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ATOMISTIC SUBSEMIRINGS OF THE LATTICE OF SUBSPACES OF AN ALGEBRA

DANIEL S. SAGE

ABSTRACT. Let A be an associative algebra with identity over a field k . An atomistic subsemiring R of the lattice of subspaces of A , endowed with the natural product, is a subsemiring which is a closed atomistic sublattice. When R has no zero divisors, the set of atoms of R is endowed with a multivalued product. We introduce an equivalence relation on the set of atoms such that the quotient set with the induced product is a monoid, called the condensation monoid. Under suitable hypotheses on R , we show that this monoid is a group and the class of $k1_A$ is the set of atoms of a subalgebra of A called the focal subalgebra. This construction can be iterated to obtain higher condensation groups and focal subalgebras. We apply these results to G -algebras for G a group; in particular, we use them to define new invariants for finite-dimensional irreducible projective representations.

1. INTRODUCTION

Let A be an associative algebra with identity over a field k , and let $S(A)$ be the complete lattice of subspaces of A . The algebra multiplication on A induces a product on $S(A)$ given by $EF = \text{span}\{ef \mid e \in E, f \in F\}$. The lattice S thus becomes an additively idempotent semiring, with $\{0\}$ and $k = k1_A$ (which we will often denote by 0 and 1) as the additive and multiplicative identities.

Let R be a closed sublattice of $S(A)$ which is also a subsemiring, i.e., R contains 0 and k and is closed under arbitrary sums and intersections and finite products. (We do not require the maximum element of R to be A .) A nonzero element $X \in R$ is called decomposable (or join-reducible) if there exists $U, V \subsetneq R$ such that $X = U + V$ and indecomposable otherwise. It is immediate that the multiplication in R is determined by the product of indecomposable elements. In other words, the semiring structure is determined by the structure constants $c_{U,V}^W$ for $U, V, W \in R$ indecomposable, where $c_{U,V}^W$ is 1 if $W \subset UV$ and 0 otherwise.

In this paper, we consider subsemirings R whose product is determined by its minimal nonzero elements—the atoms of the lattice. This means that the indecomposable elements of R are precisely the atoms, so that every nonzero element is a join of atoms, i.e., R is an atomistic lattice¹.

Definition 1.1. A subsemiring $R \subset S(A)$ is called an *atomistic subsemiring* of $S(A)$ if it is also a closed atomistic sublattice.

Note that k is always an atom in R .

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¹In the usual definition, every nonzero element of an atomistic lattice is a finite join of atoms. In this paper, we allow arbitrary joins of atoms.

Example 1.2. For any A , $S(A)$ and $\{0, 1\}$ are atomistic subsemirings.

Example 1.3. Let X be any proper subspace with $X + k1_A = A$. Then $R = \{0, 1, X, A\}$ is an atomistic subsemiring if and only if $X^2 \in R$. All four possible values for X^2 can occur. Indeed, if we let $X = k\bar{t}$ in the three two-dimensional algebras $k[t]/(t^2)$, $k[t]/(t^2 - 1)$, and $k[t]/(t^2 - t)$, we obtain X^2 equal to 0, 1, and X respectively. On the other hand, if $X = \text{span}(\bar{t}, \bar{t}^2)$ in $A = k[t]/(t^3 - 1)$, then $X^2 = A$. (Note that there are never any atomistic subsemirings of size 3.)

Example 1.4. Let V be a vector space with $\dim V \geq 2$, and suppose $(\text{char } k, \dim V) = 1$. Let $A = \text{End}(V)$, and let $X = \{x \in \text{End}(V) \mid \text{tr}(x) = 0\}$. Then $R = \{0, 1, X, A\}$ is atomistic with $X^2 = A$. To see this, simply note that every matrix unit lies in X^2 : $E_{ii} = E_{ij}E_{ji}$ and $E_{ij} = E_{ij}(E_{ii} - E_{jj})$ where $i \neq j$.

Our primary motivation for considering atomistic subsemirings comes from representation theory. Let G be a group which acts on A by algebra automorphisms. This means that A is a $k[G]$ -module such that $g \cdot 1_A = 1_A$ and $g \cdot (ab) = (g \cdot a)(g \cdot b)$ for all $g \in G$ and $a, b \in A$. We let $S_G(A) \subset S(A)$ be the set of all $k[G]$ -submodules of A . This set, called the *subrepresentation semiring* of A , is simultaneously a subsemiring and complete sublattice of $S(A)$; such semirings were introduced and studied in [9, 10]. If A is a completely reducible representation, i.e., a direct sum of irreducible representations, then $S_G(A)$ is an atomistic subsemiring. For example, this occurs when G is finite, A is finite-dimensional, and k has characteristic zero.

When $G = \text{SU}(2)$ (or more generally, G is a *quasi-simply reducible* group), then the subrepresentation semirings for the G -algebras $\text{End}(V)$ (with V a representation of G) have had important applications in materials science and physics [5, 4, 9]. The structure of such semirings is intimately related to the theory of $6j$ -coefficients from the quantum theory of angular momentum [9, 10, 11, 6].

Our goal in this paper is to study the set of atoms $\mathcal{Q}(R)$ of an atomistic subsemiring and to use it to define new invariants for appropriate R —the condensation group, the focus, the focal subalgebra, and higher analogues. Our methods are motivated by the theory of hypergroups.

We now give a brief outline of the contents of the paper. In Section 2, we define a multivalued product on the set $\mathcal{Q}(R)$ of atoms of an atomistic subsemiring R . In the next section, we introduce an equivalence relation ζ^* on $\mathcal{Q}(R)$. We show that if R has no zero-divisors, then the quotient set $\mathcal{Q}(R)/\zeta^*$ is naturally a monoid (called the condensation monoid) while if R is *weakly reproducible*, the condensation monoid is in fact a group. In Section 4, we define the focus $\varpi_R \subset \mathcal{Q}(R)$ and focal subalgebra $F(R) \subset A$ of R . The main result is Theorem 4.3, which states that if R is weakly reproducible of finite length, then $[0, F(R)]$ is an atomistic subsemiring with the same properties and whose set of atoms is ϖ_R . This allows us to iterate our construction to obtain higher order versions of our invariants. In Section 5, we prove Theorem 4.3 by analyzing *complete subsets* of $\mathcal{Q}(R)$. We apply our results to G -algebras in the final section. In particular, we show how to associate new invariants to irreducible projective representations.

2. A HYPERPRODUCT ON THE SET OF ATOMS

From now on, R will always be an atomistic subsemiring of $S(A)$. Let $\mathcal{Q}(R)$ denote the set of atoms of R . If $R = S_G(A)$ for a G -algebra A , we write $\mathcal{Q}_G(A)$

instead of $\mathcal{Q}(S_G(A))$. We make the notational convention that, unless otherwise specified, capital letters towards the end of the alphabet will denote atoms.

There is a natural operation $\mathcal{Q}(R) \times \mathcal{Q}(R) \rightarrow \mathcal{P}(\mathcal{Q}(R))$ given by $X \circ Y = \{Z \in \mathcal{Q}(R) \mid Z \subset XY\}$. Our first goal is to find a natural equivalence relation on $\mathcal{Q}(R)$ (for appropriate R) for which \circ induces a monoid (or group) structure on the set of equivalence classes.

Before proceeding, we need to recall some definitions from the theory of hypergroups. A set \mathcal{H} is called a hypergroupoid if it is endowed with a binary operation $\circ : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{P}^*(\mathcal{H})$, where $\mathcal{P}^*(\mathcal{H})$ is the set of nonempty subsets of \mathcal{H} . If this operation is associative, then \mathcal{H} is called a semihypergroup; if \mathcal{H} also satisfies the reproductive law $\mathcal{H} \circ x = \mathcal{H} = x \circ \mathcal{H}$ for all $x \in \mathcal{H}$, then \mathcal{H} is called a hypergroup. (For more details on hypergroups, see the books by Corsini [1] and Vougiouklis [12].)

An element e of the hypergroupoid \mathcal{H} is called a scalar identity if $e \circ x = \{x\} = x \circ e$ for all $x \in \mathcal{H}$; if a scalar identity exists, it is unique. For later use, we introduce a weak version of the reproductive law. A hypergroupoid with scalar identity e satisfies the *weak reproductive law* if for any $x \in \mathcal{H}$, there exists $u, v \in \mathcal{H}$ such that $e \in x \circ u \cap v \circ x$. Note that a semihypergroup that satisfies the weak reproductive law is a hypergroup. Indeed, given $y \in \mathcal{H}$, $y \in y \circ e \subset y \circ (v \circ x) = (y \circ v) \circ x$, so there exists $w \in y \circ v$ such that $y \in w \circ x$. Similarly, there exists w' such that $y \in x \circ w'$.

In general, $\mathcal{Q}(R)$ is not even a hypergroupoid. However, we have the following result:

Proposition 2.1. *Let R be an atomistic subsemiring. Then $(\mathcal{Q}(R), \circ)$ is a hypergroupoid if and only if R is an entire semiring (i.e., R has no left or right zero divisors).*

Proof. Suppose R is entire. If $X, Y \in \mathcal{Q}(R)$, then the nonzero subspace XY must contain an atom, so $X \circ Y \neq \emptyset$. Conversely, if E, F are nonzero elements of R such that $EF = 0$, then choosing $X, Y \in \mathcal{Q}(R)$ such that $X \subset E$ and $Y \subset F$ implies that $XY = 0$, i.e., $X \circ Y = \emptyset$. \square

In particular, if A has zero divisors, then $\mathcal{Q}(S(A))$ is not a hypergroupoid. We will only be interested in atomistic subsemirings R for which $\mathcal{Q}(R)$ is a hypergroupoid, so, from now on, we assume that R is entire, unless otherwise specified. Note that k is a scalar identity for $\mathcal{Q}(R)$.

We begin by considering a motivating example. We need to recall some basic properties of semisimple, multiplicity-free representations. This class of G -modules is closed under taking submodules and quotients. Any such representation V is the direct sum of its irreducible submodules, and this is the only way of decomposing V as the internal direct sum of irreducible submodules. Moreover, there is a bijection between the power set of the set of irreducible submodules of V and the set of subrepresentations of V given by $J \mapsto \sum_{X \in J} X$. It follows that if $\{V_i \mid i \in I\}$ is a collection of submodules of V and $W = \sum_{i \in I} V_i$, then for X irreducible, $X \subset W$ if and only if $X \subset V_j$ for some $j \in I$.

Proposition 2.2. *Let A be a multiplicity-free G -algebra with no proper, nontrivial left (or right) invariant ideals. Then $\mathcal{Q}_G(A)$ is a hypergroup.*

Proof. First, we show that the multiplication on $\mathcal{Q}_G(A)$ is associative. Fix $X, Y, Z \in \mathcal{Q}_G(A)$. Since A is multiplicity-free, $XY = \sum_{j \in J} U_j$, where $X \circ Y = \{U_j \mid j \in J\}$.

As discussed above, an irreducible submodule W lies in $(XY)Z = \sum U_j Z$ if and only if it is contained in $U_i Z$ for some i , i.e., $W \in U_i \circ Z$. We thus see that $(X \circ Y) \circ Z$ is the set of irreducible submodules of XYZ . A similar argument shows that the same holds for $X \circ (Y \circ Z)$.

Next, we show that $X \circ Y \neq \emptyset$ for any $X, Y \in \mathcal{Q}_G(A)$. It suffices to show that $XY \neq 0$ for all X, Y . Let $Y^\perp = \{a \in A \mid ay = 0 \text{ for all } y \in Y\}$. The subspace Y^\perp is clearly a left ideal. Moreover, it is a subrepresentation: given $g \in G, u \in Y^\perp$, $(g \cdot a)u = g \cdot (a(g^{-1} \cdot u)) = g \cdot 0 = 0$. Since $Y^\perp \neq A$, our hypothesis on invariant left ideals implies that $Y^\perp = 0$ and $XY \neq 0$ for all X .

Finally, we show that $X \circ \mathcal{Q}_G(A) = \mathcal{Q}_G(A) = \mathcal{Q}_G(A) \circ X$ for any X . The subspace AX is a nonzero left ideal which is obviously a subrepresentation, so $AX = A$. Writing A as a sum of irreducible submodules $A = \sum_{i \in I} U_i$, we have $A = \sum U_i X$. The usual multiplicity-free argument shows that each U_j lies in some $U_{i_j} X$, so $U_j \in U_{i_j} \circ X$. The other equality uses the condition on invariant right ideals. \square

Matrix algebras give an important class of examples. If V is a finite-dimensional vector space and $\text{End}(V)$ is a G -algebra, then V is naturally a projective representation of G [8]. It was further shown in [8] that $\text{End}(V)$ for such representations has no proper, nontrivial invariant left or right ideals if and only if V is irreducible. Hence, we obtain:

Corollary 2.3. *If V is a finite-dimensional irreducible projective representation of a group such that $\text{End}(V)$ is multiplicity free, then $\mathcal{Q}_G(\text{End}(V))$ is a hypergroup.*

This corollary applies, for example, to every irreducible complex representation of $\text{SU}(2)$.

The importance of Proposition 2.2 stems from the fact that there is a group naturally associated to every hypergroup. More generally, let \mathcal{H} be a semihypergroup. Consider the relation β defined by $x \beta y$ if and only if there exists $z_1, \dots, z_n \in \mathcal{H}$ such that $x, y \in z_1 \circ \dots \circ z_n$. Koskas showed that if β^* is the transitive closure of β , then the induced multiplication makes \mathcal{H}/β^* into a semigroup, and β^* is the largest equivalence relation on \mathcal{H} with this property [7]. If \mathcal{H} is a hypergroup, then Freni proved that β is automatically transitive [2]; thus, \mathcal{H}/β is a group.

We are led to the following provisional definition.

Definition 2.4. Let A be a G -algebra satisfying the hypotheses of Proposition 2.2. The group $Q_G(A) = \mathcal{Q}_G(A)/\beta$ is called the *condensation group* of A .

We will generalize this definition to a much broader class of atomistic subsemirings below. However, before continuing we provide a few examples.

Example 2.5. If k denotes the trivial G -algebra, then $Q_G(k)$ is the trivial group.

Example 2.6. If V is any irreducible representation of $\text{SU}(2)$, then $Q_{\text{SU}(2)}(\text{End}(V)) = 1$. The proof is a special case of Theorem 6.6 below.

Example 2.7. Let V be the standard representation of S_3 over the complex numbers. The corresponding S_3 -algebra decomposes as $\text{End}(V) = \mathbf{C} \oplus \sigma \oplus V$, where σ is the sign representation. Since $\sigma^2 = \mathbf{C}$, $\sigma V = V\sigma = V$, and $V^2 = \mathbf{C} \oplus \sigma$, we see that the classes of β are $\{\mathbf{C}, \sigma\}$ and $\{V\}$; hence, the condensation group has order 2.

Example 2.8. If F is a finite Galois extension of k with abelian Galois group G , then $Q_G(F) = G$.

We remark that if the relation β is replaced by Freni's relation γ [3], one gets an abelian group canonically related to any hypergroup. However, we will not attempt to generalize the abelian group $\mathcal{Q}_G(A)/\gamma$ to other atomistic subsemirings in this paper.

3. THE EQUIVALENCE RELATION ζ^*

It is not true in general that the hypergroupoid $\mathcal{Q}(R)$ is a hypergroup or even a semihypergroup. For example, the binary operation on $\mathcal{Q}_{A_4}(\text{End}(W))$ is not associative, where W is the three-dimensional irreducible representation of A_4 . Moreover, the reproductive law is not satisfied. (See Example 6.5 below.) We can thus no longer use the relation β^* to associate a monoid or group to R . Instead, we will do so by introducing a new relation ζ . This relation will coincide with β in the situation of Proposition 2.2.

Definition 3.1. The relation ζ on $\mathcal{Q}(R)$ is defined by $X \zeta Y$ if and only if there exists $Z_1, \dots, Z_n \in \mathcal{Q}(R)$ such that $X, Y \subset \prod_{i=1}^n Z_i$. We let ζ^* denote the transitive closure of ζ .

It is obvious that ζ^* is an equivalence relation. We will let \bar{X} denote the equivalence class of $X \in \mathcal{Q}(R)$.

Remark 3.2. If Z is an atom contained in the \circ product of Z_1, \dots, Z_n with any choice of parentheses, then $Z \subset \prod_{i=1}^n Z_i$. In fact, the relation β can be defined for hypergroupoids, and this observation just says that $\beta \subset \zeta$. However, the set of β^* -equivalence classes is not necessarily a monoid.

Definition 3.3. Let R be an entire, atomistic subsemiring of $S(A)$.

- (1) R is called *weakly reproducible* if the hypergroupoid $\mathcal{Q}(R)$ satisfies the weak reproductive law, i.e., for all $X \in \mathcal{Q}(R)$, there exists $Y, Z \in \mathcal{Q}(R)$ such that $k \in X \circ Y \cap Z \circ X$.
- (2) R is called *reproducible* if $\mathcal{Q}(R)$ satisfies the reproductive law, i.e., for all $X \in \mathcal{Q}(R)$, $\mathcal{Q}(R) \circ X = \mathcal{Q}(R) = X \circ \mathcal{Q}(R)$.

Remark 3.4. One can define an atomistic subsemiring R to be weakly reproducible without the assumption that R is entire. However, R is then entire automatically. Indeed, if $XY = 0$ for $X, Y \in \mathcal{Q}(R)$, then weak reproducibility implies the existence of Z such that $k \subset ZX$, so $Y = kY \subset ZXY = 0$, a contradiction.

Theorem 3.5. *Let R be an entire, atomistic semiring of $S(A)$. Then*

- (1) *The induced multiplication on classes makes $\mathcal{Q}(R) \stackrel{\text{def}}{=} \mathcal{Q}(R)/\zeta^*$ into a monoid.*
- (2) *If R is weakly reproducible, then $\mathcal{Q}(R)$ is a group.*

Definition 3.6. The monoid $\mathcal{Q}(R)$ is called the *condensation monoid* (or *group*) of R .

The following lemma shows that this terminology does not conflict with our previous definition.

Lemma 3.7. *If A satisfies the hypotheses of Proposition 2.2, then β and ζ coincide on $\mathcal{Q}_G(A)$.*

Proof. A similar argument to that used to demonstrate the associativity of $\mathcal{Q}_G(A)$ shows that $Z_1 \circ \dots \circ Z_n$ is the set of irreducible submodules of $\prod_{i=1}^n Z_i$, so $\beta = \zeta$. \square

Recall that an equivalence relation \sim on a hypergroupoid \mathcal{H} is called strongly regular if, for any x, y such that $z \sim y$ and any $w \in \mathcal{H}$, then $u \in x \circ w$ and $v \in y \circ w$ (resp. $u \in w \circ x$ and $v \in w \circ y$) implies that $u \sim v$. It is a standard fact that for such \sim, \circ induces a binary operation on \mathcal{H}/\sim via $\bar{x} \circ \bar{y} = \bar{z}$, where $z \in x \circ y$ [1]. Indeed, strong regularity implies that the set $\{\bar{z} \mid z \in x' \circ y' \text{ for some } x' \in \bar{x}, y' \in \bar{y}\}$ is a singleton.

Lemma 3.8. *The equivalence relation ζ^* is strongly regular.*

Proof. First, suppose that $X \zeta Y$, so $X, Y \subset \prod_{i=1}^n Z_i$ for some Z_i 's. If $U \in X \circ W$ and $V \in Y \circ W$, then $U \subset XW$ and $V \subset YW$. Thus, $U, V \subset (\prod_{i=1}^n Z_i)W$, i.e., $U \zeta W$. If $X \zeta^* Y$, then there exists $X_0, \dots, X_s \in \mathcal{Q}(R)$ with $X = X_0, Y = X_s$, and $X_i \zeta X_{i+1}$ for all i . Taking $U_i \in X_i \circ W$ with $U = U_0$ and $V = U_s$, the previous case shows that $U_i \zeta U_{i+1}$ for all i , i.e., $U \zeta^* V$. The opposite direction in the definition of strong regularity is proved similarly. \square

We now verify that the induced binary operation makes $Q(R)$ into a monoid. The identity is given by \bar{k} ; indeed, this follows immediately from the fact that $k \circ X = X = X \circ k$. Next, we check that $(\bar{X} \circ \bar{Y}) \circ \bar{Z} = \bar{X} \circ (\bar{Y} \circ \bar{Z})$. Choose $U \in X \circ Y$ and $V \in U \circ Z$, so that $\bar{V} = (\bar{X} \circ \bar{Y}) \circ \bar{Z}$. Since $U \subset XY, V \subset UZ \subset XYZ$. Similarly, choosing $T \in Y \circ Z$ and $W \in X \circ T$ gives $\bar{W} = \bar{X} \circ (\bar{Y} \circ \bar{Z})$ and $W \subset XT \subset XYZ$. By definition, $V \zeta W$, so $Q(R)$ is associative.

Remark 3.9. If we allow R to be an atomistic hemiring of $S(A)$, i.e., we do not require that $k \in R$, then the same argument shows that $Q(R)$ is a semigroup.

Finally, assume that R is weakly reproducible. Given $X \in \mathcal{Q}(R)$, choose Y, Z such that $k \subset XY \cap ZX$. By definition of the product on $Q(R)$, we obtain $\bar{X} \circ \bar{Y} = \bar{k} = \bar{Z} \circ \bar{X}$, so \bar{X} is left and right invertible. This shows that $Q(R)$ is a group and finishes the proof of the theorem.

Remark 3.10. Any monoid can be realized as the condensation monoid of an atomistic subsemiring. Indeed, given a monoid M , let kM be the corresponding monoid algebra over k with basis elements $\{e_x \mid x \in M\}$. Let $R = \{\text{span}\{e_x \mid x \in F\} \mid F \subset M\}$. This is an entire atomistic subsemiring of $S(kM)$ with $\mathcal{Q}(R) = \{ke_x \mid x \in M\}$. It is now easy to see that $Q(R) = M$.

4. THE FOCUS AND THE FOCAL SUBALGEBRA

Recall that if \mathcal{H} is a hypergroup, the *heart* $\omega_{\mathcal{H}}$ of \mathcal{H} is the kernel of the canonical homomorphism $\phi : \mathcal{H} \rightarrow \mathcal{H}/\beta^*$; it is a subhypergroup of \mathcal{H} . Returning to the context of Proposition 2.2, let A be a multiplicity-free G -algebra with no proper, nonzero left or right invariant ideals. We may then use the heart ω of the hypergroup $\mathcal{Q}_G(A)$ to define an invariant subalgebra with the same properties.

Proposition 4.1. *Let A be a multiplicity-free G -algebra with no proper, nontrivial one-sided invariant ideals. Then $B = \sum\{X \mid X \in \omega\}$ is a multiplicity-free G -subalgebra with no proper one-sided invariant ideals.*

Proof. It is trivial that B is a multiplicity-free subrepresentation that contains k . Moreover, if $X, Y \in \omega$ and $Z \subset XY$ is irreducible, then $\phi(Z) = \phi(X)\phi(Y) = 1$, i.e., $Z \in \omega$. This means that Z and hence XY are subspaces of B . It remains to show that $BX = B = XB$ for any $X \in \omega$. Choose $Z \in \omega$. Since $\mathcal{Q}_G(A)$ is a hypergroup,

there exists Y irreducible such that $Z \in Y \circ X$. Since $1 = \phi(Z) = \phi(Y)\phi(X) = \phi(Y)$, we see that $Y \in \omega$, so $Z \subset BX$. The proof that $Z \subset XB$ is similar. \square

This result allows us to iterate the construction of the condensation group. Indeed, the hypergroup structure on $\mathcal{Q}_G(B) = \omega$ gives rise to the group $Q_G(B)$ and an invariant subalgebra $B' \subset B$ such that $\mathcal{Q}_G(B')$ is again a hypergroup. See Section 6 for examples.

Motivated by this situation, we make the following definitions.

Definition 4.2. Let R be an entire atomistic subsemiring.

- (1) The *focus* ϖ_R of R is the kernel of the homomorphism $\psi_R : \mathcal{Q}(R) \rightarrow Q(R)$. Equivalently, it is the equivalence class of k .
- (2) The subspace $F(R) = \sum\{X \mid X \in \varpi_R\} \in R$ is called the *focal subalgebra* associated to R .

We can now state one of the main results of the paper.

Theorem 4.3. *Let R be an entire atomistic subsemiring of $S(A)$.*

- (1) *The focal subspace $F(R)$ is a unital subalgebra of A .*
- (2) *The sublattice $[0, F(R)] \subset R$ is an entire atomistic subsemiring of $S(F(A))$ with $\varpi_R \subset \mathcal{Q}([0, F(R)])$.*
- (3) *If R is weakly reproducible and has finite length, then $\mathcal{Q}([0, F(R)]) = \varpi_R$.*
- (4) *If R is weakly reproducible (resp. reproducible) of finite length, then the same holds for $[0, F(R)]$.*

We remark that part (3) is very useful in computations as it is often easier to calculate $F(R)$ than to compute ϖ_R directly.

We will only prove the first two parts of the theorem now. The proof of the other parts requires a more detailed study of the relation ζ^* and will be given at the end of Section 5.

Proof of parts (1) and (2). If $X, Y \in \varpi_R$ and $Z \in X \circ Y$, then $1 = \psi(X)\psi(Y) = \psi(Z)$. This means that $Z \in \varpi$, so $XY \subset F(R)$. Since $k \subset F(R)$, $F(R)$ is a subalgebra. This implies that $F(R)^2 = F(R)$, so if $E, E' \in [0, F(R)]$, then $E + E' \subset F(R)$ and $EE' \subset F(R)$. Thus, the closed sublattice $[0, F(R)] \subset R$ is a subsemiring of R , and it is immediate that it is entire and atomistic. The atoms of $[0, F(R)]$ are precisely the atoms of R which are contained in $F(R)$, so $\varpi_R \subset \mathcal{Q}([0, F(R)])$. \square

Corollary 4.4. *If R is weakly reproducible and has finite length, then $Q(R) = 1$ if and only if $F(R)$ is the maximum element of R , i.e., $[0, F(R)] = R$.*

Proof. If $Q(R) = 1$, then $\varpi_R = \mathcal{Q}(R)$. Thus, $F(R)$ contains every atom in R , hence is the maximum element of R . Conversely, if $F(R)$ is the maximum of R , then part (3) of the theorem implies that $\varpi_R = \mathcal{Q}(R)$. This gives $Q(R) = 1$. \square

Remark 4.5. The forward implication in the corollary holds for any entire atomistic subsemiring.

The theorem shows that we can iterate the construction of the invariants associated to R .

Definition 4.6. The higher foci, focal subalgebras, and condensation monoids (or groups) for R are defined recursively as follows:

- $\varpi_R^1 = \varpi_R$, $F^1(R) = F(R)$, and $Q^1(R) = Q(R)$;
- $\varpi_R^{n+1} = \varpi_{[0, F^n(R)]}$, $F^{n+1}(R) = F([0, F^n(R)])$, and $Q^{n+1}(R) = Q([0, F^n(R)])$.

We observe that if R is weakly reproducible and has finite length, then $Q^n(R)$ is a group for all n .

5. COMPLETE SUBSETS OF $\mathcal{Q}(R)$

In order to prove Theorem 4.3, we need a better understanding of the equivalence relation ζ^* . In this section, we define *complete subsets* of $\mathcal{Q}(R)$ and use them to investigate the ζ^* -equivalence classes. Our analysis of ζ^* follows a similar pattern to that of β^* carried out by Corsini and Freni [1, 2]. In the end, we will show that if R is weakly reproducible, then every element of ϖ_R is ζ -related (and not just ζ^* -related) to k ; this will be the key ingredient in the proof of Theorem 4.3.

Definition 5.1.

- (1) A subset $E \subset \mathcal{Q}(R)$ is called *complete* if for all $X_1, \dots, X_n \in \mathcal{Q}(R)$, if there exists $X \in E$ such that $X \subset \prod_{i=1}^n X_i$, then for any $Y \subset \prod_{i=1}^n X_i$, $Y \in E$.
- (2) If E is a nonempty subset of $\mathcal{Q}(R)$, then the intersection of all complete subsets containing E is denoted by $\mathcal{C}(E)$; it is called the *complete closure* of E .

It is obvious that $\mathcal{C}(E)$ is the smallest closed subset containing E .

Remark 5.2. This is not the usual notion of a complete subset of a semihypergroup [1, 7], though it coincides in the context of Proposition 2.2. In this paper, we only consider completeness in the sense given above.

The basic examples of closed subsets are the ζ^* -equivalence classes.

Proposition 5.3. Any ζ^* -equivalence class is closed.

Proof. Consider the class of Z . Suppose that $X \zeta^* Z$ and $X, Y \subset \prod_{i=1}^n X_i$. Then $Y \zeta X$, so $Y \zeta^* Z$. \square

The complete closure may be computed inductively. Indeed, given $E \neq \emptyset$, define a sequence of subsets $\kappa_n(E) \subset \mathcal{Q}(R)$ recursively as follows: $\kappa_1(E) = E$ and

$$\kappa_{n+1}(E) = \{X \in \mathcal{Q}(R) \mid \exists Y_1, \dots, Y_s \in \mathcal{Q}(R) \text{ and } Y \in \kappa_n(E) \text{ such that } X, Y \subset \prod_{i=1}^s Y_i\}.$$

Set $\kappa(E) = \bigcup_{n \geq 1} \kappa_n(E)$.

Proposition 5.4. For any nonempty $E \subset \mathcal{Q}(R)$, $\mathcal{C}(E) = \kappa(E)$.

Proof. Suppose $Y \in \kappa(E)$, say $Y \in \kappa_n(E)$, and $X, Y \subset \prod_{i=1}^s Y_i$. Then $X \in \kappa_{n+1}(E) \subset \kappa(E)$, so $\kappa(E)$ is complete. Since $E \subset \kappa(E)$, $\mathcal{C}(E) \subset \kappa(E)$. Conversely, suppose that $F \supset E$ and F is complete. We show inductively that $\kappa_n(E) \subset F$. This is obvious for $n = 1$. Suppose $\kappa_n(E) \subset F$. If $X \in \kappa_{n+1}(E)$, then we can find $Y_1, \dots, Y_s \in \mathcal{Q}(R)$ and $Y \in \kappa_n(E)$ such that $X, Y \subset \prod_{i=1}^s Y_i$. Completeness of F now shows that $X \in F$ as desired. \square

We can now give a new characterization of ζ^* . Define a relation κ on $\mathcal{Q}(R)$ by $X \kappa Y$ if and only if $X \in \mathcal{C}(Y)$, where $\mathcal{C}(Y) = \mathcal{C}(\{Y\})$.

Theorem 5.5. The relations κ and ζ^* coincide.

Before beginning the proof, we will need a lemma.

Lemma 5.6.

- (1) For any $X \in \mathcal{Q}(R)$ and $n \geq 2$, $\kappa_{n+1}(E) = \kappa_n(\kappa_2(X))$.
- (2) For $X, Y \in \mathcal{Q}(R)$, $X \in \kappa_n(Y)$ if and only if $Y \in \kappa_n(X)$.

Proof. Note that $\kappa_n(\kappa_2(X))$ consists of those atoms Z for which there exists Y_i 's and $Y \in \kappa_{n-1}(\kappa_2(X))$ such that $Y, Z \subset \prod_{i=1}^s Y_i$. If $n = 2$, then $\kappa_{n-1}(\kappa_2(X)) = \kappa_2(X)$, and this is precisely the defining property of $\kappa_3(X)$. If $n > 2$, then $\kappa_{n-1}(\kappa_2(X)) = \kappa_n(X)$ by inductive hypothesis, and we see that such atoms are precisely the elements of $\kappa_{n+1}(X)$. This proves part (1).

The second assertion is also proven by induction. Suppose $X \in \kappa_2(Y)$. Then there exist Y_i 's such that $X, Y \subset \prod_{i=1}^s Y_i$, so $Y \in \kappa_2(X)$. Next, assume that the statement holds for n . If $X \in \kappa_{n+1}(Y)$, then $X, Z \subset \prod_{i=1}^s Y_i$ for some Y_i 's and $Z \in \kappa_n(Y)$. By definition, $Z \in \kappa_2(X)$, and $Y \in \kappa_n(Z)$ by induction. Hence, $Y \in \kappa_n(\kappa_2(X)) = \kappa_{n+1}(X)$. \square

Proof of Theorem 5.5. First, we show that κ is an equivalence relation. It is clear that κ is reflexive. If $X \kappa Y$ and $Y \kappa Z$, then $X \in \mathcal{C}(Y)$ and $Y \in \mathcal{C}(Z)$. Since $\mathcal{C}(Z)$ is complete and contains Y , $\mathcal{C}(Y) \subset \mathcal{C}(Z)$, so $X \in \mathcal{C}(Z)$, i.e., $X \kappa Z$. Finally, if $X \kappa Y$, then Proposition 5.4 implies that $X \in \kappa_n(Y)$ for some n . By the lemma, $Y \in \kappa_n(X) \subset \kappa(X)$, and another application of Proposition 5.4 gives $Y \kappa X$.

Next, suppose that $X \zeta Y$. Then $X, Y \subset \prod_{i=1}^s X_i$ for some X_i 's, so $X \kappa Y$. Since κ is an equivalence relation, it follows that $\zeta^* \subset \kappa$.

Conversely, assume that $X \kappa Y$, say $X \in \kappa_{n+1}(Y)$. Set $X_0 = X$. We recursively construct $X_j \in \kappa_{n+1-j}(Y)$ for $0 \leq j \leq n$ satisfying $X_j \kappa X_{j+1}$. Choose $X_1 \in \kappa_n(Y)$ such that $X, X_1 \subset \prod_{i=1}^{s_1} Y_{1,i}$ for some $Y_{1,i}$'s. This means that $X \zeta X_1$. Suppose that we have constructed the desired atoms up through X_r with $r < n$. Again, we can choose $X_{r+1} \in \kappa_{n-r}(Y)$ satisfying $X_r, X_{r+1} \subset \prod_{i=1}^{s_{r+1}} Y_{r+1,i}$ for some $Y_{r+1,i}$'s, and this gives $X_r \zeta X_{r+1}$. Note that $X_n \in \kappa_1(Y) = \{Y\}$, i.e., $X_n = Y$. We conclude that $X \zeta^* Y$ as desired. \square

Corollary 5.7. For any $E \subset \mathcal{Q}(R)$ nonempty, $\psi^{-1}(\psi(E)) = \cup_{X \in E} \mathcal{C}(X) = \mathcal{C}(E)$. In particular, the ζ^* -equivalence class of X is $\mathcal{C}(X)$.

Proof. The set $\psi^{-1}(\psi(E))$ consists of those atoms equivalent to an atom in E , hence is the union of the $\zeta^* = \kappa$ equivalence classes of atoms in E . This gives the first equality. The second follows immediately from the fact that a union of closed subsets is closed. \square

To proceed further, we need to impose additional conditions on R .

Proposition 5.8.

- (1) If R is reproducible, then for all $X \in \mathcal{Q}(R)$, $\mathcal{C}(X) = \varpi_R \circ X = X \circ \varpi_R$. In particular, the subhypergroupoid ϖ_R satisfies the reproductive law.
- (2) If R is weakly reproducible, then ϖ_R satisfies the weak reproductive law.

Proof. First, assume that R is reproducible. Suppose that $Y \in \mathcal{C}(X)$, so $Y \zeta^* X$. By reproducibility, there exist U such that $Y \subset XU$, i.e., $Y \in X \circ U$. Hence, $\psi(Y) = \psi(X)\psi(U)$, so $\psi(U) = 1$. This shows that $U \in \varpi_R$, giving $Y \in X \circ \varpi_R$. Conversely, if $Z \in X \circ \varpi_R$, then $\psi(Z) = \psi(X)$. This means that $Z \in \mathcal{C}(X)$. The

equality $\mathcal{C}(X) = \varpi_R \circ X$ is proved in the same way. When $X \in \varpi_R$, the first statement says that $\varpi_R = \varpi_R \circ X = X \circ \varpi_R$, which is the reproductive law.

If R is weakly reproducible, then the argument given above (with $Y = k$ and $X \in \varpi_R$) shows that there exists $U, V \in \varpi_R$ such that $k \subset U \circ X \cap X \circ V$ as desired. \square

Given $Z \in \mathcal{Q}(R)$, define

$$M(Z) = \{X \in \mathcal{Q}(R) \mid \exists Y_1, \dots, Y_s \in \mathcal{Q}(R) \text{ such that } X, Z \subset \prod_{i=1}^s Y_i\}.$$

Lemma 5.9. *If R is reproducible (resp. weakly reproducible), then $M(Z)$ is a complete part for all Z (resp. for $Z = k$).*

Proof. Assume that R is reproducible. Take $Y \in M(Z)$, so $Y, Z \subset \prod_{i=1}^s Y_i$ for some Y_i 's. Suppose that $Y \subset \prod_{j=1}^n Z_j$. By reproducibility, choose V, W such that $Z \subset YV$ and $Z_n \subset WZ$. Now, suppose that $X \subset \prod_{j=1}^n Z_j$ also. Then

$$X \subset \prod_{j=1}^n Z_j \subset \left(\prod_{j=1}^{n-1} Z_j \right) WZ \subset \left(\prod_{j=1}^{n-1} Z_j \right) WYV \subset \left(\prod_{j=1}^{n-1} Z_j \right) W \left(\prod_{i=1}^s Y_i \right) V.$$

On the other hand,

$$Z \subset YV \subset \left(\prod_{j=1}^n Z_j \right) V \subset \left(\prod_{j=1}^{n-1} Z_j \right) WZV \subset \left(\prod_{j=1}^{n-1} Z_j \right) W \left(\prod_{i=1}^s Y_i \right) V.$$

Thus, $Y \in M(Z)$, so $M(Z)$ is complete.

If R is weakly reproducible, then the same argument works with $Z = k$. Indeed, we need only set $W = Z_n$ and use weak reproducibility to choose V such that $k \subset YV$. \square

Corollary 5.10. *If R is reproducible (resp. weakly reproducible), then $M(Z) = \varpi_R$ for any $Z \in \varpi_R$ (resp. for $Z = k$).*

Proof. Suppose that $Z \in \varpi_R$. If $X \in M(Z)$, then $X \zeta Z$ by definition, so $X \in \varpi_R$. Thus, $M(Z) \subset \varpi_R$. Conversely, the lemma shows that, under the hypothesis on R , $M(Z)$ is a complete subset containing Z , so $\mathcal{C}(Z) = \varpi_R \subset M(Z)$. \square

Theorem 5.11.

- (1) *If R is reproducible, then ζ is transitive.*
- (2) *If R is weakly reproducible, then $X \zeta^* k$ implies that $X \zeta k$.*

Proof. Assume that R is reproducible, and take $X \zeta^* Y$. By Proposition 5.8, there exists $U \in \varpi_R$ such that $Y \in X \circ U$. Since $M(k) = \varpi_R$, Corollary 5.10 implies the existence of Y_i 's such that $U, k \subset \prod_{i=1}^s Y_i$. Thus, $Y \subset XU \subset X \prod_{i=1}^s Y_i \supset Xk = X$, so $Y \zeta X$. If R is weakly reproducible, the same argument works for $Y = k$. \square

We are now ready to return to the proof of Theorem 4.3. We first state a proposition.

Proposition 5.12. *Let R be weakly reproducible of finite length. Then there exists $X_1, \dots, X_n \in \mathcal{Q}(R)$ such that $F(R) = \prod_{i=1}^n X_i$.*

Proof. First, note that if $k \subset \prod_{i=1}^n X_i$, then $\prod_{i=1}^n X_i \subset F(R)$. Indeed, if $Z \subset \prod_{i=1}^n X_i$ is an atom, then $Z \zeta k$, so $Z \subset F(R)$. The claim follows because $\prod_{i=1}^n X_i$ is the sum of the atoms it contains.

Choose $E = \prod_{i=1}^n X_i$ containing k such that $[E, F(R)]$ has minimal length. If this length is 0, then $E = F(R)$, so suppose it is positive, i.e., $E \subsetneq F(R)$. Take $Y \in \varpi_R$ such that $Y \subsetneq E$. By Theorem 5.11, $Y \zeta 1$, so there exist $Y_1, \dots, Y_m \in \mathcal{Q}(R)$ such that $k, Y \subset E' = \prod_{j=1}^m Y_j$. This implies that $E = Ek \subset EE'$ and $Y = kY \subset EE'$, and the previous paragraph shows that $EE' \subset F(R)$. We obtain $E \subsetneq EE' \subset F(R)$, contradicting the minimality of the length of $[E, F(R)]$. \square

We apply the proposition to prove part (3) of the theorem. Indeed, if $Z \in \mathcal{Q}(R)$ and $Z \subset F(R) = \prod_{i=1}^n X_i$, then $Z \zeta 1$ by definition. This means that $Z \in \varpi_R$ as desired.

Finally, we prove part (4). Suppose that R is reproducible of finite length, and $X, Y \subset F(R)$ are atoms. By part (3), $X, Y \in \varpi_R$. By hypothesis, there exists $Z \in \mathcal{Q}(R)$ such that $Y \in X \circ Z$. Since $1 = \psi(Y) = \psi(X)\psi(Z) = \psi(Z)$, $Z \subset F(R)$ and so $\mathcal{Q}([0, F(R)]) = X \circ \mathcal{Q}([0, F(R)])$. Similarly, one shows $\mathcal{Q}([0, F(R)]) = \mathcal{Q}([0, F(R)]) \circ X$, so $\mathcal{Q}([0, F(R)])$ is reproducible. The same argument applies when R is weakly reproducible; here, one takes $Y = k$. This completes the proof of Theorem 4.3.

6. APPLICATIONS TO G -ALGEBRAS

In this section, we apply our results on atomistic semirings to subrepresentation semirings. We assume throughout that A is a G -algebra which is completely reducible as a representation. We write $Q_G(A)$ (resp. $F_G(A)$) instead of $Q(S_G(A))$ (resp. $F(S_G(A))$). We will now be able to generalize our earlier results on multiplicity-free G -algebras.

Proposition 6.1. *Let A be a G -algebra in which the trivial representation has multiplicity one. Then A has no proper, nontrivial one-sided invariant ideals if and only if $S_G(A)$ is weakly reproducible.*

Remark 6.2. Note that both conditions imply that $S_G(A)$ is entire. Indeed, if the condition on invariant ideals holds, then the argument given in the proof of Proposition 2.2 shows that $S_G(A)$ is entire. The analogous statement for weak reproducibility was shown in Remark 3.4.

Proof. Assume that A has no proper, nontrivial invariant ideals. Fix $X \in \mathcal{Q}_G(A)$, and express A as a direct sum of irreducible subrepresentations $A = \bigoplus_{i \in I} Y_i$. The subspace AX is a nonzero invariant left ideal, so we obtain $A = AX = \sum_{i \in I} XY_i$. The trivial representation must accordingly be an irreducible component of some XY_j . The fact that the trivial representation has multiplicity one in A implies that $k \subset XY_j$ as desired. Similarly, since $A = XA$, there exists Y_l such that $k \subset Y_l X$.

Conversely, suppose that $0 \neq L \neq A$ is an invariant left ideal. Let $X \subset L$ be an irreducible submodule. For any $Y \in \mathcal{Q}_G(A)$, we have $YX \subset L$; since $k \cap L = 0$, $S_G(A)$ is not weakly reproducible. A similar argument works for right ideals. \square

Corollary 6.3. *The atomistic semiring $S_G(A)$ is weakly reproducible of finite length if*

- (1) *A is a finite Galois extension of k and G is the Galois group; or*
- (2) *$A = \text{End}(V)$ is a finite-dimensional G -algebra whose underlying projective representation V is irreducible.*

Proof. Schur's lemma shows that $\text{End}(V)$ contains the trivial representation with multiplicity one, and the statement about invariant ideals was proved in [8, Theorem 5.2]. The analogous verifications for the other case are obvious. \square

We are thus able to define invariants for any G -algebra satisfying the conditions of Proposition 6.1, without our earlier assumption that the G -algebra is multiplicity-free. In particular, our results determine two new sequences of invariants associated to any irreducible projective representation, namely, the condensation groups $Q_G^n(\text{End}(V))$ and the focal subalgebras $F_G^n(\text{End}(V))$.

The focal subalgebras $F_G^n(A)$ are a decreasing sequence of invariant subalgebras (i.e., subalgebras which are also subrepresentations) of A . This is particularly interesting for $A = \text{End}(V)$ with V irreducible and k algebraically closed because in this case, there is a complete classification of such invariant subalgebras in terms of representation-theoretic data [8, Theorem 3.23].

For the rest of the paper, we assume that either G is finite and k is algebraically closed of characteristic zero or G is a compact group and $k = \mathbf{C}$. We let V be an irreducible (linear) representation of G , and set $A = \text{End}(V)$. (We make these assumptions on G and k to guarantee complete reducibility of $\text{End}(V)$; the classification of invariant subalgebras described below holds in general.)

An invariant subalgebra of A is determined by data consisting of a quadruple (H, W, U, U') ; here, H is a finite index subgroup of G , W is a linear representation of H such that $V = \text{Ind}_H^G(W)$, and U, U' are a pair of projective representations of H such that $W \cong U \otimes U'$. More precisely, there is a bijection between invariant algebras and equivalence classes of such quadruples under conjugation by G . In particular, there are a finite number of invariant subalgebras.

Given such a quadruple (H, W, U, U') , we construct the corresponding invariant subalgebra as follows: Let $g_1 = e, g_2, \dots, g_n$ be a left transversal for H in G . This gives a direct sum decomposition $V = \bigoplus_{i=1}^n g_i W$ and an associated block diagonal invariant subalgebra $\text{Ind}_H^G(\text{End}(W)) \stackrel{\text{def}}{=} \bigoplus_{i=1}^n \text{End}(g_i W)$. As an algebra, this is just the direct product of n copies of $\text{End}(W)$. Next, the isomorphism $W \cong U \otimes U'$ shows that the endomorphism algebra factors (as H -algebras) into the tensor product $\text{End}(W) \cong \text{End}(U) \otimes \text{End}(U')$. It is now immediate that $\text{End}(U) \otimes k$ is an H -invariant subalgebra of $\text{End}(W)$. Finally, we obtain the invariant algebra for the quadruple: $\text{Ind}_H^G(\text{End}(U) \otimes k)$. We remark that the two obvious invariant subalgebras k and $\text{End}(V)$ correspond to (G, V, k, V) and (G, V, V, k) respectively.

It now follows that the sequence of focal subalgebras associated to the irreducible representation V gives rise to a sequence of such quadruples.

The classification of invariant subalgebras can be very helpful for computing the $F_G^n(\text{End}(V))$. For example, suppose that $\text{End}(V)$ has no nontrivial invariant subalgebras, so that any irreducible representation generates $\text{End}(V)$. In order to show that $F_G(\text{End}(V)) = \text{End}(V)$, it is only necessary to check that $F_G(\text{End}(V))$ contains a nonscalar matrix. However, it should be noted that computing the invariant subalgebras is not necessarily straightforward. Even when G is finite, it is not determined by the character table of G . In general, one needs to know the character tables of a covering group for every subgroup of G whose index divides $\dim(V)$.

Example 6.4. Let V be the standard representation of S_3 . We have already seen that $Q_G^1(\text{End}(V)) = \mathbf{Z}_2$. The focal subalgebra $F_G^1(\text{End}(V)) = \mathbf{C} \oplus \sigma$ is isomorphic

to $\mathbf{C} \oplus \mathbf{C}$ as an algebra; it comes from the quadruple $(A_3, \chi, \chi, \mathbf{C})$, where χ is either nontrivial character of A_3 . Since \mathbf{C} and σ are not ζ^* -equivalent in $\mathcal{Q}_G(F_G^1(\text{End}(V)))$, we have $Q_G^2(\text{End}(V)) = \mathbf{Z}_2$ and for $m \geq 2$, $F_G^m(\text{End}(V)) = \mathbf{C}$ (corresponding to (S_3, V, \mathbf{C}, V)). Finally, $Q_G^n(\text{End}(V)) = 1$ for $n \geq 3$,

Example 6.5. Let W be the three-dimensional irreducible representation of A_4 . We will show that $\mathcal{Q}_{A_4}(\text{End}(W))$ is not associative and does not satisfy the reproductive law.

We have the direct sum decomposition $\text{End}(W) = \mathbf{C} \oplus Z \oplus Z' \oplus X \oplus Y$, where Z and Z' correspond to the two nontrivial characters of A_4 and X and Y are isomorphic to W . We can choose a basis for W with respect to which Y (resp. X) consists of the skew-symmetric (resp. off-diagonal symmetric) matrices and the diagonal T is the direct sum of \mathbf{C} , Z , and Z' . There are an infinite number of atoms isomorphic to W , parameterized by $[a : b] \in \mathbf{P}^1(\mathbf{C})$; we set

$$U_{[a:b]} = \text{span}\{(a+b)E_{23} + (a-b)E_{32}, (a-b)E_{13} + (a+b)E_{31}, (a+b)E_{12} + (a-b)E_{21}\}.$$

In this notation, $X = U_{[1:0]}$ and $Y = U_{[0:1]}$.

Let $P = U_{[1:1]}$. It is easily checked that $P^2 = U_{[1:-1]}$. We now calculate that $P\mathbf{C} = PZ = PZ' = P$, $P(P^2) = T$, and $PU_{[a:b]} = T \oplus P^2$ for $[a : b] \neq [1 : \pm 1]$. We thus see that $\mathcal{Q}_{A_4}(\text{End}(W))$ does not satisfy the reproductive law; if $[a : b] \neq [1 : \pm 1]$, there is no V for which $U_{[a:b]} \in P \circ V$. To verify that the associative law does not hold, note that $X \in (P \circ P^2) \circ X = \{\mathbf{C}, Z, Z'\} \circ X$. However, $P \circ (P^2 \circ X) = P \circ \{\mathbf{C}, Z, Z', P\} = \{P, P^2\}$ does not contain X .

Since $PX = T \oplus P^2$, we have $\mathbf{C}, Z, Z', P^2 \in \varpi$. Also, $P^2X = T \oplus P$, so $Q \in \varpi$. This implies that $F_{A_4}^n(\text{End}(W)) = \text{End}(W)$ for all n , and by Corollary 4.4, $Q_{A_4}^n(\text{End}(W)) = 1$ for all n .

The only nontrivial invariant subalgebra of $\text{End}(W)$ is T . (It corresponds to $(H, \chi, \chi, \mathbf{C})$, where $H \cong \mathbf{Z}_2 \times \mathbf{Z}_2$ is the subgroup of order 4 and χ is any nontrivial character of H .) Thus, one knows that $F_{A_4}(\text{End}(W)) = \text{End}(W)$ as soon as one know that $P^2 \in \varpi$.

We conclude by computing the condensation groups and focal subalgebras of endomorphism algebras for simple compact Lie groups.

Theorem 6.6. *Let V be an irreducible representation of the simple compact Lie group G . Then $Q_G^n(\text{End}(V)) = 1$ and $F_G^n(\text{End}(V)) = \text{End}(V)$ for all n .*

Proof. If $V = \mathbf{C}$, the statement is trivial. Any other V has dimension at least 2. By Corollary 4.4, it suffices to show that $F_G(\text{End}(V)) = \text{End}(V)$. Moreover, by [8, Theorem 4.3], the only proper invariant subalgebra of $\text{End}(V)$ is \mathbf{C} . Hence, we need only show that $F_G(\text{End}(V))$ contains a nonscalar matrix.

Let λ be the highest weight of V . The highest weight of the dual representation V^* is $-w_0\lambda$, where w_0 is the longest element in the Weyl group. The representation $\text{End}(V) \cong V \otimes V^*$ has a unique irreducible submodule X with highest weight $\lambda - w_0\lambda$. We can write down a highest and lowest weight vector in X explicitly. Let v_λ (resp. w_λ) be a highest (resp. lowest) weight vector in V . (The highest and lowest weights are different since $\dim V \geq 2$.) Extend the set $\{v_\lambda, w_\lambda\}$ to a basis of weight vectors for V , and let v_λ^*, w_λ^* be the corresponding dual basis vectors in V^* . Then w_λ^* (resp. v_λ^*) is a highest (resp. lowest) weight vector in V^* . It follows that $v_\lambda \otimes w_\lambda^*$ (resp. $w_\lambda \otimes v_\lambda^*$) is a highest (resp. lowest) weight vector in X .

Multiplying, we obtain $z = (v_\lambda \otimes w_\lambda^*)(w_\lambda \otimes v_\lambda^*) = v_\lambda \otimes v_\lambda^* \in X^2$. The matrix z has rank one, so is not a scalar matrix. Thus, $X^2 \neq \mathbf{C}$. However, $\text{tr}(z) = 1$, so z is not orthogonal to \mathbf{C} . This implies that $\mathbf{C} \subset X^2$. We conclude that ϖ contains at least two elements, so $F_G(\text{End}(V)) \neq \mathbf{C}$. \square

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