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An Introduction to and Discussion of Automata and Sequential Machines

Joseph Philip Seab

An Introduction to and Discussion of Automata and Sequential Machines

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Joseph Philip Seab
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Introduction

This paper is an introduction to and a discussion of some mathematical machines. Specifically, the theory of finite automata will be introduced, and a sequential machine will be discussed with respect to Peano's mapping of the unit interval continuously onto the unit square,

$$f^{\text{conts}} : I \longrightarrow I^2$$

The paper is in two parts. The first part is on automata theory, the second on Peano's function and sequential machines.

For this paper it is assumed that the reader has a basic knowledge of calculus, such as may be found in R. Creighton Buck's <u>Advanced Calculus</u>; a basic knowledge of algebra, such as may be found in Hiram Paley and Paul M. Weichsel's <u>Abstract Algebra</u>; a basic knowledge of topology, such as may be found in J. L. Kelley's <u>General Topology</u>.

Peano's paper may be found in <u>Mathematische Annalen</u>, vol. 36, 1890, pp. 157-160.

Part I

The first part of this paper will be a brief introduction to finite automata and a theorem conerning the types of tapes they define. The contents of this part are based on the paper "Finite Automata and Their Decision Problems" by M. O. Rabin and D. Scott.

The following remarks are form the introduction to the paper:

"In the last few years the idea of finite automation has appeared in the literature. These are machines having only a finite number of internal states that can be used for memory and computation. The restriction of finiteness appears to give a better approximation to the idea of a physical machine.

"(In this paper)...the definition of the one-tape, one-way automaton is given. These machines are considered as 'black boxes' having only a finite number of internal states and reacting to their environment in a deterministic fashion."

The automaton, then, can be thought of as a black box which can be asked questions and will give "yes" or "no" answers. The questions are an arbitrary finite sequence of symbols. The number of questions that can be asked is infinite. For this paper, the questions can be thought of as being given on one-dimensional tapes. The machine will have a reading device that can read one symbol of the tape at a time and then advance the tape and read the next symbol. When the tape is completed, the machine gives the answer.

The internal workings of the machine will be generalized to <u>internal states</u> which represent stable states of the machine at discrete time intervals.

The answer the machine gives is determined by the final state the machine is in when it finishes the tape. Some states are in the "yes" category and some are in the "no" category. It is assumed that all states are in one category or the other but not both.

Before a formal definition of an automaton can be given, some notation and preliminary definitions must be established.

A finite alphabet Σ is given. A <u>tape</u> is any finite sequence of symbols from Σ. The empty tape, with no symbols, is denoted by λ. The class of all tapes is denoted by T. If x and y are tapes in T, then xy denotes the tape obtained by splicing x and y together, or by concatenating the two sequences. That is, if

$x = \sigma_0 \sigma_1 \ldots \sigma_n$ and $y = \tau_0 \tau_1 \ldots \tau_n$ then $xy = \sigma_0 \ldots \sigma_n \tau_0 \ldots \tau_n$

In their paper Rabin and Scott do not define what it means for two tapes x and y to be equal. Based on their use of equality of tapes, I will supply the following definition:

Let $x = \sigma_0 \sigma_1 \ldots \sigma_n$

and $y = \tau_0 \tau_1 \ldots \tau_m$

We have x = y if n = m and

$$\sigma_0 = \tau_0, \quad \sigma_1 = \tau_1, \quad \ldots, \quad \sigma_n = \tau_m$$

That is, the symbols must be identical and must come in the same order.

The two laws $\lambda x = x\lambda = x$ $\quad$ $x(yz) = (xy)z$ follow immediately since λ, containing no symbols, does not change x, and since grouping the symbols has no effect on their order.

x^n denotes xxx···x, a tape of n x's. Also, by convention, $x^0 = \lambda$.

A formal definition of an automaton will now be given.

Definition 1. A finite automaton over the alphabet Σ is a system A = (S, M, s_0, F), where S is a finite non-empty set (the internal states of A), M is a function defined on the Cartesian product S x Σ of all pairs of states and symbols with values in S (the table of transitions or moves of A), s_0 is an element of S (the initial state of A), and F is a subset of S (the designated final states of A).

Let A be an automaton. The function M can be extended from S x Σ to S x T by the following recursive definition:

$M(s,\lambda) = s$ for s in S

$M(s,x\sigma) = M(M(s,x),\sigma)$ for s in S, x in T and σ in Σ.

From this it follows immediately that

$M(s,xy) = M(M(s,x),y)$ for all s in S and x,y in T.

The set of tapes for which the automaton gives a "yes" answer is now defined:

Definition 2. The set of tapes accepted or defined by the automaton A, in symbols T(A), is the collection of all tapes x in T such that $M(s_0,x)$ in in F.

Throughout this discussion when a set of tapes is referred to as being defined by an automaton, this will mean that the tapes are strictly defined; that is, the specified tapes and only the specified tapes are defined.

Definition 3. The class of all definable sets of tapes, in symbols $\mathfrak{J}$, is the collection of all sets of the form T(A) for some automaton A.

In order to state the theorem which characterizes definable sets, I will need the following additional definitions:

Definition 4. An equivalence relation R over the set T of tapes is right invariant if whenever xRy, then xzRyz for all z in T.

There is also an analogous definition of left-invariant equivalence relations.

Definition 5. An equivalence relation over the set T is a congruence relation if it is both right and left invariant.

Thus, if R is a congruence relation, then xRz and yRw implies xyRzw.

Letting [x] denote the equivalence class containing x and [y] that containing y, the following is a definition of the product of two equivalence classes: $[x][y] = [xy]$

This definition is unambiguous since we have, from above,

$$xRx' \text{ and } yRy' \text{ imply } xyRx'y'.$$

Definition 6. An equivalence relation over T is of finite index if there are only finitely many equivalence classes under the relation.

We may now state and prove the theorem characterizing definable sets. This theorem is due to J. R. Myhill.

THEOREM 1. (Myhill) Let U be a set of tapes. The following three conditions are equivalent:

(i) U is in $\mathfrak{J}$; that is, U is a definable set of tapes.

(ii) U is the union of some of the equivalence classes of a congruence relation over T of finite index.

(iii) The explicit congruence relation $\equiv$ defined by the condition that for all x, y, in T, $x \equiv y$ if and only if for all z, w in T, whenever zxw is in U, then zyw is in U, and conversely, whenever zyw is in U, then zxw is in U, is a congruence relation of finite index.

Proof: The three conditions will be proved equivalent by showing that (i) implies (ii) implies (iii) implies (i).

I. Show (i) implies (ii).

Assume (i). Let U = T(A) for some automaton A. Define a relation as follows: xRy if and only if M(s,x) = M(s,y) for all s in S. Show R is an equivalence relation; that is, show R is reflexive, symmetric, and transitive.

(1) xRx. Pf: M(s,x) = M(s,x) implies xRx.

(2) xRy implies yRx. Pf: xRy implies M(s,x) = M(s,y), but then M(s,y) = M(s,x) implies yRx.

(3) xRy and yRz imply xRz. Pf: xRy and yRz implies M(s,x) = M(s,y) and M(s,y) = M(s,z), but this implies that M(s,x) = M(s,y) = M(s,z) and M(s,x) = M(s,z) implies xRz.

Show R is a congruence relation; that is, show R is right and left invariant.

(1) R is right invariant:
Assume xRy and let z be any tape in T. xRy implies M(s,x) = M(s,y) for all s in S. Then
M(s,xz) = M(M(s,x),z) = M(M(s,y),z) = M(s,yz), for all s in S implies xzRyz.

(2) R is left invariant:
Let M(s,z) = s'. M(s,zx) = M(M(s,z),x) = M(s',x) = M(s',y) = M(M(s,z),y) = M(s,zy) for all s in S implies zxRzy.

Therefore, R is a congruence relation over T. It must now be shown that R is of finite index.

R is of finite index (i.e., R generates only finitely many equivalence classes.) because M(s,x) can assume only finitely many different values. If x is a fixed tape and r is the number of internal states of A, then the number of equivalence classes is at most r^r. This is based on the theorem $|f: X \rightarrow Y| \leq |X|^{|Y|}$.

It remains to be shown that U is the union of some of the equivalence classes of R. It will in fact be shown that

$U = \bigcup([x] \mid \text{for all } x \text{ in } U)$ where $[x]$ denotes the equivalence class containing x under the relation R.

(1) Given y in U, we immediately have

$[y] \subseteq \bigcup([x] \mid \text{for all } x \text{ in } U)$

therefore $y \in \bigcup([x] \mid \text{for all } x \text{ in } U)$

therefore $U \subseteq \bigcup([x] \mid \text{for all } x \text{ in } U)$.

(2) Given $y \in \bigcup([x] \mid \text{for all } x \text{ in } U)$. Then

$y \in [x]$ for some $x \in U$. This implies that

xRy implies $M(s_o,x) = M(s_o,y)$.

But since $x \in U$, $M(s_o,x) \in F$. So $M(s_o,y) \in F$

and we get $y \in U$.

Therefore $\bigcup([x] \mid \text{for all } x \text{ in } U)$.

Therefore $U = \bigcup([x] \mid \text{for all } x \text{ in } U)$.

Thus, it has been shown that U is the union of the equivalence classes under R of those tapes in U. This completes the proof that (i) implies (ii).

II. Show that (ii) implies (iii).

Assume (ii). Let R stand for any congruence relation satisfying the conditions of (ii). Consider the specific relation $\equiv$ defined in (iii) in terms of U. Show $\equiv$ is a congruence relation of finite index.

Let x and y be any tapes such that xRy. Suppose that for any z,w in T zxw is in U. R is a congruence relation implies that zxwRzyw. Since U is a union of equivalence classes of R (from ii), then zyw is in U. Now, suppose that for any z, w, in T zyw is in U. xRy implies yRx, so the same argument shows zxw is also in U. Thus, the conditions for $x \equiv y$ are fulfilled. This whole argument shows that if xRy, then $x \equiv y$. In other words, $\equiv$ is a relation making fewer distinctions than the relation R.

That $\equiv$ is a congruence relation follows from its definition. That is, given $x \equiv y$, show that for any p, q in T, $pxq \equiv pyq$. $x \equiv y$ implies that for all z, p, q, w in T, (zp)x(qw) is in U if and only if (zp)y(qw) is in U. We then have z(pxq)w is in U if and only if z(pyq)w is in U. Thus, $pxq \equiv pyq$.

Therefore, since R is of finite index, $\equiv$ must necessarily be a congruence relation of finite index.

This completes the proof that (ii) implies (iii).

III. Finally, show (iii) implies (i).

Assume (iii). An automaton A must be found which strictly defines U; that is, an A must be found such that U = T(A). Let S be the set of equivalence classes under the congruence relation $\equiv$. Define M as $M([x],\sigma) = [x\sigma]$ where the brackets indicate the formation of equivalence classes under $\equiv$. Show M is well defined.

Consider $x \equiv x'$ implies $[x] = [x']$. Show $M([x],\sigma) = M([x'],\sigma)$. $x \equiv x'$ implies that $x\sigma \equiv x'\sigma$ since $\equiv$ is right invariant. Therefore $[x\sigma] = [x'\sigma]$ and $M([x],\sigma) = M([x'],\sigma)$, which was to be shown.

Let $s_o = [\lambda]$ and let F be the set of all $[x]$ where x is in U. Show U is a union of equivalence classes under $\equiv$. This means that it must be shown that $U = \bigcup([x] \mid \text{for all } x \text{ in } U)$.

(1) Pick $x \in U$. By definition $x \in \bigcup([x] \mid \text{for all } x \text{ in } U)$. Therefore, $U \subseteq \bigcup([x] \mid \text{for all } x \text{ in } U)$.

(2) Pick $y \in \bigcup([x] \mid \text{for all } x \text{ in } U)$. Then for some $x \in U$, $y \in [x]$ implies $x \equiv y$. Since $\lambda x \lambda \in U$ then $\lambda y \lambda \in U \Rightarrow y \in U$ by the definition of $\equiv$. So we have $\bigcup([x] \mid \text{for all } x \text{ in } U) \subseteq U$. Therefore, $U = \bigcup([x] \mid \text{for all } x \text{ in } U)$.

Extending M as before from $S \times \Sigma$ to $S \times T$ we get $M([x],y) = [xy]$ for all x,y in T. We now need $M(s_o,x) = M([\lambda],x) = [x]$ is in F if and only if x is in U.

(1) Given $[x]$ is in F. Then x is in U since U is the union of equivalence classes under $\equiv$.

(2) Given x is in U. Then $[x]$ is in F by definition of F.

This gives the result that $U = T(A)$.

This completes the proof that (iii) implies (i), and the proof of Theorem 1 is complete.

A simple application of Theorem 1 is the following: Show that the set U of all tapes of the form 0^n10^n for $n = 0, 1, 2, \ldots$ is not strictly definable by any automaton. Suppose to the contrary that U is in $\mathfrak{J}$; that is, suppose U is definable. Consider the relation $\equiv$ of Theorem 1(iii). This relation must be of finite index, so for some integers

$n \neq m$ we must have $0^n \equiv 0^m$. It then follows that $0^n10^m \equiv 0^n10^n$ since $\equiv$ is a congruence relation, and hence 0^n10^m is in U since U is a union of equivalence classes under $\equiv$. But this contradicts the definition of U above. Therefore, U cannot be in $\mathfrak{J}$.

Because of the limited nature of this paper, this is all of the finite automata theory that I will present.

Part II

In this part I will present Peano's paper which defines a continuous function on the unit interval, the image of which fills the unit square,

$$f^{conts} : I \longrightarrow I^2$$

I will then present a sequential machine, due to Samuel Eilenberg, which also defines this function. At the end, some other applications of Eilenberg's sequential machine will be given.

First, I will give a discussion of "Sur une courbe, qui remplit toute une aire plane" by G. Peano. Peano's paper is a definition of his function. He presents his function in the following way:

Denote the numbers in base 3 notation. Sequences will then be made up of the numbers 0, 1, and 2.

Consider a sequence $T = 0.a_1a_2a_3\ldots$

For a term a of a sequence, define the function $ka = 2-a$, the compliment of a. We then have $k0 = 2$, $k1 = 1$, and $k2 = 0$. Note that

(1) $b = ka$ implies $a = kb$. Pf: $b = 2-a$ implies $a = 2-b$.

(2) $ka \equiv a \pmod 2$. Pf: This can be easily checked by substituting in the numbers 0, 1, 2, and observing that it is true.

Let k^na represent the result of applying k to the term a n times. Also note the following:

(3) If n is even $k^na = a$. This is because $k^2a = 2-(2-a) = a$. Thus, $k^2 = I$ (the identity function). For a number n even, there exists a q such that $n = 2q$. Therefore $k^n = k_1^2k_2^2\ldots k_q^2a = I_1I_2\ldots I_qa = a$.

(4) If n is odd, $k^na = ka$. This is because for n odd there exists a q such that $n = 2q+1$. From (3) above, we have $k^na = k^{2q+1}a = k^{2q}ka = Ika = ka$.

(5) $m \equiv n \pmod 2$ implies $k^ma = k^na$. This is because $m \equiv n \pmod 2$ just says that m and n are either

both even or both odd. From (3) and (4) above, it then follows that $k^m a = k^n a$.

Define two sequences X and Y corresponding to T,

$x = 0.b_1 b_2 b_3 \ldots \qquad y = 0.c_1 c_2 c_3 \ldots$

with these relations:

$b_1 = a_1 \quad b_2 = k^{a_2} a_3 \quad b_3 = k^{a_2+a_4} a_5$

$b_n = k^{a_2+a_4+\cdots+a_{2n-2}} a_{2n-1}$

$c_1 = k^{a_1} a_2 \quad c_2 = k^{a_1+a_3} a_4 \quad c_n = k^{a_1+a_3+\cdots+a_{2n-1}} a_{2n}$

where the a_i's are terms of T.

We then have that b_n, the n^{th} term of x, is a_{2n-1}, the n^{th} odd term of T, or its compliment, depending on whether the sum $a_2+\cdots+a_{2n-2}$ of even terms preceding it is even or odd. Similarly for Y regarding the n^{th} even term.

The relations may also be written as

$a_1 = b_1 \quad a_2 = k^{b_1} c_1 \quad a_3 = k^{c_1} b_2 \quad a_4 = k^{b_1+b_2} c_2$

$a_{2n-1} = k^{c_1+c_2+\cdots+c_{n-1}} b_n \quad a_{2n} = k^{b_1+b_2+\cdots+b_n} c_n$

I will now show that this is true. To begin with, two lemmas are needed.

<u>Lemma 1</u>. $b = k^n a$ implies $a = k^n b$. Pf: If n is even $k^n a = a$ and $k^n b = b$. Therefore $a = k^n a = b = k^n b$. If n is odd, $k^n a = ka$. Therefore, $b = k^n a = ka$ and $k^n b = kb$. So we have $b = ka$ implies $a = kb = k^n b$.

<u>Lemma 2</u>. $a = k^n b$ implies $a \equiv b \pmod 2$.

Pf: If n is even $a = k^n b = b$ implies $a \equiv b \pmod 2$.

If n is odd $a = k^n b = kb$ and $kb \equiv b \pmod 2$ implies $a \equiv b \pmod 2$ by substitution.

Therefore we have

$$b_n = k^{a_2+a_4+\cdots+a_{2n-2}}\, a_{2n-1} \Rightarrow a_{2n-1} = k^{a_2+a_4+\cdots+a_{2n-2}}\, b_n$$

by Lemma 1. Also, $a_2 \equiv c_1 \pmod 2$ since $c_1 = k^{a_1}a_2$ and by Lemma 2. Continuing, $a_4 \equiv c_2 \pmod 2$, etc.

Therefore, $a_{2n-1} = k^{a_2+a_4+\cdots+a_{2n-2}}\, b_n = k^{c_1+c_2+\cdots+c_{n-1}}\, b_n$ by (5) above.

Notice that given T, X and Y are determined, or given X and Y, T is determined by the two sets of equations given above. This fact will be used later to show that Peano's function is onto.

Define the <u>value</u> of the sequence T to be the quantity (analogous to a decimal number in the same notation)

$$t = \text{val } T = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_n}{3^n} + \cdots$$

To every sequence T there corresponds a number $t \in [0,1]$. The numbers $t \in (0,1)$ are divided into two classes:

(α) The numbers different from 0 and 1 which, when multiplied by a power of 3 give a whole number. They are represented by two sequences:

$$T = 0.a_1 a_2 \ldots a_{n-1} a_n 222\ldots$$

where $a_n = 0$ or $a_n = 1$.

This sequence can be shown as follows to give a whole number when multiplied by a power of three:

$$\frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \cdots = \frac{1}{3^{n+1}}\left[2 + \frac{2}{3} + \frac{2}{3^2} + \cdots\right]$$

$$= \frac{1}{3^{n+1}}\left[\frac{2}{1-1/3}\right] = \frac{1}{3^n}.$$

Therefore, $.a_1a_2\ldots a_{n-1}a_n222\ldots = \frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_n}{3^n} + \frac{1}{3^n}$

and this number multiplied by 3^n gives a whole number.

The second sequence representing t of class α is

$$T' = 0.a_1a_2\ldots a_{n-1}a_n'000\ldots$$

where $a_n' = a_n+1$.

This sequence is of the required type because

$$3^n\left[\frac{a_1}{3} + \frac{a_2}{3^2} + \cdots + \frac{a_n'}{3^n} + 0 \cdots\right] = 3^{n-1}a_1 + \cdots + a_n' + 0 \cdots$$

which is a whole number.

The second class is

(β) The other numbers; they are represented by only one sequence.

With the correspondence between T and the pair of sequences (X,Y) and with T and T' being two sequences of different forms, we need to show that

$$\text{val } T = \text{val } T'$$
$$\text{val } X = \text{val } X'$$
$$\text{val } Y = \text{val } Y'$$

where X and Y correspond to T and X' and Y' correspond to T'. To do this we will need the following lemma:

Lemma 3. $k^n0 = k^{n+1}2$

Pf: If n is even $k^n0 = 0$ and $k^{n+1}2 = k2 = 0$.

If n is odd $k^n0 = k0 = 2$ and $k^{n+1}2 = 2$.

To show the desired correspondence, consider the series

$$T = 0.\ a_1a_2\ldots a_{2n-2}a_{2n-1}a_{2n}222\ldots$$

where a_{2n-1} and a_{2n} are not both equal to 2(If a_{2n-1} and a_{2n}

were both equal to 2, then the series would be in general the same. If, however, we have only $a_{2n} = 2$ then the series is not in general the same since a_{2n} is an even term but the first 2 of the infinite sequence of 2's above is an odd term.) This is a class α series. We have the corresponding X sequence, $x = 0.b_1 b_2 \ldots b_{n-1} b_n b_{n+1} \ldots$

with

$$b_n = k^{a_2+\cdots+a_{2n-2}} a_{2n-1}$$

$$b_{n+1} = b_{n+2} = \cdots = k^{a_2+\cdots+a_{2n-2}+a_{2n}} 2$$

$$\text{since } a_2+\cdots+a_{2n-2}+a_{2n}+2+2+\cdots \equiv a_2+\cdots+a_{2n} \pmod 2.$$

The other class α series corresponding to t is

$T' = 0.a_1 a_2 \ldots a_{2n-2} a'_{2n-1} a'_{2n} 000\ldots$ and $X' = 0.b_1 \ldots b_{n-1} b'_n b'_{n+1} \ldots$

The first 2n-2 terms of T' coincide with the first 2n-2 terms of T. Also, the n-1 terms of X' coincide with those of X. The other terms are determined as follows:

$$b'_n = k^{a_2+\cdots+a_{2n-2}} a'_{2n-1}$$

$$b'_{n+1} = b'_{n+2} = \cdots = k^{a_2+\cdots+a_{2n-2}+a'_{2n}} 0.$$

We consider two cases:

<u>Case 1</u>. $a_{2n} < 2$.

If a_{2n} is 0 or 1, $a'_{2n} = a_{2n}+1$ and $a'_{2n-1} = a_{2n-1}$

Therefore $b'_n = b_n$ by substitution.

We have that $a_2+a_4+\cdots+a_{2n-2} + a'_{2n} = a_2+\cdots+a_{2n-2}+a_{2n}+1$ so that by lemma 3

$$b'_{n+1} = b'_{n+2} = \cdots = k^{a_2+\cdots+a_{2n-2}+a'_{2n}} 0 = k^{a_2+\cdots+a_{2n}} 2$$

$$= b_{n+1} = b_{n+2} = \cdots$$

Therefore, the two series X and X' coincide in form and in value.

Case 2. $a_{2n} = 2$ and $a_{2n-1} = 0$ or 1.

We have that $a'_{2n} = 0$, $a'_{2n-1} = a_{2n-1}+1$.

Let $s = a_2+a_4+\cdots+a_{2n-2}$

Then $b_n = k^s a_{2n-1}$, $b_{n+1} = b_{n+2} = \cdots = k^s 2$

since adding $a_{2n} = 2$ to s does not change $k^s 2$ (i.e., $k^s 2 = k^{s+2} 2$). Also $b'_n = k^s a'_{2n-1}$, $b'_{n+1} = b'_{n+2} = \cdots = k^s 0$

since $a'_{2n} = 0$ and $k^s 0 = k^{s+a'_{2n}} 0$.

Note that

(6) If $a' = a+1$, then it follows that $.a222\ldots = .a'000\ldots$

We must show that $.b_1\ldots b_n b_{n+1} \ldots = .b_1\ldots b'_n b'_{n+1} \ldots$

There are 2 cases.

Case 1. s is even.

Then we have $b_n = a_{2n-1}$ $b_{n+1} = b_{n+2} = \cdots = 2$

$b'_n = a'_{2n-1}$ $b'_{n+1} = b'_{n+2} = \cdots = 0$

Therefore val X = $.b_1 b_2\ldots b_{n-1} a_{2n-1} 2222\ldots$

and val X' = $.b_1 b_2\ldots b_{n-1} a'_{2n-1} 000\ldots$

and it follows that val X = val X' from (6) above.

Case 2. s is odd.

We then have $b_n = 2-a_{2n-1}$ $b_{n+1} = b_{n+2} = \cdots = 0$

$b'_n = 2-(a_{2n-1}+1) = (2-a_{2n-1})-1$, $b'_{n+1} = b'_{n+2} = \cdots = 2$.

Therefore val X = $b_1 b_2\ldots b_{n-1}(2-a_{2n-1})000\ldots$

and val X' = $b_1 b_2\ldots b_{n-1}[(2-a_{2n-1})-1]222\ldots$

It then follows that val X = Val X' from (6) above.

Therefore, the two fractions X and X', although they have different forms, have the same value.

It has been shown, then, that in all cases val X = val X'. Similarly, it can be shown that val Y = val Y'.

Now consider the two functions

$$x(t) = \text{val } X \quad \text{and} \quad y(t) = \text{val } Y$$

where t = val T, and X and Y correspond to T. Define the following function: $(x,y)(t) = (x(t),y(t))$. Show the function $(x,y)(t)$ is continuous at any point $t_o \in [0,1]$.

Let x = val X and y = val Y. Two numbers x_1 and x_2 are close together when $|x_1 - x_2| = \delta$ is small.

Given $\epsilon > 0$. Show there exists δ such that for $|t-t_o| < \delta$, $|(x,y)(t) - (x,y)(t_o)| < \epsilon$. This is true whenever $|x(t) - x(t_o)| < \epsilon$ and $|y(t) - y(t_o)| < \epsilon$, using a square neighborhood. Say that for a given δ_n, $|t-t_o| < \delta_n$ implies t correspondes to t_o in the first 2n terms. It follows that x and y corresponding to t coincide in the first n terms to the x_o and y_o corresponding to t_o since the first n terms of x, y and x_o, y_o are determined by the first 2n terms of t and t_o respectively, and since for T and T' we have val X = Val X' and val Y = val Y'.

Pick a number ϵ_n with $\epsilon_n = .000...1$ where the 1 is in the n^{th} place. For $|t-t_o| < \delta_n$ we have

$$|x(t) - x(t_o)| < \epsilon_n \quad \text{and} \quad |y(t) - y(t_o)| < \epsilon_n$$

since x(t) coincides with $x(t_o)$ in the first n terms, and similarly for y(t) and $y(t_o)$. Therefore, for a given ϵ pick $\delta < \delta_n$ for some n such that $\epsilon_n < \epsilon$. This gives $|x(t) - x(t_o)| < \epsilon$ and $|y(t) - y(t_o)| < \epsilon$. Hence, the desired result is shown.

The function $(x,y)(t)$ is onto since for any pair of numbers (x,y) with $0 \leq x \leq 1$ and $0 \leq y \leq 1$ there corresponds a pair of sequences (X,Y). As was stated before, with X and Y determined, we can then determine T, and hence t, giving us $(x,y)(t) = (x,y)$. Therefore, $(x,y)(t)$ is onto.

Thus, $(x,y)(t)$ is continuous and maps I onto I^2.

It may help to clarify the function $(x,y)(t)$ by noting the following scheme for getting from t to $(x,y)(t)$:

$$t \begin{cases} T \longrightarrow X,Y \longrightarrow \text{val } X,\ \text{val } Y \longrightarrow x,y \\ \qquad\qquad\qquad\qquad\ \, \shortparallel \qquad\quad \shortparallel \qquad\qquad\ \shortparallel \ \ \shortparallel \\ T' \longrightarrow X',Y' \longrightarrow \text{val } X',\ \text{val } Y' \longrightarrow x',y' \end{cases} \begin{matrix} x = .a_1 a_2 \ldots \\ \\ y = .b_1 b_2 \ldots \end{matrix}$$

This concludes my discussion of Peano's paper.

Next I will discuss Eilenberg's machine which defines Peano's function. This machine is not an automaton because there are no final states and there is an output function. It is similar to an automaton in that it is a machine with a finite number of internal states, reads a finite alphabet, has an initial state, and has a function giving the state changes as a function of the letters of the alphabet.

The machine is as follows:

The input and output numbers are written base 3. The input tapes consist of base 3 representations of the elements of the unit interval, and are written as

$$T = .a_1 b_1 a_2 b_2 \ldots a_n b_n \ldots$$

The a's represent the odd terms, the b's represent the even terms.

The function k, to be used in defining the output sequences, is defined as

$$k : \{0,1,2\} \longrightarrow \{0,1,2\}$$

$$k(x) = 2-x$$

This is the same function k as appeared in Peano's paper.

The output sequences, written as

$$x = .c_1 c_2 \ldots$$

$$y = .d_1 d_2 \ldots$$

are defined as follows:

$$c_n = k^{b_1+b_2+\cdots+b_{n-1}} a_n$$

$$d_n = k^{a_1+a_2+\cdots+a_n} b_n$$

It should be noted that these definitions are the same as Peano's since the b's represent the even terms and the a's are the odd terms of the original sequence T.

The machine is defined as follows:

$S = \{(p,q) \mid p,q \text{ integers mod } 2\}$

$\Sigma = \{0,1,2\} \times \{0,1,2\}$

$M((p,q),(a,b)) = (p+a(\text{mod } 2),\ q+b(\text{mod } 2))$

$s_0 = (0,0)$

The output function U is

$U((p,q),(a,b)) = (k^q a, k^{p+a} b)$ where $k^q a$ gives a term of the X sequence and $k^{p+a} b$ gives a term of the Y sequence.

It should be noted here that the purpose of the states S, being pairs of integers mod 2, is to designate whether certain numbers are even or odd. The purpose of M is to keep track of the even or odd values of the sum of the even terms of T and the sum of the odd terms of T. That is, say p, being an integer mod 2, represents whether the sum of odd terms at a particular point in the sequence is even or odd. Adding the term a is adding the next odd term, and taking p+a mod 2 is designating p+a, the next step in the summation of odd terms, as odd or even. The same thing is happening with q+b. It should also be noted that U defines X and Y the same way as Peano does. That is, q gives the even or odd value of all the even terms before the term $\underline{a}$ in the definition, and p+a does the same for all the odd terms before the term b.

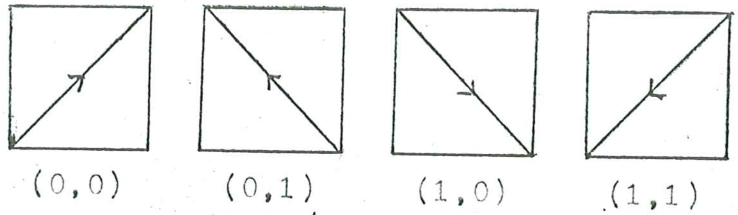
For example, remembering that $T = .a_1 b_1 \ldots$, let p represent the even or odd value of the sum of odd terms $a_1+a_2+\cdots+a_n$ and let q_n represent the even or odd value of the sum of even terms $b_1+b_2+\cdots+b_n$. Say the machine is in the state (p_n, q_n) after having read through the letter b_n of the sequence T. The next step of the machine is to read (a_{n+1}, b_{n+1}). The state change is as follows:
$M((p_n,q_n),(a_{n+1},b_{n+1})) = (p_n+a_{n+1} \pmod 2,\ q_n+b_{n+1} \pmod 2)$, $p_n+a_{n+1} \pmod 2$ being the even or odd value of the sum $a_1+a_2+\cdots+a_n+a_{n+1}$ and $q_n+b_{n+1} \pmod 2$ being the even or odd value of the sum $b_1+b_2+\cdots+b_n+b_{n+1}$. The output is as follows:
$U((p_n,q_n),(a_{n+1},b_{n+1})) = (k^{q_n} a_{n+1}, k^{p_n+a_{n+1}} b_{n+1})$, where q_n is the even or odd value of even terms preceding a_{n+1} and p_n+a_{n+1} is the even or odd value of the odd terms preceding b_{n+1} (Using p_n+a_{n+1} as it is instead of taking it mod 2 works here for the same reason it works in Peano's paper).

This defines the machine, and with the comments given above it can be seen to be the same definition as that given by Peano for his function.

Eilenberg also gave a way that the action of his machine may be pictured. It is as follows:

The four states of the machine represent the four symmetries of the square:

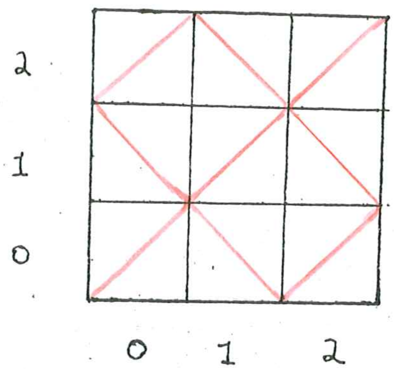
fig 1



Thus, two symmetries represent the even or odd value of the sum of the even terms and two symmetries represent the even or odd value of the sum of the odd terms.

At the first stage of the machine's action, after having read a b , the unit square may be pictured as follows:

fig 2



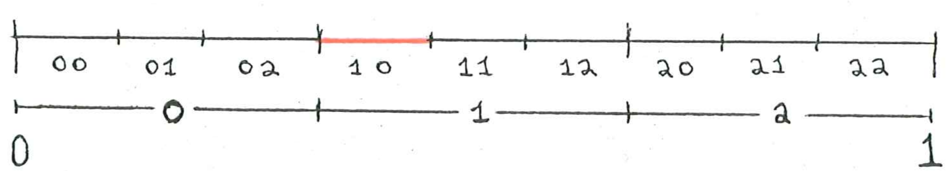
The red line shows the "path" that is to be taken in constructing the picture, beginning at the origin.

It should be noted that each of the nine blocks represents a symmetry of the square.

This picture may be thought of as representing the first approximation of a sequence of approximations to to Peano's mapping.

What the machine can be pictured as doing is the following: Reading two digits of a number written in base 3 designates a 1/9th segment of the unit interval. For example, the unit interval broken up to base 3 segments looks like this:

fig 3



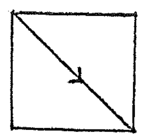
That is, .10 designates the red interval, $1/9^{th}$ of the unit interval.

Each nineth of the interval is mapped into one of the nine blocks of the unit square given in figure 2. Keeping with the same example, the segment .10 is mapped as follows:

The state function M gives which symmetry of the square the image will be. We have then

$M(s_0,(a_1,b_1)) = M((0,0),(1,0)) = (0+1(\text{mod } 2),0+0(\text{mod } 2)) = (1,0).$

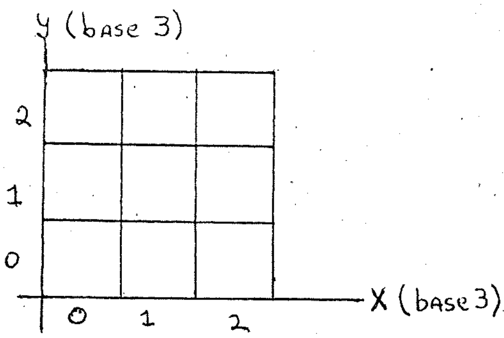
Thus, from figure 1, we know the image is the symmetry



(1,0)

Where to put this symmetry is determined by the output function U. The unit square may be thought of as set on the x,y axes of the co-ordinate plane,

fig 4



Thus, the x term of the output function U gives the x co-ordinate and the y term gives the y co-ordinate.

In our example, .10 is mapped as follows:

$U((p_0,q_0),(a_1,b_1)) = U((0,0),(1,0)) = (k^0 1,k^{0+1} 0) = (1,2).$

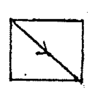
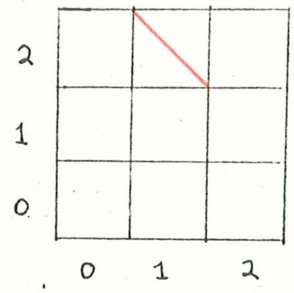
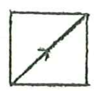
This means we put  in the block designated by (1,2):

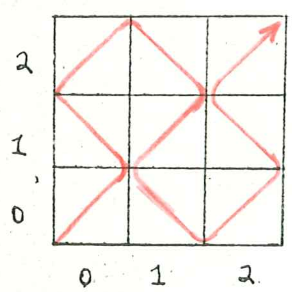
fig 5



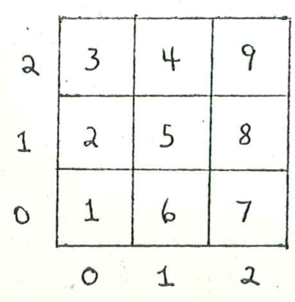
This can be seen to fit the original drawing in figure 2.

The action of the state of the machine in the output function U may be thought of in the following way: The state the machine is in is a symmetry of the square and gives an orientation to the square.

This orientation may be thought of as giving the order in which the subdivisions of the square come. For example, the machine begins in the initial state (0.0) which corresponds to the symmetry . This symmetry gives the following orientation to the blocks in the unit square:



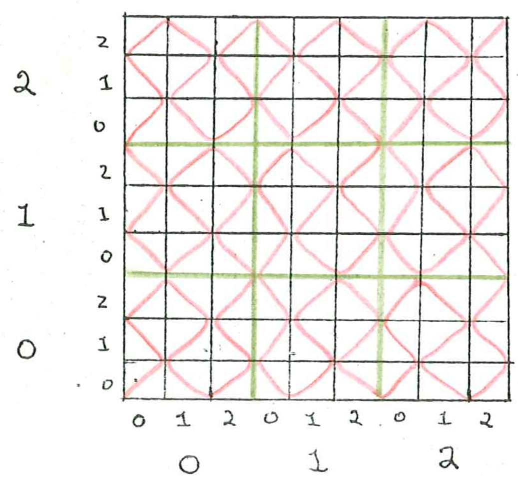
Thus, when the output function U is determining which block the next symmetry is to go in, it can be thought of as considering the blocks in this order:



With different symmetries the blocks will be considered in a different order and the function k (in the output function U) will operate so as to reflect this. The necessity for this can be seen more easily in the further discussion of the operation of the machine below.

When the first step of the machine is done for every 1/9th segment of the unit interval I one obtains the entire picture of figure 2. At the second stage of the machine's action, after it has done the mapping of the next two digits a_2b_2, the unit square may be pictured as follows:

fig 6

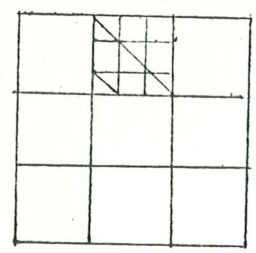


where each 1/9th block of this square is a copy of the original in figure 2 (block (0,0), for example), or a reflection of it (block (0,1), for example).

The process for finding the image of a_2b_2 is the same as before. The symmetry and the location tuple at this second stage will apply to the block designated by the operations of the machine on a_1b_1 in the same sequence. In our above example, this means the symmetry and location results for a_2b_2 will apply to block (1,2) in figure 6. After the second stage we have two digits each for X and Y. For example, X = .12 and Y = .20 represents the blue segment in figure 6.

It should be noted here that figure 6 is obtained by running the machine through two stages on 81 different sequences. Running the machine through two stages on a given sequence would produce a picture like this, showing the segments from the first and second stages:

fig 7

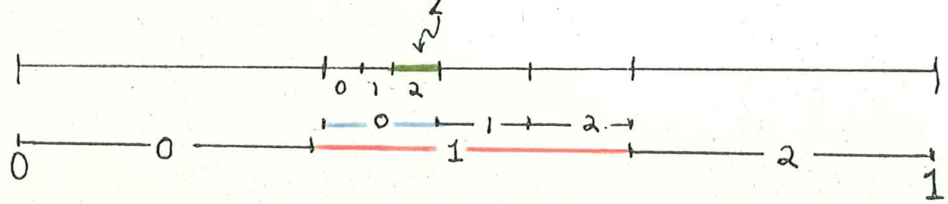


Successive stages of the machine proceed similarly; that is, each stage n+1 breaks up each block b_{n_i} of the previous stage into 9 blocks. Block b_{n_i} then takes on the appearance of figure 2 or a reflection of it.

Here one can intuitively see how the image point of a given point z in the unit interval is obtained. Using the sequential representation of a tape z*, the machine generates a nested sequence of squares whose diameters go to zero. Call the first square generated at the first stage s_1, the second square generated at the second stage s_2, etc, with

*The sequential representation of a tape z, when taken term by term, is just approximating z with successively smaller segments. For example, the approximation of z = .102... looks like this:

fig 8



Each square s contains a segment of the function f , the i^{th} approximation to the square-filling curve. Thus, one gets those segments of the approximating functions f which converge to the image point $\lim_{i\to\infty} f_i(z)$.

Returning to the machine action again, it should be noted what the effect of the function k is in the picture. In the first stage, k keeps track of whether one is moving up or down the vertical columns of the subdivided square of figure 2. That is, for x co-ordinate 0, the y co-ordinates come in the same order as they do on the unit interval; however, for x co-ordinate 1, the y co-ordinates come in reverse order since the function f_1 is then moving down the axis. Upon examining the function U, it can be seen that for $a_1 = 1$ the function k acts on the y co-ordinate, changing it to its compliment. Also note that k does not act on the y co-ordinate for $a_1 = 0$ or $a_1 = 2$.

On successive stages of the machine's action, k must also keep track of whether or not the basic block orientation of figure 2 has been reflected, thereby changing the order the x or y co-ordinates must come in. This means that at any given stage the machine must keep track of what has happened before. For this reason, the machine must keep track of the even or odd value of the <u>sum</u> of the even and odd terms rather than just of the particular term being read at any given stage of the machine's action. The stages (p,q) of the machine, then, store up a record of what has happened in the previous action of the machine. The fact should be noted that only a finite number of distinct things can happen in this machine even though it generates infinitely many different functions.

Eilenberg's machine gives another way that Peano's function may be thought of and proved to be a continuous mapping of I onto I^2. This is a rigorous development of the intuitive idea I suggested before. We noted that at each stage of the machine we got successive approximations $f_1, f_2, \ldots$ of Peano's function. Each f_i is continuous and is a function from a complete metric space to a complete metric space. The sequence $\{f_n\}$ can easily be shown to be uniformly cauchy. Let $f(x) = \lim_{n\to\infty} f_n(x)$ for all x in I.

It can then be shown in succession that:

(1) f exists

(2) $\{f_n\}$ converges uniformly to f

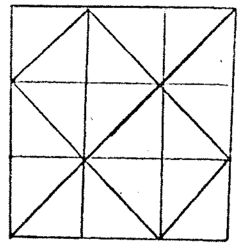
(3) f is continuous

(4) The image of f is dense in I^2 implies the image of f is equal to I^2

These are shown in the following way:

It must first be shown that $\{f_n\}$ is uniformly cauchy. It can be seen that at any stage n of the machine's action, where we have f_n, each block is of diameter $d_n = (1/3^n)\sqrt{2}$. Therefore, from the construction of the function f_n, no point of the unit square is farther than $(1/3^n)\sqrt{2}$ from some part of the function.

fig 9



First stage, $d_n = (1/3)\sqrt{2}$

Thus, every point of f_{n+1} is within $(1/3^n)\sqrt{2}$ of some point of f_n, giving $d(f_n, f_{n+1}) < (1/3^n)\sqrt{2}$. Since this holds for any n, this implies that for any ϵ there exists an N such that for n and m greater than N,

$$d(f_n(x), f_m(x)) < \epsilon \quad \text{for all x in I.}$$

This means that $\{f_n\}$ is uniformly cauchy.

(1) The $\lim_{x\to\infty} f_n(x)$ exists for all x in I because I^2 is a complete metric space, which means that each cauchy sequence in the space converges.

(2) $\{f_n\}$ converges uniformly to f because $\{f_n\}$ is a uniformly cauchy sequence. The following theorem, which may be found in Buck, is used:

Theorem: Given the sequence $\{f_n\}$ of functions defined on E. If $\lim_{\substack{n,m\to\infty \\ x\in E}} |f_n(x) - f_m(x)| = 0$, then there is a function F to which the sequence $\{f_n\}$ converges uniformly on E.

(3) f is continuous due to the following theorem, also from Buck:

Theorem: If $\{f_n\}$ converges to F, uniformly on E, and each function f_n is continuous on E, then F is continuous on E.

(4) Im f dense in I^2 implies Im f = I^2.

Three lemma's are needed to prove this.

Lemma 1. The continuous image of a compact metric is compact.

Lemma 2. A compact subset of a metric space is closed.

Lemma 3. In a compact metric space, a closed set is equal to its closure.

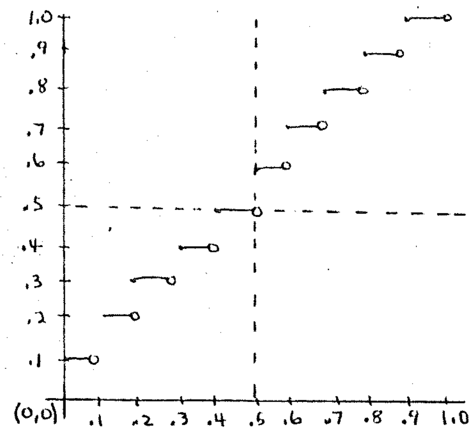
It must first be shown that Im f is dense in I^2. This is because the diameter of the blocks in the unit square goes to zero. Thus, given an open neighborhood S_ϵ of diameter ϵ in the unit square, there exists an N such that for n > N the diameter of the blocks at stage n of the machine's action is less than $(1/3)\epsilon$. This means that S_ϵ contains one of the blocks at every stage n > N, which implies that S_ϵ contains a point of Im f. Thus, Im f is dense in I^2.

Now (4) can be shown. The Im f is a continuous image of the compact set I, so Im f is compact by Lemma 1. This means that Im f is closed by Lemma 2; in fact, then, it is equal to its closure by Lemma 3. But the closure of a set dense in I^2 is I^2. Therefore, we have Im f = Cl(Im f) = I^2, which was to be shown.

Thus, it has been shown that f, defined as the limit of the sequence of functions defined by Eilenberg's machine, is continuous, and its image is I^2.

A simple example of another application of the sequential machine is in generating a step function.

Consider the step function determined by this graph:



The function may be described by the following machine:

$\Sigma = \{1,2,3,4,5,6,7,8,9,0\}$

$S = \{0,1\}$

$s_0 = 1$

$M(s,\sigma) = 0$

$U(s,\sigma) = s\sigma + s$

There is only one state change in this machine. This is because only the first term of a sequence is needed to determine its image under the step function. All the remaining terms are disregarded with the machine in the "zero" state.

For another, more complex example, consider the following:

The Peano function raised one dimension to two. Cantor's function is also a dimension raising function, raising the zero-dimensional Cantor's set to one dimension,

$$f_c : C \longrightarrow I$$

where f_c denotes Cantor's function and C denotes the Cantor ternary set, and I the unit interval.

To write a machine that will perform this function, using a mapping technique due to J. L. Kelley, take the triadic expansions of the elements of C and the dyadic expansions of the elements of I. We then have

$$f_c : C^{\text{triadic}} \longrightarrow I^{\text{dyadic}}$$

The machine is given as follows:

$\Sigma = \{0,1,2\}$

$T = \{x \mid x = \sigma_1\sigma_2 \ldots \sigma_i \in \{0,2\}\}$

$S = s_o$

$s_o = s_o$

$M(s,\sigma) = s_o$

$U(s,\sigma) = \sigma/2$

In this machine there is only one state, the initial state. This is because the machine has only one distinct action, and it performs it the same at each stage regardless of what it has done before.

To see the function of the machine, take

$$x \in C^{\text{triadic}},\ x = a_1a_2a_3\ldots = .\frac{a_1}{3}+\frac{a_2}{3^2}+\ldots$$

The output function U divides each of the terms of x by 2, giving the terms of the dyadic expansion. Thus,

$$f(x) = (\tfrac{1}{2})(\tfrac{a_1}{2} + \tfrac{a_2}{2^2} + \tfrac{a_3}{2^3} + \cdots),$$

giving the number whose dyadic expansion is $.\frac{a_1}{2}\frac{a_2}{2}\frac{a_3}{2}\ldots$

For example, let x = .022020...(triadic expansion). Then

$$f(x) = \tfrac{1}{2}\left(\tfrac{0}{2} + \tfrac{2}{2^2} + \tfrac{2}{2^3} + \tfrac{0}{2^4} + \tfrac{0}{2^5} + \cdots\right)$$

$$= .011010\ldots \text{(dyadic expansion)}.$$

It can be seen that whenever $0 < x-y < 1/3^k$ then $0 < f(x)-f(y) < 1/2^{k+1} + 1/3^k$, showing that this mapping is continuous. f is onto since for any z in I, $z = .b_1 b_2 b_3 \ldots$ (dyadic expansion), we have f(x) = z where x is in C and $x = .(2b_1)2b_2)\ldots$(triadic expansion). This machine, then, gives Cantor's function.