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Comparison between solutions of SDEs and ODEs

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COMPARISON BETWEEN SOLUTIONS OF SDES AND ODES

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A ratio-limit comparison between ξ_t, the solution of an SDE driven by a semimartingale, and H_t, the solution of an associated ODE, is proved on the set where $\lim_{t\to\infty} \xi_t = \infty$. Sufficient conditions in terms of the driving processes and the coefficients are obtained for $\lim_{t\to\infty} \xi_t$ to be ∞.

stochastic differential equations * ordinary differential equations * martingales

1. Introduction

Let $(\Omega, F, (F_t)_{t\geq 0}, P)$ be a given stochastic basis satisfying the "usual hypotheses". Let N denote a continuous local martingale and A, a continuous F_t-adapted finite variation process with $N_0 = A_0 = 0$.

In this paper, we consider the following stochastic differential equation (SDE):

$$d\xi_t = g(\xi_t)\, dA_t + g(\xi_t)\sigma(\xi_t)\, dN_t \tag{1.1}$$

with $\xi_0 = 0$.

The associated ODE is given by

$$dH_t = g(H_t)\, dA_t \quad \text{with } H_0 = 0 \tag{1.2}$$

The problem is to find conditions on g, σ, N and A so that

$$\lim_{t\to\infty} \frac{\xi_t}{H_t} = 1 \quad \text{a.s. on the set } (\lim_{t\to\infty} \xi_t = \infty).$$

Such a problem was first considered by Gihman and Skorohod [3, page 129] and recently by Keller et al. [4] for SDES driven by a Brownian motion, in which case strong Markov solutions exist. Therefore, a strengthening of well-known conditions under which $\lim_{t\to\infty} \xi_t = \infty$ with positive probability, resulted in $\lim_{t\to\infty} \xi_t / H_t = 1$. In our problem we have to first find conditions that ensure $\lim_{t\to\infty} \xi_t = \infty$ a.s. This is done in the next section. Section 3 deals with the problem of comparison between ξ_t and H_t. A discussion of some of the hypotheses made here can be found in Keller et al. [4], though the ones needed in our general set up are naturally stronger. It

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should be possible to remove the condition of boundedness on σ and impose that

$$\int_0^\infty \frac{\sigma^2(H)_s}{1+|A|_s^2}\,\mathrm{d}\langle N\rangle_s<\infty \quad \text{a.s.}$$

instead, though it is not known at present how to carry this out.

Interest in this problem arose out of a search for situations in which limit results of the type obtained by the author [7, 8] do not hold. Intuitively, the idea is to make the drift "large" so that if noticeably affects the long term behavior of the solution of (1.1).

2. Asymptotic behavior of solution of SDES

A complete characterization of the asymptotic behavior of ξ_t is well-known if $N=B$, a Brownian motion and $A_t=t$. (See [3, page 119] and [2].) Here we shall restrict our attention to the problem when $\lim_{t\to\infty}\xi_t=\infty$ a.s. Let $\langle N\rangle$ denote the Meyer increasing process [6] and $|A|_t$, the total variation up to time t of the signed measure induced by A. We make the following hypotheses:

Hypotheses H. (H.1) g and σ are such that a unique solution of (1.1) exists without explosions.

(H.2) $\langle N\rangle_\infty=\infty$. Furthermore there is a stopping time S and finite positive random variables K, K' such that for all $t\geq S$, $\langle N\rangle_t\leq K|A|_t\leq K'A_t$.

(H.3) σ is bounded.

(H.4) g is a positive continuously differentiable function such that $\int_0^\infty \mathrm{d}s/g(s)=\infty$.

Theorem 2.1. *Under hypotheses H,* $\lim_{t\to\infty}\xi_t=\infty$ *a.s. if either one of the following holds.*

(a) *g is non-increasing.*

(b) $\displaystyle\varliminf_{t\to\infty}\left(\frac{A_t}{|A|_t}-\frac{\|g'\sigma^2\|_\infty\langle N\rangle_t}{2|A|_t}\right)>0$ *a.s.*

Proof. The proof is given only under condition (a) since the rest follows very much along the same lines.

Let

$$G(x)=\int_0^x\frac{\mathrm{d}a}{g(a)}.$$

By the Ito lemma applied to G,

$$G(\xi_t)=\int_0^t\sigma(\xi_t)\,\mathrm{d}N_s+A_t-\tfrac{1}{2}\int_0^t g'(\xi_s)\sigma^2(\xi_s)\,\mathrm{d}\langle N\rangle_s, \tag{2.2}$$

$$\frac{G(\xi_t)}{|A|_t}\geq\frac{1}{|A|_t}\int_0^t\sigma(\xi_t)\,\mathrm{d}N_s+\frac{A_t}{|A|_t}, \tag{2.3}$$

by using Condition (a) of the theorem.

$\lim_{t\to\infty} 1/|A|_t \int_0^t \sigma(\xi_s)\,dN_s = 0$ a.s. provided that $\lim_{t\to\infty} \int_0^t (\sigma(\xi_s)/(1+|A|_s))\,dN_s$ exists and is finite a.s. by a stochastic Kronecker Lemma. (See Lepingle [5], Reference [7].)

Thus it suffices to show that $\int_0^\infty (\sigma^2(\xi_s)/(1+|A|_s)^2)\,d\langle N\rangle_s < \infty$ a.s. since the limit of a local martingale M exists and is finite if $\langle M\rangle_\infty < \infty$ a.s.

Let $T_1 = \inf(t: \langle N\rangle_t \geq 1)$ and $T = T_1 \wedge S$ where S is the stopping time in (H.2)

$$\begin{aligned}\int_0^\infty \frac{d\langle N\rangle_s}{(1+|A|_s)^2} &= \int_0^T \frac{d\langle N\rangle_s}{(1+|A|_s)^2} + \int_T^\infty \frac{d\langle N\rangle_s}{(1+|A|_s)^2} \\ &\leq \langle N\rangle_T + \int_T^\infty \left(\frac{1+\langle N\rangle_s}{(1+|A|_s)}\right)^2 \frac{d\langle N\rangle_s}{(1+\langle N\rangle_s)^2} \\ &\leq \langle N\rangle_T + 2(1+K^2)\int_1^\infty \frac{ds}{(1+s)^2} < \infty\end{aligned}$$

by Lebesgue's change of time formula (see Dellacherie [1, page 92]) and K is as in (H.2).

By (2.3) and (H.2),

$$\varliminf_{t\to\infty} \frac{G(\xi_t)}{|A|_t} \geq \varliminf_{t\to\infty} \frac{A_t}{|A|_t} > 0 \quad a.s.$$

Using the monotonicity of G and (H.4), we get $\lim_{t\to\infty} \xi_t = \infty$ a.s.

3. Ratio limit theorems for solutions of SDES

First we prove the following lemmas which are needed for limit comparisons.

Lemma 3.1. *Under the hypotheses of Theorem* 2.1, *if*

$$\lim_{x\to\infty} g'(x)\sigma^2(x) = 0, \quad \lim_{t\to\infty} \frac{G(\xi_t)}{A_t} = 1 \quad a.s.$$

Proof. Choose $\varepsilon > 0$. Let

$$T_n = \inf(t: \inf_{s\geq t} \xi_s > n)$$

and

$$|g'(x)\sigma^2(x)| < \varepsilon \quad \text{for all } x > n.$$

$$\frac{1}{|A|_t}\left|\int_0^t g'(\xi_s)\sigma^2(\xi_s)\,d\langle N\rangle_s\right| \leq \frac{1}{|A|_t}\int_0^{T_n} |g'(\xi_s)\sigma^2(\xi_s)|\,d\langle N\rangle_s + \frac{\varepsilon\langle N\rangle_t}{|A|_t}$$

so that

$$\lim_{t\to\infty} \frac{1}{|A|_t}\int_0^t g'(\xi_s)\sigma^2(\xi_s)\,d\langle N\rangle_s = 0.$$

Using this and (H.2) in (2.2), the Lemma is obtained.

The next lemma, though already known, is given here for ready reference.

Lemma 3.2. *If $g(x)x^{\delta-1}$ is ultimately decreasing for some $\delta \in (0, 1)$, then*

$$\lim_{\substack{t/s\to 1\\ s\to\infty}} \frac{G^{-1}(t)}{G^{-1}(s)} = 1.$$

Proof. Let $\varepsilon > 0$. It is sufficient to show that $G(t)/G(s) \not\to 1$, as $s \to \infty$, if $t \geq s(1+\varepsilon)$. Let $d > 0$ such that $g(x)x^{\delta-1}$ decreases for $x \geq d$. Then for $s \geq d$ and some suitable $c > 0$

$$\begin{aligned} G(t) - G(s) &\geq g(s)^{-1}s^{1-\delta} \int_s^t u^{\delta-1}\,du \\ &\geq cg(s)^{-1}s^{1-\delta} \int_d^s u^{\delta-1}\,du \\ &\geq c(G(s) - G(d)) \end{aligned}$$

from which the lemma follows.

Theorem 3.3. *Let the hypotheses of Theorem 2.1 hold. Let*
(i) $\lim_{x\to\infty} g'(x)\sigma^2(x) = 0$, *and*
(ii) $g(x)x^{\delta-1}$ *be ultimately decreasing for some* $\delta \in (0, 1)$. *Then*

$$\lim_{t\to\infty} \frac{\xi_t}{H_t} = 1 \quad \textit{a.s.}$$

Proof. First note that $G(H_t) = A_t$. From Lemma 3.1,

$$\lim_{t\to\infty} \frac{G(\xi_t)}{G(H_t)} = 1 \quad \text{a.s.}$$

Now applying G^{-1} to the numerator and the denominator and using Lemma 3.2 the result is obtained.

Theorem 3.4. *Let the conditions of Theorem 2.1 hold. Let*
(i) *there exist a real number x_0 such that g is monotone and σ^2 is non-increasing on $[x_0, \infty)$.*
(ii) $\lim_{x\to\infty} g'(x) = 0$.
(iii) $\int_0^\infty \sigma^2(G^{-1}(k|A|_s))\,d\langle N\rangle_s < \infty$ for all $k < 1$.
Then

$$\lim_{t\to\infty} \frac{\xi_t}{H_t} = 1 \quad \textit{a.s.}$$

Proof. By Lemma 3.1 and (H.2), $\underline{\lim}_{t\to\infty} G(\xi_t)/|A|_t > 0$ a.s. Therefore by Conditions (i) and (iii) above,

$$\int_0^\infty \sigma^2(\xi_t)\, d\langle N\rangle_t < \infty \quad \text{a.s.}$$

Using this in equation (2.2), $G(\xi_t) - A_t$ tends a.s. to a finite limit as t goes to ∞.

Upon noting that $G(H_t) = A_t$,

$$G(\xi_t) - A_t = \frac{1}{g(U_t)}(\xi_t - H_t) \tag{3.5}$$

by mean value theorem with U_t between ξ_t and H_t.

It can be shown that $\lim_{t\to\infty} g(U_t)/g(H_t) = 1$ the proof of which follows exactly along the same lines as in Keller et al. [4, page 177].

Thus (3.5) implies that $\lim_{t\to\infty}(\xi_t - H_t)/g(H)_t$ exists and is finite a.s.

The theorem follows since $g(t) = o(t)$ as a result of condition (ii).

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