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Large deviations for the two-dimensional Navier–Stokes equations with multiplicative noise

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Abstract

A Wentzell–Freidlin type large deviation principle is established for the two-dimensional Navier–Stokes equations perturbed by a multiplicative noise in both bounded and unbounded domains. The large deviation principle is equivalent to the Laplace principle in our function space setting. Hence, the weak convergence approach is employed to obtain the Laplace principle for solutions of stochastic Navier–Stokes equations. The existence and uniqueness of a strong solution to (a) stochastic Navier–Stokes equations with a small multiplicative noise, and (b) Navier–Stokes equations with an additional Lipschitz continuous drift term are proved for unbounded domains which may be of independent interest.

Keywords: Stochastic Navier–Stokes equations; Large deviations; Girsanov theorem

1. Introduction

The theory of large deviations is an active and important topic in probability theory, and has rightly received considerable attention. The framework for the theory along with important applications can be found in the book by Varadhan [28]. There are several other interesting and important books on this theory and its applications (Dembo and Zeitouni [6], Dupuis and Ellis [9], Stroock [24], to name a few). Wentzell–Freidlin type large deviation results for the two-dimensional stochastic Navier–Stokes equations (SNSE) with additive noise were proved by Chang [2] using the Girsanov transformation. In the present work, the Wentzell–Freidlin large

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deviation principle is established for SNSE with multiplicative noise for domains that can be unbounded. The methods employed in this paper are different from those in [2], and will extend to the stochastic magneto-hydrodynamic system introduced in [23].

It is worth noting that several authors have established the large deviation principle (LDP) for a class of stochastic partial differential equations (see, e.g., Chow [3], Kallianpur and Xiong [15], Sowers [22]). The proofs of LDP have usually relied on first approximating the original problem by time discretization so that LDP can be shown for the resulting simpler problems via the contraction principle, and then showing that LDP holds in the limit. The discretization method used to establish LDP was invented by Wentzell and Freidlin.

Dupuis and Ellis [9] have recently combined weak convergence methods with the stochastic control approach developed earlier by Fleming [13] for the large deviations theory. The origins of this approach can be traced to a result of Laplace which states that given an $h \in C([0,1])$,

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^1 \mathrm{e}^{-nh(x)}\, \mathrm{d}x = -\min_{x\in[0,1]} h(x). \tag{1.1}$$

Motivated by (1.1), Varadhan's Lemma and Bryc's converse show that for a family $\{(X^\epsilon)\}$ of random elements defined on a probability space $(\Omega, \mathcal{F}, P)$, and taking values in a Polish space E, LDP with rate function I is equivalent to the following:

$$\lim_{\epsilon\to 0} \epsilon \log E\{\mathrm{e}^{-\frac{1}{\epsilon}h(X^\epsilon)}\} = -\inf_{x\in E}\{h(x) + I(x)\} \tag{1.2}$$

for all bounded continuous functions h mapping E into $\mathbb{R}$. The statement (1.2) is known as the Laplace principle (LP). Thus LDP is equivalent to LP (Theorems 1.2.1 and 1.2.3 in Dupuis and Ellis [9]) if the family of random elements is Polish space valued.

In this work, we consider the two-dimensional stochastic Navier–Stokes equation (SNSE) which can be written in the abstract evolution form on a suitable function space as

$$\mathrm{d}(\mathbf{u}(t)) + \nu\mathbf{A}\mathbf{u}(t) + \mathbf{B}(\mathbf{u}(t))\mathrm{d}t = \mathbf{f}(t)\mathrm{d}t + \boldsymbol{\sigma}(t, \mathbf{u}(t))\, \mathrm{d}W(t). \tag{1.3}$$

The operators $\mathbf{A}$ and $\mathbf{B}$ are defined in Section 2. The stochastic Navier–Stokes equation has been studied by several authors (see for example, Capinsky and Gatarek [5], Flandoli and Gatarek [11], Flandoli and Maslowski [12], and Menaldi and Sritharan [18]) in recent years. If the noise coefficient $\boldsymbol{\sigma}$ in the Eq. (1.3) is replaced by $\sqrt{\epsilon}\boldsymbol{\sigma}$ for $\epsilon > 0$, then the resulting solution is denoted by $\mathbf{u}^\epsilon$. The aim of this paper is to establish LDP (equivalently, LP) for the family $\{\mathbf{u}^\epsilon\}$.

The main result of the paper is the following theorem. The spaces V, V', H, H_0 and $L_Q(H_0 : H)$ that appear in the statement of the theorem are defined in Section 2. The notation for the operators $\mathbf{A}$ and $\mathbf{B}$ and the basic function spaces mentioned above is standard in the literature on stochastic Navier–Stokes equations. The theorem stated below will be of interest to readers who are already familiar with stochastic Navier–Stokes equations.

Theorem 1.1. *Let $\{\mathbf{u}^\epsilon(\cdot)\}$ be the strong solution of the equation*

$$\mathrm{d}\mathbf{u}^\epsilon(t) + \{\mathbf{A}\mathbf{u}^\epsilon(t) + \mathbf{B}(\mathbf{u}^\epsilon(t))\}\mathrm{d}t = \mathbf{f}(t)\mathrm{d}t + \sqrt{\epsilon}\boldsymbol{\sigma}(t, \mathbf{u}^\epsilon(t))\mathrm{d}\mathbf{W}(t) \tag{1.4}$$

with $\mathbf{u}^\epsilon(0) = \xi \in H$, and W an H-valued Wiener process with a nuclear covariance operator Q. Let $\mathbf{f} \in L^4(0, T; V')$. Assume that $\boldsymbol{\sigma}$ satisfies the following hypotheses of joint continuity, Lipschitz condition and linear growth:

1. *The function $\boldsymbol{\sigma} \in C([0, T] \times V; L_Q(H_0; H))$.*

2. *For all* $t \in (0, T)$, *there exists a positive constant* K *such that for all* $\mathbf{u} \in V$, $|\boldsymbol{\sigma}(t, \mathbf{u})|_{L_Q}^2 \le K(1 + \|\mathbf{u}\|^2)$.
3. *For all* $t \in (0, T)$, *there exists a positive constant* L *such that for all* $\mathbf{u}, \mathbf{v} \in V$, $|\boldsymbol{\sigma}(t, \mathbf{u}) - \boldsymbol{\sigma}(t, \mathbf{v})|_{L_Q}^2 \le L\|\mathbf{u} - \mathbf{v}\|^2$.

Then $\{\mathbf{u}^\epsilon\}$ *satisfies the Laplace principle in* $C([0, T] : H) \cap L^2(0, T : V)$ *with a good rate function*

$$I_\xi(\mathbf{h}) = \inf_{\{\mathbf{v}\in L^2([0,T]:H_0):\mathbf{h}(t)=g^0(\int_0^{\cdot} \mathbf{v}(s)\mathrm{d}s)\}} \left\{ \frac{1}{2} \int_0^T \|\mathbf{v}(t)\|_0^2 \,\mathrm{d}t \right\} \tag{1.5}$$

with the convention that the infimum of an empty set is infinity, and where $g^0(\int_0^{\cdot} \mathbf{v}(s)\mathrm{d}s)$ *denotes the solution* $\mathbf{u_v}$ *of the equation*

$$\mathrm{d}\mathbf{u_v}(t) + [\nu \mathbf{A}\mathbf{u_v}(t) + \mathbf{B}(\mathbf{u_v}(t))]\mathrm{d}t = [\mathbf{f}(t) + \boldsymbol{\sigma}(t, \mathbf{u_v}(t))\mathbf{v}(t)]\mathrm{d}t$$

with $\mathbf{u_v}(0) = \xi$.

The organization of the paper is as follows. In Section 2, the abstract evolution equation formulation of stochastic Navier–Stokes equations is given, and the a priori estimates are proved. Using a local-monotonicity argument, the existence and uniqueness of strong solutions is shown when the noise coefficient is small. In Section 3, the large deviation principle in terms of the equivalent Laplace principle is briefly described. In Section 4, the large deviation principle is established for the two-dimensional Navier–Stokes equations perturbed by a small multiplicative noise in domains that can be unbounded.

2. Stochastic Navier–Stokes equations

Let G be an arbitrary and possibly unbounded open domain in $\mathbf{R}^2$ with a smooth boundary ∂G if the domain has a boundary. For $t \in [0, T]$, consider the stochastic Navier–Stokes equation for a viscous incompressible flow with a no-slip condition at the boundary. Displaying the external forces on the right side of the equation, we have, for $\nu > 0$,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}(t) + \sigma(t, \mathbf{u}) \frac{\mathrm{d}W(t)}{\mathrm{d}t} \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2.2}$$

with $\mathbf{u}(t, x) = 0, x \in \partial G$, and $\mathbf{u}(0, x) = \mathbf{u}_0(x), x \in G$, and $\mathbf{u}(t, x) \to 0$ as $|x| \to \infty$ if G is unbounded.

In the above, p denotes pressure and is a scalar-valued function. The process $\{W_t\}$ is a Hilbert space-valued Wiener process, and ν is the coefficient of viscosity. The solution of Eqs. (2.1) and (2.2) subject to the above boundary and initial conditions is $(\mathbf{u}, p)$ where $\mathbf{u}$ is a two-dimensional vector.

The stochastic Navier–Stokes equations can be written in the abstract evolution equation set-up (see Temam [25] or [26] for bounded domains; [14,21,27] and [16] for arbitrary domains) by introducing the following function spaces. The notation $L^2(G)$, $H^1(G)$ etc. would mean vector functions each of whose coordinates belong to $L^2(G)$, $H^1(G)$ etc.

Let $\mathcal{V}$ denote the space of $C_0^\infty(G)$ functions which are divergence free. Define the spaces H and V as the completion of $\mathcal{V}$ in $L^2(G)$ and in $H^1(G)$ norms respectively. Let $\tilde{V}$ be the space obtained by the completion of $\mathcal{V}$ in the norm $|\nabla \mathbf{u}|_{L^2}$ (the Dirichlet integral). Note that $\tilde{V}$ will in general be different from the space V, though they would coincide in the case of Poincaré domains [17]. Throughout the paper, the symbol V will denote the completion of $\mathcal{V}$ in $H^1(G)$ norm if G is a bounded domain; otherwise, V will denote $\tilde{V}$.

In the particular case of bounded domains, H and V can be characterized as follows:

$$H = \{\mathbf{u} \in L^2(G); \nabla \cdot \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n}|_{\partial G} = 0\}$$
$$V = \{\mathbf{u} \in W_0^{1,2}(G) : \nabla \cdot \mathbf{u} = 0\}$$

where $W_0^{1,2}(G) = \{u \in L^2(G) : \nabla u \in L^2(G), u|_{\partial G} = 0\}$ and $\mathbf{n}$ is the outward normal.

Let V' be the dual of V. We have the dense, continuous embedding

$$V \subset_{\rightarrow} H = H' \subset_{\rightarrow} V'.$$

Let us define the operator $\mathbf{A} : V \to V^{'}$ by $\mathbf{A}\mathbf{u} = -\Pi_H \Delta \mathbf{u}$ for $\mathbf{u} \in D(\mathbf{A})$ where Π_H is the Leray projector : $L^2(G) \to H$, and the domain $D(\mathbf{A})$ is defined as

$$D(\mathbf{A}) = \{\mathbf{u} \in H; -\Pi_H \Delta \mathbf{u} \in H\}.$$

Then by the Cattabriga–Solonnikov regularity theorem (see Chapter 3 in [16]), $D(\mathbf{A}) = W^{2,2}(G) \cap V$. The operator $\mathbf{A}$ is known as the Stokes operator and is positive, self-adjoint. Define $\|\mathbf{u}\| = |\nabla \mathbf{u}| = |\mathbf{A}^{1/2}\mathbf{u}|$ for $\mathbf{u} \in D(\mathbf{A}^{1/2})$. Note that the fractional powers of $\mathbf{A}$ can be defined by spectral resolution in unbounded domains and in $\mathbf{R}^2$ by Fourier transforms or Bessel potentials.

Notation. Throughout this article, $\|\mathbf{u}\|$ will denote $|\mathbf{A}^{1/2}\mathbf{u}|$ whereas $|\cdot|$ will denote the H-norm. Recall that on the space H, the norm is the $L^2(G)$-norm.

Define $b(\cdot, \cdot, \cdot)$: $V \times V \times V \to \mathbf{R}$ by

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \sum_{i,j=1}^{2} \int_G u_i \frac{\partial v_j}{\partial x_i} w_j \, \mathrm{d}x \tag{2.3}$$

using which we can define $\mathbf{B}$: $V \times V \to V'$ as the continuous bilinear operator such that

$$\langle \mathbf{B}(\mathbf{u}_1, \mathbf{u}_2), \ \mathbf{u}_3 \rangle = b(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \tag{2.4}$$

for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in V$. $\mathbf{B}(\mathbf{u})$ will be used to denote $\mathbf{B}(\mathbf{u}, \mathbf{u})$. Note that since $\mathbf{u} \in V$, it follows that

$$b(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -b(\mathbf{u}, \mathbf{w}, \mathbf{v}) \tag{2.5}$$

and hence $b(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0$.

The external body force $\mathbf{f}(t)$ is assumed to be $V^{'}$-valued for all t. Let Q be a positive, symmetric, trace class operator on H.

Definition 2.1. A stochastic process $\{W(t) : 0 \le t \le T\}$ is said to be an H-valued $\{\mathcal{F}_t\}$-adapted Wiener process with covariance operator Q if

(1) for each non-zero $h \in H$, $|Q^{1/2}h|^{-1} (W(t), h)$ is a standard one-dimensional Wiener process, and
(2) for any $h \in H$, $(W(t), h)$ is a martingale adapted to $\{\mathcal{F}_t\}$.

Let $H_0 = Q^{1/2}H$. Then H_0 is a Hilbert space with the inner product

$$(\mathbf{u}, \mathbf{v})_0 = (Q^{-1/2}\mathbf{u}, Q^{-1/2}\mathbf{v}) \quad \forall\, \mathbf{u}, \mathbf{v} \in H_0. \tag{2.6}$$

Let $|\cdot|_0$ denote the norm in H_0. Clearly, the imbedding of H_0 in H is Hilbert–Schmidt since Q is a trace class operator.

The noise term in the stochastic partial differential equation (2.1) in its integral form is given by $\int_0^t \boldsymbol{\sigma}(r, \mathbf{u}(r))\, \mathrm{d}W(r)$. The conditions on $\boldsymbol{\sigma}$ are given below.

Let L_Q denote the space of linear operators S such that $SQ^{1/2}$ is a Hilbert–Schmidt operator from H to H. Define the norm on the space L_Q by $|S|^2_{L_Q} = \mathrm{tr}\,(SQS^*)$. The noise coefficient $\boldsymbol{\sigma} : [0, T] \times V \to L_Q(H_0; H)$ is such that it satisfies the following hypotheses:

A.1. The function $\boldsymbol{\sigma} \in C([0, T] \times V; L_Q(H_0; H))$.
A.2. For all $t \in (0, T)$, there exists a positive constant K such that $|\boldsymbol{\sigma}(t, \mathbf{u})|^2_{L_Q} \le K(1 + \|\mathbf{u}\|^2)$.
A.3. For all $t \in (0, T)$, there exists a positive constant L such that for all $\mathbf{u}, \mathbf{v} \in V$, $|\boldsymbol{\sigma}(t, \mathbf{u}) - \boldsymbol{\sigma}(t, \mathbf{v})|^2_{L_Q} \le L\|\mathbf{u} - \mathbf{v}\|^2$.

By applying the Leray projection Π_H to each term of the Navier–Stokes system, and invoking the result of Helmholtz that $L^2(G)$ admits an orthogonal decomposition into divergence free and irrotational components, namely $L^2(G) = H + H^\perp$ where $H^\perp$ can be characterized by

$$H^\perp = \{g \in L^2(G) : g = \nabla h \text{ where } h \in W^{1,2}(G)\}, \tag{2.7}$$

we can write the system (2.1) and (2.2) as

$$\mathrm{d}\mathbf{u} + [\nu \mathbf{A}\,\mathbf{u} + \mathbf{B}(\mathbf{u})]\, \mathrm{d}t = \mathbf{f}(t)\mathrm{d}t + \boldsymbol{\sigma}(t, \mathbf{u})\, \mathrm{d}W(t). \tag{2.8}$$

A priori estimates on the solution $\mathbf{u}$ are obtained in the following propositions. The following well-known interpolation inequality of Ladyzhenskaya [16] is valid for arbitrary two-dimensional unbounded domains and an easy proof of it is included here for the convenience of the reader.

Lemma 2.1. *For any real-valued smooth functions ϕ and ψ with compact support in $\mathbf{R}^2$, the following hold:*

1. $|\phi^2\psi^2|_{L^1} \le |\phi|_{L^2}|\psi|_{L^2}|\nabla\phi|_{L^2}|\nabla\psi|_{L^2}$
2. $|\phi|^4_{L^4} \le \frac{1}{2}|\phi|^2_{L^2}\,|\nabla\phi|^2_{L^2}$.

Proof. For any (x, y), by the fundamental theorem of calculus, the function can be written as

$$\phi(x, y) = \int_{-\infty}^{x} \partial_1\phi(s, y)\mathrm{d}s = -\int_{x}^{\infty} \partial_1\phi(s, y)\mathrm{d}s$$
$$\psi(x, y) = \int_{-\infty}^{y} \partial_2\psi(x, t)\mathrm{d}t = -\int_{y}^{\infty} \partial_2\psi(x, t)\mathrm{d}t.$$

By using the first equation above,

$$|\phi(x, y)| = \frac{1}{2}\left|\left[\int_{-\infty}^{x} \partial_1\phi(s, y)\mathrm{d}s + \int_{x}^{\infty} -\partial_1\phi(s, y)\mathrm{d}s\right]\right| \le \frac{1}{2}\int_{-\infty}^{\infty} |\partial_1\phi(s, y)|\mathrm{d}s. \tag{2.9}$$

Likewise,

$$|\psi(x, y)| \le \frac{1}{2}\int_{-\infty}^{\infty} |\partial_2\psi(x, t)|\mathrm{d}t. \tag{2.10}$$

Using (2.9) and (2.10), and integrating with respect to x and y,

$$|\phi\psi|_{L^1} \leq \frac{1}{4}|\partial_1\phi|_{L^1}|\partial_2\psi|_{L^1}. \tag{2.11}$$

In the above, using ϕ^2 in place of ϕ, and ψ^2 in place of ψ, yields

$$|\phi^2\psi^2|_{L^1} \leq |\phi\partial_1\phi|_{L^1}|\psi\partial_2\psi|_{L^1}. \tag{2.12}$$

Using the Cauchy–Schwarz inequality twice,

$$|\phi^2\psi^2|_{L^1} \leq |\phi|_{L^2}|\psi|_{L^2}|\partial_1\phi|_{L^2}|\partial_2\psi|_{L^2} \tag{2.13}$$
$$\leq |\phi|_{L^2}|\psi|_{L^2}|\nabla\phi|_{L^2}|\nabla\psi|_{L^2} \tag{2.14}$$

which gives the first inequality stated in the proposition. Putting $\psi = \phi$ in the above inequality, and using Young's inequality in (2.13) yields

$$|\phi|^4_{L^4} \leq \frac{1}{2}|\phi|^2_{L^2}|\nabla\phi|^2_{L^2}$$

which finishes the proof. □

Remark. From the above proposition, it follows that for any $\mathbf{u} \in V$,

$$|\mathbf{u}|^4_{L^4} \leq |\mathbf{u}|^2\|\mathbf{u}\|^2, \tag{2.15}$$

where if $\mathbf{u} = (u_1, u_2)$, then $|\mathbf{u}|^2 = \int\int_G \{u_1^2(x_1, x_2) + u_2^2(x_1, x_2)\}\,\mathrm{d}x_1\,\mathrm{d}x_2$ and $\|\mathbf{u}\|^2 = \sum_{i=1}^2\sum_{j=1}^2 \int\int_G (\frac{\partial u_i}{\partial x_j})^2\,\mathrm{d}x_1\mathrm{d}x_2$. The bound given by (2.15) holds for both bounded and unbounded domains.

Using the inequality (2.15), it readily follows that the operator $\mathbf{B}(\mathbf{u})$ satisfies the following estimate:

$$\|\mathbf{B}(\mathbf{u})\|_{V'} \leq |\mathbf{u}|\|\mathbf{u}\|. \tag{2.16}$$

Proposition 2.2. *For a given $r > 0$, let B_r denote the $L^4(G)$ ball in V: $B_r = \{\mathbf{v} \in V : \|\mathbf{v}\|_{L^4(G)} \leq r\}$. Define the nonlinear operator F on V by $F(\mathbf{u}) := -\nu\mathbf{A}\mathbf{u} - \mathbf{B}(\mathbf{u})$. Then for any $0 < \epsilon < \frac{\nu}{2L}$ where L is the constant that appears in the condition (A.3), the pair $(F, \sqrt{\epsilon}\boldsymbol{\sigma})$ is monotone in B_r: i.e. for any $\mathbf{u} \in V$ and $\mathbf{v} \in B_r$, if $\mathbf{w}$ denotes $\mathbf{u} - \mathbf{v}$, then*

$$\langle F(\mathbf{u}) - F(\mathbf{v}), \mathbf{w}\rangle - \frac{r^4}{\nu^3}|\mathbf{w}|^2 + \epsilon|\boldsymbol{\sigma}(t, \mathbf{u}) - \boldsymbol{\sigma}(t, \mathbf{v})|^2_{L_Q} \leq 0. \tag{2.17}$$

Proof. First, it is clear that $\langle\mathbf{A}\mathbf{w}, \mathbf{w}\rangle = \|\mathbf{w}\|^2$. Using (2.5), and the bilinearity of the operator B, it follows that

$$\langle\mathbf{B}(\mathbf{u}), \mathbf{w}\rangle = -\langle\mathbf{B}(\mathbf{u}, \mathbf{w}), \mathbf{v}\rangle.$$

Likewise, $\langle\mathbf{B}(\mathbf{v}), \mathbf{w}\rangle = -\langle\mathbf{B}(\mathbf{v}, \mathbf{w}), \mathbf{v}\rangle$.

Using the two equations above, one obtains

$$\langle\mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle = -\langle\mathbf{B}(\mathbf{w}), \mathbf{v}\rangle.$$

Using the Hölder inequality, and then the estimate (2.15),

$$
\begin{aligned}
|\langle \mathbf{B}(\mathbf{u}) - \mathbf{B}(\mathbf{v}), \mathbf{w}\rangle| &\le \|\mathbf{w}\|_{L^4(G)} \|\mathbf{w}\| \, \|\mathbf{v}\|_{L^4(G)} \\
&\le |\mathbf{w}|^{1/2} \, \|\mathbf{w}\|^{3/2} \, \|\mathbf{v}\|_{L^4(G)} \\
&\le \frac{\nu}{2}\|\mathbf{w}\|^2 + \frac{27}{32\nu^3}|\mathbf{w}|^2 \, \|\mathbf{v}\|^4_{L^4(G)} \qquad (2.18)
\end{aligned}
$$

where the last inequality follows from the fact that for any two real numbers a, b, and any $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$
ab \le \frac{|a|^p}{p} + \frac{|b|^q}{q}.
$$

Using (2.18), and the definition of the operator F in the theorem yields

$$
\langle F(\mathbf{u}) - F(\mathbf{v}), \mathbf{w}\rangle \le -\frac{\nu}{2}\|\mathbf{w}\|^2 + \frac{r^4}{\nu^3}|\mathbf{w}|^2 \qquad (2.19)
$$

since $\|\mathbf{v}\|_{L^4(G)} \le r$. The proof is finished upon using condition (A.3) and that $\epsilon < \frac{\nu}{2L}$. □

Let $H_n := \text{span}\{e_1, e_2, \ldots, e_n\}$ where $\{e_j\}$ is any fixed orthonormal basis in H with each $e_j \in D(\mathbf{A})$. Let P_n denote the orthogonal projection of H to H_n. Define $W_n = P_n W$. Let $\boldsymbol{\sigma}_n = P_n \boldsymbol{\sigma}$. Define $\mathbf{u}_n^\epsilon$ as the solution of the following stochastic differential equation: For each $\mathbf{v} \in H_n$,

$$
\mathrm{d}(\mathbf{u}_n^\epsilon(t), \mathbf{v}) = \{\langle \mathbf{f}(t), \mathbf{v}\rangle + (F(\mathbf{u}_n^\epsilon(t)), \mathbf{v})\}\mathrm{d}t + \sqrt{\epsilon}(\boldsymbol{\sigma}_n(t, \mathbf{u}_n^\epsilon(t))\, \mathrm{d}W_n(t), \mathbf{v}) \qquad (2.20)
$$

with $\mathbf{u}_n^\epsilon(0) = P_n \mathbf{u}(0)$.

Proposition 2.3. *Let* $\mathbf{f}$ *be in* $L^2([0,T] : H)$ *and let* $E(|\mathbf{u}(0)|^2) < \infty$. *Let* $\mathbf{u}_n^\epsilon$ *denote the unique strong solution of the finite system of Eq. (2.20) in* $C([0,T] : H_n)$. *Then, with* K *as in condition (A.2), the following estimates hold:*

1. *For all* $\epsilon < \frac{\nu}{2K} \wedge \frac{1}{2K^2}$, *and* $0 \le t \le T$,

$$
E(|\mathbf{u}_n^\epsilon(t)|^2) + \nu \int_0^t E\|\mathbf{u}_n^\epsilon(s)\|^2 \,\mathrm{d}s \le E|\mathbf{u}(0)|^2 + \frac{2}{\nu} E \int_0^t |\mathbf{f}(s)|^2_{V'} \mathrm{d}s + \epsilon K T \qquad (2.21)
$$

and

$$
E\left(\sup_{0\le t\le T} |\mathbf{u}_n^\epsilon(t)|^2 + \nu \int_0^T \|\mathbf{u}_n^\epsilon(s)\|^2 \mathrm{d}s\right) \le C\left(E|\mathbf{u}(0)|^2, E\int_0^T |\mathbf{f}(s)|^2_{V'}\mathrm{d}s, \nu, T\right). \qquad (2.22)
$$

2. *Let* $\delta > 0$, *and* $\epsilon < \frac{\nu}{2K}$. *Then*

$$
E|\mathbf{u}_n^\epsilon(t)|^2 \,\mathrm{e}^{-\delta t} + \nu \int_0^t E\|\mathbf{u}_n^\epsilon(s)\|^2 \mathrm{e}^{-\delta s}\,\mathrm{d}s \le E|\mathbf{u}(0)|^2 + \frac{1}{\delta} E\int_0^t |\mathbf{f}(s)|^2_{V'}\mathrm{d}s + \frac{\epsilon K}{\delta}. \qquad (2.23)
$$

3. *If* $E|\mathbf{u}(0)|^4 < \infty$, $\mathbf{f}$ *is in* $L^4([0,T] : V')$, *and* $\epsilon < \frac{\nu}{4K}$, *then for all* $0 \le t \le T$,

$$
\begin{aligned}
&E|\mathbf{u}_n^\epsilon(t)|^4 \,\mathrm{e}^{-\delta t} + 3\nu \int_0^t E\|\mathbf{u}_n^\epsilon(s)\|^2 |\mathbf{u}_n^\epsilon(s)|^2 \mathrm{e}^{-\delta s}\,\mathrm{d}s \\
&\quad \le E|\mathbf{u}(0)|^4 + C_\delta E\int_0^t |\mathbf{f}(s)|^4_{V'}\mathrm{e}^{-\delta s}\mathrm{d}s + \frac{4\epsilon K}{\delta} \sup_{0\le t\le T} E(|\mathbf{u}_n^\epsilon(s)|^2). \qquad (2.24)
\end{aligned}
$$

Proof. From the Itô Lemma, one obtains

$$(d|\mathbf{u}_n^\epsilon(t)|^2 + 2\nu\,\|\mathbf{u}_n^\epsilon(t)\|^2\mathrm{d}t) = (2\langle \mathbf{f}(t), \mathbf{u}_n^\epsilon(t)\rangle + \epsilon\,\mathrm{tr}\,(\boldsymbol{\sigma}_n(t,\mathbf{u}_n^\epsilon(t))Q\boldsymbol{\sigma}_n(t,\mathbf{u}_n^\epsilon(t))))\mathrm{d}t + 2\sqrt{\epsilon}(\boldsymbol{\sigma}_n(t,\mathbf{u}_n^\epsilon(t))\mathrm{d}W_n(t), \mathbf{u}_n^\epsilon(t)). \tag{2.25}$$

Define$\tau_N = \inf\{t : |\mathbf{u}_n^\epsilon(t)|^2 + \int_0^t \|\mathbf{u}_n^\epsilon(s)\|^2\,\mathrm{d}s > N\}$. Then, by using the Young inequality,

$$\begin{aligned}|\mathbf{u}_n^\epsilon(t\wedge\tau_N)|^2 &+ 2\nu\int_0^{t\wedge\tau_N}\|\mathbf{u}_n^\epsilon(s)\|^2\mathrm{d}s\\ &= |\mathbf{u}_n^\epsilon(0)|^2 + 2\int_0^{t\wedge\tau_N}\left(\frac{1}{\nu}|\mathbf{f}(s)|_{V'}^2 + \frac{\nu}{4}\|\mathbf{u}_n^\epsilon(s)\|^2\right)\mathrm{d}s\\ &\quad + \epsilon\int_0^{t\wedge\tau_N}\mathrm{tr}\,(\boldsymbol{\sigma}_n(s,\mathbf{u}_n^\epsilon(s))Q\boldsymbol{\sigma}_n(s,\mathbf{u}_n^\epsilon(s)))\mathrm{d}s\\ &\quad + 2\sqrt{\epsilon}\int_0^{t\wedge\tau_N}(\boldsymbol{\sigma}_n(s,\mathbf{u}_n^\epsilon(s))\mathrm{d}W_n(s), \mathbf{u}_n^\epsilon(s)).\end{aligned} \tag{2.26}$$

Taking the expectation on both sides, and using condition (A.2), one obtains

$$E|\mathbf{u}_n^\epsilon(t\wedge\tau_N)|^2 + E\frac{3}{2}\nu\int_0^{t\wedge\tau_N}\|\mathbf{u}_n^\epsilon(s)\|^2\mathrm{d}s \le E|\mathbf{u}_n^\epsilon(0)|^2 + \frac{2}{\nu}E\int_0^{t\wedge\tau_N}|\mathbf{f}(s)|^2\mathrm{d}s + \epsilon K E\int_0^{t\wedge\tau_N}(1+\|\mathbf{u}_n^\epsilon(s)\|^2)\mathrm{d}s.$$

If $\epsilon < \frac{\nu}{2K}$, then

$$E(|\mathbf{u}_n^\epsilon(t\wedge\tau_N)|^2) + \nu E\int_0^{t\wedge\tau_N}\|\mathbf{u}_n^\epsilon(s)\|^2\mathrm{d}s \le E|\mathbf{u}(0)|^2 + \frac{2}{\nu}E\int_0^{t\wedge\tau_N}|\mathbf{f}(s)|_{V'}^2\mathrm{d}s + \epsilon KT. \tag{2.27}$$

Taking the supremum up to time $T\wedge\tau_N$ in Eq. (2.26), and then taking the expectation,

$$\begin{aligned}E\sup_{0\le t\le T\wedge\tau_N}&\left(|\mathbf{u}_n^\epsilon(t)|^2 + \nu\int_0^t\|\mathbf{u}_n^\epsilon(s)\|^2\mathrm{d}s\right)\\ &\le E|\mathbf{u}(0)|^2 + \frac{2}{\nu}E\int_0^{T\wedge\tau_N}|\mathbf{f}(s)|_{V'}^2\mathrm{d}s + \epsilon KT\\ &\quad + 2\sqrt{\epsilon}E\sup_{0\le t\le T\wedge\tau_N}\left|\int_0^t(\boldsymbol{\sigma}_n(s,\mathbf{u}_n^\epsilon(s))\mathrm{d}W_n(s), \mathbf{u}_n^\epsilon(s))\right|.\end{aligned} \tag{2.28}$$

By using the Davis inequality, condition (A.2) and then the Cauchy–Schwartz inequality,

$$\begin{aligned}2\sqrt{\epsilon}E&\sup_{0\le t\le T\wedge\tau_N}\left|\int_0^t(\boldsymbol{\sigma}_n(s,\mathbf{u}_n^\epsilon(s))\mathrm{d}W_n(s), \mathbf{u}_n^\epsilon(s))\right|\\ &\le \sqrt{2\epsilon}2KE\left\{\left(\int_0^{T\wedge\tau_N}(1+\|\mathbf{u}_n^\epsilon(s)\|^2)|\mathbf{u}_n^\epsilon(s)|^2\mathrm{d}s\right)^{1/2}\right\}\\ &\le \sqrt{2\epsilon}K\left(E\sup_{0\le t\le T\wedge\tau_N}|\mathbf{u}_n^\epsilon(t)|^2 + E\int_0^{T\wedge\tau_N}\|\mathbf{u}_n^\epsilon(s)\|^2\mathrm{d}s + T\right).\end{aligned}$$

Using the above estimate in (2.28), we get

$$E \sup_{0\le t\le T\wedge\tau_N} |\mathbf{u}_n^\epsilon(t)|^2 \le E|\mathbf{u}(0)|^2 + \frac{2}{\nu}\int_0^T E|\mathbf{f}(s)|_{V'}^2 \mathrm{d}s + (\epsilon K + \sqrt{2\epsilon})T + \sqrt{2\epsilon}K\left(E \sup_{0\le t\le T\wedge\tau_N} |\mathbf{u}_n^\epsilon(t)|^2 + E\int_0^{T\wedge\tau_N} \|\mathbf{u}_n^\epsilon(s)\|^2 \mathrm{d}s\right). \quad (2.29)$$

From (2.27), it is easy to see that

$$\nu E\int_0^{T\wedge\tau_N} \|\mathbf{u}_n^\epsilon(s)\|^2 \mathrm{d}s \le E|\mathbf{u}(0)|^2 + \frac{2}{\nu}\int_0^T E|\mathbf{f}(s)|_{V'}^2 \mathrm{d}s + \epsilon K T. \quad (2.30)$$

Using the above in (2.29), and that $2\epsilon K < 1$,

$$E \sup_{0\le t\le T\wedge\tau_N} |\mathbf{u}_n^\epsilon(t)|^2 \le C\left(E|\mathbf{u}(0)|^2, \int_0^T E|\mathbf{f}(s)|_{V'}^2 \mathrm{d}s, \nu, T\right). \quad (2.31)$$

The inequalities (2.27) and (2.31) yield

$$E \sup_{0\le t\le T\wedge\tau_N} |\mathbf{u}_n^\epsilon(t)|^2 + \nu\int_0^{T\wedge\tau_N} \|\mathbf{u}_n^\epsilon(s)\|^2 \mathrm{d}s \le C\left(E|\mathbf{u}(0)|^2, \int_0^T E|\mathbf{f}(s)|_{V'}^2 \mathrm{d}s, \nu, T\right). \quad (2.32)$$

The estimate (2.32) shows that $T\wedge\tau_N$ increases to T a.s. as $N\to\infty$. Taking the limit in (2.32) as $N\to\infty$ gives (2.22).

The proof of inequality (2.23) is shown since the proof of (2.21) is simpler and follows along similar lines. Let $\delta > 0$. By applying the Itô Lemma to the function $g(t,\xi) = \mathrm{e}^{-\delta t}|\xi|^2$ in the finite dimensional system (2.20),

$$\begin{aligned}
&\mathrm{e}^{-\delta t}(d|\mathbf{u}_n^\epsilon(t)|^2 + \delta\,|\mathbf{u}_n^\epsilon(t)|^2\mathrm{d}t + 2\nu\,\|\mathbf{u}_n^\epsilon(t)\|^2\mathrm{d}t)\\
&\quad = \mathrm{e}^{-\delta t}(2\langle \mathbf{f}(t), \mathbf{u}_n^\epsilon(t)\rangle + \epsilon\,\mathrm{tr}\,(\boldsymbol{\sigma}_n(t,\mathbf{u}_n^\epsilon(t))Q\boldsymbol{\sigma}_n(t,\mathbf{u}_n^\epsilon(t))))\mathrm{d}t\\
&\qquad + 2\sqrt{\epsilon}\mathrm{e}^{-\delta t}(\boldsymbol{\sigma}_n(t,\mathbf{u}_n^\epsilon(t))\mathrm{d}W_n(t), \mathbf{u}_n^\epsilon(t)).
\end{aligned} \quad (2.33)$$

Define$\tau_N = \inf\{t : \max(|\mathbf{u}_n^\epsilon(t)|^2, \int_0^t \|\mathbf{u}_n^\epsilon(s)\|^2\,\mathrm{d}s) > N\}$. Upon writing (2.33) in integral form up to time $t\wedge\tau_N$, taking expectations, and using Young's inequality,

$$\begin{aligned}
&E|\mathbf{u}_n^\epsilon(t\wedge\tau_N)|^2\,\mathrm{e}^{-\delta t\wedge\tau_N} + 2\nu E\int_0^{t\wedge\tau_N} \mathrm{e}^{-\delta s}\|\mathbf{u}_n^\epsilon(s)\|^2\mathrm{d}s\\
&\quad\le E|\mathbf{u}_n^\epsilon(0)|^2 + \frac{1}{\delta}E\int_0^{t\wedge\tau_N}\mathrm{e}^{-\delta s}|\mathbf{f}(s)|^2\mathrm{d}s + \epsilon E\int_0^{t\wedge\tau_N}\mathrm{tr}\,(\boldsymbol{\sigma}_n(s,\mathbf{u}_n^\epsilon(s))Q\boldsymbol{\sigma}_n(s,\mathbf{u}_n^\epsilon(s)))\mathrm{d}s\\
&\quad\le E|\mathbf{u}_n^\epsilon(0)|^2 + \frac{1}{\delta}E\int_0^{t\wedge\tau_N}\mathrm{e}^{-\delta s}|\mathbf{f}(s)|_{V'}^2\mathrm{d}s + \epsilon K E\int_0^{t\wedge\tau_N}\mathrm{e}^{-\delta s}(1 + \|\mathbf{u}_n^\epsilon(s)\|^2)\mathrm{d}s
\end{aligned}$$

where condition (A.2) has been used in the last display. If $\epsilon < \frac{\nu}{K}$, the above inequality yields

$$\begin{aligned}
&E(|\mathbf{u}_n^\epsilon(t\wedge\tau_N)|^2\,\mathrm{e}^{-\delta t\wedge\tau_N}) + \nu E\int_0^{t\wedge\tau_N}\mathrm{e}^{-\delta s}\|\mathbf{u}_n^\epsilon(s)\|^2\mathrm{d}s\\
&\quad\le E|\mathbf{u}(0)|^2 + \frac{1}{\delta}E\int_0^{t\wedge\tau_N}\mathrm{e}^{-\delta s}|\mathbf{f}(s)|_{V'}^2\mathrm{d}s + \frac{\epsilon K}{\delta}.
\end{aligned} \quad (2.34)$$

Allowing $N \to \infty$ in the above display, the proof (2.23) is completed upon noting that $\tau_N \to \infty$ almost surely.

To prove (2.24), the Itô Lemma is first applied to the function $h(t,\xi) = \mathrm{e}^{-\delta t}|\xi|^4$; taking the expectation and using condition (A.2) as before, one obtains

$$E\mathrm{e}^{-\delta t}|\mathbf{u}_n^\epsilon(t)|^4 + 4\nu \int_0^t E\|\mathbf{u}_n^\epsilon(s)\|^2\,|\mathbf{u}_n^\epsilon(s)|^2\,\mathrm{e}^{-\delta s}\mathrm{d}s \le E|\mathbf{u}_n^\epsilon(0)|^4 + C_\delta E\int_0^t |\mathbf{f}(s)|^4_{V'}\mathrm{e}^{-\delta s}\mathrm{d}s$$
$$+ 2\epsilon K E\int_0^t (1+\|\mathbf{u}_n^\epsilon(s)\|^2)|\mathbf{u}_n^\epsilon(s)|^2\mathrm{e}^{-\delta s}\mathrm{d}s + 2\epsilon K E\int_0^t |\mathbf{u}_n^\epsilon(s)|^2(1+\|\mathbf{u}_n^\epsilon(s)\|^2)\mathrm{e}^{-\delta s}\mathrm{d}s.$$

From (2.21), there exists a constant M_T such that $\sup_{t\in[0,T]} E|\mathbf{u}_n^\epsilon(t)|^2 < M_T$. If $4\epsilon K < \nu$, the above inequality yields

$$E\mathrm{e}^{-\delta t}|\mathbf{u}_n^\epsilon(t)|^4 + 3\nu\int_0^t E\|\mathbf{u}_n^\epsilon(s)\|^2\,|\mathbf{u}_n^\epsilon(s)|^2\,\mathrm{e}^{-\delta s}\mathrm{d}s$$
$$\le E|\mathbf{u}_n^\epsilon(0)|^4 + C_\delta E\int_0^t |\mathbf{f}(s)|^4_{V'}\mathrm{e}^{-\delta s}\mathrm{d}s + 4\epsilon K E\int_0^t |\mathbf{u}_n^\epsilon(s)|^2\mathrm{e}^{-\delta s}\mathrm{d}s$$
$$\le E|\mathbf{u}_n^\epsilon(0)|^4 + C_\delta E\int_0^t |\mathbf{f}(s)|^4_{V'}\mathrm{e}^{-\delta s}\mathrm{d}s + \frac{4\epsilon K M_T}{\delta}$$

which finishes the proof of (2.24). □

Proposition 2.4. *Let $E|\mathbf{u}(0)|^4 < \infty$. If $3\epsilon M < \nu$, then*

$$E\left\{\sup_{0\le t\le T} |\mathbf{u}_n^\epsilon(t)|^4\,\mathrm{e}^{-\delta t} + \int_0^T \|\mathbf{u}_n^\epsilon(t)\|^2\,|\mathbf{u}_n^\epsilon(t)|^2\,\mathrm{e}^{-\delta t}\,\mathrm{d}t\right\}$$
$$\le E|\mathbf{u}_n^\epsilon(0)|^4 + C_{\delta,T}\int_0^T E|\mathbf{f}(t)|^4_{V'}\mathrm{e}^{-\delta t}\,\mathrm{d}t + \frac{\epsilon M}{\delta}. \tag{2.35}$$

Proof. The proof follows by use of the Burkholder–Davis–Gundy inequality, and the estimates given in the previous proposition. □

First, a general proposition (which gives the stochastic version of a result of J.L. Lions) is stated without proof. A proof of the proposition can be found in ([19], page 127).

Proposition 2.5. *Let $p > 1$ and q be the conjugate of p. Let $V \subset\!\!\to H \subset\!\!\to V'$ be a Gelfand triple with V being a separable reflexive Banach space and V' its strong dual, and H a separable Hilbert space with dense injections. Let M be an H-valued L^2-integrable continuous martingale whose paths belong P-a.s. to $L^p(0,T;V)$ and with $M(0)=0$.*

Assume that X and U are two adapted processes whose trajectories belong a.s. to $L^p(0,T;V)$ and $L^q(0,T;V')$ such that a.s.

$$X(t) = X_0 + \int_0^t U(s)\mathrm{d}s + M(t)$$

holds for all $t\in[0,T]$ with $X_0 \in H$. Then, the paths of X are a.s. in $C([0,T];H)$, and

$$|X(t)|^2 = |X_0|^2 + 2\int_0^t \langle X(s),U(s)\rangle\mathrm{d}s + 2\int_0^t (X(s),\mathrm{d}M(s)) + \mathrm{tr}[M]_t \tag{2.36}$$

holds a.s. and for all $t\in[0,T]$.

The method of monotonicity for proving existence of strong solutions of SPDEs was initiated by Pardoux [20] (also see Metivier [19] and Chow [3]). The existence of strong solutions for Navier–Stokes equations with additive noise was first shown using the local-monotonicity method by Menaldi and Sritharan [18]. Since a multiplicative noise appears in the Navier–Stokes system (1.3) that is studied in this paper, the proof of existence and uniqueness of solutions is given in full.

Theorem 2.6. *Let $E|\mathbf{u}_0|^4 < \infty$ and $\mathbf{f} \in L^4(0, T : V')$. If ϵ is small enough that $0 < \epsilon < \frac{\nu}{L}$ and $3\epsilon K < \nu$, then under the conditions (A.1)–(A.3) on $\boldsymbol{\sigma}$, there exists a strong solution of the following stochastic Navier–Stokes equation:*

$$\mathrm{d}\mathbf{u}^\epsilon + [\nu \mathbf{A}\,\mathbf{u}^\epsilon + \mathbf{B}(\mathbf{u}^\epsilon)]\,\mathrm{d}t = \mathbf{f}(t)\mathrm{d}t + \sqrt{\epsilon}\boldsymbol{\sigma}(t, \mathbf{u}^\epsilon)\,\mathrm{d}W(t) \tag{2.37}$$

in $L^2(\Omega : C(0, T; H)) \cap L^2(\Omega \times (0, T); V)$. The solution is pathwise unique.

Proof. Let $\Omega_T := \Omega \times [0, T]$. Using the a priori estimates given in the above propositions, it follows from the Banach–Alaoglu theorem that along a subsequence, the Galerkin approximations $\{\mathbf{u}_n\}$ have the following limits:

$$\mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } L^2(\Omega_T, V)$$
$$\mathbf{u}_n \to \mathbf{u} \quad \text{weak star in } L^4(\Omega; L^\infty(0, T; H))$$

and

$$\mathbf{u}_n(T) \to \eta \quad \text{weakly in } L^2(\Omega; H).$$

Recall that $F(\mathbf{u}) := -\nu\mathbf{A}\mathbf{u} - \mathbf{B}(\mathbf{u})$. Since $F(\mathbf{u}_n)$ is bounded in $L^2(\Omega_T, V')$,

$$F(\mathbf{u}_n) \to F_0 \quad \text{weakly in } L^2(\Omega_T, V')$$

and likewise

$$\boldsymbol{\sigma}_n(\cdot, \mathbf{u}_n) \to S \quad \text{weakly in } L^2(\Omega_T, L_Q).$$

The assertion of the last statement holds since $\boldsymbol{\sigma}$ has linear growth (condition A.2) and $\mathbf{u}_n$ is bounded in $L^2(0, T; V)$ uniformly in n by the a priori estimates.

As in Chow [4], extend Eq. (2.20) to an open interval $(-\delta, T+\delta)$ by setting the terms equal to 0 outside of the interval $[0, T]$. Let $\phi(t)$ be a function in $H^1(-\delta, T+\delta)$ with $\phi(0) = 1$. Define for all integers $j \geq 1$, $e_j(t) = \phi(t)e_j$ where $\{e_j\}$ is the fixed orthonormal sequence for H.

Using the Itô formula for the function $(\mathbf{u}_n(t), e_j(t))$, one obtains

$$(\mathbf{u}_n(T), e_j(T)) = (\mathbf{u}_n(0), e_j) + \int_0^T \left(\mathbf{u}_n(s), \frac{\mathrm{d}e_j(s)}{\mathrm{d}s}\right)\mathrm{d}s + \int_0^T \langle F(s, \mathbf{u}_n(s)), e_j(s)\rangle \mathrm{d}s + \sqrt{\epsilon}\int_0^T (\boldsymbol{\sigma}_n(s, \mathbf{u}_n(s))\mathrm{d}W_n(s), e_j(s)). \tag{2.38}$$

It is possible to take the limit in (2.38) term by term as $n \to \infty$. For instance, consider the stochastic integral that appears on the right side of (2.38) with j fixed.

Let $\mathcal{P}_T$ denote the class of predictable processes with values in $L^2(\Omega_T; L_Q(H_0; H))$ with inner product given by

$$(G, J)_{\mathcal{P}_T} = E\int_0^T \mathrm{tr}\,\{G(s)QJ^*(s)\}\mathrm{d}s \quad \forall\, G, J \in \mathcal{P}_T.$$

The map $J : \mathcal{P}_T \to L^2(\Omega_T)$ defined by the real-valued integral $J_t(G) = \int_0^t (G(s)\mathrm{d}W(s), e_j(s))$ is linear and continuous. Besides that, $(\boldsymbol{\sigma}_n(s, \mathbf{u}_n(s))P_n, R)_{\mathcal{P}_T} \to (S\mathrm{d}W(s), R)_{\mathcal{P}_T}$ for any $R \in \mathcal{P}_T$ as $n \to \infty$ by the weak convergence of $\boldsymbol{\sigma}_n(s, \mathbf{u}_n(s))P_n \to S$ in $L_Q^2(H_0; H)$. Thus one can conclude that

$$\int_0^T (\boldsymbol{\sigma}_n(s, \mathbf{u}_n(s))\mathrm{d}W_n(s), e_j(s)) \to \int_0^T \phi(s)(S(s)\mathrm{d}W(s), e_j)$$

as $n \to \infty$ for each j.

Taking the limit termwise in (2.38) yields

$$-\int_0^T \left(\mathbf{u}(s), \frac{\mathrm{d}e_j(s)}{\mathrm{d}s}\right)\mathrm{d}s = (\mathbf{u}_0, e_j) + \int_0^T \langle F_0(s), e_j\rangle\phi(s)\mathrm{d}s + \sqrt{\epsilon}\int_0^T \phi(s)(S(s)\mathrm{d}W(s), e_j) - (\eta, e_j)\phi(T). \tag{2.39}$$

Choose a sequence of functions $\{\phi_k\}$ in place of ϕ such that $\phi_k \to 1_{[0,T]}$ and the time derivative of ϕ_k converges to δ_t weakly as $k \to \infty$. Using ϕ_k in (2.39) in place of ϕ and then letting $k \to \infty$, one obtains

$$(\mathbf{u}(t), e_j) = (\mathbf{u}_0, e_j) + \int_0^t \langle F_0(s), e_j\rangle\mathrm{d}s + \sqrt{\epsilon}\int_0^t (S(s)\mathrm{d}W(s), e_j) \tag{2.40}$$

for $t < T$ with $(\mathbf{u}_T, e_j) = (\eta, e_j)$ for all j. Thus,

$$\mathbf{u}(t) = \mathbf{u}(0) + \int_0^t F_0(s)\mathrm{d}s + \sqrt{\epsilon}\int_0^t S(s)\mathrm{d}W(s) \tag{2.41}$$

with $\mathbf{u}(T) = \eta$.

Let $\mathbf{v} \in L^\infty(\Omega_T; H_m)$ for $m \le n$. Define $r(t) = \frac{27}{\nu^3}\int_0^t \|\mathbf{v}(s)\|_{L^4(G)}^4 \mathrm{d}s$.

Note that by monotonicity,

$$X_n(T) := 2E\int_0^T \langle F(\mathbf{u}_n(s)) - F(\mathbf{v}(s)), \mathbf{u}_n(s) - \mathbf{v}(s)\rangle \mathrm{e}^{-r(s)}\mathrm{d}s \tag{2.42}$$

$$+ 2E\int_0^T \frac{\mathrm{d}r(s)}{\mathrm{d}t}\mathrm{e}^{-r(s)}|\mathbf{u}_n(s) - \mathbf{v}(s)|^2\mathrm{d}s + E\int_0^T \mathrm{e}^{-r(s)}|\boldsymbol{\sigma}_n(s, \mathbf{u}_n(s)) - \boldsymbol{\sigma}_n(s, \mathbf{v}(s))|_{L_Q}^2\mathrm{d}s \le 0. \tag{2.43}$$

Let $X_n(T)$ be written as $Y_n + Z_n$ where

$$Y_n = 2E\int_0^T \langle F(\mathbf{u}_n(s)), \mathbf{u}_n(s)\rangle \mathrm{e}^{-r(s)}\mathrm{d}s + 2E\int_0^T \frac{\mathrm{d}r(s)}{\mathrm{d}t}\mathrm{e}^{-r(s)}|\mathbf{u}_n(s)|^2\mathrm{d}s + E\int_0^T \mathrm{e}^{-r(s)}|\boldsymbol{\sigma}_n(s, \mathbf{u}_n(s))|_{L_Q}^2\mathrm{d}s \tag{2.44}$$

and

$$
\begin{aligned}
Z_n = {} & 2E\int_0^T (\langle F(\mathbf{v}(s)),\mathbf{v}(s)\rangle - \langle F(\mathbf{v}(s)),\mathbf{u}_n(s)\rangle - \langle F(\mathbf{u}_n(s)),\mathbf{v}(s)\rangle)\mathrm{e}^{-r(s)}\mathrm{d}s \\
& + 2E\int_0^T \frac{\mathrm{d}r(s)}{\mathrm{d}t}\mathrm{e}^{-r(s)}(|\mathbf{v}(s)|^2 - 2(\mathbf{u}_n(s),\mathbf{v}(s)))\mathrm{d}s \\
& + E\int_0^T \mathrm{e}^{-r(s)}\{|\boldsymbol{\sigma}_n(s,\mathbf{v}(s))|^2_{L_Q} - 2(\boldsymbol{\sigma}_n(\mathbf{u}_n(s),\boldsymbol{\sigma}_n(\mathbf{v}(s))))_{L_Q}\}\mathrm{d}s. \qquad (2.45)
\end{aligned}
$$

By the Itô formula,

$$
\begin{aligned}
Y_n &= E(\mathrm{e}^{-r(T)}|\mathbf{u}_n(T)|^2 - |\mathbf{u}_n(0)|^2) \\
&\geq E(\mathrm{e}^{-r(T)}|\mathbf{u}_n(T)|^2 - |\mathbf{u}_0|^2). \qquad (2.46)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\liminf_{n\to\infty} Y_n \geq {} & E\mathrm{e}^{-r(T)}|\mathbf{u}_n(T)|^2 - E|\mathbf{u}(0)|^2 \\
= {} & 2E\int_0^T \mathrm{e}^{-r(s)}\langle F_0(s),\mathbf{u}(s)\rangle\mathrm{d}s + 2E\int_0^T \frac{\mathrm{d}r(s)}{\mathrm{d}s}\mathrm{e}^{-r(s)}|\mathbf{u}(s)|^2\mathrm{d}s \\
& + E\int_0^T \mathrm{e}^{-r(s)}\|S(s)\|^2_{L_Q}\mathrm{d}s + E\int_0^T \mathrm{e}^{-r(s)}(S(s)\mathrm{d}W(s),\mathbf{u}(s)). \qquad (2.47)
\end{aligned}
$$

In Z_n, each term has a limit so that we can conclude that

$$
\begin{aligned}
& 2E\int_0^T \mathrm{e}^{-r(s)}\langle F_0(s) - F(\mathbf{v}(s)),\mathbf{u}(s)-\mathbf{v}(s)\rangle\mathrm{d}s + E\int_0^T \frac{\mathrm{d}r}{\mathrm{d}s}\mathrm{e}^{-r(s)}|\mathbf{u}(s)-\mathbf{v}(s)|^2\mathrm{d}s \\
& + E\int_0^T \|S(s) - \boldsymbol{\sigma}(s,\mathbf{v}(s))\|^2_{L_Q}\mathrm{e}^{-r(s)}\mathrm{d}s \leq \liminf_{n\to\infty} X_n \leq 0.
\end{aligned}
$$

Take $\mathbf{v} = \mathbf{u}$ to see that $Z(s) = \boldsymbol{\sigma}(s,\mathbf{u}(s))$. Take $\mathbf{v} = \mathbf{u} - \lambda\mathbf{w}$ with $\lambda > 0$. Then,

$$
\lambda E\int_0^T \langle F_0(s) - F(\mathbf{u}(s)+\lambda\mathbf{w}(s)),\mathbf{w}(s)\rangle\mathrm{d}s + \lambda^2 E\int_0^T \frac{\mathrm{d}r(s)}{\mathrm{d}s}\mathrm{e}^{-r(s)}|\mathbf{w}(s)|^2\mathrm{d}s \leq 0.
$$

Dividing by λ on both sides of the inequality above, and letting λ go to 0, one obtains

$$
E\int_0^T \langle F_0(s) - F(\mathbf{u}(s)),\mathbf{w}(s)\rangle\mathrm{d}s \leq 0.
$$

Since $\mathbf{w}$ is arbitrary, F_0 can be identified with $F(\mathbf{u}(s))$. Thus the existence of a strong solution has been proved.

In what follows, the proof of pathwise uniqueness is sketched. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be two solutions of the SNSE (2.8). For $i = 1, 2$, and $N > 0$, define $\tau_N^i := \inf\{t \leq T : |\mathbf{u}_i(t)| \geq N\}$. Let $\tau_N = \min\{\tau_N^1, \tau_N^2\}$.

$$
P\{\tau_N < T\} \leq P\{\max\{\sup_{t\in[0,T]}|\mathbf{u}_1(t)|, \sup_{t\in[0,T]}|\mathbf{u}_2(t)|\} \geq N\} \leq \frac{C}{N^2} \qquad (2.48)
$$

by the Chebyshev inequality. Thus $\lim_{N\to\infty}\tau_N = T$ a.s.

Let $\mathbf{w} = \mathbf{u}_1 - \mathbf{u}_2$, and $\boldsymbol{\sigma}_{12} = \boldsymbol{\sigma}(\mathbf{u}_1) - \boldsymbol{\sigma}(\mathbf{u}_2)$. By the energy equality,

$$|\mathbf{w}(t \wedge \tau_N)|^2 + 2\int_0^{t\wedge\tau_N} \|\mathbf{w}(s)\|^2 \,\mathrm{d}s \le -2\int_0^{t\wedge\tau_N} b(\mathbf{w}(s), \mathbf{u}_1(s), \mathbf{w}(s))\,\mathrm{d}s + \int_0^{t\wedge\tau_N} \mathrm{tr}(\boldsymbol{\sigma}_{12}(s) Q \boldsymbol{\sigma}_{12}(s))\mathrm{d}s + 2\int_0^{t\wedge\tau_N} (\mathbf{w}, \boldsymbol{\sigma}_{12}(s)\mathrm{d}W(s))$$

so that by an application of the Itô Lemma, with k as any positive constant,

$$\begin{aligned}
&\mathrm{e}^{-k\int_0^{t\wedge\tau_N} \|\mathbf{u}_1(s)\|^2\mathrm{d}s}|\mathbf{w}(t\wedge\tau_N)|^2 + 2\int_0^{t\wedge\tau_N} \mathrm{e}^{-k\int_0^{s\wedge\tau_N} \|\mathbf{u}_1(r)\|^2\mathrm{d}r} \|\mathbf{w}(s)\|^2\mathrm{d}s \\
&\le \int_0^{t\wedge\tau_N} \mathrm{e}^{-k\int_0^{s\wedge\tau_N} \|\mathbf{u}_1(r)\|^2\mathrm{d}r} [-2\,b(\mathbf{w}(s), \mathbf{u}_1(s), \mathbf{w}(s)) + \mathrm{tr}(\boldsymbol{\sigma}_{12}(s) Q \boldsymbol{\sigma}_{12}(s)) \\
&\quad - k\|\mathbf{u}_1(s)\|^2\,|\mathbf{w}|^2\,]\mathrm{d}s + 2\int_0^{t\wedge\tau_N} \mathrm{e}^{-k\int_0^{s\wedge\tau_N} \|\mathbf{u}_1(r)\|^2\mathrm{d}r} (\mathbf{w}(s), \boldsymbol{\sigma}_{12}(s)\,\mathrm{d}W(s)) \\
&\le \int_0^{t\wedge\tau_N} \mathrm{e}^{-k\int_0^{s\wedge\tau_N} \|\mathbf{u}_1(r)\|^2\mathrm{d}r} [2\epsilon\|\mathbf{w}(s)\|^2 + C_\epsilon\,\|\mathbf{u}_1\|^2\,|\mathbf{w}(s)|^2 + L\mathrm{tr}Q\,|\mathbf{w}|^2 \\
&\quad - k\|\mathbf{u}_1(s)\|^2\,|\mathbf{w}(s)|^2]\,\mathrm{d}s + \int_0^{t\wedge\tau_N} \mathrm{e}^{-k\int_0^{s\wedge\tau_N} \|\mathbf{u}_1(r)\|^2\mathrm{d}r} 2(\mathbf{w}(s), \boldsymbol{\sigma}_{12}(s)\mathrm{d}W(s)).
\end{aligned}$$

Note that for each fixed N, the process

$$M_N(t) = \int_0^{t\wedge\tau_N} \mathrm{e}^{-k\int_0^{s\wedge\tau_N} \|\mathbf{u}_1(r)\|^2\mathrm{d}r} 2(\mathbf{w}(s), \boldsymbol{\sigma}_{12}(s)\mathrm{d}W(s))$$

is a martingale since

$$E\int_0^{t\wedge\tau_N} |\mathbf{w}(s)|^2 |\boldsymbol{\sigma}_{12}(s)|^2\,\mathrm{d}s \le L\int_0^{t\wedge\tau_N} |\mathbf{w}(s)|^4\,\mathrm{d}s \le 8LTN^4 < \infty.$$

On setting $k = 2C_\epsilon$ where $\epsilon < 1$, and taking the expectation,

$$E\mathrm{e}^{-k\int_0^{t\wedge\tau_N} \|\mathbf{u}_1(r)\|^2\mathrm{d}r}\,|\mathbf{w}(t\wedge\tau_N)|^2 \le L\,\mathrm{tr}\,Q\,E\int_0^{t\wedge\tau_N} \mathrm{e}^{-k\int_0^{s\wedge\tau_N} \|\mathbf{u}_1(r)\|^2\mathrm{d}r}\,|\mathbf{w}(s)|^2\,\mathrm{d}s.$$

An application of Gronwall's Lemma implies that $\mathbf{w}(t \wedge \tau_N) = 0$ a.s. Since $\tau_N \to T$ as $N \to \infty$, we get that $\mathbf{u}_1 = \mathbf{u}_2$ for all $t \in [0, T]$ a.s. □

3. Large deviation principle

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with an increasing family $\{\mathcal{F}_t\}_{0\le t\le T}$ of sub-σ-fields of $\mathcal{F}$ satisfying the usual conditions of right continuity and P-completeness.

In what follows, the notation and terminology are built in order to state the large deviations result of Budhiraja and Dupuis [1] for Polish space valued random elements:

Let $\mathcal{A}$ denote the class of H_0-valued $\{\mathcal{F}_t\}$-predictable processes ϕ which satisfy $\int_0^T |\phi(s)|_0^2\,\mathrm{d}s < \infty$ a.s.

Let $S_N = \{v \in L^2([0, T] : H_0) : \int_0^T |v(s)|_0^2\,\mathrm{d}s \le N\}$. The set S_N endowed with the weak topology is a Polish space [8]. Define $\mathcal{A}_N = \{\phi \in \mathcal{A} : \phi(\omega) \in S_N,\ P\text{-}a.s.\}$

Let E denote a Polish space, and let $g^\epsilon : C([0, T]; H) \to E$ be a measurable map. Define $X^\epsilon = g^\epsilon(W(\cdot))$. We are interested in the large deviation principle for X^ϵ as $\epsilon \to 0$. Since $\{X^\epsilon\}$

are Polish space valued random elements, the Laplace principle and the large deviation principle are equivalent.

Definition 3.1. A function I mapping E to $[0, \infty]$ is called a rate function if I is lower semicontinuous. A rate function I is called a good rate function if for each $M < \infty$, the level set $\{x \in E : I(x) \leq M\}$ is compact.

Definition 3.2. Let I be a rate function on E. A family $\{X^\epsilon : \epsilon > 0\}$ of E-valued random elements is said to satisfy the Laplace principle on E with rate function I if for each real-valued, bounded and continuous function h defined on E,

$$\lim_{\epsilon \to 0} \epsilon \log E \left\{ \exp \left[-\frac{1}{\epsilon} h(X^\epsilon) \right] \right\} = - \inf_{x \in E} \{h(x) + I(x)\}. \tag{3.1}$$

Hypothesis 3.1. There exists a measurable map $g^0 : C([0, T] : H) \to E$ such that the following hold:

1. Let $\{v^\epsilon : \epsilon > 0\} \subset \mathcal{A}_M$ for some $M < \infty$. Let v^ϵ converge in distribution as S_M-valued random elements to v. Then $g^\epsilon(W(\cdot) + \frac{1}{\sqrt{\epsilon}} \int_0^\cdot v^\epsilon(s)\, ds)$ converges in distribution to $g^0(\int_0^\cdot v(s)\, ds)$.
2. For every $M < \infty$, the set $K_M = \{g^0(\int_0^\cdot v(s)\, ds) : v \in S_M\}$ is a compact subset of E.

For each $f \in E$, define

$$I(f) = \inf_{\{v \in L^2([0,T]:H_0): f = g^0(\int_0^\cdot v(s) ds)\}} \left\{ \frac{1}{2} \int_0^T |v(s)|_0^2\, ds \right\} \tag{3.2}$$

where the infimum over an empty set is taken as ∞.

The following theorem was proven by Budhiraja and Dupuis [1].

Theorem 3.1. *Let $X^\epsilon = g^\epsilon(W(\cdot))$. If $\{g^\epsilon\}$ satisfies the Hypothesis 3.1, then the family $\{X^\epsilon : \epsilon > 0\}$ satisfies the Laplace principle in E with the rate function I given by (3.2).*

4. The large deviations result

Consider the stochastic Navier–Stokes equations given by

$$d\mathbf{u}^\epsilon + [\nu \mathbf{A}\, \mathbf{u}^\epsilon + \mathbf{B}(\mathbf{u}^\epsilon)]\, dt = \mathbf{f}(t) dt + \sqrt{\epsilon} \boldsymbol{\sigma}(t, \mathbf{u}^\epsilon)\, d\mathbf{W}(t) \tag{4.1}$$

with $\mathbf{u}(0) = \xi \in H$, and $\int_0^T |\mathbf{f}(t)|^4\, dt < \infty$ and ϵ as in the statement of Theorem 2.6. Recall that there exists a strong solution of (4.1) with values in $C([0, T]; H) \cap L^2(0, T; V)$, and it is pathwise unique. It follows that (see [1]) there exists a Borel-measurable function $\mathbf{g}^\epsilon : C([0, T]; H) \to C([0, T] : H) \cap L^2(0, T : V)$ such that $\mathbf{u}^\epsilon(\cdot) = \mathbf{g}^\epsilon(\mathbf{W}(\cdot))$ a.s. The aim of this section is to prove the large deviation principle for $\mathbf{u}^\epsilon$. The following Lemmas show that the family $\{\mathbf{g}^\epsilon\}$ satisfies the Hypothesis 3.1 so that Theorem 3.1 can be invoked to prove the main result.

Lemma 4.1. *Let the family $\{\mathbf{g}^\epsilon\}$ be defined as above. For any $\mathbf{v} \in \mathcal{A}_M$ where $0 < M < \infty$, let $\mathbf{g}^\epsilon(\mathbf{W}(\cdot) + \frac{1}{\sqrt{\epsilon}} \int_0^\cdot \mathbf{v}(s)\, ds)$ be denoted by $\mathbf{u}^\epsilon_\mathbf{v}$. Then $\mathbf{u}^\epsilon_\mathbf{v}$ is the unique strong solution of*

$$d\mathbf{u}^\epsilon_\mathbf{v} + [\nu \mathbf{A} \mathbf{u}^\epsilon_\mathbf{v} + \mathbf{B}(\mathbf{u}^\epsilon_\mathbf{v})] dt = [\mathbf{f} + \boldsymbol{\sigma}(t, \mathbf{u}^\epsilon_\mathbf{v})\mathbf{v}] dt + \sqrt{\epsilon} \boldsymbol{\sigma}(t, \mathbf{u}^\epsilon_\mathbf{v}) d\mathbf{W}(t) \tag{4.2}$$

with $\mathbf{u}^\epsilon_\mathbf{v}(0) = \xi$.

Proof. Since $\mathbf{v} \in \mathcal{A}_M$, $\int_0^T |\mathbf{v}(s)|_0^2\,\mathrm{d}s < M$ a.s., $\tilde{\mathbf{W}}(\cdot) := \mathbf{W}(\cdot) + \frac{1}{\sqrt{\epsilon}} \int_0^{\cdot} \mathbf{v}(s)\mathrm{d}s$ is a Wiener process with covariance form Q under the probability measure

$$\mathrm{d}\tilde{P}_{\mathbf{v}}^{\epsilon} := \exp\left\{ -\frac{1}{\sqrt{\epsilon}} \int_0^T \mathbf{v}(s)\mathrm{d}\mathbf{W}(s) - \frac{1}{2\epsilon} \int_0^T |\mathbf{v}(s)|_0^2\,\mathrm{d}s \right\} \mathrm{d}P.$$

A Girsanov argument can be used to complete the proof as follows: Let $\mathbf{w}$ be the unique solution of (4.1) on $(\Omega, \mathcal{F}, \tilde{P}_{\mathbf{v}}^{\epsilon})$ with $\tilde{W}$ in place of W. Then $\mathbf{w}$ solves (4.2) P-a.s., and $\mathbf{w}(\cdot) = \mathbf{g}^{\epsilon}(\tilde{\mathbf{W}}(\cdot))$.

If $\mathbf{u}_{\mathbf{v}}^{\epsilon}$ and $\mathbf{w}$ are two solutions of (4.2) on $(\Omega, \mathcal{F}, P)$, then $\mathbf{u}_{\mathbf{v}}^{\epsilon}$ and $\mathbf{w}$ would solve (4.1) on $(\Omega, \mathcal{F}, \tilde{P})$ with $\tilde{\mathbf{W}}$ in place of $\mathbf{W}$. Thus $\mathbf{u}_{\mathbf{v}}^{\epsilon} = \mathbf{w}\tilde{P}_{\mathbf{v}}$-a.s. so that $\mathbf{u}_{\mathbf{v}}^{\epsilon} = \mathbf{w}$, P-a.s. Thus uniqueness of solutions to (4.2) is obtained. □

The proof of the following lemma can be found in [1].

Lemma 4.2. *Let $\{\mathbf{v}_n\}$ be a sequence of elements from $\mathcal{A}_M$ for some finite $M > 0$. Let $\mathbf{v}_n \to \mathbf{v}$ in distribution as S_M-valued random elements. Then, $\int_0^{\cdot} \mathbf{v}_n(s)\,\mathrm{d}s$ converges in distribution as $C([0,T] : H)$-valued processes to $\int_0^{\cdot} \mathbf{v}(s)\,\mathrm{d}s$ as $n \to \infty$.*

The next theorem is a powerful result of De Simon [7] (see Fattorini [10], Page 438) which we use below. It is stated here for the reader's convenience:

Theorem 4.3. *Let A be the infinitesimal generator of an analytic semigroup $T(t)$ in a Hilbert space H, and let $f(\cdot) \in L^p(0,T;H)$ where $p \in (1,\infty)$. Then the function*

$$Y(t) = \int_0^t T(t-\tau) f(\tau)\mathrm{d}\tau$$

has a derivative $Y'(t) \in L^p(0,T;H)$. Moreover, $Y(t) \in D(A)$ a.e., and

$$\|Y\|_{L^p(0,T;D(A))} \le C\|u\|_{L^p(0,T;H)},$$
$$\|Y'\|_{L^p(0,T;H)} \le C\|u\|_{L^p(0,T;H)}$$

and $Y'(t) = AY(t) + f(t)$ a.e. in $0 \le t \le T$.

The next result is a minor extension of a standard result on mild solutions of the Navier–Stokes equations.

Theorem 4.4. *Let $\mathbf{v} \in L^2([0,T] : H_0)$ and fix $\xi \in H$. Let $\mathbf{f} \in L^4(0,T : V')$ and $\boldsymbol{\sigma}$ satisfy Hypotheses A. Then there exists a unique strong solution $\mathbf{w} \in C([0,T] : H) \cap L^2(0,T : V)$ of the equation*

$$\mathbf{w}(t) + \int_0^t \{\nu\,\mathbf{A}\mathbf{w}(s) + \mathbf{B}(\mathbf{w}(s))\}\,\mathrm{d}s = \xi + \int_0^t \mathbf{f}(s)\,\mathrm{d}s + \int_0^t \boldsymbol{\sigma}(s,\mathbf{w}(s))\mathbf{v}(s)\,\mathrm{d}s. \tag{4.3}$$

Proof. Let $S(t)$ denote the semigroup generated by the Stokes operator. Define

$$Y_1(t) = S(t)\xi$$
$$Y_2(t) = \int_0^t S(t-r)\mathbf{B}\mathbf{w}(r)\mathrm{d}r$$
$$Y_3(t) = \int_0^t S(t-r)\boldsymbol{\sigma}(\mathbf{w}(r))\mathbf{v}(r)\mathrm{d}r$$
$$Y_4(t) = \int_0^t S(t-r)\mathbf{f}(r)\mathrm{d}r.$$

The first step consists in showing that Y_i defined on $X := L^2(0, T; V) \cap C([0, T] : H)$ maps X into itself and is a contraction for small enough T. Consider first $Y_1(t)$. For $\xi \in H$, a result of Masuda (see Fattorini [10], Lemma 8.6.3, Page 423) shows that $Y_1 \in L^2(0, T : V)$, and the fact that $Y_1 \in C([0, T] : H)$ is standard. In fact, note that Masuda's result can be argued as follows.

Let $E(\cdot)$ stand for the spectral measure (resolution of identity) for $\mathbf{A}$. The spectral resolution of $\mathbf{A}$ is given by $\int_0^\infty \lambda \mathrm{d}E(\lambda)$ which accrues because $\mathbf{A}$ is a positive self-adjoint operator. So,

$$|\mathbf{A}^\alpha S(t)\mathbf{v}|_H^2 \le \int_0^\infty |\lambda^\alpha \exp(-t\lambda)|^2 \mathrm{d}(E(\lambda)\mathbf{v}, \mathbf{v}).$$

Integrating with respect to t from 0 to T and setting $\alpha = 1/2$, we get that the right hand side is $\frac{1}{2}|\mathbf{v}|_H^2$.

Note that $|\mathbf{A}^{-1/2}\mathbf{B}(\mathbf{w})| \le C|\mathbf{w}|\,\|\mathbf{w}\|$ with $|\mathbf{w}(\cdot)| \in L^\infty(0, T)$ and $\|\mathbf{w}(\cdot)\| \in L^2(0, T)$. Therefore $\mathbf{A}^{-1/2}\mathbf{B}(\mathbf{w}(t)) \in L^2(0, T : H)$. The result of De Simon applies so that $Y_2(\cdot) \in L^2(0, T : V) \cap C([0, T] : H)$ follows.

Likewise consider $g_2(t) = \boldsymbol{\sigma}(\mathbf{w}(t))\mathbf{v}(t)$. Clearly $\|g_2(t)\| \le c\|\mathbf{w}\|\,|\mathbf{v}|$, and $g_2 \in L^2(0, T : H)$. Again the result of De Simon applies. The term $Y_4(t)$ is simpler to handle, and thus $\sum_{i=1}^4 Y_i(\cdot)$ maps $X := L^2(0, T : V) \cap C([0, T] : H)$ into itself.

To show that the map J defined by $J(\mathbf{w}) = \sum_{i=1}^4 Y_i(\mathbf{w})$ is a contraction map, consider $J(\mathbf{u}_1) - J(\mathbf{u}_2)$ for $\mathbf{u}_1, \mathbf{u}_2 \in X$. We will start with

$$\begin{aligned}|Y_3(\mathbf{u}_1)(t) - Y_3(\mathbf{u}_2)(t)| &\le C \int_0^t |S(t-r)|\,\|\mathbf{u}_1(r) - \mathbf{u}_2(r)\|\mathrm{d}r \\ &\le C(T)\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^2(0,T;V)}.\end{aligned}$$

Also, if $g(t)$ denotes $(\boldsymbol{\sigma}(\mathbf{u}_1(t)) - \boldsymbol{\sigma}(\mathbf{u}_2(t)))\mathbf{v}(t)$, then

$$|g(t)| \le L\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\| \in L^2(0, T)$$

which implies that $g(\cdot) \in L^2(0, T; H)$. Therefore $Y_3 \in L^2(0, T; D(\mathbf{A}))$. Thus

$$|Y_3(\mathbf{u}_1) - Y_3(\mathbf{u}_2)|_{L^2(0,T:V)} \le C(T)\,\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^2(0,T;V)}$$

which is a contraction for small T.

Let us now consider the nonlinear inertial term Y_2. Consider

$$Y_2(\mathbf{u}_1)(t) - Y_2(\mathbf{u}_2)(t) = \int_0^t S(t-r)(\mathbf{B}(\mathbf{u}_1(r)) - \mathbf{B}(\mathbf{u}_2(r)))\mathrm{d}r.$$

We know that

$$\mathbf{B}(\mathbf{u}_1) - \mathbf{B}(\mathbf{u}_2) = \mathbf{B}(\mathbf{u}_1 - \mathbf{u}_2, \mathbf{u}_1) + \mathbf{B}(\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)$$

so that

$$\begin{aligned}|\mathbf{A}^{-1/2}(\mathbf{B}(\mathbf{u}_1) - \mathbf{B}(\mathbf{u}_2))| &\le |\mathbf{u}_1 - \mathbf{u}_2|^{1/2}\,\|\mathbf{u}_1 - \mathbf{u}_2\|^{1/2}\,|\mathbf{u}_1|^{1/2}\,\|\mathbf{u}_1\|^{1/2} \\ &\quad + |\mathbf{u}_2|^{1/2}\,\|\mathbf{u}_2\|^{1/2}\,|\mathbf{u}_1 - \mathbf{u}_2|^{1/2}\,\|\mathbf{u}_1 - \mathbf{u}_2\|^{1/2} \\ &\le |\mathbf{u}_1 - \mathbf{u}_2|^{1/2}\,\|\mathbf{u}_1 - \mathbf{u}_2\|^{1/2}(|\mathbf{u}_1|^{1/2}\,\|\mathbf{u}_1\|^{1/2} + |\mathbf{u}_2|^{1/2}\,\|\mathbf{u}_2\|^{1/2}).\end{aligned}$$

Thus as before, one arrives at the bound

$$|Y_2(\mathbf{u}_1) - Y_2(\mathbf{u}_2)|_{L^2(0,T:V)} \le C(T)\,\|\mathbf{u}_1 - \mathbf{u}_2\|_X \tag{4.4}$$

where $X = L^2(0, T : V) \cap C([0, T] : H)$. The terms Y_1 and Y_4 are easier to handle than Y_2 and Y_3. Thus the map J is a contraction on X for a small time T. By the fixed point theorem, a local solution is thus obtained. This fact along with the energy estimate gives the global solution.

Note that De Simon's theorem along with Masuda's Lemma (see Fattorini [10], Page 423) automatically implies that the mild solution obtained is the strong solution of

$$\frac{d\mathbf{u}}{dt} + \nu\mathbf{A}\mathbf{u} + \mathbf{B}(\mathbf{u}) = \mathbf{f} + \boldsymbol{\sigma}(\mathbf{u})\mathbf{v} \tag{4.5}$$

in $L^2(0, T : V')$. □

Lemma 4.5. *Let M be any fixed finite positive number. Let*

$$K_M := \{\mathbf{u}_\mathbf{v} \in C([0, T] : H) \cap L^2(0, T : V) : \mathbf{v} \in S_M\}$$

where $\mathbf{u}_\mathbf{v}$ is the unique solution in $C([0, T] : H) \cap L^2(0, T : V)$ of the equation

$$\mathbf{u}_\mathbf{v}(t) + [\nu\mathbf{A}\mathbf{u}_\mathbf{v}(t) + \mathbf{B}(\mathbf{u}_\mathbf{v}(t))]dt = [\mathbf{f}(t) + \boldsymbol{\sigma}(t, \mathbf{u}_\mathbf{v}(t))\mathbf{v}(t)]dt \tag{4.6}$$

with $\mathbf{u}_\mathbf{v}(0) = \xi \in H$. Then K_M is compact in $C([0, T] : H) \cap L^2(0, T : V)$.

Proof. Let $\{\mathbf{u}_n\}$ be a sequence in K_M where $\mathbf{u}_n$ corresponds to the solution of (4.6) with $\mathbf{v}_n \in S_M$ in place of $\mathbf{v}$. By weak compactness of S_M, there exists a subsequence of $\{\mathbf{v}_n\}$ which converges to a limit $\mathbf{v}$ weakly in $L^2(0, T : H_0)$. The subsequence is indexed by n for ease of notation. Define $\mathbf{u}$ as the solution of the equation

$$\mathbf{u}(t) + [\nu\mathbf{A}\mathbf{u}(t) + \mathbf{B}(\mathbf{u}(t))]dt = [\mathbf{f}(t) + \boldsymbol{\sigma}(t, \mathbf{u}(t))\mathbf{v}(t)]dt. \tag{4.7}$$

By the energy equality,

$$\begin{aligned}
|\mathbf{u}(t)|^2 + 2\nu\int_0^t \|\mathbf{u}(s)\|^2 ds &= |\xi|^2 + 2\int_0^t \{\langle\mathbf{f}(s), \mathbf{u}(s)\rangle + (\boldsymbol{\sigma}(s, \mathbf{u}(s))\mathbf{v}(s), \mathbf{u}(s))\}\, ds \\
&\le |\xi|^2 + 2\int_0^t \{|\boldsymbol{\sigma}(s, \mathbf{u}(s))\mathbf{v}(s)| + |\mathbf{f}(s)|_{V'}\}\|\mathbf{u}(s)\|\, ds \\
&\le |\xi|^2 + \frac{1}{\nu}\int_0^t |\mathbf{f}(s)|^2 ds + \nu\int_0^t |\mathbf{u}(s)|^2 ds + \int_0^t |\boldsymbol{\sigma}(\mathbf{u}(s))|_{L_Q}|\mathbf{v}(s)|_0|\mathbf{u}(s)| ds
\end{aligned}$$

by using Young's inequality in the last step above; using condition (A.2), and continuing,

$$\begin{aligned}
|\mathbf{u}(t)|^2 + \nu\int_0^t \|\mathbf{u}(s)\|^2\, ds \le |\xi|^2 &+ \frac{1}{\nu}\int_0^t |\mathbf{f}(s)|^2_{V'} ds + \frac{\nu}{2}\int_0^t (1 + \|\mathbf{u}(s)\|^2)\, ds \\
&+ \frac{K}{\nu}\int_0^t |\mathbf{v}(s)|_0|\mathbf{u}(s)|^2\, ds.
\end{aligned} \tag{4.8}$$

By the Gronwall Lemma, for any T,

$$\sup_{0\le t\le T} |\mathbf{u}(t)|^2 \le \left(|\xi|^2 + \frac{1}{\nu}\int_0^T |\mathbf{f}(s)|^2_{V'}\, ds\right) e^{\frac{K}{\nu}\int_0^T |\mathbf{v}(s)|_0^2 ds}. \tag{4.9}$$

Using the above bound in (4.8), it follows that

$$\int_0^T \|\mathbf{u}(s)\|^2 ds \le C\left(\nu, |\xi|^2, \int_0^T |\mathbf{f}(s)|^2_{V'} ds, M, K, \sup_{0\le s\le T} |\mathbf{u}(s)|^2\right). \tag{4.10}$$

Thus

$$\sup_{0\le t\le T}|\mathbf{u}(t)|^2+\int_0^T\|\mathbf{u}(t)\|^2\mathrm{d}s\le C_1\left(\nu,|\xi|^2,\int_0^T|\mathbf{f}(s)|^2_{V'}\mathrm{d}s,M,K,\sup_{0\le s\le T}|\mathbf{u}(s)|^2\right). \tag{4.11}$$

By (4.9), the above bound can be written as

$$\sup_{0\le t\le T}|\mathbf{u}(t)|^2+\int_0^T\|\mathbf{u}(t)\|^2\mathrm{d}s\le C_2\left(\nu,|\xi|^2,\int_0^T|\mathbf{f}(s)|^2_{V'}\mathrm{d}s,M,K\right) \tag{4.12}$$

so that the bound is uniform in n. Let $\mathbf{w}_n=\mathbf{u}_n-\mathbf{u}$. It suffices to show that $\mathbf{w}_n\to 0$ in $C([0,T]:H)\cap L^2(0,T:V)$ as $n\to\infty$.

$$\begin{aligned}\mathbf{w}_n(t)&+\int_0^t\{\nu\mathbf{A}\mathbf{w}_n(s)+\mathbf{B}(\mathbf{u}_n(s))-\mathbf{B}(\mathbf{u}(s))\}\mathrm{d}s\\&=\int_0^t\{\boldsymbol{\sigma}(s,\mathbf{u}_n(s))\mathbf{v}_n(s)-\boldsymbol{\sigma}(s,\mathbf{u}(s))\mathbf{v}(s)\}\mathrm{d}s\end{aligned}$$

so that

$$\begin{aligned}\frac12|\mathbf{w}_n(t)|^2&+\nu\int_0^t\|\mathbf{w}_n(s)\|^2\,\mathrm{d}s+\int_0^t\{b(\mathbf{u}_n(s),\mathbf{u}_n(s),\mathbf{w}_n(s))-b(\mathbf{u}(s),\mathbf{u}(s),\mathbf{w}_n(s))\}\,\mathrm{d}s\\&=\int_0^t\{((\boldsymbol{\sigma}(s,\mathbf{u}_n(s))-\boldsymbol{\sigma}(s,\mathbf{u}(s)))\mathbf{v}_n(s),\mathbf{w}_n(s))\\&\quad+(\boldsymbol{\sigma}(s,\mathbf{u}(s))(\mathbf{v}_n(s)-\mathbf{v}(s)),\mathbf{w}_n(s))\}\,\mathrm{d}s.\end{aligned} \tag{4.13}$$

The estimates that will be used in (4.13) are explained in what follows: Using the properties of the function b introduced in (2.3),

$$\begin{aligned}b(\mathbf{u}_n,\mathbf{u}_n,\mathbf{w}_n)-b(\mathbf{u},\mathbf{u},\mathbf{w}_n)&=b(\mathbf{w}_n,\mathbf{u}_n,\mathbf{w}_n)+b(\mathbf{u},\mathbf{w}_n,\mathbf{w}_n)\\&=b(\mathbf{w}_n,\mathbf{u},\mathbf{w}_n)\end{aligned}$$

so that

$$|b(\mathbf{u}_n,\mathbf{u}_n,\mathbf{w}_n)-b(\mathbf{u},\mathbf{u},\mathbf{w}_n)|\le|\mathbf{w}_n|\,\|\mathbf{w}_n\|\,\|\mathbf{u}\|. \tag{4.14}$$

The first term in the integrand on the right side of (4.13) can be bounded by $L\,\|\mathbf{w}_n(s)\|\,|\mathbf{w}_n(s)|\,|v_n(s)|_0$, while the second term in the integrand of (4.13) can be bounded by $|\boldsymbol{\sigma}(s,\mathbf{u}(s))\,(\mathbf{v}_n(s)-\mathbf{v}(s))|\,|\mathbf{w}_n(s)|$. The above estimates and (4.14) used in (4.13) yield

$$\begin{aligned}\frac12|\mathbf{w}_n(t)|^2+\nu\int_0^t\|\mathbf{w}_n(s)\|^2\,\mathrm{d}s&\le\int_0^t|\mathbf{w}_n(s)|\,\|\mathbf{w}_n(s)\|\,\|\mathbf{u}(s)\|\,\mathrm{d}s\\&+\int_0^t L\,\|\mathbf{w}_n(s)\|\,|\mathbf{w}_n(s)|\,|\mathbf{v}_n(s)|_0\mathrm{d}s+\int_0^t|\boldsymbol{\sigma}(s)(\mathbf{v}_n(s)-\mathbf{v}(s))|\,|\mathbf{w}_n(s)|\,\mathrm{d}s.\end{aligned}$$

By using Young's inequality,

$$\begin{aligned}\frac12|\mathbf{w}_n(t)|^2+\frac{\nu}{2}\int_0^t\|\mathbf{w}_n(s)\|^2\,\mathrm{d}s&\le\frac{2(L+1)}{\nu}\int_0^t|\mathbf{w}_n(s)|^2\,(\|\mathbf{u}(s)\|+|\mathbf{v}_n(s)|_0)^2\,\mathrm{d}s\\&+\int_0^t|\boldsymbol{\sigma}(s,\mathbf{u}(s))(\mathbf{v}_n(s)-\mathbf{v}(s))|^2\mathrm{d}s\end{aligned}$$

so that by the Gronwall inequality,

$$\frac{1}{2}|\mathbf{w}_n(t)|^2 + \frac{\nu}{2}\int_0^t \|\mathbf{w}_n(s)\|^2 \, \mathrm{d}s$$
$$\leq \left(\int_0^T |\boldsymbol{\sigma}(s, \mathbf{u}(s))(\mathbf{v}_n(s) - \mathbf{v}(s))|^2 \mathrm{d}s\right) \mathrm{e}^{\frac{2(L+1)}{\nu}\int_0^t (\|\mathbf{u}(r)\| + |\mathbf{v}_n(r)|_0)^2 \, \mathrm{d}r}.$$

Note that $\sigma(\cdot, \cdot)Q^{1/2}$ is a Hilbert–Schmidt operator on H, and hence a compact operator on H. Using the estimate (4.12), and the weak convergence of $\mathbf{v}_n \to \mathbf{v}$ in S_M, one obtains

$$\sup_{0 \leq t \leq T} |\mathbf{w}_n(t)|^2 + \int_0^T \|\mathbf{w}_n(s)\|^2 \, \mathrm{d}s \to 0$$

as $n \to \infty$. □

Theorem 4.6. *Let $\{\mathbf{u}^\epsilon(\cdot)\}$ be the solution of the equation*

$$\mathrm{d}\mathbf{u}^\epsilon(t) + \{\mathbf{A}\mathbf{u}^\epsilon(t) + \mathbf{B}(\mathbf{u}^\epsilon(t))\}\mathrm{d}t = \mathbf{f}(t)\mathrm{d}t + \sqrt{\epsilon}\boldsymbol{\sigma}(t, \mathbf{u}^\epsilon(t))\mathrm{d}\mathbf{W}(t) \tag{4.15}$$

with $\mathbf{u}^\epsilon(0) = \xi \in H$. Then $\{\mathbf{u}^\epsilon\}$ satisfies the Laplace principle in $C([0, T] : H) \cap L^2(0, T : V)$ with a good rate function

$$I_\xi(\mathbf{h}) = \inf_{\{\mathbf{v} \in L^2([0,T]:H_0) : \mathbf{h}(t) = \mathbf{g}^0(\int_0^\cdot \mathbf{v}(s)\mathrm{d}s)\}} \left\{\frac{1}{2}\int_0^T \|\mathbf{v}(t)\|_0^2 \, \mathrm{d}t\right\} \tag{4.16}$$

with the convention that the infimum of an empty set is infinity.

Proof. Let $\mathbf{v}_\epsilon$ converge to $\mathbf{v}$ in distribution as random elements taking values in S_M where S_M is equipped with the weak topology. Let $\mathbf{u}_{\mathbf{v}_\epsilon}$ solve

$$\begin{aligned}\mathrm{d}\mathbf{u}_{\mathbf{v}_\epsilon}(t) + [\nu\mathbf{A}\mathbf{u}_{\mathbf{v}_\epsilon}(t) + \mathbf{B}(\mathbf{u}_{\mathbf{v}_\epsilon}(t))]\mathrm{d}t &= [\mathbf{f}(t) + \boldsymbol{\sigma}(t, \mathbf{u}_{\mathbf{v}_\epsilon}(t))\mathbf{v}_\epsilon(t)]\mathrm{d}t \\ &\quad + \sqrt{\epsilon}\boldsymbol{\sigma}(t, \mathbf{u}_{\mathbf{v}_\epsilon}(t))\mathrm{d}\mathbf{W}(t)\end{aligned}$$

with $\mathbf{u}_{\mathbf{v}_\epsilon}(0) = \xi$. Strictly speaking, the solution should have been denoted as $\mathbf{u}^\epsilon_{\mathbf{v}_\epsilon}$. The slight abuse of notation makes it easier to read. Let $\mathbf{u}_\mathbf{v}$ be the solution of

$$\mathrm{d}\mathbf{u}_\mathbf{v}(t) + [\nu\mathbf{A}\mathbf{u}_\mathbf{v}(t) + \mathbf{B}(\mathbf{u}_\mathbf{v}(t))]\mathrm{d}t = [\mathbf{f}(t) + \boldsymbol{\sigma}(t, \mathbf{u}_\mathbf{v}(t))\mathbf{v}(t)]\mathrm{d}t \tag{4.17}$$

with $\mathbf{u}_\mathbf{v}(0) = \xi$. Since pathwise unique strong solutions exist for the above equations, the Borel measurable function $\mathbf{g}^\epsilon$ mentioned earlier satisfies the equality $\mathbf{g}^\epsilon(W(\cdot) + \frac{1}{\sqrt{\epsilon}}\int_0^\cdot \mathbf{v}_\epsilon(s)\,\mathrm{d}s) = \mathbf{u}_{\mathbf{v}_\epsilon}$. For all $\mathbf{v} \in L^2([0, T] : H_0)$, note that $\int_0^\cdot \mathbf{v}(s)\mathrm{d}s \in C([0, T] : H_0)$. Define $\mathbf{g}^0 : C([0, T] : H_0) \to C([0, T] : H) \cap L^2(0, T : V)$ by

$$\mathbf{g}^0(h) = \mathbf{u}_\mathbf{v} \quad \text{if } \mathbf{h} = \int_0^\cdot \mathbf{v}(s)\,\mathrm{d}s \quad \text{for some } \mathbf{v} \in L^2([0, T] : H_0).$$

If $\mathbf{h}$ cannot be represented as above, then define $\mathbf{g}^0(h) = 0$. To prove the theorem, it suffices to verify that the first of the Hypothesis 3.1 is satisfied, since the second has already been verified by the previous lemma. Since S_M is Polish, the Skorokhod representation theorem can be invoked to construct processes $(\tilde{\mathbf{v}}_\epsilon, \tilde{\mathbf{v}}, \tilde{\mathbf{W}}_\epsilon)$ such that the joint distribution of $(\tilde{\mathbf{v}}_\epsilon, \tilde{\mathbf{W}}_\epsilon)$ is the same as that of $(\mathbf{v}_\epsilon, \mathbf{W})$, and the distribution of $\tilde{\mathbf{v}}$ coincides with that of $\mathbf{v}$, and $\tilde{\mathbf{v}}_\epsilon \to \tilde{\mathbf{v}}$ a.s. in the topology.

Define $\mathbf{w}_\epsilon(t) := \mathbf{u}_{\mathbf{v}_\epsilon}(t) - \mathbf{u}_\mathbf{v}(t)$. The notation $|\cdot|_{H.S.}$ will denote the Hilbert–Schmidt norm in what follows.

An application of the Itô Lemma due to Gyöngy and Krylov yields

$$\begin{aligned}
&\frac{1}{2}|\mathbf{w}_\epsilon(t)|^2 + \nu \int_0^t \|\mathbf{w}_\epsilon(s)\|^2\,\mathrm{d}s \\
&= -\int_0^t b(\mathbf{w}_\epsilon(s), \mathbf{u}_\mathbf{v}(s), \mathbf{w}_\epsilon(s))\mathrm{d}s \\
&\quad + \int_0^t (\boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s))\mathbf{v}_\epsilon(s) - \boldsymbol{\sigma}(s, \mathbf{u}_\mathbf{v}(s))\mathbf{v}(s), \mathbf{w}_\epsilon(s))\mathrm{d}s \\
&\quad + \sqrt{\epsilon}\int_0^t (\mathbf{w}_\epsilon(s), \boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s))\mathrm{d}\mathbf{W}(s)) + \frac{\epsilon}{2}\int_0^t |\boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s))Q^{1/2}|^2_{H.S.}\,\mathrm{d}s \\
&\le \int_0^t |\mathbf{w}_\epsilon(s)|\,\|\mathbf{w}_\epsilon(s)\|\,\|\mathbf{u}_\mathbf{v}(s)\|\mathrm{d}s \\
&\quad + \int_0^t |\boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s)) - \boldsymbol{\sigma}(s, \mathbf{u}_\mathbf{v}(s))Q^{1/2}|_{H.S.}\,|\mathbf{v}_\epsilon(s)|_0\,|\mathbf{w}_\epsilon(s)|\mathrm{d}s \\
&\quad + \int_0^t |\boldsymbol{\sigma}(s, \mathbf{u}_\mathbf{v}(s))(\mathbf{v}_\epsilon(s) - \mathbf{v}(s))\|\mathbf{w}_\epsilon(s)|\mathrm{d}s \\
&\quad + \sqrt{\epsilon}\int_0^t (\mathbf{w}_\epsilon(s), \boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s))\mathrm{d}\mathbf{W}(s)) + \frac{\epsilon}{2}\int_0^t |\boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s))Q^{1/2}|_{H.S.}\,\mathrm{d}s.
\end{aligned}$$

From the above inequality, it follows that

$$\begin{aligned}
&\frac{1}{2}|\mathbf{w}_\epsilon(t)|^2 + \frac{3\nu}{4}\int_0^t \|\mathbf{w}_\epsilon(s)\|^2\,\mathrm{d}s \\
&\le \frac{1}{\nu}\int_0^t |\mathbf{w}_\epsilon(s)|^2\,\|\mathbf{u}_\mathbf{v}(s)\|^2\mathrm{d}s + L\int_0^t \|\mathbf{w}_\epsilon(s)\|\,|\mathbf{v}_\epsilon(s)|_0\,|\mathbf{w}_\epsilon(s)|\mathrm{d}s \\
&\quad + \int_0^t |\boldsymbol{\sigma}(s, \mathbf{u}_\mathbf{v}(s))(\mathbf{v}_\epsilon(s) - \mathbf{v}(s))|\,|\mathbf{w}_\epsilon(s)|\,\mathrm{d}s + \frac{\epsilon}{2}K\int_0^t (1 + \|\mathbf{u}_{\mathbf{v}_\epsilon}(s)\|^2)\mathrm{d}s \\
&\quad + \sqrt{\epsilon}\left|\int_0^t (\mathbf{w}_\epsilon(s), \boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s))\mathrm{d}\mathbf{W}(s))\right| \\
&\le \frac{1}{\nu}\int_0^t |\mathbf{w}_\epsilon(s)|^2\,\|\mathbf{u}_\mathbf{v}(s)\|^2\mathrm{d}s + \frac{\nu}{4}\int_0^t \|\mathbf{w}_\epsilon(s)\|^2\,\mathrm{d}s \\
&\quad + \frac{L^2}{\nu}\int_0^t |\mathbf{v}_\epsilon(s)|_0^2\,|\mathbf{w}_\epsilon(s)|^2\mathrm{d}s + \frac{1}{\nu}\int_0^t |\boldsymbol{\sigma}(s, \mathbf{u}_\mathbf{v}(s))(\mathbf{v}_\epsilon(s) - \mathbf{v}(s))|^2\,\mathrm{d}s \\
&\quad + \frac{\nu}{4}\int_0^t \|\mathbf{w}_\epsilon(s)\|^2\mathrm{d}s + \frac{\epsilon}{2}K\int_0^t (1 + \|\mathbf{u}_{\mathbf{v}_\epsilon}(s)\|^2)\mathrm{d}s \\
&\quad + \sqrt{\epsilon}\left|\int_0^t (\mathbf{w}_\epsilon(s), \boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s))\mathrm{d}W(s))\right|.
\end{aligned} \tag{4.18}$$

Define

$$\begin{aligned}
\tau_{N,\epsilon} := T \wedge \inf\Bigg\{ t : \int_0^t \{\|\mathbf{u}_\mathbf{v}(s)\|^2 + \|\mathbf{u}_{\mathbf{v}_\epsilon}(s)\|^2\}\mathrm{d}s > N \\
\text{or } \sup_{0\le s\le t} |\mathbf{u}_\mathbf{v}(s)|^2 > N \text{ or } \sup_{0\le s\le t} |\mathbf{u}_{\mathbf{v}_\epsilon}(s)| > N \Bigg\}.
\end{aligned}$$

Let T_0 be any member of $[0, T]$. Taking the supremum in (4.18) over the interval $[0, T_0 \wedge \tau_{N,\epsilon}]$ yields

$$\begin{aligned}
&\frac{1}{2}\Big(\sup_{0\le t\le T_0\wedge\tau_{N,\epsilon}} |\mathbf{w}_\epsilon(t)|^2\Big) + \frac{\nu}{4}\int_0^{T_0\wedge\tau_{N,\epsilon}} \|\mathbf{w}_\epsilon(s)\|^2 \mathrm{d}s \\
&\le \frac{1}{\nu}\int_0^{T_0\wedge\tau_{N,\epsilon}} |\mathbf{w}_\epsilon(s)|^2(\|\mathbf{u}_\mathbf{v}(s)\|^2 + L^2|\mathbf{v}_\epsilon(s)|_0^2)\mathrm{d}s \\
&\quad + \frac{1}{\nu}\int_0^{T\wedge\tau_{N,\epsilon}} |\boldsymbol{\sigma}(s, \mathbf{u}_\mathbf{v}(s))(\mathbf{v}_\epsilon(s) - \mathbf{v}(s))|^2 \mathrm{d}s + \frac{\epsilon}{2}K(T+N) \\
&\quad + \sqrt{\epsilon}\left\{\sup_{0\le t\le T\wedge\tau_{N,\epsilon}} \left|\int_0^t (\mathbf{w}_\epsilon(s), \boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s))\mathrm{d}W(s))\right|\right\}.
\end{aligned} \tag{4.19}$$

The Burkhölder–Davis–Gundy inequality allows us to bound the expectation of the last term on the right side of (4.19) by

$$\begin{aligned}
&2\sqrt{\epsilon}E\left[\left\{\int_0^{T\wedge\tau_{N,\epsilon}} |\mathbf{w}_\epsilon(s)|^2\, |\boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s))|_{L_Q}^2\, \mathrm{d}s\right\}^{1/2}\right] \\
&\le 2\sqrt{\epsilon}E\left[\sup_{0\le s\le T\wedge\tau_{N,\epsilon}} |\mathbf{w}_\epsilon(s)| \left\{\int_0^{T\wedge\tau_{N,\epsilon}} |\boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s))|_{L_Q}^2\, \mathrm{d}s\right\}^{1/2}\right] \\
&\le \sqrt{\epsilon}\left(E\left[\sup_{0\le s\le T\wedge\tau_{N,\epsilon}} |\mathbf{w}_\epsilon(s)|^2 + \int_0^{T\wedge\tau_{N,\epsilon}} |\boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s))|_{L_Q}^2\, \mathrm{d}s\right]\right) \\
&\le \sqrt{\epsilon}\left(E\left[\sup_{0\le s\le T\wedge\tau_{N,\epsilon}} |\mathbf{w}_\epsilon(s)|^2\right] + 2K^2(T+N)\right) < \infty
\end{aligned} \tag{4.20}$$

where condition (A.2) has been used in the last step. Using the Gronwall inequality in (4.19) and the definition of $\tau_{N,\epsilon}$ yields

$$\begin{aligned}
&\frac{1}{2}\sup_{0\le t\le T\wedge\tau_{N,\epsilon}} |\mathbf{w}_\epsilon(t)|^2 + \frac{\nu}{4}\int_0^{T\wedge\tau_{N,\epsilon}} \|\mathbf{w}_\epsilon(s)\|^2\, \mathrm{d}s \\
&\le \left\{\frac{1}{\nu}\int_0^{T\wedge\tau_{N,\epsilon}} |\boldsymbol{\sigma}(s, \mathbf{u}_\mathbf{v}(s))(\mathbf{v}_\epsilon(s) - \mathbf{v}(s))|^2\, \mathrm{d}s + \frac{\epsilon}{2}K\int_0^{T\wedge\tau_{N,\epsilon}} (1 + \|\mathbf{u}_{\mathbf{v}_\epsilon}(s)\|^2)\mathrm{d}s \right.\\
&\quad \left. +\ \sqrt{\epsilon}\left\{\sup_{0\le t\le T\wedge\tau_{N,\epsilon}} \left|\int_0^t (\mathbf{w}_\epsilon(s), \boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s))\mathrm{d}W(s))\right|\right\}\right\} \mathrm{e}^{\frac{1}{\nu}N + L^2 M}.
\end{aligned} \tag{4.21}$$

Let N be fixed. It is easy to show that for a suitable constant C,

$$\liminf_{\epsilon\to 0} P\{\tau_{N,\epsilon} = T\} \ge 1 - \frac{C}{N}.$$

Note that (4.20) shows that $\sqrt{\epsilon}\,\{\sup_{0\le t\le T\wedge\tau_{N,\epsilon}} |\int_0^t (\mathbf{w}_\epsilon(s), \boldsymbol{\sigma}(s, \mathbf{u}_{\mathbf{v}_\epsilon}(s))\mathrm{d}W(s))|\}$ converges in probability to 0 as ϵ tends to 0. These two observations along with the weak convergence of

$\mathbf{v}_\epsilon \to \mathbf{v}$ in S_M, when used in (4.21), yield

$$\frac{1}{2}\sup_{0\le t\le T}|\mathbf{w}_\epsilon(t)|^2 + \frac{\nu}{4}\int_0^T \|\mathbf{w}_\epsilon(s)\|^2\,\mathrm{d}s \to 0$$

in probability as $\epsilon \to 0$. The proof is thus completed. □

Remark. If one considers the three-dimensional stochastic Navier–Stokes equations with a smoothed out inertial term of the form $\mathbf{B}(k * \mathbf{u}, \mathbf{u})$ in place of $\mathbf{B}(\mathbf{u})$ where $k * \mathbf{u}(x) = \int_{\mathbb{R}^3} k(x-y)\mathbf{u}(y)\mathrm{d}y$, then note that

$$|b(k * \mathbf{u}, \mathbf{v}, \mathbf{u})| \le |k * \mathbf{u}|_6 |\nabla \mathbf{v}|\,|\mathbf{u}|_3 \le |k|_{6/5}\,|\mathbf{u}|_3^2\,|\nabla \mathbf{v}| \tag{4.22}$$

where the last inequality follows by the Young inequality for convolutions. Thus if $k \in L^{6/5}(G)$, then the last estimate combined with the fact that in $\mathbb{R}^3$, $|\mathbf{u}|_3 \le |\mathbf{u}|^{1/2}|\nabla \mathbf{u}|^{1/2}$, yields a priori estimates of the same type as ones that are employed in this paper. The results of this work therefore apply to smoothed out versions (as explained above) of the three-dimensional stochastic Navier–Stokes equations.

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References

[1] A. Budhiraja, P. Dupuis, A variational representation for positive functionals of infinite dimensional Brownian motion, Probab. Math. Statist. 20 (2000) 39–61.
[2] M.-H. Chang, Large deviation for Navier–Stokes equations with small stochastic perturbation, Appl. Math. Comput. 76 (1996) 65–93.
[3] P.-L. Chow, Some parabolic Ito equations, Comm. Pure Appl. Math. 45 (1992) 97–120.
[4] P.-L. Chow, Introduction to Stochastic Partial Differential Equations, Lecture Notes, Wayne State University, 2000.
[5] M. Capinsky, D. Gatarek, Stochastic equations in Hilbert space with application to Navier–Stokes equations in any dimension, J. Funct. Anal. 126 (1994) 26–35.
[6] A. Dembo, O. Zeitouni, Large Deviations Techniques and Applications, Springer-Verlag, New York, 2000.
[7] L. De Simon, Un applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine, Rend. Sem. Mat. Univ. Padova 34 (1964) 205–223.
[8] N. Dunford, J. Schwartz, Linear Operators, Interscience Publishers, John Wiley and Sons Inc., 1958.
[9] P. Dupuis, R.S. Ellis, A Weak Convergence Approach to the Theory of Large Deviations, Wiley-Interscience, New York, 1997.
[10] H.O. Fattorini, Infinite Dimensional Optimization and Control Theory, Cambridge University Press, 1999.
[11] F. Flandoli, D. Gatarek, Martingale and stationary solutions for stochastic Navier–Stokes equations, Probab. Theory Related Fields 102 (1995) 367–391.
[12] F. Flandoli, B. Maslowski, Ergodicity of the 2-D Navier–Stokes equation under random perturbations, Commun. Math. Phys. 171 (1995) 119–141.
[13] W.H. Fleming, A stochastic control approach to some large deviations problems, in: C. Dolcetta, W.H. Fleming, T. Zolezzi (Eds.), Recent Mathematical Methods in Dynamic Programming, in: Springer Lecture Notes in Math., vol. 1119, 1985, pp. 52–66.
[14] G.P. Galdi, An Introduction to the Theory of the Navier–Stokes Equations, vols. 1 and 2, Springer-Verlag, New York, 1993.
[15] G. Kallianpur, J. Xiong, Stochastic Differential Equations in Infinite Dimensional Spaces, Institute Math. Stat., 1996.

[16] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, New York, 1969.
[17] O.A. Ladyzhenskaya, V.A. Solonnikov, On the solvability of boundary value problems and initial–boundary value problems for the Navier–Stokes equations in regions with noncompact boundaries, Vestnik Leningrad Univ. Math. 10 (1977) 271–279.
[18] J.L. Menaldi, S.S. Sritharan, Stochastic 2-D Navier–Stokes equation, Appl. Math. Optim. 46 (2002) 31–53.
[19] M. Metivier, Stochastic Partial Differential Equations in Infinite Dimensional Spaces, Quaderni, Scuola Normale Superiore, Pisa, 1988.
[20] E. Pardoux, Equations aux derivées partielles stochastiques non linéaires monotones, Etude des solutions fortes de type Itô, Thesis, Université de Paris Sud. Orsay, 1975.
[21] H. Sohr, The Navier–Stokes Equations: An elementary Functional Analytic Approach, Birkhäuser, Boston, 2001.
[22] R. Sowers, Large Deviations for a reaction diffusion equation with non-gaussian perturbations, Ann. Probab. 20 (1992) 504–537.
[23] S.S. Sritharan, P. Sundar, The stochastic magneto-hydrodynamic System, Infinite Dimensional Analysis, Quantum Probab. Related Topics 2 (1999) 241–265.
[24] D. Stroock, An Introduction to the Theory of Large Deviations, Springer-Verlag, Universitext, New York, 1984.
[25] R. Temam, Navier–Stokes Equations, Theory and Numerical Analysis, North-Holland, Amsterdam, 1984.
[26] R. Temam, Navier–Stokes Equations and Nonlinear Functional Analysis, CBMS-NSF 41, SIAM, Philadelphia, PA, 1983.
[27] W. von Wahl, The Equations of Navier–Stokes and Abstract Parabolic Equations, Friedr. Vieweg and Sohn, Braunschweig, 1985.
[28] S.R.S. Varadhan, Large Deviations and its Applications, in: CBMS-NSF Series in Applied Mathematics, vol. 46, SIAM, Philadelphia, 1984.