density of  $I_\varepsilon$ . By taking the  $\liminf$  all over the sequences  $(\tau, z, s) \rightarrow (t, x, r)$  in  $I_\varepsilon \times \mathbb{R}^{n+1}$  and proceeding in identical way as in the proof of the “if” part of Theorem 5.10, we see the following inequalities hold:

$$\begin{aligned} 1 - h_{(-G)}\left(t, (x, r), -\frac{\xi}{\rho}\right) &\geq 1 + \liminf_{y \rightarrow (-\xi/\rho)x} h_{(-F)}\left(y, \frac{\xi}{\rho}\right) \\ &= 1 + \liminf_{y \rightarrow (-\xi/\rho)x} h_F\left(y, -\frac{\xi}{\rho}\right). \end{aligned} \quad (5.22)$$

We can again extend the last argument all over  $I$  as in Theorem 5.10, and so we find a set of null measure  $A_{(-G)} \subset I$  for which (5.22) is satisfied for all  $t \in I \setminus A_{(-G)}$ , and all  $\zeta \in N_E^P(x)$ . As it was seen in (5.20), it follows that

$$h_{\tilde{G}}(t, (x, r), (\xi, \rho)) = \rho + h_{(-G)}(t, (x, r), \xi). \quad (5.23)$$

Multiplying (5.22) by  $-\rho$ , the last equality yields

$$-h_{\tilde{G}}(t, (x, r), (\xi, \rho)) = -\rho \left( 1 - h_{(-G)}\left(t, (x, r), -\frac{\xi}{\rho}\right) \right) \quad (5.24)$$

$$\geq -\rho \left( 1 + \liminf_{y \rightarrow (-\xi/\rho)x} h_F\left(y, -\frac{\xi}{\rho}\right) \right). \quad (5.25)$$

Notice that indeed  $-\xi/\rho \in \partial_P \theta(x)$  since  $N_E^P(x, \theta(x))$  is a cone and therefore  $(-\xi/\rho, -1) = (-1/\rho)(\xi, \rho) \in N_E^P(x, \theta(x))$ . The hypothesis together with inequality (5.25) imply  $-h_{\tilde{G}}(t, (x, r), (\xi, \rho)) \geq 0$  for all  $t \in I \setminus A_{(-G)}$ .

$\rho = 0$ : in this case we also have that  $(\xi, 0) \in N_E^P(x, \theta(x))$ . By Rockafellar's horizontality Theorem (see Theorem 1.1) there exist sequences  $x_i \rightarrow x$ ,  $\theta(x_i) \rightarrow \theta(x)$ ,  $\xi_i \rightarrow \xi$ ,  $\rho_i \uparrow 0$ , with  $\rho_i < 0$ , and  $-\xi_i/\rho_i \in \partial_P \theta(x_i)$ . Let  $\tilde{I} \subset I$  be a compact interval, and  $\tilde{A}_{(-G)} := \tilde{I} \setminus \left( \bigcup_{\varepsilon > 0} \tilde{J}_\varepsilon \right)$  be a set of null measure such that the restriction of  $h_{(-G)}$  on each  $J_r \times \mathbb{R}^{2n+1}$  is lower semicontinuous, and  $\tilde{J}_\varepsilon$  is the set of density points of  $J_\varepsilon$ . Since for each  $i$  we have  $\rho_i < 0$ , from the previous case (see equality (5.24)) we obtain

$$-\rho_i - h_{(-G)}(t, (x_i, r_i), \xi_i) = -h_{\tilde{G}}(t, (x_i, r_i), (\xi_i, \rho_i)) \geq 0, \quad (5.26)$$

for all sequence  $r_i$  and all  $t \in \tilde{I} \setminus \tilde{A}_{(-G)}$ . The lower semicontinuity of  $h_{(-G)}(\cdot, \cdot, \cdot, \cdot)$  on each  $J_r \times \mathbb{R}^{2n+1}$  and (5.26) imply that for each  $t \in \tilde{I} \setminus \tilde{A}_{(-G)}$  we have

$$\begin{aligned} -h_{(-G)}(t, (x, r), \xi) &\geq - \liminf_{(\tau, (y, s), \eta) \rightarrow (t, (x, r), \xi)} h_{(-G)}(\tau, (y, s), \eta) \\ &\geq - \liminf_{((x_i, r_i), \xi_i) \rightarrow ((x, r), \xi)} h_{(-G)}(t, (x_i, r_i), \xi_i) \\ &= \limsup_{i \rightarrow \infty} \left( -\rho_i - h_{(-G)}(t, (x_i, r_i), \xi_i) \right) \\ &\geq 0. \end{aligned} \quad (5.27)$$

Therefore,  $-h_{\tilde{G}}(t, (x, r), (\xi, 0)) \geq 0$  for all  $t \in \tilde{I} \setminus \tilde{A}_{(-G)}$ , and this almost everywhere condition clearly extends over the whole  $I$ . Hence,  $(E, \tilde{G})$  is weakly invariant.  $\square$

## 5.5 Approximate HJ Inequalities

Given a closed set  $S \subseteq \mathbb{R}^n$ , we now characterize  $T_S(\cdot)$  as a semisolution to an approximate Hamilton-Jacobi equation on  $S^c$ , which also satisfies certain approximate boundary conditions.

**Theorem 5.13.** *Suppose  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is upper semicontinuous, satisfies (SH) and (5.9). Then, there exists a unique lower semicontinuous function  $\theta : \mathbb{R}^n \rightarrow (-\infty, \infty]$  bounded below on  $\mathbb{R}^n$  and satisfying the following.*



(HJI) For each  $x \notin S$  and  $\xi \in \partial_P \theta(x)$ , we have

$$1 + h_F(x, \xi) \leq 0, \quad \text{and} \quad 1 + \liminf_{y \rightarrow_\xi x} h_F(y, \xi) \geq 0. \quad (5.28)$$

(AABC) Each  $x \in S$  satisfies  $\theta(x) = 0$  and

$$1 + \liminf_{y \rightarrow_\xi x} h_F(y, \xi) \geq 0, \quad (5.29)$$

whenever  $\xi \in \partial_P \theta(x)$ . The unique such function is  $\theta(\cdot) = T_S(\cdot)$ .

**Remark 5.14.** In contrast with Theorem 3.2 of [52], here (HJI) are the first inequality and the second approximate Hamilton-Jacobi inequality in (5.28), which involves a liminf taken along  $\xi$ , and (AABC) stands for approximate analytic boundary condition, which is given in (5.29).

**Proof.**  $T_S(\cdot)$  is bounded below by zero by definition, and it is lower semicontinuous by Proposition 5.5. Proposition 5.6 and 5.12(a) combine to imply that

$$1 + h_F(x, \zeta) \leq 0, \quad \text{for all } x \notin S \text{ and } \zeta \in \partial_P T_S(x).$$

In the same way, Proposition 5.6 and 5.12(b) combine to imply that

$$1 + \liminf_{y \rightarrow_\xi x} h_F(x, \zeta) \geq 0, \quad \text{for all } x \in \mathbb{R}^n \text{ and } \zeta \in \partial_P T_S(x),$$

and in particular, the last approximate inequality holds for  $x \notin S$  and  $x \in S$ . This establishes that  $\theta(\cdot) = T_S(\cdot)$  satisfies (HJI) and (AABC). The uniqueness follows from the combined application of Proposition 5.12 and 5.7.  $\square$

Theorem 5.13 evidences how the upper semicontinuity and the monotone Lipschitz property are not sufficient ingredients to generate proximal solutions to exact HJ equations. Nevertheless, we believe the asymptotic conditions satisfied by the lower Hamiltonians in (5.28) and in (5.29) are the most accurate complementary inequalities the proximal subgradients of the minimal time function can satisfy

under our mild hypotheses. The following consequence of Theorem 5.13 expresses that by adding continuity to  $F$  it is possible to recover proximal solutions to the HJ equation.

**Corollary 5.15.** *Additionally to the assumptions given in Theorem 5.13 suppose  $F$  is continuous. The minimal time function  $T_S(\cdot)$  is the unique lower semicontinuous function  $\theta : \mathbb{R}^n \rightarrow (-\infty, \infty]$ , which is bounded below on  $\mathbb{R}^n$ , and satisfying the following.*

(HJE) *For each  $x \notin S$  and  $\xi \in \partial_P \theta(x)$ , we have*

$$1 + h_F(x, \xi) = 0. \quad (5.30)$$

(ABC) *Each  $x \in S$  satisfies  $\theta(x) = 0$  and*

$$1 + h_F(x, \xi) \geq 0, \quad (5.31)$$

*whenever  $\xi \in \partial_P \theta(x)$ .* □

## 5.6 A Particular Control System

We now study a particular dynamic of some independent interest in which the  $\liminf$  in (5.14) can be calculated directly. We consider a system that can be decomposed as  $F = M + G$ , where both  $M$  and  $G$  satisfy (SH), and the following reinforce conditions:

- $M$  is upper semicontinuous and monotone;
- $G(\cdot)$  is continuous and monotone Lipschitz.

We continue to use the Hamiltonians  $h_F$  defined as before, but also have the Hamiltonians  $h_M$ , and  $h_G$  associated to the summands  $M$  and  $G$ , respectively. We recall that from Proposition 1.9 we have  $h_F(x, p) = h_M(x, p) + h_G(x, p)$  for all  $(x, p)$ .

Recall that  $M : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is *monotone* if

$$h_M(y, y - x) + h_M(x, x - y) \geq 0, \quad (5.32)$$

for all  $x, y \in \mathbb{R}^n$ , and  $G(\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is continuous provided  $x \mapsto h_G(x, p)$  is continuous with modulus of continuity independent over  $p \in \mathbb{R}^n$  in a bounded set. Invariance properties of multifunctions with decomposition  $F = -(M + G)$  have been studied already in the second and third chapters using techniques of proximal aiming feedback and continuous selections, but here, we deduce the following result from Theorem 5.13 by utilizing the special character of monotone maps. We mention that with  $M$  monotone and  $G$  satisfying (5.9), then it is immediate that  $F = M + G$  also satisfies (5.9) with same constant  $m$  as  $G$ . In particular, therefore, Theorem 5.13 is applicable.

**Theorem 5.16.** *Suppose  $F = M + G$  satisfies (SH) and the reinforce conditions given above. Then, there exists a unique lower semicontinuous function  $\theta : \mathbb{R}^n \rightarrow (-\infty, \infty]$  bounded below on  $\mathbb{R}^n$  and satisfying the following.*

(HJI) *For each  $x \notin S$  and  $\xi \in \partial_P \theta(x)$ , we have*

$$1 + h_M(x, \xi) + h_G(x, \xi) \leq 0, \quad \text{and} \quad 1 + h_G(x, \xi) - h_M(x, -\xi) \geq 0. \quad (5.33)$$

(ABC) *Each  $x \in S$  satisfies  $\theta(x) = 0$  and*

$$1 + h_G(x, \xi) - h_M(x, -\xi) \geq 0 \quad (5.34)$$

*whenever  $\xi \in \partial_P \theta(x)$ . The unique such function is  $\theta(\cdot) = T_S(\cdot)$ .*

**Proof.** We show the  $\liminf$  in the approximate inequalities given in (5.28) and (5.29) equal the left hand side of the second inequality in (5.33) and in (5.34) respectively, and the result then follows from Theorem 5.13. As it is pointed out in Remark 5.11, we can consider only  $y \rightarrow_\zeta x$  of the form  $y = x + \delta\zeta$  with  $\delta \rightarrow 0$ . We

observe from the positive homogeneity of the Hamiltonians in the last component and condition (5.32) that

$$-h_M(x, -\zeta) = -\frac{1}{\delta}h_M(x, x - y) \leq \frac{1}{\delta}h_M(y, y - x) = h_M(y, \zeta).$$

Therefore,

$$-h_M(x, -\zeta) \leq \liminf_{y \rightarrow \zeta x} h_M(y, \zeta) \leq \limsup_{y \rightarrow \zeta x} h_M(y, \zeta). \quad (5.35)$$

On the other hand, the upper semicontinuity of  $-h_M(\cdot, -\zeta) = H_M(\cdot, \zeta)$ , and the fact that  $H_M \geq h_M$ , imply

$$\begin{aligned} -h_M(x, -\zeta) &\geq \limsup_{y \rightarrow x} -h_M(y, -\zeta) \\ &= \limsup_{y \rightarrow x} H_M(y, \zeta) \\ &\geq \limsup_{y \rightarrow \zeta x} h_M(y, \zeta). \end{aligned} \quad (5.36)$$

From (5.35) and (5.36) it follows

$$-h_M(x, -\zeta) = \liminf_{y \rightarrow \zeta x} h_M(y, \zeta) = \limsup_{y \rightarrow \zeta x} h_M(y, \zeta). \quad (5.37)$$

The continuity of  $G$  and (5.37) then imply

$$\begin{aligned} -h_M(x, -\zeta) + h_G(x, \zeta) &= \liminf_{y \rightarrow \zeta x} h_M(y, \zeta) + \liminf_{y \rightarrow \zeta x} h_G(y, \zeta) \\ &\leq \liminf_{y \rightarrow \zeta x} (h_M(y, \zeta) + h_G(y, \zeta)) \\ &= \liminf_{y \rightarrow \zeta x} h_F(y, \zeta), \end{aligned} \quad (5.38)$$

and also yield to

$$\begin{aligned} \liminf_{y \rightarrow \zeta x} h_F(y, \zeta) &\leq \limsup_{y \rightarrow \zeta x} (h_M(y, \zeta) + h_G(y, \zeta)) \\ &\leq \limsup_{y \rightarrow \zeta x} h_M(y, \zeta) + \limsup_{y \rightarrow \zeta x} h_G(y, \zeta) \\ &= -h_M(x, -\zeta) + h_G(x, \zeta). \end{aligned} \quad (5.39)$$

The combination of (5.38) and (5.39) leads to

$$\liminf_{y \rightarrow_{\zeta} x} h_F(y, \zeta) = -h_M(x, -\zeta) + h_G(x, \zeta),$$

which completes the proof. □

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