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INFINITE GRAPHS AND BICIRCULAR MATROIDS

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B-matroids are a class of pre-independence spaces which retain many important properties of independence spaces. Higgs has shown that an infinite generalization of the cycle matroid of a finite graph which admits two-way infinite paths as circuits need not be a B-matroid. In this note it is shown that a similar generalization of the finite bicircular matroid is always a B-matroid.

1. Infinite matroids and graphs

The operator approach to infinite matroids has been examined by Klee [5] and Higgs [1–3] and the link between their investigations is discussed in [7]. As far as possible Klee's terminology will be followed here and, where necessary, results of Higgs will be restated.

Let S be a non-empty set and f be a function from 2^S, the power set of S, into 2^S. Then f is an *operator* on S if and only if the following conditions are satisfied for all $X \subseteq Y \subseteq S$.

(i) $X \subseteq f(X)$.

(ii) $f(X) \subseteq f(Y)$.

If f is an operator on S, define a function f^* by $f^*(X) = X \cup \{x : x \notin f(S \backslash (X \cup x))\}$ for all $X \subseteq S$, where a one-element subset $\{x\}$ of S is denoted x. Then f^* is an operator on S called the *dual* of f, and $(f^*)^* = f$. If $T \subseteq S$ and f_T is defined by $f_T(X) = f(X) \cap T$ for all $X \subseteq T$, then f_T is an operator on T called the *restriction* of f to T.

The closure operator of a finite matroid is an operator in the above sense. Such operators satisfy certain additional conditions including the following, which hold for all elements p, x of S and all subsets X, Y of S:

(wI) If $x \in f(Y)$, then $f(x \cup Y) = f(Y)$.

(I) If $X \subseteq f(Y)$, then $f(X \cup Y) = f(Y)$.

(E) If $p \in f(Y)$ and $p \notin f(Y \backslash X)$, then $x \in f((Y \backslash x) \cup p)$ for some x in X.

Any operator satisfying (wI) and (E) will be called simply a wIE-operator. Similar abbreviations will be used for other types of operators.

Note that (I) implies (wI) and that (I) is equivalent to the condition that

$f(f(Y)) = f(Y)$. Moreover, (I) and (E) form a dual pair in the sense that f satisfies (I) if and only if f^* satisfies (E).

Again, for f an operator on S and X a subset of S, X is *f-independent* provided that, for all x in X, $x \notin f(X \backslash x)$; X is *f-spanning* if $f(X) = S$ and X is an *f-base* if X is both f-independent and f-spanning. An *f-circuit* is a minimal f-dependent subset of S. The operator prefix in the above terms will be dropped where ambiguity will not arise.

A *B-matroid*, (S, f), is an IE-operator f on a non-empty set S such that, for all $T \subseteq S$, every f_T-independent subset of T is contained in an f_T-base. If (S, f) is a B-matroid, then so is (S, f^*), the dual of (S, f) [1]. Although B-matroids need not have finite circuits, they retain many familiar properties of independence spaces (see [1] and [2]).

This note will be concerned with wIE-operators satisfying the further condition:

(C) If $p \in f(Y)$ and $p \notin Y$, then there is a circuit X contained in $Y \cup p$ such that $p \in X$.

From [1], if (S, f) is a B-matroid, then f is a wIEC-operator on S. The next result [5] characterizes wIEC-operators in terms of their collections of circuits. This should be compared with a similar, more familiar result for independence spaces.

Theorem 1.1. *The collection Γ of circuits of a* wIEC-*operator satisfies the following conditions.*

(1.1) *No element of Γ properly contains another.*

(1.2) *If C_1 and C_2 are in Γ, $p \in C_1 \backslash C_2$ and $q \in C_1 \cap C_2$, then there is an element C_3 of Γ such that $p \in C_3 \subseteq (C_1 \cup C_2) \backslash q$.*

Conversely, if Γ is a collection of subsets of S satisfying (1.1) *and* (1.2), *then there is a unique* wIEC-*operator on S having Γ as its collection of circuits. This operator is given by*

$$\gamma_\Gamma(X) = X \cup \{x : x \in C \subseteq X \cup x \text{ for some } C \text{ in } \Gamma\} \text{ for all } X \subseteq S.$$

If, in the above, every element of Γ is finite, then the collection $\mathcal{I}$ of γ_Γ-independent subsets of S is an independence structure on S. Moreover, for all subsets X of S, $\gamma_\Gamma(X)$ is the *closure* or *span* (see, for example, [4]) of X in $(S, \mathcal{I})$.

Let G be an infinite undirected graph which we allow to have loops and parallel edges. We may use Theorem 1.1 to define four wIEC-operators on the edge set $E(G)$ of G.

Example 1.2. From the set $\mathscr{M}(G)$ of finite cycles of G, we obtain a wIEC-operator $\gamma_{\mathscr{M}(G)}$ on $E(G)$, and, as above, an independence structure — the familiar cycle matroid of an infinite graph.

Example 1.3. From the set of subgraphs of G homeomorphic to one of the graphs (a), (b) or (c) in Fig. 1, we obtain an independence structure on $E(G)$. This is an obvious extension of the bicircular matroid of a finite graph (see [8] or [6]).

The next two examples generalize the first two by admitting infinite circuits.

Example 1.4. (see Klee [5] or Higgs [3]). If $\mathscr{C}(G)$ consists of all finite cycles of G together with all two-way infinite paths, then $\gamma_{\mathscr{C}(G)}$ is a wIEC-operator on $E(G)$.

Example 1.5. (see Klee [5]). Let $\mathscr{B}(G)$ be the collection of subgraphs of G homeomorphic to one of the five graphs shown in Fig. 1 (where an arrow denotes a one-way infinite path). Then $\gamma_{\mathscr{B}(G)}$ is a wIEC-operator on $E(G)$ and $(E(G), \gamma_{\mathscr{B}(G)})$ will be called the *infinite-bicircular matroid* of G.

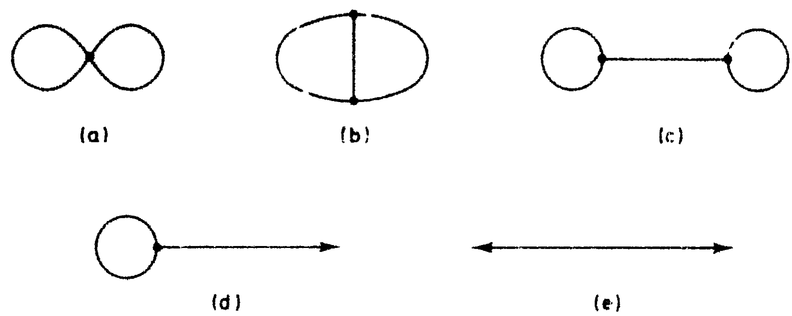

Fig. 1.

Higgs [3] has given an excluded subgraph characterization of graphs G for which $(E(G), \gamma_{\mathscr{C}(G)})$ is a B-matroid, showing further that $(E(G), \gamma_{\mathscr{C}(G)})$ is a B-matroid if and only if $\gamma_{\mathscr{C}(G)}$ is an IE-operator. These results prompt the questions as to when $\gamma_{\mathscr{B}(G)}$ is an IE-operator and when $(E(G), \gamma_{\mathscr{B}(G)})$ is a B-matroid. The main result of this note answers these questions. (A similar problem was posed by Simões-Pereira [9].)

Theorem 1.6. *Let G be a graph. Then the infinite-bicircular matroid $(E(G), \gamma_{\mathscr{B}(G)})$ is a B-matroid.*

2. Infinite-bicircular matroids and B-matroids

The proof of Theorem 1.6 will use two lemmas.

Lemma 2.1. *Let G be a graph. Then $\gamma_{\mathscr{B}(G)}$ is an* IE*-operator on $E(G)$.*

Proof. We need only show that $\gamma_{\mathscr{B}(G)}$ satisfies (I). In this proof we shall write γ for $\gamma_{\mathscr{B}(G)}$ and $\mathscr{B}$ for $\mathscr{B}(G)$.

Suppose that $X \subseteq E(G)$ and that $x \in \gamma(\gamma(X)) \setminus \gamma(X)$. Then $x \in C \subseteq \gamma(X) \cup x$ for some C in $\mathscr{B}$. We distinguish two cases: when $C \cap (\gamma(X) \setminus X)$ is finite, and when $C \cap (\gamma(X) \setminus X)$ is infinite. In each case we shall get a contradiction, thereby showing that $\gamma(\gamma(X)) \setminus \gamma(X)$ is empty and hence that $\gamma(\gamma(X)) = \gamma(X)$.

If $C \cap (\gamma(X) \setminus X)$ is finite and $y \in C \cap (\gamma(X) \setminus X)$, then, since $y \in \gamma(X) \setminus X$, there is an element C_y of $\mathscr{B}$ such that $y \in C_y \subseteq X \cup y$. Now $y \in C \cap C_y$ and $x \in C \setminus C_y$, hence, by (1.2), there is an element C' of $\mathscr{B}$ such that $x \in C' \subseteq (C \cup C_y) \setminus y$. Since $|C' \cap (\gamma(X) \setminus X)| < |C \cap (\gamma(X) \setminus X)|$, after finitely many steps we get an element $C^{(n)}$ of $\mathscr{B}$ such that $x \in C^{(n)} \subseteq X \cup x$. Therefore, $x \in \gamma(X)$; a contradiction.

Next suppose that $C \cap (\gamma(X) \setminus X)$ is infinite. Then C is homeomorphic to one of the graphs (d) or (e) in Fig. 1. Let the edge x have endpoints u and v. Then either there is an edge y in $C \cap (\gamma(X) \setminus X)$ and a finite path P_y in $X \cap C$ joining u to an endpoint of y, or there is no such path in $X \cap C$ from u to an edge of $C \cap (\gamma(X) \setminus X)$. In the latter case, $X \cap C$ contains either a one-way infinite path from u, or a cycle joined to u by a path of finite (possibly zero) length. In the former case, since $y \in \gamma(X) \setminus X$, there is an element C'_y of $\mathscr{B}$ such that $y \in C'_y \subseteq X \cup y$. But both P_y and $C'_y \setminus y$ are contained in X. A routine check of the five possibilities for C'_y shows that this time X contains either a one-way infinite path from u, or a cycle joined to u by a finite path in X.

The argument for u may be repeated for v and then a check of the various possibilities (including the case $u = v$) gives that $\mathscr{B}$ contains an element C_x containing x such that $C_x \subseteq X \cup x$. Thus, when $C \cap (\gamma(X) \setminus X)$ is infinite, $x \in \gamma(X)$; the required contradiction.

The following result is proved by Higgs [3, (5)(i)].

Lemma 2.2. *Let G be a graph. Then every $\gamma_{\mathscr{C}(G)}$-independent set is contained in a $\gamma_{\mathscr{C}(G)}$-base.*

We now complete the proof of the main theorem.

Proof of Theorem 1.6. By Lemma 2.1, $\gamma_{\mathscr{B}(G)}$ is an IE-operator. Since every restriction of $(E(G), \gamma_{\mathscr{B}(G)})$ is isomorphic to $(E(F), \gamma_{\mathscr{B}(F)})$ for some subgraph F of G, it suffices to show that every independent set of $(E(G), \gamma_{\mathscr{B}(G)})$ is contained in a base.

Suppose that A is a $\gamma_{\mathscr{B}(G)}$-independent subset of $E(G)$ and let $\{A_k : k \in K\}$ be the set of connected components of the subgraph of G induced by A. Let

$$H = \{k \in K : A_k \text{ contains a cycle of } G\}.$$

Modify G to obtain a graph G' as follows. For each h in H, using the Axiom of Choice, select an edge e_h of the cycle contained in A_h; delete the edge e_h and insert a one-way infinite path P_h (of new vertices and edges) from an endpoint of e_h. Let

$$A'_h = (A_h \setminus e_h) \cup P_h \quad \text{for each } h \text{ in } H, \text{ and let}$$

$$A' = \Big(\bigcup_{h \in H} A'_h\Big) \cup \Big(\bigcup_{k \in K \setminus H} A_k\Big).$$

Now apply Lemma 2.2 in G'. As A' is a $\gamma_{\mathscr{C}(G')}$-independent set, there is a $\gamma_{\mathscr{C}(G')}$-base T' containing A'. Let T be the subgraph of G corresponding to T', that is, T is obtained from T' by deleting the path P_h and inserting the edge e_h for each h in H.

By construction, T is $\gamma_{\mathscr{B}(G)}$-independent, and moreover, if e is an edge of G joining vertices in distinct components of the subgraph T, then $T \cup e$ is $\gamma_{\mathscr{B}(G)}$-dependent. Let $\{T_i : i \in I\}$ be the set of components of T and $J = \{i \in I : T_i$ contains no finite cycle or one-way infinite path$\}$. Now recall that $\gamma_{\mathscr{M}(G)}(T_i)$ is the closure of T_i in the cycle matroid of G (see Example 1.2). If $j \in J$ and $\gamma_{\mathscr{M}(G)}(T_j) \backslash T_j$ is non-empty, then a single element from $\gamma_{\mathscr{M}(G)}(T_j) \backslash T_j$ may be added to T_j without forming a $\gamma_{\mathscr{B}(G)}$-circuit. For each such j, using the Axiom of Choice, select such an element f_j. Finally let

$$B = T \cup \{f_j : j \in J \text{ and } \gamma_{\mathscr{M}(G)}(T_j) \backslash T_j \text{ is non-empty}\}.$$

It is easy to check that B is $\gamma_{\mathscr{B}(G)}$-independent and $\gamma_{\mathscr{B}(G)}$-spanning; hence, as required, B is a $\gamma_{\mathscr{B}(G)}$-base containing A.

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