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MODULI SPACES OF IRREGULAR SINGULAR CONNECTIONS

CHRISTOPHER L. BREMER AND DANIEL S. SAGE

Abstract. In the geometric version of the Langlands correspondence, irregular singular point connections play the role of Galois representations with wild ramification. In this paper, we develop a geometric theory of fundamental strata to study irregular singular connections on the projective line. Fundamental strata were originally used to classify cuspidal representations of the general linear group over a local field. In the geometric setting, fundamental strata play the role of the leading term of a connection. We introduce the concept of a regular stratum, which allows us to generalize the condition that a connection has regular semisimple leading term to connections with nonintegral slope. Finally, we construct a moduli space of meromorphic connections on the projective line with specified formal type at the singular points.

1. Introduction

A fundamental problem in the theory of differential equations is the classification of first order singular differential operators up to gauge equivalence. An updated version of this problem, rephrased into the language of algebraic geometry, is to study the moduli space of meromorphic connections on an algebraic curve C/k, where k is an algebraically closed field of characteristic 0. This problem has been studied extensively in recent years due to its relationship with the geometric Langlands correspondence. To elaborate, the classical Langlands conjecture gives a bijection between automorphic representations of a reductive group G over the adèles of a global field K and Galois representations taking values in the Langlands dual group $G^\vee$. By analogy, meromorphic connections (or, to be specific, flat $G^\vee$-bundles) play roughly the same role in the geometric setting as Galois representations (see [11, 12] for more background). Naively, one would like to find a description of the moduli space of meromorphic connections that resembles the space of automorphic representations of a reductive group.

A more precise statement is that the geometric Langlands data on the Galois side does not strictly depend on the connection itself, but rather on the monodromy representation determined by the connection. When the connection has regular singularities, i.e., when there is a basis in which the matrix of the connection has simple poles at each singular point, the Riemann-Hilbert correspondence states that the monodromy representation is simply a representation of the fundamental group. However, when the connection is irregular singular, the monodromy has a more subtle description due to the Stokes phenomenon. The irregular monodromy data consists of a collection of Stokes matrices at each singular point, which characterize

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the asymptotic expansions of a horizontal section on sectors around each irregular singular point (see [24], or [25] for a modern treatment.)

The irregular Riemann-Hilbert map from moduli spaces of connections to the space of Stokes matrices is well understood in the following situation. Let $V \cong \mathcal{O}_C^n$ be a trivializable rank n vector bundle, and let ∇ be a meromorphic connection on V with an irregular singular point at x. After fixing a local parameter t at x, suppose that ∇ has the following local description:

$$\nabla = d + M_r \frac{dt}{t^{r+1}} + M_{r-1} \frac{dt}{t^r} + \dots,$$

where $M_j \in \mathfrak{gl}_n(\mathbb{C})$ and the leading term M_r has pairwise distinct eigenvalues. Then, we say that ∇ has a regular semisimple leading term at x. Under this assumption, Jimbo, Miwa and Ueno [16] classify the deformations of ∇ that preserve the Stokes data by showing that they satisfy a system of differential equations (the so-called isomonodromy equations). In principle, the isomonodromy equations give a foliation of the moduli space of connections, with each leaf corresponding to a single monodromy representation. Indeed, the Riemann-Hilbert map has surprisingly nice geometric properties for connections with regular semisimple leading terms. Consider the moduli space $\mathscr{M}$ of connections on $\mathbb{P}^1$ which have singularities with regular semisimple leading terms at $\{x_1, \dots, x_m\}$ and which belong to a fixed formal isomorphism class at each singular point. Boalch, whose paper [5] is one of the primary inspirations for this project, demonstrates that $\mathscr{M}$ is the quotient of a smooth, symplectic manifold $\widetilde{\mathscr{M}}$ by a torus action. Moreover, the space of Stokes data has a natural symplectic structure which makes the Riemann-Hilbert map symplectic.

However, many irregular singular connections that arise naturally in the geometric Langlands program do not have a regular semisimple leading term. In [25, Section 6.2], Witten considers a connection of the form

$$\nabla = d + \begin{pmatrix} 0 & t^{-r} \\ t^{-r+1} & 0 \end{pmatrix} dt,$$

which has a nilpotent leading term. Moreover, it is not even locally gauge-equivalent to a connection with regular semisimple leading term unless one passes to a ramified cover. A particularly important example is described by Frenkel and Gross in [13]. They construct a flat G-bundle on $\mathbb{P}^1$, for arbitrary reductive G, that corresponds to a 'small' supercuspidal representation of G at ∞ and the Steinberg representation at 0. In the GL_n case, the result is a connection (due originally to Katz [17]) with a regular singular point at 0 (with unipotent monodromy) and an irregular singular point with nilpotent leading term at ∞. This construction suggests that connections with singularities corresponding to cuspidal representations of G, an important case in the geometric Langlands correspondence, do not have regular semisimple leading term in the sense above. These examples lead us to one of our main questions: is there a natural generalization of the notion of a regular semisimple leading term which allows us to extend the results of Boalch, Jimbo, Miwa and Ueno?

The solution to this problem is again motivated by analogy with the classical Langlands correspondence. Suppose that F is a local field and W is a ramified representation of $\mathrm{GL}_n(F)$. Let $P \subset \mathrm{GL}_n(F)$ be a parahoric subgroup with a decreasing filtration by congruence subgroups $\{P^j\}$, and suppose that β is an irreducible representation of P^r on which P^{r+1} acts trivially. We say that W contains the *stratum* (P, r, β) if the restriction of W to P^r has a subrepresentation isomorphic to β.

In the language of Bushnell and Kutzko [7, 8, 18], the data (P, r, β) is known as a *fundamental stratum* if β satisfies a certain non-degeneracy condition (see Section 2.4). If we write e_P for the period of the lattice chain stabilized by P, an equivalent condition is that (P, r, β) attains the minimal value r/e_P over all strata contained in W ([7, Theorem 1]). It was proved independently by Howe and Moy [15] and Bushnell [7] that every irreducible admissible representation of $\mathrm{GL}_n(F)$ contains a fundamental stratum. It was further shown in [8] and [18] that fundamental strata play an important role in the classification of supercuspidal representations, especially in the case of wild ramification.

As a tool in representation theory, fundamental strata play much the same role as the leading term of a connection in the cases considered above. Therefore, we are interested in finding an analogue of the theory of strata in the context of meromorphic connections in order to study moduli spaces of connections with cuspidal type singularities.

In this paper, we develop a geometric theory of strata and apply it to the study of meromorphic connections. We introduce a class of strata called *regular strata* which are particularly well-behaved: connections containing a regular stratum have similar behavior to connections with regular semisimple leading term. More precisely, a regular stratum associated to a formal meromorphic connection allows one to "diagonalize" the connection so that it has coefficients in the Cartan subalgebra of a maximal torus T. We call the diagonalized form of the connection a T-formal type. In Section 4.4, we show that two formal connections that contain regular strata are isomorphic if and only if their formal types lie in the same orbit of the relative affine Weyl group of T.

The perspective afforded by a geometric theory of strata has a number of benefits.

(1) The description of formal connections obtained in terms of fundamental strata translates well to global connections on curves; one reason for this is that, unlike the standard local classification theorem [20, Theorem III.1.2], one does not need to pass to a ramified cover. In the second half of the paper, we use regular strata to explicitly construct moduli spaces of irregular connections on $\mathbb{P}^1$ with a fixed formal type at each singular point. In particular, we obtain a concrete description of the moduli space of connections with singularities of "supercuspidal" formal type.
(2) Fundamental strata provide an illustration of the wild ramification case of the geometric Langlands correspondence; specifically, in the Bushnell-Kutzko theory [8, Theorem 7.3.9], refinements of fundamental strata correspond to induction data for admissible irreducible representations of GL_n.
(3) The analysis of the irregular Riemann-Hilbert map due to Jimbo, Miwa, Ueno, and Boalch [16, 5] generalizes to a much larger class of connections. Specifically, one can concretely describe the isomonodromy equations for families of connections that contain regular strata [6].
(4) Since the approach is purely Lie-theoretic, it can be adapted to study flat G-bundles (where G is an arbitrary reductive group) using the Moy-Prasad theory of minimal K-types [21].

Here is a brief outline of our results. In Section 2, we adapt the classical theory of fundamental strata to the geometric setting. Next, in Section 3, we introduce the notion of regular strata; these are strata that are centralized (in a graded sense defined below) by a possibly non-split maximal torus T. The major result of this section is Theorem 3.8, which states that regular strata split into blocks corresponding to the minimal Levi subgroup L containing T.

In Section 4, we show how to associate a stratum to a formal connection with coefficients in a Laurent series field F or, equivalently, to a flat GL_n-bundle over the formal punctured disk $\mathrm{Spec}(F)$. By Theorem 4.10, every formal connection (V, ∇) contains a fundamental stratum (P, r, β), and the quantity r/e_P for any fundamental stratum contained in (V, ∇) is precisely the slope. Moreover, Theorem 4.12 states that any splitting of (P, r, β) induces a splitting of (V, ∇). In particular, any connection containing a regular stratum has a reduction of structure to the Levi subgroup L defined above. In Section 4.3, we show that the matrix of any connection containing a regular stratum is gauge-equivalent to a matrix in $\mathfrak{t} = \mathrm{Lie}(T)$, which we call a formal type. We show in Section 4.4 that the set of formal types associated to an isomorphism class of formal connections corresponds to an orbit of the relative affine Weyl group in the space of formal types.

In Section 5, we construct a moduli space $\mathscr{M}$ of meromorphic connections on $\mathbb{P}^1$ with specified formal type at a collection of singular points as the symplectic reduction of a product of smooth varieties that only depend on local data. By Theorem 5.6, there is a symplectic manifold $\widetilde{\mathscr{M}}$ which resolves $\mathscr{M}$ by symplectic reduction. Finally, Theorem 5.26 relaxes the regularity condition on strata at regular singular points so that it is possible to consider connections with unipotent monodromy. In particular, this construction contains the GL_n case of the flat G-bundle described by Frenkel and Gross.

2. Strata

In this section, we describe an abstract theory of fundamental strata for vector spaces over a Laurent series field in characteristic zero. Strata were originally developed to classify cuspidal representations of GL_n over non-Archimedean local fields [7, 8, 18]. We will show that there is an analogous geometric theory with applications to the study of flat connections with coefficients in F. In Section 3, we introduce a novel class of fundamental strata of "regular uniform" type. These strata will play an important role in describing the moduli space of connections constructed in section 5.

2.1. Lattice Chains and the Affine Flag Variety.

Let k be an algebraically closed field of characteristic zero. Here, $\mathfrak{o} = k[[t]]$ is the ring of formal power series in a variable t, $\mathfrak{p} = t\mathfrak{o}$ is the maximal ideal in $\mathfrak{o}$, and $F = k((t))$ is the field of fractions.

Suppose that V is an n-dimensional vector space over F. An $\mathfrak{o}$-lattice $L \subset V$ is a finitely generated $\mathfrak{o}$-module with the property that $L \otimes_{\mathfrak{o}} F \cong V$. If we twist L by powers of t,

$$L(m) = t^{-m} L,$$

then every $L(m)$ is an $\mathfrak{o}$-lattice as well.

Definition 2.1. A *lattice chain* $\mathscr{L}$ is a collection of lattices $(L^i)_{i\in\mathbb{Z}}$ with the following properties:

(1) $L^i \supsetneq L^{i+1}$; and
(2) $L^i(m) = L^{i-me}$ for some fixed $e = e_{\mathscr{L}}$.

Notice that a shift in indexing $(L[j])^i := L^{i+j}$ produces a (trivially) different lattice chain $\mathscr{L}[j]$. The lattice chain $\mathscr{L}$ is called *complete* if $e = n$; equivalently, L^i/L^{i+1} is a one dimensional k-vector space for all i.

Definition 2.2. A parahoric subgroup $P \subset \mathrm{GL}(V)$ is the stabilizer of a lattice chain $\mathscr{L}$, i.e., $P = \{g \in \mathrm{GL}(V) \mid gL^i = L^i \text{ for all } i\}$. The Lie algebra of P is the parahoric subalgebra $\mathfrak{P} \subset \mathfrak{gl}(V)$ consisting of $\mathfrak{P} = \{p \in \mathfrak{gl}(V) \mid pL^i \subset L^i \text{ for all } i\}$. Note that $\mathfrak{P}$ is in fact an associative subalgebra of $\mathfrak{gl}(V)$. An Iwahori subgroup I is the stabilizer of a complete lattice chain, and an Iwahori subalgebra $\mathfrak{I}$ is the Lie algebra of I.

There are natural filtrations on P (resp. $\mathfrak{P}$) by congruence subgroups (resp. ideals). For $r \in \mathbb{Z}$, define the $\mathfrak{P}$-module $\mathfrak{P}^r$ to consist of $X \in \mathfrak{P}$ such that $XL^i \subset L^{i+r}$ for all i; it is an ideal of $\mathfrak{P}$ for $r \geq 0$ and a fractional ideal otherwise. The congruence subgroup $P^r \subset P$ is then defined by $P^0 = P$ and $P^r = \mathrm{I}_n + \mathfrak{P}^r$ for $r > 0$. Define $e_P = e_{\mathscr{L}}$; then, $t\mathfrak{P} = \mathfrak{P}^{e_P}$. Finally, P is *uniform* if $\dim L^i/L^{i+1} = n/e$ for all i. In particular, an Iwahori subgroup I is always uniform.

Proposition 2.3 ([7, Proposition 1.18])**.** *The Jacobson radical of the parahoric subalgebra $\mathfrak{P}$ is $\mathfrak{P}^1$. Moreover, when P is uniform, there exists an element $\varpi_P \in \mathfrak{P}$ such that $\varpi_P \mathfrak{P} = \mathfrak{P} \varpi_P = \mathfrak{P}^1$.*

As an example, suppose that $V = V_k \otimes_k F$ for a given k-vector space V_k. There is a distinguished lattice $V_{\mathfrak{o}} = V_k \otimes_k \mathfrak{o}$, and an evaluation map

$$\rho : V_{\mathfrak{o}} \to V_k$$

obtained by setting $t = 0$. Any subspace $W \subset V_k$ determines a lattice $\rho^{-1}(W) \subset V$. Thus, if $\mathscr{F} = (V_k = V^0 \supset V^1 \supset \ldots \supset V^e = \{0\})$ is a flag in V_k, then $\mathscr{F}$ determines a lattice chain by

$$\mathscr{L}_{\mathscr{F}} = (\ldots \supset t^{-1}\rho^{-1}(V^{n-1}) \supset V_{\mathfrak{o}} \supset \rho^{-1}(V^1) \supset \ldots \supset \rho^{-1}(V^{n-1}) \supset tV_{\mathfrak{o}} \supset \ldots).$$

We call such lattice chains (and their associated parahorics) *standard*. Thus, if $\mathscr{F}_0$ is the complete flag determined by a choice of Borel subgroup B, then $\rho^{-1}(B)$ is the standard Iwahori subgroup which is the stabilizer of $\mathscr{L}_{\mathscr{F}_0}$. Similarly, the partial flag in V_k determined by a parabolic subgroup Q gives rise to a standard parahoric subgroup $\rho^{-1}(Q)$ which is the stabilizer of the corresponding standard lattice chain. In particular, the maximal parahoric subgroup $\mathrm{GL}_n(\mathfrak{o})$ is the stabilizer of the standard lattice chain associated to $(V_k \supset \{0\})$.

In this situation, the obvious $\mathrm{GL}_n(F)$-action acts transitively on the space of complete lattice chains, so we may identify this space with the affine flag variety $\mathrm{GL}_n(F)/I$, where I is a standard Iwahori subgroup. More generally, every lattice chain is an element of a partial affine flag variety $\mathrm{GL}_n(F)/P$ for some standard parahoric P. For more details on the relationship between affine flag varieties and lattice chains in general, see [22].

For any maximal subfield $E \subset \mathfrak{gl}(V)$, there is a unique Iwahori subgroup I_E such that $\mathfrak{o}_E^\times \subset I_E$.

Lemma 2.4. *Suppose that P is a parahoric subgroup of $\mathrm{GL}(V)$ that stabilizes a lattice chain $\mathscr{L}$. Let E/F be a degree $n = \dim V$ field extension with a fixed* *embedding in $\mathfrak{gl}(V)$ such that $\mathfrak{o}_E^\times \subset P$. Then, there exists a complete lattice chain $\mathscr{L}_E \supset \mathscr{L}$ with stabilizer I_E such that $\mathfrak{o}_E^\times \subset I_E$ and $I_E \subset P$; it is unique up to translation of the indexing. In particular, $\mathfrak{o}_E^\times$ is contained in a unique Iwahori subgroup.*

Proof. We may identify V with E as an E-module. Since $\mathfrak{o}_E^\times \subset P$, it follows that $\mathfrak{o}_E \subset \mathfrak{P}$. Therefore, we may view $\mathscr{L}$ as a filtration of E by nonzero $\mathfrak{o}_E$-fractional ideals. Since $\mathfrak{o}_E$ is a discrete valuation ring, there is a maximal saturation $\mathscr{L}_E$ of $\mathscr{L}$, unique up to indexing, consisting of all the nonzero fractional ideals, and it is clear that $\mathfrak{o}_E^\times \subset I_E \subset P$. The final statement follows by taking P to be the stabilizer of the lattice $\mathfrak{o}_E$.

□

2.2. **Duality.** Let $\Omega^1_{F/k}$ be the space of one-forms on F, and let $\Omega^\times \subset \Omega^1_{F/k}$ be the $F^\times$-torsor of non-zero one-forms. If $\nu \in \mathfrak{o}^\times \frac{dt}{t^\ell} \subset \Omega^\times$, its *order* is defined by $\operatorname{ord}(\nu) = -\ell$. Any $\nu \in \Omega^\times$ defines a nondegenerate invariant symmetric k-bilinear form $\langle , \rangle_\nu$ on $\mathfrak{gl}_n(F)$ by

$$\langle A, B\rangle_\nu = \operatorname{Res}\left[\operatorname{Tr}(AB)\nu\right],$$

where Res is the usual residue on differential forms. In most contexts, one can take ν to be $\frac{dt}{t}$.

Let $\mathfrak{P}$ be the parahoric subalgebra that preserves a lattice chain $\mathscr{L}$.

Proposition 2.5 (Duality). *Fix $\nu \in \Omega^\times$. Then,*

$$(\mathfrak{P}^s)^\perp = \mathfrak{P}^{1-s-(1+\operatorname{ord}(\nu))e_P},$$

and, if $r \leq s$,

$$(\mathfrak{P}^r/\mathfrak{P}^s)^\vee \cong \mathfrak{P}^{1-s-(1+\operatorname{ord}(\nu))e_P}/\mathfrak{P}^{1-r-(1+\operatorname{ord}(\nu))e_P};$$

here, the superscript $^\vee$ denotes the k-linear dual.

This is shown in Proposition 1.11 and Corollary 1.13 of [7]. In particular, when $\operatorname{ord}(\nu) = -1$, $(\mathfrak{P}^s)^\perp = \mathfrak{P}^{1-s}$ and $(\mathfrak{P}^r/\mathfrak{P}^{r+1})^\vee = \mathfrak{P}^{-r}/\mathfrak{P}^{-r+1}$.

Observe that any element of $\mathfrak{P}^r$ induces an endomorphism of the associated graded $\mathfrak{o}$-module $\operatorname{gr}(\mathscr{L})$ of degree r; moreover, two such elements induce the same endomorphism of $\operatorname{gr}(\mathscr{L})$ if and only if they have the same image in $\mathfrak{P}^r/\mathfrak{P}^{r+1}$.

The following lemma gives a more precise description of the quotients $\mathfrak{P}^r/\mathfrak{P}^{r+1}$. Let $\bar{G} = \operatorname{GL}(L^0/tL^0) \cong \operatorname{GL}_n(k)$ with $\bar{\mathfrak{g}}$ the corresponding Lie algebra. Note that there is a natural map from $P \to \bar{G}$ whose image is a parabolic subgroup Q; its unipotent radical U is the image of P^1. Analogous statements hold for the Lie algebras $\operatorname{Lie}(Q) = \mathfrak{q}$ and $\operatorname{Lie}(U) = \mathfrak{u}$.

Lemma 2.6.

(1) *There is a canonical isomorphism of $\mathfrak{o}$-modules*

$$\mathfrak{P}^r/\mathfrak{P}^{r+1} \cong \bigoplus_{i=0}^{e_P-1} \operatorname{Hom}(L^i/L^{i+1}, L^{i+r}/L^{i+r+1}).$$

(2) *In the case $r = 0$, this isomorphism gives an algebra isomorphism between $\mathfrak{P}/\mathfrak{P}^1$ and a Levi subalgebra $\mathfrak{h}$ for Lie(Q) $= \mathfrak{q}$ (defined up to conjugacy by U). Moreover, $\mathfrak{P}$ is a split extension of $\mathfrak{h}$ by $\mathfrak{P}^1$.*

(3) *Similarly, if H is a Levi subgroup for Q, then $P \cong H \ltimes P^1$.*

Proof. There is a natural $\mathfrak{o}$-module map $\mathfrak{P}^r \to \bigoplus_{i=0}^{e_P-1} \mathrm{Hom}(L^i/L^{i+1}, L^{i+r}/L^{i+r+1})$; it is an algebra homomorphism when $r = 0$. It is clear that $\mathfrak{P}^{r+1}$ is the kernel, since any $\mathfrak{o}$-module map that takes L^i to L^{i+r+1} for $0 \leq i \leq e_P - 1$ must lie in $\mathfrak{P}^{r+1}$.

Now, suppose that $(\phi_i) \in \bigoplus_{i=0}^{e_P-1} \mathrm{Hom}(L^i/L^{i+1}, L^{i+r}/L^{i+r+1})$. Let $\mathscr{F}$ be the partial flag in $L^0/tL^0 = L^0/L^{e_P}$ given by $\{L^i/L^{e_P} \mid 0 \leq i \leq e_P\}$. We may choose an ordered basis $\mathbf{e}$ for L^0 that is compatible with $\mathscr{F}$ modulo L^{e_P}. This means that there is a partition $\mathbf{e} = \mathbf{e}_0 \cup \cdots \cup \mathbf{e}_{e_P-1}$ such that $W_j = \mathrm{span}(\mathbf{e}_j) \subset L^j$ is naturally isomorphic to L^j/L^{j+1}. In this basis, the groups $\mathrm{Hom}(L^i/L^{i+1}, L^{i+r}/L^{i+r+1})$ appear as disjoint blocks in $\mathfrak{P}^r$ (with exactly one block in each row and column of the array of blocks), so it is clear that we can construct a lift $\tilde{\phi} \in \mathfrak{P}^r$ that maps to (ϕ_i).

Note that when $r = 0$, the image of this isomorphism is a Levi subalgebra $\mathfrak{h}$ for the parabolic subalgebra $\mathfrak{q}$. The choice of basis gives an explicit embedding $\mathfrak{h} \cong \mathfrak{gl}(W_1) \oplus \cdots \oplus \mathfrak{gl}(W_{e_P-1}) \subset \mathrm{GL}(V)$, so the extension is split. The proof in the group case is similar. □

Remark 2.7. The same proof gives an isomorphism

$$\mathfrak{P}^r/\mathfrak{P}^{r+1} \cong \bigoplus_{i=m}^{m+e_P-1} \mathrm{Hom}(L^i/L^{i+1}, L^{i+r}/L^{i+r+1}) \tag{2.1}$$

for any m. However, if $\mathrm{Hom}(L^i/L^{i+1}, L^{i+r}/L^{i+r+1})$ and $\mathrm{Hom}(L^j/L^{j+1}, L^{j+r}/L^{j+r+1})$ for $i \equiv j \mod e_P$ are identified via homothety, the image of an element of $\mathfrak{P}^r/\mathfrak{P}^{r+1}$ is independent of m up to cyclic permutation. Indeed, this follows immediately from the observation that if $m = se_P + j$ for $0 \leq j < e_P$, then $t^{s+1}\mathbf{e}_0 \cup \cdots \cup t^{s+1}\mathbf{e}_{j-1} \cup t^s\mathbf{e}_j \cup t^s\mathbf{e}_{e_P-1}$ is a basis for L^m. In particular, $\mathfrak{P}/\mathfrak{P}^1$ is isomorphic to a Levi subalgebra in $\mathfrak{gl}(L^m/tL^m)$

Remark 2.8. Any element $\bar{x} \in \mathfrak{P}/\mathfrak{P}^1$ determines a canonical $GL(L^0/tL^0)$-orbit in $\mathfrak{gl}(L^0/tL^0)$, and similarly for P/P^1. To see this, note that any choice of ordered basis for L^0 compatible with $\mathscr{L}$ maps $\bar{x}$ onto an element of a Levi subalgebra of $\mathfrak{gl}(L^0/tL^0)$; a different choice of compatible basis will conjugate this image by an element of Q. In fact, this orbit is also independent of the choice of base point L^0 in the lattice chain. Indeed, an ordered basis for L^0 compatible with $\mathscr{L}$ gives a compatible ordered basis for L^m by multiplying basis elements by appropriate powers of t and then permuting cyclically. Using the corresponding isomorphism $L^0 \to L^m$ to identify $\mathfrak{gl}(L^0/tL^0)$ and $\mathfrak{gl}(L^m/tL^m)$, the images of $\bar{x}$ are the same. Accordingly, it makes sense to talk about the characteristic polynomial or eigenvalues of $\bar{x}$.

Notational Conventions. Let $\nu \in \Omega^\times$, and let V be an F-vector space. Suppose that $P \subset \mathrm{GL}(V)$ is a parahoric subgroup with Lie algebra $\mathfrak{P}$ that stabilizes a lattice chain $(L^i)_{i\in\mathbb{Z}}$. We will use the following conventions throughout the paper:

1. $\bar{L}^i = L^i/L^{i+1}$.
2. $\bar{P} = P/P^1$, $\bar{\mathfrak{P}} = \mathfrak{P}/\mathfrak{P}^1$.
3. $\bar{P}^\ell = P^\ell/P^{\ell+1}$, $\bar{\mathfrak{P}}^\ell = \mathfrak{P}^\ell/\mathfrak{P}^{\ell+1}$.
4. If $X \in \mathfrak{P}^s$, then $\bar{X}$ will denote its image in $\bar{\mathfrak{P}}^s$ and the corresponding degree s endomorphism of $\mathrm{gr}(\mathscr{L})$.
5. If $\alpha \in (\mathfrak{P}^r)^\vee$, then $\alpha_\nu \in \mathfrak{gl}(V)$ with denote an element such that $\alpha = \langle \alpha_\nu, \cdot \rangle_\nu$.

(6) If $\beta \in (\bar{\mathfrak{P}}^r)^\vee$, then β_ν will denote an element of $\mathfrak{P}^{r-(1+\mathrm{ord}(\nu))e_P}$ such that $\bar{\beta}_\nu \in \bar{\mathfrak{P}}^{r-(1+\mathrm{ord}(\nu))e_P}$ is the coset determined by the isomorphism in Proposition 2.5.
(7) Let $X \in \mathfrak{P}^s$. Then, $\delta_X : \bar{\mathfrak{P}}^i \to \bar{\mathfrak{P}}^{i+s}$ is the map induced by $\mathrm{ad}(X)$.
(8) Let $\mathfrak{a} \subset \mathrm{End}(V)$ be a subalgebra. We define $\mathfrak{a}^i = \mathfrak{a} \cap \mathfrak{P}^i$ and $\bar{\mathfrak{a}}^i = \mathfrak{a}^i/\mathfrak{a}^{i+1}$.
(9) If $A \subset \mathrm{GL}(V)$ is a subgroup, define $A^i = A \cap P^i$ and $\bar{A}^i = A^i/A^{i+1}$.

2.3. **Tame Corestriction.** In this section, we first suppose that $\mathscr{L}$ is a complete (hence uniform) lattice chain in V with corresponding Iwahori subgroup I. By Proposition 2.3, $\mathfrak{I}^1$ is a principal ideal generated by ϖ_I; similarly, each fractional ideal $\mathfrak{I}^\ell$ is generated by ϖ_I^ℓ. Choose an ordered basis $(e_0, \dots, e_{n-1})$ for V indexed by $\mathbb{Z}_n$, so that $e_{n+i} = e_i$ for all i. Furthermore, we may choose the basis to be compatible with $\mathscr{L}$: if $r = qn - s$, $0 \le s < n$, then L^r is spanned by $\{t^{q-1}e_0, \dots, t^{q-1}e_{s-1}, t^q e_s, \dots, t^q e_{n-1}\}$. Thus, if we let $\bar{e}_i$ denote the image of e_i in L^0/tL^0, then $\mathscr{L}$ corresponds to the full flag in L^0/tL^0 determined by the ordered basis $(\bar{e}_0, \dots \bar{e}_{n-1})$.

In this basis, we may take

$$\varpi_I = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & 1 \\ t & 0 & \cdots & 0 \end{pmatrix}. \tag{2.2}$$

Notice that the characteristic polynomial of ϖ_I is equal to $\lambda^n - t$, which is irreducible over F. Thus, $F[\varpi_I]$ is a degree n field extension isomorphic to $F[t^{1/n}]$.

Remark 2.9. If P is a parahoric subgroup stabilizing the lattice chain $\mathscr{L}$, we say that a basis for L^0 is compatible with $\mathscr{L}$ if it is a basis that is compatible as above for any complete lattice chain extending $\mathscr{L}$. Note that any pullback of a compatible basis for the induced partial flag in L^0/tL^0 is such a basis.

We first examine the kernel and image of the map $\delta_{\varpi_I^r} : \bar{\mathfrak{I}}^l \to \bar{\mathfrak{I}}^{l-r}$. Let $\nu \in \Omega^\times$ have order -1. We define $\psi_\ell(X) = \langle X, \varpi_I^{-\ell} \rangle_\nu$. Note that $\psi_\ell(\mathfrak{I}^{\ell+1}) = 0$; we let $\bar{\psi}_\ell$ be the induced functional on $\bar{\mathfrak{I}}^\ell$.

Let $\mathfrak{d} \subset \mathfrak{gl}_n(k)$ be the subalgebra of diagonal matrices. By Lemma 2.6, the Iwahori subgroup and subalgebra have semidirect product decompositions: $I = \mathfrak{d}^* \rtimes I^1$ and $\mathfrak{I}$ is a split extension of $\mathfrak{d}$ by $\mathfrak{I}^1$. Accordingly, any coset in $\mathfrak{I}^\ell/\mathfrak{I}^{\ell+1}$ has a unique representative $x\varpi_I^\ell$ with $x = \mathrm{diag}(x_0, x_1, \dots, x_{n-1}) \in \mathfrak{d}$.

Lemma 2.10. *The image of* $\delta_{\varpi_I^{-r}}$ *in* $\bar{\mathfrak{I}}^{\ell-r}$ *is contained in* $\ker(\bar{\psi}_{\ell-r})$*, and the kernel of* $\delta_{\varpi_I^{-r}}$ *in* $\bar{\mathfrak{I}}^\ell$ *contains the one-dimensional subspace spanned by* $\overline{\varpi_I^\ell}$*. Equality in both cases happens if and only if* $\gcd(r, n) = 1$.

Proof. Take $X = x\varpi_I^\ell$ with $x \in \mathfrak{d}$ as above. By direct calculation,

$$[X, \varpi_I^{-r}] = x'\varpi_I^{\ell-r}, \text{ with } x' = \mathrm{diag}(x_0 - x_{-r}, \dots, x_{n-1} - x_{n-1-r}). \tag{2.3}$$

Therefore,

$$\begin{aligned}\psi_{\ell-r}(x'\varpi_I^{\ell-r}) &= \operatorname{Res}(\operatorname{Tr}(x'\varpi_I^{\ell-r}\varpi_I^{r-\ell})\nu)\\ &= \operatorname{Res}(\nu)\operatorname{Tr}(x')\\ &= \operatorname{Res}(\nu)\sum_{i=0}^{n-1}(x_i - x_{i-r}) = 0.\end{aligned}$$

It follows that $\delta_{\varpi_I^{-r}}(\bar{\mathfrak{I}}^\ell) \subset \ker(\bar{\psi}_{\ell-r})$.

The kernel of $\delta_{\varpi_I^{-r}}$ satisfies the equations $x_i - x_{i-r} = 0$ for $0 \leq i \leq n-1$. If we set $x_0 = \alpha$, then $x_{-r} = x_{-2r} = x_{-3r} = \cdots = \alpha$. When $\gcd(r,n) = 1$, $j \equiv -mr \pmod{n}$ is solvable for any j, and it follows that $x_j = \alpha$ for all j. Therefore, the kernel is just the span of $\overline{\varpi_I^\ell}$. Otherwise, the dimension of the kernel is at least 2. This implies that the image of $\delta_{\varpi_I^{-r}}$ has codimension 1 if and only if $\gcd(r,n) = 1$. □

For future reference, we remark that there is a similar formula to (2.3) for Ad. Any element of $\bar{I} \cong \mathfrak{d}^*$ is of the form $\bar{p}$ for $p = \operatorname{diag}(p_0, \ldots, p_{n-1})$.

Then,

$$\operatorname{Ad}(\bar{p})(\overline{\varpi_I^{-r}}) = \overline{p'\varpi_I^{-r}} \in \bar{\mathfrak{I}}^{-r}, \text{ where } p' = \operatorname{diag}\left(\frac{p_0}{p_{-r}}, \frac{p_1}{p_{1-r}}, \ldots, \frac{p_{n-1}}{p_{n-1-r}}\right). \tag{2.4}$$

In particular, when $\gcd(r,n) = 1$, every generator of $\mathfrak{I}^{-r}$ lies modulo $\mathfrak{I}^{-r+1}$ in the $\operatorname{Ad}(I)$-orbit of $a\varpi_I^{-r}$ for some $a \in k^*$.

Next, we consider more general uniform parahorics. Let E/F be a degree m extension; it is unique up to isomorphism. Now, identify $V \cong E^{n/m}$ as an F vector space. We will view E as a maximal subfield of $\mathfrak{gl}_m(F)$: if we define $\varpi_E = \varpi_I \in \mathfrak{gl}_m(F)$ as in (2.2), then E is the centralizer of ϖ_E, which is in fact a uniformizing parameter for E. Since $m|n$, define a Cartan subalgebra $\mathfrak{t} \cong E^{n/m}$ in $\mathfrak{gl}(V)$ as the block diagonal embedding of n/m copies of $E \subset \mathfrak{gl}_m(F)$. Let $\gcd(r,m) = 1$, and take $\xi = (a_1\varpi_E^{-r}, \ldots, a_{n/m}\varpi_E^{-r})$ with the a_i's pairwise unequal elements of k. This implies that ξ is regular semisimple with centralizer $\mathfrak{t}$.

Let $\mathscr{L}_E$ be the complete lattice chain in F^m stabilized by $\mathfrak{o}_E$; we let I_E be the corresponding Iwahori subgroup. We define a lattice chain $\mathscr{L} = \bigoplus_{i=1}^{n/m} \mathscr{L}_E$ in V with associated parahoric subgroup P. It is clear that P is uniform with $e_P = m$. Moreover, elements of $\mathfrak{P}^\ell$ are precisely those $n/m \times n/m$ arrays of $m \times m$ blocks with entries in $\mathfrak{I}_E^\ell$; in particular, we can take $\varpi_P = (\varpi_E, \ldots, \varpi_E)$. Note that $\mathfrak{t} \cap \mathfrak{gl}(L) \subset \mathfrak{P}$ for any lattice L in $\mathscr{L}$.

Proposition 2.11 (Tame Corestriction)**.** *There is a morphism of $\mathfrak{t}$-bimodules $\pi_\mathfrak{t} : \mathfrak{gl}(V) \to \mathfrak{t}$ satisfying the following properties:*

1. *$\pi_\mathfrak{t}$ restricts to the identity on $\mathfrak{t}$;*
2. *$\pi_\mathfrak{t}(\mathfrak{P}^\ell) \subset \mathfrak{P}^\ell$;*
3. *the kernel of the induced map*

$$\bar{\pi}_\mathfrak{t} : (\mathfrak{t} + \mathfrak{P}^{\ell-r})/\mathfrak{P}^{\ell-r+1} \to \mathfrak{t}/(\mathfrak{t} \cap \mathfrak{P}^{\ell-r+1})$$

   is given by the image of $\operatorname{ad}(\mathfrak{P}^\ell)(\xi)$ modulo $\mathfrak{P}^{\ell-r+1}$;
4. *if $z \in \mathfrak{t}$ and $X \in \mathfrak{gl}(V)$, then $\langle z, X\rangle_\nu = \langle z, \pi_\mathfrak{t}(X)\rangle_\nu$;*
5. *$\pi_\mathfrak{t}$ commutes with the action of the normalizer $N(T)$ of T.*

Proof. First, take $\nu = \frac{dt}{t}$. Let $\epsilon_i \in \mathfrak{gl}(V)$ be the identity element in the i^{th} copy of E in $\mathfrak{t}$ and 0 elsewhere. Define $\psi_s^i(X) = \frac{1}{m}\langle \varpi_E^{-s}\epsilon_i, X\rangle_\nu$ and

$$\pi_{\mathfrak{t}}(X) = \sum_{s=-\infty}^{\infty}\sum_{i=1}^{n/m} \psi_s^i(X)\varpi_E^s\epsilon_i.$$

It is easily checked that for $s \ll 0$, $\psi_s^i(X) = 0$ and that $\pi_{\mathfrak{t}}$ is a $\mathfrak{t}$-map. A direct calculation shows that for $n \in N(T)$, $\pi_{\mathfrak{t}}(\mathrm{Ad}(n)X) = \mathrm{Ad}(n)\pi_{\mathfrak{t}}(X)$. Since $\pi_{\mathfrak{t}}$ is defined using traces, it is immediate that it vanishes on the off-diagonal blocks $\epsilon_i\,\mathfrak{gl}(V)\epsilon_j$ for $i \neq j$. Moreover, $\pi_{\mathfrak{t}}$ is the identity on $\mathfrak{t}$, since $\psi_s^i(\varpi_E^j\epsilon_k) = 1$, if $j = s$ and $i = k$, and equals 0 otherwise.

We note that $\pi_{\mathfrak{t}}(\mathfrak{P}^\ell) \subset \mathfrak{P}^\ell \cap \mathfrak{t}$, so the induced map $\bar{\pi}_{\mathfrak{t}}$ makes sense. Let $V_{ij}^{\ell-r} = \epsilon_i\mathfrak{P}^{\ell-r}\epsilon_j$ so that $\bar{V}_{ij}^{\ell-r} \subset \ker(\bar{\pi}_{\mathfrak{t}})$ for $i \neq j$. By regularity, $\delta_\xi : \bar{V}_{ij}^{\ell} \to \bar{V}_{ij}^{\ell-r}$ is an isomorphism whenever $i \neq j$. This proves that the off-diagonal part of $\ker(\bar{\pi}_{\mathfrak{t}})$ is of the desired form. We may now reduce without loss of generality to the case of a single diagonal block, i.e., $\mathfrak{t} = E$ and $\mathfrak{P} = \mathfrak{I}$.

Since $\pi_{\mathfrak{t}}$ is the identity on $\mathfrak{t}$, the kernel of $\bar{\pi}_{\mathfrak{t}}$ is contained in $\bar{\mathfrak{I}}^{\ell-r} = \varpi_I^{\ell-r}\bar{\mathfrak{I}}$. Notice that when $s < \ell - r$, $\varpi_I^{-s}\varpi_I^{\ell-r}\mathfrak{I} \subset \mathfrak{I}^1$; therefore, $\psi_s(\mathfrak{I}^{\ell-r}) = 0$. It is trivial that $\psi_s(X)\varpi_E^s \in \mathfrak{I}^{\ell-r+1}$ for $s > \ell - r$. It follows that $\ker(\bar{\pi}_{\mathfrak{t}}) = \ker(\bar{\psi}_{\ell-r})$. By Lemma 2.10, $\mathrm{ad}(\mathfrak{I}^\ell)(\varpi_E^{-r}) = \ker(\bar{\psi}_{\ell-r})$. This completes the proof of the third part of the proposition.

Finally, for arbitrary $\nu' = f\nu$, $\langle z, X\rangle_{\nu'} = \langle z, fX\rangle_\nu$. Since $f \in F \subset T$, and $\pi_{\mathfrak{t}}$ is a $\mathfrak{t}$-map, it suffices to prove the fourth part when $\nu = \frac{dt}{t}$. Although $z \in \mathfrak{t}$ is an infinite sum of the form $\sum_{s\geq q}\sum_{i=1}^{n/m} a_{si}\varpi_E^s\epsilon_i$ for some $a_{si} \in k$, only a finite number of terms contribute to the inner products. Hence, it suffices to consider $z = \varpi_E^s\epsilon_i$. Observing that $\langle \varpi_E^s\epsilon_i, \varpi_E^{-r}\epsilon_j\rangle_\nu = m\delta_{ij}\delta_{rs}$, we see that

$$\langle \varpi_E^s\epsilon_i, X\rangle_\nu = m\psi_{-s}^i(X) = \langle \varpi_E^s\epsilon_i, \psi_{-s}^i(X)\varpi_E^{-s}\epsilon_i\rangle_\nu = \langle \varpi_E^s\epsilon_i, \pi_{\mathfrak{t}}(X)\rangle_\nu,$$

as desired. □

Remark 2.12. Suppose that $P \subset \mathrm{GL}(V)$ is a uniform parahoric that stabilizes a lattice chain $\mathscr{L}$. Let $H \subset P$ be the Levi subgroup that splits $P \to P/P^1$ as in Lemma 2.6, and let $\mathfrak{h} \subset \mathfrak{P}$ be the corresponding subalgebra. We will show that there is a generator ϖ_P for $\mathfrak{P}^1$ that is well-behaved with respect to H, akin to $\varpi_I \in \mathfrak{I}^1$. In the notation used in the proof of Lemma 2.6, $\mathfrak{h}$ is determined by an ordered basis $\mathbf{e}$ for L^0 partitioned into e_P equal parts: $\mathbf{e} = \bigcup_{j=0}^{e_P-1}\mathbf{e}_j$. Setting $W_j = \operatorname{span}\mathbf{e}_j \cong \bar{L}^j$, we have $\mathfrak{h} = \bigoplus_{j=0}^{e_P-1}\mathfrak{gl}(W_j)$ and $H = \prod_{j=0}^{e_P-1}\mathrm{GL}(W_j)$.

Now, let $\mathscr{L}'$ be the complete lattice chain determined by the ordered basis for L^0 given above, and let I be the corresponding Iwahori subgroup. If ϖ_I is the generator of $\mathfrak{I}^1$ constructed in (2.2), define $\varpi_P = \varpi_I^m$, where $m = n/e_P$. This matrix is an $e_P \times e_P$ block matrix of the same form as (2.2), but with scalar $m \times m$ blocks. Evidently, $\varpi_P(L^i) = L^{i+1}$, so ϖ_P generates $\mathfrak{P}^1$. Furthermore, $\varpi_P(W_j) = W_{j+1}$ for $0 \leq j < e_P - 1$, and $\varpi_P(t^{-1}W_{e_P-1}) = W_0$. It follows that ϖ_P normalizes H, and $\mathrm{Ad}(\varpi_P)(\mathfrak{h}) \subset \mathfrak{h}$. In fact, if $A = \mathrm{diag}(A_0, \ldots, A_{e_P-1}) \in \mathfrak{h}$, then $\mathrm{Ad}(\varpi_P^r)(A) = \mathrm{diag}(A_r, \ldots, A_{r+e_P-1})$, with the indices understood modulo e_P.

2.4. **Strata.** For the remainder of Section 2, $\nu \in \Omega^\times$ will be a fixed one-form of order -1.

Definition 2.13. Let V be an F vector space. A *stratum* in $\mathrm{GL}(V)$ is a triple (P, r, β) consisting of

- $P \subset \mathrm{GL}(V)$ a parahoric subgroup;
- $r \in \mathbb{Z}_{\geq 0}$;
- $\beta \in (\bar{\bar{\mathfrak{P}}}^r)^\vee$.

Proposition 2.5 states that $(\bar{\mathfrak{P}}^r)^\vee = \bar{\mathfrak{P}}^{-r}$. Therefore, we may choose a representative $\beta_\nu \in \mathfrak{P}^{-r}$ for β. Explicitly, a stratum is determined by a triple $(\mathscr{L}, r, \beta_\nu)$, where $\mathscr{L}$ is the lattice chain preserved by P, and β_ν is a degree $-r$ endomorphism of $\mathscr{L}$. The triples $(\mathscr{L}, r, \beta_\nu)$ and $(\mathscr{L}', r', \beta'_\nu)$ give the stratum if and only if $r = r'$, $\mathscr{L}'$ is a translate of $\mathscr{L}$, and β_ν and β'_ν induce the same maps on $\mathrm{gr}(\mathscr{L})$, i.e., $\bar{\beta}_\nu = \bar{\beta}_\nu$.

We say that (P, r, β) is *fundamental* if $\beta_\nu + \mathfrak{P}^{-r+1}$ contains no nilpotent elements of $\mathfrak{gl}_n(F)$. By [7, Lemma 2.1], a stratum is non-fundamental if and only if $(\beta_\nu)^m \in \mathfrak{P}^{1-rm}$ for some m.

Remark 2.14. A stratum (P, r, β) is fundamental if and only if $\bar{\beta}_\nu \in \mathrm{End}(\mathrm{gr}(\mathscr{L}))$ is non-nilpotent in the usual sense. In particular, if $\beta_\nu(L^i) = L^{i-r}$ for all $i \in \mathbb{Z}$, then (P, r, β) is necessarily fundamental.

Definition 2.15. Let (P, r, β) be a stratum in $\mathrm{GL}(V)$. A reduction of (P, r, β) is a $\mathrm{GL}(V)$-stratum (P', r', β') with the following properties: $\left(\beta'_\nu + (\mathfrak{P}')^{1-r'}\right) \cap \left(\beta_\nu + \mathfrak{P}^{1-r}\right) \neq \emptyset$, $\beta_\nu + \mathfrak{P}^{1-r} \subset (\mathfrak{P}')^{-r'}$, and there exists a lattice L that lies in both of the associated lattice chains $\mathscr{L}$ and $\mathscr{L}'$.

Let (P', r', β') be a reduction of (P, r, β). The first property allows one to choose $\beta_\nu \in \mathfrak{gl}_n(F)$ to represent both β and β'. The second implies that any representative β_ν for β determines an element of $(\bar{\mathfrak{P}}')^{r'}$. Note that it is possible to have two different reductions (P', r', β'_1) and (P', r', β'_2) with the same P' and r', if $\mathfrak{P}^{1-r} \not\subseteq (\mathfrak{P}')^{1-r'}$.

An important invariant of a stratum (P, r, β) is its *slope*, which is defined by $\mathrm{slope}(P, r, \beta) = r/e_P$. The following theorem describes the relationship between slope and fundamental strata.

Theorem 2.16. *Suppose that $r \geq 1$. Then (P, r, β) is a non-fundamental stratum if and only if there is a reduction (P', r', β') with* $\mathrm{slope}(P, r, \beta) < \mathrm{slope}(P', r', \beta')$.

This is proved in Theorem 1 and Remark 2.9 of [7].

Definition 2.17. A stratum (P, r, β) is called *uniform* if it is fundamental, P is a uniform parahoric subgroup, and $\gcd(r, e_P) = 1$. The stratum is *strongly uniform* if it is uniform and $\beta_\nu(L^i) = L^{i-r}$ for all $L^i \in \mathscr{L}$.

Remark 2.18. A uniform stratum (P, r, β) is strongly uniform if and only if the induced maps $\bar{\beta}_\nu : \bar{L}^i \to \bar{L}^{i-r}$ are isomorphisms for each i. The forward implication follows since the $\bar{L}^i$'s have the same dimension. For the converse, note that if the $\bar{\beta}^i_\nu$'s are isomorphisms, then, in particular, $L^{i-r+j} = \beta_\nu(L^{i+j}) + L^{i-r+j+1}$ for $0 \leq j < e_P$. Substituting gives $L^{i-r} = \sum_{j=0}^{e_P-1} \beta_\nu(L^{i+j}) + L^{i-r+e_P} = \beta_\nu(L^i) + tL^{i-r}$, so $\beta_\nu(L^i) = L^{i-r}$ by Nakayama's Lemma.

Any fundamental stratum has a reduction with $\gcd(r, e_P) = 1$.

Lemma 2.19. *If (P, r, β) is a fundamental stratum, there is a fundamental reduction (P', r', β') with the property that* $\gcd(r', e_P) = 1$.

Proof. Let $g = \gcd(r, e_P)$ and $r' = r/g$. Let $(L')^j = L^{jg}$, and set $\mathscr{L}' = ((L')^j)$. This is the sub-lattice chain of $\mathscr{L}$ consisting of all lattices of the form L^{ae_P+br} with $a, b \in \mathbb{Z}$. If we choose a representative β_ν for β, $\beta_\nu((L')^j) \subset (L')^{j-r'}$. Thus, $\beta_\nu + \mathfrak{P}^{1-r} \subset (\mathfrak{P}')^{-r'}$. Let $\beta' \in ((\bar{\mathfrak{P}}')^r)^\vee$ be the functional determined by the image of β_ν in $(\bar{\mathfrak{P}}')^{-r'}$. If $\beta_\nu^N \in (\mathfrak{P}')^{-Nr'+1}$, then $\beta_\nu^N \in \mathfrak{P}^{-Nr+1}$: if $\beta_\nu(L')^j \subset (L')^{j-Nr'+1}$, then for any $j \in \mathbb{Z}$ and $0 < m < g$, $\beta_\nu(L^{jg+m}) \subset \beta_\nu(L^{jg}) \subset L^{jg-Nr+g} \subset L^{jg-Nr+m+1}$. Thus, if (P, r, β) is fundamental, so is (P', r', β'). □

2.5. **Split Strata.** We now generalize the notion of a 'split stratum' given in [18, Section 2] and [8, Section 2.3] to the geometric setting. Suppose that (P, r, β) is a stratum in V and that $\mathscr{L} = (L^i)_{i\in\mathbb{Z}}$ is the lattice chain stabilized by P. Let $V = V_1 \oplus V_2$ with $V_1, V_2 \neq \{0\}$. Define $L_j^i = L^i \cap V_j$ for $j = 1, 2$. Note that L_j^i has maximal rank in V_j, so it is indeed a lattice. Let $\mathscr{L}_j$ be the lattice chain consisting of (L_j^i) omitting repeats. We denote the parahoric associated to $\mathscr{L}_j$ by P_j. Note that if $L^i = L_1^i \oplus L_2^i$ for all i, then each L_j^i is automatically a lattice in V_j.

Definition 2.20. We say that (V_1, V_2) *splits* P if

(1) $L^i = L_1^i \oplus L_2^i$ for all i, and
(2) $\mathscr{L}_1$ is a uniform lattice chain with $e_{P_1} = e_P$.

In addition, (V_1, V_2) splits β at level r if $\beta_\nu(L_j^i) \subset L_j^{i-r} + L^{i-r+1}$.

Note that the above definition is independent of the choice of representative β_ν. However, it is possible to choose a 'split' representative for β_ν. Let $\pi_j : V \to V_j$ be the projection, $\iota_j : V_j \to V$ the inclusion, and $\epsilon_j = \iota_j \circ \pi_j$. Set $\beta_{j\nu} = \pi_j \circ \beta_\nu \circ \iota_j$ and $\beta'_\nu = \beta_{1\nu} \oplus \beta_{2\nu}$. Whenever (V_1, V_2) splits β and P, $\beta'_\nu \in \beta_\nu + \mathfrak{P}^{1-r}$. Thus, by replacing β_ν by β'_ν, we may assume without loss of generality that the representative β_ν is "block-diagonal", i.e., it satisfies $\beta'_\nu(L_j^i) \subset L_j^{i-r}$. If β_j is the functional induced by $\beta_{j\nu}$, then (P_j, r, β_j) is a stratum in $\mathrm{GL}(V_j)$.

Remark 2.21. If P is uniform and (V_1, V_2) splits P, then P_2 is also uniform with $e_{P_2} = e_P$, since $\bar{L}^i \cong \bar{L}_1^i \oplus \bar{L}_2^i$. Furthermore, if (P, r, β) is strongly uniform, (V_1, V_2) splits β, and the first part of the splitting condition for P is satisfied, then (V_1, V_2) splits P. Since $\gcd(r, e_P) = 1$ and $\beta_\nu(L^i) = L^{i-r}$ for all i, we may choose integers a and b such that $\alpha_\nu = t^a \beta_\nu^b$ generates the $\mathfrak{P}$-module $\mathfrak{P}^1$. Thus, if we choose β_ν such that $\beta_\nu(V_j) \subset V_j$ as above, it is clear that $\alpha_\nu(L_1^i) = L_1^{i+1}$ and $\bar{\alpha}_\nu(\bar{L}_1^i) = \bar{L}_1^{i+1}$. Accordingly, the subquotients $\bar{L}_1^i$ have the same dimension for all i and there are no repeats in the lattice chain. This implies that $\mathscr{L}_1$ is uniform and $e_{P_1} = e_P$. In fact, we see that (P_1, r, β_1) is strongly uniform.

Define V_{12} to be the vector space $\mathrm{Hom}_F(V_2, V_1)$, and let ∂_{β_ν} be the operator

$$\begin{aligned} \partial_{\beta_\nu} : V_{12} &\to V_{12} \\ x &\mapsto \beta_{1\nu} x - x \beta_{2\nu}. \end{aligned}$$

We remark that if we embed V_{12} in $\mathfrak{gl}(V)$ in the obvious way and assume that β_ν is block-diagonal, then $\partial_{\beta_\nu}(x) = [\beta_\nu, x] = \delta_{\beta_\nu}(x)$.

The map ∂_{β_ν} is a degree $-r$ endomorphism of the $\mathfrak{o}$-lattice chain $\mathscr{M} = (M^j)$ defined by

$$M^j = \{x \in V_{12} \mid xL_2^i \subset L_1^{i+j} \text{ for all } i\}.$$

By [18, Lemma 2.2], $\mathscr{M}$ is a uniform lattice chain with period e_P. The functional on $\bar{\mathfrak{P}}$ induced by ∂_{β_ν} is independent of the choice of representative; we denote the corresponding stratum by $(P_{12}, r, \partial_\beta)$.

Recall from Remark 2.8 that any element of $\mathfrak{P}/\mathfrak{P}^1$ determines a conjugacy class in $\mathfrak{gl}_n(k)$. Accordingly, if $(P, 0, \beta)$ is a stratum with $r = 0$, it makes sense to refer to the 'eigenvalues' of $\bar{\beta}_\nu$. If the stratum splits at level 0, then the eigenvalues of the diagonal blocks $(\bar{\beta}_1)_\nu$ and $(\bar{\beta}_2)_\nu$ are well-defined.

Definition 2.22. We say that (V_1, V_2) splits the fundamental stratum (P, r, β) if

(1) (V_1, V_2) splits P and β at level r;
(2) (P_1, r, β_1) and $(P_{12}, r, \partial_\beta)$ are strongly uniform; and
(3) when $r = 0$, the eigenvalues of $\bar{\beta}_{1\nu}$ are distinct from the eigenvalues of $\bar{\beta}_{2\nu}$ modulo $\mathbb{Z}$.

Remark 2.23. The congruence subgroup P^1 acts on the set of splittings of (P, r, β), i.e., if $g \in P^1$ and (V_1, V_2) splits (P, r, β), then so does (gV_1, gV_2). First, note that P^1 stabilizes this stratum. Next, given $g \in P^1$, it is clear that $L^i = gL^i = gL_1^i \oplus gL_2^i$ and that $g\mathscr{L}_1$ is uniform with the same period as $\mathscr{L}_1$. Thus, (gV_1, gV_2) splits P; it also splits β at level r, since $gx \in x + L^{i+1}$ for any $x \in L^i$. It is obvious that the induced strata on gV_1 and gV_{12} are strongly uniform. Finally, note that when viewed as subalgebras of $\bar{\mathfrak{P}}$ is the natural way, $\bar{\mathfrak{P}}_j$ and $g\mathfrak{P}_j/g\mathfrak{P}_j^1$ are the same. It follows that the eigenvalues of $g\beta_{j\nu} + g\mathfrak{P}_j^1$ and $\bar{\beta}_{j\nu}$ are the same, so the last condition also holds.

If P_1 is a uniform parahoric, then P_{12} is as well, with the same period. To see this, note that there is an isomorphism

$$\bar{M}^j \to \bigoplus_{\ell=0}^{e_P-1} \operatorname{Hom}(\bar{L}_2^\ell, \bar{L}_1^{\ell+j}). \tag{2.5}$$

Since $\dim_k(\bar{L}_1^\ell) = \dim_k(L_1^0/tL_1^0)/e_{P_1}$ for all ℓ,

$$\dim_k(\bar{M}^j) = \dim_k(L_1^0/tL_1^0)\dim_k(L_2^0/tL_2^0)/e_{P_1},$$

and P_{12} is uniform. Furthermore, since $\dim_k(M^j/tM^j) = \dim_k(L_1^0/tL_1^0)\dim_k(L_2^0/tL_2^0)$, it follows that $tM^j = M^{j+e_{P_1}}$, i.e., $e_{P_{12}} = e_{P_1} = e_P$.

Let $V_{21} = \operatorname{Hom}_F(V_1, V_2)$. Define a lattice chain $\mathscr{N} = \{N^i\}$ in V_{21} in the same way as for $\mathscr{M}$. An argument similar to that given above shows that $\mathscr{N}$ is uniform with period e_P and that the operator ∂'_{β_ν} on V_{21} defined by $\partial'_{\beta_\nu}(x) = \beta_{2\nu}x - x\beta_{1\nu}$ is an endomorphism of $\mathscr{N}$ of degree $-r$. We let $(P_{21}, r, \partial'_\beta)$ be the associated stratum.

Lemma 2.24. *The stratum $(P_{21}, r, \partial'_\beta)$ is strongly uniform.*

Proof. It only remains to show that $\partial'_\beta(N^i) = N^{i-r}$ for all i. First, observe that there is a natural injection $\bar{M}^i \hookrightarrow \bar{\mathfrak{P}}^i$, so by Proposition 2.5, we have a surjection $\bar{\mathfrak{P}}^{-i} \cong (\bar{\mathfrak{P}}^i)^\vee \to (\bar{M}^i)^\vee$. Since the kernel of this map consists of the image of the "block upper triangular" matrices, we see that $(\bar{M}^i)^\vee \cong \bar{N}^{-i}$. Next, if $x \in M^i$ and

$y \in N^{-i+r}$,

$$\begin{aligned}\langle \beta_1 x - x\beta_2, y\rangle_\nu &= \operatorname{Res}(\operatorname{Tr}(\beta_1 xy)\nu - \operatorname{Tr}(x\beta_2 y)\nu)\\ &= \operatorname{Res}(\operatorname{Tr}(xy\beta_1)\nu - \operatorname{Tr}(x\beta_2 y)\nu)\\ &= -\langle x, \beta_2 y - y\beta_1\rangle_\nu.\end{aligned}$$

Since $\bar{\partial}_{\beta_\nu} : \bar{M}^i \to \bar{M}^{i-r}$ is an isomorphism by Remark 2.18, it follows that $\bar{\partial}'_\beta(\bar{N}^{-i+r}) = \bar{N}^{-i}$, so $\partial_\beta(N^i) = N^{i-r}$ by the same remark.

□

Suppose that (V_1, V_2) splits a uniform stratum (P, r, β) as above. By Remark 2.21, $e_{P_1} = e_{P_2} = e_P$. Thus, it is never the case that $L^i_j = L^{i+1}_j$ using the indexing convention in Definition 2.20, and indeed $\mathscr{L}_j = (L^i_j)_{i\in\mathbb{Z}}$. In this setting, it makes sense to think of (P, r, β) as the direct sum of (P_1, r, β_1) and (P_2, r, β_2).

In the following, let J be a finite indexing set, and suppose $V_J = \bigoplus_{j\in J} V_j$, with each $V_j \neq \{0\}$. Let (P_j, r, β_j) be a stratum in $\mathrm{GL}(V_j)$ corresponding to a uniform parahoric P_j, and let $e_{P_j} = e_{P_k}$ for all j, k. Define $L^i_J = \bigoplus_{j\in J} L^i_j$ and $\mathscr{L}_J = (L^i_J)_{i\in\mathbb{Z}}$, and let $P_J \subset \mathrm{GL}(V)$ be the parahoric subgroup that stabilizes $\mathscr{L}_J$. Finally, let $\beta_J = \bigoplus_{j\in J} \beta_j$.

Definition 2.25. Under the assumptions of the previous paragraph:

(1) When $J = \{1, 2\}$, we say that $(P_J, r, \beta_J) = (P_1, r, \beta_1) \oplus (P_2, r, \beta_2)$ if (V_1, V_2) splits (P_J, r, β_J).
(2) When $J = \{1, \dots, m\}$, we define the direct sum recursively by

$$\bigoplus_{j\in J}(P_j, r, \beta_j) = (P_1, r, \beta_1) \oplus ((P_2, r, \beta_2) \oplus (\ldots \oplus (P_m, r, \beta_m)))\,.$$

Note that if we set $J_\ell = \{\ell, \dots, m\}$ for $\ell \in J$, then $(V_\ell, V_{J_{\ell+1}})$ must split $(P_{J_\ell}, r, \beta_{J_\ell})$ for all ℓ.
(3) We say that a uniform stratum $(P, r, \beta) \in \mathrm{GL}(V)$ splits into the direct sum $\bigoplus_{j\in J}(P_j, r, \beta_j)$ if there is an isomorphism $V \cong V_J$ under which (P, r, β) and (P_J, r, β_J) are equivalent.

Remark 2.26. It is clear that the direct sum operation is associative. It is not symmetric because Definition 2.22(2) implies that (P_j, r, β_j) is strongly uniform whenever $j \neq m$. The definition is easily modified to make it symmetric, but we will not do so here.

We can determine if a stratum has a splitting by considering the characteristic polynomial of (P, r, β). Fix a parameter $t \in F$ and let $g = \gcd(r, e_P)$. Define an element

$$y_\beta = \beta_\nu^{e_P/g} t^{r/g} + \mathfrak{P}^1 \in \bar{\mathfrak{P}}. \tag{2.6}$$

Recall from Remark 2.8 that y_β determines a conjugacy class in $\mathfrak{gl}_n(k)$.

Definition 2.27. We define the characteristic polynomial $\phi_\beta \in k[X]$ of the stratum (P, r, β) to be the characteristic polynomial of y_β.

The local field version of the following proposition is in [18, Proposition 3.4].

Proposition 2.28. *Suppose that* $\gcd(r, e_P) = 1$ *and* $r > 0$*. The stratum* (P, r, β) *is fundamental if and only if* $\phi_\beta(X)$ *has a non-zero root. If* (P, r, β) *is fundamental, it splits if* $\phi_\beta(X) = g(X)h(X)$ *for* $g, h \in k[X]$ *relatively prime of positive degree.*

Remark 2.29. Given any fundamental stratum (P, r, β), one can always find a reduction that satisfies the condition $\gcd(r, e_P) = 1$ by Lemma 2.19.

Proof. Note that $\bar{\beta}_\nu$ and $y_\beta = t^r \bar{\beta}_\nu^{e_P}$, viewed as endomorphisms of $\mathrm{gr}(\mathscr{L})$, are either simultaneously nilpotent or not. Using the identification of $\bar{\mathfrak{P}}$ and a Levi subalgebra of $\mathfrak{gl}_n(k)$, we see that the latter is nilpotent if and only its characteristic polynomial $\phi_\beta(X)$ equals X^n. Since (P, r, β) is not fundamental if and only if $\bar{\beta}_\nu$ is nilpotent, we see that (P, r, β) is fundamental if and only if $\phi_\beta(X)$ has a nonzero root.

Let $\tilde{\phi}_\beta \in F[X]$ be the characteristic polynomial of $\tilde{y} = t^r \beta_\nu^{e_P}$. Then, $\tilde{\phi}_\beta$ necessarily has coefficients in $\mathfrak{o}$ and $\tilde{\phi}_\beta \equiv \phi_\beta \pmod{\mathfrak{p}}$. Hensel's lemma states that $\tilde{\phi}(X) = \tilde{g}(X)\tilde{h}(X)$, where $\tilde{h} \equiv h \pmod{\mathfrak{p}}$ and $\tilde{g} \equiv g \pmod{\mathfrak{p}}$.

We take $V_1 = \ker(g(\tilde{y}))$ and $V_2 = \ker(h(\tilde{y}))$. By Lemmas 3.5 and 3.6 of [18], (V_1, V_2) splits P and β at level r, $\beta_1(L_1^i) = L_1^{i-r}$ for all i, and $\partial_\beta(M^j) = M^{j-r}$ for all j. Therefore, (V_1, V_2) splits (P, r, β). □

Corollary 2.30. *Suppose that (P, r, β) is a uniform stratum that is not strongly uniform. Then, (P, r, β) splits into the direct sum of two strata (P_1, r, β_1) and (P_2, r, β_2), where (P_1, r, β_1) is strongly uniform and (P_2, r, β_2) is non-fundamental.*

Proof. Factor $\phi_\beta(X) = g(X)h(X)$ so that $h(X) = X^m$ and $g(0) \neq 0$, and let V_1 and V_2 be the subspaces from the proof of the previous proposition. Since (P, r, β) is fundamental, $\deg(g) > 0$ and V_1 is nontrivial. Moreover, Remark 2.18 implies that $V_1 \neq V$; if not, $y_\beta = t^r \bar{\beta}_\nu^{e_P}$ (and hence $\bar{\beta}_\nu$) would be invertible endomorphisms of $\mathrm{gr}(\mathscr{L})$, contradicting the fact that (P, r, β) is not strongly uniform.

Since y_β (and hence $\bar{\beta}_\nu$) restricts to an automorphism of $\mathrm{gr}(\mathscr{L}_1)$, the same remark shows that (P_1, r, β_1) is strongly uniform. On the other hand, since y_β restricts to a nilpotent endomorphism of $\mathrm{gr}(\mathscr{L}_2)$, Remark 2.14 shows that (P_2, r, β_2) is not fundamental. □

3. Regular Strata

In this section, we make precise the notion of a stratum with regular semisimple "leading term". We introduce the concept of a regular stratum; this is a stratum which is "graded-centralized" by a maximal torus. Regular strata do not appear in the theory of strata for local fields. However, they play an important role in the geometric theory.

3.1. **Classification of regular strata.** Consider a stratum (P, r, β). Recall that the congruence subgroups P^i act on β_ν by the adjoint action. In particular,

$$\mathrm{Ad}(P^i)(\beta_\nu) \subset \beta_\nu + \mathfrak{P}^{i-r},$$

since $p \in P^i$ implies that p and p^{-1} act trivially on L^j/L^{i+j}. Define $Z^i(\beta_\nu) \subset \bar{P}^i$ to be the stabilizer of $\beta_\nu \pmod{\mathfrak{P}^{i-r+1}}$. Notice that this is independent of the choice of representative β_ν for β: if $\beta'_\nu = \beta_\nu + \gamma$, for some $\gamma \in \mathfrak{P}^{-r+1}$, then $\mathrm{Ad}(P^i)(\gamma) \subset \gamma + \mathfrak{P}^{i-r+1}$.

Let $T \subset \mathrm{GL}(V)$ be a maximal torus. The corresponding Cartan subalgebra $\mathfrak{t} \subset \mathfrak{gl}(V)$ is the centralizer of a regular semisimple element and is therefore an associative subalgebra. In particular, since $\mathfrak{t}$ must have the structure of a commutative semisimple algebra, $\mathfrak{t}$ is the product of field extensions of F: $\mathfrak{t} = E_1 \times E_2 \times \ldots \times E_\ell$ and $T = E_1^\times \times E_2^\times \times \ldots \times E_\ell^\times$. We let $\mathfrak{o}_j$ be the ring of integers of E_j and $\mathfrak{p}_j$ its maximal ideal. Let $s_j = [E_j : F]$. The field E_j contains a uniformizer which is an

s_j^{th} root of t; we let $\omega_j \in \mathfrak{t}$ denote this uniformizer supported on the j^{th} summand. We will also denote the identity of the j^{th} Wedderburn component of $\mathfrak{t}$ by χ_j.

There is a map $N_T : T \to (F^\times)^\ell$ obtained by taking the norm on each summand. Define $T(\mathfrak{o}) = N_T^{-1}(\mathfrak{o}^\times)^\ell = \prod_j \mathfrak{o}_j^\times$. Similarly, there is a trace map $\mathrm{Tr}_\mathfrak{t} : \mathfrak{t} \to F^\ell$, and we set $\mathfrak{t}(\mathfrak{o}) = \mathrm{Tr}_\mathfrak{t}^{-1}(\mathfrak{o}) = \prod_j \mathfrak{o}_j$. We also define a finite-dimensional k-toral subalgebra and k-torus: $\mathfrak{t}^\flat \subset \mathfrak{t}(\mathfrak{o})$ is the k-linear span of the χ_j's and $T^\flat \stackrel{\mathrm{def}}{=} (\mathfrak{t}^\flat)^\times \subset T(\mathfrak{o})$. Of course, $T^\flat \cong (k^\times)^\ell$ and $\mathfrak{t}^\flat \cong k^\ell$.

We will be concerned with tori which are compatible with a given parahoric subgroup in the sense that $T(\mathfrak{o}) \subset P$ or equivalently $\mathfrak{t}(\mathfrak{o}) \subset \mathfrak{P}$.

Lemma 3.1. *If $T(\mathfrak{o}) \subset P$, then $T \cap P^i = T(\mathfrak{o}) \cap P^i$ and $\mathfrak{t} \cap \mathfrak{P}^i = \mathfrak{t}(\mathfrak{o}) \cap \mathfrak{P}^i$ for all $i \geq 0$.*

Proof. It suffices to show that $T \cap P = T(\mathfrak{o})$ and $\mathfrak{t} \cap \mathfrak{P} = \mathfrak{t}(\mathfrak{o})$; moreover, the first statement follows from the second by taking units. Since the central primitive idempotents χ_j are contained in $\mathfrak{t} \cap \mathfrak{P}$, it is enough to check that if $x\chi_j \in E_j \cap \mathfrak{P}$, then $x \in \mathfrak{o}_j$. Suppose $x \notin \mathfrak{o}_j$, so that $x\chi_j = \omega_j^q f$ for some $f \in \mathfrak{o}_j^\times$ and $q < 0$. Since $\mathfrak{o}_j^\times \chi_j \in P \subset \mathfrak{P}$, we see that $\omega_j^q \in \mathfrak{P}$. This implies that $t^q \chi_j = \omega_j^{qm} \in \mathfrak{P}$. We deduce that $t^s \chi_j \in \mathfrak{P}$ for all $s \in \mathbb{Z}$, which is absurd. □

Definition 3.2. A uniform stratum (P, r, β) is called *regular* if there exists a maximal torus T (possibly non-split) with the following properties:

- $T(\mathfrak{o}) \subset P$;
- $\bar{T}^i = Z^i(\beta_\nu)$ for all i;
- $y_\beta \in \bar{P}$ (defined as in (2.6)) is semisimple;
- in the case $r = 0$ (and thus $e_p = 1$), the eigenvalues of $\bar{\beta}_\nu \in \mathfrak{gl}(\bar{L}^0)$ are distinct modulo $\mathbb{Z}$.

We say that T centralizes (P, r, β). If $T \cong E^\times$ for some field extension E/F, the stratum is called *pure*.

Remark 3.3. Suppose that (P, r, β) is a regular stratum centralized by T, and L is a lattice with $P \subset \mathrm{GL}(L)$. Then, for any $g \in \mathrm{GL}(L)$, $(gPg^{-1}, r, \mathrm{Ad}^*(g)\beta)$ is a regular stratum centralized by gTg^{-1}.

Remark 3.4. If T centralizes a regular stratum (P, r, β), then any conjugate of T by an element of P^1 also centralizes (P, r, β). Thus, T is not unique.

It will be useful to have a variation of Definition 3.2 in terms of the graded action of $\mathfrak{t}$ on β. Define $\mathfrak{z}^i(\beta_\nu) \subset \bar{\mathfrak{P}}^i$ to be the image of $\{z \in \mathfrak{P}^i \mid \mathrm{ad}(z)(\beta_\nu) \in \mathfrak{P}^{-r+i+1}\}$.

Proposition 3.5. *Let P be a uniform parahoric, and let (P, r, β) be a regular stratum centralized by the torus T. Then $\bar{T}^i = Z^i(\beta_\nu)$ if and only if $\bar{\mathfrak{t}}^i = \mathfrak{z}^i(\beta_\nu)$ for each $i \geq 0$.*

Proof. First, we take $i = 0$. Suppose $\bar{\mathfrak{t}}^0 = \mathfrak{z}^0(\beta_\nu)$. Given $z \in P$, it is clear that $z\beta_\nu z^{-1} - \beta_\nu \in \mathfrak{P}^{-r+1}$ if and only if $\mathrm{ad}(z)\beta_\nu \in \mathfrak{P}^{-r+1}$. This immediately gives $\bar{T}^0 \subset Z^0(\beta_\nu)$. It also implies that if $zP^1 \in Z^0(\beta_\nu)$, then $z + \mathfrak{P}^1 \in \mathfrak{z}^0(\beta_\nu)$. By assumption, this means that there exists $s \in \mathfrak{t}(\mathfrak{o})$ such that $z - s \in \mathfrak{P}^1$, and we obtain $s \in P \cap \mathfrak{t}(\mathfrak{o}) = T(\mathfrak{o})$. Consequently, $s^{-1}z \in P^1$, i.e., $zP^1 \in \bar{T}^0$.

Next, suppose that $Z^0(\beta_\nu) = \bar{T}^0$. Recall that a finite-dimensional k-algebra is spanned by its units. (Let A be such an algebra with Jacobson radical J. Since

$1+x \in A^\times$ for $x \in J$, J is in the span of $A^\times$. Moreover, $a \in A^\times$ if and only if $\bar{a} \in (A/J)^\times$. The result now follows because A/J is a product of matrix algebras, and hence is spanned by its units.) We show that $(\mathfrak{z}^0(\beta_\nu))^\times = (\bar{\mathfrak{t}}^0)^\times$. Suppose $y \in (\mathfrak{z}^0(\beta_\nu))^\times \subset \bar{\mathfrak{P}}^\times$. Since $\mathfrak{P}^1$ is the Jacobson radical of $\mathfrak{P}$, any $z \in \mathfrak{P}$ lifting y is invertible, hence lies in P. The argument above shows that $zP^1 \in Z^0(\beta_\nu)$, so we can assume $z \in T$, i.e., $y \in (\bar{\mathfrak{t}}^0)^\times$. A similar argument (using the fact that a lift of $y \in (\bar{\mathfrak{t}}^0)^\times$ to $\mathfrak{t}(\mathfrak{o})$ actually lies in $T(\mathfrak{o})$) gives the reverse inclusion. We conclude that $\mathfrak{z}^0(\beta_\nu) = \operatorname{span}\left((\mathfrak{z}^0(\beta_\nu))^\times\right) = \operatorname{span}\left((\bar{\mathfrak{t}}^0)^\times\right) = \bar{\mathfrak{t}}^0$.

Now suppose $i > 0$. There is an isomorphism $\bar{\mathfrak{P}}^i \to \bar{P}^i$ induced by $X \mapsto 1+X$. Since $\operatorname{Ad}(1+X)(\beta_\nu) \in \beta_\nu + \operatorname{ad}(X)(\beta_\nu) + \mathfrak{P}^{-r+i+1}$ for $X \in \mathfrak{P}^i$, it is clear that this map restricts to give an isomorphism between $\mathfrak{z}^i(\beta)$ and $Z^i(\beta)$. Since this same map takes $\bar{\mathfrak{t}}^i$ to $\bar{T}^i$, the proof is complete.

□

Remark 3.6. If T centralizes (P, r, β), then in fact $\bar{\mathfrak{t}}^i = \mathfrak{z}^i(\beta_\nu)$ for all $i \in \mathbb{Z}$. For $i \geq 0$, this has been shown in the proposition. On the other hand, if $i < 0$ and s is any integer such that $i + se_P \geq 0$, then the result follows because multiplication by t^s induces isomorphisms $\bar{\mathfrak{t}}^i \cong \bar{\mathfrak{t}}^{i+se_P}(\beta_\nu)$ and $\mathfrak{z}^i \cong \mathfrak{z}^{i+se_P}(\beta_\nu)$. Since $\bar{\beta}_\nu \in \mathfrak{z}^{-r}(\beta_\nu)$, $\beta_\nu \in \mathfrak{t} \cap \mathfrak{P}^{-r} + \mathfrak{P}^{-r+1}$; it follows that we can always choose the representative $\beta_\nu \in \mathfrak{t} \cap \mathfrak{P}^{-r}$.

Corollary 3.7. *Let (P, r, β) be a regular stratum centralized by T and let $X \in \mathfrak{P}^\ell$. If $\beta_\nu \in \mathfrak{t}^{-r}$ is a representative for β, and $\operatorname{ad}(X)(\beta_\nu) \in \mathfrak{P}^{-r+j}$, then $X \in \mathfrak{t}^\ell + \mathfrak{P}^j$.*

Proof. When $\ell \geq j$, there is no content. We note that the case $j = \ell + 1$ follows from Proposition 3.5 and Remark 3.6. By induction on $j > \ell$, suppose that the statement is true for $j-1$. Let $Y \in \mathfrak{P}^{j-1}$ satisfy $X - Y \in \mathfrak{t}$. Then, $X \in \mathfrak{t}^\ell + \mathfrak{P}^j$ if and only if $Y \in \mathfrak{t}^{j-1} + \mathfrak{P}^j$. The latter statement follows from the base step above. □

The main goal of this section is to give a structure theorem for regular strata.

Theorem 3.8. *Let (P, r, β) be a regular stratum.*

(1) *If (P, r, β) is pure, then $e_P = \dim V$.*
(2) (a) *If (P, r, β) is strongly uniform, then it splits into a direct sum of pure strata (necessarily of the same dimension).*
  (b) *If (P, r, β) is not strongly uniform, then $e_P = 1$ and (P, r, β) is the direct sum of a regular, strongly uniform stratum and a non-fundamental stratum of dimension* 1*.*

In each case, the splitting coincides with the splitting induced by a P^1-conjugate of T.

Corollary 3.9. *Let (P, r, β) be a regular stratum centralized by T. Take E/F to be the unique (up to isomorphism) field extension of degree e_P. Then, $T \cong (E^\times)^{n/e_P}$. Moreover, the maps $T^\flat \to \bar{T}^0$ and $\mathfrak{t}^\flat \to \bar{\mathfrak{t}}^0$ are isomorphisms.*

Remark 3.10. We note that if (P, r, β) is not strongly uniform, Theorem 3.8 implies that $e_P = 1$. By the corollary, this can only happen when T is totally split.

The following proposition allows us to make sense of what it means for an element β_ν to have regular semisimple leading term.

Proposition 3.11. *If (P, r, β) is regular, then every representative β_ν for β is regular semisimple.*

By Remark 3.6, we may choose $\beta_\nu \in \mathfrak{t} \cap \mathfrak{P}^{-r}$. Corollary 3.9 implies that β_ν is a block diagonal matrix with entries in $F[\varpi_E]^\times$. Then, the *leading term* β'_ν is the matrix consisting of the degree $-r/e_P$ terms from each block diagonal entry in β_ν (after identifying ϖ_E with t^{1/e_P}).

It suffices to check that $\beta_\nu - \beta'_\nu \in \mathfrak{P}^{-r+1}$, in which case Proposition 3.11 implies that β'_ν is regular semisimple. If $e_P = 1$, this is clear. When $e_P > 1$, we may assume without loss of generality that the splittings for T and (P, r, β) are induced by the same splitting of V. In particular, the j^{th} block-diagonal entry $\beta_{j\nu}$ is a representative for the j^{th} summand of (P, r, β). Therefore, by Theorem 3.8, we may reduce to the case where (P, r, β) is pure and $e_P = \dim V$. In this case, ϖ_E generates $\mathfrak{P}^1$, so it is clear that $\beta_\nu - \beta'_\nu \in \mathfrak{P}^{-r+1}$.

We call a maximal torus *uniform* if it isomorphic to $(E^\times)^\ell$ for some field extension E. Given a fixed lattice L and a uniform maximal torus T with $T(\mathfrak{o}) \subset \mathrm{GL}(L)$, we can associate a corresponding parahoric subgroup $P_{T,L} \subset \mathrm{GL}(L)$ containing $T(\mathfrak{o})$ as follows. The isomorphism $\mathfrak{t} \cong E^\ell$ induces splittings $V = \oplus V_j$ and $L = \oplus L_j$. Lemma 2.4 states that there is a unique complete lattice chain $(L^i_j)_{i\in\mathbb{Z}}$ in V_j up to indexing; we normalize it so that $L^0_j = L_j$. Let $\mathscr{L}_T$ be the lattice chain with $L^i = \oplus L^i_j$, and let $P_{T,L}$ be its stabilizer. Since $L_0 = L$, we have $P_{T,L} \subset \mathrm{GL}(L)$ as desired. It is obvious that $T(\mathfrak{o}) \subset P_{T,L}$. Note that $e_{P_{T,L}} = n/\ell = [E : F]$.

Given a uniform torus T, there is a canonical $\mathbb{Z}$-grading on its Cartan subalgebra $\mathfrak{t}$; the i^{th} graded piece is given by $\bigoplus k\varpi_E^i\chi_i$, where ϖ_E is a uniformizer in E which is an $[E : F]^{th}$ root of t. We denote the corresponding filtration by $\mathfrak{t}^{(i)} = \bigoplus \mathfrak{o}_E\varpi_E^i\chi_i$. There is a corresponding canonical $\mathbb{N}$-filtration on $T(\mathfrak{o})$ given by $T^{(0)} = T(\mathfrak{o})$ and $T^{(i)} = 1 + \mathfrak{t}^{(i)}$.

Proposition 3.12. *Let L be a fixed lattice, and let $T \cong (E^\times)^\ell$ be a uniform maximal torus with $T(\mathfrak{o}) \subset \mathrm{GL}(L)$.*

1. *If $P \subset \mathrm{GL}(L)$ is a parahoric subgroup for which there exists a regular stratum (P, r, β) centralized by T, then $P = P_{T,L}$.*
2. *Let $r \geq 0$ satisfy $(r, n/\ell) = 1$, and suppose $x \in \mathfrak{t}^{-r}$ has regular semisimple leading term. Then, there is a unique regular stratum (P, r, β) with $P \subset \mathrm{GL}(L)$ which has x as a representative.*
3. *The canonical filtrations on $\mathfrak{t}$ and $T(\mathfrak{o})$ coincide with the filtrations induced by $P_{T,L}$, i.e., $\mathfrak{t}^{(i)} = \mathfrak{t} \cap \mathfrak{P}^i_{T,L}$ and $T^{(i)} = T \cap P^i_{T,L}$.*

Proof. Let (P, r, β) be a regular stratum as in the first statement. Since $P \subset \mathrm{GL}(L)$, we may take $L^0 = L$ in the corresponding lattice chain $\mathscr{L}$. By Corollary 3.9, $\mathfrak{t} \cong E^{n/e_P}$, so $e_P = n/\ell$. Theorem 3.8 implies that the splitting $V = \oplus V_j$ induced by $\mathfrak{t} \cong E^{n/e_P}$ splits (P, r, β) into a direct sum of pure strata when $e_P > 1$ and a sum of one-dimensional strata (with at most one non-fundamental summand) when $e_P = 1$. In either case, each V_j is an E-vector space of dimension one, and $L^i = \oplus(L^i \cap V_j)$. However, Lemma 2.4 states that there is a unique complete lattice chain $(L^i_j)_{i\in\mathbb{Z}}$ in V_j up to indexing, and we know that $L^0_j = L \cap V_j$. By definition, $\mathscr{L} = \mathscr{L}_T$, so $P = P_{T,L}$.

The uniqueness part of the second statement is now immediate. For existence, it is clear from the construction of $P_{T,L}$ that $(a_1\varpi_E^{-r}, \ldots, a_\ell\varpi_E^{-r}) \in E^\ell \cong \mathfrak{t}$ determines a regular stratum with parahoric subgroup $P_{T,L}$ if $a_i \neq a_j$ whenever $i \neq j$ and r is coprime to n/ℓ. Since the leading term of x is regular semisimple, we obtain a regular stratum $(P_{T,L}, r, \beta)$ with the leading term of x, and hence x itself, as a representative.

Finally, $\mathfrak{t}|_{V_j} \cong E$. It follows that $L^i_j = \varpi^j_E L^0_j$. We deduce that $\mathfrak{t} \cap \mathfrak{P}^i_{T,L} = \bigoplus(\mathfrak{t}|_{V_j} \cap \mathfrak{P}^i_{T_{V_j}, L^0_j}) = \bigoplus \mathfrak{o}_E \varpi^i_E \chi_j = \mathfrak{t}^{(i)}$. The fact that $T^{(i)} = T \cap P^i_{T,L}$ is an immediate consequence when $i \geq 1$. The $i = 0$ case is obtained by taking units in $\mathfrak{t}^{(0)} = \mathfrak{t} \cap \mathfrak{P}_{T,L}$.

□

If (P, r, β) is a regular stratum centralized by T, then the proposition shows that $\{\mathfrak{t}^i\}$ and $\{T^i\}$ are actually the canonical filtrations.

Remark 3.13. Given a fixed F-isomorphism $V \xrightarrow{\sim} F^n$, we can choose a standard representative of each conjugacy class of uniform maximal tori. Indeed, if the torus is isomorphic to $(F((t^{1/e}))^\times)^{n/e}$, then under the identification $\mathrm{GL}(V) \cong \mathrm{GL}_n(F)$, we can choose a block diagonal representative T (and $\mathfrak{t}$) with each uniformizer $t^{1/e}$ mapping to the $e \times e$ matrix ϖ_I from (2.2) in the corresponding block. In this case, P_T is the standard uniform parahoric subgroup that is 'block upper-triangular modulo t'.

3.2. **Lemmas and proofs.** We now give proofs of the results described above. We also include some lemmas that will be needed later. We remark that this section is largely technical in nature.

Lemma 3.14. *The homomorphism $T^\flat \to \bar{T}^0$ is an injection, and if U is the unipotent radical of $\bar{T}^0$, then the induced map $T^\flat \to \bar{T}^0/U$ is an isomorphism. Similarly, the map $\mathfrak{t}^\flat \to \bar{\mathfrak{t}}^0$ is an injection which induces an isomorphism $\mathfrak{t}^\flat \cong \bar{\mathfrak{t}}^0/\mathfrak{n}$, where $\mathfrak{n} \subset \bar{\mathfrak{t}}^0$ is the Jacobson radical.*

Proof. It is immediate from the definitions that $T^\flat \subset P$ and $T^\flat \cap P^1 = \{1\}$, so $T^\flat \to \bar{T}^0$ is injective. Moreover, the unipotent radical U of $\bar{T}^0$ is the image of $\prod_j (1 + \mathfrak{p}_j)$, whence the isomorphism $T^\flat \to \bar{T}^0/U$. A similar proof works for $\mathfrak{t}^\flat$.

□

Lemma 3.15. *Suppose that P is a uniform parahoric and (P, r, β) is a non-fundamental stratum in an F-vector space V. If (P, r, β) satisfies the first three conditions of Definition 3.2 and $\gcd(r, e_P) = 1$, then V must have dimension one.*

Proof. Since (P, r, β) is non-fundamental, it follows that there is a minimal $m > 0$ such that $\beta^m_\nu \in \mathfrak{P}^{-rm+1}$. Without loss of generality, we may assume $\beta^m_\nu = 0$. Indeed, after choosing a basis, Lemma 2.6 shows that we may take the representative β_ν to be the product of ϖ_P^{-r} with an element $D \in \mathfrak{h}$. By Remark 2.12, we may assume $\mathrm{Ad}(\varpi_P)(\mathfrak{h}) \subset \mathfrak{h}$. Therefore, $(D\varpi_P^{-r})^m = D'\varpi_P^{-rm}$ for some $D' \in \mathfrak{h}$. Since $D'\varpi_P^{-rm} \in \mathfrak{P}^{-rm+1}$, $D' \in \mathfrak{P}^1 \cap \mathfrak{h} = \{0\}$.

First, we claim that β_ν is regular nilpotent. Let $\mathfrak{z}$ be the centralizer of β_ν in $\mathfrak{P}$. Note that $\mathfrak{z}$ is a free $\mathfrak{o}$-module of rank equal to the dimension of the centralizer of β_ν in $\mathfrak{gl}(V)$, hence is at least n. Since Nakayama's lemma implies that $\mathrm{rank}(\mathfrak{z}) = \dim_k(\mathfrak{z}/t\mathfrak{z})$, to show that β_ν is regular, it now suffices to show that $\dim_k(\mathfrak{z}/t\mathfrak{z}) \leq n$.

By Lemma 3.1, $\mathfrak{t}^0/\mathfrak{t}^{e_P} = \mathfrak{t}(\mathfrak{o})/t\mathfrak{t}(\mathfrak{o})$, which clearly has k-dimension n. Also, recalling our convention that $\bar{\mathfrak{z}}^i$ is the projection of $\mathfrak{z}$ in $\bar{\mathfrak{P}}^i$, we have $\bar{\mathfrak{z}}^i \subset \mathfrak{z}^i(\beta)$. It follows that

$$\dim_k(\mathfrak{z}/t\mathfrak{z}) \leq \sum_{i=0}^{e_P-1} \dim_k(\mathfrak{z}^i(\beta)) = \sum_{i=0}^{e_P-1} \dim_k(\bar{\mathfrak{t}}^i) = \dim_k(\mathfrak{t}(\mathfrak{o})/t\mathfrak{t}(\mathfrak{o})) = n$$

as desired. Note that this argument actually shows that any coset representative β_ν is regular.

Since the index of nilpotency of a regular nilpotent matrix is n, we have $m = n$. In particular, since $y_\beta = t^r \beta_\nu^{e_P} + \mathfrak{P}^1 \in \bar{\mathfrak{P}}$ is nilpotent, y_β is semisimple only if $t^r \beta_\nu^{e_P} \in \mathfrak{P}^1$. This implies that $\beta_\nu^{e_P} \in \mathfrak{P}^{-re_P+1}$, so $n \geq e_P \geq m$, i.e., $e_P = n$. Thus, $P = I$ is an Iwahori subgroup, and $\gcd(n, r) = 1$.

In the notation of Section 2.3, we write $\beta_\nu = x\varpi_I^{-r}$ where $x = \mathrm{diag}(x_0, \dots, x_{n-1}) \in \mathfrak{d}$. Define $\sigma^q(x) = (x_{-qr}, x_{1-qr}, \dots, x_{n-1-qr})$ to be the cyclic shift of the coefficients of x by $-qr$ places (with indexing in $\mathbb{Z}_n$). It is immediate from (2.3) that $\mathrm{Ad}(\varpi_I^q)(x) = \sigma^q(x)$. Therefore,

$$\beta_\nu^s = x\,\mathrm{Ad}(\varpi_I^{-r})(x) \dots \mathrm{Ad}(\varpi_I^{-(s-1)r})(x)\varpi_I^{-rs} = x\sigma^1(x) \dots \sigma^{s-1}(x).$$

By assumption, $\beta_\nu^{n-1} \neq 0$. Thus, one of the components $x'_j = x_j x_{j-r} \dots x_{j-(n-2)r}$ of $x' = x\sigma^1(x) \dots \sigma^{n-2}(x)$ is non-zero; since $\gcd(r, n) = 1$, x'_j is the product of all but one of the components of x. Moreover, $\beta_\nu^n = 0$, so $x'_j x_{j-(n-1)r} = 0$ and we conclude that exactly one component of x is equal to 0.

Without loss of generality, assume that $x_0 = 0$. Then, if $\bar{p} \in Z^0(\beta)$, we may choose a representative $p = \mathrm{diag}(p_0, \dots, p_{n-1}) \in \mathfrak{d}^*$. Equation (2.4) shows that $p_i = p_{i-r}$ for all i; again, since $\gcd(r, n) = 1$, this implies that $p_0 = p_1 = \cdots = p_{n-1}$. It follows that $Z^0(\beta)$ has dimension 1. Lemma 3.14 now implies that $T^\flat$ also has dimension 1, so $T = E^\times$, where E/F is a field extension of degree n.

By Lemma 2.4, $\mathfrak{L}$ is a saturated chain of $\mathfrak{o}_E$-lattices, so may assume that ϖ_I is a uniformizer in E. Applying (2.3), we see that $\mathrm{ad}(x)(\varpi_I) = x'\varpi_I$ where $x' = \mathrm{diag}(x_0 - x_1, \dots)$. However, $x_0 = 0$ and $x_1 \neq 0$ whenever $n > 1$, so $E^\times$ only centralizes (P, r, β) when $n = 1$. □

Lemma 3.16. *Let (P, r, β) be a regular stratum centralized by a torus T. If (V_1, V_2) is a splitting of (P, r, β), then (P_1, r, β_1) is regular, and (P_2, r, β_2) is either regular or non-fundamental. In the latter case, V_2 has dimension 1 and $e_P = 1$. Moreover, there exists $p \in P^1$ such that (V_1, V_2) splits the torus pTp^{-1} (which also centralizes (P, r, β)) into $T_1 \times T_2$; here, T_1 centralizes (P_1, r, β_1) and T_2 centralizes (P_2, r, β_2) when this stratum is regular.*

Proof. Let $Z_j^i(\beta_\nu) = Z^i(\beta_{j\nu})$, the centralizer of $\beta_{j\nu}$ in $\bar{P}_j^i$; similarly, let $\mathfrak{z}_j^i(\beta_\nu) = \mathfrak{z}^i(\beta_{j\nu})$. First, we claim that $Z^i(\beta_\nu) = Z_1^i(\beta_\nu) \times Z_2^i(\beta_\nu)$, which is embedded in $\bar{P}^i$ by diagonal blocks. It is clear that $Z_1^i(\beta_\nu) \times Z_2^i(\beta_\nu) \subset Z^i(\beta_\nu)$. Recall that ∂_{β_ν} (resp. ∂'_{β_ν}) has trivial kernel in $\bar{M}^i$ (resp. $\bar{N}^i$) by Definition 2.22 (resp. by Lemma 2.24). If we identify $\bar{M}^i$ and $\bar{N}^i$ with the upper and lower off-diagonal components of $\bar{\mathfrak{P}}^i$, then δ_{β_ν} preserves each of these subspaces and restricts to ∂_{β_ν} (resp. ∂'_{β_ν}) on $\bar{M}^i$ (resp. $\bar{N}^i$). Since δ_{β_ν} also preserves the diagonal blocks, its kernel lies in $\bar{\mathfrak{P}}_1^i \times \bar{\mathfrak{P}}_2^i$.

We handle the cases $i > 0$ and $i = 0$ separately. When $i = 0$, $Z^0(\beta_\nu) \subset \bar{P} \subset \bar{\mathfrak{P}}$. It is clear that $Z^0(\beta_\nu)$ lies in the kernel of δ_{β_ν}. Therefore, $Z^0(\beta_\nu)$ is supported on the diagonal blocks, so $Z^0(\beta_\nu) = Z_1^0(\beta_\nu) \times Z_2^0(\beta_\nu)$. When $i > 0$, there is an isomorphism $\bar{\mathfrak{P}}^i \to \bar{P}^i$ induced by $x \mapsto 1+x$. Moreover, $\mathrm{Ad}(1+x)(\beta_\nu) \in \beta_\nu + \mathrm{ad}(x)(\beta_\nu) + \mathfrak{P}^{i-r+1}$. Thus, $Z^i(\beta_\nu))$ must lie in $1 + \ker(\delta_{\beta_\nu})$. The same argument as above implies that

$Z^i(\beta_\nu)$ is supported on the diagonal blocks and equals $Z^i_1(\beta_\nu)\times Z^i_2(\beta_\nu)$. Similarly, one shows that $\mathfrak{z}^i(\beta_\nu)=\mathfrak{z}^i_1(\beta_\nu)\times\mathfrak{z}^i_2(\beta_\nu)$.

Let $\mathfrak{u}$ be the Jacobson radical of $\mathfrak{z}^0(\beta_\nu)$. If $\epsilon_j\in\mathfrak{P}$ is the idempotent corresponding to the identity in $\mathfrak{P}_j$, its image $\bar{\epsilon}_j$ in $\bar{\mathfrak{P}}$ lies in $\mathfrak{z}^0_j(\beta_\nu)$. Therefore, $\mathfrak{u}=\bar{\epsilon}_1\mathfrak{u}+\bar{\epsilon}_2\mathfrak{u}$, and the splitting $\mathfrak{z}^0(\beta_\nu)=\mathfrak{z}^0_1(\beta_\nu)\times\mathfrak{z}^0_2(\beta_\nu)$ induces a splitting on $\mathfrak{z}^0(\beta_\nu)/\mathfrak{u}$. Moreover, this splitting is non-trivial, since ϵ_j has non-trivial image in $\mathfrak{z}^0(\beta_\nu)/\mathfrak{u}$. Lemma 3.14 implies that $\mathfrak{t}^\flat\cong\mathfrak{z}^0(\beta_\nu)/\mathfrak{u}$, so $\mathfrak{t}^\flat$ is split. Let ϵ'_1 and ϵ'_2 be the idempotents corresponding to the identity in each summand of $\mathfrak{t}^\flat$. The same lemma implies that $\bar{\epsilon}'_j\in\bar{\mathfrak{P}}$ is simply the identity matrix in the corresponding diagonal block; thus, $\epsilon'_j\in\epsilon_j+\mathfrak{P}^1$. These idempotents determine a splitting of $\mathfrak{t}$, and thus of T. Write $T=T'_1\times T'_2$. We claim that $Z^i_j(\beta_\nu)=(\bar{T}'_j)^i$. Since $\bar{T}^i=Z^i(\beta_\nu)$, it suffices to show that $(\bar{T}'_j)^i$ maps to $Z^i_j(\beta_\nu)$. When $i=0$, this is clear. In the case $i>0$, $\epsilon'_j\mathfrak{t}\cap\mathfrak{P}^i\subset\epsilon_j\mathfrak{P}^i\epsilon_j+\mathfrak{P}^{i+1}$. Since $(T'_j\times 1)\cap P^i=1+\epsilon'_j\mathfrak{t}\cap\mathfrak{P}^i$ and $\epsilon_j\mathfrak{P}^i\epsilon_j$ is the image of $\mathfrak{P}^i_j$ embedded as a diagonal block, we see that $(\bar{T}'_j)^i$ maps to $Z^i_j(\beta_\nu)$ as desired.

Let (V'_1,V'_2) be the splitting of V determined by $V'_j=\epsilon'_jV$. Let $p=\epsilon_1\epsilon'_1+\epsilon_2\epsilon'_2$, so $p(V'_j)\subset V_j$. The map p induces the identity map on $\bar{L}^i$, and since the kernel of p lies in $\cap_{i\in\mathbb{Z}}L^i=\{0\}$, we deduce that $p\in P^1$. It is clear that pTp^{-1} centralizes (P,r,β) (indeed, this is true for any $p\in P^1$ by remark 3.4), and that (V_1,V_2) splits pTp^{-1} into a product $T_1\times T_2$. Moreover, y_β is semisimple if and only if y_{β_1} and y_{β_2} are, since $y_\beta=t^r(\beta_{1\nu}+\beta_{2\nu})^{e_P}=t^r\beta_{1\nu}^{e_P}+t^r\beta_{2\nu}^{e_P}=y_{\beta_1}+y_{\beta_2}$. The fact that T_1 centralizes (P_1,r,β_1) follows from the previous paragraph, so (P_1,r,β_1) is regular. The first part of Remark 2.21 implies that P_2 is a uniform parahoric with $e_{P_2}=e_P$. If (P_2,r,β_2) is fundamental, we conclude in the same way that it is regular and centralized by T_2. Finally, if P_2 is non-fundamental, (P_2,r,β_2) satisfies the conditions of Lemma 3.15. Thus, V_2 has dimension 1, and moreover $e_P=e_{P_2}=1$. □

Lemma 3.17. *If (P,r,β) is a pure stratum, then $e_P=n$.*

Proof. Set $m=n/e_P$, and assume that $m>1$. Let $T=E^\times$ be a torus centralizing (P,r,β). By Lemma 2.4, we can find a saturation $\mathscr{L}_E=\{L^i_E\}$ of $\mathscr{L}$ that is stabilized by $\mathfrak{o}_E$. We index $\mathscr{L}_E$ so that $L^{mi}_E=L^i$, and let I be the Iwahori subgroup that stabilizes $\mathscr{L}_E$. We fix a uniformizer ϖ_E for E; we can assume that $\varpi_P=\varpi_E^m$. Recall that $\mathfrak{I}^1_E=\varpi_E\mathfrak{I}=\mathfrak{I}\varpi_E$ by Proposition 2.3. Thus, $\varpi_E^jL^i=L_E^{mi+j}$, and furthermore $\varpi_E^j\in\mathfrak{P}^{\lfloor\frac{j}{m}\rfloor}$. By Proposition 3.5, $\operatorname{ad}(\varpi_E^j)(\beta_\nu)\in\mathfrak{P}^{-r+\lfloor\frac{j}{m}\rfloor+1}$.

First, we show that $\beta_\nu\in\mathfrak{I}^{-rm}$. By assumption, $\beta_\nu(L^{im}_E)\subset L^{im-rm}_E$. Now, take $0<j<m$. We have

$$\beta_\nu(L^{mi+j}_E)=\beta_\nu\varpi_E^j(L^i)=(\varpi_E^j\beta_\nu-\operatorname{ad}(\varpi_E^j)(\beta_\nu))L^i\subset L^{im+j-rm}_E,$$

since $\operatorname{ad}(\varpi_E^j)(\beta_\nu)L^i\subset L^{i-r+1}=L^{im+m-rm}_E$. Thus, $\beta_\nu\in\mathfrak{I}^{-rm}$.

We next show that $\operatorname{ad}(\varpi_E)(\beta_\nu)\in\mathfrak{I}^{-rm+2}$. The calculation above actually showed that $\beta_\nu\varpi_E^jv\equiv\varpi_E^j\beta_\nu v\pmod{L_E^{-rm+j+1}}$ for any $v\in L^0$ and $0\le j<m$. In particular, this gives

$$\varpi_E\beta_\nu\varpi_E^jv\equiv\varpi_E^{j+1}\beta_\nu v\equiv\beta_\nu\varpi_E\varpi_E^jv\pmod{L_E^{-rm+j+2}},$$

for $0<j<m-1$, with the first congruence also holding for $j=m-1$. However, the second congruence is also true when $j=m-1$; in this case, $\varpi_E^{j+1}=\varpi_P$, and the

congruence follows from $\mathrm{ad}(\varpi_P)(\beta_\nu)L^0 \subset L^{-r+2} = L_E^{-rm+2m} \subset L_E^{-rm+(m-1)+2}$. The congruences also hold trivially for $j=0$, so $\mathrm{ad}(\varpi_E)\beta_\nu \in \mathfrak{I}^{-rm+2}$ as desired.

By Lemma 2.10, $\beta_\nu \in E + \mathfrak{I}^{-rm+1}$. Thus, there exists $\beta'_\nu \in \mathfrak{I}^{-rm} \cap E$ such that $\beta'_\nu \in \beta_\nu + \mathfrak{I}^{-rm+1}$. Let $\beta_{-rm} = \alpha\varpi_E^{-rm}$, with $\alpha \in k^\times$, be the homogeneous degree $-rm$ term of β'_ν, so that $X = \beta_\nu - \beta_{-rm} \in \mathfrak{I}^{-rm+1}$. Clearly, $\mathrm{ad}(\beta_{-rm})(\beta_\nu) = \mathrm{ad}(\beta_{-rm})(X)$. Moreover, since E centralizes (P, r, β), the remark after Proposition 3.5 shows that $\mathrm{ad}(\beta_{-rm})(X) \in \mathfrak{P}^{-2r+1}$. It follows that X and β_{-rm} commute up to a term in $\mathfrak{P}^{-2r+1}$, so

$$t^r\beta_\nu^{e_P} = \alpha^{e_P}1 + e_P t^r X\beta_{-rm}^{e_P-1} + \text{higher order terms.} \tag{3.1}$$

If $X \in \mathfrak{I}^{-rm+j}$ for $1 \le j < e_P$, then $e_P t^r X\beta_{-rm}^{e_P-1} \in \mathfrak{I}^j$ and the higher order terms of (3.1) lie in $\mathfrak{I}^{2j}$. In particular, if $X \notin \mathfrak{P}^{-r+1}$, there exists $1 \le j < e_P$ such that $X \in \mathfrak{I}^{-rm+j} \setminus \mathfrak{I}^{-rm+j+1}$, and it now follows that $N = t^r\beta_\nu^{e_P} - \alpha^{e_P} \in \mathfrak{I}^{-rm+j} \setminus \mathfrak{I}^{-rm+j+1}$. It is obvious that $\bar{N} \in \bar{\mathfrak{P}}$ is a nonzero nilpotent operator, so y_β has Jordan decomposition $y_\beta = \overline{t^r\beta_\nu^{e_P}} = \alpha^{e_P}1 + \bar{N}$. This contradicts the semisimplicity of y_β, so $X \in \mathfrak{P}^{-r+1}$.

On the other hand, if $X \in \mathfrak{P}^{-r+1}$, then $\bar{\mathfrak{t}}^0 = \mathfrak{z}^0(\beta_\nu) = \mathfrak{z}^0(\beta_{-rm})$. By Lemma 2.10, $\mathfrak{z}^0(\beta_{-rm})$ is one-dimensional if and only if $m = \gcd(-rm, n) = 1$, contradicting our assumption that $m > 1$. Hence, $m = 1$ and $e_P = n$. □

Proof of Theorem 3.8. First, assume that (P, r, β) is strongly uniform. Suppose that we have a nontrivial splitting $T = T_1 \times T_2$, with corresponding idempotents ϵ_j. Setting $V_1 = V^{1\times T_2}$ and $V_2 = V^{T_1\times 1}$, we show that (V_1, V_2) splits P and β at level r. Note that $\epsilon_j \in \mathfrak{P}$, since $\epsilon_j \in \mathfrak{t}(\mathfrak{o})$. Therefore, $L^i_j = \epsilon_j L^i$, and $L^i = L^i_1 \oplus L^i_2$. By Remark 2.21, in order to see that (V_1, V_2) splits P and β, it suffices to show that $\epsilon_1\beta_\nu\epsilon_2$ and $\epsilon_2\beta_\nu\epsilon_1$ are in $\mathfrak{P}^{-r+1}$.

Note that $\epsilon_j \in T^\flat$; indeed, it is a (nonempty) sum of the primitive idempotents χ_i for $\mathfrak{t}$. By Lemma 3.14, $a_1\epsilon_1 + a_2\epsilon_2 \in Z^0(\beta_\nu)$ for any $a_1, a_2 \in k^\times$. This implies that $\mathrm{Ad}(a_1\epsilon_1 + a_2\epsilon_2)(\epsilon_1\beta_\nu\epsilon_2) = \frac{a_1}{a_2}\epsilon_1\beta_\nu\epsilon_2 \equiv \epsilon_1\beta_\nu\epsilon_2 \pmod{\mathfrak{P}})^{-r+1}$; accordingly, $\epsilon_1\beta_\nu\epsilon_2 \in \mathfrak{P}^{-r+1}$. Similarly, $\epsilon_2\beta_\nu\epsilon_2 \in \mathfrak{P}^{-r+1}$.

Let (P_j, r, β_j) be the stratum corresponding to V_j. By Remark 2.21, each (P_j, r, β_j) is strongly uniform. We next show that $(P_{12}, r, \partial_\beta)$ is uniform. Using notation from the previous section, ∂_β determines a map from $\bar{M}^j \to \bar{M}^{j-r}$. It has already been established in (2.5) that $\mathscr{M} = \{M^j\}$ is uniform. It remains to show that ∂_β has trivial kernel in $\bar{M}^j$, say for all $j \ge 0$. If $x \in \ker(\partial_\beta)$, then $(\beta_1)_\nu x \equiv x(\beta_2)_\nu \pmod{M^{j-r+1}}$. Therefore, $\mathrm{Ad}(1 + \iota_1 x\pi_2)(\beta_\nu) \equiv \beta_\nu \pmod{\mathfrak{P}^{j-r+1}}$, so $1 + \iota_1 x\pi_2 \in Z^j(\beta_\nu)$. However, $(T_1 \times T_2) \cap (1 + \iota_1 M^j\pi_2) = 1$, implying that $x \in M^{j+1}$. We note that in the case $r = 0$, the eigenvalues of $\bar{\beta}_\nu$ are pairwise distinct modulo $\mathbb{Z}$ by Definition 3.2. A fortiori, the eigenvalues of $(\bar{\beta}_1)_\nu$ are distinct from the eigenvalues of $(\bar{\beta}_2)_\nu$ modulo $\mathbb{Z}$. We have thus shown that (P, r, β) is the direct sum of two strongly uniform strata, and Lemma 3.16 shows that these strata are regular (centralized by the T_j's).

We can iterate this procedure until (P, r, β) is the direct sum of regular, strongly uniform strata each of which is centralized by a rank one torus, i.e., by the units of a field. Therefore, (P, r, β) splits into a sum of pure strata.

Finally, suppose that (P, r, β) is not strongly uniform. When $r > 0$, Corollary 2.30 implies that (P, r, β) splits into a strongly uniform stratum (P_1, r, β_1) and a non-fundamental stratum (P_2, r, β_2). By Lemma 3.16, V_2 has dimension one and $e_P = 1$. When $r = 0$, Definition 3.2 implies that the kernel of $\bar{\beta}_\nu$ has dimension one and that the non-zero eigenvalues of $\bar{\beta}_\nu$ are not integers. Write $\bar{L}^0 = \bar{V}_1 \oplus \bar{V}_2$, where $\bar{V}_2 = \ker(\bar{\beta}_\nu)$ and $\bar{V}_1$ is the span of the other eigenvectors. It is easily checked that any lift of this splitting to L^0 induces a splitting $V = V_1 \oplus V_2$, and (V_1, V_2) splits $(P, 0, \beta)$. □

Proof of Corollary 3.9. When (P, r, β) is strongly uniform, Theorem 3.8 states that (P, r, β) splits into a sum of pure strata (P_i, r, β_i) with $e_{P_i} = e_P$. Therefore, by Lemma 3.17, (P_i, r, β_i) is centralized by a torus isomorphic to $E^\times$, and each component $V_i \subset V$ has dimension e_P. It follows from Lemma 3.16 that $T \cong (E^\times)^{n/e_P}$. Otherwise, $e_P = 1$ and (P, r, β) splits into a strongly uniform stratum (P_1, r, β_1) and a one-dimensional non-fundamental stratum (P_2, r, β_2). In particular, by Lemma 3.16, this gives a splitting of a conjugate of T into $T_1 \times T_2$, where $T_2 \cong F^\times$. Since $e_P = 1$, it follows from the theorem that T_1 also splits into rank one factors.

We now prove the last statement. By Lemma 3.14, we know that $\mathfrak{t}^\flat \cong \bar{\mathfrak{t}}^0/\mathfrak{n}$, where $\mathfrak{n}$ is the image in $\bar{\mathfrak{P}}$ of $\prod \mathfrak{p}_E$. However, we have already seen that $(\varpi_E, \ldots, \varpi_E)$ generates $\mathfrak{P}^1$, so $\mathfrak{n} = \{0\}$. The proof for $T^\flat$ is similar.

□

We can now prove Proposition 3.11. First, we need a lemma.

Lemma 3.18. *Let (P, r, β) be a regular stratum centralized by T, and suppose that $\beta_\nu \in \mathfrak{t} + \mathfrak{P}^{-r+m}$. Then, β_ν is conjugate to an element of $\mathfrak{t}$ by an element of P^m.*

Proof. Let E/F be a field extension of degree e_P, and let ϖ_E be a uniformizer in E. By Corollary 3.9, (P, r, β) splits into a sum of pure strata (P_i, r, β_i), each of which is centralized by a torus isomorphic to $E^\times$. In particular, we can choose a block-diagonal representative $\beta'_\nu = (a_1 \varpi_E^{-r}, a_2 \varpi_E^{-r}, \ldots, a_{n/e_P} \varpi_E^{-r}) \in \prod_i \mathfrak{P}_i^{-r}$. Denote the summands of V by V_i. We may identify the $(\ell - j)^{th}$ off-diagonal block with $\operatorname{Hom}_F(V_j, V_\ell)$. Let $\mathfrak{n} \subset \mathfrak{gl}(V)$ be the subalgebra of matrices in the $(\ell - j)^{th}$ off-diagonal block corresponding to $\operatorname{Hom}_E(V_j, V_\ell)$. If $a_\ell = a_j$, then $1 + \mathfrak{n}$ centralizes β_ν. Since $((1 + \mathfrak{n}) \cap P^i) P^{i+1} \not\subseteq T^i P^{i+1}$, this is a contradiction. Thus, the a_j's are pairwise distinct.

By Proposition 2.11, there exists $X_m \in \mathfrak{P}^m$ such that $\beta_\nu - \pi_{\mathfrak{t}}(\beta_\nu) \equiv \operatorname{ad}(\beta'_\nu)(X_m) \equiv \operatorname{ad}(\beta_\nu)(X_m) \pmod{\mathfrak{P}^{2-r}}$. Therefore, $\operatorname{Ad}(1 + X_m)(\beta_\nu) \equiv \pi_{\mathfrak{t}}(\beta_\nu) \pmod{\mathfrak{P}^{2-r}}$. Inductively, we can find $X_j \in \mathfrak{P}^{j+1}$ so that, setting $p_j = (1 + X_j)(1 + X_{j-1}) \ldots (1 + X_m)$, $\operatorname{Ad}(p_j)(\beta_\nu) \in \mathfrak{t} + \mathfrak{P}^{j+1-r}$ and $p_j \equiv p_{j-1} \pmod{P^{j-1}}$. If we let $p \in P^1$ be the inductive limit of the p_j's, we see that $\operatorname{Ad}(p)(\beta_\nu) \in \mathfrak{t}$. □

Remark 3.19. We note that, in the argument above, it is not necessarily the case that β_ν is conjugate to $\pi_{\mathfrak{t}}(\beta_\nu)$.

Proof of Proposition 3.11. It was shown in the proof of Lemma 3.15 that any representative β_ν is regular. To show that β_ν is also semisimple, it suffices to show that it is conjugate to an element of a Cartan subalgebra $\mathfrak{t}$.

First, suppose (P, r, β) is strongly uniform. By Theorem 3.8 and Corollary 3.9, there exists a splitting $V = V_1 \oplus \cdots \oplus V_{n/e_P}$ with $\dim V_i = e_P$ for each i and a block diagonal $\beta'_\nu = (\beta_{i\nu}) \in \prod_{i=1}^{n/e_P} \mathfrak{gl}(V_i)$ such that $\beta_\nu \in \beta'_\nu + \mathfrak{P}^{-r+1}$ and the (P_i, r, β_i)'s are pure strata. Moreover, by Lemma 3.16, we can choose a maximal torus T centralizing (P, r, β) such that the splitting of V induces a splitting $T = T_1 \times \cdots \times T_{n/e_P}$, with T_i centralizing (P_i, r, β_i). Since T_i is isomorphic to the units of the field extension E/F of degree e_P, Lemma 2.10 implies that $\beta_{i\nu} \in T_i \cap \mathfrak{P}_i^{-r} + \mathfrak{P}_i^{1-r}$. Therefore, $\beta_\nu \in \mathfrak{t} + \mathfrak{P}^{1-r}$. By Lemma 3.18, β_ν is conjugate to an element of $\mathfrak{t}$.

If (P, r, β) is not strongly uniform, then $e_P = 1$ by the second part of Theorem 3.8. As above, we can choose a splitting $V = V_1 \oplus \cdots \oplus V_n$, a diagonal representative $\beta'_\nu \in \prod_{i=1}^n \mathfrak{gl}(V_i)$, and a compatibly split torus T which centralizes (P, r, β). In this case, $\dim V_i = 1$ for all i, so $\mathfrak{t} = \prod_{i=1}^n \mathfrak{gl}(V_i)$ and $\beta'_\nu \in \mathfrak{t}$. In particular, $\beta_\nu \in \mathfrak{t}^{-r} + \mathfrak{P}^{1-r}$, and Lemma 3.18 again implies that β_ν is conjugate to an element of $\mathfrak{t}$. □

We conclude this section with two lemmas that will be needed in Section 5. We recall from Remark 3.6 that if (P, r, β) is a regular stratum centralized by the maximal torus T, then one can choose $\beta_\nu \in \mathfrak{t}$. We next show that if two such representatives are conjugate, then they are the same.

Lemma 3.20. *Suppose that (P, r, β) is a regular stratum. Choose representatives $\beta_\nu, \beta'_\nu \in \mathfrak{t}$ for β. If $\mathrm{Ad}(g)(\beta_\nu) = \beta'_\nu$ for some $g \in \mathrm{GL}(V)$, then $\beta'_\nu = \beta_\nu$.*

Proof. By Proposition 3.11, β_ν is regular semisimple. Since $\mathrm{Ad}(g)(\beta_\nu) \in \mathfrak{t}$, g lies in the normalizer of T. Let w be the image of g in the relative Weyl group $W = N(T)/T$. It suffices to show that w is the identity.

First, we show that $\mathrm{Ad}(g)(\mathfrak{t}^i) \subset \mathfrak{t}^i$. Recall from Corollary 3.9 that $\mathfrak{t} \cong \prod_{i=1}^{n/e_P} E$, so $\mathfrak{t}$ splits over E. Let ω_j be a uniformizer of E supported on the j^{th} summand of $\mathfrak{t}$ as before; we have $\omega_j \mathfrak{P}_j = \mathfrak{P}_j^1$ in the splitting determined by Theorem 3.8. Therefore, $\mathfrak{t}^i$ consists of those $(x_j) \in \prod_{i=1}^{n/e_P} E$ such that $x_j \in \omega_j^i \mathfrak{o}_E$. These are precisely the F-rational points of $\mathfrak{t}_E$ with eigenvalues of degree at least i/e_P. Since the action of W permutes the eigenvalues, it follows that $\mathrm{Ad}(g)(\mathfrak{t}^i) \subset (\mathfrak{t}^i)$.

Suppose that $s \in \mathfrak{t}^{1-r}$. The previous paragraph shows that $\mathrm{Ad}(g)(\beta_\nu + s) = \beta'_\nu + \mathrm{Ad}(g)(s) \in \beta_\nu + \mathfrak{t}^{1-r}$. By induction on i, $\mathrm{Ad}(g^i)(\beta_\nu) \in \beta_\nu + \mathfrak{t}^{1-r}$, so $\mathrm{Ad}(g^i)(\beta_\nu)$ is a representative for β. Let m be the order of w. Then, $\beta''_\nu = \frac{1}{m}(\sum_{i=0}^{m-1} \mathrm{Ad}(g^i)(\beta_\nu))$ is a representative for β fixed by the action of w. Since β''_ν is regular semisimple, w must be the identity.

□

Lemma 3.21. *Suppose that (P, r, β) is a regular stratum centralized by T and that $\beta_\nu \in \mathfrak{t}^{-r}$. Let $A \in \mathfrak{gl}(V)^\vee$ be the functional determined by β_ν and ν. Then, A determines an element $A_i \in (\mathfrak{P}^i)^\vee$ by restriction, and the stabilizer of A_i under the coadjoint action of P^i is given by $T^i P^{r+1-i}$ whenever $r \geq 2i$, and P^i whenever $r < 2i$.*

Proof. Recall from Proposition 2.5 that $(\mathfrak{P}^i)^\perp = \mathfrak{P}^{1-i}$. Thus, $\mathrm{Ad}^*(p)(A_i) = A_i$ if and only if $\mathrm{Ad}(p)(\beta_\nu) \in \beta_\nu + \mathfrak{P}^{1-i}$. If $r < 2i$, then $-r + i \geq -i + 1$. Therefore, since $\mathrm{Ad}(p)(\beta_\nu) \in \beta_\nu + \mathfrak{P}^{-r+i}$ for any $p \in P^i$, P^i lies in the stabilizer of A_i in this case.

Suppose now that $\mathrm{Ad}^*(p)(A_i) = A_i$ and $r \geq 2i$. The image of p in $\bar{P}^i$ must lie in $Z^i(\beta_\nu)$; therefore $p = tp' \in T^i P^{i+1}$. Assume, inductively, that $p = tq \in T^i P^j$ with $j < r + 1 - i$.

Since $\mathrm{Ad}(t^{-1})(\beta_\nu) = \beta_\nu$, q stabilizes A_i. Moreover, $\mathrm{Ad}(P^j)(\beta_\nu) \subset \beta_\nu + \mathfrak{P}^{j-r}$; since $j - r < 1 - i$, the image of q in $\bar{P}^j$ lies in $Z^j(\beta_\nu)$. Therefore, $q \in T^j P^{j+1} \subset T^i P^{j+1}$. We conclude that $p \in T^i P^{r+1-i}$. When $j = r + 1 - i$, $\mathrm{Ad}(P^j)(\beta_\nu) \subset \beta_\nu + \mathfrak{P}^{1-i}$, so P^j stabilizes A_i. It is now clear that $T^i P^{r+1-i}$ stabilizes A_i. □

Note that although the functionals A_i depend on the choice of $\beta_\nu \in \mathfrak{t}^{-r}$, their stabilizers do not.

4. Connections and Strata

In this section, we describe how to associate a stratum to a formal connection. The local theory of irregular singular point connections is well understood; an elegant classification is given in [20, Theorem III.1.2]. The geometric theory of strata provides a Lie-theoretic interpretation of elements in the classical theory. In particular, the combinatorics of fundamental strata may be used to determine the slope of a connection, and the theory of strata makes precise the notion of the leading term of a connection with noninteger slope. Moreover, a split stratum induces a splitting on the level of formal connections.

First, we recall some notation and basic facts. As before, k is an algebraically closed field of characteristic 0, $\mathfrak{o}$ is the ring of formal power series in a parameter t, and F is the field of formal Laurent series.

(1) $\mathcal{D}_F$ (resp. $\mathcal{D}_\mathfrak{o}$) is the ring of formal differential operators on F (resp. $\mathfrak{o}$). $\mathcal{D}_F$ is generated as an F-algebra by $\partial_t = \frac{d}{dt}$ and contains the Lie algebra of k-derivations (i.e., vector fields) on F: $\mathrm{Der}_k(F) = F\partial_t$.

(2) $\Omega^\times \subset \Omega^1_{F/k}$ is the $F^\times$-torsor of non-zero one forms in $\Omega^1_{F/k}$; if $\omega, \nu \in \Omega^\times$, then $\frac{\omega}{\nu} \in F^\times$ is the unique element such that $\frac{\omega}{\nu}\nu = \omega$.

(3) If $\nu \in \Omega^\times$, there is a unique vector field $\tau_\nu \in \mathcal{D}_F$ whose inner derivation takes ν to 1, i.e., $\iota_{\tau_\nu}(\nu) = 1$. For example, $\tau_{\frac{dt}{t}} = t\partial_t$, and $\tau_{df} = \frac{dt}{df}\partial$.

(4) A connection ∇ on an F-vector space V is a k-linear derivation

$$\nabla : V \to V \otimes_F \Omega^1_{F/k}.$$

The connection ∇ gives V the structure of a $\mathcal{D}_F$-module: if $v \in V$, and $\xi \in \mathrm{Der}_k(F)$, then $\xi(v) = \nabla_\xi(v) \stackrel{\mathrm{def}}{=} \iota_\xi(\nabla(v))$.

(5) A connection ∇ is *regular singular* if there exists an $\mathfrak{o}$-lattice $L \subset V$ with the property that $\nabla(L) \subset L \otimes_\mathfrak{o} \Omega^1_{\mathfrak{o}/k}(1)$. Equivalently, if $\nu \in \Omega^\times$ has order -1, $\nabla_\tau(L) \subset L$. Otherwise, ∇ is irregular.

(6) Suppose V has dimension n. Let $V^{\mathrm{triv}} = F^n$ be the trivial vector space with standard basis. If we fix a trivialization $\phi : V \xrightarrow{\sim} V^{\mathrm{triv}}$, then ∇ has the matrix presentation

$$\nabla = d + [\nabla]_\phi \tag{4.1}$$

where $[\nabla]_\phi \in \mathfrak{gl}_n(F) \otimes_F \Omega^1_{F/k}$. The space of trivializations is a left $\mathrm{GL}_n(F)$-torsor, and multiplication by g changes the matrix of $[\nabla]_\phi$ by the usual gauge change formula

$$g \cdot [\nabla]_\phi := g[\nabla]_\phi g^{-1} - (dg)g^{-1}. \tag{4.2}$$

Thus, $[\nabla]_{g\phi} = g \cdot [\nabla]$. We note that the matrix form of ∇_τ is given by $[\nabla_\tau]_\phi = \iota_\tau([\nabla]_\phi)$, with gauge change formula $g \cdot [\nabla_\tau]_\phi := g[\nabla_\tau]_\phi g^{-1} - (\tau g)g^{-1}$. We will drop the subscript ϕ whenever there is no ambiguity about the trivialization.

4.1. **Strata contained in connections.** Let ∇ be a connection on an n-dimensional F-vector space. Fix $\nu \in \Omega^\times$, and set $\tau = \tau_\nu$. Suppose that $\mathscr{L}$ is a lattice chain with the property that $\nabla_\tau(L^i) \subset L^{i-r-(1+\mathrm{ord}(\nu))e_P}$. We define $\mathrm{gr}^i(\nabla_\tau)$ to be the following map induced by ∇_τ:

$$\mathrm{gr}^i(\nabla_\tau) : \bigoplus_{j=0}^{e_P-1} \bar{L}^{i+j} \to \bigoplus_{j=0}^{e_P-1} \bar{L}^{i+j-r-(1+\mathrm{ord}(\nu))e_P}.$$

By Lemma 2.6, this determines an element of $\bar{\mathfrak{P}}^{-r-(1+\mathrm{ord}(\nu))e_P}$. Equivalently, if we fix a trivialization ϕ of L^i (viewed as a trivialization of V taking L^i to $\mathfrak{o}^n$), then $\mathrm{gr}^i(\nabla_\tau)$ equals the image of $\phi^{-1}[\nabla_\tau]\phi$ in $\bar{\mathfrak{P}}^{-r-(1+\mathrm{ord}(\nu))e_P}$.

Definition 4.1. We say that (V, ∇) contains the stratum (P, r, β) if P stabilizes the lattice chain (L^i), $\nabla_\tau(L^i) \subset L^{i-r-(1+\mathrm{ord}(\nu))e_P}$ for all i, and $\mathrm{gr}^j(\nabla_\tau) = \bar{\beta}_\nu \in \bar{\mathfrak{P}}^{-r-(1+\mathrm{ord}(\nu))e_P}$ for some j.

Proposition 4.2. *The stratum (P, r, β) is independent of $\nu \in \Omega^\times$.*

Proof. Take $\nu' = f\nu$ for some $f \in F^\times$, so that $\tau' = \tau_{\nu'} = \frac{1}{f}\tau_\nu$. Since $\nabla_{\tau'} = \frac{1}{f}\nabla_\tau$, it is clear that $\nabla_{\tau'}(L^i) \subset L^{i-r-(1+\mathrm{ord}(\nu'))e_P}$ if and only if the analogous inclusion holds for ν; in addition, $\mathrm{gr}^i(\nabla_{\tau'}) = \frac{1}{f}\,\mathrm{gr}^i(\nabla_\tau)$. On the other hand, if $\beta_\nu \in \mathfrak{P}^{-r-(1+\mathrm{ord}(\nu))e_P}$ is a representative for the functional β on $\bar{\mathfrak{P}}^r$, then $\langle \beta_\nu, X\rangle_\nu = \langle f^{-1}\beta_\nu, X\rangle_{\nu'}$ for all $X \in \mathfrak{P}^r$ implies that one can take $\beta_{\nu'} = \frac{1}{f}\beta_\nu$. Hence, $\mathrm{gr}^i(\nabla_{\tau'}) = \bar{\beta_{\nu'}}$. □

For the rest of Section 4, we will fix $\nu \in \Omega^\times$ of order -1.

Proposition 4.3. *Suppose that $r \geq 1$. Then, the coset in $\bar{\mathfrak{P}}^{-r}$ determined by $\mathrm{gr}^\ell(\nabla_\tau)$ under the isomorphism* (2.1) *is independent of ℓ.*

Proof. The maps $\mathrm{gr}^\ell(\nabla_\tau)$ and $\mathrm{gr}^0(\nabla_\tau)$ determine the same element on $\bar{\mathfrak{P}}^{-r}$ when they "coincide up to homothety". More precisely, we must show that if $0 \leq i < e_P$ and $\ell \leq i + je_P < \ell + e_P$, then $\nabla_\tau(t^j v) \equiv t^j \nabla_\tau(v) \pmod{L^{i+je_P+1}}$ for all $v \in L^i$. By the Leibniz rule, $\nabla_\tau(t^j v) = \frac{\tau(t^j)}{t^j}(t^j v) + t^j \nabla_\tau(v)$. However, $\frac{\tau(t^j)}{t^j} \in \mathfrak{o}$, so for $r \geq 1$, $\frac{\tau(t^j)}{t^j}(t^j v) \in L^{\ell+je_P} \subset L^{\ell+je_P-r+1}$ as desired. □

In other words, if $r \geq 1$, the $\mathrm{gr}^\ell(\nabla_\tau)$'s determine a unique coset $\mathrm{gr}(\nabla_\tau) \in \bar{\mathfrak{P}}^{-r}$; viewed as a degree $-r$ endomorphism of $\mathrm{gr}(\mathscr{L})$, $\mathrm{gr}(\nabla_\tau)(\bar{x}) = \mathrm{gr}^\ell(\nabla_\tau)(\bar{x})$ for $x \in \bar{L}^i$ and any ℓ with $\ell \leq i < \ell + e_P$.

The following lemma will be used in Sections 4.3 and 5.

Lemma 4.4. *If $P \subset \mathrm{GL}_n(\mathfrak{o})$ is a parahoric subgroup, then $\tau(\mathfrak{P}^\ell) \subset \mathfrak{P}^\ell$. Moreover, if $p \in P$, then $\tau(p)p^{-1} \in \mathfrak{P}^1$.*

Proof. Let $\mathscr{L}$ be the lattice chain stabilized by P such that $L^0 = \mathfrak{o}^n$. We may choose a basis $e_1, \dots, e_n$ for $\mathfrak{o}^n$ that is compatible with $\mathscr{L}$ (as in Remark 2.9). It is clear that $\tau(L^j) \subset L^j$ for any j, since $\tau(e_i) \subset t\mathfrak{o}^n$. Therefore, if $v \in L^j$ and $X \in \mathfrak{P}^\ell$, $\tau(X)v = \tau(Xv) - X\tau(v) \in \mathfrak{P}^{j+\ell}$. It follows that $\tau(X) \in \mathfrak{P}^\ell$.

By Lemma 2.6, there exists $H \subset \mathrm{GL}_n(k)$ for which $P = P^1 \ltimes H$. Hence, it suffices to prove the second statement for $p \in P^1$. Since P^1 is topologically unipotent, $\exp : \mathfrak{P}^1 \to P^1$ is surjective. If $p = \exp(X)$, we obtain

$$\tau(\exp(X))\exp(-X) = \tau(X)\exp(X)\exp(-X) = \tau(X) \in \mathfrak{P}^1.$$

□

We now investigate the strata contained in a connection.

Lemma 4.5. *Every connection (V, ∇) contains a stratum.*

Proof. Take any lattice $L \subset V$ with stabilizer P, and let $\mathscr{L}$ be the lattice chain $(L^i = t^i L)$. It is obvious that $\nabla_\tau(L) \subset L^{-r}$ for some $r \geq 0$. The Leibniz rule calculation from the proof of Proposition 4.3 shows that $\nabla_\tau(L^i) \subset L^{i-r}$, so (V, ∇) contains (P, r, β), where β corresponds to $\mathrm{gr}^0(\nabla_\tau)$. □

One of the standard ways to study irregular singular connections is to find a good lattice pair. The theory of good lattice pairs bears a superficial resemblance to the theory of fundamental strata. However, we will see that there are only a few cases in which there is a direct relationship between the two theories.

Lemma 4.6. [9, Lemme II.6.21] *Given a connection (V, ∇), there exist two $\mathfrak{o}$-lattices $M^1 \subset M^2 \subset V$ with the following properties:*

(1) $\nabla(M^1) \subset M^2 \otimes_{\mathfrak{o}} \Omega^1_{\mathfrak{o}/k}(1)$
(2) *For all $\ell > 0$, ∇ induces an isomorphism*

$$\bar{\nabla} : M^1(\ell)/M^1(\ell - 1) \cong \left[M^2 \otimes_{\mathfrak{o}} \Omega^1_{\mathfrak{o}/k}(\ell + 1)\right] / \left[M^2 \otimes_{\mathfrak{o}} \Omega^1_{\mathfrak{o}/k}(\ell)\right]. \tag{4.3}$$

The connection ∇ is regular singular if and only if $M^1 = M^2$.

We call M^1 and M^2 a *good lattice pair* for (V, ∇).

Remark 4.7. In the regular singular case, the data of a good lattice pair ($\bar{\nabla}$, M^1, M^2) is equivalent to a strongly uniform stratum contained in (V, ∇): take $\mathscr{L} = (t^i M^1)_{i \in \mathbb{Z}}$, and β such that the image of $\bar{\beta}_\nu$ under the appropriate isomorphism (2.1) is $\iota_\tau(\bar{\nabla}) = \mathrm{gr}^{-1}(\nabla_\tau)$. However, it is not immediately possible to construct a fundamental stratum from a good lattice pair in general.

Given a good lattice pair M^1 and M^2, one might naively construct a lattice chain $\mathscr{L}$ as follows. Set $L^0 = M^1$. Choose $s \in \mathbb{Z}_{\geq 0}$ such that $L^0(s) \supset M^2$ but $L^0(s-1) \not\supseteq M^2$. First, we suppose that $M^2 = L^0(s)$. Define $\mathscr{L}$ to be the chain $(L^i = t^i L^0)$. In this case, $e_P = 1$. Take β such that $\bar{\beta}_\nu = \mathrm{gr}^{-1}(\nabla_\tau)$ as above. Equation (4.3) implies that (V, ∇) contains $(\mathrm{GL}(L^0), s, \beta)$, and this stratum is fundamental (in fact, strongly uniform).

The naive generalization of the construction above does not necessarily produce a fundamental stratum. Set $L^1 = M^2(-s) + L^0(-1)$. Since $L^0(s) \supsetneq M^2$, it follows using Nakayama's Lemma that the map $M^2(-s) \to L^0/L^0(-1)$ is not surjective. We conclude that $L^0 \supsetneq L^1 \supsetneq L^0(-1)$. This extends to a lattice chain $\mathscr{L}$ with $e_P = 2$.

Finally, it is clear that there exists a minimal $r \geq 0$ such that $\nabla_\tau(L^i) \subset L^{i-r}$ for $i = 0, 1$. The usual Leibniz rule argument shows that $\nabla_\tau(L^i) \subset L^{i-r}$ for all i. Choosing $\beta_\nu \in \mathfrak{P}^{-r}$ whose coset corresponds to $\mathrm{gr}^j(\nabla_\tau)$ (for some fixed j), we have the data necessary to give a stratum contained in (V, ∇).

Notice that the stratum constructed above is not necessarily fundamental. For instance, suppose that

$$[\nabla_\tau] = \begin{pmatrix} 0 & t^{-3} \\ 1 & 0 \end{pmatrix}$$

in $V = V^{\mathrm{triv}}$. Set $M^1 = \mathfrak{o}e_1 + \mathfrak{o}e_2$ and $M^2 = \mathfrak{p}^{-3}e_1 + \mathfrak{o}e_2$. It is easy to check that this is a good lattice pair for (V, ∇), and our construction gives $L^0 = M^1$ and

$L^1 = \mathfrak{o}e_1 + \mathfrak{p}e_2$. However, the coset in $\bar{\mathfrak{P}}^{-5}$ corresponding to $\mathrm{gr}^0(\nabla_\tau)$ contains the nilpotent operator $\left(\begin{smallmatrix} 0 & t^{-3} \\ 0 & 0 \end{smallmatrix}\right)$.

In Theorem 2.16, we showed that a stratum is fundamental if and only if it can not be reduced to a stratum with smaller slope. The main goal of this section is to show that the slope of a connection is the same as the slope of any fundamental stratum contained in it. First, we define the slope of a connection.

Fix a lattice $L \subset V$. If $\mathbf{e} = \{e_j\}$ is a finite collection of vectors in V, we define $v(\mathbf{e}) = m$ if m is the greatest integer such that $\mathbf{e} \subset t^m L$. Take $\mathbf{e}$ to be a basis for V. An irregular connection (V, ∇) has slope σ, for σ a positive rational number, if the subset of $\mathbb{Q}$ given by

$$\{\left|(\nabla_\tau^i \mathbf{e}) + \sigma i\right| \mid i > 0\}$$

is bounded. Here, $\nabla_\tau^i \mathbf{e} = \{\nabla_\tau^i(e_j)\}$. By a theorem of Katz [9, Theorem II.1.9], every irregular singular connection has a unique slope, and the slope is independent both of the choice of basis $\mathbf{e}$ and the choice of ν of order -1. We define the slope of a regular singular connection to be 0.

Lemma 4.8. *A connection (V, ∇) contains a stratum of slope 0 if and only if it is regular singular, i.e., has slope 0. Moreover, a regular singular connection contains a fundamental stratum with slope 0 and no fundamental stratum with positive slope.*

Proof. If (V, ∇) contains $(P, 0, \beta)$, then there exists a lattice $L \in \mathscr{L}$ such that $\nabla_\tau(L) \subset L$. Thus, ∇ is regular singular. Now, suppose that (V, ∇) is regular singular. Lemma 4.6 gives a lattice $L = M^1 = M^2$ for which $\nabla_\tau(L) \subset L$. Thus, ∇_τ preserves the lattice chain $(L^i = t^i L)$, and the corresponding stratum has $r = 0$. Moreover, (4.3) implies that $\mathrm{gr}^{-1}(\nabla_\tau)(L^{-1}/L^0) = L^{-1}/L^0$, so the stratum is fundamental.

Suppose that the same connection (V, ∇) contains a stratum (P, r, β) with $r > 0$. Let (L^i) be the associated lattice chain. We will show that $\bar{\beta} \in \bar{\mathfrak{P}}^{-r}$ is a nilpotent operator on $\mathrm{gr}(\mathscr{L})$. With the lattice L as above, choose i and $m > 0$ such that $L^{i-mr+e_P} \supset L \supset L^i$. For $0 \leq j < e_P$, we obtain $(\nabla_\tau)^m(L^{i+j}) \subset L \subset L^{i+j-mr+1}$. By Proposition 4.3, the endomorphism $\bar{\beta}_\nu$ coincides with each $\mathrm{gr}^\ell(\nabla_\tau)$ on the latter's domain, so $\bar{\beta}_\nu^m(\bar{L}^{i+j}) = \mathrm{gr}^{i-(m-1)r}(\nabla_\tau) \circ \cdots \circ \mathrm{gr}^i(\nabla_\tau)(\bar{L}^{i+j}) = 0$. Therefore, (P, r, β) is not fundamental. □

The following corollary describes the relationship between a fundamental stratum contained in (V, ∇) and its slope.

Proposition 4.9. *If (V, ∇) contains the fundamental stratum (P, r, β), then* $\mathrm{slope}(\nabla) = r/e_P$.

Proof. When (V, ∇) is regular singular, $r/e_P = 0 = \mathrm{slope}(\nabla)$ by Lemma 4.8, so we assume that (V, ∇) is irregular singular. Let $L = L^0$. Choose an ordered basis $\mathbf{e}$ for L that is compatible with $\mathscr{L}$. We use the notation from the proof of Lemma 2.6: $\mathbf{e} = \bigsqcup_{j=0}^{e_P-1} \mathbf{e}_j$ with $W_j = \mathrm{span}(\mathbf{e}_j) \subset L^j$ naturally isomorphic to L^j/L^{j+1}. For $w \in W_j$, $(\nabla_\tau)^i(w) \in \beta_\nu^i(w) + L^{-ir+j+1}$ by repeated application of Proposition 4.3. Since (P, r, β) is fundamental, $\beta_\nu^i \notin \mathfrak{P}^{-ri+1}$. Therefore, $\beta_\nu^i(\mathbf{e}_j) \nsubseteq L^{-ri+j+1}$ for some $\mathbf{e}_j$. It follows that $\nabla_\tau^i \mathbf{e}_j \nsubseteq L^{-ri+j+1}$, and thus $\nabla_\tau^i \mathbf{e} \nsubseteq L^{-ri+e_P}$.

Let $\lceil \frac{ri}{e_P} \rceil$ (resp. $\lfloor \frac{ri}{e_P} \rfloor$) be the integer ceiling (resp. floor) of $\frac{ri}{e_P}$. Then, $\nabla_\tau^i \mathbf{e} \subseteq L^{-ri} \subseteq t^{-\lceil \frac{ri}{e_P} \rceil} L^0$. However, $\nabla_\tau^i \mathbf{e} \nsubseteq t^{-\lfloor \frac{ri}{e_P} \rfloor + 1} L^0$, since $t^{-\lfloor \frac{ri}{e_P} \rfloor + 1} L^0 \subseteq L^{-ri+e_P}$. We conclude that $v(\nabla_\tau^i \mathbf{e})$ is equal to either $-\lfloor \frac{ri}{e_P} \rfloor$ or $-\lceil \frac{ri}{e_P} \rceil$. In particular, $\left| v(\nabla_\tau^i \mathbf{e}) + \frac{ri}{e_P} \right| < 1$, which proves the proposition. □

We are now ready to state our main theorem on the relationship between slopes of connections and fundamental strata. In the context of the representation theory of local fields, the analogous theorem is due to Bushnell [7, Theorem 2].

Theorem 4.10. *Any stratum (P, r, β) contained in (V, ∇) has slope greater or equal to* $\mathrm{slope}(\nabla)$*. Moreover, the set of strata contained in ∇ with slope equal to* $\mathrm{slope}(\nabla)$ *is nonempty and consists precisely of the fundamental strata contained in ∇. In particular, every connection contains a fundamental stratum.*

Proof. The regular singular case is dealt with in Lemma 4.8, so assume that (V, ∇) is irregular singular. By Proposition 4.9, any fundamental stratum contained in ∇ has slope equal to $\mathrm{slope}(\nabla)$. Now, let (P_0, r_0, β_0) be any stratum contained in ∇. Assume that this stratum is not fundamental. We show that the stratum has a reduction (P_1, r_1, β_1) with strictly smaller slope which is also contained in ∇.

By Theorem 2.16, there is a reduction of (P_0, r_0, β_0) to (P_1, r_1, β_1') such that $r_1/e_{P_1} < r_0/e_{P_0}$. Let $\mathscr{L}_0 = (L_0^i)$ and $\mathscr{L}_1 = (L_1^i)$ be the lattice chains corresponding to P_0 and P_1 respectively. By definition, there is a lattice $L \in \mathscr{L}_0 \cap \mathscr{L}_1$; reindexing the lattice chains if necessary, we can assume without loss of generality that $L = L_0^0 = L_1^0$.

Choose a basis $\{e_1, \ldots, e_n\}$ for L, and write $\nabla_\tau(v) = [\nabla_\tau](v) + \tau(v)$. In particular, if $v \in L$, then $\tau(v) \in tL$, since $\tau(f e_j) = t f' e_j \in tL$ for $f \in \mathfrak{o}$. Thus, for any $v \in L_0^i$, $0 \leq i < e_{P_0}$ (resp. $w \in L_1^\ell$, $0 \leq \ell < e_{P_1}$), $\nabla_\tau(v) - [\nabla_\tau](v) \in L_0^{e_{P_0}}$ (resp. $\nabla_\tau(w) - [\nabla_\tau](w) \in L_1^{e_{P_1}}$). Therefore, $[\nabla_\tau]$ is a representative for both $\mathrm{gr}_0^0(\nabla_\tau)$ and $\mathrm{gr}_1^0(\nabla_\tau)$. By Proposition 4.3, we conclude that $[\nabla_\tau]$ is a representative for β_0.

By definition of a reduction, $[\nabla_\tau] \in \mathfrak{P}_1^{-r_1}$. If we let $\beta_1 \in (\bar{\mathfrak{P}}_1^r)^\vee$ be the corresponding functional, it is immediate that (P_1, r_1, β_1) is also a reduction of (P_0, r_0, β_0) with strictly smaller slope.

Finally, consider the collection of all strata contained in ∇; this set is nonempty by Lemma 4.5. Since $e_P \leq n$ (where $n = \dim V$, these slopes are all contained in $\frac{1}{n!}\mathbb{Z}$. Thus, the set of these slopes has a minimum value $s > 0$. (This value is positive by Lemma 4.8.) The argument given above shows that any stratum with slope s is fundamental, so ∇ contains a fundamental stratum and $s = \mathrm{slope}(\nabla)$. □

4.2. **Splittings.** Recall that the connection (V, ∇) is split by the direct sum decomposition $V = V_1 \oplus V_2$ if $\nabla_\tau(V_1) \subset V_1$ and $\nabla_\tau(V_2) \subset V_2$. In this section, we will show that any time a connection (V, ∇) contains a fundamental stratum (P, r, β) that splits, then there is an associated splitting of the connection itself. We note that, in the language of flat GL_n-bundles, this corresponds to a reduction of structure of (V, ∇) to a Levi subgroup.

Lemma 4.11. *Suppose that (V, ∇) contains a fundamental stratum $(P, 0, \beta)$ that is split by (V_1, V_2). Then, there exists $q \in P^1$ and a fundamental stratum $(P', 0, \beta')$ contained in ∇ such that $e_{P'} = 1$, $P'^1 \subset P^1$, and (qV_1, qV_2) splits both strata*

Proof. Let $\mathscr{L}$ be the lattice chain stabilized by P, and, without loss of generality assume that β is determined by $\mathrm{gr}^0(\nabla_\tau)$. Choose a trivialization for L^0, and let $[\nabla_\tau] \in \mathfrak{P}$ be the corresponding F-endomorphism. Now let $\mathscr{L}' = (L_i' = t^i L^0)$

with corresponding stabilizer $P' = \mathrm{GL}(L^0)$. It is obvious that $P'^1 \subset P^1$. Setting $\beta' \in (\bar{\mathfrak{P}}')^\vee$ equal to the functional induced by $[\nabla_\tau]$, we see that $(P', 0, \beta')$ is a stratum contained in (V, ∇) with $e_{P'} = 1$. It will be convenient to denote $[\nabla_\tau]$ by β_ν or β'_ν depending on whether it is being viewed as a representative of β or β'. With this convention, it follows from Lemma 2.6 that there is a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{gl}(\bar{L}^0)$ with Levi subalgebra $\mathfrak{h}$ such that $\bar{\beta}'_\nu \in \mathfrak{q}$ and $\bar{\beta}_\nu$ is the projection of $\bar{\beta}'_\nu$ onto $\mathfrak{h}$.

Since $(P, 0, \beta)$ is fundamental, there exists $0 \leq j < e_P$ such that $\bar{\beta}_\nu \in \mathrm{End}(\bar{L}^j)$ is non-nilpotent. It follows that $(P', 0, \beta')$ is fundamental. The splitting $V = V_1 \oplus V_2$ for β does not necessarily split β' at level 0: using the notation of Section 2.5, it is possible that $\epsilon_1 \beta'_\nu \epsilon_2 \in \mathfrak{P}^1$ has non-trivial image in $\bar{\mathfrak{P}}'$. Identify $\bar{M}^j$ with $\epsilon_1 \mathfrak{P}^j \epsilon_2 / \epsilon_1 \mathfrak{P}^{j+1} \epsilon_2$ and $\bar{N}^j$ with $\epsilon_2 \mathfrak{P}^j \epsilon_1 / \epsilon_2 \mathfrak{P}^{j+1} \epsilon_1$. By Definition 2.22 and Lemma 2.22, $\bar{\partial}_{\beta'_\nu}$ (resp. $\bar{\partial}'_{\beta'_\nu}$) is an automorphism of each $\bar{M}^j$ (resp. $\bar{M}^j$). In particular, there exists $X_1 \in \epsilon_1 \mathfrak{P}^1 \epsilon_2$ and $Y_1 \in \epsilon_2 \mathfrak{P}^1 \epsilon_1$ such that $\mathrm{ad}(X_1)(\beta'_\nu) \in -\epsilon_1 \beta'_\nu \epsilon_2 + \mathfrak{P}^2$ and $\mathrm{ad}(Y_1)(\beta'_\nu) \in -\epsilon_2 \beta'_\nu \epsilon_1 + \mathfrak{P}^2$. It follows that $\epsilon_i \operatorname{Ad}(1+Y_1) \operatorname{Ad}(1+X_1)(\beta'_\nu)\epsilon_j \in \mathfrak{P}^2$ for $i \neq j$. Continuing this process, we construct $p = (1+Y_{e_P-1})(1+X_{e_P-1}) \ldots (1+Y_1)(1+X_1) \in P^1$ such that $\epsilon_i \operatorname{Ad}(p)(\beta'_\nu)\epsilon_j \in \mathfrak{P}^{e_P}$ for $i \neq j$. Since $\mathfrak{P}^{e_P} \subset \mathfrak{P}'$, this implies that (V_1, V_2) splits $\operatorname{Ad}(p)(\beta'_\nu)$ at level 0, so $(p^{-1}V_1, p^{-1}V_2)$ splits β'_ν at level 0. Clearly, $(p^{-1}V_2, p^{-1}V_2)$ still splits $(P, 0, \beta_\nu)$; $q = p^{-1}$ will be the desired element of P^1.

Without loss of generality, we may assume that V_1 and V_2 split $\mathscr{L}'$ and β'_ν at level 0. Since $\bar{\beta}'_\nu$ projects to $\bar{\beta}_\nu \in \mathfrak{h}$, it is clear that these matrices have the same eigenvalues (as do $\bar{\beta}_{j\nu}$ and $\bar{\beta}'_{j\nu}$ for $j = 1, 2$), so $(P', 0, \beta')$ satisfies conditions (1) and (3) of Definition 2.22. By assumption $(P_1, 0, \beta_1)$ and $(P_{12}, 0, \partial_\beta)$ are strongly uniform. Since $(V_i \cap \mathscr{L}')$ is a sub-lattice chain of $\mathscr{L}_i$, it is clear that $(P'_1, 0, \beta'_1)$ is strongly uniform. It remains to show that the induced map $\bar{\partial}_{\beta'_\nu}$ is an automorphism of $\mathrm{End}(\bar{M}')$.

Let F^i be the image of $M^i \cap (M')^0$ in $\bar{M}'$. It is easy to see that $(M')^0 \subset M^{1-e_P}$ and $M^{e_P} \subset (M')^1$, implying that $F^{e_P} = \{0\}$ and $\bar{M}^0 = F^{-e_P+1}$. Moreover, $\partial_{\beta'_\nu}$ preserves the flag $\{F^i\}$, so F^i/F^{i+1} is a $\bar{\partial}_{\beta'_\nu}$-invariant subspace of $\bar{M}^i$. Since $(P_{12}, 0, \partial_\beta)$ is strongly uniform, $\bar{\partial}_{\beta'_\nu} \in \mathrm{Aut}(\bar{M}^i)$ for all i, hence the restrictions to F^i/F^{i+1} are also automorphisms. It follows that $\bar{\partial}_{\beta'_\nu}$ gives an automorphism of $\bar{M}'$. □

Theorem 4.12. *Suppose that (V, ∇) contains a fundamental stratum (P, r, β). Let (V_1, V_2) split (P, r, β). Then, there exists $p \in P^1$ such that (pV_1, pV_2) splits both (P, r, β) and ∇.*

The case when $e_P = 1$ is well known (see [19, Lemma 2]).

Proof. We first recall from Remark 2.23 that (pV_1, pV_2) splits (P, r, β) for any $p \in P^1$.

Let $V' = F^n$, and let $V' = V'_1 \oplus V'_2$ be the standard splitting of F^n into subspaces with $\dim V'_j = \dim V_j$. Let $\epsilon'_j \in \mathfrak{gl}_n(F)$ be the corresponding idempotents. By (4.1), (W_1, W_2) splits ∇ with $\dim W_j = \dim V_j$ if and only if there exists a trivialization $\psi : V \to V'$ such that

$$\epsilon'_1 [\nabla_\tau]_\psi \epsilon'_2 = \epsilon'_2 [\nabla_\tau]_\psi \epsilon'_1 = 0, \tag{4.4}$$

and $W_j = \psi^{-1}(V'_j)$.

Since ∇ contains (P, r, β), there exist j such that $\mathrm{gr}^j(\nabla_\tau) = \bar{\beta}_\nu$; reindexing the lattice chain if necessary, we can assume that $j = 0$. Fix a trivialization ϕ such that $\phi(L^0) = \mathfrak{o}^n$ and $\phi(V_j) = V'_j$. We set $\mathfrak{P}'^m = {}^\phi\mathfrak{P}^m \subset \mathfrak{gl}_n(F)$ and similarly for P'^m. Setting $\beta'_\nu = {}^\phi\beta_\nu$, we have $[\nabla_\tau]_\phi \equiv \beta'_\nu \pmod{\mathfrak{P}'^{-r+1}}$. By (4.2), it suffices to find $h \in P'^1$ such that $h \cdot [\nabla_\tau]_\phi$ satisfies (4.4); then (pV_1, pV_2) splits ∇, where $p = \phi^{-1}h^{-1}\phi \in P^1$.

Inductively, we construct $h_m \in P'^1$ such that $h_m \equiv h_{m-1} \pmod{\mathfrak{P}'^{m-1}}$ and $\epsilon'_i(h_m \cdot [\nabla_\tau]_\phi)\epsilon'_j \equiv 0 \pmod{\mathfrak{P}'^{-r+m+1}}$ for $i \neq j$. The limit $h = \lim h_m \in P^1$ will then satisfy (4.4). For $m = 0$, we can take $h_0 = I$, since (V_1, V_2) splits (P, r, β). Now, suppose $m \geq 1$ and we have already constructed h_{m-1}. Let $Q = h_{m-1} \cdot [\nabla_\tau]_\phi$ and $Q_{ij} = \epsilon'_i Q \epsilon'_j$, so that $Q_{12}, Q_{21} \in P'^{-r+m}$. We will find $g = I + \epsilon'_1 X \epsilon'_2 + \epsilon'_2 Y \epsilon'_1 \in P'^m$ with $X \in V_{12}$ and $Y \in V'_{21}$ satisfying

$$(g \cdot Q)_{12} \equiv (g \cdot Q)_{21} \equiv 0 \pmod{\mathfrak{P}'^{-r+m+1}}. \tag{4.5}$$

The element $h_m = gh_{m-1} \in P^1$ will then have the desired properties.

Given g as above, the gauge change formula $g \cdot Q = gQg^{-1} - \tau(g)g^{-1}$ immediately leads to the equation

$$\begin{pmatrix} \mathrm{I} & X \\ Y & \mathrm{I} \end{pmatrix}\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} - \tau\begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} = \begin{pmatrix} (g \cdot Q)_{11} & (g \cdot Q)_{12} \\ (g \cdot Q)_{21} & (g \cdot Q)_{22} \end{pmatrix}\begin{pmatrix} \mathrm{I} & X \\ Y & \mathrm{I} \end{pmatrix}.$$

Since XQ_{21} and YQ_{12} lie in $\mathfrak{P}'^{-r+2m} \subset \mathfrak{P}'^{-r+m+1}$, the congruences (4.5) are equivalent to the system of congruences

$$\left.\begin{aligned} Q_{11} - (g \cdot Q)_{11} &\equiv 0 \\ Q_{22} - (g \cdot Q)_{22} &\equiv 0 \\ -\tau X + XQ_{22} - (g \cdot Q)_{11}X + Q_{12} &\equiv 0 \\ -\tau Y + YQ_{11} - (g \cdot Q)_{22}Y + Q_{21} &\equiv 0, \end{aligned}\right. \quad \pmod{\mathfrak{P}'^{-r+m+1}}$$

where the first two automatically hold for any g of the given form. Suppose that $r \geq 1$. In this case, τX and τY are in $\mathfrak{P}'^m \subset \mathfrak{P}' - r + m + 1$, so these terms drop out of the congruences. Substituting using the first two congruences, the problem is reduced to finding X and Y such that

$$\begin{aligned} Q_{11}X - XQ_{22} &\equiv Q_{12} \pmod{\mathfrak{P}'^{-r+m+1}} \\ Q_{22}Y - YQ_{11} &\equiv Q_{21} \pmod{\mathfrak{P}'^{-r+m+1}}. \end{aligned}$$

However, since $Q \equiv \beta'_\nu \pmod{\mathfrak{P}'^{-r+1}}$, the first equation is equivalent to $\partial_{\beta'}(X) \equiv Q_{12} \pmod{\mathfrak{P}'^{-r+m+1}}$, and a solution X exists since $(P_{12}, r, \partial_\beta)$ is strongly uniform. Similarly, Lemma 2.24 guarantees the existence of a solution Y to the second equation.

When $r = 0$, Lemma 4.11 shows that there exists $q \in P^1$ such that (qV_1, qV_2) splits a fundamental stratum $(\hat{P}, 0, \hat{\beta})$ with $\hat{P}^1 \subset P^1$ and $e_{\hat{P}} = 1$. We are thus in the classical case of lattice chains with period 1, and there exists $q' \in \hat{P}^1$ such that $(q'qV_1, q'qV_2)$ splits ∇ by [19, Lemma 2]. The desired element of P^1 is thus given by $p = q'q$. □

4.3. **Formal Types.** Suppose that (V, ∇) is a formal connection which contains a regular stratum. We fix a trivialization $\phi : V \to F^n$. In this section, we will show that the matrix of (V, ∇) in this trivialization can be diagonalized by a gauge transformation into a uniform torus $\mathfrak{t} \subset \mathfrak{gl}_n(F)$. The diagonalization of $[\nabla_\tau]$ determines a functional $A \in (\mathfrak{t}^0)^\vee$ called a *formal type*, and any two connections on V with the same formal type are isomorphic.

In the following, let (P, r, β) be a regular stratum in $\mathrm{GL}_n(F)$ with $P \subset \mathrm{GL}_n(\mathfrak{o})$, and let $T \subset \mathrm{GL}_n(F)$ centralize (P, r, β). We denote $\phi^{-1}P\phi \subset \mathrm{GL}(V)$ by P^ϕ, and write β^ϕ for the pullback of β to $(\bar{\mathfrak{P}}^\phi)^{-r}$. Suppose that (V, ∇) contains (P^ϕ, r, β^ϕ), and that β^ϕ is determined by $\mathrm{gr}^0(\nabla_\tau)$. (By Lemma 4.3, the second condition is superfluous when $r > 0$.) The goal of this section is the following theorem:

Theorem 4.13. *Fix ν of order -1. There exists $p \in P^1$ and a regular element $A_\nu \in \mathfrak{t}^{-r}$ such that $p \cdot [\nabla_\tau]_\phi = A_\nu$ and A_ν is a representative for β. Furthermore, the orbit of A_ν under P^1-gauge transformations contains $A_\nu + \mathfrak{P}^1$, and A_ν is unique modulo $\mathfrak{t}^1$.*

The obvious analogue of this theorem holds for an arbitrary ν.

Remark 4.14. The above theorem implies that after passing to a ramified cover (specifically, the splitting field for T), any connection containing a regular stratum is formally gauge equivalent to a direct sum of line bundles of slope less than or equal to r (with equality in all but at most one factor, with inequality only possible when $e = 1$). Moreover, the associated rank one connections have pairwise distinct leading terms. These properties could be used as an ad hoc way of defining the class of connections which are the primary topic of this paper. However, the perspective gained from our intrinsic approach via regular strata will prove essential below. The Lie-theoretic nature of this approach also suggests that it can be adapted to study flat G-bundles for G a reductive group.

We also remark that A_ν satisfies a stronger condition than regular semisimplicity. Suppose that $A_\nu = (a_1, \dots, a_{n/d}) \in E^{n/d}$, where $d = e_P$. Then, $a_j = a_{j,-r}\varpi_E^{-r} + a_{j,-r+1}\varpi_E^{-r+1} + \dots$, with $a_j \neq 0$ except possibly for a single j when $d = 1$. The leading term $A'_\nu = (a_1^{-r}\varpi_E^{-r+1}, \dots a_{n/d}^{-r}\varpi_E^{-r+1})$ is a representative for β_ν, since the higher order terms lie in $\mathfrak{P}^{-r+1}$. By Proposition 3.11, A'_ν is regular semisimple, and we see that A_ν has regular leading term.

In the following definition, let $T \subset \mathrm{GL}_n(F)$ be a uniform maximal torus such that $T(\mathfrak{o}) \subset \mathrm{GL}_n(\mathfrak{o})$. We set $P = P_{T,\mathfrak{o}^n}$ as defined before Proposition 3.12. We also allow ν to have arbitrary order.

Definition 4.15. A functional $A \in (\mathfrak{t}^0)^\vee$ is called a *T-formal type of depth r* if

1. $\mathfrak{t}^{r+1}$ is the smallest congruence ideal contained in $A^\perp$; and
2. the stratum (P, r, β) is regular and centralized by T, where $\beta \in (\bar{\mathfrak{P}}^r)^\vee$ is the functional induced by $\pi_{\mathfrak{t}}^*(A) \in \mathfrak{P}^\vee$.

We denote the space of T-formal types of depth r by $\mathcal{A}(T, r) \subset (\mathfrak{t}^0/\mathfrak{t}^{r+1})^\vee$. A T-formal type is any element of $\mathcal{A}(T) = \cup_{r \geq 0} \mathcal{A}(T, r)$.

We will always use the notation A_ν for a representative of A in $\mathfrak{t}^{-r-(1+\mathrm{ord}(\nu))e_P}$.

Remark 4.16. There is an embedding of $\mathcal{A}(T, r)$ into $\mathfrak{t}^{-r-(1+\mathrm{ord}(\nu))e_P}/\mathfrak{t}^{1-(1+\mathrm{ord}(\nu))e_P}$ determined by the pairing $\langle,\rangle_\nu$. For simplicity, we only describe it when $\nu = \frac{dt}{t}$. First, recall that $\mathfrak{t}$ has a natural grading so that $\mathfrak{t}^{-r}/\mathfrak{t}^1 \cong \oplus_{i=-r}^0 \bar{\mathfrak{t}}^i$. If $r > 0$, then $\mathcal{A}(T, r)$ is isomorphic to the open subspace of $\mathfrak{t}^{-r}/\mathfrak{t}^1$ with degree $-r$ term regular. If $r = 0$, then $\mathfrak{t}$ is split and $\mathfrak{t}^0/\mathfrak{t}^1 \cong \mathfrak{t}^\flat \cong k^n$. In this case, $\mathcal{A}(T, r)$ corresponds to $\sum a_i\chi_i \in \mathfrak{t}^\flat$ with the a_i's distinct modulo $\mathbb{Z}$. This is not a Zariski-open subset of $\mathfrak{t}^\flat$. However, if $k = \mathbb{C}$, it is open in the complex topology.

To be even more explicit, assume that $\mathfrak{t}$ is the block-diagonal Cartan subalgebra of $\mathfrak{gl}_n(F)$ as in Remark 3.13. If $\mathfrak{t}$ is split (and $r > 0$), there is a bijection between formal types A and representatives of the form $A_\nu = \sum_{i=0}^r t^{-i} D_i$ with $D_i \in \mathfrak{gl}_n(k)$ diagonal and D_r regular. In the pure case, there is a similar bijection between formal types and representatives $A_\nu = q(\varpi_I^{-1})$ where $q \in k[X]$ has degree r. Throughout Section 5, we will assume that $\mathfrak{t}$ has such a block diagonal embedding into $\mathfrak{gl}_n(F)$.

Remark 4.17. An element of $\mathcal{A}(T, r)$ may also be viewed as a functional on $\mathfrak{P}/\mathfrak{P}^{r+1}$ (resp. $\mathfrak{gl}_n(\mathfrak{o})^\vee$) for which the corresponding stratum (P, r, β) is regular and all of whose representatives lie in $\mathfrak{t}^{-r-(1+\mathrm{ord}(\nu))e_P} + \mathfrak{P}^{1-(1+\mathrm{ord}(\nu))e_P}$.

The notion of a T-formal type actually depends only on the conjugacy class of T. Indeed, set $L = \mathfrak{o}^n \subset F^n$. If T and S are conjugate tori with $T(\mathfrak{o}), S(\mathfrak{o}) \subset \mathrm{GL}_n(\mathfrak{o})$, then Lemma 4.18 below states that there exists $h \in \mathrm{GL}(L)$ such that ${}^hT = S$. It is evident that ${}^h\mathfrak{P}^j_{T,L} = \mathfrak{P}^j_{S,L}$ and ${}^h\mathfrak{t}^j = \mathfrak{s}^j$ for all j. Applying Remark 3.3, we conclude that $\mathrm{Ad}(h)(A_\nu) \in \mathfrak{s}^{-r}$ determines a regular stratum $(P_{S,L}, r, \beta')$ centralized by S.

We say that a lattice $L \subset V$ is *compatible* with T if $T(\mathfrak{o}) \subset \mathrm{GL}(L)$.

Lemma 4.18. *Suppose that T is a uniform maximal torus.*

(1) *The set of lattices L that are compatible with T is a single $N(T)$-orbit.*
(2) *If S is conjugate to T in $\mathrm{GL}(V)$, and L is compatible with both S and T, then S is conjugate to T in $\mathrm{GL}(L)$.*

Proof. Suppose that L and L' are compatible with T, and let $g \in \mathrm{GL}(V)$ satisfy $gL = L'$. In particular, this implies that $S = g^{-1}Tg$ is compatible with L. Choose $x \in \mathfrak{t}^{-r}$ with regular leading term. By Proposition 3.12, there exist parahoric subgroups $P_{T,L}, P_{S,L} \subset \mathrm{GL}(L)$ such that x and $\mathrm{Ad}(g^{-1})(x)$ determine regular strata (P_T, r, β) and (P_S, r, β'). By Theorem 3.8, $e_{P_T} = e_{P_S}$.

The same theorem states that (P_T, r, β) and (P_S, r, β') induce splittings of L, and it is easily checked that there exists an element of $h \in \mathrm{GL}(L)$ taking the components of the T-splitting to the S-splitting. Replacing g with gh, we may assume that the splittings induced by S and T are the same. Thus, we may reduce to the pure case.

Suppose that (P_T, r, β) and (P_S, r, β') are pure. We may choose $h' \in \mathrm{GL}(L)$ such that $h'P_T(h')^{-1} = P_S$, so by a similar argument, we may assume $P_T = P_S = P$. By (2.4), there exists $p \in P$ such that $\mathrm{Ad}(p)(\mathrm{Ad}(g^{-1})(x)) \in \mathfrak{t} + \mathfrak{P}^{-r+1}$. Finally, Lemma 3.18 implies that there exists $p' \in P^1$ such that $\mathrm{Ad}(p'pg^{-1})(x)) \in \mathfrak{t}$. It follows that $p'pg^{-1} = n^{-1} \in N(T)$. It is now clear that $nL = L'$, since p' and p are in $\mathrm{GL}(L)$.

We now prove the second statement. Suppose that $S = gTg^{-1}$ for $g \in \mathrm{GL}_n(F)$. Then, L and gL are compatible with S. By the first part, $gL = nL$ for some $n \in N(S)$. It follows that there exists $h = n^{-1}g \in \mathrm{GL}(L)$ such that $S = hTh^{-1}$. □

We continue to fix $T \subset \mathrm{GL}_n(F)$ as in Definition 4.15.

Definition 4.19. The set $\mathcal{A}_T^{(V,\nabla)} \subset \mathcal{A}(T)$ of T-formal types associated to (V, ∇) consists of those A for which there is a trivialization $\phi : V \to F^n$ such that (V, ∇) contains the stratum (P^ϕ, r, β^ϕ) and the matrix $[\nabla_\tau]_\phi$ is formally gauge equivalent to an element of $A_\nu + \mathfrak{P}^{1-(1+\mathrm{ord}(\nu))e_P}$ by an element of P^1.

By Theorem 4.13, the last statement is equivalent to the condition $[\nabla_\tau]_\phi$ is formally gauge equivalent to A_ν.

Proposition 4.20. *Let ∇ be a connection containing a T-formal type A_T. If S is a maximal torus with $S(\mathfrak{o}) \subset \mathrm{GL}_n(\mathfrak{o})$, then ∇ has an S-formal type if and only if S is $\mathrm{GL}_n(F)$-conjugate to T. Moreover, if $h \in \mathrm{GL}_n(\mathfrak{o})$ conjugates T to S, then $\mathrm{Ad}^*(h^{-1})$ gives a bijection from $\mathcal{A}_T^{(V,\nabla)}$ to $\mathcal{A}_S^{(V,\nabla)}$.*

Proof. Set $L = \mathfrak{o}^n$. By Lemma 4.18, there exists $h \in \mathrm{GL}_n(\mathfrak{o})$ such that $S = hTh^{-1}$. We may choose a trivialization $\phi : V \to F^n$ such that $[\nabla_\tau]_\phi = A_\nu \in \mathfrak{t}$ by Theorem 4.13. We now observe that, by Lemma 4.4,

$$h \cdot [\nabla_\tau]_\phi \in \mathrm{Ad}(h)(A_\nu) + t\,\mathfrak{gl}_n(\mathfrak{o}) \subset \mathfrak{s} + t\,\mathfrak{gl}_n(\mathfrak{o}).$$

After dualizing, A_ν and $\mathrm{Ad}(h)(A_\nu)$ determine functionals A_T and A_S (respectively) in $\mathfrak{gl}_n(\mathfrak{o})^\vee$ and $\mathrm{Ad}^*(h^{-1})(A_T) = A_S$. Moreover, A_S is an S formal type corresponding to (V, ∇): it is clear that any representative for A_S lies in $S = \mathrm{Ad}(h)(T) + \mathfrak{P}_S^1$, since $t\,\mathfrak{gl}_n(\mathfrak{o}) \subset \mathfrak{P}_S^1$, and (V, ∇) contains the regular stratum $(P_S^{h\phi}, r, \beta')$. Here, β' is the functional on $\mathfrak{P}_S^r/\mathfrak{P}_S^{r+1}$ determined by A_S. □

Before proving Theorem 4.13, we give two corollaries.

Corollary 4.21. *Suppose that $A_\nu \in \mathfrak{t}$ is a representative of a T-formal type. If $g \in \mathrm{GL}_n(F)$ satisfies $g \cdot A_\nu = A_\nu$, then $g \in T^\flat$.*

Proof. We assume without loss of generality that $\nu = \frac{dt}{t}$, so $\tau = t\frac{d}{dt}$. Since g is invertible, it will suffice to show that $g \in \mathfrak{t}^\flat$. Choose a regular stratum (P, r, β) corresponding to A, and consider the exhaustive filtration $\mathfrak{t}^\flat + \mathfrak{P}^i$ of $\mathfrak{gl}_n(F)$. Suppose that $g \notin \mathfrak{t}^\flat$, and let ℓ be the largest integer such that $g \in \mathfrak{t}^\flat + \mathfrak{P}^\ell$. By assumption, $[g, A_\nu] = \tau(g) \in \mathfrak{P}^\ell$.

First, assume that $r > 0$. Note that $\ell \neq 0$, since $g \in \mathfrak{P}$ implies that $g \in \mathfrak{t}^\flat + \mathfrak{P}^1$ by Proposition 3.5. Suppose $\ell < 0$. Corollary 3.7 gives $g = s + h$ with $s \in \mathfrak{t}^\ell$ and $h \in \mathfrak{P}^{\ell+r}$. Since $\pi_\mathfrak{t}([g, A_\nu]) \in \mathfrak{t}^{\ell+1}$ by Proposition 2.11(3), we also get $\pi_\mathfrak{t}(\tau(s)) \in \mathfrak{t}^{\ell+1}$. Lemma 4.26 now gives $s \in \mathfrak{P}^{\ell+1}$, a contradiction. Hence, $\ell \geq 1$, and we have $g = s_0 + x$ with $s_0 \in \mathfrak{t}^\flat$ and $x \in \mathfrak{P}^\ell$. Since $[x, A_\nu] = \tau(x) \in \mathfrak{P}^\ell$, we get the contradiction $x \in \mathfrak{P}^{\ell+1}$ by the same argument as in the $\ell < 0$ case.

When $r = 0$, we may assume A_ν is a regular diagonal matrix in $\mathfrak{gl}_n(k)$ satisfying the last condition of Definition 3.2. In other words, $-\mathrm{ad}(A_\nu)$ has no non-zero integer eigenvectors in $\mathfrak{gl}_n(k)$. Write $g = t^\ell g_\ell + t^{\ell+1} g_{\ell+1} + \ldots$ with $g_j \in \mathfrak{gl}_n(k)$. Then, $[g, A_\nu] = \tau(g)$ implies that $-\mathrm{ad}(A_\nu)(g_j) = j g_j$. We deduce that $g_j = 0$ except when $j = 0$. Moreover, since A_ν is regular, $[g_0, A_\nu] = 0$ implies that $g_0 \in \mathfrak{t}^\flat$. Thus, $g \in \mathfrak{t}^\flat$.

□

Corollary 4.22. *Let A be a formal type. Any two connections with formal type A are formally isomorphic. Furthermore, the set of formal types associated to a connection is independent of choice of $\nu \in \Omega^\times$.*

Proof. Independence of ν follows by the argument given in Proposition 4.2 and the remark above. Thus, fix ν with order -1. Suppose (V, ∇) and (V', ∇') have formal type A, and let ϕ (resp. ϕ') be the given trivialization for V (resp. V'). By Theorem 4.13, there exists $A_\nu \in \mathfrak{t}^{-r}$ and $p, p' \in P^1$ such that $p \cdot [\nabla_\tau]_\phi = A_\nu = p' \cdot [\nabla'_\tau]_{\phi'}$. It is easily checked that the composition $(\phi')^{-1} \circ (p')^{-1} \circ p \circ \phi : V \to V'$ takes ∇ to ∇'.

□

We begin the proof of Theorem 4.13. Throughout, we will suppress the fixed trivialization ϕ from the notation. We may assume that $\nu = \frac{dt}{t}$, so $\tau = t\frac{d}{dt}$. First, we show that if the result holds for the trivialization ϕ and the regular stratum (P, r, β) centralized by T, then, for any $g \in \mathrm{GL}_n(\mathfrak{o})$, it holds for the trivialization $g\phi$, the regular stratum $({}^gP, r, {}^g\beta)$, and its centralizing torus ${}^gT$.

Suppose that $g \in \mathrm{GL}_n(\mathfrak{o})$. By Lemma 4.4, $\tau(g)g^{-1} \in t\,\mathfrak{gl}_n(\mathfrak{o})$. In particular, $\tau(g)g^{-1} \in \mathfrak{P}^1$. Therefore, if the theorem holds for (P, r, β), then there exists $p \in P^1$ such that $p \cdot [\nabla_\tau] = A_\nu + \mathrm{Ad}(g^{-1})(\tau(g)g^{-1})$. It follows that ${}^gp \cdot (g \cdot [\nabla_\tau]) = g \cdot (p \cdot [\nabla_\tau]) = \mathrm{Ad}(g)(A_\nu)$. Thus, the first part of the theorem still holds after changing the trivialization by g. The second and third parts follows from a similar argument.

Without loss of generality, we henceforth assume that $\mathfrak{t}$ embeds into the $d \times d$ diagonal blocks of $\mathfrak{gl}_n(F)$ and in each diagonal block the matrix ϖ_I from (2.2) is a uniformizer for the corresponding copy of E.

By Theorem 3.8, this splitting of V splits (P, r, β) into pure strata, plus at most one non-fundamental stratum in the case $e_P = 1$. Therefore, Theorem 4.12 shows that ∇ splits into a direct sum of connections containing a pure stratum when $e_P > 1$ and into a direct sum of connections in dimension 1 when $e_P = 1$. Moreover, the splitting for ∇ maps to the splitting determined by T by an automorphism $p \in P^1$. In other words, $p \cdot [\nabla_\tau]$ lies in $\bigoplus_{j=1}^{n/d} \mathfrak{gl}(V_j)$.

First, we consider the case $e_P = 1$ (which includes the case $r = 0$). By the above discussion, we may reduce to the case where $\dim V = 1$. In this case, $[\nabla_\tau] \in F$ and $g \cdot [\nabla_\tau] \in [\nabla_\tau] + \mathfrak{p}^1$ for all $g \in 1 + \mathfrak{p}^1$. This proves the first statement and the statement about uniqueness. It suffices to show that the orbit of $[\nabla_\tau]$ under gauge transformations contains $[\nabla_\tau] + \mathfrak{p}^1$. Suppose $X \in \mathfrak{p}^1$. Since $\tau : \mathfrak{p}^1 \to \mathfrak{p}^1$ and $\log : (1 + \mathfrak{p}^1) \to \mathfrak{p}^1$ are surjective, there exists $g \in 1 + \mathfrak{p}^1$ such that $\tau(\log(g)) = X$. Therefore, $g \cdot ([\nabla_\tau] + X) = [\nabla_\tau]$, and the assertion follows.

When $e_P > 1$, it suffices to prove the theorem in the case when (P, r, β) is pure. In particular, $P = I$ is an Iwahori subgroup and $T \cong E^\times$. Take $\beta_\nu = [\nabla_\tau]$. By Remark 3.6, $\beta_\nu \in \mathfrak{t} \cap \mathfrak{I}^{-r} + \mathfrak{I}^{1-r}$.

The following two lemmas prove Theorem 4.13 in the pure case with $e_P > 1$ and thus complete the proof of the theorem.

Lemma 4.23. *Let ψ_ℓ be defined as in Section 2.3. When $\ell \geq 1$, $\tau(\mathfrak{I}^\ell) \subset \mathfrak{I}^\ell$ and $\psi_\ell(\tau(\varpi_I^\ell)) \neq 0$. Furthermore,*

$$\left[\tau(1 + \alpha\varpi_I^\ell)\right](1 + \alpha\varpi_I^\ell)^{-1} \equiv \alpha\tau(\varpi_I^\ell) \pmod{\mathfrak{I}^{\ell+1}}$$

for any $\alpha \in k$.

Proof. Suppose that $\ell = qn + z$, for $0 \leq z < n$. The matrix coefficients of ϖ_I^ℓ are

$$(\varpi_I^\ell)_{ij} = \begin{cases} t^{q+1} & \text{if } j = i + z - n; \\ t^q & \text{if } j = i + z; \\ 0 & \text{if } j \not\equiv i + z \pmod{n}. \end{cases}$$

Let x be the diagonal matrix with $x_{jj} = q$ when $j \leq n - z$ and $q + 1$ otherwise. Then, $\tau(\varpi_I^\ell) = x\varpi_I^\ell$. Moreover, $\tau(\mathfrak{I}) \subset \tau(\mathfrak{gl}_n(\mathfrak{o})) \subset t\,\mathfrak{gl}_n(\mathfrak{o}) \subset \mathfrak{I}$. The Leibniz rule and the fact that $\mathfrak{I}^\ell = \varpi_I^\ell \mathfrak{I}$ now imply that $\tau(\mathfrak{I}^\ell) \subset \mathfrak{I}^\ell$ for all $\ell \geq 1$. The first assertion of the lemma follows, since $\psi_\ell(\tau(\varpi_I^\ell))$ is the trace of x, which is non-zero for $\ell \neq 0$.

To see the second statement, observe that $(1+\alpha\varpi_I^\ell)^{-1} = 1-\alpha\varpi_I^\ell + y$, with $y \in \mathfrak{I}^{\ell+1}$. Therefore,

$$\begin{aligned}\left[\tau(1+\alpha\varpi_I^\ell)\right](1+\alpha\varpi_I^\ell)^{-1} &= \left[\tau(1+\alpha\varpi_I^\ell)\right](1-\alpha\varpi_I^\ell+y)\\ &= \alpha\tau(\varpi_I^\ell)(1-\alpha\varpi_I^\ell+y)\\ &\equiv \alpha\tau(\varpi_I^\ell) \pmod{\mathfrak{I}^{\ell+1}}.\end{aligned}$$

□

Lemma 4.24. *Suppose that (V,∇) contains the pure stratum (I,r,β) with $n\geq 2$ (so $r\geq 1$). Then, there is a unique $q(x)\in k[x]$ such that $[\nabla_\tau]$ is formally gauge equivalent to $q(\varpi_I^{-1})$ by an element of I^1. If $B_\nu \in q(\varpi_I^{-1})+\mathfrak{I}^1$, then B_ν is formally gauge equivalent to $[\nabla_\tau]$ by an element of I^1.*

Proof. By the remarks made before Lemma 4.23, $[\nabla_\tau]=\beta_\nu = q_r\varpi_I^{-r}+y$ with $y\in\mathfrak{I}^{-r+1}$. Moreover, since $\beta_\nu\notin\mathfrak{I}^{-r+1}$, $q_r\neq 0$. We need to find $p\in I^1$ with the property

$$p\cdot\beta_\nu = q(\varpi_I^{-1}), \tag{4.6}$$

for $q\in k[x]$ as in the statement of the lemma.

Inductively, we construct $g_\ell\in I^1$ and $q^\ell\in k[x]$ of degree r such that $g_\ell\equiv g_{\ell-1} \pmod{\mathfrak{I}^{\ell-1}}$, $\deg(q^\ell-q^{\ell-1})\leq r-\ell+1$, and $g_\ell\cdot\beta_\nu\in q^\ell(\varpi_I^{-1})+\mathfrak{I}^{\ell-r}$. Moreover, we will show that $q^\ell(\varpi_I^{-1})$ is unique modulo $\mathfrak{t}^{\ell-r}$. Note that q^ℓ is independent of ℓ for $\ell>r+1$. If we set $p=\lim g_\ell$ and $q=q^\ell$ for large ℓ, (4.6) is satisfied.

We start by taking $g_1=1$ and $q_1=q_rx^r$. Suppose that we have constructed g_ℓ and q^ℓ; note that $q_r^\ell=q_r$. We will find $g=1+X\in I^\ell$ such that $g_{\ell+1}=gg_\ell\in I^1$ has the required properties. Obviously, $g_{\ell+1}\equiv g_\ell \pmod{\mathfrak{I}^\ell}$.

To construct g, first, note that $\tau(g)g^{-1}=\tau(X)g^{-1}\in\mathfrak{I}^\ell$ by Lemma 4.23. Moreover, $g^{-1}\equiv 1-X \pmod{\mathfrak{I}^{\ell+1}}$. If $\ell-r\leq 0$, it suffices to find $g\in I^\ell$ such that

$$\mathrm{Ad}(g)(g_\ell\cdot\beta_\nu)\in\mathfrak{t}+\mathfrak{I}^{-r+\ell+1}.$$

We see that

$$\begin{aligned}\mathrm{Ad}(g)(g_\ell\cdot\beta_\nu) &\equiv (1+X)(g_\ell\cdot\beta_\nu)(1-X) \pmod{\mathfrak{I}^{\ell-r+1}}\\ &\equiv g_\ell\cdot\beta_\nu + q_r\delta_X(\varpi_I^{-r}) \pmod{\mathfrak{I}^{\ell-r+1}}.\end{aligned}$$

Thus, we need to solve the equation

$$g_\ell\cdot\beta_\nu + q_r\delta_X(\varpi_I^{-r})\equiv Y \pmod{\mathfrak{I}^{\ell-r+1}}.$$

for $Y\in\mathfrak{t}$. Since $q_r\neq 0$, Proposition 2.11(3) implies that a solution for X exists if and only if $Y\in\pi_{\mathfrak{t}}(g_\ell\cdot\beta_\nu)+\mathfrak{I}^{\ell-r+1}$. Letting $q^{\ell+1}(\varpi_I^{-1})$ denote the terms of nonpositive degree in $\pi_{\mathfrak{t}}(g_\ell\cdot\beta_\nu)$ (where $q^{\ell+1}\in k[x]$), we see that $\deg(q^{\ell+1}-q^\ell)\leq r-l$. Moreover, $q^{\ell+1}$ is uniquely determined.

Now, suppose $\ell-r>0$. The first part of Lemma 4.23 implies that $\pi_{\mathfrak{t}}(\tau(\varpi_I^{\ell-r}))\notin\mathfrak{I}^{\ell-r+1}$. The argument above implies that we may choose $s\in I^\ell$ with the property

$$\mathrm{Ad}(s)(g_\ell\cdot\beta_\nu)\equiv q^\ell(\varpi_I^{-1})+\alpha\pi_{\mathfrak{t}}\tau(\varpi_I^{\ell-r}) \pmod{\mathfrak{I}^{\ell-r+1}}.$$

for some $\alpha\in k$. Again, Proposition 2.11 implies that there exists $h\in I^\ell$ such that

$$\mathrm{Ad}(h)(q^\ell(\varpi_I^{-1})+\alpha\tau(\varpi_I^{\ell-r}))\equiv q^\ell(\varpi_I^{-1})+\alpha\pi_{\mathfrak{t}}\tau(\varpi_I^{\ell-r}) \pmod{\mathfrak{I}^{\ell-r+1}}.$$

Thus, by the second part of Lemma 4.23,

$$\mathrm{Ad}(h^{-1}s)(g_\ell\cdot\beta_\nu)\equiv q^\ell(\varpi_I^{-1})+\left[\tau(1+\alpha\varpi_I^{\ell-r})\right](1+\alpha\varpi_I^{\ell-r})^{-1} \pmod{\mathfrak{I}^{\ell-r+1}}.$$

Since $1+\alpha\varpi_I^{\ell-r}$ commutes with $q^\ell(\varpi_I^{-1})$ and $\tau(h^{-1}s)s^{-1}h \in \mathfrak{I}^\ell \subset \mathfrak{I}^{\ell-r+1}$, it follows that

$$(1+\alpha\varpi_I^\ell)\cdot\left[(h^{-1}s)\cdot(g_\ell\cdot\beta_\nu)\right] \equiv q^\ell(\varpi_I^{-1}) \pmod{\mathfrak{I}^{\ell-r+1}}.$$

Setting $g_{\ell+1} = (1+\alpha\varpi_I^\ell)h^{-1}s$ and $q^{\ell+1} = q^\ell$ completes the induction.

The same inductive argument (beginning with $\ell = r+1$) shows that for any $B_\nu \in q(\varpi_I^{-1}) + \mathfrak{I}^1$, there exists $h \in I^\ell$ such that $h \cdot B_\nu = q(\varpi_I^{-1})$. This completes the proof of the second statement of the lemma.

□

4.4. **Formal Types and Formal Isomorphism Classes.** In this section, we describe the relationship between formal types and isomorphism classes of formal connections. In particular, we show that formal types are the isomorphism classes in the category of *framed formal connections*. This category is the disjoint union of the categories of T-*framed formal connections* as T runs over conjugacy classes of uniform maximal tori. Moreover, there is an action of the relative affine Weyl group of T on the set of T-formal types, and the forgetful functor to the category of formal connections sets up a bijection between orbits of T-formal types and isomorphism classes of formal connections containing a regular stratum of the form (P_T, r, β). We also exhibit an intermediate category whose isomorphism classes correspond to relative Weyl group orbits.

Given a conjugacy class of uniform maximal tori and a fixed lattice L, we can choose a representative T such that $T(\mathfrak{o}) \subset \mathrm{GL}(L)$. Setting $P = P_{T,L}$, we will further have $T(\mathfrak{o}) \subset P \subset \mathrm{GL}(L)$ and $T \cong (E^\times)^{n/e_P}$ with E/F a degree e_P ramified extension. Upon choosing a basis for L, we can assume without loss of generality that $T(\mathfrak{o}) \subset P \subset \mathrm{GL}_n(\mathfrak{o})$ and that T is the standard block diagonal torus described in Remark 3.13. Throughout this section, we will fix a form $\nu = \frac{dt}{t}$ and the corresponding derivation $\tau = t\frac{d}{dt}$.

Let $W_T = N(T)/T$ and $W_T^{\mathrm{aff}} = N(T)/T(\mathfrak{o})$ be the relative Weyl group and the relative affine Weyl group associated to T. Note that W_T^{aff} is a semi-direct product of W_T with the free abelian group $T/T(\mathfrak{o})$, i.e., $W_T^{\mathrm{aff}} \cong W_T \ltimes T/T(\mathfrak{o})$. Furthermore, if we write Σ_{n/e_P} for the group of permutations on the $E^\times$-factors of T and C_{e_P} for the Galois group of E/F, then $W_T \cong \Sigma_{n/e_P} \ltimes (C_{e_P}^{n/e_P})$. Here, Σ_{n/e_P} acts on $C_{e_P}^{n/e_P}$ by permuting the factors. We note that $N(T) \cap \mathrm{GL}_n(\mathfrak{o}) \subset P_{T,\mathfrak{o}^n}$, since C_{e_P} and Σ_{n/e_P} both preserve the filtration determined in Proposition 3.12.

Any element of W_T has a representative in $\mathrm{GL}_n(k) \subset \mathrm{GL}_n(F)$. Therefore, $W_T \cong (N(T) \cap \mathrm{GL}_n(k))/T^\flat$. In fact, W_T can be embedded as a subgroup of $\mathrm{GL}_n(k)$ as follows. The centralizer of $T^\flat$ in $\mathrm{GL}_n(k)$ is a Levi subgroup isomorphic to $\prod_{i=1}^{n/e_P} \mathrm{GL}_{e_P}(k)$.

Let D_i (resp. $\mathfrak{d}_i$) denote the diagonal subgroup (resp. subalgebra) in each component. Fix a primitive e_P^{th} root of unity ξ. We view Σ_{n/e_p} as the subgroup of permutation matrices that permute the factors of this Levi subgroup while the i^{th} copy of C_{e_P} maps to the cyclic subgroup of D_i generated by $\mathrm{diag}(1,\xi,\xi^2,\dots,\xi^{e_P-1})$.

We now define an action ϱ of W_T^{aff} on $\mathcal{A}(T,r)$. Taking $w \in \mathrm{GL}_n(k)$ a representative for $wT \in W_T$, $s = (s_1,\dots,s_{n/e_P}) \in T$, and $A \in (\mathfrak{t}^0/\mathfrak{t}^{r+1})^\vee$, we obtain actions of W_T and $T(F)/T(\mathfrak{o})$ on $(\mathfrak{t}^0/\mathfrak{t}^{r+1})^\vee$ via

$$\varrho(wT)(A) = \mathrm{Ad}^*(w)(A)$$
$$\varrho(sT(\mathfrak{o}))(A) = A - \sum_{i=1}^{n/e_P} \frac{\deg_E s_i}{e_P}\chi_i^\vee.$$

Here, $\chi_i^\vee$ is the functional induced by $\chi_i \frac{dt}{t}$, where χ_i is the identity of the i^{th} component of $\mathfrak{t}$. It is easy to see that these two actions give rise to a unique action of W_T^{aff}.

To check that this action restricts to an action on $\mathcal{A}(T,r)$, consider the action of W_T^{aff} on $\mathfrak{t}$ defined by the similar formulas $\varrho_\nu(wT)(x) = \mathrm{Ad}(w)(x)$ and $\varrho_\nu(s)(x) = x - \sum_{i=1}^{n/e_P} \frac{\deg_E s_i}{e_P}\chi_i$. The induced action on $\mathfrak{t}^{-r}/\mathfrak{t}^1$ corresponds to ϱ under the isomorphism $\mathfrak{t}^{-r}/\mathfrak{t}^1 \cong (\mathfrak{t}^0/\mathfrak{t}^{r+1})^\vee$ determined by $\nu = \frac{dt}{t}$, and for any $\hat{w} \in W_T^{\mathrm{aff}}$, $\varrho_\nu(\hat{w})(A_\nu)$ is a representative for $\rho(\hat{w})(A)$. If $A \in \mathcal{A}(T,r)$, then the leading term of A_ν is regular, with distinct eigenvalues modulo $\mathbb{Z}$ when $r = 0$. If $r > 0$, it is clear that $\varrho_\nu(\hat{w})(A_\nu)$ also has regular leading term. If $r = 0$, then the action permutes and adds integers to the eigenvalues, so again the condition for being a formal type is preserved.

Proposition 4.25. *Suppose that $g \in N(T)$ and $A \in \mathcal{A}(T,r)$. If $A_\nu \in \mathfrak{t}$ is a representative for A, then $g \cdot A_\nu$ is formally gauge equivalent to $\varrho_\nu(gT(\mathfrak{o}))(A_\nu)$ by an element of P^1. Furthermore, $\varrho_\nu(gT(\mathfrak{o}))(A_\nu) \in \pi_{\mathfrak{t}}(g \cdot A_\nu) + \mathfrak{t}^1$.*

Proof. The case when $r = 0$ is easily checked since T is the usual split torus, so we assume that $r > 0$. First, consider the case $g = s \in T$, so $s \cdot A_\nu = A_\nu - (\tau s)s^{-1}$. Recall that the intersection of P with the block-diagonal Levi subgroup is a product of Iwahori subgroups $I_i \subset \mathrm{GL}_{e_P}(F)$. Each P_i determines an ordering on the roots of D_i. We take $H_i \in \mathfrak{d}_i$ to be the half sum of positive coroots, and $H = (H_1, \ldots, H_{n/e_P}) \in \mathfrak{P}$.

Lemma 4.26. *Suppose that $s \in \mathfrak{t}^r$. If H is defined as above, then*

$$\tau(s) + \frac{1}{e_P}\mathrm{ad}(H)(s) - \frac{r}{e_P}s \in \mathfrak{P}^{1+r}.$$

Moreover, $\pi_{\mathfrak{t}}(\tau(s)) \in \frac{r}{e_P}s + \mathfrak{P}^{1+r}$.

Proof. The first statement follows from the observation $\tau(\varpi_E^i) = \frac{i}{e_P}\varpi_E^i - \frac{1}{e_P}\mathrm{ad}(H_j)(\varpi_E^i)$. We then obtain the second statement from Proposition 2.11(3). □

Setting $s = (s_1, \ldots, s_{n/e_P})$, the lemma gives

$$(\tau s)s^{-1} \in \sum_{i=1}^{n/e_P} \frac{\deg_E s_i}{e_P}\chi_i - \frac{1}{e_P}\mathrm{ad}(H)(s)s^{-1} + \mathfrak{P}^1.$$

Observe that each term on the right of this expression lies in $\mathfrak{P}$. Applying Proposition 2.11, we obtain $X \in \mathfrak{P}^r$ such that $\mathrm{ad}(X)(A_\nu) \in \pi_{\mathfrak{t}}((\tau s)s^{-1}) - (\tau s)s^{-1} + \mathfrak{P}^1$. Taking $h = 1 - X$, we see that $h \cdot (s \cdot A_\nu) \in \pi_{\mathfrak{t}}(s \cdot A_\nu) + \mathfrak{P}^1$. By Theorem 4.13, it follows that $s \cdot A_\nu$ is gauge equivalent to $\pi_{\mathfrak{t}}(s \cdot A_\nu)$.

Since $\pi_{\mathfrak{t}}$ is a $\mathfrak{t}$-bimodule map, we deduce from Lemma 2.10 that $\pi_{\mathfrak{t}}(\mathrm{ad}(H)(s)s^{-1}) \in \mathfrak{t}^1$. Therefore, $\pi_{\mathfrak{t}}(A_\nu - (\tau s)s^{-1}) \in \varrho_\nu(sT(\mathfrak{o}))(A_\nu) + \mathfrak{t}^1$. Note that if $s \in T(\mathfrak{o})$, Lemma 4.4 implies that $(\tau s)s^{-1} \in \mathfrak{P}^1$; thus, $T(\mathfrak{o})$ does not affect the P^1-gauge equivalence class.

For the general case, take $g = sn$ with $n \in N(T) \cap \mathrm{GL}_n(k)$ and $s \in T$. The result now follows by applying the case above to the formal type $\varrho_\nu(nT(\mathfrak{o})(A_\nu) = \mathrm{Ad}(n)(A_\nu) = n \cdot A_\nu = \pi_{\mathfrak{t}}(n \cdot A_\nu) \in \mathcal{A}(T, r)$.

□

Lemma 4.27. *Suppose that $A, A' \in \mathcal{A}(T, r)$ with representatives $A_\nu, A'_\nu \in \mathfrak{t}$. If A_ν and A'_ν are $\mathrm{GL}_n(F)$-gauge equivalent modulo $\mathfrak{t}^1$, then there exists a unique $\hat{w} \in W_T^{\mathrm{aff}}$ such that $\varrho(\hat{w})(A) = A'$.*

Proof. Since W_T^{aff} acts freely on $\mathcal{A}(T, r)$, it suffices to show existence of $\hat{w}$. First, take $r = 0$, so $\mathfrak{t}$ is the space of diagonal matrices. Write $A_\nu(0)$ for the image of A_ν under the evaluation map $t \mapsto 0$. Then, A_ν and A'_ν are gauge equivalent modulo t if and only if there exists $f \in \mathrm{GL}_n(\mathbb{C})$ such that $\mathrm{Ad}(f)\exp(2\pi i A_\nu(0)) = \exp(2\pi i A'_\nu(0))$. (This follows from the Riemann-Hilbert correspondence and [24, Theorem 5.5 and Section 17.1]). Therefore, f lies in the normalizer of T, and $\mathrm{Ad}(f)(A_\nu(0))$ differs from $A'_\nu(0)$ by a diagonal matrix with integer entries. In particular, A and A' lie in the same W_T^{aff} orbit.

Now, assume $r > 0$. Suppose that $h \cdot A'_\nu = A_\nu + x$ for some $x \in \mathfrak{P}^1$. Fix a split torus D with $D(\mathfrak{o}) \subset P$. Using the affine Bruhat decomposition, we may write $h = p_1 n p_2$, where $p_1, p_2 \in P$ and $n \in N(D)$. We see that

$$A_\nu + x = \mathrm{Ad}(h)(A'_\nu - p_2^{-1}\tau p_2) - (\tau p_1)p_1^{-1} - \mathrm{Ad}(p_1)((\tau n)n^{-1}).$$

By Lemma 4.4, $p_2^{-1}\tau p_2$ and $(\tau p_1)p_1^{-1}$ both lie in $\mathfrak{P}^1$. Moreover, there exists $d \in D$ and $\sigma \in N(D) \cap \mathrm{GL}_n(\mathbb{C})$ such that $n = d\sigma$, so $(\tau n)n^{-1} = (\tau d)d^{-1} \in \mathfrak{d}(\mathfrak{o}) \subset \mathfrak{P}$. In particular, $\mathrm{Ad}(h)(A'_\nu - p_2^{-1}\tau p_2) \in A_\nu + \mathrm{Ad}(p_1)((\tau d)d^{-1}) + \mathfrak{P}^1$. By Lemma 3.18, there exist $q_1 \in P^r$ and $q_2 \in P^{r+1}$ such that $\mathrm{ad}(q_2)(A'_\nu - p_2^{-1}\tau p_2)$ and $\mathrm{ad}(q_1^{-1})(A_\nu + x + (\tau p_1)p_1^{-1} + \mathrm{Ad}(p_1)((\tau n)n^{-1}))$ lie in $\mathfrak{t}$. Since $A'_\nu - p_2^{-1}\tau p_2$ is regular by Proposition 3.11, it follows that $h \in q_1 N(T) q_2$.

Set $g = q_1^{-1} h q_2^{-1}$ and $\hat{w} = gT(\mathfrak{o})$. We will show that $\varrho(\hat{w})(A) = A'$. The element $A_\nu^1 = \pi_{\mathfrak{t}}(q_2 \cdot A_\nu) \in \mathfrak{t}$ is a valid representative for A, since $\pi_{\mathfrak{t}}(\mathrm{Ad}(q_2)(A_\nu)) \in A_\nu + \mathfrak{P}^1$ and $(dq_2)q_2^{-1} \in \mathfrak{P}^1$. Moreover, the fact that $\pi_{\mathfrak{t}}$ is a $N(T)$-map implies that $\pi_{\mathfrak{t}}((gq_2) \cdot A_\nu) = \pi_{\mathfrak{t}}(g \cdot A_\nu^1)$.

By Proposition 4.25, $\pi_{\mathfrak{t}}(g \cdot A_\nu^1) \in \varrho_\nu(\hat{w})(A_\nu^1) + \mathfrak{t}^1$, so $\pi_{\mathfrak{t}}((gq_2) \cdot A_\nu) \in \varrho_\nu(\hat{w})(A_\nu^1) + \mathfrak{t}_1$. Write $q_1 = 1 + X$ for $X \in \mathfrak{P}^r$ so that $(gq_2) \cdot A_\nu \in A'_\nu - \mathrm{ad}(X)(A'_\nu) + \mathfrak{P}^1$. It follows that $\pi_{\mathfrak{t}}((gq_2) \cdot A_\nu) \in A'_\nu + \mathfrak{t}^1$ by Proposition 2.11(3). Thus, $\hat{w} = gT(\mathfrak{o})$ satisfies the Lemma. □

Theorem 4.28. *Suppose (V, ∇) is a formal connection. If $A \in \mathcal{A}_T^{(V,\nabla)} \cap \mathcal{A}(T, r)$ and $A' \in \mathcal{A}_{T'}^{(V,\nabla)} \cap \mathcal{A}(T', r')$, then T and T' are $\mathrm{GL}_n(\mathfrak{o})$-conjugate and $r = r'$. Moreover, if $h \in \mathrm{GL}_n(\mathfrak{o})$ satisfies ${}^hT = T'$, then there exists a unique $\hat{w} \in W_T^{\mathrm{aff}}$ such that $A' = \mathrm{Ad}^*(h^{-1})\varrho(\hat{w})(A)$.*

Proof. Proposition 4.20 shows that $r = r'$ and allows us to assume without loss of generality that $T = T'$. By definition of formal types, any choice of representatives A_ν and A'_ν are formally gauge equivalent modulo $\mathfrak{P}^1$. The theorem now follows from Lemma 4.27.

□

Corollary 4.29. *Suppose that $A \in \mathcal{A}_T^{(V,\nabla)}$. Let ϕ be an associated trivialization, let (P^ϕ, r, β^ϕ) be the associated stratum (in $\mathrm{GL}(V)$), and let $L = \phi^{-1}(\mathfrak{o}^n)$. Suppose $A' \in \mathcal{A}_T^{(V,\nabla)}$ has associated trivialization ϕ', and choose $\hat{w} \in W_T^{\mathrm{aff}}$ such that $A' = \varrho(\hat{w})(A)$.*

1. *If $\phi'^{-1}(\mathfrak{o}^n) = L$, then $\hat{w} \in W_T$.*
2. *If, in addition, $(P^\phi, r, \beta^\phi) = (P^{\phi'}, r, \beta^{\phi'})$, then $\hat{w}$ is the identity.*

Proof. By Lemma 4.27, there exists $\hat{w} \in W_T^{\mathrm{aff}}$ such that $\varrho(\hat{w})(A) = A'$. Recall that $W_T^{\mathrm{aff}} \cong T/T(\mathfrak{o}) \ltimes W_T$, and $W_T \cong (N(T) \cap \mathrm{GL}_n(k))/T^\flat$. Observe that the set of trivializations of satisfying $\phi'^{-1}(\mathfrak{o}^n) = L$ is a single $\mathrm{GL}_n(\mathfrak{o})$-orbit. Moreover, $P^\phi = P^{\phi'}$ by Proposition 3.12. Since P is its own normalizer in $\mathrm{GL}_n(\mathfrak{o})$, it follows that A_ν and A'_ν are gauge equivalent by an element $p \in P$ in both cases.

In the second case, the stabilizer of $\beta + \mathfrak{P}^{1-r}$ in P is equal to $P^1 T^\flat$. Without loss of generality, take $p \in P^1$. Now, the uniqueness statement in Theorem 4.13 implies that $A = A'$.

In the first case, as long as $r > 0$, we have $A'_\nu \in \mathrm{Ad}(n)(A_\nu) + \mathfrak{P}^{-r+1}$, where $n \in N(T)$ is a representative for $\hat{w}$. Since $n \in mT(\mathfrak{o})$ for some $m \in N(T) \cap \mathrm{GL}_n(k) \subset P$, we also $A'_\nu \in \mathrm{Ad}(m)(A_\nu) + \mathfrak{P}^{-r+1}$. We deduce that $p \in mP^1$, and the same uniqueness argument shows that $A'_\nu = \mathrm{Ad}(m)(A_\nu)$. This implies that $A' = \varrho(mT(\mathfrak{o}))(A)$, and simple transitivity of the W_T^{aff}-action gives $\hat{w} = \varrho(mT(\mathfrak{o})) \in W_T$. If $r = 0$, then let $m \in \mathrm{GL}_n(k)$ be the image of p modulo t. Since $A'_\nu \in \mathrm{Ad}(p)(A_\nu) + t\,\mathfrak{gl}_n(\mathfrak{o})$, we have $A'_\nu = \mathrm{Ad}(m)(A_\nu)$. We conclude that $\hat{w} = \varrho(mT(\mathfrak{o})) \in W_T$ as before.

□

We can now make precise the relationship between formal types and formal isomorphism classes in terms of moduli spaces of certain categories of formal connections. In the following, we consider three related categories of formal connections. We define $\mathscr{C}$ to be the full subcategory of formal connections (V, ∇) of rank n such that (V, ∇) contains a regular stratum. Let $\mathscr{C}^{lat}$ be the category of triples (V, ∇, L), where $V \in \mathscr{C}$ and $L \subset V$ is a distinguished $\mathfrak{o}$-lattice such such that $P \subset \mathrm{GL}(L)$ for some regular stratum (P, r, β) contained in V. Morphisms in $\mathrm{Hom}_{\mathscr{C}^{lat}}((V, \nabla, L), (V', \nabla', L'))$ in $\mathscr{C}^{lat}$ consist of homomorphisms $\phi : V \to V'$ (in the category $\mathscr{C}$) such that $L' \cap \phi(V) = \phi(L)$. Note that if ϕ is an isomorphism, this implies that $\phi(L) = L'$. Finally, $\mathscr{C}^{fr}$ is the category of *framed* connections. This consists of objects in $\mathscr{C}^{lat}$ where $\mathrm{Hom}_{\mathscr{C}^{fr}}((V, \nabla, L), (V', \nabla', L'))$ is the set of isomorphisms $\phi \in \mathrm{Hom}_{\mathscr{C}^{lat}}((V, \nabla, L), (V', \nabla', L'))$ such that $(\phi^{-1}(P'), r, \phi^*(\beta')) = (P, r, \beta)$.

Fix a uniform torus $T \subset \mathrm{GL}_n(F)$ satisfying $T(\mathfrak{o}) \subset \mathrm{GL}_n(\mathfrak{o})$ and an integer $r \geq 0$. We denote the full subcategory of $\mathscr{C}$ (resp. $\mathscr{C}^{lat}$, $\mathscr{C}^{fr}$) of connections that have formal type in $\mathcal{A}(T, r)$ by $\mathscr{C}(T, r)$ (resp. $\mathscr{C}^{lat}(T, r)$, $\mathscr{C}^{fr}(T, r)$). Proposition 4.20 implies that the subcategory $\mathscr{C}(T, r)$ only depends on the conjugacy class of T in $\mathrm{GL}_n(F)$. It follows from Theorem 4.28 that the set of objects in $\mathscr{C}$ is the disjoint union of objects in $\mathscr{C}(T_i, r)$, taken over a set of representatives T_i as above for the conjugacy classes of uniform tori. The analogous statement holds for $\mathscr{C}^{lat}$ and $\mathscr{C}^{fr}$.

Corollary 4.30. *Fix a uniform torus $T \subset \mathrm{GL}_n(F)$ with $T(\mathfrak{o}) \subset \mathrm{GL}_n(\mathfrak{o})$ and $r \in \mathbb{Z}^{\geq 0}$. Then, $\mathcal{A}(T, r)$ is the moduli space for $\mathscr{C}^{fr}(T, r)$, $\mathcal{A}(T, r)/W_T$ is the moduli space for $\mathscr{C}^{lat}(T, r)$, and $\mathcal{A}(T, r)/W_T^{\mathrm{aff}}$ is the moduli space for the category $\mathscr{C}(T, r)$.*

5. Moduli Spaces

In this section, we will describe the moduli space $\mathscr{M}(A^1,\dots,A^m)$ of 'framed' connections on $C=\mathbb{P}^1(\mathbb{C})$ with singular points $\{x_1,\dots,x_m\}$ and formal type A^i at x_i. In our explicit construction, we show that this moduli space is the Hamiltonian reduction of a symplectic manifold via a torus action.

Set $k=\mathbb{C}$. We denote by $F_x\cong F$ the field of Laurent series at $x\in C$ and $\mathfrak{o}_x\subset F_x$ the ring of power series. Let V be a trivializable rank n vector bundle on $\mathbb{P}^1$; thus, there is a noncanonical identification of V with the trivial rank n vector bundle $V^{\mathrm{triv}}\cong\mathcal{O}^n_{\mathbb{P}^1}$. The space of global trivializations of V is a $\mathrm{GL}_n(\mathbb{C})$-torsor, so we will fix a base point and identify each trivialization ϕ with an element $g\in\mathrm{GL}_n(\mathbb{C})$. Thus, we will write $[\nabla]$ for the matrix of ∇ in the fixed trivialization, and $g\cdot[\nabla]$ for $[\nabla]_\phi$.

Define $V_x=V\otimes_{\mathcal{O}_C}F_x$ and $L_x=V\otimes_{\mathcal{O}_C}\mathfrak{o}_x$. The inclusion $V_{\mathbb{C}}=\Gamma(\mathbb{P}^1;V)\subset V_x$ gives V_x a natural $\mathbb{C}$-structure. Furthermore, L_x determines a unique maximal parahoric $G_x=\mathrm{GL}(L_x)\cong\mathrm{GL}_n(\mathfrak{o})$. In particular, by the remarks preceding Lemma 2.6, there is a one-to-one correspondence between parahoric subgroups $P\subset G_x$ and parabolic subgroups $Q\subset\mathrm{GL}(V_{\mathbb{C}})$, where $Q=P/G^1_x$.

Let T_x be a uniform torus in $\mathrm{GL}_n(F_x)$ such that $T_x(\mathfrak{o}_x)\subset\mathrm{GL}_n(\mathfrak{o}_x)$, and set $P_x=P_{T_x,\mathfrak{o}^n_x}$. In the following, (V,∇) is a connection on C, and $A_x\in\mathcal{A}(T_x,r)$ is a formal type associated to (V,∇) at x. This means that the formal completion (V_x,∇_x) at x has formal type A_x. We denote the corresponding $\mathrm{GL}_n(F_x)$-stratum by (P_x,r,β_x). We may assume, by Proposition 4.20, that T_x has a block-diagonal embedding in $\mathrm{GL}_n(F_x)$ as in Remark 4.16. We write $U_x=P^1_x/G^1_x$. Furthermore, if $g\in\mathrm{GL}_n(\mathbb{C})$, $P^g_x\subset\mathrm{GL}(V)$ and β^g_x are the pullbacks of P and β, respectively, under the corresponding trivialization (as in Section 4.3).

Definition 5.1. A *compatible framing* for ∇ at x is an element $g\in\mathrm{GL}_n(\mathbb{C})$ with the property that ∇ contains the $\mathrm{GL}(V_x)$-stratum (P^g_x,r,β^g_x) defined above. We say that ∇ is *framable* at x if there exists such a g.

For example, suppose that $e_{P_0}=1$. Choose $\nu\in\Omega^\times_0$ of order -1. By Remark 4.16, $A_{0\nu}=\frac{1}{t^r}D_r+\frac{1}{t^{r-1}}D_{r-1}+\dots$ where $D_j\in\mathrm{GL}_n(\mathbb{C})$ are diagonal matrices and D_r is regular. It follows that g is a compatible framing for (V,∇) at 0 if and only if

$$g\cdot[\nabla]=\frac{1}{t^r}D_r\nu+\frac{1}{t^{r-1}}M_{r-1}\nu+\dots,$$

with $M_j\in\mathfrak{gl}_n(\mathbb{C})$.

Now, let $\mathbf{A}=(A_1,\dots,A_m)$ be a collection of formal types A_i at points $x_i\in\mathbb{P}^1$.

Definition 5.2. The category $\mathscr{C}^*(\mathbf{A})$ of framable connections with formal types $\mathbf{A}$ is the category whose objects are connections (V,∇), where

- V is a trivializable rank n vector bundle on $\mathbb{P}^1$;
- ∇ is a meromorphic connection on V with singular points $\{x_i\}$;
- ∇ is framable and has formal type A^i at x_i;

and whose morphisms are vector bundle maps compatible with the connections. The moduli space of this category is denoted by $\mathscr{M}^*(\mathbf{A})$.

By Corollary 4.22, any two objects in $\mathscr{C}^*(\mathbf{A})$ correspond to connections that are formally isomorphic at each x_i. Note that $\mathscr{C}^*(\mathbf{A})$ is not a full subcategory of the category of meromorphic connections. However, the next proposition show that the moduli space of this full subcategory coincides with $\mathscr{M}^*(\mathbf{A})$, so this moduli space may be viewed as a well-behaved subspace of the moduli stack of meromorphic connections.

Proposition 5.3. *Suppose that (V,∇) and (V',∇') are framable connections in $\mathscr{C}^*(\mathbf{A})$. If they are isomorphic as meromorphic connections, then they are isomorphic as framable connections.*

Proof. Choose trivializations for V and V'. Then, (V,∇) and (V',∇') are isomorphic as meromorphic connections if and only if there exists a meromorphic section g of the trivial $\mathrm{GL}_n(\mathbb{C})$-bundle such that $g\cdot[\nabla]=[\nabla']$. Moreover, g is necessarily regular at all points of $\mathbb{P}^1\backslash\{x_1,\dots,x_m\}$. It suffices to show that g is regular at each of the the singular points of ∇. Thus, we may reduce to the following local problem: if ∇ and ∇' are formal framed connections, $g\cdot[\nabla]=[\nabla']$, and ∇ and ∇' have the same formal type, then $g\in\mathrm{GL}_n(\mathfrak{o})$.

Fix $\nu=\frac{dt}{t}$. By Theorem 4.13, there exist $g_1,g_2\in\mathrm{GL}_n(\mathfrak{o})$ such that $g_1\cdot[\nabla_\tau]=g_2\cdot[\nabla'_\tau]=A_\nu$. Therefore, $(g_2gg_1^{-1})\cdot A_\nu=A_\nu$. By Corollary 4.21, this implies that $g_2gg_1^{-1}\in T^\flat$. It follows that $g\in\mathrm{GL}_n(\mathfrak{o})$. □

We will construct $\mathscr{M}^*(\mathbf{A})$ using symplectic reduction, so in general $\mathscr{M}^*(\mathbf{A})$ will not be a manifold. Following Section 2 of [5], we define an extended moduli space $\widetilde{\mathscr{M}}^*(\mathbf{A})$ that resolves $\mathscr{M}^*(\mathbf{A})$.

Definition 5.4. The category $\widetilde{\mathscr{C}}^*(\mathbf{A})$ of *framed* connections with formal types $\mathbf{A}$ has objects consisting of triples $(V,\nabla,\mathbf{g})$, where

- (V,∇) satisfies the first two conditions of Definition 5.2;
- $\mathbf{g}=(U_{x_1}g_1,\dots,U_{x_m}g_m)$, where g_i is a compatible framing for ∇ at x_i;
- the formal type $(A')^i$ of ∇ at x_i satisfies $(A')^i|_{\mathfrak{t}^1}=A^i|_{\mathfrak{t}^1}$.

A morphism between $(V,\nabla,\mathbf{g})$ and $(V',\nabla',\mathbf{g}')$ is a vector bundle isomorphism $\phi:V\to V'$ that is compatible with ∇ and ∇', with the added condition that $(\phi_{x_i}^{-1}(P'^{g'_i}_{x_i}),r,\phi^*_{x_i}((\beta'_{x_i})^{g'_i}))=(P^{g_i}_{x_i},r,\beta^{g_i}_{x_i})$ for all i. We let $\widetilde{\mathscr{M}}^*(\mathbf{A})$ denote the corresponding moduli space.

Remark 5.5. Define $W^{\mathrm{aff}}_{T_{x_i}}$ and $W_{T_{x_i}}$ as in Section 4.4. The groups $\mathbf{W}=\prod_{x_i}W_{T_{x_i}}$ and $\mathbf{W}^{\mathrm{aff}}=\prod_{x_i}W^{\mathrm{aff}}_{T_{x_i}}$ act componentwise on $\prod_{x_i}\mathcal{A}(T_{x_i},r_{x_i})$. We note that a global connection (V,∇) lies in $\mathscr{C}^*(\mathbf{A})$ if (V_{x_i},∇_{x_i}) is isomorphic to the diagonalized connection $(F^n_{x_i},d+A^i_\nu\nu)$ in $\mathscr{C}^{lat}$. It follows from Corollary 4.30 that the categories $\mathscr{C}^*(\mathbf{A}')$ and $\mathscr{C}^*(\mathbf{A})$ have the same objects if and only if $\mathbf{A}'=\mathbf{w}\mathbf{A}$ for some $\mathbf{w}\in\mathbf{W}$. In particular, $\mathscr{M}^*(\mathbf{w}\mathbf{A})\cong\mathscr{M}^*(\mathbf{A})$. If we let j denote the injection of these spaces into the moduli space of meromorphic connections, then $j(\mathscr{M}^*(\mathbf{w}\mathbf{A}))=j(\mathscr{M}^*(\mathbf{A}))$. On the other hand, if $\mathbf{A}'$ is not in the $\mathbf{W}^{\mathrm{aff}}$-orbit of $\mathbf{A}$, then $j(\mathscr{M}^*(\mathbf{A}'))$ and $j(\mathscr{M}^*(\mathbf{A}))$ are disjoint. This is because connections in the corresponding categories are not even formally isomorphic by Theorem 4.28.

Now, suppose that $\mathbf{s}=(s_1,\dots,s_m)\in\mathbf{W}^{\mathrm{aff}}$ and $s_i\in T_{x_i}(F)/T_{x_i}(\mathfrak{o})$. In this case, $\mathscr{C}^*(\mathbf{s}\mathbf{A})\neq\mathscr{C}^*(\mathbf{A})$ unless s_i is the identity. However, it is clear that $\varrho(s_i)(A^i)|_{\mathfrak{t}^1}=A^i|_{\mathfrak{t}^1}$. We deduce that $\widetilde{\mathscr{C}}^*(\hat{\mathbf{w}}\mathbf{A})=\widetilde{\mathscr{C}}^*(\mathbf{A})$, and $\widetilde{\mathscr{M}}^*(\hat{\mathbf{w}}\mathbf{A})\cong\widetilde{\mathscr{M}}^*(\mathbf{A})$ for all $\hat{\mathbf{w}}\in\mathbf{W}^{\mathrm{aff}}$. (Indeed, $\widetilde{\mathscr{C}}^*(\mathbf{A}')=\widetilde{\mathscr{C}}^*(\mathbf{A})$ if and only if for every x_i there exists $\hat{w}_i\in W^{\mathrm{aff}}_{T_{x_i}}$ such that $\varrho(\hat{w}_i)((A')^i)|_{\mathfrak{t}^1}=A^i|_{\mathfrak{t}^1}$.)

Let X be a symplectic variety with a Hamiltonian action of a group G. There is a moment map $\mu_G : X \to \mathfrak{g}^\vee$. If $\alpha \in \mathfrak{g}^\vee$ lies in $[\mathfrak{g}, \mathfrak{g}]^\perp$, so that the coadjoint orbit of α is a singleton, then the symplectic reduction $X /\!\!/_\alpha G$ is defined to be the quotient $\mu^{-1}(\alpha)/G$.

In Section 5.1, we will use the formal type A_i at x_i to define *extended orbits* $\mathscr{M}_i$ and $\widetilde{\mathscr{M}_i}$. These are smooth symplectic manifolds with a Hamiltonian action of $\mathrm{GL}_n(\mathbb{C})$. The following theorem generalizes [5, Proposition 2.1]:

Theorem 5.6. *Let $\mathscr{M}^*(\mathbf{A})$, $\widetilde{\mathscr{M}^*}(\mathbf{A})$ be the moduli spaces defined above.*

(1) *The moduli space $\mathscr{M}^*(\mathbf{A})$ is a symplectic reduction of $\prod_i \mathscr{M}_i$:*

$$\mathscr{M}^*(\mathbf{A}) \cong (\prod_i \mathscr{M}_i) /\!\!/_0 \mathrm{GL}_n(\mathbb{C}).$$

(2) *Similarly,*

$$\widetilde{\mathscr{M}^*}(\mathbf{A}) \cong (\prod_i \widetilde{\mathscr{M}_i}) /\!\!/_0 \mathrm{GL}_n(\mathbb{C}).$$

Moreover, $\widetilde{\mathscr{M}^*}(\mathbf{A})$ is a symplectic manifold.

(3) *Let $T_i = T_{x_i}$. There is a Hamiltonian action of $T_i^\flat$ on $\widetilde{\mathscr{M}^*}(\mathbf{A})$, and $\mathscr{M}^*(\mathbf{A})$ is naturally a symplectic reduction of $\widetilde{\mathscr{M}^*}(\mathbf{A})$ by the group $\prod_i T_i^\flat$.*

This theorem will be proved in Section 5.2.

Remark 5.7. We also obtain a version of this theorem when additional singularities corresponding to regular singular points are allowed (Theorem 5.26). In the case of the Katz-Frenkel-Gross connection [17, 13], the moduli space reduces to a point, consistent with the rigidity of this connection. Theorem 5.26 also allows one to construct many other examples of connections with singleton moduli spaces, which are thus plausible candidates for rigidity.

Remark 5.8. It is not surprising that these moduli spaces are symplectic: it is conceivable that this fact might be proved independently using the abstract methods of [10, Section 6]. The advantage of Theorem 5.6 is that it gives explicit constructions of $\mathscr{M}^*(\mathbf{A})$ and $\widetilde{\mathscr{M}^*}(\mathbf{A})$ in a number of important, novel cases (including connections with 'supercuspidal' type singularities). Moreover, the fact that $\widetilde{\mathscr{M}^*}(\mathbf{A})$ is smooth allows one to generalize the work of Jimbo, Miwa, and Ueno [16] and explicitly calculate the isomondromy equations in these cases (see [6]).

5.1. **Extended Orbits.** In this section, we will construct symplectic manifolds, called extended orbits, which will be "local pieces" of the moduli spaces $\mathscr{M}^*(\mathbf{A})$ and $\widetilde{\mathscr{M}^*}(\mathbf{A})$. Without loss of generality, we will take our singular point to be $x = 0$, and we will suppress the subscript x from F_x, P_x, A_x, etc.

Our study of extended orbits is motivated by the relationship between coadjoint orbits and gauge transformations. In the following, fix $\nu \in \Omega_0^\times$.

Proposition 5.9. *The map $\mathfrak{gl}_n(F) \to \mathfrak{P}^\vee$ determined by ν intertwines the gauge action of P on $\mathfrak{gl}_n(F)$ with the coadjoint action of P on $\mathfrak{P}^\vee$.*

Proof. Recall from Lemma 2.6 that $P = H \ltimes P^1$ for $H \cong Q/U$ a Levi subgroup of $\mathrm{GL}_n(\mathbb{C})$. Thus, we may write any element of P as $p = hu$, for $h \in H$ and $u \in P^1$. Without loss of generality, we may assume that ν has order -1. Thus,

$$p \cdot X = h \cdot (\mathrm{Ad}(u)(X) - \tau(u)u^{-1}) = \mathrm{Ad}(p)(X) - \mathrm{Ad}(h)(\tau(u)u^{-1}).$$

Lemma 4.4 shows that $\operatorname{Ad}(h)(\tau(u)u^{-1}) \in \mathfrak{P}^1 = \mathfrak{P}^\perp$. Applying Proposition 2.5, we see that $\operatorname{Ad}^*(p)(\langle X, \cdot\rangle_\nu|_{\mathfrak{P}}) = \langle \operatorname{Ad}(p)(X), \cdot\rangle_\nu|_{\mathfrak{P}} = \langle p \cdot X, \cdot\rangle_\nu|_{\mathfrak{P}}$. □

From now on, we assume ν has order -1. We suppose that A is a formal type at 0 stabilized by a torus T with $T(\mathfrak{o}) \subset P$ and that the corresponding regular stratum (P, r, β) has $r > 0$. In particular, any connection with formal type A is irregular singular. Denote the projection of $\pi_{\mathfrak{t}}^*(A)$ onto $(\mathfrak{P}^1)^\vee$ by A^1. Let $G = \mathrm{GL}_n(\mathfrak{o})$ be the maximal standard parahoric subgroup at 0 with congruence subgroups (resp. fractional ideals) G^i (resp. $\mathfrak{g}^i$). Then, $G^1 \subset P \subset G$, and $P/G^1 \cong Q$. For any subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$, there is a natural projection $\pi_{\mathfrak{h}} : \mathfrak{g}^\vee \to (\mathfrak{h})^\vee$ obtained by restricting functionals to $\mathfrak{h} \subset \mathfrak{g}$. Denote the P-coadjoint orbit of $\pi_{\mathfrak{t}}^*(A)$ by $\mathscr{O}$, and the P^1-coadjoint orbit of A^1 by $\mathscr{O}^1$.

Definition 5.10. Let A be a formal type at 0 with irregular singularity, and let U be the unipotent radical of Q. We define the extended orbits $\mathscr{M}(A)$ and $\widetilde{\mathscr{M}}(A)$ by

- $\mathscr{M}(A) \subset (Q\backslash \mathrm{GL}_n(\mathbb{C})) \times \mathfrak{g}^\vee$ is the subvariety defined by

$$\mathscr{M}(A) = \{(Qg, \alpha) \mid \pi_{\mathfrak{P}}(\operatorname{Ad}^*(g)(\alpha)) \in \mathscr{O}\}; \tag{5.1}$$

- $\widetilde{\mathscr{M}}(A) \subset (U\backslash \mathrm{GL}_n(\mathbb{C})) \times \mathfrak{g}^\vee$ is defined by

$$\widetilde{\mathscr{M}}(A) = \{(Ug, \alpha) \mid \pi_{\mathfrak{P}^1}(\operatorname{Ad}^*(g)(\alpha)) \in \mathscr{O}^1)\};$$

Proposition 5.11. *The extended orbits $\mathscr{M}(A)$ and $\widetilde{\mathscr{M}}(A)$ are isomorphic to symplectic reductions of $T^*G \times \mathscr{O}$ and $T^*G \times \mathscr{O}^1$ respectively:*

$$\mathscr{M}(A) \cong T^*G \times \mathscr{O} /\!\!/_0 P$$
$$\widetilde{\mathscr{M}}(A) \cong T^*G \times \mathscr{O}^1 /\!\!/_0 P^1.$$

In particular, the natural symplectic form on $T^*G \times \mathscr{O}$ descends to both $\mathscr{M}(A)$ and $\widetilde{\mathscr{M}}(A)$. Moreover, $\mathscr{M}(A)$ and $\widetilde{\mathscr{M}}(A)$ are smooth symplectic manifolds.

Remark 5.12. Note that T^*G is not finite dimensional. However, for ℓ sufficiently large, $A \in (\mathfrak{g}^\ell)^\perp$. Since $G^\ell \subset P^1$, we see that $G/P^1 \cong (G/G^\ell)/(P^1/G^\ell)$. Thus, in Proposition 5.11, it suffices to consider $T^*(G/G^\ell) \times \mathscr{O} /\!\!/_0 P$ (resp. $T^*(G/G^\ell) \times \mathscr{O}^1 /\!\!/_0 P^1$). This fact, although concealed in our notation, ensures that we are always applying results from algebraic and symplectic geometry to finite-dimensional varieties.

Proof. The proof in each case is similar, so we will prove the second isomorphism. The group P^1 acts on T^*G by the usual left action $p(g, \alpha) = (pg, \alpha)$ and on $\mathscr{O}^1$ by the coadjoint action. Moreover, on each factor, the action of P^1 is Hamiltonian with respect to the standard symplectic form. The moment map for the diagonal action of P^1 is the sum of the two moment maps:

$$\mu_{P^1} : T^*G \times \mathscr{O}^1 \to (\mathfrak{P}^1)^\vee$$
$$\mu_{P^1}(g, \alpha, \beta) = \pi_{\mathfrak{P}^1}(-\operatorname{Ad}^*(g)(\alpha)) + \beta.$$

In particular,

$$\mu_{P^1}^{-1}(0) = \{(g, \alpha, \beta) \mid \pi_{\mathfrak{P}^1}(\operatorname{Ad}^*(g)(\alpha)) = \beta\}.$$

We will show that $\mu_{P^1}^{-1}(0)$ is smooth. Let $\varphi : \mu_{P^1}^{-1}(0) \to G \times \mathscr{O}^1$ be defined by $\varphi(g, \alpha, \beta) = (g, \beta)$. Choose a local section $f : \mathscr{O}^1 \to \mathscr{O}^1 + (\mathfrak{P}^1)^\perp \subset \mathfrak{g}^\vee$. Then, $\varphi^{-1}(g, \beta) = \{(g, \operatorname{Ad}^*(g^{-1})(f(\beta) + X), \beta) \mid X \in (\mathfrak{P}^1)^\perp\}$. Therefore, $\mu_{P^1}^{-1}(0)$ is an affine bundle over $G \times \mathscr{O}^1$ with fibers isomorphic to $\mathfrak{P}/\mathfrak{g}^1$. It follows that $\mu_{P^1}^{-1}(0)$ is smooth.

Since $U \cong P^1/G^1$, $U\backslash \mathrm{GL}_n(\mathbb{C}) \cong P^1\backslash G$. Therefore, the map $(g, \alpha, \beta) \mapsto (P^1 g, \alpha)$ takes $\mu_{P^1}^{-1}(0)$ to $\widetilde{\mathscr{M}}(A)$; moreover, the fibers are P^1 orbits, so the map identifies $P^1\backslash\mu_{P^1}^{-1}(0) \cong \widetilde{\mathscr{M}}(A)$.

Finally, choose a local section $\zeta : P^1\backslash G \to G$ with domain W, and let $W' = \widetilde{\mathscr{M}}(A) \cap (W \times \mathfrak{g}^\vee)$. There is a section $\zeta' : W' \to \mu_{P^1}^{-1}(0)$ given by $\zeta'(P^1 g, \alpha) = (\zeta'(P^1 g), \alpha, \pi_{\mathfrak{P}^1}(\mathrm{Ad}^*(\zeta(P^1 g))(\alpha)))$. This shows that $\mu_{P^1}^{-1}(0) \to \widetilde{\mathscr{M}}(A)$ is a principal P^1-bundle, since P^1 acts freely on the fibers. In particular, $\widetilde{\mathscr{M}}(A)$ is smooth, and the symplectic form on $T^*G \times \mathscr{O}^1$ descends to $\widetilde{\mathscr{M}}(A)$. □

Let $\mathrm{res} : \mathfrak{g}^\vee \to \mathfrak{gl}_n(\mathbb{C})^\vee$ be the restriction map dual to the inclusion $\mathfrak{gl}_n(\mathbb{C}) \to \mathfrak{g}$. Notice that if we fix a representative $\alpha_\nu \in \mathfrak{gl}_n(F)$ for $\alpha \in \mathfrak{g}^\vee$, then $\mathfrak{gl}_n(\mathbb{C})^\vee \cong \mathfrak{gl}_n(\mathbb{C})$ under the trace pairing and $\mathrm{res}(\alpha)$ corresponds to the ordinary residue of $\alpha_\nu \nu$.

There is a Hamiltonian left action of $\mathrm{GL}_n(\mathbb{C})$ on T^*G defined by

$$\rho(h)(g, \alpha) = (gh^{-1}, \mathrm{Ad}^*(h)\alpha). \tag{5.2}$$

The moment map μ_ρ is given by $\mu_\rho(g, \alpha) = \mathrm{res}(\alpha)$. To see this, observe that ρ is the restriction to $\mathrm{GL}_n(\mathbb{C})$ of the usual left action of G on T^*G (via inversion composed with right multiplication). Hence, the map μ_ρ is just the composition of the moment map for right multiplication $\mu(g, \alpha) = \alpha$ with res.

The action ρ defines left actions of $\mathrm{GL}_n(\mathbb{C})$ on the first components of $T^*G \times \mathscr{O}$ and $T^*G \times \mathscr{O}^1$ respectively. These actions commute with the left actions of P and P^1, and it is clear that μ_P and μ_{P^1} are $\mathrm{GL}_n(\mathbb{C})$-equivariant.

Lemma 5.13. *Let G_1 and G_2 act on a symplectic manifold X via Hamiltonian actions, and let μ_1 and μ_2 be the corresponding moment maps. If μ_2 is G_1-invariant on $\mu_1^{-1}(\lambda)$, then there is a natural Hamiltonian action of G_2 on $X /\!\!/_\lambda G_1$. Furthermore, if $\iota_\lambda : \mu_1^{-1}(\lambda) \to X$ and $\pi_\lambda : \mu_1^{-1}(\lambda) \to X /\!\!/_\lambda G_1$ are the natural maps, then the induced moment map $\bar{\mu}_2$ on $X /\!\!/_\lambda G_1$ is the unique map satisfying $\mu_2 \circ \iota_\lambda = \bar{\mu}_2 \circ \pi_\lambda$.*

This follows from [1, Theorem 4.3.5]. Thus, ρ descends to natural Hamiltonian actions on $\mathscr{M}(A)$ and $\widetilde{\mathscr{M}}(A)$. For example, if $(Qg, \alpha) \in \mathscr{M}(A)$ and $h \in \mathrm{GL}_n(\mathbb{C})$, then

$$h(Qg, \alpha) = (Qgh^{-1}, \mathrm{Ad}^*(h)(\alpha)).$$

Proposition 5.14. *The moment map for the action of $\mathrm{GL}_n(\mathbb{C})$ on $\mathscr{M}(A)$ is given by*

$$\mu_{\mathrm{GL}_n}(Qg, \alpha)) = \mathrm{res}(\alpha).$$

The action of $\mathrm{GL}_n(\mathbb{C})$ on $\widetilde{\mathscr{M}}(A)$ has the analogous moment map $\tilde{\mu}_{\mathrm{GL}_n}$.

Proof. This follows directly from Lemma 5.13. □

Lemma 5.15. *The moment map $\tilde{\mu}_{\mathrm{GL}_n} : \widetilde{\mathscr{M}}(A) \to \mathfrak{gl}_n(\mathbb{C})$ is a submersion.*

Proof. By Proposition 5.11, $\widetilde{\mathscr{M}}(A)$ is smooth. We will show that the differential map $d\tilde{\mu}_{\mathrm{GL}_n}$ on tangent spaces is surjective. Note that $\tilde{\mu}_{\mathrm{GL}_n}(Ugg', \mathrm{Ad}^*(g'^{-1})\alpha) = g'^{-1}\tilde{\mu}_{\mathrm{GL}_n}(Ug, \alpha)$. Therefore, it suffices to show that the tangent map is surjective at points $s = (U, \alpha)$ in the subvariety S defined by taking g to be the identity.

Let $\mathfrak{u} = \mathrm{Lie}(U)$, so $\mathfrak{u}^\perp \subset \mathfrak{gl}_n(\mathbb{C})^\vee$. Indeed, $\mathfrak{u}^\perp \cong (\mathfrak{P}^1)^\perp \subset \mathfrak{g}^\vee$. If we choose a section $f : (\mathfrak{P}^1)^\vee \to \mathfrak{g}^\vee$, we see that $\mathscr{O}^1 \times \mathfrak{u}^\perp \cong S$ by the map $(\gamma, y) \mapsto (U, f(\gamma) + y)$.

Here, the image of y is identified with its image in $\mathfrak{g}^\vee$. In particular, the image of $d\tilde{\mu}_{\mathrm{GL}_n}(T_s\widetilde{\mathscr{M}}(A))$ contains $\mathfrak{u}^\perp \subset \mathfrak{gl}_n(\mathbb{C})^\vee$. Therefore, it suffices to show that the composition of $d\tilde{\mu}_{\mathrm{GL}_n}$ with the quotient $\mathfrak{gl}_n(\mathbb{C})^\vee \to \mathfrak{u}^\vee$ is surjective. Observe that tangent vectors to $\mathscr{O}^1$ are of the form $\mathrm{ad}^*(X)(\alpha)$ for $X \in \mathfrak{P}^1$.

First, suppose that $r > e_P$. In this case, we will show that $\mathrm{ad}^*(\mathfrak{P}^{1-e_P+r})(\alpha) \subset T_s\mathscr{O}^1$ surjects onto $\mathfrak{u}^\vee$. More precisely, we will construct a filtration $\mathfrak{u}^\vee = \mathfrak{u}^1 \supset \mathfrak{u}^2 \supset \cdots \supset \mathfrak{u}^{e_P} = \{0\}$ such that the map $X \mapsto \mathrm{ad}^*(X)(\alpha)$ induces a surjection $\bar{\pi}^j_{\mathfrak{u}} : \bar{\mathfrak{P}}^{j-e_P+r} \to \mathfrak{u}^j/\mathfrak{u}^{j+1}$ for each j. Since $\mathfrak{u} \cong \mathfrak{P}^1/\mathfrak{g}^1$, we see that $\mathfrak{u}^\vee \cong \mathfrak{g}/\mathfrak{P}$ under the duality isomorphism. We now obtain the desired filtration on $\mathfrak{u}^\vee$ by subspaces of the form $\mathfrak{u}^j \cong (\mathfrak{P}^{j-e_P} \cap \mathfrak{g})/\mathfrak{P}$. More explicitly, $\mathfrak{u}^j$ is the restriction of $(\mathfrak{P}^{e_P-j+1})^\perp \subset (\mathfrak{P}^1)^\vee$ to $\mathfrak{u}$. Note that the map $\bar{\pi}^j_{\mathfrak{u}} = \tau^j \circ (-\delta_{\alpha_\nu})$, where $\tau^j : \bar{\mathfrak{P}}^{j-e_P} \to \mathfrak{u}^j/\mathfrak{u}^{j+1}$ is the surjection defined by $\tau^j(X) = (\langle X, \cdot\rangle_\nu|_{\mathfrak{u}}) + \mathfrak{u}^{j+1}$. Furthermore, $\bar{\pi}^j_{\mathfrak{u}}$ depends only on the coset $\alpha_\nu + \mathfrak{P}^{-r+1}$.

By assumption, $\alpha_\nu \in A_\nu + \mathfrak{P}^{-r+1}$. Proposition 2.11 shows that $\delta_{\alpha_\nu}(\bar{\mathfrak{P}}^{j-e_P+r}) = \ker(\bar{\pi}_{\mathfrak{t}})$. Since $\bar{\pi}_{\mathfrak{t}} : \bar{\mathfrak{P}}^{j-e_P} \to \bar{\mathfrak{t}}^{j-e_P}$ is a surjection, a diagram chase shows that $\tau^j|_{\ker(\bar{\pi}_{\mathfrak{t}})}$ is surjective if and only if $\bar{\pi}_{\mathfrak{t}}|_{\ker(\tau^j)}$ is surjective. We now verify this last statement. In the case $e_P = n$, recall the description of ϖ_E from (2.2). It is easily checked that $Y_{j\nu} = t^{-1}\operatorname{Res}(\varpi_E^{j-e_P}\,dt)$ corresponds to a non-zero element Y_j of $\ker(\bar{\pi}^j_{\mathfrak{u}})$ (since $1 \le j < e_P$) and $\pi_{\mathfrak{t}}(Y_{j\nu}) = \frac{e_P - j}{e_P}\varpi_E^{j-e_P}$. Therefore, the span of $Y_{j\nu}$ surjects onto the one-dimensional space $\bar{\mathfrak{t}}^{j-e_P}$. A similar proof works for $e_P < n$, using the observation in Corollary 3.9 that $\mathfrak{t} \cong E^{n/e_P}$.

Now, assume that $1 \le r \le e_P$. The above argument shows that $\bar{\pi}^j_{\mathfrak{u}}(\bar{\mathfrak{P}}^{j-e_P+r}) = \mathfrak{u}^j/\mathfrak{u}^{j+1}$; however, in this case, we can only conclude that the image of $d\tilde{\mu}_{\mathrm{GL}_n}|_{T_s\mathscr{O}^1}$ contains $\mathfrak{u}^{e_P-r+1}$. Let $\mathfrak{w}^j = \mathfrak{P}^{j-e_P+r} \cap \mathfrak{gl}_n(\mathbb{C})$; it follows that $\mathfrak{w}^j/\mathfrak{w}^{j+1}$ determines a well-defined subset of $\bar{\mathfrak{P}}^{j-e_P+r}$. We claim that $\bar{\pi}^j_{\mathfrak{u}}(\mathfrak{w}^j/\mathfrak{w}^{j+1}) = \mathfrak{u}^j/\mathfrak{u}^{j+1}$ for $1 \le j \le e_P - r$. Observe that $t^{-1}\mathfrak{g} \supset \mathfrak{P}^\ell \supset \mathfrak{P}^{\ell+1} \supset t\mathfrak{g}$ for $-e_P \le \ell \le 0$. Therefore, we may take a representative $\beta_\nu \in t^{-1}\,\mathfrak{gl}_n(\mathbb{C}) + \mathfrak{gl}_n(\mathbb{C})$ for $\alpha_\nu + \mathfrak{P}^{-r+1}$. Similarly, choose a representative $X + t^{-1}X'$ for $\bar{X} \in \bar{\mathfrak{P}}^{j-e_P+r}$, where $X, X' \in \mathfrak{gl}_n(\mathbb{C})$. It follows that $\langle \mathrm{ad}(X + t^{-1}X')\beta_\nu, Y\rangle_\nu = \langle \mathrm{ad}(X)\beta_\nu, Y\rangle_\nu$ whenever $Y \in \mathfrak{u}$. This proves the claim, and we conclude that $\mathrm{ad}^*(\mathfrak{w}^1)(\alpha)$ surjects onto $\mathfrak{u}^\vee/\mathfrak{u}^{e_P-r+1}$.

Finally, let $X \in \mathfrak{w}^1$. The action of $\mathrm{GL}_n(\mathbb{C})$ on $\widetilde{\mathscr{M}}(A))$ gives rise to a map $\mathfrak{gl}_n(\mathbb{C}) \to T_s\widetilde{\mathscr{M}}(A) \subset \mathfrak{u}\backslash\,\mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{g}^\vee$; explicitly, $X \mapsto (-X, \mathrm{ad}^*(X)\alpha)$, which is sent to $\mathrm{res}(\mathrm{ad}^*(X)(\alpha))$ by $d\tilde{\mu}_{\mathrm{GL}_n}$. Therefore, $d\tilde{\mu}_{\mathrm{GL}_n}$ maps tangent vectors coming from $\mathfrak{w}^1 \subset \mathfrak{gl}_n(\mathbb{C})$ surjectively onto $\mathfrak{u}^\vee/\mathfrak{u}^{e_P-r+1}$. It follows that the image of $d\tilde{\mu}_{\mathrm{GL}_n}$ contains $\mathfrak{u}^\vee$, so $\tilde{\mu}_{\mathrm{GL}_n}$ is a submersion. □

Lemma 5.16. $\mathrm{GL}_n(\mathbb{C})$ *acts freely on* $\widetilde{\mathscr{M}}(A)$.

Proof. Suppose that $h \in \mathrm{GL}_n(\mathbb{C})$ fixes (Ug, α). In particular, $Ugh = Ug$, so ${}^g h \in U$. To show that $h = 1$, it suffices to show that ${}^g h = 1$, so without loss of generality, we may assume that $g = 1$ and $h \in U$. By Proposition 2.5, there exists a representative $\alpha_\nu \in \mathfrak{gl}_n(F)$ for α with terms only in nonpositive degrees. The fact that $\mathrm{Ad}^*(h^{-1})(\alpha) = \alpha$ implies that $\mathrm{Ad}(h^{-1})(\alpha_\nu) = \alpha_\nu + X$ for $X \in \mathfrak{g}^1$. Since $h \in \mathrm{GL}_n(\mathbb{C})$, $X = 0$, and we see that $\mathrm{Ad}(h^{-1})(\alpha_\nu) = \alpha_\nu$.

We will show that h is P^1-conjugate to an element of $T(\mathfrak{o})$. In particular, since $P^1 \cong U \ltimes G^1$, we see that h is U-conjugate to an element of $T(\mathfrak{o})G^1 \cap \mathrm{GL}_n(\mathbb{C}) = T^\flat$. Since $T^\flat \cap P^1$ is trivial, Corollary 3.9 implies that $h = 1$.

Take $p \in P^1$ such that $\operatorname{Ad}^*(p)(\alpha) = A^1$; thus, $\operatorname{Ad}(p)(\alpha_\nu) \in \mathfrak{t}+\mathfrak{P}$. By Lemma 3.18, there exists $p' \in P^1$ and a representative $A^1_\nu \in \mathfrak{t}^{-r}$ of A^1 such that $\operatorname{Ad}(p')(\operatorname{Ad}(p)(\alpha_\nu)) = A^1_\nu$. Therefore, setting $q = p'p \in P^1$, $\operatorname{Ad}(({}^q h)^{-1})A^1_\nu = A^1_\nu$. By Lemma 3.11, ${}^q h \in T \cap G = T(\mathfrak{o})$. □

Lemma 5.17. *If (Qg_1, α) and (Qg_2, α) both lie in $\mathscr{M}(A)$, then $g_2 = pg_1$ for some $p \in Q$. Moreover, if (Ug_1, α) and (Ug_2, α) both lie in $\widetilde{\mathscr{M}}(A)$, then $g_2 = usg_1$ for some $u \in U$ and $s \in T^\flat$.*

Proof. Notice that $(Q, \operatorname{Ad}^*(g_1)\alpha)$ and $(Qg_2(g_1^{-1}), \operatorname{Ad}^*(g_1)\alpha)$ satisfy the conditions of the first statement. There is a similar reformulation of the second statement. Thus, we may assume without loss of generality that g_1 is the identity; we set $g_2 = g$.

In the first case, note that by Lemmas 3.18 and 3.20, there exist $p_1, p_2 \in P$ such that $\operatorname{Ad}(p_1 g)(\alpha_\nu) = A_\nu = \operatorname{Ad}(p_2)(\alpha_\nu)$ for some $A_\nu \in \mathfrak{t}$. Since $p_1 g p_2^{-1}$ centralizes the regular semisimple element A_ν, $p_1 g p_2^{-1} \in T \cap G = T(\mathfrak{o})$. We conclude that $g \in P \cap \mathrm{GL}_n(\mathbb{C}) = Q$.

In the second case, the same argument shows that whenever $\operatorname{Ad}^*(g)(\alpha) \in \mathscr{O}^1$, $g = p_1^{-1} s p_2$ for some $s \in T(\mathfrak{o})$ and $p_i \in P^1$. Since P^1 is normal in P, $g = us$ for some $u \in P^1$. By Corollary 3.9, we may assume that $s \in T^\flat$. This implies that $u \in \mathrm{GL}_n(\mathbb{C}) \cap P^1 = U$ as desired. □

Lemma 5.18. *Let $\alpha \in \mathfrak{g}^\vee$ be a functional such that $\pi_{\mathfrak{P}^1}(\alpha) = A^1$. Then, if $s \in T(\mathfrak{o})$, $\pi_{\mathfrak{P}^1}(\operatorname{Ad}^*(s)\alpha) = A^1$.*

Proof. Since any representative of A^1 lies in $\mathfrak{t}(\mathfrak{o})$, $\alpha_\nu \in \mathfrak{t} + \mathfrak{P}$. The lemma is now clear, since $T(\mathfrak{o})$ preserves $\mathfrak{P}$ and stabilizes $\mathfrak{t}$. □

We are now ready to describe the relationship between $\mathscr{M}(A)$ and $\widetilde{\mathscr{M}}(A)$. Recall, from Lemma 3.14, that $T^\flat = T(\mathfrak{o}) \cap \mathrm{GL}_n(\mathbb{C})$. There is a left action of $T^\flat$ on $\widetilde{\mathscr{M}}(A)$ given by $s(Ug, \alpha) = (Usg, \alpha)$. To see this, note that by assumption, $\pi_{\mathfrak{P}^1}(\operatorname{Ad}^*(g)(\alpha)) \in \mathscr{O}^1$, so there exists $u \in P^1$ such that $\operatorname{Ad}^*(u)(\pi_{\mathfrak{P}^1}(\operatorname{Ad}^*(g)(\alpha))) = A^1$. We wish to show that there exists $u' \in P^1$ such that $\operatorname{Ad}^*(u')(\pi_{\mathfrak{P}^1}(\operatorname{Ad}^*(sg)(\alpha))) = A^1$. However,

$$\operatorname{Ad}^*(u^s)(\pi_{\mathfrak{P}^1}(\operatorname{Ad}^*(sg)(\alpha))) = \pi_{\mathfrak{P}^1}(\operatorname{Ad}^*({}^s u)\operatorname{Ad}^*(sg)(\alpha)) = \pi_{\mathfrak{P}^1}(\operatorname{Ad}^*(s)\operatorname{Ad}^*(ug)(\alpha)) = A^1, \tag{5.3}$$

where the last equality follows from Lemma 5.18. In particular, $s(Ug, \alpha) \in \widetilde{\mathscr{M}}(A)$.

We will show that this action is Hamiltonian with moment map $\mu_{T^\flat}$ defined as follows. Take $(Ug, \alpha) \in \widetilde{\mathscr{M}}(A)$. There exists $u \in P^1$ such that

$$\pi_{\mathfrak{P}^1}(\operatorname{Ad}^*(ug)(\alpha)) = A^1. \tag{5.4}$$

Define a map

$$\mu_{T^\flat}(Ug, \alpha) = -(\operatorname{Ad}^*(ug)(\alpha))|_{T^\flat}.$$

We need to show that this map is well-defined. Let $\tilde{A} = \operatorname{Ad}^*(ug)(\alpha)$. Suppose that $u' \in P^1$ satisfies (5.4). Observe that $\operatorname{Ad}^*(u'u^{-1})(A^1) = A^1$. By Lemma 3.21, $u'u^{-1} \in T(\mathfrak{o})P^r$. It suffices to show that whenever $s \in T(\mathfrak{o})$ and $p \in P^r$, $(\operatorname{Ad}^*(sp)(\tilde{A}))|_{T^\flat} = (\tilde{A})|_{T^\flat}$. In fact, we will prove the stronger statement:

$$\pi_{\mathfrak{t}\cap\mathfrak{P}}(\operatorname{Ad}^*(sp)(\tilde{A})) = \pi_{\mathfrak{t}\cap\mathfrak{P}}(\tilde{A}). \tag{5.5}$$

Fix a representative $\tilde{A}_\nu \in \mathfrak{P}^{-r}$. By Proposition 2.11(4), the projection $(\mathfrak{gl}(F))^\vee \to \mathfrak{t}^\vee$ corresponds to tame corestriction $\pi_\mathfrak{t} : \mathfrak{gl}_n(F) \to \mathfrak{t}$ after dualizing. Thus, $\pi_\mathfrak{t}(\mathrm{Ad}(sp)(\tilde{A}_\nu))$ is a representative of $\pi_{\mathfrak{t}\cap\mathfrak{P}}(\mathrm{Ad}^*(sp)\tilde{A})$. Since $\pi_\mathfrak{t}$ commutes with the action of $\mathfrak{t}$, $\pi_\mathfrak{t}(\mathrm{Ad}(sp)(\tilde{A}_\nu)) = \pi_\mathfrak{t}(\mathrm{Ad}(p)(\tilde{A}_\nu))$. By Proposition 2.11(3), $\pi_\mathfrak{t}(\mathrm{Ad}(p)(\tilde{A}_\nu)) - \pi_\mathfrak{t}(\tilde{A}_\nu) \in \mathfrak{P}^1 = \mathfrak{P}^\perp$, so $\pi_\mathfrak{t}(\tilde{A}_\nu)$ is a representative for both functionals in (5.5) as desired.

The following lemma generalizes [5, Lemma 2.3]. The proof is more complicated, due to the absence of a 'decoupling' map in the general case.

Proposition 5.19. *Let $\Lambda = A|_{\mathfrak{t}^\flat}$. The action of $T^\flat$ on $\widetilde{\mathscr{M}}(A)$ is Hamiltonian with moment map $\mu_{T^\flat}$. Moreover,*

$$\mathscr{M}(A) \cong \widetilde{\mathscr{M}}(A) /\!\!/_{-\Lambda} T^\flat.$$

Proof. Recall that $\mathscr{O}^1$ be the P^1-coadjoint orbit containing A^1. If $\beta \in \mathscr{O}^1$, we may take $\alpha \in \mathfrak{g}^\vee$ such that $\pi_{\mathfrak{P}^1}(\alpha) = \beta$. The torus $T^\flat$ acts on $\mathscr{O}^1$ by $s \cdot \beta = \pi_{\mathfrak{P}^1}((\mathrm{Ad}^*(s)(\alpha)))$. (One sees that this element is in $\mathscr{O}^1$ by an argument similar to the one used to show (5.3), and it is easily checked that it is independent of the choice of α.)

We construct a moment map for this action. Consider the semi-direct product $T^\flat \ltimes P^1 \subset P$, and lift $A^1 \in \mathscr{O}^1$ to $\tilde{A} \in (\mathfrak{t}^\flat)^\perp \subset (\mathfrak{t}^\flat \times \mathfrak{P}^1)^\vee$. Let $\tilde{\mathscr{O}} \subset (\mathfrak{t}^\flat \times \mathfrak{P}^1)^\vee$ be the coadjoint orbit of $\tilde{A}$. Since $T^\flat$ stabilizes A^1 by Lemma 5.18, it is clear that it stabilizes $\tilde{A}$ as well. In particular, P^1 acts transitively on $\tilde{\mathscr{O}}$. We will prove in Lemma 5.20 that the natural map $\tilde{\pi} : \tilde{\mathscr{O}} \to \mathscr{O}^1$ is a $T^\flat$-equivariant symplectic isomorphism. Therefore, the moment map $\tilde{\mu} : \mathscr{O}^1 \to (\mathfrak{t}^\flat)^\vee$ is given by

$$\tilde{\mu}(\beta) = \pi_{T^\flat}(\tilde{\pi}^{-1}(\beta)),$$

where $\pi_{T^\flat}$ is the projection $(\mathfrak{t}^\flat \times \mathfrak{P}^1)^\vee \to (\mathfrak{t}^\flat)^\vee$.

We remark that if a different lift of A^1 is chosen, say $\tilde{A}+\gamma$ for $\gamma \in (\mathfrak{P}^1)^\perp \cong (\mathfrak{t}^\flat)^\vee$, then

$$(\mathrm{Ad}^*(u)(\tilde{A} + \gamma))(z) = \mathrm{Ad}^*(u)(\tilde{A})(z) + \gamma(z) \tag{5.6}$$

for $u \in P^1$ and $z \in \mathfrak{t}^\flat$. In particular, this changes $\widetilde{\mu}$ by a constant γ.

The action of $T^\flat$ on $\widetilde{\mathscr{M}}(A)$ descends from a Hamiltonian action of $T^\flat$ on $T^*G \times \mathscr{O}^1$. Indeed, if $(g, \alpha, \beta) \in T^*G \times \mathscr{O}^1$, then $s(g, \alpha, \beta) = (sg, \alpha, s \cdot \beta)$ defines a Hamiltonian action; the moment map μ' is given by the sum of the natural moment map on T^*G and $\tilde{\mu}$. Moreover, $T^\flat$ preserves $\mu_{P^1}^{-1}(0)$, and the map from $\mu_{P^1}^{-1}(0) \to \widetilde{\mathscr{M}}(A)$ is $T^\flat$-equivariant.

We will show that the restriction of μ' to $\mu_{P^1}^{-1}(0)$ is P^1-invariant. Let $(g, \alpha, \beta) \in \widetilde{\mathscr{M}}(A)$, and define $\phi(g, \alpha)$ to be the projection of $\mathrm{Ad}^*(g)(\alpha)$ onto $(\mathfrak{t}^\flat \times \mathfrak{P}^1)^\vee$. Then, if $u \in P^1$,

$$\begin{aligned} \mu'(u(g, \alpha, \beta)) &= \mu'(ug, \alpha, \mathrm{Ad}^*(u)\beta) \\ &= \pi_{T^\flat}\left(-\mathrm{Ad}^*(u)\phi(g, \alpha) + \mathrm{Ad}^*(u)\tilde{\pi}^{-1}(\beta)\right) \end{aligned}$$

However, $\beta = \pi_{\mathfrak{P}^1}(\mathrm{Ad}^*(g)(\alpha))$ lies in $\mathscr{O}^1$, so $\phi(g, \alpha)$ must lie in a coadjoint orbit containing $\tilde{A} - \gamma$ for some $\gamma \in (\mathfrak{t}^\flat)^\vee$. Equation (5.6) implies that

$$\pi_{T^\flat}(-\phi(g, \alpha) + \tilde{\pi}^{-1}(\beta)) = \pi_{T^\flat}\left(-\mathrm{Ad}^*(u)\phi(g, \alpha) + \mathrm{Ad}^*(u)\tilde{\pi}^{-1}(\beta)\right) = \gamma.$$

Thus, μ' is P^1-invariant. By Lemma 5.13, the action of $T^\flat$ on $\widetilde{\mathscr{M}}(A)$ is Hamiltonian, and the moment map descends from μ'.

It remains to show that $\gamma = \mu_{T^\flat}(Ug, \alpha)$. By P^1-invariance, it suffices to consider the case where $\pi_{\mathfrak{P}^1}(\mathrm{Ad}^*(g)(\alpha)) = A^1$. By construction, $\tilde{\mu}(A^1) = 0$, so $\gamma = -\mathrm{Ad}^*(g)(\alpha)|_{\mathfrak{t}^\flat} = \mu_{T^\flat}(Ug, \alpha)$.

We now prove that $\mathscr{M}(A) \cong \widetilde{\mathscr{M}}(A) /\!\!/_{-\Lambda} T^\flat$. First, we show that if $\mu_{T^\flat}(Ug, \alpha) = -\Lambda$, then $(Qg, \alpha) \in \mathscr{O}$. Let $u \in P^1$ satisfy $\pi_{\mathfrak{P}^1}(\mathrm{Ad}^*(ug)\alpha) = A^1$. Choosing a representative α_ν for α, we have $\mathrm{Ad}(ug)(\alpha_\nu) \in A_\nu + \mathfrak{P}$. Applying $\pi_{\mathfrak{t}}$, we see that $\pi_{\mathfrak{t}}(\mathrm{Ad}(ug)(\alpha_\nu)) = A_\nu + z$ for some $z \in \mathfrak{t}(\mathfrak{o})$. In fact, $z \in \mathfrak{t} \cap \mathfrak{P}^1$ because the restrictions of $\mathrm{Ad}^*(ug)\alpha$ and A to $(\mathfrak{t}^\flat)^\vee$ agree. By Proposition 2.11(3), there exists $X \in \mathfrak{P}^r$ such that $\mathrm{Ad}(1+X)\,\mathrm{Ad}(ug)\alpha_\nu) \in A_\nu + \mathfrak{P}^1$, so $\pi_{\mathfrak{P}}(\mathrm{Ad}^*((1+X)ug)\alpha) = \pi_{\mathfrak{t}}^*(A)$, i.e., $\mathrm{Ad}^*(g)(\alpha) \in \mathscr{O}$. Thus, we have a map $\mu_{T^\flat}^{-1}(-\Lambda) \to \mathscr{M}(A)$ given by $(Ug, \alpha) \mapsto (Qg, \alpha)$. Lemma 5.17 shows that the fibers of the map are $T^\flat$-orbits, and we obtain the desired isomorphism. □

Lemma 5.20. *In the notation from the previous proof, the map $\tilde{\pi} : \tilde{\mathscr{O}} \to \mathscr{O}^1$ is a $T^\flat$-equivariant symplectic isomorphism.*

Proof. First, we show $T^\flat$-equivariance. We have already observed that there is a transitive action of P^1 on $\tilde{\mathscr{O}}$ and that $T^\flat$ stabilizes $\tilde{A}$. For any $s \in T^\flat$ and $u \in P^1$, we calculate

$$\begin{aligned}\tilde{\pi}(\mathrm{Ad}^*(s)\,\mathrm{Ad}^*(u)\tilde{A}) = \tilde{\pi}(\mathrm{Ad}^*({}^s u)\tilde{A}) &= \mathrm{Ad}^*({}^s u)\tilde{\pi}(\tilde{A}) \\ &= \pi_{\mathfrak{P}^1}(\mathrm{Ad}^*(s)\,\mathrm{Ad}^*(u)\,\mathrm{Ad}^*(s^{-1})(\pi_{\mathfrak{t}}^*(A))) \\ &= \pi_{\mathfrak{P}^1}(\mathrm{Ad}^*(s)\,\mathrm{Ad}^*(u)(\pi_{\mathfrak{t}}^*(A))) = s \cdot (\mathrm{Ad}^*(u)(\pi_{\mathfrak{t}}^*(A))).\end{aligned}$$

Next, we show that the stabilizer of $\tilde{A}$ in P^1 is the same as the stabilizer of $\pi_{\mathfrak{t}}^*(A)$. Let $\tilde{A}_\nu \in \mathfrak{P}^{-r}$ be a representative for $\tilde{A}$. In fact, $\tilde{A}_\nu \in (\mathfrak{t} + \mathfrak{P}) \cap (\mathfrak{t}^\flat)^\perp$. By Lemma 3.21, the stabilizer of $\pi_{\mathfrak{t}}^*(A)$ is precisely $(T(\mathfrak{o}) \cap P^1)P^r$ if $r \geq 2$ or P^1 if $r = 1$ (in which case $\pi_{\mathfrak{t}}^*(A)$ is a singleton orbit). It suffices to show that this group stabilizes $\tilde{A}$, as the stabilizer of $\tilde{A}$ is a subgroup of the stabilizer of $\pi_{\mathfrak{t}}^*(A)$. Since $\tilde{A}_\nu \in \mathfrak{t} + \mathfrak{P}$, $T \cap P^1$ stabilizes $\tilde{A}$. Now, take $u \in P^r$, $z \in \mathfrak{t}^\flat$ and $X \in \mathfrak{P}^1$. We see that

$$\mathrm{Ad}^*(u)(\tilde{A})(z + X) = \tilde{A}(\mathrm{Ad}(u^{-1})z + \mathrm{Ad}(u^{-1})X) = \tilde{A}(\mathrm{Ad}(u^{-1})z) + \tilde{A}(X),$$

so we need only check that $\mathrm{Ad}^*(u)\tilde{A}(z) = \tilde{A}(z)$. However, by Proposition 2.11(4), $\mathrm{Ad}^*(u)\tilde{A}(z) = \langle \mathrm{Ad}(u)\tilde{A}_\nu, z\rangle_\nu = \langle \pi_{\mathfrak{t}}(\mathrm{Ad}(u)\tilde{A}_\nu), z\rangle_\nu$. Proposition 2.11(3) implies that $\pi_{\mathfrak{t}}(\mathrm{Ad}(u)\tilde{A}_\nu) \equiv \pi_{\mathfrak{t}}(\tilde{A}_\nu) \pmod{\mathfrak{P}^1}$. It follows that the stabilizer of $\tilde{A}$ is, indeed, the same as the stabilizer of $\pi_{\mathfrak{t}}^*(A)$; moreover, since $\tilde{\pi}$ is a P^1-map, it follows that $\tilde{\pi}$ is an isomorphism.

Finally, we need to show that $\tilde{\pi}$ preserves the natural symplectic form on each coadjoint orbit. In particular, since $\tilde{\mathscr{O}}$ and $\mathscr{O}^1$ are P^1 orbits, it suffices by transitivity to show that the symplectic forms are the same at $\tilde{A}$ and A^1. In other words, we need to show that $\tilde{A}([X_1 + z_1, X_2 + z_2]) = A^1([X_1, X_2])$ for $z_j \in \mathfrak{t}^\flat$ and $X_j \in \mathfrak{P}^1$. This is clear, since the restriction of $\tilde{A}$ to $\mathfrak{P}^1$ is exactly A^1, and $\mathfrak{t}^\flat$ lies in the kernel of the symplectic form at $\tilde{A}$. □

5.2. **Proof of the theorem.** Let V be a trivializable vector bundle on $\mathbb{P}^1$, and let ∇ be a meromorphic connection with singularities at $\{x_1, \ldots, x_m\}$. We assume that ∇ has compatible framings $\{g_1, \ldots, g_m\}$ at each of the singular points and that ∇ has formal type $A_i \in \mathfrak{t}_i^\vee$ at x_i. We define $\mathscr{O}_i \subset \mathfrak{P}_i$ (resp. $\mathscr{O}_i^1 \subset \mathfrak{P}_i^1$) to be the coadjoint orbit of $\pi_\mathfrak{t}^*(A_i)$ under P_i (resp. P_i^1). We fix a global trivialization as in the beginning of the section; as usual, we will use this fixed trivialization to identify subgroups of $\mathrm{GL}(V_x)$ and $\mathrm{GL}_n(F_x)$, etc.

Definition 5.21. The *principal part* $[\nabla_x]^{pp}$ of ∇ at x is the image of $[\nabla_x]$ in $\mathfrak{g}_x^\vee$ by the residue-trace pairing.

To give an example, if $[\nabla_0] = M_{-1}\frac{dt}{t^2} + M_0\frac{dt}{t} + M_1 dt + M_2 t dt + \ldots$, with the $M_i \in \mathrm{GL}_n(\mathbb{C})$, then $[\nabla_0]^{pp}(X) = \mathrm{Res}(\mathrm{Tr}((M_{-1}\frac{dt}{t^2} + M_0\frac{dt}{t})X))$ for any $X \in \mathfrak{g}_0$.

We set $[\nabla_i]^{pp}$ to be the principal part of ∇ at x_i. It is a consequence of the duality theorem ([23, Theorem II.2]) that ∇ is uniquely determined by the collection $\{[\nabla_i]^{pp}\}$. Moreover, the residue theorem ([23, Proposition II.6]) shows that $\sum_i \mathrm{res}([\nabla_i]^{pp}) = 0$.

If g_i is a compatible framing for ∇ at x_i,

$$\pi_{\mathfrak{P}_i}((\mathrm{Ad}^*(g_i)[\nabla_i]^{pp}) \in \mathscr{O}_i \qquad \text{and} \qquad \pi_{\mathfrak{P}_i^1}(\mathrm{Ad}^*(g_i)[\nabla_i])^{pp}) \in \mathscr{O}_i^1.$$

This follows from the observation that $g_i \cdot [\nabla_i]^{pp} = \mathrm{Ad}^*(g_i)[\nabla_i]^{pp}$ and Proposition 5.9. In particular, since $g_i \cdot [\nabla_i]$ is formally gauge equivalent to $\pi_\mathfrak{t}^*(A)$ by an element of $p_i \in P_i^1$, it follows that $\mathrm{Ad}^*(p_i)\,\mathrm{Ad}^*(g_i)[\nabla_i]^{pp} = \pi_\mathfrak{t}^*(A)$.

Finally, we need to define extended orbits $\mathscr{M}(A)$ and $\widetilde{\mathscr{M}}(A)$ in the case where A is a regular singular formal type. In particular, the corresponding uniform stratum is of the form $(G, 0, \beta)$. Since $\pi_\mathfrak{t}^*(A)$ is a functional on $\mathfrak{g}$ that kills $\mathfrak{g}^1$, we may think of $\pi_\mathfrak{t}^*(A)$ as an element of $\mathfrak{gl}_n(\mathbb{C})^\vee$. We define $(\mathfrak{t}^\flat)' \subset \mathfrak{gl}_n(\mathbb{C})^\vee$ to be the set of functionals of the form $\phi(X) = \mathrm{Tr}(DX)$, where $D \in \mathfrak{t}^\flat$ is a diagonal matrix with distinct eigenvalues modulo $\mathbb{Z}$.

The following definition comes from Section 2 of [5].

Definition 5.22. Let A be a formal type corresponding to a stratum $(G, 0, \beta)$. Define $\mathscr{M}(A) = \mathscr{O}_A$, the coadjoint orbit of $\pi_\mathfrak{t}^*(A)$ in $\mathfrak{g}^\vee$. Moreover, let

$$\widetilde{\mathscr{M}}(A) := \{(g, \alpha) \in \mathrm{GL}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C})^\vee \mid \mathrm{Ad}^*(g)\alpha \in (\mathfrak{t}^\flat)'\} \subset G \times \mathfrak{g}^\vee.$$

Remark 5.23. This definition of $\mathscr{M}(A)$ coincides with the definition for $r > 0$ given in (5.1) (where now $Q = \mathrm{GL}_n(\mathbb{C})$), but this is not true for $\widetilde{\mathscr{M}}(A)$. Indeed, $\widetilde{\mathscr{M}}(A)$ is independent of formal type when $r = 0$.

However, the essential results of Section 5.1 remain true in the regular singular case. By [14, Theorem 26.7], $\widetilde{\mathscr{M}}(A)$ is a symplectic submanifold of $T^*\mathrm{GL}_n(\mathbb{C})$. Moreover, $T^\flat$ (resp. $\mathrm{GL}_n(\mathbb{C})$) acts on $\widetilde{\mathscr{M}}(A)$ by left multiplication (resp. inversion composed with right multiplication). The moment map for $T^\flat$ is simply $(g, X) \mapsto -\mathrm{Ad}^*(g)(X)$, and the map $(g, \alpha) \mapsto \alpha$ induces an isomorphism $\widetilde{\mathscr{M}}(A) /\!\!/_{-A} T^\flat \cong \mathscr{M}(A)$.

Proof of Theorem 5.6. For each x_i, set $\mathscr{M}_i = \mathscr{M}(A^i)$, and $\widetilde{\mathscr{M}_i} = \widetilde{\mathscr{M}}(A^i)$. As above, a meromorphic connection ∇ on $\mathbb{P}^1$ is uniquely determined by the principal parts at its singular points $\{x_i\}$. Moreover, any collection $\{M_i\}$, where $M_i \in \mathfrak{g}_i^\vee$, that also satisfies the residue condition

$$\sum_i \mathrm{res}(M_i) = 0 \tag{5.7}$$

determines a unique connection with singularities only at the x_i's and with principal part at x_i given by M_i.

There is a map $\mathscr{M}_i \to \mathfrak{g}_i^\vee$ obtained by taking $(Q_i g, \alpha_i)$ to α_i. Lemma 5.17 implies that this map is one-to-one, and it identifies elements of $\mathscr{M}_i$ with the principal part of a framed connection at x_i with formal type A_i. We conclude that any element of $\prod_i \mathscr{M}_i$ satisfying (5.7) uniquely determines a connection ∇ with framed formal type A_i at x_i.

The action of $\mathrm{GL}_n(\mathbb{C})$ on $\prod_i \mathscr{M}_i$ induced by its action on global trivializations of V is the product of the left actions on $\mathscr{M}_i$ given in (5.2). Therefore, it follows from Proposition 5.14 that the moment map of this action is simply

$$\mu : \prod_i (Q_i g_i, \alpha_i) \mapsto \sum_i \mathrm{res}(\alpha_i).$$

The moment map $\tilde{\mu} : \prod_i \widetilde{\mathscr{M}_i} \to \mathfrak{gl}_n(\mathbb{C})^\vee$ is defined similarly. The residue condition (5.7) now translates into an m-tuple lying in $\mu^{-1}(0)$, so

$$\mathscr{M}^*(\mathbf{A}) \cong \left(\prod_i \mathscr{M}_i\right) /\!\!/_0 \, \mathrm{GL}_n(\mathbb{C}).$$

Similarly, $\widetilde{\mathscr{M}^*}(\mathbf{A}) \cong \left(\prod_i \widetilde{\mathscr{M}_i}\right) /\!\!/_0 \, \mathrm{GL}_n(\mathbb{C})$: the map $\tilde{\mu}^{-1}(0) \to \widetilde{\mathscr{M}^*}(\mathbf{A})$ takes $(U_i g_i, \alpha_i)$ to the data $(V, \nabla, \mathbf{g})$, where ∇ has principal part α_i at x_i and $\mathbf{g} = (U_i g_i)$.

By Lemma 5.16, $\mathrm{GL}_n(\mathbb{C})$ acts freely on $\widetilde{\mathscr{M}_i}$, so the action on $\prod_i \widetilde{\mathscr{M}_i}$ is free. Moreover, Lemma 5.15 states that $\tilde{\mu}$ is a submersion on each factor, so $\tilde{\mu}$ is a submersion. Therefore, $\tilde{\mu}^{-1}(0)$ is smooth. It follows that $\widetilde{\mathscr{M}^*}(\mathbf{A})$ is a smooth symplectic variety.

Finally, let $\Lambda_i = A_i|_{\mathfrak{t}_i^\flat}$. The action of $\prod_i T_i^\flat$ on $\prod_i \widetilde{\mathscr{M}_i}$ commutes with the action of $\mathrm{GL}_n(\mathbb{C})$, so by Lemma 5.13, there is a natural Hamiltonian action of $\prod_i T_i^\flat$ on $\widetilde{\mathscr{M}^*}(\mathbf{A})$. Similarly, there is a Hamiltonian action of $\mathrm{GL}_n(\mathbb{C})$ on

$$\prod_i (\widetilde{\mathscr{M}_i} /\!\!/_{-\Lambda_i} T_i^\flat) \cong (\prod_i \widetilde{\mathscr{M}_i}) /\!\!/_{\prod_i (-\Lambda_i)} \prod T_i^\flat.$$

We now see that taking the iterated symplectic reduction of the product of local data by $\mathrm{GL}_n(\mathbb{C})$ and the product of the local tori is independent of order:

$$\left((\prod_i \widetilde{\mathscr{M}_i}) /\!\!/_0 \, \mathrm{GL}_n(\mathbb{C}) \right) /\!\!/_{\prod_i (-\Lambda_i)} \prod_i T_i^\flat \cong \prod_i (\widetilde{\mathscr{M}_i} /\!\!/_{-\Lambda_i} T_i^\flat) /\!\!/_0 \, \mathrm{GL}_n(\mathbb{C});$$

indeed, both are isomorphic to the symplectic reduction via the product action: $\prod \widetilde{\mathscr{M}_i} /\!\!/_{(0, \prod_i (-\Lambda_i))} (\mathrm{GL}_n(\mathbb{C}) \times \prod T_i^\flat)$. By Proposition 5.19, it follows that

$$\widetilde{\mathscr{M}^*}(\mathbf{A}) /\!\!/_{\prod_i (-\Lambda_i)} \prod_i T_i^\flat \cong \mathscr{M}^*(\mathbf{A}).$$

□

Remark 5.24. In the case $m > 1$ above, we only require μ to be a submersion on one factor in $\prod_{i=1}^m \widetilde{\mathscr{M}}(A_i)$. In particular, the residue map on $\widetilde{\mathscr{M}}(A_1) \times \prod_{i=2}^m \mathscr{M}(A_i)$ is submersive. Moreover, by Lemma 5.16, the action of $\mathrm{GL}_n(\mathbb{C})$ on $\widetilde{\mathscr{M}}(A_1) \times \prod_{i=2}^m \mathscr{M}(A_i)$ is free. Therefore, $\mathscr{M}'(\mathbf{A}) = \left(\widetilde{\mathscr{M}}(A_1) \times \prod_{i=2}^m \mathscr{M}(A_i)\right) /\!\!/_0 \, \mathrm{GL}_n(\mathbb{C})$ is smooth, and $\mathscr{M}'(\mathbf{A}) /\!\!/_{-\Lambda_1} T_1^\flat \cong \mathscr{M}(\mathbf{A})$.

We state here a more general version of Theorem 5.6. The proof is essentially the same; however, it allows us to consider regular singular points with arbitrary monodromy. In particular, this construction includes the GL_n case of the flat G-bundle constructed in [13].

Let $\{\hat{\mathscr{O}}_j\}$ be a collection of 'non-resonant' adjoint orbits in $\mathfrak{gl}_n(\mathbb{C})$; this means that the distinct eigenvalues of elements $\hat{\mathscr{O}}_j$ do not differ by nonzero integers. Using the trace pairing, we may identify $\hat{\mathscr{O}}_j$ with a coadjoint orbit $\mathscr{O}_j \subset \mathfrak{gl}_n(\mathbb{C})^\vee$. Thus, we say that a connection ∇ on the trivial bundle V over C has residue in $\mathscr{O}_j$ at $y_j \in C$ if the principal part at y_j corresponds to an element of $\mathscr{O}_j$ in $\mathfrak{gl}_n(\mathbb{C})^\vee$. Equivalently, $[\nabla_{y_j}]^{pp} = X\frac{dt}{t}$ for some $X \in \hat{\mathscr{O}}_j$. By the standard theory of regular singular point connections (see, for example, [24, Chapter II]), if a connection (V, ∇) has non-resonant residue $X \in \hat{\mathscr{O}}_j$, then (V, ∇) is formally equivalent to $d + X\frac{dt}{t}$.

Let $\mathbf{B} = \{\mathscr{O}_j\}$ be a finite collection of non-resonant adjoint orbits corresponding to a collection of regular singular points $\{y_j\} \subset C$, and let $\mathbf{A} = \{A_i\}$ be a finite collection of formal types at $\{x_i\} \subset C$, disjoint from $\{y_j\}$.

Definition 5.25. Define $\mathscr{M}(\mathbf{A}, \mathbf{B})$ to be the moduli space of connections ∇ on the trivial bundle V with the following properties:

(1) (V, ∇) is compatibly framed at each x_i, with formal type A_i;
(2) (V, ∇) is regular singular and has residue in $\mathscr{O}_j$ at each y_j.

If $\mathbf{A}$ is nonempty, we define the extended moduli space $\mathscr{M}(\mathbf{A}, \mathbf{B})$ of isomorphism classes of data $(V, \nabla, \mathbf{g})$, where (V, ∇) satisfy the conditions above and $\mathbf{g} = (g_i)$ is a collection of local compatible framings at the x_i's.

We omit the proof of the following theorem, since it is almost identical to the proof of Theorem 5.6. We note that a similar construction is used in [4] and [3] in the case where the A_i are totally split.

Theorem 5.26.

(1) *The moduli space $\mathscr{M}(\mathbf{A}, \mathbf{B})$ is a symplectic reduction of the product of local data:*
$$\mathscr{M}(\mathbf{A}, \mathbf{B}) \cong \left[\left(\prod_i \mathscr{M}(A_i)\right) \times \left(\prod_j \mathscr{O}_j\right)\right] /\!\!/_0 \, \mathrm{GL}_n(\mathbb{C}).$$
(2) *If $\mathbf{A}$ is nonempty, then $\widetilde{\mathscr{M}}(\mathbf{A}, \mathbf{B})$ is a symplectic manifold, and*
$$\widetilde{\mathscr{M}}(\mathbf{A}, \mathbf{B}) = \left[\left(\prod_i \widetilde{\mathscr{M}}(A_i)\right) \times \left(\prod_j \mathscr{O}_j\right)\right] /\!\!/_0 \, \mathrm{GL}_n(\mathbb{C}).$$
(3)
$$\mathscr{M}(\mathbf{A}, \mathbf{B}) \cong \widetilde{\mathscr{M}}(\mathbf{A}, \mathbf{B}) /\!\!/_{\prod_i(-\Lambda_i)} \left(\prod_i T_i^\flat\right).$$

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